A TORSION PROJECTIVE CLASS FOR A GROUP ALGEBRA

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1. *Introduction*

Let G be the group given by the following presentation:

$$G = \langle a, b, c \ : \ a^2 = b^2 = c^3 = 1, \ c^a = c^b = c^2 \rangle. \tag{1.1}$$

The subgroup generated by ab is infinite-cyclic and is normal, with quotient the dihedral group of order 6, and so G is cyclic-by-finite. The subgroups $H = \langle a, c \rangle$ and $K = \langle b, c \rangle$ are both dihedral of order 6, and G is isomorphic to the free product of H and K amalgamating $L = H \cap K$. We study $K_0(kG)$, the Grothendieck group of isomorphism classes of finitely generated projective kG-modules, and in particular the dependence of $K_0(kG)$ on the choice of field k. As usual, let $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ stand for the rationals, reals and complex numbers respectively. We prove

THEOREM 1. *There is an element of order two in $K_0(\mathbb{Q}G)$, whose image in $K_0(\mathbb{R}G)$ is non-zero, but whose image in $K_0(\mathbb{C}G)$ is zero.*

There are two published accounts of groups G and fields k for which $K_0(kG)$ contains torsion. P. H. Kropholler and B. Moselle exhibited crystallographic groups G for which $K_0(kG)$ contains elements of order two, for any field k of characteristic zero [**3**]. They used F. Waldhausen's work on the algebraic K-theory of free products with amalgamation [**11**]. More recently, M. Lorenz has exhibited crystallographic groups for which $K_0(kG)$ contains 3- and 4-torsion for any field k of characteristic zero [**5**]. Lorenz's techniques appear to be unable to detect torsion that is annihilated by field extensions. Moselle's Ph. D. thesis contained examples of crystallographic groups with torsion in $K_0(\mathbb{Q}G)$ but not in $K_0(\mathbb{C}G)$ [**7**], but relied on a theorem of F. Quinn [**8**] for which no full proof has appeared.

It seems to have gone unnoticed, or at least unremarked, that results due to G. M. Bergman [**1**], W. Dicks [**2**], and F. Waldhausen [**11**] can be used to exhibit groups G having torsion of any order in $K_0(kG)$, and torsion in $K_0(\mathbb{Q}G)$ that dies in $K_0(\mathbb{C}G)$. Examples of this kind are to be given in [**4**]. Our purpose here is to give a topological proof of Theorem 1, using the following theorem of R. G. Swan:

THEOREM 2 (Swan, [**10**]). *Let X be a compact Hausdorff space, and let R be the ring of real-valued continuous functions on X. The functor from real vector-bundles over X to R-modules taking a bundle to its sections induces an isomorphism from the real topological K-group $KO^0(X)$ to the algebraic K-group $K_0(R)$.*

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We also use the Morita invariance of K_0: for any R, the non-unital ring homomorphism including R in the top left corner of the matrix ring $M_n(R)$ induces an isomorphism from $K_0(R)$ to $K_0(M_n(R))$. The class of the free R-module of rank one is mapped by this isomorphism to the class of the module of column vectors, R^n. As a general reference for K-theory, including proofs of Swan's theorem and Morita invariance, we recommend [**9**].

2. *The proof*

Throughout this section, G stands for the group presented in equation 1.1, and H, K are the subgroups generated by $\{a, c\}$ and $\{b, c\}$ respectively.

Proof. The groups H and K are both isomorphic to the dihedral group of order 6, so have 3 isomorphism classes of irreducible representations over $\mathbb{Q}$. Let α (resp. α') denote the faithful irreducible 2-dimensional representation of H (resp. of K). The element of $K_0(\mathbb{Q}G)$ whose existence is claimed in Theorem 1 is ξ defined by

$$\xi = \mathrm{Ind}_H^G(\alpha) - \mathrm{Ind}_K^G(\alpha'). \tag{2.1}$$

In terms of idempotents, α may be represented by $\frac{1}{6}(1 + a)(2 - c - c^2)$ in $\mathbb{Q}H$, and $\mathrm{Ind}_H^G(\alpha)$ by the same element in $\mathbb{Q}G$. Similarly, α' and $\mathrm{Ind}_K^G(\alpha')$ are represented by the element $\frac{1}{6}(1 + b)(2 - c - c^2)$.

Over $\mathbb{C}$, there is a 1-dimensional faithful representation γ of $L = H \cap K = \langle c \rangle$ such that $\alpha = \mathrm{Ind}_L^H(\gamma)$ and $\alpha' = \mathrm{Ind}_L^K(\gamma)$. It follows that in $K_0(\mathbb{C}G)$,

$$\xi = \mathrm{Ind}_H^G(\alpha) - \mathrm{Ind}_K^G(\alpha') = \mathrm{Ind}_L^G(\gamma) - \mathrm{Ind}_L^G(\gamma) = 0. \tag{2.2}$$

The same argument applies over any field of characteristic zero containing ω, a primitive cube root of 1. In terms of idempotents, the idempotents representing α and α' are both conjugate to $\frac{1}{3}(1 + \omega c + \omega^2 c^2)$ in $\mathbb{C}G$.

The representation γ is not defined over every field of characteristic zero, but there is a 2-dimensional representation β of L defined over $\mathbb{Q}$ such that $2\alpha = \mathrm{Ind}_L^H(\beta)$ and similarly for α'. It follows that in $K_0(\mathbb{Q}G)$,

$$2\xi = 2\mathrm{Ind}_H^G(\alpha) - 2\mathrm{Ind}_K^G(\alpha') = \mathrm{Ind}_L^G(\beta) - \mathrm{Ind}_L^G(\beta) = 0. \tag{2.3}$$

It remains to show that the image of ξ in $K_0(\mathbb{R}G)$ is non-zero. Let R denote the ring of continuous functions from the circle $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$. The matrix ring $M_2(R)$ may be identified with the ring of continuous functions from the circle to $M_2(\mathbb{R})$. Define a function ϕ from the generators of G to $M_2(R)$ by

$$\begin{aligned}
\phi(a) &= \left(t \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \\
\phi(b) &= \left(t \mapsto \begin{pmatrix} \cos 2\pi t & \sin 2\pi t \\ \sin 2\pi t & -\cos 2\pi t \end{pmatrix}\right) \\
\phi(c) &= \left(t \mapsto \begin{pmatrix} \cos 2\pi/3 & -\sin 2\pi/3 \\ \sin 2\pi/3 & \cos 2\pi/3 \end{pmatrix}\right)
\end{aligned} \tag{2.4}$$

The group relations are satisfied by the images, and ϕ extends $\mathbb{R}$-linearly to a homomorphism $\phi : \mathbb{R}G \to M_2(R)$. View $M_2(R)$ as the endomorphism ring of R^2, and view R^2 as the sections of a 2-dimensional trivial vector bundle over the circle. The idempotents representing α and α' are sent by ϕ to a projection onto a 1-dimensional trivial sub-bundle and to a projection onto a Möbius band sub-bundle respectively. It follows that α and α' represent distinct elements of $K_0(\mathbb{R}G)$, since the Möbius bundle over the circle is not stably trivial [**6**].

The above argument shows that $\xi \in K_0(kG)$ is non-zero when k embeds in $\mathbb{R}$, and is zero when k contains a primitive cube root of 1. There are subfields of $\mathbb{C}$ that satisfy neither of these conditions. An argument using either Bergman's or Waldhausen's exact sequence for the K-theory of free products [**1**, **11**] shows that $\xi \in K_0(kG)$ is zero if and only if k contains a primitive cube root of 1 [**4**].

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