

# TORSION IN LOOP SPACE HOMOLOGY OF RATIONALLY CONTRACTIBLE SPACES

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ABSTRACT. Let  $R$  be a torsion free principal ideal domain. We study the growth of torsion in loop space homology of simply-connected  $DGR$ -coalgebras  $C$ , whose homology admits an exponent  $r$  in  $R$ . Here by loop space homology we mean the homology of the loop algebra construction on  $C$ . We compute a bound on the growth of torsion in such objects and show that in general this bound is best possible. Our methods are applied to certain simply-connected spaces associated with classifying spaces of finite groups, where we are able to deduce the existence of global exponents in loop space homology.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

The growth of torsion in loop space homology is known to be rather hard to control in general. A remarkable example of this was discovered by D. Anick [2], who introduced a finite dimensional CW complex  $X$  with the property that the integral loop space homology  $H_*(\Omega X; \mathbb{Z})$  has torsion of all orders. A similar example was also constructed by L. Avramov [3]. On the other extreme, Y. Felix, S. Halperin and J-C. Thomas showed in [6] that if  $Y$  is a space of finite Lusternik-Schnirelmann category, with finite type  $\mathbb{Z}_{(p)}$ -homology and such that the dimension of the graded vector space  $H_*(\Omega Y; \mathbb{F}_p)$  grows at most polynomially, then the torsion in  $H_*(\Omega Y; \mathbb{Z}_{(p)})$  has an exponent, namely there exists an integer  $r$  such that  $p^r$  annihilates all the torsion in the  $p$ -local loop space homology of  $Y$ .

The spaces constructed by both Anick and Avramov have the additional feature that their integral homology is torsion free. The theorem in [6] on the other hand is concerned with neither spaces of infinite category nor with such, where the mod- $p$  loop space homology growth rate is larger than polynomial.

The motivation for this paper comes from considering spaces of the form  $BG_p^\wedge$ , where  $G$  is a finite  $p$ -perfect group and by  $(-)_p^\wedge$  we denote the  $\mathbb{F}_p$ -completion functor of Bousfield and Kan [4]. In [10] we show that for such groups  $G$ , the integral loop space homology of  $BG_p^\wedge$  has an exponent, given by the order of the Sylow  $p$ -subgroup of  $G$ . It is always the case that  $BG_p^\wedge$  has infinite category when  $G$  is finite and in [11, 12] we present examples of finite  $p$ -perfect groups  $G$  such that the mod- $p$  loop space homology of  $BG_p^\wedge$  grows exponentially. Thus, in general, spaces of this form do not fit in the context of the theorems mentioned above.

We specialize to rationally contractible spaces with a reduced integral homology exponent and attempt to understand the growth of torsion in their loop space homology. Clearly

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*Date:* March 31 1995.

*1991 Mathematics Subject Classification.* Primary 55R35, Secondary 55R40, 55Q52.

The author is supported by a DFG grant.

one should not expect an exponent result in general, even when restricting to this family of spaces. For example the integral homology of the double loop space on a mod- $p^r$  Moore space has  $p$ -torsion of arbitrarily high order, even though the homology of the single loop space has an exponent  $p^r$ . Indeed, our study here may suggest that it is hardly ever the case that an infinite dimensional rationally contractible space  $X$ , which admits a homology exponent, still has one in its loop space homology. We refer the reader to Section 9 below for further discussion.

In the preceding we will abbreviate the words “differential graded” by the letters  $\mathcal{DG}$ . Let  $R$  be a commutative ring with a unit. The major tool we use is the loop algebra functor  $\Omega(-)$  from the category of  $\mathcal{DGR}$ -coalgebras to  $\mathcal{DGR}$ -algebras [7] (this functor is more commonly known as the Adams’ cobar construction [1]). We shall generally assume that the ground ring  $R$  is a torsion free principal ideal domain (by torsion free we mean that the natural map from the integers to  $R$  is injective). For every  $\mathcal{DGR}$ -module, the underlying graded  $R$ -module is assumed to be dimension-wise free and graded by the non-negative integers. A  $\mathcal{DGR}$ -module  $(M; d)$  is said to have a homology exponent if there exists a non-zero element  $r \in R$ , such that  $r \cdot H_i(M; d) = 0$  for all  $i \geq 0$ . All topological spaces considered in this note are assumed to be pointed, connected and to have the homotopy type of a CW complex. For precise definitions of the terminology used below, we refer the reader to section 2.

For a supplemented  $\mathcal{DGR}$ -coalgebra  $C$ , let  $JC$  denote the augmentation coideal of  $C$ . For a supplemented  $\mathcal{DGR}$ -algebra  $A$ , let  $IA$  denote the augmentation ideal of  $A$ . A coalgebra  $C$ , as above, is said to have a homology exponent  $r \in R$  if  $r$  is an exponent for  $H_*(JC)$ . Similarly one defines what it means for a supplemented  $\mathcal{DGR}$  algebra to have a homology exponent.

**Theorem 1.1.** *Let  $C$  be a  $c$ -connected  $\mathcal{DGR}$ -coalgebra,  $c \geq 1$ . Suppose that  $C$  admits a homology exponent  $r \in R$ . Let  $d$  and  $e$  be positive integers such that  $e \geq \frac{d-c+1}{c}$ . Then  $r^e$  annihilates  $H_i(\Omega C)$  for each  $1 \leq i \leq d$ .*

We construct an example over the integers, which shows that in general the approximation given by Theorem 1.1 is best possible. Indeed if  $C$  is precisely  $(2k - 1)$ -connected, for some positive integer  $k$  and has a homology exponent  $r \in R$ , then the theorem claims that  $r^{e-1}$  annihilates  $H_i(\Omega C)$  for  $1 \leq i \leq e(2k - 1) - 1$ . Thus the least  $d$ , for which  $r^{e-1}$  might fail to be an exponent for  $H_d(\Omega C)$ , is  $d = e(2k - 1)$ .

**Theorem 1.2.** *For each pair of positive integers  $t$  and  $k$ , there exists a  $(2k - 1)$ -connected  $\mathcal{DGZ}$ -coalgebra  $C^{k,t}$  with a homology exponent  $t$ , such that for every positive integer  $e$ ,  $H_{e(2k-1)}(\Omega C^{k,t})$  contains an element of order  $t^e$ .*

An interesting fact about the  $\mathcal{DGZ}$ -coalgebra  $C^{k,t}$  is that its dual algebra  $(C^{k,t})^*$  is commutative and nilpotent of degree 3, in the sense that the third power of its augmentation ideal vanishes. Indeed, it is a triviality that if the dual  $\mathcal{DGR}$ -algebra  $C^*$  of a  $\mathcal{DGR}$ -coalgebra  $C$  is nilpotent of degree 2 (i.e.  $C$  has a trivial diagonal) and  $C$  has a homology exponent  $r$  then so does  $\Omega C$ . Our example shows that this fails to hold once the nilpotency assumption is relaxed.

It is known and in fact admits a rather easy proof that if  $X$  is a rationally contractible finite dimensional CW complex, which admits an integral homology exponent (trivially it

does if it has only finitely many cells), then  $\Omega X$  admits an integral homology exponent. However the usual proof, using the Serre spectral sequence for the path-loop fibration over  $X$ , fails to give the best approximation. The methods used in the proof of Theorem 1.1 enables us to improve the existing bound.

**Proposition 1.3.** *Let  $C$  be a  $c$ -connected  $\mathcal{DGR}$ -coalgebra of homological dimension  $d$  with a homology exponent  $r \in R$ . Then  $r^e$  annihilates  $H_i(\Omega C)$  for every  $e \geq \frac{d-c-1}{c}$  and  $i > 0$ .*

Our bound here should be compared with the corresponding result using the Serre spectral sequence which gives  $r^{d-c}$  as a bound, under the same hypotheses. At the end of Section 5 below, we consider finite sub  $\mathcal{DGR}$ -coalgebra of those constructed in Theorem 1.2, for which the bound given in Proposition 1.3 is best possible.

Next we study an application of the techniques developed here. Recall the construction  $P_n(-)$  on  $\mathcal{DGR}$ -coalgebras given in [10]. Given any  $\mathcal{DGR}$ -coalgebra  $C$ , we produce an  $n$ -connected quotient  $\mathcal{DGR}$ -coalgebra  $P_n(C)$ , such that the natural projection from  $C$  to  $P_n(C)$  induces an isomorphism in homology above dimension  $n$ . This algebraic construction, being motivated by the Quillen “plus” construction, admits a topological analogue, which in a sense is an easy generalization of Quillen’s idea.

**Proposition 1.4.** *Let  $X$  be a connected CW-complex and let  $n$  be a positive integer. Then there exists an  $n$ -connected CW-complex  $P_n X$  together with a map  $\iota_n : X \rightarrow P_n X$ , inducing an isomorphism on integral homology above dimension  $n$ . Furthermore, if  $C = S_*(X)$ , the integral singular chain coalgebra on  $X$ , then  $P_n(C)$  and  $S_*(P_n X)$  are quasi isomorphic as  $DG\mathbb{Z}$ -coalgebras.*

With the terminology of Proposition 1.4 we now restrict attention to classifying spaces  $BG$ , where  $G$  is a finite group.

**Theorem 1.5.** *Let  $G$  be a finite group of order  $N$ . Then  $\tilde{H}_*(\Omega P_n B G; \mathbb{Z})$  is annihilated by  $N^q$ , where  $q = 3$  if  $n = 1$  and  $q = 2$  otherwise.*

If  $G$  is a finite  $p$ -perfect group then  $(P_1 B G)_p^\wedge$  is homotopy equivalent to  $BG_p^\wedge$ . Thus Theorem 1.5 is, in a sense, a generalization of [10, I, thm. 1], although in the last the exponent is given by the order of the group rather than a power of it.

All the results above are motivated by topology. However the loop algebra functor is a purely algebraic construction and so one could consider loop space homology torsion in a context that might have nothing to do with topology. We look at the loop algebra functor  $\Omega(-)$  together with its left adjoint, the classifying coalgebra functor  $B(-)$  [7]. The adjunct morphisms here are equivalences. Thus one could consider a  $\mathcal{DGR}$ -algebra  $A$ , for which the classifying construction  $B(A)$  admits a homology exponent and ask whether the same holds for  $A$  itself. An example of this is given by the following.

**Theorem 1.6.** *Let  $A$  be a nilpotent  $\mathcal{DGR}$ -algebra of nilpotency rank  $n$ . Suppose that  $B(A)$  admits a homology exponent  $r \in R$ . Then  $r^{n-1}$  is a homology exponent for  $A$ .*

In section 8 below we observe that the mod- $p$  homology of a  $p$ -local loop space  $X$  is a nilpotent algebra if and only if it is finite dimensional, in which case it cannot be rationally contractible. Thus Theorem 1.6 does not correspond to a topological situation.

The paper is organized as follows. Sections 2 and 3 are preliminary in nature and the techniques developed are applied in Section 4 for the proof of Theorem 1.1 and Proposition 1.3. In Section 5 we prove Theorem 1.2. Section 6 is dedicated to the proof of an algebraic analogue of Theorem 1.5. The theorem follows using Proposition 1.4 which we prove in Section 7. In Section 8 we prove Theorem 1.6. We conclude the paper in Section 9 by making some remarks and speculations on the subject.

The author is grateful to the Mathematisches Institut Der Universität Heidelberg for its kind hospitality and support and to Hans-Werner Henn for some useful discussions.

## 2. EXTENDED MAPS ON $\mathcal{DGR}$ -ALGEBRAS

Our terminology is mostly borrowed from [7]. We begin this section by recalling the basic definitions we need.

Let  $R$  be a commutative ring with a unit. Recall that a  $\mathcal{DGR}$ -module  $M$  is a graded  $R$ -module, also denoted by  $M$ , together with a differential  $d_M$  of degree  $-1$ . A morphism of  $\mathcal{DGR}$ -modules is a morphism of graded modules, which commutes with the differentials. Morphisms of graded  $R$ -modules (differential or not) will generally be assumed to have degree 0. However we shall frequently use morphisms of non-zero degree, in which case the degree will be explicitly spelled out.

A  $\mathcal{DGR}$ -module  $M$  is said to be positive if  $M_n = 0$  for  $n < 0$ . The suspension functor  $\Sigma(-)$ , given by  $(\Sigma M)_n = M_{n-1}$  and  $(d_{\Sigma M}) = -(d_M)_{n-1}$ , is an automorphism on the category of  $\mathcal{DGR}$ -modules and preserves the subcategory of positive  $\mathcal{DGR}$ -modules.

We shall generally assume that  $R$  is an integrally torsion free principal ideal domain. Let  $\mathcal{GR}m$  denote the category of positive graded  $R$ -modules  $M$ , such that  $M_n$  is a free  $R$ -module for every  $n$ . Similarly let  $\mathcal{DGR}m$  denote the category of  $\mathcal{DGR}$ -modules such that for every object  $M$  the underlying graded  $R$ -module is an object of  $\mathcal{GR}m$ . From this point and on by a  $\mathcal{GR}$ -module ( $\mathcal{DGR}$ -module) we shall always mean an object of  $\mathcal{GR}m$  ( $\mathcal{DGR}m$ ).

A  $\mathcal{DGR}$ -coalgebra  $C$  is a positive  $\mathcal{DGR}$ -module, also denoted by  $C$ , together with a diagonal  $\Delta_C : C \rightarrow C \otimes C$  and a morphism  $\epsilon_C : C \rightarrow R$ , such that  $\Delta$  is associative and  $\epsilon$  is a counit for  $\Delta$ . A morphism  $f : C \rightarrow D$  of  $\mathcal{DGR}$ -coalgebras is a morphism of  $\mathcal{DGR}$ -modules, which commutes with the diagonals and such that  $\epsilon_D f = \epsilon_C$ . Notice that the ground ring  $R$  admits a unique structure of a  $\mathcal{DGR}$ -coalgebra.

A supplemented  $\mathcal{DGR}$ -coalgebra is a  $\mathcal{DGR}$ -coalgebra  $C$ , together with a map  $\eta_C : R \rightarrow C$  of  $\mathcal{DGR}$ -coalgebras called an augmentation. A morphism  $f : C \rightarrow D$  between two such objects is a map of  $\mathcal{DGR}$ -coalgebras, which in addition satisfies  $f\eta_C = \eta_D$ . For an integer  $n \geq 0$ , a  $\mathcal{DGR}$ -coalgebra  $C$  is said to be  $n$ -connected if  $(\epsilon_C)_q$  is an isomorphism for all  $q \leq n$ . For a supplemented  $\mathcal{DGR}$ -coalgebra  $C$  let the augmentation coideal  $J_C$  denote the cokernel of  $\eta_C$ .

The dual terminology applies to define the concept of a  $\mathcal{DGR}$ -algebra. Thus the structure maps on a supplemented  $\mathcal{DGR}$ -algebra  $A$  are a multiplication  $\mu_A$ , a unit  $R \xrightarrow{\eta_A} A$  and an augmentation  $A \xrightarrow{\epsilon_A} R$ . For a supplemented  $\mathcal{DGR}$ -algebra  $A$ , let the augmentation ideal  $IA$  denote the kernel of  $\epsilon_A$ . We let  $\mathcal{DGR}c$  and  $\mathcal{DGR}a$  denote the categories of supplemented  $\mathcal{DGR}$ -coalgebras and algebras respectively, such that for each object the underlying graded

$R$ -module is an object of  $\mathcal{GR}m$ . As before, by  $\mathcal{DGR}$ -coalgebras and algebras we shall always mean, from now on, objects of  $\mathcal{DGR}c$  and  $\mathcal{DGR}a$  respectively.

For a  $\mathcal{GR}$ -module  $M$ , let  $T(M)$  denote the tensor algebra on  $M$ . Let  $j : M \rightarrow T(M)$  denote the natural inclusion map into elements of tensor filtration 1. For an element  $x \in M$  let  $j(x)$  be denoted by  $[x]$ . The module  $T(M)$  is naturally a bigraded object. Namely, let  $bideg([x]) = (-1, deg(x))$ , and if  $bideg(y_i) = (n_i, k_i)$ ,  $i = 1, 2$ , then  $bideg(y_1 y_2) = (n_1 + n_2, k_1 + k_2)$ .

**Definition 2.1.** *Let  $M$  be a  $\mathcal{GR}$ -module and let  $F : IT(M) \rightarrow IT(M)$  be a map of  $\mathcal{GR}$ -modules of some non-negative degree. We say that  $F$  is right extended if it is a map of right  $T(M)$ -modules. The map  $F$  is said to be a right extension of  $f : M \rightarrow IT(M)$  given by  $f = F \circ j$ .*

The following two lemmas are rather elementary. The first appears in [10, I, chp. 4] and the second is an observation made to the author by John C. Moore. Short proofs are included for the convenience of the reader.

**Lemma 2.2.** *For  $M \in \mathcal{GR}m$ , let  $f : M \rightarrow IT(M)$  be a map of  $\mathcal{GR}$ -modules of some non-negative degree. Then there exists a unique right extension  $F : IT(M) \rightarrow IT(M)$  of  $f$ .*

*Proof.* For  $x \in M$ , define  $F[x] = f(x)$  and inductively

$$F([x_1 | \cdots | x_n]) = F([x_1])[x_2 | \cdots | x_n].$$

Using the fact that  $M_i$  is a free  $R$ -module for each  $i$ , one observes immediately that  $F$  is a well defined right extension of  $f$  in the sense given above. Uniqueness is immediate from the definitions.  $\square$

For any  $R$ -module  $M$  and a non-zero element  $r \in R$ , let  $\phi_r$  denote the endomorphism of  $M$  given by multiplication by  $r$ .

**Lemma 2.3.** *Let  $M$  be a  $\mathcal{DGR}$ -module, with a homology exponent  $r \in R$ . Then  $\phi_r$  is null-homotopic on  $M$ . Moreover, there is a choice of a null-homotopy  $s$  for  $\phi_r$ , such that  $s^2 = 0$ .*

*Proof.* Since  $M$  is a free  $R$ -module in each dimension and since  $R$  is assumed to be a principal ideal domain, the short exact sequence of  $\mathcal{DGR}$ -modules

$$0 \rightarrow Z(M) \xrightarrow{i} M \xrightarrow{d} \Sigma B(M) \rightarrow 0$$

is split in  $\mathcal{GR}m$ . Here  $Z(M)$  denotes the submodule of cycles and  $\Sigma B(M)$  means the suspension of the submodule of boundaries, where in both  $Z(M)$  and  $\Sigma B(M)$  the differentials are taken to be trivial.

Let  $\sigma$  denote any right inverse for  $d$  and let  $j$  denote the left inverse for  $i$ , given by  $j(m) = i^{-1}(m - \sigma d(m))$ . Notice that  $m - \sigma d(m)$  is a cycle for every  $m \in M$ , so  $i^{-1}$  is defined on it. Since  $ij(m)$  is a cycle for every  $m \in M$ , one has by hypothesis that  $r \cdot ij(m)$  is a boundary. Define

$$s(m) = \sigma(r \cdot ij(m)).$$

The reader can easily verify that  $s$  is a null-homotopy for  $\phi_r$  and that  $s^2 = 0$ .  $\square$

Let  $C$  be a simply-connected  $\mathcal{DGR}$ -coalgebra. Recall that the loop algebra functor  $\Omega C$  is given as a  $\mathcal{GR}$ -algebra by  $\Omega C = T(\Sigma^{-1}JC)$ . As a differential graded module,  $\Omega C$  is bigraded and has two differentials; an internal differential  $d_I$  of bidegree  $(0, -1)$ , given by  $d_I[x] = -[dx]$ , where  $d$  is the differential of  $C$ , and an external differential  $d_E$  of bidegree  $(-1, 0)$ , given by  $d_E[x] = \sum_i (-1)^{|x'_i|} [x'_i | x''_i]$ , where  $\bar{\Delta}x = \sum_i x'_i \otimes x''_i$  is the reduced diagonal. Both differentials are required to be derivations of the algebra structure and the total differential  $d_T = d_I + d_E$  gives  $\Omega C$  the structure of a  $\mathcal{DGR}$ -algebra.

Let  $C$  be a simply-connected  $\mathcal{DGR}$ -coalgebra with a homology exponent  $r \in R$ . Let  $s : C \rightarrow C$  be a choice of a null-homotopy for  $\phi_r$ . Let  $s_1 : \Omega C \rightarrow \Omega C$  denote the right extension of the composite

$$\Sigma^{-1}JC \xrightarrow{-s} \Sigma^{-1}JC \xrightarrow{j} \Omega C$$

in the sense of Lemma 2.2. It is easy to see then that  $s_1$  is a null-homotopy for  $\phi_r$  with respect to the internal differential  $d_I$  [10, I, chp. 4].

### 3. THE OBSTRUCTION MAP $\theta$ AND ITS ITERATIONS

In some sense, could the map  $s_1$ , constructed in the previous section, be a map of  $\mathcal{DGR}$ -modules with respect to the external differential, it would then be a null-homotopy for  $\phi_r$  with respect to the total differential  $d_T$ . However, this is hardly ever the case, due to the fact that in general the null-homotopy  $s$  on  $C$ , ‘‘inducing’’  $s_1$ , fails to be, in a sense that can be made precise, a map of comodules over  $C$ . We now discuss a map  $\theta$  of  $\mathcal{DGR}$ -modules on  $\Omega C$ , which can be considered as the obstruction for  $s_1$  to be a null-homotopy of  $\phi_r$ .

Given the null-homotopy  $s$  for  $\phi_r$  on  $C$ , construct  $s_1$  on  $\Omega C$ , as in Section 2, and define  $\theta = \theta(s) : \Omega C \rightarrow \Omega C$  by

$$\theta = d_E s_1 + s_1 d_E.$$

The following are basic properties of  $\theta$ . The proofs are straight forward from the definition and are left to the reader.

**Proposition 3.1.** *For any null-homotopy  $s$  for  $\phi_r$  on  $C$ , let  $\theta$  denote the corresponding map on  $\Omega C$ . For each  $k \geq 1$  let  $\theta^k$  denote the  $k$ -th iteration of  $\theta$ . Then*

- (1)  $\theta^k$  is a map of  $\mathcal{DGR}$ -modules with respect to both differentials on  $\Omega C$ .
- (2)  $\theta^k$  is homotopic to  $(-1)^k \phi_{r^k}$  with respect to the total differential on  $\Omega C$ .
- (3)  $\theta^k$  is right extended.
- (4)  $\theta^k$  has bidegree  $(-k, k)$ .

**Corollary 3.2.** *Let  $k \geq 1$  and suppose that  $\theta^k$  vanishes on elements of tensor filtration 1. Then  $\theta^k$  vanishes on  $\Omega C$ . Moreover the map  $\phi_{r^k}$  is null-homotopic on  $\Omega C$  if and only if  $\theta^k$  is. Thus if  $\theta^k = 0$ , for some  $k \geq 1$ , then  $\Omega C$  has a homology exponent  $r^k$ .*

Define  $s_k : \Omega C \rightarrow \Omega C$  by  $s_k = \theta^{k-1} \circ s_1$ . Define  $\sigma_k : \Omega C \rightarrow \Omega C$  inductively by  $\sigma_1 = s_1$  and for every  $k \geq 2$ ,  $\sigma_k = r^{k-1} s_1 - r^{k-2} s_2 + \dots + (-1)^{k-1} s_k$ . The following equalities provide some more insight into the structure of  $\theta^k$  and can be proven by an easy induction.

**Proposition 3.3.** *With the notation above, the following relations hold.*

- (1)  $s_k d_I + d_I s_k = r \theta^{k-1}$ .
- (2)  $\theta^k = \sum_{i=0}^k (d_E s_1)^i (s_1 d_E)^{k-i}$ .

- (3) If  $s^2 = 0$  then  $\theta^k = (d_E s_1)^k + (s_1 d_E)^k$ .  
(4)  $\theta^k = d_E s_k + s_k d_E$ .  
(5)  $\phi_{r^k} = d\sigma_k + \sigma_k d + (-1)^k \theta^k$ .

#### 4. PROOF OF THEOREM 1.1 AND PROPOSITION 1.3

Let  $C$  be a  $c$ -connected  $\mathcal{DGR}$ -coalgebra,  $c \geq 1$ , such that  $H_*(JC)$  has an exponent  $r \in R$ . Let  $s$  be a null-homotopy for  $\phi_r$  on  $C$ , such that  $s^2 = 0$ . Let  $T$  denote the loop algebra construction  $\Omega C$  and let  $s_1$  and  $\theta$  be the maps constructed in the previous sections with respect to  $s$ .

Since  $C$  is  $c$ -connected,  $T$  is  $(c-1)$ -connected and so  $T_{p,q} = 0$  for  $q \leq -p(c+1) - 1$ . In particular  $T_{-1,q} = 0$  if  $q \leq c$ . For each positive integer  $i$  let  $T(i)$  denote the  $i$ -th skeleton of  $IT$ .

**Lemma 4.1.** *Suppose  $e \geq \frac{d-c+1}{c}$ . Then  $\theta^e$  vanishes on  $T(i)$ , for  $i \leq d$ .*

*Proof.* By Corollary 3.2 it suffices to check that  $\theta^e$  vanishes on elements of tensor filtration  $-1$ , namely on  $T_{-1,i}$ ,  $i \leq d+1$ . Since  $\text{bideg}(\theta^e) = (-e, e)$ , it suffices to verify that  $T_{-1-e, d+e+1} = 0$ , for each  $c \leq d \leq c(e+1) - 1$ . Indeed, by the connectivity argument above  $T_{-1-e, q} = 0$  if  $q \leq ec + e + c$ , which is satisfied for  $q = d + e + 1$ .  $\square$

Now suppose that  $C$  is  $d$ -dimensional for some  $d \geq c$ . Then we have  $T_{-1,q} = 0$  if  $q > d$ . By the same argument as in Lemma 4.1 it follows that the map  $\theta^e$  vanishes on  $IT$  if  $e \geq \frac{d-c}{c}$ . However this can be slightly improved.

**Lemma 4.2.** *Suppose that  $C$  is  $c$ -connected and  $d$ -dimensional for some  $d > c$ . Then  $\theta^e$  vanishes on  $IT$  for  $e \geq \frac{d-c-1}{c}$ .*

*Proof.* It suffices to show that  $\theta^e$  vanishes on  $T_{-1,d}$ . Indeed, since  $E_{-1, d+1}^0 = 0$  and  $s^2 = 0$ , it follows from Lemma 3.3 that  $\theta^e = (s_1 d_E)^e$ . But  $(s_1 d_E)^e = s_1((d_E s_1)^{e-1} d_E)$ . Hence the restriction of  $\theta^e$  on  $T_{-1,d}$  factors through  $T_{-1-e, d+e-1}$ , which vanishes since  $d+e-1 \leq ec + e + c$ . Thus  $\theta^e = 0$  on  $T_{-1,*}$  and the lemma follows.  $\square$

We are now ready to prove Theorem 1.1 and Proposition 1.3. Let  $C$  be an  $c$ -connected  $\mathcal{DGR}$ -coalgebra with a homology exponent  $r \in R$ . Let  $T$  denote  $\Omega C$ . By Lemma 4.1,  $\theta^e$  vanishes on  $T(i)$  for  $i \leq d$ , if  $e \geq \frac{d+c+1}{c}$ . Hence for any  $x \in T(i)$ ,  $i \leq d$ , we have by (5) of Proposition 3.3

$$\phi_{r^e}(x) = (d\sigma_e + \sigma_e d)(x).$$

Thus  $r^e \cdot H_i(C) = 0$  for  $0 < i \leq d$ . This proves Theorem 1.1.

Next suppose that  $C_q = 0$  for  $q > d$ . By Lemma 4.2,  $\theta^e$  vanishes on  $IT$ , if  $e \geq \frac{d-c-1}{c}$ . Thus  $\phi_{r^e}$  is null-homotopic on  $IT$  and Proposition 1.3 is proved.

#### 5. EXAMPLES

For each pair of positive integers  $(k, t)$  define  $C^{k,t}$  to be the graded free abelian group, generated by elements  $\{x_n\}_{n \geq 2}$  and  $\{y_n\}_{n \geq 3}$ , where dimensions of the generators are given below.

$$|x_{2n}| = (2k-1)(2n-1) + 1,$$

$$\begin{aligned} |x_{2n+1}| &= (2k-1)(2n-1) + 2, \\ |y_{2n+1}| &= (2k-1)2n + 1, \\ |y_{2n+2}| &= (2k-1)2n + 2. \end{aligned}$$

Define a diagonal map  $\Delta : C^{k,t} \longrightarrow C^{k,t} \otimes C^{k,t}$  by

$$\begin{aligned} \bar{\Delta}x_{2n} &= \bar{\Delta}y_{2n+1} = 0 \\ \bar{\Delta}x_{2n+1} &= \sum_{i=1}^{n-1} x_{2i} \otimes y_{2(n-i)+1} + y_{2(n-i)+1} \otimes x_{2i} \\ \bar{\Delta}y_{2n+2} &= \sum_{i=1}^n x_{2i} \otimes x_{2(n-i+1)} \end{aligned}$$

where  $\bar{\Delta}$  denotes the reduced diagonal and extends to  $\Delta$  in the obvious way.

Define a differential  $d$  on  $C^{k,t}$  by

$$dx_{2n} = dy_{2n+1} = 0, \quad dx_{2n+1} = tx_{2n}, \quad \text{and} \quad dy_{2n+2} = ty_{2n+1}$$

for each  $i \geq 1$ .

Define a map  $s$  of degree  $+1$  on  $C^{k,t}$  by

$$sx_{2n+1} = sy_{2n+2} = 0, \quad sx_{2n} = x_{2n+1} \quad \text{and} \quad sy_{2n+1} = y_{2n+2}.$$

Notice that  $s^2 = 0$ .

One verifies by inspection that the triple  $(C^{k,t}, d, \Delta)$  is a  $(2k-1)$ -connected  $\mathcal{D}\mathcal{G}\mathbb{Z}$ -coalgebra with a homology exponent  $t$  and a null-homotopy for  $\phi_t$  given by  $s$ . Construct  $s_1$  and  $\theta$  on  $I\Omega C^{k,t}$  with respect to the null-homotopy  $s$ .

**Proposition 5.1.** *For each positive integer  $n$  the following equalities hold in  $\Omega C^{k,t}$ .*

$$\begin{aligned} (1_n) \quad \theta^{2n-1}[y_{2n+1}] &= (-1)^n [x_2]^{2n}. \\ (2_n) \quad \theta^{2n}[x_{2n+2}] &= (-1)^n [x_2]^{2n+1}. \end{aligned}$$

*Proof.* Since  $x_{2n}$  and  $y_{2n+1}$  are primitive for all  $n \geq 1$  it follows from Proposition 3.3 that  $\theta^k[x_{2n}] = (d_E s_1)^k[x_{2n}]$  and  $\theta^k[y_{2n+1}] = (d_E s_1)^k[y_{2n+1}]$ .

One verifies directly that  $\theta[y_3] = -[x_2]^2$  and that  $\theta^2[x_4] = -[x_2]^3$ . Thus the statement is true for  $n = 1$ . Assume that  $\theta^{2n-3}[y_{2n-1}] = (-1)^{n-1}[x_2]^{2n-2}$  and  $\theta^{2n-2}[x_{2n}] = (-1)^{n-1}[x_2]^{2n-1}$ , namely that  $(1_{n-1})$  and  $(2_{n-1})$  hold. We prove  $(1_n)$  and  $(2_n)$ .

$$\begin{aligned} \theta^{2n-1}[y_{2n+1}] &= -\theta^{2n-2}d_E[y_{2n+2}] = \\ &= -\theta^{2n-2} \sum_{i=1}^n [x_{2i}|x_{2(n-i)+1}] = \\ &= -\sum_{i=1}^n (\theta^{2n-2}[x_{2i}])[x_{2(n-i)+1}] \end{aligned}$$

By Lemma 4.1 all but the last summand vanish and hence by the inductive hypothesis

$$\theta^{2n-1}[y_{2n+1}] = -(\theta^{2n-2}[x_{2n}])[x_2] = (-1)^n [x_2]^{2n}$$

and  $(1_n)$  is proved. Next

$$\begin{aligned} \theta^{2n}[x_{2n+2}] &= -\theta^{2n-1}d_E[x_{2n+3}] = \\ &= \theta^{2n-1} \sum_{i=1}^n [y_{2(n-i)+1}|x_{2i}] - [x_{2i}|y_{2(n-i)+1}] = \\ &= \sum_{i=1}^n (\theta^{2n-1}[y_{2(n-i)+1}])[x_{2i}] - (\theta^{2n-1}[x_{2i}])[y_{2(n-i)+1}]. \end{aligned}$$

By Lemma 4.1 all summands vanish except for possibly

$$(\theta^{2n-1}[y_{2n+1}])[x_2] \quad \text{and} \quad (\theta^{2n-1}[x_{2n}])[y_{2(n-i+1)+1}].$$

But

$$\theta^{2n-1}[x_{2n}] = \theta\theta^{2n-2}[x_{2n}] = (-1)^{n-1}\theta[x_2]^{2n-1} = 0.$$

Hence

$$\theta^{2n}[x_{2n+2}] = (\theta^{2n-1}[y_{2n+1}])[x_2] = (-1)^n[x_2]^{2n}[x_2] = (-1)^n[x_2]^{2n+1}.$$

This completes the proof.  $\square$

**Proposition 5.2.** *For any positive integer  $m$ , the element  $[x_2]^m$  represents a non-zero homology class in  $H_{m(2k-1)}(C^{k,t})$ .*

Theorem 1.2 follows at once from Proposition 5.2. Indeed, set  $c = 2k - 1$ . We must show that  $H_{cc}(\Omega C^{k,t})$  contains an element of order  $t^e$ . For  $e = 2n$ , consider the class  $\alpha_e \in H_{cc}(\Omega C^{k,t})$ , represented by the cycle  $[y_{e+1}]$ . Since  $[y_{e+1}]$  cannot possibly be a boundary,  $\alpha_e \neq 0$ . By propositions 5.1 and 5.2,  $\theta_*^{e-1}\alpha_e \neq 0$ . Hence by Proposition 3.1,  $t^{e-1}\alpha_e \neq 0$ . On the other hand by Theorem 1.1,  $t^e\alpha_e = 0$ . Hence  $\alpha_e$  is an element of order  $t^e$ .

For  $e = 2n + 1$ , consider the class  $\beta_e \in H_{cc}(\Omega C^{k,t})$ , represented by the cycle  $[x_{e+1}]$ . Again  $\beta_e$  is evidently non-zero. Propositions 5.1 and 5.2 imply that  $\theta_*^{e-1}\beta_e \neq 0$ . Hence  $t^{e-1}\beta_e \neq 0$ . On the other hand, by Theorem 1.1  $t^e\beta_e = 0$ . Hence  $\beta_e$  is an element of order  $t^e$ . This completes the proof of Theorem 1.2.

We proceed by proving Proposition 5.2. Let  $k$  and  $t$  be fixed and let  $C$  denote  $C^{k,t}$ . Let  $C^* = \text{Hom}(C; \mathbb{Z})$  denote the dual of  $C$ . Thus  $C^*$  is a  $\mathcal{DGZ}$ -algebra with product given by dualizing the diagonal map on  $C$ . By abuse of notation, we denote the duals of  $x_i, y_j$  in  $C^*$  again by  $x_i$  and  $y_j$  respectively. Let  $C^*(j)$  denote the sub-algebra of  $C^*$  generated by  $x_2, x_3, \dots, x_{j+2}, y_3, y_4, \dots, y_{j+1}$  if  $j$  is odd and by  $x_2, \dots, x_{j+1}, y_3, \dots, y_{j+2}$  if  $j$  is even. Define  $C(j)$  to be  $(C^*(j))^*$ . Thus  $C(j)$  is a quotient  $\mathcal{DGZ}$ -coalgebra of  $C$  and for every  $j \geq 1$  there are natural projections

$$C \xrightarrow{\pi_{j+1}} C(j+1) \xrightarrow{\varphi_{j+1}} C(j).$$

The following lemma is immediate from the definitions.

**Lemma 5.3.** *For each  $n \geq 2$ , the submodule  $\text{Ker}(\varphi_n)$  of  $C(n)$  is  $n(2k-1)$ -connected. Thus so is the submodule  $\text{Ker}(\pi_n)$  of  $C$ .*

Since the element  $[x_2] \in \Omega C$  is preserved under each of the maps  $\Omega\pi_n$ , it suffices to show that  $[x_2]^n$  is not a boundary in  $\Omega C(n)$  to complete the proof of Proposition 5.2.

**Proposition 5.4.** *The element  $[x_2]^n$  is not a boundary in  $\Omega C(n)$  but  $[x_2]^{n+1}$  is. In fact for each  $m \geq 1$  there exist decomposable elements  $A_m \in \Omega C(2m)$  and  $B_m \in \Omega C(2m-1)$  such that*

$$(1_m) \quad d(t^{2m-2}[y_{2m+2}] + B_m) = [x_2]^{2m} \text{ in } \Omega C(2m-1),$$

$$(2_m) \quad d(t^{2m-1}[x_{2m+3}] + A_m) = [x_2]^{2m+1} \text{ in } \Omega C(2m).$$

Furthermore, if  $z'_n \in \Omega C(n)$  is such that  $dz'_n = [x_2]^{n+1}$ , then  $z'_n$  is indecomposable.

*Proof.* We first show that  $[x_2]^n$  is not a boundary in  $\Omega C(n)$ . This certainly holds for  $n = 1$ . Thus assume  $[x_2]^j$  is not a boundary in  $\Omega C(j)$  for  $j \leq n - 1$  and suppose  $n = 2m$ . The kernel of  $\varphi_{2m}$  through dimension  $(2k - 1)2m + 2 = |y_{2m+2}|$  is given by the element  $y_{2m+1}$ . Thus the kernel of  $\Omega\varphi_{2m}$  through dimension  $(2k - 1)2m + 1$  is generated by the element  $[y_{2m+1}]$  if  $k > 1$  and by  $[y_{2m+1}]$ ,  $[y_{2m+1}|x_2]$  and  $[x_2|y_{2m+1}]$  if  $k = 1$ . Notice that in either case  $\text{Ker}(\Omega\varphi_{2m})$  through dimension  $(2k - 1)2m + 1$  consists of cycles.

Suppose  $z \in \Omega C(2m)$  is such that  $dz = [x_2]^{2m}$ . Then by naturality  $d\Omega\varphi_{2m}z = [x_2]^{2m}$  in  $\Omega C(2m - 1)$ . But by our induction hypothesis it follows that  $\Omega\varphi_{2m}z$  is indecomposable in  $\Omega C(2m - 1)$ . Hence it contains an essential summand of tensor filtration 1. Since  $\Omega\varphi_{2m}$  maps the single element of tensor filtration 1 and total dimension  $(2k - 1)2m + 1$ ,  $[y_{2m+2}]$ , non-trivially, it follows that  $z$  itself is indecomposable. But in  $\Omega C(2m)$ ,  $[y_{2m+2}]$  is not a cycle with respect to the internal differential. This yields a contradiction. The case  $n = 2m + 1$  follows by analogy.

Next we must show that  $[x_2]^{n+1}$  is a boundary in  $\Omega C(n)$ . Notice that  $d[y_4] = [x_2]^2$  in  $\Omega C(1)$  and that  $d([x_5] - [y_3|x_3] - [x_2|y_4]) = [x_2]^3$  in  $\Omega C(2)$ . Thus  $(1_m)$  and  $(2_m)$  hold for  $m = 1$ . Assume that for each  $j \leq m - 1$  there are decomposable elements  $A'_j$  and  $B'_j$  in  $\Omega C$  such that

$$\begin{aligned} (1'_j) \quad & d(t^{2j-2}[y_{2j+2}] + B'_j) = [x_2]^{2j} - t^{2j-1}[y_{2j+1}], \\ (2'_j) \quad & d(t^{2j-1}[x_{2j+3}] + A'_j) = [x_2]^{2j+1} - t^{2j}[x_{2j+2}]. \end{aligned}$$

Notice that  $(1'_j)$  and  $(2'_j)$  are satisfied for  $j = 1$  with respect to  $A'_1 = -([y_3|x_3] + [x_2|y_4])$  and  $B'_1 = 0$ . Proving  $(1_m)$  and  $(2_m)$  amounts to showing that there are decomposable elements  $A'_m$  and  $B'_m$  in  $\Omega C$ , such that  $(1'_m)$  and  $(2'_m)$  hold. Indeed, reducing  $(1'_m)$  to  $\Omega C(2m - 1)$  and  $(2'_m)$  to  $\Omega C(2m)$  yields  $(1_m)$  and  $(2_m)$ . For a graded algebra  $A$  and  $x, y \in A$ , write  $[x, y]$  for the difference  $xy - (-1)^{|x||y|}yx$ . We write down the formulas and leave it for the reader to verify that the required equations hold.

Define, for  $m \geq 2$

$$B'_{m,1} = -([x_2]A'_{m-1} + t^{2m-3}([x_2|x_{2m+1}] + [x_{2m}|x_3])).$$

$$B'_{m,2} = \sum_{2 \leq j < \frac{m+1}{2}} t^{2(m-j)-1}[t^{2j-2}[x_{2j+1}] + tA'_{j-1} + [x_3][x_2]^{2j-2}, [x_{2(m-j+1)}]].$$

For  $m$  even define  $B'_{m,3} = 0$  and for  $m$  odd define

$$B'_{m,3} = t^{m-2}([x_3][x_2]^{m-1}[x_{m+1}] + t^{m-1}[x_{m+2}|x_{m+1}] + tA'_{\frac{m-1}{2}}[x_{m+1}]).$$

Next define

$$A'_{m,1} = -(t^{2m-2}[x_2|y_{2m+2}] + [x_2]B'_m + t^{2m-2}[y_{2m+1}|x_3]).$$

$$A'_{m,2} = \sum_{j=2}^m t^{2(m-j)}[t^{2j-2}[x_{2j+1}] + tA'_{j-1} + [x_3][x_2]^{2j-2}, [y_{2(m-j)+3}]].$$

Let

$$B'_m = B'_{m,1} + B'_{m,2} + B'_{m,3} \quad \text{and} \quad A'_m = A'_{m,1} + A'_{m,2}.$$

Finally we must show that if  $b'_n \in \Omega C(n)$  is such that  $db'_n = [x_2]^{n+1}$ , then  $b'_n$  is indecomposable. This is evident for  $n = 1$ . Let  $b_n \in \Omega C(n)$  denote the element constructed above such that  $db_n = [x_2]^{n+1}$  in  $\Omega C(n)$ . Assume that  $b'_n$  is a decomposable element with the same property. Then by naturality  $d\Omega\varphi_n b'_n = [x_2]^{n+1}$ . Hence the element  $\Omega\varphi_n b'_n - b_{n-1}[x_2]$  is a cycle in  $\Omega C(n-1)$ . However  $|b_{n-1}| = (2k-1)n+1$  and  $\Omega\varphi_n$  is an epimorphism in this dimension with kernel given by cycles. Thus  $b_{n-1}$  can be pulled back to  $\Omega C(n)$  and the lift is unique up to a cycle. Hence the element  $b'_n - b_{n-1}[x_2]$  is a cycle in  $\Omega C(n)$ . However by  $(1'_n)$  and  $(2'_n)$ ,  $db_{n-1} = [x_2]^n - t^{n-1}\alpha$  in  $\Omega C(n)$ , where  $\alpha = [y_{n+1}]$  if  $n$  is even and  $[x_{n+1}]$  otherwise. Thus  $d(b'_n - b_{n-1}[x_2]) = t^{n-1}\alpha[x_2] \neq 0$ . This yields a contradiction and thus completes the proof.  $\square$

For each triple of positive integers  $(k, t, n)$ , we consider two sub- $\mathcal{D}\mathcal{G}\mathcal{Z}$ -coalgebras  $A^{k,t,n}$  and  $B^{k,t,n}$  of  $C^{k,t}$ , generated as free abelian group by the sets  $\{x_i\}_{i=2}^{2n+1} \cup \{y_j\}_{j=3}^{2n}$  and  $\{x_i\}_{i=2}^{2n+1} \cup \{y_j\}_{j=3}^{2n+2}$  respectively. One immediately observes that  $A^{k,t,n} \subset B^{k,t,n}$  as a sub- $\mathcal{D}\mathcal{G}\mathcal{Z}$ -coalgebra, for each  $n \geq 1$  and that both  $A^{k,t,n}$  and  $B^{k,t,n}$  are closed under the null-homotopy  $s$  for  $\phi_t$ . Furthermore,  $A^{k,t,n}$  and  $B^{k,t,n}$  are  $((2n-1)(2k-1)+2)$  and  $(2n(2k-1)+2)$ -dimensional respectively. By Proposition 1.3  $t^e \cdot H_*(A^{k,t,n}) = 0$  if  $e \geq 2n-1$  and  $t^e \cdot H_*(B^{k,t,n}) = 0$  if  $e \geq 2n$ .

On the other hand since both  $A^{k,t,n}$  and  $B^{k,t,n}$  are sub- $\mathcal{D}\mathcal{G}\mathcal{Z}$ -coalgebras of  $C^{k,t}$  and since  $[x_2]$  represents an element of infinite height in  $H_*(\Omega C^{k,t})$ , it follows that the same holds in the loop space homology of  $A^{k,t,n}$  and  $B^{k,t,n}$ . Thus by the argument above  $[x_{2n}]$  represents an element of order  $t^{2n-1}$  in  $H_{(2n-1)(2k-1)}(\Omega A^{k,t,n})$  and that  $[y_{2n+1}]$  represents an element of order  $t^{2n}$  in  $H_{2n(2k-1)}(\Omega B^{k,t,n})$ . Hence the approximation of Proposition 1.3 is best possible in these examples.

## 6. TRANSFER TYPE NULL-HOMOTOPIES

In this section we restrict attention to a particular family of  $\mathcal{D}\mathcal{G}\mathcal{R}$ -coalgebras, namely those  $\mathcal{D}\mathcal{G}\mathcal{R}$ -coalgebras which appear as a suitable modification of the classifying construction  $B(A)$  on a  $\mathcal{D}\mathcal{G}\mathcal{R}$ -algebra  $A$ , concentrated in degree 0.

We start by recalling the construction  $P_n(-)$  on  $\mathcal{D}\mathcal{G}\mathcal{R}$ -coalgebras as introduced in [10]. Let  $C$  be a  $\mathcal{D}\mathcal{G}\mathcal{R}$ -coalgebra. For some integer  $n > 0$  consider the differential  $d_n : C_{n+1} \rightarrow C_n$ . Since  $R$  is assumed a principal ideal domain and  $C_n$  is free, the image  $B_n(C)$  of  $d_n$  is a free submodule of  $C_n$ . Thus there exists a right inverse  $\epsilon_n : B_n(C) \rightarrow C_{n+1}$  for  $d_n$ . Let  $C'_{n+1}$  denote the image of  $\epsilon_n$  in  $C_{n+1}$ . Let  $D$  denote the sub- $\mathcal{D}\mathcal{G}\mathcal{R}$ -module of  $C$  given by  $D_i = C_i$  for  $0 < i \leq n$ ,  $D_{n+1} = C'_{n+1}$  and  $D_i = 0$  otherwise. Define  $P_n C$  to be the quotient  $\mathcal{D}\mathcal{G}\mathcal{R}$ -module  $C/D$ . Notice that  $P_n C$  is a  $\mathcal{D}\mathcal{G}\mathcal{R}$ -coalgebra by construction and that the projection  $\iota_n : C \rightarrow P_n C$  is a map of  $\mathcal{D}\mathcal{G}\mathcal{R}$ -coalgebras. The following are elementary properties of  $P_n C$  and are proven in [10].

**Proposition 6.1.** *Let  $C$  be a  $\mathcal{D}\mathcal{G}\mathcal{R}$ -coalgebra. Then for every  $n > 0$*

- (1) *the projection  $\iota_n$  is a split epimorphism in  $\mathcal{D}\mathcal{G}\mathcal{R}m$ .*
- (2)  *$\iota_n$  induces a homology isomorphism above dimension  $n$ .*
- (3)  *$P_n C$  is  $n$ -connected.*

Let  $C$  be a  $\mathcal{DGR}$ -coalgebra and let  $s$  denote a self map of  $C$  of degree  $+1$ . Let  $n$  be a fixed positive integer. Consider the projection  $\iota = \iota_n : C \rightarrow P_n C$ , where  $P_n C$  is constructed with respect to some right inverse  $\epsilon$  for the  $n$ -th differential  $d$  as above. Denote elements of  $P_n C$  by  $\iota x$ , where  $x \in C$  is some pre-image. A right inverse  $\eta$  of  $\iota$  in  $\mathcal{DGR}m$  is given by the identity in dimensions higher than  $n+1$  and for  $\iota x \in (P_n C)_{n+1}$ , define  $\eta(\iota x) = x - \epsilon dx$ . Then  $\tilde{s} = \iota s \eta$  can be identified with  $s$  in dimensions higher than  $n+1$  and for  $\iota x \in (P_n C)_{n+1}$  one has  $\tilde{s}(\iota x) = s(x) - s\epsilon d(x)$ .

The self map of  $C$  given by  $f_s = ds + sd$  is a map of  $\mathcal{DGR}$ -modules, for which  $s$  is a null-homotopy. Suppose that  $f_s = \phi_r$  for some  $r \in R$ . Then it is easy to verify that  $\tilde{s}$  is a null-homotopy for the self map of  $P_n C$ , given by  $\tilde{f}_s = \iota f_s \eta$ .

We now specialize to the case mentioned above. Let  $A$  be a  $\mathcal{DGR}$ -algebra concentrated in degree 0. Let  $B(A)$  denote the classifying construction [7] for  $A$ . A self map  $s$  of degree  $+1$  on  $B(A)$  is said to be a transfer type null-homotopy, if for each  $n > 1$

$$s([x_1|x_2|\cdots|x_n]) = \sum_i a_i [x'_i|x''_i|x_2|\cdots|x_n],$$

whenever  $s([x_1]) = \sum_i a_i [x'_i|x''_i]$ . Notice that transfer type null-homotopies  $s$  are a priori unlikely to satisfy  $s^2 = 0$ .

**Proposition 6.2.** *Let  $A$  be a  $\mathcal{DGR}$ -algebra concentrated in degree 0, which is free as an  $R$ -module. For some  $r \in R$ , let  $s$  be a transfer type null-homotopy for  $\phi_r$  on  $B(A)$ . For some  $n \geq 1$  let  $\theta$  be the corresponding map on  $\Omega P_n B(A)$ . Then for every  $k > n$*

$$\begin{aligned} \theta([\iota[x_1|\cdots|x_k]]) = \\ (-1)^n([\iota s[x_1|\cdots|x_n]|\iota[x_{n+1}|\cdots|x_k]]) - [\iota s\epsilon d[x_1|\cdots|x_{n+1}]|\iota[x_{n+2}|\cdots|x_k]]. \end{aligned}$$

Thus  $\theta^2 = 0$  if  $n > 1$  and  $\theta^3 = 0$  if  $n = 1$ .

*Proof.* The proof amounts to calculating  $\theta$  on the algebra generators for  $\Omega P_n B(A)$  given by  $[\iota[x_1|x_2|\cdots|x_k]]$ ,  $x_j \in A$ , observing the fact that  $\theta$  is right extended. In computing  $d_E s_1$  on a typical generator  $[x]$  of homological degree  $k$ , we may assume that  $k \geq 2n+1$ , since otherwise the element  $\tilde{s}(x)$  is primitive in the coalgebra structure of  $P_n B(A)$  and thus  $d_E$  vanishes.

$$\begin{aligned} d_E s_1[\iota[x_1|x_2|\cdots|x_k]] = -d_E \sum_i a_i [\iota[x'_i|x''_i|x_2|\cdots|x_k]] = \\ \sum_i a_i (-1)^n [\iota[x'_i|x''_i|x_2|\cdots|x_n]|\iota[x_{n+1}|\cdots|x_k]] + \\ \sum_i a_i \sum_{j=1}^{k-2n-1} (-1)^{n+j} [\iota[x'_i|x''_i|x_2|\cdots|x_{n+j}]|\iota[x_{n+j+1}|\cdots|x_k]] = \\ (-1)^n [\iota s[x_1|x_2|\cdots|x_n]|\iota[x_{n+1}|\cdots|x_k]] + \\ (-1)^{n+1} [\iota s[x_1|x_2|\cdots|x_{n+1}]|\iota[x_{n+2}|\cdots|x_k]] + \\ \sum_{j=2}^{k-2n-1} (-1)^{n+j+1} s_1[\iota[x_1|x_2|\cdots|x_{n+j}]|\iota[x_{n+j+1}|\cdots|x_k]]. \end{aligned}$$

On the other hand

$$\begin{aligned}
s_1 d_E[\iota[x_1|x_2|\cdots|x_k]] &= \\
&(-1)^{n+1} s_1[\iota[x_1|x_2|\cdots|x_{n+1}]|\iota[x_{n+2}|\cdots|x_k]] + \\
&\sum_{j=2}^{k-2n-1} (-1)^{n+j} s_1[\iota[x_1|x_2|\cdots|x_{n+j}]|\iota[x_{n+j+1}|\cdots|x_k]].
\end{aligned}$$

Thus adding the two equations we get

$$\begin{aligned}
\theta[\iota[x_1|x_2|\cdots|x_k]] &= \\
&(-1)^n [\iota s[x_1|x_2|\cdots|x_n]|\iota[x_{n+1}|\cdots|x_k]] + \\
&(-1)^{n+1} [\iota s[x_1|x_2|\cdots|x_{n+1}]|\iota[x_{n+2}|\cdots|x_k]] + \\
&(-1)^{n+1} s_1[\iota[x_1|x_2|\cdots|x_{n+1}]|\iota[x_{n+2}|\cdots|x_k]] = \\
&(-1)^n [\iota s[x_1|x_2|\cdots|x_n]|\iota[x_{n+1}|\cdots|x_k]] + \\
&(-1)^{n+1} [\iota s[x_1|x_2|\cdots|x_{n+1}]|\iota[x_{n+2}|\cdots|x_k]] + \\
&(-1)^n [\iota s[x_1|x_2|\cdots|x_{n+1}]|\iota[x_{n+2}|\cdots|x_k]] + \\
&(-1)^{n+1} [\iota s \epsilon d[x_1|x_2|\cdots|x_{n+1}]|\iota[x_{n+2}|\cdots|x_k]] = \\
&(-1)^n ([\iota s[x_1|\cdots|x_n]|\iota[x_{n+1}|\cdots|x_k]]) - [\iota s \epsilon d[x_1|\cdots|x_{n+1}]|\iota[x_{n+2}|\cdots|x_k]],
\end{aligned}$$

as claimed.

Notice that  $\theta[\iota[x_1|\cdots|x_k]] = 0$  if  $k \leq 2n$ . Since  $\theta$  is right extended, it follows that  $\theta^2 = 0$  if  $n > 1$ . If  $n = 1$  then  $\theta^2$  might not vanish but then  $\theta^3$  certainly does. This completes the proof.  $\square$

**Corollary 6.3.** *Let  $G$  be a finite group of order  $p^r m$  with  $(p, m) = 1$ . Let  $B[G]$  denote the classifying construction for the group ring  $\mathbb{Z}_{(p)}[G]$ . Then  $p^{2r}$  annihilates  $\tilde{H}_*(\Omega P_n B[G])$  if  $n \geq 2$  and  $p^{3r}$  annihilates  $\tilde{H}_*(\Omega P_1 B[G])$ .*

*Proof.* For a finite group  $G$  of order  $N$  there is a canonical null-homotopy  $s$  for the map  $\phi_N$  on  $B[G]$  given by

$$s([x_1|x_2|\cdots|x_n]) = \sum_{x \in G} [x|x_1|x_2|\cdots|x_n].$$

The map  $s$  is easily seen to satisfy the hypotheses of Proposition 6.2. The result follows from Corollary 3.2.  $\square$

Theorem 1.5 now follows by combining Corollary 6.3 and Proposition 1.4, which we prove in the next section.

## 7. GEOMETRIC REALIZATION OF THE CONSTRUCTION $P_n$

Throughout this section whenever we say a space, we mean a pointed connected CW complex for which the cellular chain complex admits a natural strictly coassociative diagonal. This can of course always be obtained by geometrically realizing the associated singular simplicial set. This restriction is imposed here for simplicity and has no further significance.

Let  $X^{(n)}$  denote the  $n$ -skeleton of a given space  $X$ . Consider the space  $P'_n X$  obtained from  $X$  by collapsing  $X^{(n)}$  to the base point. Obviously the natural map  $\iota'_n : X \rightarrow P'_n X$  induces

an isomorphism on homology groups  $H_i(-; \mathbb{Z})$  for  $i \geq n + 2$ . Furthermore,  $H_n(X^{(n)}; \mathbb{Z})$  is free abelian and thus it follows by inspection of the associated long exact homology sequence that  $H_{n+1}(P'_n X; \mathbb{Z}) \cong H_{n+1}(X; \mathbb{Z}) \oplus F$ , where  $F$  is a free abelian group and that  $\iota'_n$  induces the obvious split monomorphism in this dimension. Notice that the free abelian group  $F$  is no other but the subgroup of boundaries  $B_n(S_*(X; \mathbb{Z}))$ , where by  $S_*(-; \mathbb{Z})$  we denote the cellular chain complex on  $X$ .

Next the Hurewicz isomorphism theorem implies that  $\pi_{n+1} P'_n X \cong H_{n+1}(X; \mathbb{Z}) \oplus F$ . Choose generators  $\{f_i\}_{i \in I}$  for  $F$  and realize each  $f_i$  as a map  $S^{n+1} \xrightarrow{f_i} P'_n X$ . Define  $P_n X$  to be the cofibre in the sequence

$$\bigvee_{i \in I} S_i^{n+1} \xrightarrow{\bigvee f_i} P'_n X \longrightarrow P_n X.$$

It is immediate that the natural map  $\iota_n : X \longrightarrow P_n X$  induces an isomorphism on homology groups in dimensions larger than or equal to  $n + 1$ . This completes the proof of the first part of Proposition 1.4.

To prove the second part an intermediate step is needed. For a  $\mathcal{DGR}$ -coalgebra  $C$ , define  $\bar{P}_n C$  as follows. Let  $F$  denote a free  $R$ -module on as many generators as the submodule of boundaries  $B_n(C)$ . Without loss of generality we may identify  $F$  with  $B_n(C)$ , thus the differential  $d : C_{n+1} \longrightarrow F$  is an epimorphism and one can choose a right inverse  $\epsilon$  for it. Let  $\bar{P}_n C$  be the  $\mathcal{DGR}$ -coalgebra given by dividing  $C$  by its  $n$ -skeleton with the additional modification that  $(\bar{P}_n C)_{n+2} = C_{n+2} \oplus F$ . Define  $\bar{d}(x, y) = dx + \epsilon y$  to be the differential on  $(\bar{P}_n C)_{n+2}$ . Require that every element in  $(\bar{P}_n C)_{n+2}$  is primitive. It is thus easy to verify that the obvious map  $\bar{\iota} : C \longrightarrow \bar{P}_n C$  is a map of  $\mathcal{DGR}$ -coalgebras inducing an isomorphism on homology in dimensions larger than  $n$ . If  $C = S_* X$ , then making compatible choices for the attaching maps defining  $P_n X$  out of  $P'_n X$  and the right inverse  $\epsilon$ , we immediately get that there is an isomorphism of  $\mathcal{DGR}$ -coalgebras  $\bar{P}_n S_* X \cong S_* P_n X$ .

Next for any  $\mathcal{DGR}$ -coalgebra  $C$ , one can define  $P_n C$  and  $\bar{P}_n C$  with respect to the same right inverse  $\epsilon$  for the differential  $C_{n+1} \xrightarrow{d} B_n(C)$ . Thus it is immediate that there is a quasi isomorphism of  $\mathcal{DGR}$ -coalgebras  $\bar{P}_n C \longrightarrow P_n C$ . Applying this for  $C = S_* X$ , completes the proof of Proposition 1.4.

We end this section by pointing out that although  $\bar{P}_n(-)$  and  $P_n(-)$  are equivalent constructions on  $\mathcal{DGR}$ -coalgebras,  $P_n(-)$  behaves nicer with respect to algebraic maps due to the fact that the projection  $C \xrightarrow{\iota_n} P_n C$  is a split epimorphism in the category of  $\mathcal{DGR}$ -modules. On the other hand  $\bar{P}_n(-)$  corresponds more naturally to the geometric analogue.

## 8. NILPOTENT $\mathcal{DGR}$ -ALGEBRAS

In this section we prove Theorem 1.6 and observe that it does not correspond to a geometric situation. We begin with a proof of the theorem.

Let  $A$  be a  $\mathcal{DGR}$ -algebra and assume that the classifying construction  $B(A)$  admits a homology exponent  $r \in R$ . Assume further that  $A$  is nilpotent as an algebra, namely that some power of its augmentation ideal  $IA$  vanishes.

There is a quasi isomorphism of  $\mathcal{DGR}$ -algebras [7]

$$\alpha(A) : \Omega B(A) \longrightarrow A.$$

Since  $\alpha(A)$  is multiplicative,  $\alpha(A)(I\Omega B(A))^n \subset (IA)^n$ , for every  $n \geq 1$ .

Next, by hypothesis  $B(A)$  admits a homology exponent  $r \in R$ . Thus we have a null-homotopy  $s$  for  $\phi_r$  on  $B(A)$ . Let  $\theta$  denote the map constructed on  $\Omega B(A)$  with respect to  $s$ . Then the  $k$ -th iteration  $\theta^k$  of  $\theta$  increases tensor filtration by  $k$  and is homotopic to  $\phi_{r,k}$  up to a sign. Thus for every  $k \geq 1$  we get a commutative diagram

$$\begin{array}{ccccc}
 I\Omega B(A) & \xrightarrow{\theta^k} & (I\Omega B(A))^{k+1} & \xrightarrow{\alpha(A)} & (IA)^{k+1} \\
 \downarrow = & & \downarrow inc & & \downarrow inc \\
 I\Omega B(A) & \xrightarrow{\theta^k} & I\Omega B(A) & \xrightarrow{\alpha(A)} & IA
 \end{array}$$

By hypothesis  $(IA)^n = 0$ , which implies Theorem 1.6.

Next we observe that Theorem 1.6 does not refer to a topological situation, but rather to a purely algebraic one. The content of this remark is contained in the following

**Proposition 8.1.** *Let  $X$  be a simply-connected, rationally contractible space of finite type. Then  $H_*(\Omega X; \mathbb{F}_p)$  is not a nilpotent algebra.*

*Proof.* First observe that a finite loop space cannot possibly have a rationally contractible classifying space. Indeed R. Kane showed [8, pages 256 and 300] that the loop space homology of simply-connected finite  $H$ -spaces is torsion free and hence rationally non-trivial. Thus by  $C$ -class theory of Serre, a finite  $H$ -space cannot have a rationally contractible classifying space. Hence we may assume that  $\Omega X$  is infinite dimensional.

Consider the Hopf algebra  $H_*(\Omega X; \mathbb{F}_p)$ . By assumption it is an infinite dimensional, cocommutative Hopf algebra, which is nilpotent as an algebra. But such Hopf algebras do not exist by Proposition 8.2 below.  $\square$

**Proposition 8.2.** *Let  $A$  be a cocommutative Hopf algebra of finite type and infinite dimension over a field  $k$  of characteristic  $p > 0$ . Then  $A$  is not a nilpotent algebra.*

The rest of this section is devoted to the proof of Proposition 8.2. The proposition is rather elementary and should be well known to the expert. We include a proof for the convenience of the reader. Graded vector spaces are always assumed to be of finite type.

**Lemma 8.3.** *Let  $A$  be a cocommutative Hopf algebra of finite type over a field  $k$  of characteristic  $p > 0$ , which is nilpotent as an algebra. Then the underlying algebra of the dual Hopf algebra  $A^*$  is locally nilpotent, namely there exists no element  $x \in A$  of infinite height.*

*Proof.* Let  $A$  be a cocommutative Hopf algebra over a field  $k$  of characteristic  $p > 0$ , which is nilpotent as an algebra. Assume that the dual commutative Hopf algebra  $A^*$  is not locally nilpotent as an algebra. Then we claim that  $A^*$  contains a primitive element  $y$  of infinite height. Indeed choose an element  $x$  of infinite height and minimal degree. Then, since every element of lower degree is nilpotent, some power  $y = x^{p^k}$  must be primitive.

Let  $y \in A^*$  be a primitive element of infinite height. Then there is a monomorphism of Hopf algebras

$$P[y] \xrightarrow{\phi} A^*.$$

Dualizing we get an epimorphism of Hopf algebras

$$A \xrightarrow{\phi^*} \Gamma[y],$$

where  $\Gamma[y]$  denotes the divided polynomial algebra over  $k$  on one generator. In particular  $\Gamma[y]$  contains products of arbitrary length and thus so does  $A$  contradicting its nilpotency.  $\square$

**Lemma 8.4.** *Let  $A$  be a commutative Hopf algebra of infinite dimension over a field  $k$  of characteristic  $p > 0$ , which as an algebra is locally nilpotent. Then the dual Hopf algebra  $A^*$  contains non-vanishing products of arbitrary length.*

*Proof.* Since  $A$  is commutative the Borel structure theorem applies and  $A$  is isomorphic as an algebra to a tensor product of monogenic algebras. Assuming that  $A$  is locally nilpotent implies that the Borel decomposition for  $A$  does not contain polynomial factors. Hence it must be infinitely generated as an algebra. Consequently the module of primitives in the dual Hopf algebra  $P(A^*)$  is infinite dimensional. But a product of an arbitrary number of distinct primitives in a Hopf algebra is never zero. The result follows.  $\square$

Lemmas 8.3 and 8.4 imply that a cocommutative Hopf algebra over  $k$ , for which the underlying algebra is nilpotent must be finite dimensional, which completes the proof of Proposition 8.2.

## 9. SPECULATIONS, FINITENESS PROPERTIES

In the study of loop space homology torsion, it seems reasonable to try and relate certain finiteness conditions on the objects under consideration to the existence of exponents. One may take two different approaches here. Fix a  $\mathcal{DGR}$ -coalgebra  $C$  with a reduced homology exponent  $r \in R$ .

The first approach is to assume certain finiteness conditions on  $C$  and try to gain control on the growth of torsion in loop space homology. In particular one might be interested in finding a loop space homology exponent. Thus one may assume for instance that  $C$  is of finite rank over  $R$ , in which case we have shown that  $\Omega C$  admits a homology exponent. If  $C$  has a trivial coproduct then it is immediate that  $C$  and  $\Omega C$  have the same homology exponent as is the case when  $C$  is the chain coalgebra for a rationally contractible suspension space. One might thus expect that nilpotency of the  $Hom$ -dual of a  $\mathcal{DGR}$ -coalgebra  $C$  should imply the existence of a loop space homology exponent. However by Theorem 1.2 this is not nearly the case. Indeed the  $\mathcal{DGZ}$ -coalgebras  $C^{k,t}$  constructed in the theorem have the property that their duals are nilpotent of degree 3.

A finiteness condition on  $C$ , which we did not consider here is the existence of only finitely many primitives in  $C$ . Namely one may conjecture that if  $C$  contains only finitely many primitives, then  $\Omega C$  admits a homology exponent. In fact we are not aware of a counter example to this.

The second approach to the problem is taken in Section 8 above and arises by recalling the fact that for every  $\mathcal{DGR}$ -algebra  $A$ , the natural map  $\Omega B(A) \rightarrow A$  is a homotopy equivalence.

Thus one may like to consider  $\mathcal{DGR}$ -coalgebras of the form  $B(A)$ , which satisfy our basic hypothesis of having a homology exponent and impose additional finiteness conditions on  $A$ .

If  $A$  is concentrated in degree 0, then the study of its homology doesn't make too much sense. But in this case one might like to study the associated  $\mathcal{DGR}$ -algebras  $\Omega P_n B(A)$ . If  $A$  is of finite rank over  $R$ , as is the case if  $A = R[G]$  for a finite group  $G$ , then it is not hard to construct a condition which imply that  $B(A)$  admits a transfer type null-homotopy for  $\phi_r$ , where  $r$  is the image of the rank of  $A$  in  $R$  under the canonical map from the integers. However the existence of such a null-homotopy appears to impose severe restrictions on the algebra structure of  $A$ . We thus find it reasonable to conjecture that there exist algebras  $A$ , concentrated in degree 0, which are finitely generated as algebras or even of finite rank over  $R$ , such that  $B(A)$  has a homology exponent but  $\Omega P_n B(A)$  does not have one.

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