EQUIVARIANT UNIVERSAL COEFFICIENT AND KÜNNETH SPECTRAL SEQUENCES

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Abstract. We construct hyper-homology spectral sequences of $\mathbb{Z}$-graded and $RO(G)$-graded Mackey functors for Ext and Tor over G-equivariant S-algebras (A_∞ ring spectra) for finite groups G. These specialize to universal coefficient and Künneth spectral sequences.

Introduction

In the non-equivariant context, universal coefficient and Künneth spectral sequences provide important tools for computing generalized homology and cohomology. EKMM [5] constructs examples of these types of spectral sequences for theories that come from "S-algebras" (or, equivalently, A_∞ ring spectra). These spectral sequences are special cases of the general "hyper-homology" spectral sequences for computing "Tor" (the homotopy groups of the derived smash product) and "Ext" (the homotopy groups of the derived function module) for the category of modules over an S-algebra. The purpose of this paper is to construct versions of these hyper-homology spectral sequences for the category of modules over an G-equivariant S-algebra indexed on a complete universe, where G is a finite group. In Section 1, we derive a number of equivariant universal coefficient and Künneth spectral sequences from these equivariant hyper-homology spectral sequences.

In this context, the homotopy groups of a G-spectrum X form a graded Mackey functor. There is an abelian group $(\underline{\pi}_q X)(G/H) = \pi_q^H X = \pi_q(X^H)$ for each subgroup H of G, and for each subgroup K of H there are maps $\pi_q^H X \to \pi_q^K X$ (induced by the inclusion of fixed points) and $\pi_q^K X \to \pi_q^H X$ (the transfer), satisfying various natural relations. This "homotopy Mackey functor" can be regarded as graded over either the integers or the real representation ring of G. Both choices lead to equivariant generalizations of the spectral sequences of EKMM [5]. Although these generalizations appear formally similar and are developed here in parallel, the spectral sequences have a very different calculational feel.

As reviewed in Section 2, the categories of $\mathbb{Z}$-graded and of $RO(G)$-graded Mackey functors have symmetric monoidal products denoted "$\Box_\#$" and "$\Box_*$" respectively. The products have adjoint function object functors "$\langle -,-\rangle_\#$" and "$\langle -,-\rangle_*$". There is an obvious notion of a graded Mackey functor ring (in either context), which consists of a graded Mackey functor $\underline{R}_*$ with an associative and unital multiplication $\underline{R}_* \Box \underline{R}_* \to \underline{R}_*$. We have the evident notion of a left (resp. right) $\underline{R}_*$-module as a graded Mackey functor $\underline{M}_*$ with an associative and

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unital map $\underline{R}_* \,\Box\, \underline{M}_* \to \underline{M}_*$ (resp. $\underline{N}_* \,\Box\, \underline{R}_* \to \underline{N}_*$). The usual coequalizer defines a functor taking a right $\underline{R}_*$-module $\underline{N}_*$ and a left $\underline{R}_*$-module $\underline{M}_*$ to a graded Mackey functor $\underline{N}_* \,\Box_{\underline{R}_*}\, \underline{M}_*$. Similarly, the usual equalizer defines a functor taking a pair of left $\underline{R}_*$-modules $\underline{L}_*$ and $\underline{M}_*$ to the graded Mackey functor $\langle \underline{L}_*, \underline{M}_* \rangle^{\underline{R}_*}$. In Section 4, standard homological algebra is used to construct the associated derived functors. We define $\underline{\mathrm{Tor}}_s^{\underline{R}_*}(\underline{N}_*, \underline{M}_*)$ as the s-th left derived functor of $\underline{N}_* \,\Box_{\underline{R}_*}\, \underline{M}_*$ and $\underline{\mathrm{Ext}}^s_{\underline{R}_*}(\underline{L}_*, \underline{M}_*)$ as the s-th right derived functor of $\langle \underline{L}_*, \underline{M}_* \rangle^{\underline{R}_*}$. These are regarded as bigraded Mackey functors with the usual conventions: $\underline{\mathrm{Tor}}^{\underline{R}_*}_{s,\tau}(\underline{N}_*, \underline{M}_*) = (\underline{\mathrm{Tor}}_s^{\underline{R}_*}(\underline{N}_*, \underline{M}_*))_\tau$ and $\underline{\mathrm{Ext}}^{s,\tau}_{\underline{R}_*}(\underline{L}_*, \underline{M}_*) = (\underline{\mathrm{Ext}}^s_{\underline{R}_*}(\underline{L}_*, \underline{M}_*))_{-\tau}$. The homological grading in s is always over the non-negative integers whereas the internal grading in τ is over $\mathbb{Z}$ or $RO(G)$. When $\underline{R}_*$ is commutative, $\Box_{\underline{R}_*}$ becomes a closed symmetric monoidal product on the category of left $\underline{R}_*$-modules, and $\underline{\mathrm{Tor}}_s^{\underline{R}_*}$ and $\underline{\mathrm{Ext}}^s_{\underline{R}_*}$ become graded $\underline{R}_*$-modules. More generally, when M is an $\underline{R}_*$-bimodule (has commuting left and right module structures), $\underline{\mathrm{Tor}}_s^{\underline{R}_*}(\underline{N}_*, \underline{M}_*)$ and $\underline{\mathrm{Ext}}^s_{\underline{R}_*}(\underline{L}_*, \underline{M}_*)$ become graded right $\underline{R}_*$-modules via the "unused" $\underline{R}_*$-module structure on $\underline{M}_*$. Similarly, when $\underline{N}_*$ and $\underline{L}_*$ are $\underline{R}_*$-bimodules, $\underline{\mathrm{Tor}}_s^{\underline{R}_*}(\underline{N}_*, \underline{M}_*)$ and $\underline{\mathrm{Ext}}^s_{\underline{R}_*}(\underline{L}_*, \underline{M}_*)$ become graded left $\underline{R}_*$-modules.

The equivariant stable category has a symmetric monoidal product "$\wedge$", and the homotopy Mackey functor $\underline{\pi}_*$ is a lax symmetric monoidal functor into either $\mathbb{Z}$-graded or $RO(G)$-graded Mackey functors. In other words, there is a suitably associative, commutative, and unital natural transformation

$$\underline{\pi}_* X \,\Box\, \underline{\pi}_* Y \longrightarrow \underline{\pi}_*(X \wedge Y).$$

For formal reasons, we obtain an adjoint natural transformation

$$\underline{\pi}_* F(X, Y) \longrightarrow \langle \underline{\pi}_* X, \underline{\pi}_* Y \rangle,$$

where F denotes the function spectrum. Further, when R is a homotopical ring spectrum (i.e., a ring spectrum in the equivariant stable category), $\underline{\pi}_* R$ is a Mackey functor ring, commutative when R is. Likewise, when M is a homotopical R-module, $\underline{\pi}_* M$ becomes a $\underline{\pi}_* R$-module. When R is an equivariant S-algebra, we have a smash product over R and an R-function spectrum functor. In this context, it is convenient to extend the conventions of EKMM [5] by using $\underline{\mathrm{Tor}}^R_*(N, M)$ and $\underline{\mathrm{Ext}}^*_R(L, M)$ for the homotopy Mackey functors of the derived functors. The natural transformations above descend to natural transformations

$$\underline{N}_* \,\Box_{\underline{R}_*}\, \underline{M}_* \to \underline{\mathrm{Tor}}^R_*(N, M) \quad \text{and} \quad \underline{\mathrm{Ext}}^*_R(L, M) \to \langle \underline{L}_*, \underline{M}_* \rangle^{\underline{R}_*},$$

where $\underline{R}_* = \underline{\pi}_* R$, $\underline{L}_* = \underline{\pi}_* L$, $\underline{M}_* = \underline{\pi}_* M$, and $\underline{N}_* = \underline{\pi}_* N$. When M is a weak bimodule, i.e., has a homotopical right R-module structure in the category of left R-modules, $\underline{\mathrm{Tor}}^R_*(N, M)$ and $\underline{\mathrm{Ext}}^{-*}_R(L, M)$ are naturally right $\underline{R}_*$-modules and the maps above are maps of right R-modules. Similar statements hold for L and N. Our main results are the following two theorems.

Theorem. *(Hyper-Tor Spectral Sequences) Let G be a finite group, R be an equivariant S-algebra indexed on a complete universe, and M and N be a left and a right R-module. There are strongly convergent spectral sequences*

$$E^2_{s,\tau} = \underline{\mathrm{Tor}}^{\underline{R}_*}_{s,\tau}(\underline{N}_*, \underline{M}_*) \quad \Longrightarrow \quad \underline{\mathrm{Tor}}^R_{s+\tau}(N, M),$$

of $\mathbb{Z}$-graded Mackey functors and of $RO(G)$-graded Mackey functors which are natural in M and N. The edge homomorphisms are the usual natural transformations

$\underline{N}_* \,\square_{\underline{R}_*}\, \underline{M}_* \to \underline{\mathrm{Tor}}^R_*(N, M)$. *When R is commutative or M is a weak bimodule, these are spectral sequences of right $\underline{R}_*$-modules. When N is a weak bimodule, they are spectral sequences of left $\underline{R}_*$-modules.*

Theorem. *(Hyper-Ext Spectral Sequences) Let G be a finite group, R be an equivariant S-algebra indexed on a complete universe, and L and M be left R-modules. There are conditionally convergent spectral sequences*

$$E_2^{s,\tau} = \underline{\mathrm{Ext}}^{s,\tau}_{\underline{R}_*}(\underline{L}_*, \underline{M}_*) \quad \Longrightarrow \quad \underline{\mathrm{Ext}}^{s+\tau}_R(L, M),$$

of $\mathbb{Z}$-graded Mackey functors and of $RO(G)$-graded Mackey functors which are natural in L and M. The edge homomorphisms are the usual natural transformations $\underline{\mathrm{Ext}}^*_R(L, M) \to \langle \underline{L}_*, \underline{M}_* \rangle^{\underline{R}_*}$. When R is commutative or M is a weak bimodule, these are spectral sequences of right $\underline{R}^*$-modules. When L is a weak bimodule, these are spectral sequences of left $\underline{R}^*$-modules.

To be precise about the differentials, the Hyper-Tor spectral sequence has r-th differential

$$d^r_{s,t} \colon E^r_{s,t} \longrightarrow E^r_{s-r,t+r-1}.$$

Thus, in the $\mathbb{Z}$-graded context, it is a homologically graded right half-plane spectral sequence of Mackey functors. In the $RO(G)$-graded context, it essentially consists of one homologically graded right half-plane spectral sequence of Mackey functors for each element of the subgroup $\widetilde{RO}(G)$ of $RO(G)$ consisting of virtual representations of dimension zero. The Hyper-Ext spectral sequence has r-th differential

$$d_r^{s,t} \colon E_r^{s,t} \longrightarrow E_r^{s+r,t-r+1}.$$

In the $\mathbb{Z}$-graded context, it is a cohomologically graded right half-plane spectral sequence of Mackey functors. In the $RO(G)$-graded context, it consists of one cohomologically graded right half-plane spectral sequence of Mackey functors for each element of $\widetilde{RO}(G)$.

In the $\mathbb{Z}$-graded context, "strong" convergence means that for every subgroup H of G, the spectral sequence of abelian groups $E^r_{s,t}(G/H)$ converges strongly. In the $RO(G)$-graded context, it means that for every subgroup H of G and every $\tilde{\tau}$ in $\widetilde{RO}(G)$, the spectral sequence consisting of the abelian groups $E^r_{s,\tilde{\tau}+t}(G/H)$ converges strongly. Conditional convergence is defined analogously in terms of the abelian groups $E_r^{s,t}(G/H)$ and $E_r^{s,\tilde{\tau}+t}(G/H)$.

We emphasize again that, although we construct them in parallel, the $\mathbb{Z}$-graded and $RO(G)$-graded versions of these spectral sequences are typically very different. For example, if $L = R \wedge S^V$ for a finite dimensional G-representation V, then $\underline{L}_*$ is projective as an $RO(G)$-graded Mackey functor $\underline{R}_*$-module, but is usually not projective as a $\mathbb{Z}$-graded Mackey functor $\underline{R}_*$-module. The $RO(G)$-graded spectral sequence collapses at E_2 and is concentrated in homological degree zero; this typically does not happen in the $\mathbb{Z}$-graded spectral sequence.

The peculiarities in the homological algebra of Mackey functors that occur when G is not finite or when the universe is not complete (cf. Lewis [7]) make it worthwhile to observe that the homological behavior of the graded Mackey functors in the present context is perfectly standard.

Remark. Let $\underline{R}_*$ be a $\mathbb{Z}$-graded or $RO(G)$-graded Mackey functor ring.

(a) If $\underline{M}_*$ is a projective left $\underline{R}_*$-module or $\underline{N}_*$ is a projective right $\underline{R}_*$-module, then $\underline{\mathrm{Tor}}_s^{\underline{R}_*}(\underline{N}_*, \underline{M}_*) = 0$ in homological degree $s > 0$. In this case, the Hyper-Tor spectral sequence above collapses at E^2 and the edge homomorphism is an isomorphism.

(b) If $\underline{L}_*$ is a projective left $\underline{R}_*$-module or $\underline{M}_*$ is an injective left $\underline{R}_*$-module, then $\underline{\mathrm{Ext}}^s_{\underline{R}_*}(\underline{L}_*, \underline{M}_*) = 0$ in cohomological degree $s > 0$. In this case, the Hyper-Ext spectral sequence above collapses at E_2 and the edge homomorphism is an isomorphism.

(c) For any left $\underline{R}_*$-modules $\underline{L}_*$, $\underline{M}_*$, the usual (Yoneda) Ext groups can be identified as $\mathrm{Ext}^s_{\underline{R}_*}(\underline{L}_*, \underline{M}_*) = (\underline{\mathrm{Ext}}^{s,0}_{\underline{R}_*}(\underline{L}_*, \underline{M}_*))(G/G)$. Evaluation of a Mackey functor at G/G is exact. Thus, our Hyper-Ext spectral sequence specializes to a conditionally convergent spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\underline{R}_*}(\underline{L}_*, \underline{M}_*) \quad \Longrightarrow \quad \mathrm{Ho}G\mathscr{M}_R^{s+t}(L, M).$$

Here, $\mathrm{Ho}G\mathscr{M}_R^{s+t}(L, M)$ is the abelian group of G-maps from L to $\Sigma^{s+t}M$ in the homotopy category of R-modules.

The final topic considered in this paper is that of composition pairings. The internal function object $F_R(-,-)$ has a composition pairing

$$F_R(M, N) \wedge F_R(L, M) \longrightarrow F_R(L, N).$$

that induces a pairing

$$\underline{\mathrm{Ext}}^s_R(M, N) \,\square\, \underline{\mathrm{Ext}}^t_R(L, M) \longrightarrow \underline{\mathrm{Ext}}^{s+t}_R(L, N)$$

If R is commutative, then this composition pairing descends to $\wedge_R$, becoming a map of R-modules, and the pairing on $\underline{\mathrm{Ext}}$ descends to $\square_{\underline{R}^*}$, becoming a map of $\underline{R}^*$-modules. Likewise, for graded Mackey functor modules, the internal function object $\langle -,- \rangle^{\underline{R}_*}$ has a composition pairing that induces a "Yoneda pairing" on $\underline{\mathrm{Ext}}$-objects

$$\underline{\mathrm{Ext}}^s_{\underline{R}_*}(\underline{M}_*, \underline{N}_*) \,\square\, \underline{\mathrm{Ext}}^t_{\underline{R}_*}(\underline{L}_*, \underline{M}_*) \longrightarrow \underline{\mathrm{Ext}}^{s+t}_{\underline{R}_*}(\underline{L}_*, \underline{N}_*).$$

We prove the following theorem regarding these pairings.

Theorem. *The Yoneda pairing on $\underline{\mathrm{Ext}}^*_{\underline{R}_*}$ induces a pairing of Hyper-Ext spectral sequences that converges (conditionally) to the composition pairing on $\underline{\mathrm{Ext}}^*_R$. When R is commutative, this is a pairing of spectral sequences of $\underline{R}^*$-modules.*

In the case when G is the trivial group, our argument corrects an error in EKMM [5, IV§5].

Organization. In section 1, we construct various universal coefficient and Künneth spectral sequences from the spectral sequences described above. Section 2 contains a brief review of the categories of graded Mackey functors, graded box products, and graded function Mackey functors. The definitions of graded Mackey functor rings and modules and the definitions and basic properties of the graded box product and function object over a graded Mackey functor ring are reviewed in Section 3. In Section 4, we discuss projective and injective graded Mackey functor modules and use them to define $\underline{\mathrm{Ext}}$ and $\underline{\mathrm{Tor}}$. Our work in equivariant stable homotopy theory begins in Section 5 with a discussion of R-modules whose homotopy Mackey functors are projective or injective $\underline{R}_*$-modules. The spectral sequences introduced above are constructed in Section 6. Our results on the naturality of these spectral sequences and on the Yoneda pairing are proven in Section 7. Finally, in the appendix, we prove the folk theorem that the $RO(G)$-graded homotopy Mackey functor is a lax symmetric monoidal functor, i.e., that the smash product of equivariant spectra is compatible with the graded box product.

Notation and conventions. Throughout this paper, G is a fixed finite group, and the "equivariant stable category" means the derived category (obtained by formally inverting the weak equivalences) of one of the modern categories of equivariant spectra, i.e., the category of equivariant S-modules, the category of equivariant orthogonal spectra, or the category of equivariant symmetric spectra. Except in Section 1, which is written from the point of view of homology and cohomology theory, the indexing universe for spectra is always understood to be complete. The term "G-spectrum" is used as an abbreviation for "object in the equivariant stable category".

Except in Section 7, we phrase all statements and arguments in the equivariant stable category and in derived categories of modules. The undecorated symbol "$\wedge$" means the smash product of spectra in the equivariant stable category or the smash product with a spectrum in the derived category of R-modules. We write "$\wedge_S$" when we mean the point-set smash product. Likewise $F(-,-)$ denotes the derived function spectrum. The point-set level functor adjoint to $\wedge_S$ is denoted $F_S(-,-)$. If R is an equivariant S-algebra, then $\mathrm{Tor}^R(N,M)$ is the derived balanced smash product of a left R-module M and a right R-module N. Also, $\mathrm{Ext}_R(L,M)$ is the derived function spectrum (or R-module) of left R-modules L and M. Further, $\underline{\mathrm{Tor}}^R_*(M,N) = \underline{\pi}_*(\mathrm{Tor}^R(N,M))$ and $\underline{\mathrm{Ext}}^*_R = \underline{\pi}_{-*}(\mathrm{Ext}_R(L,M))$. We reserve $(-)\wedge_R (-)$ and $F_R(-,-)$ for the point-set level functors.

Grading conventions. The usual conventions for homology and cohomology theories and for derived functors require the introduction of "homological" and "cohomological" grading. For a (homologically) graded Mackey functor $\underline{M}_*$, the corresponding cohomologically graded Mackey functor $\underline{M}^*$ is defined by $\underline{M}^\alpha = \underline{M}_{-\alpha}$ for all α. When $\underline{M}_*$ is a left $\underline{R}_*$-module, $\underline{M}^*$ is a left $\underline{R}^*$-module. The choice of whether to regard the category of left $\underline{R}_*$-modules and the category of left $\underline{R}^*$-modules as different categories or the same category with two different grading conventions is a philosophical one: The same formulas hold provided we are consistent about which grading we use in the latter case. For example, there are canonical natural isomorphisms

$$\underline{\mathrm{Ext}}^{s,\tau}_{\underline{R}^*}(\underline{L}^*,\underline{M}^*) \cong \underline{\mathrm{Ext}}^{s,\tau}_{\underline{R}_*}(\underline{L}_*,\underline{M}_*)$$
$$\underline{\mathrm{Tor}}^{\underline{R}^*}_{s,\tau}(\underline{N}^*,\underline{M}^*) \cong \underline{\mathrm{Tor}}^{\underline{R}_*}_{s,\tau}(\underline{N}_*,\underline{M}_*)$$

(where $\underline{\mathrm{Tor}}^{\underline{R}^*}_{s,\tau}(\underline{N}^*,\underline{M}^*) = (\underline{\mathrm{Tor}}^{\underline{R}^*}_{s}(\underline{N}^*,\underline{M}^*))^{-\tau}$). For this reason, except in the statements of the spectral sequences in Section 1 and in the final construction of them in Section 6, the main body of the paper treats graded Mackey functors exclusively in homologically graded terms.

1. Universal Coefficient and Künneth Spectral Sequences

The hyper-homology spectral sequences of the introduction lead formally to spectral sequences for computing equivariant homology and cohomology. In particular, we obtain universal coefficient spectral sequences for computing $\underline{E}_*(X)$ and $\underline{E}^*(X)$ when either the theory E is represented by an R-module or the spectrum X is represented by an R-module. Likewise, there are Künneth spectral sequences for computing $\underline{R}_*(X\wedge Y)$ and $\underline{R}^*(X\wedge Y)$. Since the equivariant homology and cohomology theories involved are $RO(G)$-graded, the arguments apply to the construction of both $RO(G)$-graded and $\mathbb{Z}$-graded spectral sequences. The resulting $RO(G)$-graded spectral sequences are new even in the case when the theories involved are ordinary (Bredon) homology and cohomology. In fact, the motivation for the research leading to this paper was the need for such spectral sequences in [6]. The Künneth spectral sequences are new for equivariant K-theory.

In the statements below, R is a fixed equivariant S-algebra indexed on a complete universe. Given a G-spectrum E indexed on a complete universe and a G-spectrum X indexed on any universe U', the E-homology of X is the $RO(G)$-graded homotopy Mackey functor of $(i_*X)\wedge E$. Here, i_* is the left adjoint of the forgetful functor from the equivariant stable category indexed on the complete universe to the equivariant stable category indexed on U'. The spectrum $(i_*X)\wedge E$ is sometimes denoted $X\wedge E$ to compactify notation. The E-cohomology of X is the $RO(G)$-graded Mackey functor

$$\underline{E}^\alpha(X)(G/H) = [X\wedge G/H_+, \Sigma^\alpha E]_G \cong [X, \Sigma^\alpha E]_H$$

or more accurately $[i_*X\wedge G/H_+, \Sigma^\alpha E]_G$. Of course, all functors used here should be understood as the derived functors.

The forgetful functor from the derived category of left R-modules to the equivariant stable category on a complete universe has a left adjoint functor that we denote as $\mathbb{R}$. We write $\mathbb{R}^{\mathrm{op}}$ for the analogous functor into the derived category of right R-modules. These functors have the usual properties expected of derived free R-module functors (q.v. [8]). For example, since R-modules are necessarily homotopical R-modules, we have natural maps

$$R\wedge X\to \mathbb{R}X \qquad \text{and} \qquad X\wedge R\to \mathbb{R}^{\mathrm{op}}X$$

in the equivariant stable category (on a complete universe). These maps are always isomorphisms in the equivariant stable category. We can compose the unit map $X\to\mathbb{R}X$ with the canonical comparison maps for the derived smash product and derived function spectra to get natural maps

$$X\wedge M\to \mathrm{Tor}^R(\mathbb{R}^{\mathrm{op}}X, M) \quad \text{and} \quad \mathrm{Ext}_R(\mathbb{R}X, M)\to F(X,M)$$

in the equivariant stable category for any left R-module M. These maps are also always isomorphisms. Just as $R\wedge X$ and $X\wedge R$ are naturally homotopical R-bimodules (R-bimodules in the equivariant stable category), $\mathbb{R}X$ and $\mathbb{R}^{\mathrm{op}}X$ are naturally weak bimodules in their respective categories: $\mathbb{R}X$ is a homotopical right R-module in the derived category of left R-modules and $\mathbb{R}^{\mathrm{op}}X$ is a homotopical left R-module in the derived category of right R-modules. Moreover, the four maps in the equivariant stable category displayed above are maps of homotopical R-modules.

As a consequence, by taking $N=\mathbb{R}^{\mathrm{op}}X$ in the Hyper-Tor spectral sequence and $L=\mathbb{R}X$ in the Hyper-Ext spectral sequence, we obtain the following spectral sequences of $\underline{R}_*$-modules.

Theorem 1.1. *(Universal Coefficient Spectral Sequence) Let X be a G-spectrum and M be a left R-module. There is a natural strongly convergent homology spectral sequence of $\underline{R}_*$-modules*

$$E^2_{s,\tau} = \underline{\mathrm{Tor}}^{\underline{R}_*}_{s,\tau}(\underline{R}_*X, \underline{M}_*) \quad\Longrightarrow\quad \underline{M}_{s+\tau}X$$

and a natural conditionally convergent cohomology spectral sequence of $\underline{R}_*$-modules

$$E_2^{s,\tau} = \underline{\mathrm{Ext}}^{s,\tau}_{\underline{R}_*}(\underline{R}_*X, \underline{M}_*) \quad \Longrightarrow \quad \underline{M}^{s+\tau}X.$$

The forgetful functor from the derived category of left R-modules to the equivariant stable category also has a right adjoint functor. This adjoint is denoted $\mathbb{R}^\sharp$ and has the usual properties expected from a derived cofree R-module functor [8]. In particular, if E is a G-spectrum indexed on a complete universe, then the canonical map $\mathbb{R}^\sharp E \to F(R, E)$ and the canonical comparison map of derived function spectra

$$\mathrm{Ext}_R(M, \mathbb{R}^\sharp E) \longrightarrow F(M, E)$$

are isomorphisms in the equivariant stable category. Once again, $\mathbb{R}^\sharp E$ is naturally a weak bimodule and the above maps in the equivariant stable category are maps of homotopical R-modules. Plugging $\mathbb{R}^\sharp E$ into the Hyper-Ext spectral sequence then gives the cohomological spectral sequence in the theorem below. Under the natural isomorphism

$$\underline{R}_*E = \underline{\pi}_*(E \wedge R) \cong \underline{\pi}_*(R \wedge E) = \underline{E}_*R,$$

the homological spectral sequence in this theorem coincides with the homological spectral sequence of the previous theorem.

Theorem 1.2. *Let E be a G-spectrum indexed on a complete universe and M be a left R-module. There is a natural strongly convergent homology spectral sequence of $\underline{R}_*$-modules*

$$E^2_{s,\tau} = \underline{\mathrm{Tor}}^{\underline{R}_*}_{s,\tau}(\underline{E}_*R, \underline{M}_*) \quad \Longrightarrow \quad \underline{E}_{s+\tau}M$$

and a natural conditionally convergent cohomology spectral sequence of $\underline{R}_*$-modules

$$E_2^{s,\tau} = \underline{\mathrm{Ext}}^{s,\tau}_{\underline{R}^*}(\underline{M}^*, \underline{E}^*R) \quad \Longrightarrow \quad \underline{E}^{s+\tau}M.$$

Künneth spectral sequences for computing the homology and cohomology of $X \wedge Y$ arise as special cases of our universal coefficient spectral sequences. Let X and Y be G-spectra indexed on the same universe U'. The change of universe functor i_* is symmetric monoidal; that is, there is an isomorphism $i_*X \wedge i_*Y \cong i_*(X \wedge Y)$ in the equivariant stable category. The observations preceding Theorem 1.1 yield a canonical isomorphism

$$i_*(X \wedge Y) \wedge R \cong (i_*X) \wedge (i_*Y) \wedge R \cong (i_*X) \wedge R \wedge (i_*Y) \cong \mathrm{Tor}^R(\mathbb{R}^{\mathrm{op}}(i^*X), \mathbb{R}(i^*Y))$$

in the equivariant stable category. Taking $M = \mathbb{R}i_*Y$ in the first spectral sequence of Theorem 1.1 gives the first spectral sequence in Theorem 1.3 below. There is a similar canonical isomorphism

$$\mathrm{Ext}_R(\mathbb{R}i_*X, F(i_*Y, R)) \cong F(i_*X, F(i_*Y, R)) \cong F(i_*(X \wedge Y), R),$$

in the equivariant stable category. Via this isomorphism, the second spectral sequence of Theorem 1.1 for $M = F(i_*Y, R)$ gives the second spectral sequence below.

Theorem 1.3. *(Künneth Theorem) Let X and Y be G-spectra indexed on the same universe. There is a natural strongly convergent homology spectral sequence of $\underline{R}_*$-modules*

$$E^2_{s,\tau} = \underline{\mathrm{Tor}}^{\underline{R}_*}_{s,\tau}(\underline{R}_*X, \underline{R}_*Y) \quad \Longrightarrow \quad \underline{R}_{s+\tau}(X \wedge Y),$$

and a natural conditionally convergent cohomology spectral sequence of $\underline{R}_*$-modules

$$E_2^{s,\tau} = \underline{\mathrm{Ext}}^{s,\tau}_{\underline{R}^*}(\underline{R}_{-*}X, \underline{R}^*Y) \quad \Longrightarrow \quad \underline{R}^{s+\tau}(X \wedge Y).$$

Plugging $M = \mathbb{R}X$ into the spectral sequences of Theorem 1.2 yields the following spectral sequences.

Theorem 1.4. *Let X be a G-spectrum and let E be a G-spectrum indexed on the complete universe. There is a natural strongly convergent homology spectral sequence of $\underline{R}_*$-modules*

$$E^2_{s,\tau} = \underline{\mathrm{Tor}}^{\underline{R}_*}_{s,\tau}(\underline{E}_* R, \underline{R}_* X) \quad \Longrightarrow \quad \underline{E}_{s+\tau}(R \wedge X),$$

and a natural conditionally convergent cohomology spectral sequence of $\underline{R}_*$-modules

$$E_2^{s,t} = \underline{\mathrm{Ext}}^{s,t}_{\underline{R}^*}(\underline{R}_{-*}X, \underline{E}^* R) \quad \Longrightarrow \quad \underline{E}^{s+t}(R \wedge X).$$

The spectral sequences above can be combined with Spanier-Whitehead duality to obtain additional spectral sequences. If X is a finite G-spectrum, then the Spanier-Whitehead dual of $i_* X$ is a G-spectrum DX (indexed on the complete universe). There are canonical isomorphisms $E_* DX \cong E^{-*}X$ and $E^* DX \cong E_{-*}X$, natural in E and in X. Plugging DX into the spectral sequences of Theorem 1.1 gives us cohomology to cohomology and cohomology to homology universal coefficient spectral sequences.

Theorem 1.5. *(Dual Universal Coefficient Spectral Sequence) Let X be a finite G-spectrum, and M a left R-module. There is a natural strongly convergent homology spectral sequence of $\underline{R}_*$-modules*

$$E^2_{s,\tau} = \underline{\mathrm{Tor}}^{\underline{R}^*}_{s,\tau}(\underline{R}^* X, \underline{M}^*) \quad \Longrightarrow \quad \underline{M}^{-(s+\tau)}X$$

and a natural conditionally convergent cohomology spectral sequence of $\underline{R}_*$-modules

$$E_2^{s,\tau} = \underline{\mathrm{Ext}}^{s,\tau}_{\underline{R}_*}(\underline{R}^{-*}X, \underline{M}_*) \quad \Longrightarrow \quad \underline{M}_{-(s+\tau)}X.$$

These dual universal coefficient spectral sequences imply the following Künneth spectral sequences.

Theorem 1.6. *(Dual Künneth Theorem) Let X and Y be G-spectra indexed on the same universe, and assume that X is finite. There is a natural strongly convergent homology spectral sequence of $\underline{R}_*$-modules*

$$E^2_{s,\tau} = \underline{\mathrm{Tor}}^{\underline{R}^*}_{s,\tau}(\underline{R}^* X, \underline{R}^* Y) \quad \Longrightarrow \quad \underline{R}^{-(s+\tau)}(X \wedge Y),$$

and a natural conditionally convergent cohomology spectral sequence of $\underline{R}_*$-modules

$$E_2^{s,\tau} = \underline{\mathrm{Ext}}^{s,\tau}_{\underline{R}_*}(\underline{R}^{-*}X, \underline{R}_* Y) \quad \Longrightarrow \quad \underline{R}_{-(s+\tau)}(X \wedge Y).$$

2. Graded Mackey functors

This section introduces the categories of $RO(G)$-graded and $\mathbb{Z}$-graded Mackey functors. Box products and function objects for these categories are defined in terms of box products and function objects for ungraded Mackey functors. In this discussion, some familiarity with (ungraded) Mackey functors is assumed. However, certain key definitions are reviewed in order to fix notation or to explain our perspective.

Recall that the Burnside category $\mathfrak{B}_G$ may be defined as the essentially small additive category whose objects are the finite G-sets and whose abelian group of morphisms between the finite G-sets X and Y is the abelian group

$$\mathfrak{B}_G(X, Y) = [\Sigma^\infty X_+, \Sigma^\infty Y_+]_G,$$

of morphisms between the associated suspension spectra in the equivariant stable category. Disjoint union of finite G-sets provides the direct sum operation giving $\mathfrak{B}_G$ its additive structure. A rather simple, purely algebraic description of the morphisms in $\mathfrak{B}_G$ can be found in [9, §V.9]; however, the approach to the category of Mackey functors best suited to our purposes is in terms of the stable homotopy theoretic description of $\mathfrak{B}_G$.

Definition 2.1. The category $\mathfrak{M}$ of Mackey functors is the category of contravariant additive functors from the Burnside category $\mathfrak{B}_G$ to the category $\mathfrak{Ab}$ of abelian groups.

Spanier–Whitehead duality provides an isomorphism of categories between $\mathfrak{B}_G$ and $\mathfrak{B}_G^{\mathrm{op}}$ that is the identity on the objects. The category of Mackey functors could therefore be defined equivalently as the category of covariant functors from $\mathfrak{B}_G$ to $\mathfrak{Ab}$. Mackey functors can also be described in terms of orbits. Since every finite G-set is a disjoint union of orbits, every object of $\mathfrak{B}_G$ is isomorphic to a direct sum of the canonical orbits G/H. Let $\mathfrak{O}_G$ be the full subcategory of $\mathfrak{B}_G$ whose objects are the orbits G/H associated to the subgroups H of G. The inclusion of $\mathfrak{O}_G$ into $\mathfrak{B}_G$ induces an equivalence between the category of contravariant additive functors from $\mathfrak{O}_G$ into $\mathfrak{Ab}$ and the category $\mathfrak{M}$ of Mackey functors. Thus, the category of Mackey functors could be defined equivalently as the category of contravariant additive functors from $\mathfrak{O}_G$ to $\mathfrak{Ab}$. Other equivalent definitions may be found in [9, §V.9] and [14].

Definition 2.2. An $RO(G)$-graded Mackey functor $\underline{M}_*$ consists of a Mackey functor $\underline{M}_\tau$ for each $\tau \in RO(G)$. A map of $RO(G)$-graded Mackey functors $\underline{M}_* \to \underline{N}_*$ consists of a map of Mackey functors $\underline{M}_\tau \to \underline{N}_\tau$ for each $\tau \in RO(G)$. $\mathbb{Z}$-graded Mackey functors and maps of $\mathbb{Z}$-graded Mackey functors are defined analogously. The categories of $RO(G)$-graded and $\mathbb{Z}$-graded Mackey functors are denoted by $\mathfrak{M}_*$ and $\mathfrak{M}_\#$, respectively.

The "shift" functor Σ^α takes a graded Mackey functor $\underline{M}_*$ to the graded Mackey functor given by

$$(\Sigma^\alpha \underline{M}_*)_\tau = \underline{M}_{\tau-\alpha},$$

for $\tau \in RO(G)$ (or $\tau \in \mathbb{Z}$).

The categories of $RO(G)$-graded Mackey functors and $\mathbb{Z}$-graded Mackey functors have all limits and colimits; these are formed degree-wise and object-wise. In other words, for any diagram $d \mapsto \underline{M}_*(d)$ of graded Mackey functors,

$$\left(\mathrm{Colim}_d\, \underline{M}_*(d)\right)_\tau (X) = \mathrm{Colim}_d \left(\underline{M}_\tau(d)(X)\right),$$

and likewise for the limit.

Proposition 2.3. *The categories of $RO(G)$-graded and $\mathbb{Z}$-graded Mackey functors are complete and cocomplete abelian categories satisfying $AB5$.*

The box product of graded Mackey functors is constructed from the box product of Mackey functors, which we now review briefly. The cartesian product of finite G-sets together with the canonical isomorphism $\Sigma^\infty X_+ \wedge \Sigma^\infty Y_+ \cong \Sigma^\infty (X \times Y)_+$ in the equivariant stable category provides $\mathfrak{B}_G$ with a symmetric monoidal structure. The unit for this structure is the one-point G-set G/G. The category $\mathfrak{M}$ inherits a symmetric monoidal closed structure from $\mathfrak{B}_G$. This product is most easily described by the coend formula

$$(\underline{M} \,\Box\, \underline{N})(X) = \int^{Y,Z \in \mathfrak{B}_G} \underline{M}(Y) \otimes \underline{N}(Z) \otimes \mathfrak{B}_G(X, Y \times Z).$$

This box product is characterized by the universal property that maps of Mackey functors from $\underline{M} \,\Box\, \underline{N}$ to $\underline{P}$ are in one-to-one correspondence with natural transformations of contravariant functors $\mathfrak{B}_G \times \mathfrak{B}_G \to \mathfrak{Ab}$ from $\underline{M} \otimes \underline{N}$ to $\underline{P} \circ \times$. The unit for the box product is the Burnside ring Mackey functor $\underline{B} = \mathfrak{B}_G(-, G/G)$.

The internal Hom functor adjoint to the box product is denoted $\langle \underline{M}, \underline{N} \rangle$ and is given by the end formula

$$\langle \underline{M}, \underline{N} \rangle(X) = \int_{Y,Z \in \mathfrak{B}_G} \operatorname{Hom}(\underline{M}(Y) \otimes \mathfrak{B}_G(Z, X \times Y), \underline{N}(Z)).$$

It may also be computed using the formulae

$$\langle \underline{M}, \underline{N} \rangle(X) \cong \mathfrak{M}(\underline{M}, \underline{N}_X) \cong \mathfrak{M}(\underline{M}_X, \underline{N}),$$

where $\underline{N}_X$ denotes the Mackey functor $\underline{N}(- \times X)$. Day's general results [4] about monoidal structures on functor categories specialize to give that the box product $\Box$ and the internal function object $\langle -, - \rangle$ provide the category $\mathfrak{M}$ with a closed symmetric monoidal structure.

Definition 2.4. Let $\underline{M}_*$ and $\underline{N}_*$ be $RO(G)$-graded Mackey functors. Define the $RO(G)$-graded Mackey functor $\underline{M}_* \,\Box_*\, \underline{N}_*$ by

$$(\underline{M}_* \,\Box_*\, \underline{N}_*)_\tau = \bigoplus_{\alpha+\beta=\tau} \underline{M}_\alpha \,\Box\, \underline{N}_\beta.$$

Define the $RO(G)$-graded Mackey functor $\langle \underline{M}_*, \underline{N}_* \rangle_*$ by

$$\langle \underline{M}_*, \underline{N}_* \rangle_\tau = \prod_{\beta-\alpha=\tau} \langle \underline{M}_\alpha, \underline{N}_\beta \rangle.$$

For $\mathbb{Z}$-graded Mackey functors, the functors $(-) \,\Box_\#\, (-)$ and $\langle -, - \rangle_\#$ are defined analogously.

Graded box products and graded function objects provide the categories $\mathfrak{M}_*$ and $\mathfrak{M}_\#$ with symmetric monoidal closed structures. The unit is the $RO(G)$-graded (or $\mathbb{Z}$-graded) Mackey functor $\underline{B}_*$ which is $\underline{B}$ in degree zero and zero in all other degrees. The unit and associativity isomorphisms for the graded box products are induced by the unit and associativity isomorphisms for the ordinary box product. Further, the adjunction relating graded box products and graded function objects is easily obtained from the adjunction relating ordinary box products and function objects. This describes the monoidal closed structure on $\mathfrak{M}_*$ and $\mathfrak{M}_\#$. For $\mathbb{Z}$-graded Mackey functors, the symmetry isomorphism is just as straightforward. On the summands of $(\underline{M}_\# \Box_\# \underline{N}_\#)_t$ and $(\underline{N}_\# \Box_\# \underline{M}_\#)_t$ for $t = m+n$, this isomorphism is the composite of the Mackey functor symmetry isomorphism $\underline{M}_m \Box \underline{N}_n \cong \underline{N}_n \Box \underline{M}_m$ with multiplication by $(-1)^{mn}$.

For $RO(G)$-graded Mackey functors, the symmetry isomorphism is more delicate because more complicated "signs" are needed for compatibility with our topological applications. These signs are units in the Burnside ring $\underline{B}(G/G)$ of G. Any element a in the Burnside ring of G defines a natural transformation in the category of Mackey functors from the identity functor to itself. The Yoneda lemma identifies

$\underline{B}(G/G)$ as the endomorphism ring of $\underline{B}$, and the natural transformation on a Mackey functor $\underline{M}$ associated to a is the composite

$$\underline{M} \cong \underline{M} \,\square\, \underline{B} \xrightarrow{\mathrm{id}_{\underline{M}} \square a} \underline{M} \,\square\, \underline{B} \cong \underline{M}.$$

When a is a unit in $\underline{B}(G/G)$, this natural transformation is an isomorphism. Each unit a in $\underline{B}(G/G)$ satisfies $a^2 = 1$ so these units can be thought of as generalized signs.

Given a function σ from $RO(G) \times RO(G)$ to the group of units of $\underline{B}(G/G)$, we can define a natural isomorphism

$$c_\sigma : \underline{M}_* \,\square_*\, \underline{N}_* \cong \underline{N}_* \,\square_*\, \underline{M}_*$$

by taking the map on the summands associated to $\alpha + \beta = \tau$ to be the composite of the symmetry isomorphism $\underline{M}_\alpha \,\square\, \underline{N}_\beta \cong \underline{N}_\beta \,\square\, \underline{M}_\alpha$ and the automorphism $\sigma(\alpha, \beta)$. An easy diagram chase shows that c_σ makes the box product into a symmetric monoidal product exactly when σ is (anti)symmetric, that is,

$$\sigma(\alpha, \beta)\sigma(\beta, \alpha) = 1$$

and bilinear, that is,

$$\sigma(\alpha + \beta, \gamma) = \sigma(\alpha, \gamma)\sigma(\beta, \gamma).$$

The appropriate choice of σ is dictated by the definition of the homotopy Mackey functor $\underline{\pi}_*$ and depends on the topological choices in that definition. Because of the required bilinearity of σ, it suffices to specify $\sigma(\alpha, \beta)$ in only those cases where α and β are irreducible real G-representations. In Appendix A, we show that, if α and β are non-isomorphic, then we can assume that $\sigma(\alpha, \beta) = 1$. The element $\sigma(\alpha, \alpha)$ of $\underline{B}(G/G)$ is the automorphism of $S = \Sigma^\infty G/G_+$ obtained by stabilizing the multiplication by -1 map on S^α. This definition of σ completes the construction of the symmetry isomorphism in the $RO(G)$-graded case and so finishes the construction of the symmetric monoidal product.

Proposition 2.5. *The categories $\mathfrak{M}_*$ and $\mathfrak{M}_\#$ are closed symmetric monoidal abelian categories.*

3. Graded Mackey functor rings and modules

This section is devoted to a discussion of rings and modules in the categories of graded Mackey functors. In particular, the "box over $\underline{R}_*$" ($\square_{\underline{R}_*}$) and the "$\underline{R}_*$-function" ($\langle -, - \rangle^{\underline{R}_*}$) constructions for modules over a graded Mackey functor ring $\underline{R}_*$ are defined, and their basic properties are described. For simplicity, all definitions are given in the context of the category $\mathfrak{M}_*$ of $RO(G)$-graded Mackey functors. However, with the obvious notational modifications, these definitions apply equally well to the category $\mathfrak{M}_\#$ of $\mathbb{Z}$-graded Mackey functors. We begin with the (usual) definitions of rings and modules in a symmetric monoidal abelian category.

Definition 3.1. An $RO(G)$-graded Mackey functor ring consists of an $RO(G)$-graded Mackey functor $\underline{R}_*$ together with unit $i\colon \underline{B}_* \to \underline{R}_*$ and multiplication

$\mu\colon \underline{R}_* \,\square_*\, \underline{R}_* \to \underline{R}_*$ maps for which the unit and associativity diagrams

$$\begin{tikzcd} \underline{B}_* \,\square_*\, \underline{R}_* \arrow[r, "i\square_*\mathrm{id}"] \arrow[dr, "\cong"'] & \underline{R}_* \,\square_*\, \underline{R}_* \arrow[d, "\mu"] & \underline{R}_* \,\square_*\, \underline{B}_* \arrow[l, "\mathrm{id}\,\square_* i"'] \arrow[dl, "\cong"] \\ & \underline{R}_* & \end{tikzcd} \qquad \begin{tikzcd} \underline{R}_* \,\square_*\, \underline{R}_* \,\square_*\, \underline{R}_* \arrow[r, "\mu\square_*\mathrm{id}"] \arrow[d, "\mathrm{id}\,\square_*\mu"'] & \underline{R}_* \,\square_*\, \underline{R}_* \arrow[d, "\mu"] \\ \underline{R}_* \,\square_*\, \underline{R}_* \arrow[r, "\mu"'] & \underline{R}_* \end{tikzcd}$$

commute. The ring $\underline{R}_*$ is said to be commutative if the symmetry diagram

$$\begin{tikzcd} \underline{R}_* \,\square_*\, \underline{R}_* \arrow[rr, "\cong"] \arrow[dr, "\mu"'] & & \underline{R}_* \,\square_*\, \underline{R}_* \arrow[dl, "\mu"] \\ & \underline{R}_* & \end{tikzcd}$$

also commutes. In these diagrams, the unlabeled isomorphisms are the unit and symmetry isomorphisms of the symmetric monoidal category $\mathfrak{M}_*$.

Definition 3.2. A left module over an $RO(G)$-graded Mackey functor ring $\underline{R}_*$ consists of an $RO(G)$-graded Mackey functor $\underline{M}_*$ and an action map $\xi\colon \underline{R}_* \,\square\, \underline{M}_* \to \underline{M}_*$ for which the unit and associativity diagrams

$$\begin{tikzcd} \underline{B}_* \,\square_*\, \underline{M}_* \arrow[r, "i\square_*\mathrm{id}"] \arrow[dr, "\cong"'] & \underline{R}_* \,\square_*\, \underline{M}_* \arrow[d, "\xi"] \\ & \underline{M}_* \end{tikzcd} \qquad \begin{tikzcd} \underline{R}_* \,\square_*\, \underline{R}_* \,\square_*\, \underline{M}_* \arrow[r, "\mu\square_*\mathrm{id}"] \arrow[d, "\mathrm{id}\,\square_*\xi"'] & \underline{R}_* \,\square_*\, \underline{M}_* \arrow[d, "\xi"] \\ \underline{R}_* \,\square_*\, \underline{M}_* \arrow[r, "\xi"'] & \underline{M}_* \end{tikzcd}$$

commute. A right module $\underline{M}_*$ over $\underline{R}_*$ is defined analogously in terms of an action map $\zeta\colon \underline{M}_* \,\square\, \underline{R}_* \to \underline{M}_*$. The categories of left and right modules over an $RO(G)$-graded Mackey functor ring $\underline{R}_*$ are denoted $\underline{R}_*$-Mod and Mod-$\underline{R}_*$, respectively.

For our study of the additional structure carried by some of our spectral sequences, we also need the notion of $\underline{R}_*$-bimodule.

Definition 3.3. An $\underline{R}_*$-bimodule is a graded Mackey functor $\underline{M}_*$ having left and right $\underline{R}_*$-actions (ξ and ζ, respectively) for which the diagram

$$\begin{tikzcd} \underline{R}_* \,\square_*\, \underline{M}_* \,\square_*\, \underline{R}_* \arrow[r, "\xi\square_*\mathrm{id}"] \arrow[d, "\mathrm{id}\,\square_*\zeta"'] & \underline{M}_* \,\square_*\, \underline{R}_* \arrow[d, "\zeta"] \\ \underline{R}_* \,\square_*\, \underline{M}_* \arrow[r, "\xi"'] & \underline{M}_* \end{tikzcd}$$

commutes. If $\underline{R}_*$ is commutative, then every left module $\underline{M}_*$ carries a bimodule structure in which the right action map ζ is the composite

$$\underline{M}_* \,\square_*\, \underline{R}_* \cong \underline{R}_* \,\square_*\, \underline{M}_* \xrightarrow{\xi} \underline{M}_*.$$

A bimodule structure of this form is called symmetric.

When $\underline{R}_*$ is an $RO(G)$-graded Mackey functor ring and $\underline{K}_*$ is an $RO(G)$-graded Mackey functor, the $RO(G)$-graded Mackey functors $\underline{R}_* \,\square_*\, \underline{K}_*$ and $\langle \underline{R}_*, \underline{K}_* \rangle_*$ carry $\underline{R}_*$-bimodule structures coming from the left and right actions of $\underline{R}_*$ on itself. Regarded simply as left $\underline{R}_*$-modules, $\underline{R}_* \,\square_*\, \underline{K}_*$ and $\langle \underline{R}_*, \underline{K}_* \rangle_*$ are called the free and cofree left $\underline{R}_*$-modules generated by $\underline{K}_*$. The free and cofree right $\underline{R}_*$-modules generated by $\underline{K}_*$ are constructed similarly. A purely formal argument indicates that free and cofree $\underline{R}_*$-modules have the expected universal properties.

Proposition 3.4. *Let $\underline{R}_*$ be an $RO(G)$-graded Mackey functor ring. The functor $\underline{R}_* \Box_* (-)\colon \mathfrak{M}_* \to \underline{R}_*\text{-Mod}$ is left adjoint to the forgetful functor $\underline{R}_*\text{-Mod} \to \mathfrak{M}_*$. The functor $\langle \underline{R}_*, -\rangle_*\colon \mathfrak{M}_* \to \underline{R}_*\text{-Mod}$ is right adjoint to the forgetful functor $\underline{R}_*\text{-Mod} \to \mathfrak{M}_*$.*

These two adjunctions may be used to identify the category of left $\underline{R}_*$-modules as both the category of algebras over a monad on $\mathfrak{M}_*$ and the category of coalgebras over a comonad on $\mathfrak{M}_*$. These identifications together with the corresponding identifications for the category of right $\underline{R}_*$-modules and the category of $\underline{R}_*$-bimodules are the key to proving the following result.

Proposition 3.5. *Let $\underline{R}_*$ be an $RO(G)$-graded Mackey functor ring. The categories of left $\underline{R}_*$-modules, right $\underline{R}_*$-modules, and $\underline{R}_*$-bimodules are complete and cocomplete abelian categories that satisfy $AB5$.*

We now move on to the main constructions of this section, the functors $(-) \Box_{\underline{R}_*} (-)$ and $\langle -, -\rangle^{\underline{R}_*}$.

Definition 3.6. Let $\underline{L}_*$ and $\underline{M}_*$ be left $\underline{R}_*$-modules and let $\underline{N}_*$ be a right $\underline{R}_*$-module.

(a) The graded Mackey functor $\underline{N}_* \Box_{\underline{R}_*} \underline{M}_*$ is defined by the coequalizer diagram

$$\underline{N}_* \Box_* \underline{R}_* \Box_* \underline{M}_* \underset{\mathrm{id}\,\Box_* \xi}{\overset{\zeta \Box_* \mathrm{id}}{\rightrightarrows}} \underline{N}_* \Box_* \underline{M}_* \xrightarrow{q} \underline{N}_* \Box_{\underline{R}_*} \underline{M}_*.$$

Here ξ is the action map for $\underline{M}_*$ and ζ is the action map for $\underline{N}_*$.

(b) The $RO(G)$-graded Mackey functor $\langle \underline{L}_*, \underline{M}_*\rangle^{\underline{R}_*}$ is defined by the equalizer diagram

$$\langle \underline{L}_*, \underline{M}_*\rangle^{\underline{R}_*} \xrightarrow{j} \langle \underline{L}_*, \underline{M}_*\rangle_* \underset{\tilde{\xi}_*}{\overset{\nu^*}{\rightrightarrows}} \langle \underline{R}_* \Box_* \underline{L}_*, \underline{M}_*\rangle_* \cong \langle \underline{L}_*, \langle \underline{R}_*, \underline{M}_*\rangle_*\rangle_*.$$

Here the map ν^* is induced from the action map ν for $\underline{L}_*$, and the map $\tilde{\xi}_*$ is induced from the coaction map $\tilde{\xi}\colon \underline{M}_* \to \langle \underline{R}_*, \underline{M}_*\rangle_*$ adjoint to the action map for $\underline{M}_*$.

The following proposition follows directly from these definitions and the properties of $\Box_*$ and $\langle -, -\rangle_*$.

Proposition 3.7. *Let $\underline{M}_*$ and $\underline{N}_*$ be a left and a right $\underline{R}_*$-module, respectively.*

(a) *The functors $\underline{N}_* \Box_{\underline{R}_*} (-)\colon \underline{R}_*\text{-Mod} \to \mathfrak{M}_*$ and $(-) \Box_{\underline{R}_*} \underline{M}_*\colon \text{Mod-}\underline{R}_* \to \mathfrak{M}_*$ preserve all colimits and are therefore right exact.*

(b) *The functor $\langle \underline{M}_*, -\rangle^{\underline{R}_*}\colon \underline{R}_*\text{-Mod} \to \mathfrak{M}_*$ preserves all limits and is therefore left exact.*

(c) *The functor $\langle -, \underline{M}_*\rangle^{\underline{R}_*}\colon \underline{R}_*\text{-Mod}^{\mathrm{op}} \to \mathfrak{M}_*$ converts colimits in $\underline{R}_*\text{-Mod}$ into limits in $\mathfrak{M}_*$ and is therefore left exact.*

For any graded Mackey functors $\underline{M}_*$, $\underline{M}'_*$, and $\underline{M}''_*$, the closed symmetric monoidal structure on $\mathfrak{M}_*$ provides a composition pairing

$$\langle \underline{M}'_*, \underline{M}''_*\rangle_* \Box_* \langle \underline{M}_*, \underline{M}'_*\rangle_* \longrightarrow \langle \underline{M}_*, \underline{M}''_*\rangle_*$$

that is both associative and unital. This pairing restricts to a pairing for the category of left $\underline{R}_*$-modules.

Proposition 3.8. *Let $\underline{M}_*$, $\underline{M}'_*$, and $\underline{M}''_*$ be left $\underline{R}_*$-modules. The composition pairing on $\langle -,-\rangle_*$ restricts to a pairing*

$$\langle \underline{M}'_*, \underline{M}''_*\rangle^{\underline{R}_*} \,\square_*\, \langle \underline{M}_*, \underline{M}'_*\rangle^{\underline{R}_*} \longrightarrow \langle \underline{M}_*, \underline{M}''_*\rangle^{\underline{R}_*}$$

that is both associative and unital.

Although the constructions $\underline{L}_* \,\square_{\underline{R}_*}\, \underline{M}_*$ and $\langle \underline{M}_*, \underline{N}_*\rangle^{\underline{R}_*}$ typically yield only $RO(G)$-graded Mackey functors, in some important special cases they yield $\underline{R}_*$-modules.

Proposition 3.9. *Let $\underline{L}_*$ and $\underline{M}_*$ be left $\underline{R}_*$-modules, and let $\underline{N}_*$ be a right $\underline{R}_*$-module.*

(a) *If $\underline{L}_*$ is an $\underline{R}_*$-bimodule, then $\langle \underline{L}_*, \underline{M}_*\rangle^{\underline{R}_*}$ is naturally a left $\underline{R}_*$-module.*

(b) *If $\underline{M}_*$ is an $\underline{R}_*$-bimodule, then $\underline{N}_* \,\square_{\underline{R}_*}\, \underline{M}_*$ and $\langle \underline{L}_*, \underline{M}_*\rangle^{\underline{R}_*}$ are naturally right $\underline{R}_*$-modules.*

(c) *If $\underline{N}_*$ is an $\underline{R}_*$-bimodule, then $\underline{N}_* \,\square_{\underline{R}_*}\, \underline{M}_*$ is naturally a left $\underline{R}_*$-module.*

When $\underline{R}_*$ is commutative, we can always consider the symmetric $\underline{R}_*$-bimodule structure on any (left or right) $\underline{R}_*$-module to obtain $\underline{R}_*$-module structures on $(-) \,\square_{\underline{R}_*}\, (-)$ and $\langle -,-\rangle^{\underline{R}_*}$. In fact, regarding either $\underline{M}_*$ or $\underline{N}_*$ as the bimodule yields the same $\underline{R}_*$-module structure on $\underline{M}_* \,\square_{\underline{R}_*}\, \underline{N}_*$, and similarly for $\langle \underline{M}_*, \underline{N}_*\rangle^{\underline{R}_*}$. In this context the following stronger version of the previous results hold.

Proposition 3.10. *Let $\underline{R}_*$ be a commutative graded Mackey functor ring.*

(a) *The module category $\underline{R}_*$-*Mod *is a closed symmetric monoidal abelian category with product $\square_{\underline{R}_*}$ and function object $\langle -,-\rangle^{\underline{R}_*}$.*

(b) *The free functor $\underline{R}_* \,\square_*\, (-)\colon \mathfrak{M}_* \to \underline{R}_*$-*Mod *is strong symmetric monoidal and the forgetful functor $\underline{R}_*$-*Mod $\to \mathfrak{M}_*$ *is lax symmetric monoidal.*

(c) *The composition pairing*

$$\langle \underline{M}'_*, \underline{M}''_*\rangle^{\underline{R}_*} \,\square_{\underline{R}_*}\, \langle \underline{M}_*, \underline{M}'_*\rangle^{\underline{R}_*} \longrightarrow \langle \underline{M}_*, \underline{M}''_*\rangle^{\underline{R}_*}$$

*coming from the closed symmetric monoidal structure on $\underline{R}_*$-*Mod *is the obvious one derived from the composition pairing of Proposition 3.8.*

4. The homological algebra of graded Mackey functor modules

This section is devoted to homological algebra for the categories of Mackey functor modules over a graded Mackey functor ring. Our first objective is to show that these categories have enough projectives and injectives. These objects are then used to construct the derived functors $\underline{\mathrm{Tor}}^{\underline{R}_*}_*$ and $\underline{\mathrm{Ext}}^*_{\underline{R}_*}$ in terms of resolutions and to show that they have the expected properties. Some notation is needed to construct the desired injective and projective objects.

Definition 4.1. For each finite G-set X, let $\underline{B}^X$ denote the Mackey functor $\mathfrak{B}_G(-,X)$. For any abelian group E, let $\underline{\mathcal{I}}(X,E)$ denote the Mackey functor $\mathrm{Hom}(\mathfrak{B}_G(X,-),E)$. The corresponding graded Mackey functors concentrated in degree zero are denoted $\underline{B}^X_*$ and $\underline{\mathcal{I}}(X,E)_*$.

The enriched Yoneda Lemma gives natural isomorphisms

$$\mathfrak{M}(\underline{B}^X, \underline{M}) \cong \underline{M}(X) \qquad \mathfrak{M}(\underline{M}, \underline{\mathcal{I}}(X,E)) \cong \mathrm{Hom}(\underline{M}(X), E)$$

$$\langle \underline{B}^X, \underline{M}\rangle(Y) \cong \underline{M}(X\times Y) \qquad \langle \underline{M}, \underline{\mathcal{I}}(X,E)\rangle(Y) \cong \mathrm{Hom}(\underline{M}(X\times Y), E)$$

of abelian groups. A coend argument dual to the end argument used to prove the Yoneda Lemma provides the well-known isomorphism

$$(\underline{B}^X \,\Box\, \underline{M})(Y) \cong \underline{M}(X \times Y).$$

In the graded context, these isomorphisms give the following proposition:

Proposition 4.2. *Let $\underline{M}_*$ be a graded Mackey functor. There are natural isomorphisms of abelian groups*

$$\langle \Sigma^\tau \underline{B}^X_*, \underline{M}_* \rangle_\alpha (Y) \cong \underline{M}_{\tau+\alpha}(X \times Y)$$
$$(\Sigma^\tau \underline{B}^X_* \,\Box_*\, \underline{M}_*)_\alpha (Y) \cong \underline{M}_{-\tau+\alpha}(X \times Y)$$
$$\langle \underline{M}_*, \Sigma^\tau \underline{\mathcal{I}}(X, E)_* \rangle_\alpha (Y) \cong \mathrm{Hom}(\underline{M}_{\tau-\alpha}(X \times Y), E).$$

In particular, $\Sigma^\tau \underline{B}^X_*$ is a projective object in $\mathfrak{M}_*$ and the functors $\langle \Sigma^\tau \underline{B}^X_*, - \rangle_*$ and $\underline{B}^X_* \,\Box_*\, (-)$ are exact. Also, if E is an injective abelian group, then $\Sigma^\tau \underline{\mathcal{I}}(X, E)_*$ is an injective object in $\mathfrak{M}_*$, and the functor $\langle -, \Sigma^\tau \underline{\mathcal{I}}(X, E)_* \rangle$ is exact.

It follows from this proposition that the category of graded Mackey functors has enough projectives and injectives. Let $\underline{M}_*$ be a graded Mackey functor. For each τ in $RO(G)$ and each subgroup H of G, choose a surjection $P_{\tau,H} \to \underline{M}_\tau(G/H)$ whose domain is a free abelian group and an injection $\underline{M}_\tau(G/H) \to I_{\tau,H}$ whose range is an injective abelian group $I_{\tau,H}$. Then the previous proposition provides an epimorphism

$$\underline{P}_* = \bigoplus_{\tau,H} \Sigma^\tau \underline{B}^{G/H}_* \otimes P_{\tau,H} \longrightarrow \underline{M}_*$$

and a monomorphism

$$\underline{M}_* \longrightarrow \prod_{\tau,H} \Sigma^\tau \underline{\mathcal{I}}(G/H, I_{\tau,H})_* = \underline{I}_*.$$

When $\underline{M}_*$ is a left $\underline{R}_*$-module for some graded Mackey functor ring $\underline{R}_*$, then the induced epimorphism $\underline{R}_* \,\Box_*\, \underline{P}_* \to \underline{M}_*$ of left $\underline{R}_*$-modules has domain a projective left $\underline{R}_*$-module. Likewise, the induced monomorphism $\underline{M}_* \to \langle \underline{R}_*, \underline{I}_* \rangle_*$ of left $\underline{R}_*$-modules has codomain an injective left $\underline{R}_*$-module. Similar observations apply in the case of right modules. Thus, we have proven:

Proposition 4.3. *Let $\underline{R}_*$ be a graded Mackey functor ring. The categories of left and right $\underline{R}_*$-modules have enough projectives and injectives.*

Since an epimorphism from one projective onto another or a monomorphism from one injective into another is split, the preceding argument also provides characterizations of projective and injection modules.

Proposition 4.4. *An $\underline{R}_*$-module is projective if and only if it is a direct summand of a direct sum of $\underline{R}_*$-modules of the form $\underline{R}_* \,\Box_*\, \Sigma^\tau \underline{B}^{G/H}_*$. Also, an $\underline{R}_*$-module is injective if and only if it is a direct summand of a product of $\underline{R}_*$-modules of the form $\langle \underline{R}_*, \Sigma^\tau \underline{\mathcal{I}}(G/H, I)_* \rangle_*$ in which I is an injective abelian group.*

Proposition 4.2 and this characterization of projectives and injectives have some important implications for the exactness of the functors $\Box_{\underline{R}_*}$ and $\langle -, - \rangle^{\underline{R}_*}$. In the terminology of Lewis [7], these exactness results assert that projectives and injectives are respectively internal projective and internal injective and that projective implies flat.

Theorem 4.5. *Let $\underline{R}_*$ be a Mackey functor ring.*

(a) *If $\underline{P}_*$ is a projective left $\underline{R}_*$-module, then $\langle \underline{P}_*, -\rangle^{\underline{R}_*}$ is an exact functor from left $\underline{R}_*$-modules to graded Mackey functors.*

(b) *If $\underline{I}_*$ is an injective left $\underline{R}_*$-module, then $\langle -, \underline{I}_*\rangle^{\underline{R}_*}$ is an exact functor from left $\underline{R}_*$-modules to graded Mackey functors.*

(c) *If $\underline{P}_*$ is a projective left $\underline{R}_*$-module, then $(-) \square_{\underline{R}_*} \underline{P}_*$ is an exact functor from right $\underline{R}_*$-modules to graded Mackey functors.*

(d) *If $\underline{Q}_*$ is a projective right $\underline{R}_*$-module, then $\underline{Q}_* \square_{\underline{R}_*} (-)$ is an exact functor from left $\underline{R}_*$-modules to graded Mackey functors.*

In the category of left $\underline{R}_*$-modules, as in any abelian category with enough projectives and injectives, every object has both projective and injective resolutions. To be consistent with the usual grading conventions for spectral sequences, we use the inner degree for the resolution degree in projective resolutions. Thus, for any projective resolution $\underline{P}_{*,*}$, $\underline{P}_{s,*}$ is an $\underline{R}_*$-module for each $s \geq 0$ and $\underline{P}_{*,\tau}$ is a chain complex of Mackey functors for each $\tau \in RO(G)$ (or $\mathbb{Z}$). When $\underline{I}^*_*$ is an injective resolution, $\underline{I}^s_*$ is an $\underline{R}_*$-module for each $s \geq 0$ and $\underline{I}^*_\tau$ is a cochain complex of Mackey functors for each $\tau \in RO(G)$ (or $\mathbb{Z}$).

If $\underline{P}_{*,*}$ is a projective resolution of a left $\underline{R}_*$-module $\underline{M}_*$ and $\underline{Q}_{*,*}$ is a projective resolution of a right $\underline{R}_*$-module $\underline{N}_*$, then the maps

$$\underline{N}_* \square_{\underline{R}_*} \underline{P}_{*,*} \longleftarrow \underline{Q}_{*,*} \square_{\underline{R}_*} \underline{P}_{*,*} \longrightarrow \underline{Q}_{*,*} \square_{\underline{R}_*} \underline{M}_*$$

are quasi-isomorphisms of chain complexes of graded Mackey functors by Theorem 4.5. The graded Mackey functor $\underline{\mathrm{Tor}}^{\underline{R}_*}_s(\underline{N}_*, \underline{M}_*)$ is defined to be the s-th homology of any of these cochain complexes. The Mackey functor $(\underline{\mathrm{Tor}}^{\underline{R}_*}_s(\underline{N}_*, \underline{M}_*))_\tau$ is usually denoted $\underline{\mathrm{Tor}}^{\underline{R}_*}_{s,\tau}(\underline{N}_*, \underline{M}_*)$.

If $\underline{O}_{*,*}$ is a projective resolution of a left $\underline{R}_*$-module $\underline{L}_*$ and $\underline{I}^*_*$ is an injective resolution of a left $\underline{R}_*$-module $\underline{M}_*$, then the maps

$$\langle \underline{O}_{*,*}, \underline{M}_*\rangle^{\underline{R}_*} \longrightarrow \langle \underline{O}_{*,*}, \underline{I}^*_*\rangle^{\underline{R}_*} \longleftarrow \langle \underline{L}_*, \underline{I}^*_*\rangle^{\underline{R}_*}$$

are quasi-isomorphisms of cochain complexes of graded Mackey functors by Theorem 4.5. The graded Mackey functor $\underline{\mathrm{Ext}}^s_{\underline{R}_*}(\underline{L}_*, \underline{M}_*)$ is defined to be the s-th cohomology of any of these cochain complexes. The Mackey functor $(\underline{\mathrm{Ext}}^s_{\underline{R}_*}(\underline{L}_*, \underline{M}_*))_{-\tau}$ is usually denoted $\underline{\mathrm{Ext}}^{s,\tau}_{\underline{R}_*}(\underline{L}_*, \underline{M}_*)$.

The standard homological arguments imply that $\underline{\mathrm{Tor}}^{\underline{R}_*}_*$ and $\underline{\mathrm{Ext}}^*_{\underline{R}_*}$ behave as one would expect.

Proposition 4.6. *The constructions $\underline{\mathrm{Tor}}^{\underline{R}_*}_*$ and $\underline{\mathrm{Ext}}^*_{\underline{R}_*}$ are well-defined, natural in each variable, and convert short exact sequences in each variable to long exact sequences. Moreover, there are canonical natural isomorphisms*

$$\underline{\mathrm{Tor}}^{\underline{R}_*}_0(\underline{L}_*, \underline{M}_*) \cong \underline{L}_* \square_{\underline{R}_*} \underline{M}_*$$

and

$$\underline{\mathrm{Ext}}^0_{\underline{R}_*}(\underline{M}_*, \underline{N}_*) \cong \langle \underline{M}_*, \underline{N}_*\rangle^{\underline{R}_*}.$$

As with $\square_{\underline{R}_*}$ and $\langle -, -\rangle^{\underline{R}_*}$, if one of the arguments of $\underline{\mathrm{Tor}}^{\underline{R}_*}_*$ or $\underline{\mathrm{Ext}}^*_{\underline{R}_*}$ is an $\underline{R}_*$-bimodule, then $\underline{\mathrm{Tor}}^{\underline{R}_*}_*$ or $\underline{\mathrm{Ext}}^*_{\underline{R}_*}$ inherits an $\underline{R}_*$-module structure from that bimodule. In the context of ordinary rings, this assertion is a purely formal consequence of the functoriality of $\underline{\mathrm{Tor}}^{\underline{R}_*}_*$ and $\underline{\mathrm{Ext}}^*_{\underline{R}_*}$. However, since $\underline{R}_*$ is a graded Mackey functor ring, to obtain this result, we must either show that the functors $\underline{\mathrm{Tor}}^{\underline{R}_*}_*$

and $\underline{\mathrm{Ext}}^*_{\underline{R}_*}$ are enriched over $\mathfrak{M}_*$ or provide a direct construction of the $\underline{R}_*$-action. Either of these approaches requires the following lemma. Its proof, which uses the right exactness of $\square_{\underline{R}_*}$, is essentially identical to that of the corresponding result for chain complexes of abelian groups.

Lemma 4.7. *Let $\underline{C}_{*,*}$ and $\underline{D}_{*,*}$ be chain complexes of graded Mackey functors, and $\underline{C}^*_*$ and $\underline{D}^*_*$ be cochain complexes of graded Mackey functors. Then there are natural transformations*

$$H_*(\underline{C}_{*,*}) \square_* H_*(\underline{D}_{*,*}) \longrightarrow H_*(\underline{C}_{*,*} \square_* \underline{D}_{*,*})$$

and

$$H^*(\underline{C}^*_*) \square_* H^*(\underline{D}^*_*) \longrightarrow H^*(\underline{C}^*_* \square_* \underline{D}^*_*)$$

which are unital and associative in the appropriate sense.

Taking one of the complexes in this lemma to be $\underline{R}_*$ (concentrated in homological or cohomological degree zero) and the other to be the complex used to compute $\underline{\mathrm{Tor}}^{\underline{R}_*}_*$ or $\underline{\mathrm{Ext}}^*_{\underline{R}_*}$, we obtain the desired actions of $\underline{R}_*$.

Theorem 4.8. *Let $\underline{L}_*$ and $\underline{M}_*$ be left $\underline{R}_*$-modules, and let $\underline{N}_*$ be a right $\underline{R}_*$-module.*

(a) *If $\underline{L}_*$ is an $\underline{R}_*$-bimodule, then $\underline{\mathrm{Ext}}^s_{\underline{R}_*}(\underline{L}_*, \underline{M}_*)$ is naturally a right $\underline{R}_*$-module.*

(b) *If $\underline{M}_*$ is an $\underline{R}_*$-bimodule, then $\underline{\mathrm{Tor}}^{\underline{R}_*}_s(\underline{N}_*, \underline{M}_*)$ and $\underline{\mathrm{Ext}}^s_{\underline{R}_*}(\underline{L}_*, \underline{M}_*)$ are naturally right $\underline{R}_*$-modules.*

(c) *If $\underline{N}_*$ is an $\underline{R}_*$-bimodule, then $\underline{\mathrm{Tor}}^{\underline{R}_*}_s(\underline{N}_*, \underline{M}_*)$ is naturally a left $\underline{R}_*$-module.*

The second natural transformation in Lemma 4.7 may also be used to construct the Yoneda pairing for $\underline{\mathrm{Ext}}^*_{\underline{R}_*}$. Let $\underline{M}_*$, $\underline{M}'_*$, and $\underline{M}''_*$ be left $\underline{R}_*$-modules, $\underline{P}_{*,*}$ be a projective resolution of $\underline{M}_*$, and $\underline{I}^*_*$ be an injective resolution of $\underline{M}''_*$. Then the Yoneda pairing

$$\underline{\mathrm{Ext}}^*_{\underline{R}_*}(\underline{M}'_*, \underline{M}''_*) \square_* \underline{\mathrm{Ext}}^*_{\underline{R}_*}(\underline{M}_*, \underline{M}'_*) \longrightarrow \underline{\mathrm{Ext}}^*_{\underline{R}_*}(\underline{M}_*, \underline{M}''_*)$$

is the composite

$$\begin{aligned} H^*(\langle \underline{M}'_*, \underline{I}^*_* \rangle^{\underline{R}_*}) \square H^*(\langle \underline{P}_{*,*}, \underline{M}'_* \rangle^{\underline{R}_*}) \quad &\longrightarrow \quad H^*(\langle \underline{M}'_*, \underline{I}^*_* \rangle^{\underline{R}_*} \square_* \langle \underline{P}_{*,*}, \underline{M}'_* \rangle^{\underline{R}_*}) \\ &\longrightarrow \quad H^*(\langle \underline{P}_{*,*}, \underline{I}^*_* \rangle^{\underline{R}_*}). \end{aligned}$$

Here, the first map is the second natural transformation from the lemma and the second map comes from the composition pairing of Lemma 3.8. The usual homological arguments imply that this pairing on $\underline{\mathrm{Ext}}^*_{\underline{R}_*}$ behaves as expected.

Proposition 4.9. *The Yoneda pairing is well-defined and associative. Moreover, in cohomological degree zero, it agrees with the usual composition pairing.*

5. Homotopy Mackey functors and the homological algebra of equivariant R-modules

This section is devoted to a discussion of the functor $\underline{\pi}_*$ from the equivariant stable category to the category of graded Mackey functors. This functor connects the derived smash product and function object constructions for modules over an equivariant S-algebra R to the corresponding $\square_{\underline{\pi}_* R}$ and $\langle -,- \rangle^{\underline{\pi}_* R}$ constructions. As a first step in our analysis of the link between the homotopy theory of R-modules and the homological algebra of $\underline{\pi}_* R$-modules, we consider here those R-modules whose homotopy Mackey functors are projective or injective as $\underline{\pi}_* R$-modules.

We begin by reviewing the construction of $\underline{\pi}_*$. The first step in this process is selecting a model S^τ of the τ-sphere for each element τ of $RO(G)$. It is convenient to take $S^0 = S$. More generally, when τ is the trivial representation of dimension $n > 0$, S^τ is taken to be the smash product of S with the standard n-sphere space. For all other τ, the object S^τ may be chosen arbitrarily from the appropriate homotopy class. Then, $\underline{\pi}_*$ is defined for any G-spectrum M and any finite G-set X by

$$\underline{\pi}_\tau(M)(X) = [S^\tau \wedge X_+, M]_G.$$

The naturality of this construction in stable maps of X gives $\underline{\pi}_\tau(M)$ the structure of a Mackey functor. Further, its naturality in M makes $\underline{\pi}_*$ a functor from the equivariant stable category to $\mathfrak{M}_*$. Restricting the index τ so that it lies in $\mathbb{Z}$ rather than $RO(G)$ gives a functor, which we also denote $\underline{\pi}_*$, from the equivariant stable category to $\mathfrak{M}_\#$.

Appendix A contains a complete proof of the following folk theorem:

Theorem 5.1. *The functor $\underline{\pi}_*$ is a lax symmetric monoidal functor from the equivariant stable category to the category of $RO(G)$-graded (or $\mathbb{Z}$-graded) Mackey functors.*

In other words, we have a suitably associative, symmetric, and unital natural transformation $\underline{\pi}_* N \,\Box_*\, \underline{\pi}_* M \to \underline{\pi}_*(N \wedge M)$. By comparing the diagrams used to define homotopical ring and module spectra in the equivariant stable category with those used to define graded Mackey functor rings and modules, it is easy to see that $\underline{\pi}_*$ takes homotopical ring and module spectra to graded Mackey functor rings and modules. We apply this to equivariant S-algebras and modules over an equivariant S-algebra R in a modern category of spectra, which pass to homotopical ring and module spectra in the equivariant stable category. A weak bimodule in the category of left R-modules is defined to be left R-module together with a homotopical right R-module structure in the derived category of left R-modules. Clearly, the underlying spectrum of a weak bimodule is a homotopical bimodule. Weak bimodules in the category of right R-modules are defined analogously; their underlying spectra are also homotopical bimodules. These observations are summarized in the following corollary of Theorem 5.1:

Corollary 5.2. *Let R be an equivariant S-algebra. Then $\underline{\pi}_* R$ is a graded Mackey functor ring which is commutative if R is. The functor $\underline{\pi}_*$ refines to a functor from the category of left (or right) R-modules to the category of left (or right) $\underline{\pi}_* R$-modules. Both refinements take weak bimodules to $\underline{\pi}_* R$-bimodules.*

Let M and N be left and right modules, respectively, over an equivariant S-algebra R. Recall that the functor Tor^R is the derived functor of $\wedge_R$ and that $\underline{\mathrm{Tor}}^R_*(N, M)$ denotes $\underline{\pi}_* \mathrm{Tor}^R(N, M)$. There is a canonical comparison map

$$N \wedge M \to \mathrm{Tor}^R(N, M)$$

derived from the natural map $N \wedge_S M \to N \wedge_R M$. The two composites

$$N \wedge R \wedge M \to N \wedge M \to \mathrm{Tor}^R(N, M)$$

coming from the actions of R on M and N coincide. Thus, the universal property defining $\Box_{\underline{\pi}_* R}$ provides a natural transformation

$$\underline{\pi}_* N \,\Box_{(\underline{\pi}_* R)}\, \underline{\pi}_* M \longrightarrow \underline{\mathrm{Tor}}^R_*(N, M).$$

There is a canonical natural isomorphism

$$\mathrm{Tor}^R(N,M)\wedge X \longrightarrow \mathrm{Tor}^R(N,M\wedge X)$$

in the equivariant stable category. This isomorphism is associative in the obvious sense and makes the diagram

$$\begin{tikzcd}[column sep=small] & N\wedge M\wedge X \arrow[dl] \arrow[dr] & \\ \mathrm{Tor}^R(N,M)\wedge X \arrow[rr] & & \mathrm{Tor}^R(N,M\wedge X) \end{tikzcd}$$

commute. There is an analogous isomorphism in the other variable with analogous properties. These isomorphisms, and their asserted properties, come from the fact that $\mathrm{Tor}^R(N,-)$ is enriched functor over the equivariant stable category [8]. Taking $X=R$, we see that, when M or N is a weak bimodule, $\mathrm{Tor}^R(N,M)$ is naturally a homotopical R-module and the comparison map $N\wedge M\to \mathrm{Tor}^R(N,M)$ is a map of homotopical R-modules. The implications of these observations are summarized in the following result:

Theorem 5.3. *Let R be an equivariant S-algebra, M be a left R-module, and N be a right R-module. There is a natural transformation of graded Mackey functors*

$$\underline{\pi}_*N\,\Box_{(\underline{\pi}_*R)}\,\underline{\pi}_*M \longrightarrow \underline{\mathrm{Tor}}^R_*(N,M).$$

If M is a weak bimodule, this is a map of right $\underline{\pi}_*R$-modules. If N is an weak bimodule, it is a map of left $\underline{\pi}_*R$-modules.

Likewise, the functor Ext_R is the derived functor of F_R, and $\underline{\mathrm{Ext}}_R^{-*}(L,M)=\underline{\pi}_*\,\mathrm{Ext}_R(L,M)$ for left R-modules L and M. The natural map $F_R(L,M)\to F_S(L,M)$ induces a map $\mathrm{Ext}_R(L,M)\to F(L,M)$. The two maps from $\mathrm{Ext}_R(L,M)$ to $F(R\wedge L,M)$ coming from the actions of R on L and M coincide and so induce a natural transformation

$$\underline{\mathrm{Ext}}_R^{-*}(L,M)\longrightarrow \langle \underline{\pi}_*L,\underline{\pi}_*M\rangle^{(\underline{\pi}_*R)}.$$

The functors $\mathrm{Ext}_R(L,-)$ and $\mathrm{Ext}_R(-,M)$ are also enriched over the equivariant stable category, and so we have natural maps

$$\mathrm{Ext}_R(L,M)\wedge X\to \mathrm{Ext}_R(L,M\wedge X)\quad\text{and}\quad \mathrm{Ext}_R(L\wedge X,M)\to F(X,\mathrm{Ext}_R(L,M))$$

in the equivariant stable category. The second of these maps is always an isomorphism, but the first map generally is not. These maps satisfy the evident associativity conditions and are compatible with the canonical maps

$$F(L,M)\wedge X\to F(L,M\wedge X)\quad\text{and}\quad F(L\wedge X,M)\to F(X,F(L,M)).$$

These observations yield the following result:

Theorem 5.4. *Let R be an equivariant S-algebra, and L and M be left R-modules. There is a natural transformation of graded Mackey functors*

$$\underline{\mathrm{Ext}}_R^{-*}(L,M)\longrightarrow \langle \underline{\pi}_*L,\underline{\pi}_*M\rangle^{(\underline{\pi}_*R)}.$$

If L is a weak bimodule, this is a map of left $\underline{\pi}_*R$-modules. If M is a weak bimodule, it is a map of right $\underline{\pi}_*R$-modules.

The remainder of this section is devoted to the statements and proofs of two results about R-modules whose homotopy Mackey functors are projective or injective as $\underline{R}_*$-modules. The behavior of the natural maps of Theorems 5.3 and 5.4 for such R-modules is of particular interest to us. For this discussion, we denote $\underline{\pi}_* R$ by $\underline{R}_*$.

Theorem 5.5.

(a) *If $\underline{P}_*$ is a projective left $\underline{R}_*$-module, then there exists a left R-module P such that $\underline{\pi}_* P \cong \underline{P}_*$.*

(b) *If $\underline{Q}_*$ is a projective right $\underline{R}_*$-module, then there exists a right R-module Q such that $\underline{\pi}_* Q \cong \underline{Q}_*$.*

(c) *If $\underline{I}_*$ is an injective left $\underline{R}_*$-module, then there exists a left R-module I such that $\underline{\pi}_* I \cong \underline{I}_*$.*

Theorem 5.6. *Let L and M be left R-modules, and let N be a right R-module.*

(a) *If $\underline{\pi}_* L$ is projective or $\underline{\pi}_* M$ is injective as a left $\underline{R}_*$-module, then the natural map $\underline{\mathrm{Ext}}_R^{-*}(L, M) \to \langle \underline{\pi}_* L, \underline{\pi}_* M \rangle^{\underline{R}_*}$ is an isomorphism.*

(b) *If $\underline{\pi}_* M$ is a projective left $\underline{R}_*$-module or $\underline{\pi}_* N$ is a projective right $\underline{R}_*$-module, then the natural map $\underline{\pi}_* N \mathbin{\Box}_{\underline{R}_*} \underline{\pi}_* M \to \underline{\mathrm{Tor}}^R_*(N, M)$ is an isomorphism.*

For any two left R-modules L and M, $\underline{\mathrm{Ext}}^0_R(L, M)(G/G)$ is canonically isomorphic to the abelian group of maps from L to M in the derived category of left R-modules. Likewise. for any left $\underline{R}_*$-modules $\underline{L}_*$ and $\underline{M}_*$, $\langle \underline{L}_*, \underline{M}_* \rangle_0^{\underline{R}_*}(G/G)$ is canonically isomorphic to the abelian group of maps from $\underline{L}_*$ to $\underline{M}_*$ in the category of left $\underline{R}_*$-modules. This implies the following corollary of Theorem 5.6.

Corollary 5.7. *Let L and M be left R-modules. If $\underline{\pi}_* L$ is projective or $\underline{\pi}_* M$ is injective as a left $\underline{R}_*$-module, then maps from L to M in the derived category of left R-modules are in one-to-one correspondence with maps from $\underline{\pi}_* L$ to $\underline{\pi}_* M$ in the category of left $\underline{R}_*$-modules.*

We begin the proof of Theorems 5.5 and 5.6 with the following special case. We state and prove it in the case of left R-modules but the analogous result holds for right R-modules.

Lemma 5.8. *Let X be a finite G-set and let τ be an element of $RO(G)$. Then there exists a left R-module $R^\tau[X]$ with $\underline{\pi}_* R^\tau[X] \cong \underline{R}_* \mathbin{\Box}_* \Sigma^\tau \underline{B}^X_*$. Moreover, for any left R-module M and any right R-module N, the natural maps*

$$(\underline{\pi}_* N) \mathbin{\Box}_{\underline{R}_*} (\underline{\pi}_* R^\tau[X]) \longrightarrow \underline{\mathrm{Tor}}^R_*(N, R^\tau[X]),$$
$$\underline{\mathrm{Ext}}_R^{-*}(R^\tau[X], M) \longrightarrow \langle \underline{\pi}_* R^\tau[X], \underline{\pi}_* M \rangle^{\underline{R}_*}$$

are isomorphisms.

Proof. By taking $R^\tau[X] = R^0[X] \wedge S^\tau$, it suffices to consider the case when $\tau = 0$. Then we take $R[X] = R^0[X] = \mathbb{R}\Sigma^\infty X_+$. Theorem 5.1 gives us a canonical map of $\underline{R}_*$-bimodules

$$\underline{R}_* \mathbin{\Box}_* \underline{B}^X_* = \underline{\pi}_* R \mathbin{\Box}_* \underline{\pi}_0(\Sigma^\infty X_+) \longrightarrow \underline{\pi}_* R[X],$$

and an easy Spanier–Whitehead duality argument shows that this map is an isomorphism. The argument generalizes to show that the map $\underline{\pi}_* N \Box_* \underline{B}^X_* \to \underline{\pi}_*(N \wedge X_+)$ is an isomorphism for all N, and the rest of the proof is an easy check of diagrams. $\square$

We generalize this to other projective modules in the following lemma.

Lemma 5.9. *Let $\underline{P}_*$ be a projective left $\underline{R}_*$-module. There exists a left R-module P with $\underline{\pi}_* P \cong \underline{P}_*$ such that the natural maps $\underline{\pi}_*(-) \,\square_{\underline{R}_*} \underline{P}_* \to \underline{\mathrm{Tor}}_*^R(-,P)$ and $\underline{\mathrm{Ext}}_R^{-*}(P,-) \to \langle \underline{P}_*, \underline{\pi}_*(-)\rangle^{\underline{R}_*}$ are isomorphisms.*

Proof. Using Proposition 4.4, we can find an epimorphism $f\colon \underline{F}_* \to \underline{P}_*$, where $\underline{F}_*$ is a direct sum of $\underline{R}_*$-modules of the form $\underline{R}_* \,\square\, \Sigma^\tau \underline{B}_*^X$. Choose a splitting map $g\colon \underline{P}_* \to \underline{F}_*$ for f. By the previous lemma, there is a left R-module F, which is a wedge of modules of the form $R^\tau[X]$, such that $\underline{\pi}_* F \cong \underline{F}_*$. Also, there is a self-map $h\colon F \to F$ in the derived category of left R-modules that induces $g \circ f$ on $\underline{\pi}_*$. Form the left R-module $P = h^{-1}F$ as the telescope of the self-map h. Then $\underline{\pi}_* P \cong \underline{P}_*$. For any right $\underline{R}_*$-module $\underline{N}_*$ and any right R-module N, the natural maps

$$\operatorname{Colim} \underline{N}_* \,\square_{\underline{R}_*} \underline{F}_* \longrightarrow \underline{N}_* \,\square_{\underline{R}_*} (\operatorname{Colim} \underline{F}_*) \cong \underline{N}_* \,\square_{\underline{R}_*} \underline{P}_*,$$

$$\operatorname{Colim} \underline{\mathrm{Tor}}_*^R(N,F) \longrightarrow \underline{\mathrm{Tor}}_*^R(N, \operatorname{Tel} F) = \underline{\mathrm{Tor}}_*^R(N,P)$$

associated to the sequential colimits over the self-maps $g\circ f$ and h are isomorphisms. By Lemma 5.8, the natural map $\underline{\pi}_*(-) \,\square_{\underline{R}_*} \underline{F}_* \to \underline{\mathrm{Tor}}_*^R(-,F)$ is an isomorphism, and so the natural map $\underline{\pi}_*(-) \,\square_{\underline{R}_*} \underline{P}_* \to \underline{\mathrm{Tor}}_*^R(-,P)$ is also an isomorphism.

For any left R-module M, we have the usual short exact sequences of homotopy groups

$$0 \to \mathrm{Lim}^1\, \pi_{*+1}^H \operatorname{Ext}_R(F,M) \to \pi_*^H \operatorname{Ext}_R(P,M) \to \operatorname{Lim} \pi_*^H \operatorname{Ext}_R(F,M) \to 0.$$

associated to a telescope. Since $(g \circ f) \circ (g \circ f) = (g \circ f)$, Lemma 5.8 implies that $h \circ h = h$. It follows that the towers of abelian groups in question are Mittag-Leffler, and so $\mathrm{Lim}^1 = 0$. Thus, the natural map

$$\underline{\mathrm{Ext}}_R^{-*}(P,M) \to \operatorname{Lim} \underline{\mathrm{Ext}}_R^{-*}(F,M)$$

is an isomorphism. For any left $\underline{R}_*$-module $\underline{M}_*$, the functor $\langle -, \underline{M}_*\rangle^{\underline{R}_*}$ converts colimits to limits, and so the natural map $\langle \underline{P}_*, \underline{M}_*\rangle^{\underline{R}_*} \to \operatorname{Lim} \langle \underline{F}_*, \underline{M}_*\rangle^{\underline{R}_*}$ is an isomorphism. By Lemma 5.8 again, the natural map $\underline{\mathrm{Ext}}_R^{-*}(F,-) \to \langle \underline{F}_*, \underline{\pi}_*(-)\rangle^{\underline{R}_*}$ is an isomorphism. The natural map $\underline{\mathrm{Ext}}_R^{-*}(P,-) \to \langle \underline{P}_*, \underline{\pi}_*(-)\rangle^{\underline{R}_*}$ is therefore also an isomorphism. □

If P' is any other left R-module with $\underline{\pi}_* P'$ isomorphic to a projective left $\underline{R}_*$-module $\underline{P}_*$, then P' is isomorphic in the derived category of left R-modules to the left R-module P of the previous lemma. To see this, note that a special case of the isomorphism $\underline{\mathrm{Ext}}_R^{-*}(P,P') \cong \langle \underline{\pi}_* P, \underline{\pi}_* P'\rangle^{\underline{R}_*}$ of the previous lemma indicates that there is a one-to-one correspondence between maps from P to P' in the derived category of left R-modules and maps from $\underline{\pi}_* P$ to $\underline{\pi}_* P'$ in the category of left $\underline{R}_*$-modules. Choosing an isomorphism $\underline{\pi}_* P \cong \underline{P}_* \cong \underline{\pi}_* P'$, we obtain a map $P \to P'$ inducing an isomorphism on homotopy groups. This proves the following proposition.

Proposition 5.10. *If P is a left R-module such that $\underline{P}_* = \underline{\pi}_* P$ is a projective left $\underline{R}_*$-module, then the natural maps $\underline{\pi}_*(-) \,\square_{\underline{R}_*} \underline{P}_* \to \underline{\mathrm{Tor}}_*^R(-,P)$ and $\underline{\mathrm{Ext}}_R^{-*}(P,-) \to \langle \underline{P}_*, \underline{\pi}_*(-)\rangle^{\underline{R}_*}$ are isomorphisms.*

This gives half of part (a) of Theorem 5.6. The other half is given by the following lemma.

Lemma 5.11. *If I is a left R-module such that $\underline{I}_* = \underline{\pi}_* I$ is an injective left $\underline{R}_*$-module, then the natural map $\underline{\mathrm{Ext}}_R^{-*}(-,I) \to \langle \underline{\pi}_*(-), \underline{I}_*\rangle^{\underline{R}_*}$ is an isomorphism.*

Proof. Let $\mathscr{C}_I$ denote the class of left R-modules L for which the map $\underline{\mathrm{Ext}}_R^{-*}(L, I) \to \langle \underline{\pi}_* L, \underline{I}_* \rangle^{\underline{R}_*}$ is an isomorphism. Clearly $\mathscr{C}_I$ is closed under arbitrary wedge products and under suspension by any element τ of $RO(G)$. Also, an R-module L is in $\mathscr{C}_I$ if any R-module isomorphic to L in the derived category is in $\mathscr{C}_I$. By Lemma 5.8, the modules $R^n[G/H]$ are in $\mathscr{C}_I$. Theorem 4.5 indicates that the functor $\langle -, \underline{I}_* \rangle^{\underline{R}_*}$ is exact. Thus, if

$$\cdots \to \Sigma^{n-1} C \to \Sigma^n A \to \Sigma^n B \to \Sigma^n C \to \Sigma^{n+1} A \to \cdots$$

is a cofibration sequence, then applying either of the functors $\langle \underline{\pi}_*(-), \underline{I}_* \rangle^{\underline{R}_*}$ or $\underline{\mathrm{Ext}}_R^{-*}(-, I)$ to this sequence produces a long exact sequence of graded Mackey functors. It follows that the cofiber of a map between modules in $\mathscr{C}_I$ is a module in $\mathscr{C}_I$. From this, we conclude that every left R-module is in $\mathscr{C}_I$. $\square$

The right module parts of Theorems 5.5 and 5.6 can be proven by arguments analogous to those already given for left R-modules. Thus, to complete the proofs of these two results, it suffices to prove part (c) of Theorem 5.5. The construction of the required R-modules with injective homotopy groups is completely analogous to the construction of Brown–Comenetz dual spectra.

Let $\underline{I}_*$ be an injective left $\underline{R}_*$-module. We define a contravariant functor F_I from the derived category of left R-modules to the category of abelian groups by letting $F_I(M)$ be the abelian group of maps of left $\underline{R}_*$-modules from $\underline{\pi}_* M$ to $\underline{I}_*$. The derived category of left R-modules and the functor F_I satisfy the hypotheses for the abstract form of the Brown representability theorem in Brown [3]. It follows that there exists a left R-module I representing F_I, i.e., the abelian group of maps in the derived category of left R-modules from M to I is naturally isomorphic to $F_I(M)$. In particular, letting M range over the left R-modules $R^\tau[X]$, we see that $\underline{\pi}_* I \cong \underline{I}_*$. This completes the proof of Theorem 5.5. Note that Lemma 5.11 implies that the R-module I is unique up to isomorphism in the derived category of left R-modules.

6. The Construction of the Spectral Sequences

The two spectral sequences described in the introduction are constructed in this section. Throughout this construction, R is a fixed equivariant S-algebra and $\underline{R}_* = \underline{\pi}_* R$. The results in the previous section allow us to construct "resolutions" of an R-module M in the derived category of R-modules corresponding to projective and injective $\underline{R}_*$-module resolutions of $\underline{\pi}_* M$. These resolutions are the equivariant generalization of the resolutions constructed in Section IV.5 of [5].

The following definition formalizes the relationship between our topological resolutions of M and algebraic resolutions of $\underline{\pi}_* M$.

Definition 6.1. Let M be a left R-module, and let $\underline{M}_* = \underline{\pi}_* M$. A projective topological resolution of M consists of collections of left R-modules M_s and P_s together with cofiber sequences

$$\Sigma^s P_s \xrightarrow{j_s} M_s \xrightarrow{i_{s+1}} M_{s+1} \xrightarrow{k_s} \Sigma^{s+1} P_s,$$

in the derived category of left R-modules for all $s \geq 0$. These objects must satisfy the conditions that $M_0 = M$, each $\underline{\pi}_* P_s$ is a projective left $\underline{R}_*$-module, and $\underline{\pi}_* j_s$ is an epimorphism.

A projective topological resolution (M_s, P_s) of M is compatible with a projective resolution $\underline{P}_{*,*}$ of $\underline{M}_*$ if there are isomorphisms $\underline{P}_{s,*} \to \underline{\pi}_* P_s$ under which $\underline{\pi}_* j_0$

coincides with the augmentation $\underline{P}_{0,*} \to \underline{M}_*$ and $\underline{\pi}_*(k_s \circ j_{s+1})$ coincides with the suspension $\Sigma^{s+1}d_{s+1}$ of the differential $d_{s+1}\colon \underline{P}_{s+1,*} \to \underline{P}_{s,*}$.

An injective topological resolution of M consists of collections of left R-modules M^s and I^s together with fiber sequences

$$\Omega^{s+1}I^s \xrightarrow{k_s} M^{s+1} \xrightarrow{i^{s+1}} M^s \xrightarrow{j^s} \Omega^s I^s,$$

in the derived category of left R-modules for all $s \geq 0$. These must satisfy the conditions that $M^0 = M$, each $\underline{\pi}_* I^s$ is an injective left $\underline{R}_*$-module, and $\underline{\pi}_* j^s$ is a monomorphism.

An injective topological resolution (M^s, I^s) of M is compatible with a given injective resolution $\underline{I}_*^*$ of $\underline{M}_*$ if there are isomorphisms $\underline{I}_*^s \to \underline{\pi}_* I^s$ under which $\underline{\pi}_* j^0$ coincides with the augmentation $\underline{M}_* \to \underline{I}_*^0$ and $\underline{\pi}_*(j^{s+1} \circ k^s)$ coincides with the desuspension $\Omega^{s+1}d^s$ of the differential $d^s\colon \underline{I}_*^s \to \underline{I}_*^{s+1}$.

Projective topological resolutions of right R-modules are defined analogously. In the projective context, the modules M_s are analogous to the quotients X/X^{s-1} for a nice filtration $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$ of a space X. As discussed in Boardman [1, 12.5ff], the cohomological spectral sequence constructed from the cofiber sequences

$$X/X^{s-1} \longrightarrow X/X^s \longrightarrow X^s/X^{s-1}$$

is isomorphic to the more usual spectral sequence constructed from the cofiber sequences

$$X^{s-1} \longrightarrow X^s \longrightarrow X^s/X^{s-1},$$

but has better structural properties. In the derived category context, Verdier's octahedral axiom converts "filtrations" (like the sequence $X^0 \to X^1 \to X^2 \to \cdots$ or the sequence $\bar{M}_0 \to \bar{M}_1 \to \bar{M}_2 \to \cdots$ used in the next section) into "resolutions" (like $X \to X/X^0 \to X/X^1 \to \cdots$ or $M_0 \to M_1 \to M_2 \to \cdots$) and vice-versa, but not canonically.

As noted below, the requirement in the definition of projective topological resolution that each $\underline{\pi}_* j_s$ is an epimorphism ensures that projective topological resolutions induce projective algebraic resolutions. An analogous observation applies to injective topological resolutions.

Proposition 6.2. *If (M_s, P_s) is a projective topological resolution of M, then $\underline{\pi}_* P_s$ is a complex of $\underline{R}_*$-modules with differential $d_{s+1} = \Sigma^{-(s+1)}\underline{\pi}_*(k_s \circ j_{s+1})$ and is a projective $\underline{R}_*$-module resolution of $\underline{\pi}_* M$ with augmentation $\underline{\pi}_* j_0$.*

It is somewhat less obvious but quite important for us that there is a topological resolution corresponding to every projective or injective algebraic resolution.

Lemma 6.3. *Let M be a left R-module. Then every projective $\underline{R}_*$-module resolution of $\underline{M}_* = \underline{\pi}_* M$ has a compatible projective topological resolution. Also, every injective $\underline{R}_*$-module resolution of $\underline{M}_*$ has a compatible injective topological resolution.*

Proof. We treat the projective case in detail; the injective case is entirely analogous. Let $\underline{P}_{*,*}$ be a projective resolution of $\underline{M}_*$. First apply Theorem 5.5 to choose left R-modules P_s with $\underline{\pi}_* P_s \cong \underline{P}_{s,*}$. By Corollary 5.7, there is a unique map $j_0\colon P_0 \to M_0 = M$ in the derived category that corresponds on passage to homotopy Mackey functors to the augmentation $\underline{P}_{0,*} \to \underline{M}_*$. Take M_1 to be the cofiber of j_0, and let i_1 and k_0 be the appropriate maps from the resulting cofiber sequence. Now assume by induction that the resolution has been constructed up to M_s and that $\underline{\pi}_* k_{s-1}$ is injective and induces an isomorphism of $\underline{\pi}_* M_s$ with the submodule $\Sigma^s \operatorname{Im}(d_s)$ of $\Sigma^s \underline{P}_{s-1,*}$. Then, by Corollary 5.7, there exists a map $j_s \colon \Sigma^s P_s \to M_s$ whose induced map on homotopy Mackey functors corresponds to the map $d_s \colon \underline{P}_{s,*} \to \operatorname{Im}(d_s)$ under the chosen isomorphisms. Take M_{s+1} to be the cofiber of j_s, and let i_{s+1} and k_s be the induced maps. Since the map j_s induces an epimorphism on homotopy Mackey functors, the long exact sequence associated to this cofiber sequence is short exact. It follows that the map on homotopy Mackey functors induced by k_s provides an isomorphism between $\underline{\pi}_* M_{s+1}$ and $\Sigma^{s+1} \operatorname{Im}(d_{s+1})$. This completes the induction. $\square$

Now we are ready to construct the spectral sequences. Let M be a left R-module and N be a right R-module. Let $\underline{P}_{*,*}$ be a projective resolution of $\underline{M}_* = \underline{\pi}_* M$, and let (M_s, P_s) be a projective topological resolution of M compatible with $\underline{P}_{*,*}$. Extend the collection of cofiber sequences to negative s by setting $P_s = *$ and $M_s = M$ for $s < 0$ (with $i_{s+1} = \mathrm{id}$ and j_s and k_s the trivial map). Applying the functor $\underline{\mathrm{Tor}}^R_*(N, -)$ to these cofiber sequences yields a homologically graded exact couple with

$$\underline{D}_{s,\tau} = \underline{\mathrm{Tor}}^R_{s+\tau}(N, M_s) \qquad \underline{E}_{s,\tau} = \underline{\mathrm{Tor}}^R_{\tau}(N, P_s). \tag{6.4}$$

The maps

$$\begin{tikzcd}
\cdots \arrow[r, "i"] & \underline{D}_{s-1,*} \arrow[rr, "i"] & & \underline{D}_{s,*} \arrow[r, "i"] \arrow[dl, "k"] & \underline{D}_{s+1,*} \arrow[rr, "i"] \arrow[dl, "k"] & & \underline{D}_{s+2,*} \arrow[r, "i"] \arrow[dl, "k"] & \cdots \\
\cdots & & \underline{E}_{s-1,*} \arrow[ul, "j"] & \underline{E}_{s,*} \arrow[u, "j"] & & \underline{E}_{s+1,*} \arrow[ul, "j"] & & \cdots
\end{tikzcd}$$

in this exact couple come from the analogously named maps in the topological resolution. Both i and j preserve the total degree $s + \tau$, and k lowers it by one. Alternatively, given a projective resolution $\underline{Q}_{*,*}$ of $\underline{N}_* = \underline{\pi}_* N$ and a compatible projective topological resolution (N_s, Q_s) of N, setting

$$\underline{D}_{s,\tau} = \underline{\mathrm{Tor}}^R_{s+\tau}(N_s, M) \qquad \underline{E}_{s,\tau} = \underline{\mathrm{Tor}}^R_{\tau}(Q_s, M) \tag{6.5}$$

gives another homologically graded exact couple of exactly the same form.

For the Ext spectral sequence, let L and M be left R-modules. Also, let $\underline{O}_{*,*}$ be a projective resolution of $\underline{L}_* = \underline{\pi}_* L$ and let (L_s, O_s) be a compatible projective topological resolution. Again set $L_s = L$, $O_s = *$ for $s < 0$. Apply the functor $\underline{\mathrm{Ext}}^*_R(-, M)$ to the cofiber sequences relating O_s and L_s, and let

$$\underline{D}^{s,\tau} = \underline{\mathrm{Ext}}^{s+\tau}_R(L_s, M) \qquad \underline{E}^{s,\tau} = \underline{\mathrm{Ext}}^{\tau}_R(O_s, M). \tag{6.6}$$

This yields a cohomologically graded exact couple of the form

$$\begin{tikzcd}
\cdots \arrow[r, "i"] & \underline{D}^{s+2,*} \arrow[rr, "i"] & & \underline{D}^{s+1,*} \arrow[r, "i"] \arrow[dl, "j"] & \underline{D}^{s,*} \arrow[rr, "i"] \arrow[dl, "j"] & & \underline{D}^{s-1,*} \arrow[r, "i"] \arrow[dl, "j"] & \cdots \\
\cdots & & \underline{E}^{s+1,*} \arrow[ul, "k"] & \underline{E}^{s,*} \arrow[u, "k"] & & \underline{E}^{s-1,*} \arrow[ul, "k"] & & \cdots .
\end{tikzcd}$$

In this exact couple, the maps i and j preserve the total cohomological degree $s+\tau$, and k raises it by one. Alternatively, let $\underline{I}^*_*$ be an injective resolution of $\underline{M}_*$ and (M^s, I^s) be a compatible injective topological resolution of M. Extend the fiber sequences of the topological resolution by setting $I^s = *$ and $M^s = M$ for $s < 0$. Applying $\underline{\mathrm{Ext}}^*_R(L, -)$ to the fiber sequences relating M^s and I^s and setting

$$\begin{aligned} \underline{D}^{s,\tau} &= \underline{\mathrm{Ext}}^{s+\tau}_R(L, M^s) \\ \underline{E}^{s,\tau} &= \underline{\mathrm{Ext}}^{\tau}_R(L, I^s) \end{aligned} \tag{6.7}$$

gives a cohomologically graded exact couple of the same form as above.

These exact couples lead to spectral sequences in the usual way. These spectral sequences are clearly natural in the unresolved variable. Theorem 5.6 identifies the E^1- and E^2-terms. When the unresolved variable is a weak bimodule, Theorems 5.3 and 5.4 imply that the exact couple is an exact couple of $\underline{R}_*$ or $\underline{R}^*$-modules. The resulting spectral sequence is therefore a spectral sequence of $\underline{R}_*$ or $\underline{R}^*$-modules. These assertions are summarized in the following result:

Theorem 6.8. *Let L and M be left R-modules and N be a right R-module.*

(a) *The spectral sequence derived from the exact couple (6.4) has E^1 complex canonically isomorphic to $\underline{N}_* \,\Box_{\underline{R}_*}\, \underline{P}_{*,*}$ and $E^2_{s,\tau}$-term canonically isomorphic to $\underline{\mathrm{Tor}}^{\underline{R}_*}_{s,\tau}(\underline{N}_*, \underline{M}_*)$. The spectral sequence is natural in N. If N is a weak bimodule, this is a spectral sequence of left $\underline{R}_*$-modules.*

(b) *The spectral sequence derived from the exact couple (6.5) has E^1 complex canonically isomorphic to $\underline{Q}_{*,*} \,\Box_{\underline{R}_*}\, \underline{M}_*$ and $E^2_{s,\tau}$-term canonically isomorphic to $\underline{\mathrm{Tor}}^{\underline{R}_*}_{s,\tau}(\underline{N}_*, \underline{M}_*)$. The spectral sequence is natural in M. If M is a weak bimodule, this is a spectral sequence of right $\underline{R}_*$-modules.*

(c) *The spectral sequence derived from the exact couple (6.6) has E_1 complex canonically isomorphic to $\langle \underline{Q}_{*,*}, \underline{M}_*\rangle^{\underline{R}_*}$ and $E_2^{s,\tau}$-term canonically isomorphic to $\underline{\mathrm{Ext}}^{s,\tau}_{\underline{R}_*}(\underline{L}_*, \underline{M}_*)$. The spectral sequence is natural in M. If M is a weak bimodule, this is a spectral sequence of right $\underline{R}^*$-modules.*

(d) *The spectral sequence derived from the exact couple (6.7) has E_1 complex canonically isomorphic to $\langle \underline{L}_*, \underline{I}^*_*\rangle^{\underline{R}_*}$ and $E_2^{s,\tau}$-term canonically isomorphic to $\underline{\mathrm{Ext}}^{s,\tau}_{\underline{R}_*}(\underline{L}_*, \underline{M}_*)$. The spectral sequence is natural in L. If L is a weak bimodule, this is a spectral sequence of left $\underline{R}^*$-modules.*

This theorem leaves unresolved the question of whether our spectral sequences are natural in the resolved variable. It is also not obvious that these spectral sequences are independent of the resolution chosen to form them. To settle these questions, we prove the following result in the next section.

Theorem 6.9. *The identity map on $\underline{\mathrm{Tor}}^{\underline{R}_*}_{*,*}(\underline{N}_*, \underline{M}_*)$ induces an isomorphism between the spectral sequences derived from (6.4) and (6.5). Similarly, the identity map on $\underline{\mathrm{Ext}}^{*,*}_{\underline{R}_*}(\underline{L}_*, \underline{M}_*)$ induces an isomorphism between the spectral sequences derived from (6.6) and (6.7).*

The issue of convergence for these spectral sequences must still be discussed. Since limits and colimits of Mackey functors are formed object-wise, convergence of spectral sequences of Mackey functors works just like convergence of spectral sequences of abelian groups. The spectral sequences derived from (6.4) and (6.5) are homological right half-plane spectral sequences. Thus, by Boardman [1, 6.1], to prove that they converge strongly to

$$\underline{\mathrm{Tor}}^R_\tau(N,M) = \mathrm{Lim}_s\, \underline{D}_{-s,s+\tau},$$

it suffices to show that $\mathrm{Colim}_s\, \underline{D}_{s,-s+\tau} = 0$. In the case of (6.4), this amounts to showing that $\mathrm{Colim}_s\, \underline{\mathrm{Tor}}^R_\tau(N, M_s) = 0$. For this, consider the cofiber sequence

$$\bigvee M_s \longrightarrow \bigvee M_s \longrightarrow \mathrm{Tel}\, M_s$$

defining the telescope of a sequence of maps in the derived category. The canonical map $\mathrm{Colim}\, \underline{\pi}_* M_s \to \underline{\pi}_*(\mathrm{Tel}\, M_s)$ is an isomorphism, and it follows that $\mathrm{Tel}\, M_s$ is trivial. Since the derived smash product over R preserves cofiber sequences, $\mathrm{Tel}\,\mathrm{Tor}^R(N, M_s)$ is isomorphic to $\mathrm{Tor}^R(N, \mathrm{Tel}\, M_s)$ and is therefore trivial. In particular, $\mathrm{Colim}\, \underline{\mathrm{Tor}}^R_*(N, M_s) = 0$. Since the edge homomorphism in this case is induced by j_0, we obtain the following result.

Theorem 6.10. *The spectral sequence derived from the exact couple (6.4) converges strongly to $\underline{\mathrm{Tor}}^R_*(N,M)$. Its edge homomorphism is the canonical map $\underline{N}_* \Box_{\underline{R}_*} \underline{M}_* \to \underline{\mathrm{Tor}}^R_*(N,M)$.*

For cohomologically graded right half-plane spectral sequences like those derived from (6.6) and (6.7), conditional convergence to

$$\underline{\mathrm{Ext}}^\tau_R(L,M) = \mathrm{Colim}_s\, \underline{D}^{-s,s+\tau}$$

is defined (see, for example, Boardman [1, 5.10]) to mean that

$$\mathrm{Lim}_s\, \underline{D}^{s,-s+\tau} = 0 \quad \text{and} \quad \mathrm{Lim}^1_s\, \underline{D}^{s,-s+\tau} = 0.$$

In the context of (6.7), $\underline{D}^{s,-s+\tau} = \underline{\mathrm{Ext}}^\tau_R(L, M^s)$. The limit and Lim^1 term that must vanish are defined by an exact sequence

$$0 \to \mathrm{Lim}_s\, \underline{D}^{s,-s+\tau} \to \prod \underline{\mathrm{Ext}}^\tau_R(L, M^s) \to \prod \underline{\mathrm{Ext}}^\tau_R(L, M^s) \to \mathrm{Lim}^1_s\, \underline{D}^{s,-s+\tau} \to 0.$$

Consider the fiber sequence

$$\mathrm{Mic}\, M^s \longrightarrow \prod M^s \longrightarrow \prod M^s$$

defining the microscope of a sequence of maps in the derived category. Each map $i^{s+1}\colon M^{s+1} \to M^s$ induces the zero map on homotopy Mackey functors, and so $\mathrm{Mic}\, M^s$ is trivial. The derived function spectrum functor $\mathrm{Ext}_R(L, -)$ preserves fiber sequences, and so

$$\mathrm{Ext}_R(L, \mathrm{Mic}\, M^s) \longrightarrow \prod \mathrm{Ext}_R(L, M^s) \longrightarrow \prod \mathrm{Ext}_R(L, M^s)$$

is a fiber sequence. Since $\mathrm{Mic}\, M^s$ is trivial, so is $\mathrm{Ext}_R(L, \mathrm{Mic}\, M^s)$. The induced long exact sequence on homotopy Mackey functors now gives conditional convergence. Since the edge homomorphism in this context is induced by j^0, we obtain the following result.

Theorem 6.11. *The spectral sequence derived from the exact couple of (6.7) converges conditionally to $\underline{\mathrm{Ext}}^*_R(L,M)$. Its edge homomorphism is the canonical map $\underline{\mathrm{Ext}}^*_R(L,M) \to \langle \underline{L}_*, \underline{M}_* \rangle^{\underline{R}_*}$.*

This completes the construction of the Hyper-Tor and Hyper-Ext spectral sequences described in the introduction.

7. Uniqueness, Naturality, and The Yoneda Pairing

In the previous section, we constructed a pair of Hyper-Tor spectral sequences and a pair of Hyper-Ext spectral sequences. This section contains the proof of Theorem 6.9, which asserts that the pair of Hyper-Tor spectral sequences are isomorphic and so are the pair of Hyper-Ext spectral sequences. The technical work needed to prove this result also suffices to construct construct the Yoneda pairing of Hyper-Ext spectral sequences mentioned in the introduction. This construction is described in Theorem 7.12.

The formation of our spectral sequences in the last section did not require any constructions more complicated than cofiber and fiber sequences, telescopes, and microscopes. However, here we need more complicated homotopy colimits and limits. Because of this, the constructions described in this section must be carried out in a point set category rather than the associated derived category. Nevertheless, these constructions are of such a general nature that they can be carried out in any of the modern categories of equivariant spectra (i.e., [5, 11, 12]).

For the remainder of the section, fix left R-modules L and M and a right R-module N. Without loss of generality, it can be assumed that each of these objects is cofibrant and fibrant in the appropriate module category. Also, fix projective topological resolutions (L_s, O_s), (M_s, P_s), and (N_s, Q_s) of L, M, and N, respectively, and an injective topological resolution (M^s, I^s) of M. We prove Theorem 6.9 by constructing isomorphisms between the pairs of spectral sequences derived from these specific resolutions.

As noted in the remarks preceding the definition of a projective topological resolution, the sequence of R-modules

$$M = M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_s \longrightarrow \cdots$$

is analogous to the sequence of quotients of M by a sequence of progressively larger submodules of M,

$$M = M/\bar{M}_{-1} \longrightarrow M/\bar{M}_0 \longrightarrow \cdots \longrightarrow M/\bar{M}_{s-1} \longrightarrow \cdots .$$

To form the homotopy limits and colimits needed for the proof of Theorem 6.9, we must "reconstruct" these missing submodules. Specifically, we construct a sequence of (point-set level) cofibrations of R-modules

$$* = \bar{M}_{-1} \longrightarrow \bar{M}_0 \xrightarrow{\bar{\imath}_1} \bar{M}_1 \longrightarrow \cdots \longrightarrow \bar{M}_{s-1} \xrightarrow{\bar{\imath}_s} \bar{M}_s \longrightarrow \cdots$$

together with compatible point-set level R-module maps $f_{s-1}\colon \bar{M}_{s-1} \to M$ whose behavior in the derived category is what one would expect from an appropriate filtration of M with quotients M_s. In particular, for each s, we choose isomorphisms $C(f_{s-1}) \cong M_s$ and $C(\bar{\imath}_s) \cong \Sigma^s P_s$ in the derived category that are compatible in the following sense: They provide an isomorphism in the derived category between the induced maps of homotopy cofibers

$$C(\bar{\imath}_s) \longrightarrow C(f_{s-1}) \longrightarrow C(f_s) \longrightarrow C(\Sigma\bar{\imath}_s) \simeq \Sigma C(\bar{\imath}_s) \tag{7.1}$$

and the cofiber sequence

$$\Sigma^s P_s \longrightarrow M_s \longrightarrow M_{s+1} \longrightarrow \Sigma^{s+1} P_s \tag{7.2}$$

that is a part of our chosen projective topological resolution. Here and for the rest of the section, we understand the point-set model for homotopy cofibers to be formed using the usual cone construction; that is,

$$C(\bar{\imath}_s) = \bar{M}_s \cup_{\bar{M}_{s-1}} (\bar{M}_{s-1} \wedge I_+) \cup_{\bar{M}_{s-1}} *$$

and

$$C(f_{s-1}) = M \cup_{\bar{M}_{s-1}} (\bar{M}_{s-1} \wedge I_+) \cup_{\bar{M}_{s-1}} *.$$

Note that since the model categories in [5, 11, 12] are simplicial and the objects $\bar{M}_{s-1}$ and M are cofibrant, these are cofibrant objects and are cofibers in the sense of Quillen. Moreover, sequence (7.1) represents a cofiber sequence in the derived category. The apparent shift in indexing in the cofiber sequence

$$\bar{M}_{s-1} \xrightarrow{f_{s-1}} M \longrightarrow M_s \longrightarrow \Sigma \bar{M}_{s-1}$$

(M_s is the cofiber of $\bar{M}_{s-1}$) is for consistency with the grading conventions of Boardman [1, §12]. This shift also leads to cleaner formulas in the work below.

The "reconstruction" of the modules $\bar{M}_s$ essentially amounts to a point-set refinement of Verdier's octahedral axiom. The standard proof of this axiom is actually strong enough to provide the refinement we need. Let $\bar{M}_{-1} = *$. Next choose a cofibrant R-module $\bar{M}_0$ weakly equivalent to P_0 and a point-set map $f_0 \colon \bar{M}_0 \to M$ of R-modules such that the induced cofiber sequence

$$\bar{M}_0 \longrightarrow M \longrightarrow C(f_0) \longrightarrow \Sigma \bar{M}_0$$

is isomorphic in the derived category to the given cofiber sequence

$$P_0 \longrightarrow M \longrightarrow M_1 \longrightarrow \Sigma P_0$$

by an isomorphism that is the identity on M. For the inductive step, assume that the sequence of cofibrations and compatible maps into M has been constructed up to the R-module $\bar{M}_s$. Choose a fibrant R-module X and a point-set level R-module map $C(f_s) \to X$ such that the induced cofiber sequence is isomorphic in the derived category to the given cofiber sequence

$$M_{s+1} \longrightarrow M_{s+2} \longrightarrow \Sigma^{s+2} P_{s+1} \to \Sigma M_{s+1}$$

by an isomorphism that restricts to the previously chosen isomorphism $C(f_s) \to M_{s+1}$. Let F be the homotopy fiber of the composite $M \to C(f_s) \to X$ defined using the usual path space construction,

$$F = M \times_X X^{I_+} \times_X *.$$

Since M and X are fibrant, this represents the fiber in the sense of Quillen. By construction, the composite $\bar{M}_s \to M \to X$ factors through the composite $\bar{M}_s \to C\bar{M}_s \to C(f_s)$. This factorization provides a null homotopy of the map from $\bar{M}_s$ to X and so a lift $\bar{M}_s \to F$ of the map $\bar{M}_s \to M$. Choose $\bar{M}_{s+1}$ by factoring the map $\bar{M}_s \to F$ as a cofibration $\bar{\imath}_{s+1} \colon \bar{M}_s \to \bar{M}_{s+1}$ followed by an acyclic fibration $\bar{M}_{s+1} \xrightarrow{\sim} F$. Let $f_{s+1} \colon \bar{M}_{s+1} \to M$ be the composite map $\bar{M}_{s+1} \to F \to M$. The objects and maps we have chosen fit into the diagram

$$\begin{tikzcd}
\bar{M}_s \arrow[r, "f_s"] \arrow[d, "\bar\imath_{s+1}"'] & M \arrow[r] \arrow[d, equal] & C(f_s) \arrow[d] \\
\bar{M}_{s+1} \arrow[r, "f_{s+1}"] \arrow[d, dashed, "\sim"'] & M \arrow[r] \arrow[dr, dashed] & C(f_{s+1}) \arrow[d, dashed, "\sim"] \\
F \arrow[ur, dashed] & & X,
\end{tikzcd}$$

which commutes on the point-set level. The map $C(\bar\imath_{s+1}) \to X$ is a weak equivalence by [5, I.6.4], [11, 3.5.(vi)], or [12, 5.8]. Thus, this procedure constructs an isomorphism in the derived category between cofiber sequence (7.1) and cofiber sequence (7.2) that restricts to the previously chosen isomorphism $C(f_s) \to M_{s+1}$. This completes our construction, which is described formally in the following proposition.

Proposition 7.3. *Let (M_s, P_s) be a projective topological resolution of an R-module M. Then there is a sequence*

$$* = \bar{M}_{-1} \longrightarrow \bar{M}_0 \xrightarrow{\bar\imath_1} \bar{M}_1 \longrightarrow \cdots \longrightarrow \bar{M}_{s-1} \xrightarrow{\bar\imath_s} \bar{M}_s \longrightarrow \cdots$$

of cofibrations of R-modules together with R-module maps $f_{s-1}\colon \bar{M}_{s-1} \to M$, compatible on the point-set level, such that $C(f_{s-1}) \cong M_s$ and $C(\bar\imath_s) \cong \Sigma^s P_s$ in the derived category. Moreover, these isomorphisms are compatible in the sense that they provide an isomorphism in the derived category between the cofiber sequences

$$C(\bar\imath_s) \longrightarrow C(f_{s-1}) \longrightarrow C(f_s) \longrightarrow C(\Sigma\bar\imath_s) \simeq \Sigma C(\bar\imath_s)$$

and

$$\Sigma^s P_s \longrightarrow M_s \longrightarrow M_{s+1} \longrightarrow \Sigma^{s+1} P_s.$$

In this same manner, choose sequences of cofibrations

$$* = L_{-1} \longrightarrow \bar{L}_0 \xrightarrow{\bar\imath_1} \bar{L}_1 \longrightarrow \cdots \longrightarrow \bar{L}_{s-1} \xrightarrow{\bar\imath_s} \bar{L}_s \longrightarrow \cdots$$

and

$$* = N_{-1} \longrightarrow \bar{N}_0 \xrightarrow{\bar\imath_1} \bar{N}_1 \longrightarrow \cdots \longrightarrow \bar{N}_{s-1} \xrightarrow{\bar\imath_s} \bar{N}_s \longrightarrow \cdots$$

together with compatible maps $f_s\colon \bar{L}_s \to L$ and $f_s\colon \bar{N}_s \to N$ consistent with the chosen projective resolutions of L and N. When it is desirable to indicate which of the objects L, M, and N is associated to a particular map $\bar\imath_s$ or f_s, the symbols $\bar\imath_s^M$, f_s^M, etc., are employed.

An analogous sequence of fibrations

$$\cdots \longrightarrow M^s \xrightarrow{\bar\imath^{s-1}} M^{s-1} \longrightarrow \cdots \longrightarrow \bar{M}^0 \xrightarrow{\bar\imath^1} \bar{M}^0 \longrightarrow M^{-1} = *$$

together with compatible maps $f^s\colon M \to M^s$ can be constructed using the "Eckmann–Hilton" dual of the argument above (that is, reverse the direction of all of the arrows, and switch cofibrations and fibrations, colimits and limits, and $(-)\wedge I_+$ and $(-)^{I_+}$).

To make use of the R-modules just constructed, we must introduce a family of very simple categories.

Definition 7.4. Let $\mathscr{D}$ denote the category which has as objects the ordered pairs of natural numbers (s,t) and as maps a unique map $(s,t)\to(s',t')$ whenever $s\le s'$ and $t\le t'$. Let $\mathscr{D}_n$ denote the full subcategory of $\mathscr{D}$ consisting of objects (s,t) with $s+t\le n$

The (point-set) smash products $\bar{N}_s\wedge_R\bar{M}_t$ may be regarded as a functor from $\mathscr{D}$ to equivariant S-modules, orthogonal spectra, or symmetric spectra, as appropriate. Likewise, the function spectra $F_R(\bar{L}_s,\bar{M}^t)$ form a contravariant functor from $\mathscr{D}$ to equivariant S-modules, orthogonal spectra, or symmetric spectra. Our main tools in this section are homotopy limits and colimits of these functors. Our argument requires constructions of homotopy limits and colimits that are functorial on the point-set level. Any functorial constructions should be adequate. However, at some points in the argument, we assume that the "bar construction" models of homotopy limits and colimits (described, for example, in [5, X§3]) are used in order to fill in certain details.

Definition 7.5. Let $T_n=\operatorname{Hocolim}_{\mathrm{D}_n}(\bar{N}_s\wedge_R\bar{M}_t)$ and $U_n=\operatorname{Holim}_{\mathrm{D}_n}F_R(\bar{L}_s,\bar{M}^t)$. There are canonical maps

$$g_{n+1}\colon T_n\longrightarrow T_{n+1}\qquad\text{and}\qquad g^{n+1}\colon U^{n+1}\longrightarrow U^n.$$

induced by the inclusion of categories $\mathscr{D}_n\to\mathscr{D}_{n+1}$. Similarly, there are maps

$$h_n\colon T_n\to N\wedge_R M\qquad\text{and}\qquad h^n\colon F_R(L,M)\to U^n$$

induced by the maps $\bar{M}_s\to M$ and the analogous maps for the $\bar{N}_s$, $\bar{L}_s$, and $\bar{M}^s$ sequences.

There are also maps

$$\begin{array}{cc} T_n\longrightarrow N\wedge_R\bar{M}_n & T_n\longrightarrow\bar{N}_n\wedge_R M,\\ F_R(\bar{L}_n,M)\longrightarrow U^n & F_R(L,\bar{M}^n)\longrightarrow U^n\end{array}$$

induced by the maps $\bar{M}_s\to\bar{M}_{s+1}$ and the analogous maps for the $\bar{N}_s$, $\bar{L}_s$, and $\bar{M}^s$ sequences. Because we are using a functorial construction of homotopy colimits and limits, these are point-set level maps. Moreover, the diagrams

$$\begin{tikzcd} T_n \arrow[r] \arrow[d, "g_{n+1}"'] & N\wedge_R\bar{M}_n \arrow[d, "\mathrm{id}\wedge\bar{\imath}^M_{n+1}"] \\ T_{n+1} \arrow[r] & N\wedge_R\bar{M}_{n+1} \end{tikzcd}\qquad\begin{tikzcd} T_n \arrow[r] \arrow[d, "g_{n+1}"'] & \bar{N}_n\wedge_R M \arrow[d, "\bar{\imath}^N_{n+1}\wedge\mathrm{id}"] \\ T_{n+1} \arrow[r] & \bar{N}_{n+1}\wedge_R M, \end{tikzcd}$$

and the analogous diagrams for the homotopy limits U^n commute on the point-set level.

Since the diagrams above commute on the point-set level, they induce canonical maps

$$\begin{tikzcd} C(g_{s+1}) \arrow[r] \arrow[d] & C(h_s) \arrow[r] \arrow[d] & C(h_{s+1}) \arrow[r] \arrow[d] & \Sigma C(g_{s+1}) \arrow[d] \\ C(\mathrm{id}_N\wedge\bar{\imath}^M_{s+1}) \arrow[r] & C(\mathrm{id}_N\wedge f^M_s) \arrow[r] & C(\mathrm{id}_N\wedge f^M_{s+1}) \arrow[r] & \Sigma C(\mathrm{id}_N\wedge\bar{\imath}^M_{s+1}) \end{tikzcd}$$

and

$$\begin{tikzcd} C(g_{s+1}) \arrow[r] \arrow[d] & C(h_s) \arrow[r] \arrow[d] & C(h_{s+1}) \arrow[r] \arrow[d] & \Sigma C(g_{s+1}) \arrow[d] \\ C(\bar{\imath}^N_{s+1} \wedge \mathrm{id}_M) \arrow[r] & C(f^N_s \wedge \mathrm{id}_M) \arrow[r] & C(f^N_{s+1} \wedge \mathrm{id}_M) \arrow[r] & \Sigma C(\bar{\imath}^N_{s+1} \wedge \mathrm{id}_M) \end{tikzcd}$$

of cofiber sequences in the derived category. The constructions $N \wedge_R (-)$ and $(-) \wedge_R M$ preserve cofiber sequences. Thus, the isomorphisms characterizing the terms in cofiber sequence (7.1) can be used to identify the bottom cofiber sequences in the two diagrams above with those defining the exact couples (6.4) and (6.5), respectively. Setting

$$\begin{aligned} \underline{D}_{s,\tau} &= \underline{\pi}_{s+\tau} C(h_s) \\ \underline{E}_{s,\tau} &= \underline{\pi}_{s+\tau} C(g_s) \end{aligned} \tag{7.6}$$

gives a third homologically graded exact couple. Moreover, the commuting diagrams above provide maps of exact couples from (7.6) to both (6.4) and (6.5).

The following result about these three exact couples implies the part of Theorem 6.9 applicable to exact couples (6.4) and (6.5).

Theorem 7.7. *The spectral sequence derived from (7.6) has its E^1 complex canonically isomorphic to the total complex of $\underline{Q}_{*,*} \Box_{\underline{R}_*} \underline{P}_{*,*}$. Moreover, under the canonical isomorphisms, the maps of this E^1 complex to the E^1 complexes of (6.4) and (6.5) coincide with the augmentations*

$$\underline{Q}_{*,*} \Box_{\underline{R}_*} \underline{P}_{*,*} \longrightarrow \underline{N}_* \Box_{\underline{R}_*} \underline{P}_{*,*} \quad \text{and} \quad \underline{Q}_{*,*} \Box_{\underline{R}_*} \underline{P}_{*,*} \longrightarrow \underline{Q}_{*,*} \Box_{\underline{R}_*} \underline{M}_*,$$

respectively.

In a similar fashion, by setting

$$\begin{aligned} \underline{D}^{s,\tau} &= \underline{\pi}_{-s-\tau} F(h^s) \\ \underline{E}^{s,\tau} &= \underline{\pi}_{-s-\tau} F(g^s), \end{aligned} \tag{7.8}$$

we obtain a cohomologically graded exact couple and maps of exact couples from both (6.6) and (6.7) to (7.8). The relation between these three exact couples is described by the following result, which implies the remaining claims made in Theorem 6.9.

Theorem 7.9. *The spectral sequence derived from (7.8) has its E_1 complex canonically isomorphic to the total complex of $\langle \underline{O}_{*,*}, \underline{I}^*_* \rangle^{\underline{R}_*}$. Moreover, under the canonical isomorphisms, the maps into this complex from the E_1 complexes of (6.6) and (6.7) coincide with the augmentations*

$$\langle \underline{O}_{*,*}, \underline{M}_* \rangle^{\underline{R}_*} \longrightarrow \langle \underline{O}_{*,*}, \underline{I}^*_* \rangle^{\underline{R}_*} \quad \text{and} \quad \langle \underline{L}_*, \underline{I}^*_* \rangle^{\underline{R}_*} \longrightarrow \langle \underline{O}_{*,*}, \underline{I}^*_* \rangle^{\underline{R}_*},$$

respectively.

For the proofs of these results, recall that, in the bar construction model of the homotopy colimit, T_n is the geometric realization of a simplicial object which in simplicial degree d is the wedge product, indexed on the set of d composable arrows

$$(s_0, t_0) \leftarrow \cdots \leftarrow (s_d, t_d)$$

in $\mathscr{D}_n$, of the summands $\bar{N}_{s_d} \wedge_R \bar{M}_{t_d}$. The face maps are given by dropping arrows on either end, composing arrows in the middle, and by the action of $\mathscr{D}_n$ on $\bar{N}_{s_d} \wedge_R \bar{M}_{t_d}$. The degeneracy maps simply insert identity maps. Likewise, the homotopy limit U^n

is the geometric realization (or Tot) of a cosimplicial object which in cosimplicial degree d is the product, indexed on the set of d composable arrows in $\mathscr{D}_n$, of the factors $F_R(\bar{L}_{s_d}, \bar{M}^{t_d})$. Here, the face maps are given by dropping or composing arrows and by the contravariant action of $\mathscr{D}_n$ on $F_R(\bar{L}_{s_d}, \bar{M}^{t_d})$. The degeneracy maps again simply insert identity maps.

Proof of Theorems 7.7 and 7.9. We give the details of the proof of Theorem 7.7. The proof of Theorem 7.9 is quite similar. The canonical maps from the homotopy colimits to the point-set colimits induce a map $T_0 \to \bar{M}_0 \wedge_R \bar{N}_0$ and, for $s > 0$, maps

$$T_s/T_{s-1} \longrightarrow (\bar{N}_0) \wedge_R (\bar{M}_s/\bar{M}_{s-1}) \vee \cdots \vee (\bar{N}_s/\bar{N}_{s-1}) \wedge_R (\bar{M}_0).$$

These maps induce maps

$$C(g_s) \longrightarrow \bigvee_{n+m=s} \operatorname{Tor}^R(C(\bar{\imath}^N_n), C(\bar{\imath}^M_m)) \cong \bigvee_{n+m=s} \operatorname{Tor}^R(\Sigma^n Q_n, \Sigma^m P_m). \tag{7.10}$$

This gives a canonical map from E^1 to $\underline{Q}_{*,*} \,\square_{\underline{R}_*}\, \underline{P}_{*,*}$. It is easy to see that this is a map of chain complexes and that the maps

$$\underline{\pi}_* C(g_s) \longrightarrow \underline{\pi}_* C(\mathrm{id}_N \wedge \bar{\imath}^M_s) \cong \underline{N}_* \,\square_{\underline{R}_*}\, \underline{P}_{s,*},$$
$$\underline{\pi}_* C(g_s) \longrightarrow \underline{\pi}_* C(\bar{\imath}^N_s \wedge \mathrm{id}_M) \cong \underline{Q}_{s,*} \,\square_{\underline{R}_*}\, \underline{M}_*$$

factor through the augmentations

$$\underline{Q}_{*,*} \,\square_{\underline{R}_*}\, \underline{P}_{*,*} \longrightarrow \underline{N}_* \,\square_{\underline{R}_*}\, \underline{P}_{*,*} \quad \text{and} \quad \underline{Q}_{*,*} \,\square_{\underline{R}_*}\, \underline{P}_{*,*} \longrightarrow \underline{Q}_{*,*} \,\square_{\underline{R}_*}\, \underline{M}_*$$

as required.

It remains to see that (7.10) induces an isomorphism on homotopy groups. Let $\mathscr{D}_{n,m}$ denote the full subcategory of $\mathscr{D}_n$ whose objects are the pairs (s,t) with $s \leq m$. Also, let $T_{n,m} = \operatorname{Hocolim}_{\mathrm{D}_{n,m}} \bar{N}_s \wedge_R \bar{M}_t$. Standard homotopy theory arguments show that the canonical map from the homotopy colimit to the colimit induces weak equivalence of the cofiber of $T_{s-1,0} \to T_{s,0}$ with $\bar{N}_0 \wedge \bar{M}_s$ and the cofiber of the map $T_{s,m-1} \to T_{s,m}$ with $(\bar{N}_m/\bar{N}_{m-1}) \wedge_R \bar{M}_{s-m}$. A filtration argument now finishes the proof. $\square$

We close this section with an explanation of the Yoneda pairing of Hyper-Ext spectral sequences. Assume that K is yet another left R-module, and let $\underline{K}_* = \underline{\pi}_* K$. Also, let $\underline{E}^{s,\tau}_r(L,K)$ denote the spectral sequence for $\underline{\operatorname{Ext}}^*_R(L,K)$ derived from (6.6), $\underline{E}^{s,\tau}_r(K,M)$ denote the spectral sequence for $\underline{\operatorname{Ext}}^*_R(K,M)$ derived from (6.7), and $\underline{E}^{s,\tau}_r(L,M)$ denote the spectral sequence for $\underline{\operatorname{Ext}}^*_R(L,K)$ derived from (7.8). A map of complexes

$$\bigoplus_{\ell+m=s} \underline{E}^{m,\tau}_r(K,M) \,\square\, \underline{E}^{\ell,\tau}_r(L,K) \longrightarrow \underline{E}^{s,\tau}_r(L,M) \tag{7.11}$$

induces in the usual way a map

$$\begin{aligned}
&\bigoplus_{\ell+m=s} \underline{E}^{m,\tau}_{r+1}(K,M) \,\square\, \underline{E}^{\ell,\tau}_{r+1}(L,K) \\
&\qquad = \bigoplus_{\ell+m=s} H^m(\underline{E}^{*,\tau}_r(K,M)) \,\square\, H^\ell(\underline{E}^{*,\tau}_r(L,K)) \\
&\qquad \longrightarrow H^s(\underline{E}^{*,\tau}_r(K,M) \,\square\, \underline{E}^{*,\tau}_r(L,K)) \\
&\qquad \longrightarrow H^s(\underline{E}^{*,\tau}_r(L,M)) \\
&\qquad = \underline{E}^{s,\tau}_{r+1}(L,M).
\end{aligned}$$

of graded Mackey functors for E_{r+1}. A pairing of spectral sequences is map of complexes (7.11) for E_1 such that the induced map on E_2 is a map of complexes, and, inductively, the induced map on each E_{r+1} is a map of complexes.

Theorem 7.12. *The composition pairing*

$$\langle \underline{K}_*, \underline{I}^m_* \rangle^{\underline{R}_*} \,\Box\, \langle \underline{O}_{\ell,*}, \underline{K}_* \rangle^{\underline{R}_*} \longrightarrow \langle \underline{O}_{\ell,*}, \underline{I}^m_* \rangle^{\underline{R}_*}$$

induces a pairing of the Hyper-Ext spectral sequences. On E_2, this pairing agrees with the Yoneda pairing. Also, this pairing agrees on E_∞ with the associated graded of the composition pairing for $\underline{\mathrm{Ext}}^*_R$.

Proof. The maps

$$(7.13) \qquad F_R(K, \bar{M}^m) \wedge F_R(\bar{L}_\ell, K) \longrightarrow F(\bar{L}_\ell, \bar{M}^m) \longrightarrow U^{\ell+m}$$

of S-modules, orthogonal spectra, or symmetric spectra induce maps

$$\mathrm{Ext}_R(K, \Omega^m I^m) \wedge_S \mathrm{Ext}_R(\Sigma^\ell O_\ell, K) \longrightarrow F(g^{\ell+m})$$

in the stable category. On homotopy groups, these maps induce the pairing on E_1 described in the theorem, which on E_2 is the Yoneda pairing, by definition. The composition maps (7.13) also induce maps

$$\begin{aligned}\mathrm{Ext}_R(K, F(\bar{M}^m \to \bar{M}^{m-r})) \wedge \mathrm{Ext}_R(C(\bar{L}_{\ell-r} \to \bar{L}_\ell), K) \\ \longrightarrow F(U^{\ell+m} \to U^{\ell+m-r})\end{aligned}$$

in the stable category relating the fiber and cofiber of the canonical maps $\bar{M}^m \to \bar{M}^{m-r}$ and $\bar{L}_{\ell-r} \to \bar{L}_\ell$, respectively, to the fiber of the canonical map $U^{\ell+m} \to U^{\ell+m-r}$. An inductive argument using these maps indicates that the pairing on E^r preserves the differential. The statement about E_∞ is clear. ◻

When R is a commutative S-algebra, the pairing described in Theorem 7.12 descends to a pairing defined in terms of $\Box_{\underline{R}_*}$ rather than $\Box$. Moreover, the identification of the pairing on the E_∞ level respects the $\underline{R}_*$-module structures. Theorem 7.12, together with these observations, establishes the properties of the Yoneda pairing of spectral sequences noted in the introduction.

Appendix A. The smash product and the box product

The purpose of this appendix is to prove Theorem 5.1, which asserts that the homotopy Mackey functor is lax symmetric monoidal. Since the argument in the case of $\mathbb{Z}$-graded Mackey functors is well-known, we concentrate on the $RO(G)$-graded case.

As in Section 5, choose a model S^τ for the τ-sphere for each element τ of $RO(G)$ with the restrictions that $S^0 = S$ and that S^τ is the smash product of S with the standard n-sphere space whenever τ is the trivial representation of dimension $n > 0$. For all other τ, the object S^τ may be chosen arbitrarily from the appropriate homotopy class. When $\tau = \alpha+\beta$, $S^\alpha \wedge S^\beta$ and S^τ are isomorphic in the equivariant stable category. Thus, we can choose an isomorphism

$$f_{\alpha,\beta} \colon S^{\alpha+\beta} \longrightarrow S^\alpha \wedge S^\beta.$$

Such a choice gives a homomorphism

$$\begin{aligned}[S^\alpha \wedge X_+, M]_G \otimes [S^\beta \wedge Y_+, N]_G \quad &\longrightarrow \quad [S^\alpha \wedge X_+ \wedge S^\beta \wedge Y_+, M \wedge N]_G \\ &\longrightarrow \quad [S^{\alpha+\beta} \wedge (X \times Y)_+, M \wedge N]_G,\end{aligned}$$

which is natural in M and N in the equivariant stable category and X and Y in the Burnside category. The universal property of the box product then gives a natural map

$$\phi\colon \underline{\pi}_* M \,\Box_*\, \underline{\pi}_* N \longrightarrow \underline{\pi}_*(M \wedge N).$$

There is also a canonical natural map $\iota\colon \underline{B}_* \to \underline{\pi}_* S$ that includes $\underline{B}$ as $\underline{\pi}_0 S$. This map does not require any choices. Theorem 5.1 is merely a less detailed version of the following result.

Theorem A.1. *The maps $f_{\alpha,\beta}$ may be chosen so that ϕ together with $\iota\colon \underline{B}_* \to \underline{\pi}_* S$ provide the functor $\underline{\pi}_*$ with a lax symmetric monoidal structure.*

In other words, the maps $f_{\alpha,\beta}$ may be chosen so that the unit diagrams

(A.2)
$$\begin{tikzcd}
\underline{B}_* \Box_* \underline{\pi}_* N \arrow[r, "\mathrm{id}\,\Box_*\iota"] \arrow[d, "\lambda"'] & \underline{\pi}_* S \Box_* \underline{\pi}_* N \arrow[d, "\phi"] \\
\underline{\pi}_* N & \underline{\pi}_*(S \wedge N) \arrow[l, "\underline{\pi}_*\lambda"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\pi}_* M \Box_* \underline{B}_* \arrow[r, "\iota\Box_*\mathrm{id}"] \arrow[d, "\rho"'] & \underline{\pi}_* M \Box_* \underline{\pi}_* S \arrow[d, "\phi"] \\
\underline{\pi}_* M & \underline{\pi}_*(M \wedge S), \arrow[l, "\underline{\pi}_*\rho"]
\end{tikzcd}$$

the associativity diagram

(A.3)
$$\begin{tikzcd}
& \underline{\pi}_*(L\wedge M)\Box_*\underline{\pi}_*N \arrow[r, "\phi"] & \underline{\pi}_*((L\wedge M)\wedge N) \arrow[dr, "\cong"] & \\
(\underline{\pi}_*L\Box_*\underline{\pi}_*M)\Box_*\underline{\pi}_*N \arrow[ur, "\phi\Box_*\mathrm{id}"] \arrow[dr, "\cong"'] & & & \underline{\pi}_*(L\wedge(M\wedge N)), \\
& \underline{\pi}_*L\Box_*(\underline{\pi}_*M\Box_*\underline{\pi}_*N) \arrow[r, "\mathrm{id}\,\Box_*\phi"'] & \underline{\pi}_*L\Box_*\underline{\pi}_*(M\wedge N) \arrow[ur, "\phi"'] &
\end{tikzcd}$$

and the symmetry diagram

(A.4)
$$\begin{tikzcd}
\underline{\pi}_* M \Box_* \underline{\pi}_* N \arrow[r, "\cong"] \arrow[d, "\phi"'] & \underline{\pi}_* N \Box_* \underline{\pi}_* M \arrow[d, "\phi"] \\
\underline{\pi}_*(M\wedge N) \arrow[r, "\cong"'] & \underline{\pi}_*(N\wedge M)
\end{tikzcd}$$

commute. The redundant right unit diagram has been included because, with it, diagrams (A.2) and (A.3) together imply that $\underline{\pi}_*$ is lax monoidal. This is verified before the question of commutativity is addressed. A direct argument in terms of specifying how to choose the isomorphisms $f_{\alpha,\beta}$ appears to be possible, but would require checking a long list of complicated details. Instead we take a more abstract approach that reduces to checking the vanishing of a certain characteristic cohomology class defined in [2, §7]. The definition of that class in our specific case is reviewed below.

Since $S^0 = S$ has been chosen as the unit for the smash product in the equivariant stable category, the canonical choices for the maps $f_{0,\tau}$ and $f_{\tau,0}$ are the inverses of the unit isomorphisms $\lambda\colon S^0 \wedge S^\tau \to S^\tau$ and $\rho\colon S^\tau \wedge S^0 \to S^\tau$. This is the unique choice making the unit diagrams (A.2) commute. This choice is assumed in the remainder of our argument.

It is easy to verify that the required associativity diagram (A.3) commutes if and only if the diagram

$$\text{(A.5)}\qquad \begin{tikzcd} & S^{\alpha+\beta+\gamma} \arrow[dl, "f_{\alpha,\beta+\gamma}"'] \arrow[dr, "f_{\alpha+\beta,\gamma}"] & \\ S^{\alpha}\wedge S^{\beta+\gamma} \arrow[d, "\mathrm{id}\wedge f_{\beta,\gamma}"'] & & S^{\alpha+\beta}\wedge S^{\gamma} \arrow[d, "f_{\alpha,\beta}\wedge \mathrm{id}"] \\ S^{\alpha}\wedge (S^{\beta}\wedge S^{\gamma}) \arrow[rr, "\cong"'] & & (S^{\alpha}\wedge S^{\beta})\wedge S^{\gamma} \end{tikzcd}$$

commutes for all α,β, and γ in $RO(G)$. In general, the failure of this diagram to commute can be measured by a unit in the Burnside ring. To see this, recall that, in any symmetric monoidal additive category, the abelian group of maps between any two objects is a module over the ring of endomorphisms of the unit object (see for example the discussion of "signs" in Section 2). The abelian group $[S^{\alpha+\beta+\gamma}, S^{\alpha}\wedge S^{\beta}\wedge S^{\gamma}]_G$ is a one-dimensional free module over $[S,S]_G = \underline{B}(G/G)$ and is generated by any isomorphism. The right-hand composite isomorphism in diagram (A.5) is therefore the product of left-hand composite isomorphism and a well-defined unit $\mathfrak{a}_{\alpha,\beta,\gamma}$ in $\underline{B}(G/G)$. Equivalently, $\mathfrak{a}_{\alpha,\beta,\gamma}$ can be defined as the unique unit in $\underline{B}(G/G)$ such that

$$\text{(A.6)}\qquad \mathfrak{a}_{\alpha,\beta,\gamma}\cdot \mathrm{id}_{S^{\alpha+\beta+\gamma}} = f^{-1}_{\alpha,\beta+\gamma}\circ(\mathrm{id}\wedge f_{\beta,\gamma})^{-1}\circ a\circ (f_{\alpha,\beta}\wedge \mathrm{id})\circ f_{\alpha+\beta,\gamma}.$$

Here, a denotes the associativity isomorphism in the equivariant stable category. Clearly, diagram (A.5) commutes if and only if $\mathfrak{a}_{\alpha,\beta,\gamma}=1$. Thus, we have proven:

Proposition A.7. *The associativity diagram (A.3) commutes for all L, M, N if and only if $\mathfrak{a}_{\alpha,\beta,\gamma}=1$ for all $\alpha,\beta,\gamma\in RO(G)$.*

The isomorphism $\mathfrak{a}_{\alpha,\beta,\gamma}\cdot\mathrm{id}_{S^{\alpha+\beta+\gamma}}$ has a straight-forward structural interpretation in terms of the choice of models S^τ and isomorphisms $f_{\alpha,\beta}$. Consider the full subcategory $\mathfrak{S}$ of the equivariant stable category consisting of objects isomorphic to spheres S^τ (for $\tau\in RO(G)$), and let $\mathfrak{C}$ be the full subcategory consisting of the chosen models S^τ. The inclusion of $\mathfrak{C}$ in $\mathfrak{S}$ is obviously an equivalence of categories. The smash product on the equivariant stable category restricts to a symmetric monoidal product on $\mathfrak{S}$. The isomorphisms $f_{\alpha,\beta}$ can be used to construct an equivalent symmetric monoidal smash product on $\mathfrak{C}$. There, the smash product of S^α and S^β is $S^{\alpha+\beta}$. For maps $g\colon S^\alpha\to S^{\alpha'}$ and $h\colon S^\beta\to S^{\beta'}$, the induced map $g\wedge h\colon S^{\alpha+\beta}\to S^{\alpha'+\beta'}$ in $\mathfrak{C}$ is the composite

$$S^{\alpha+\beta}\xrightarrow{f_{\alpha,\beta}} S^{\alpha}\wedge S^{\beta}\xrightarrow{g\wedge h} S^{\alpha'}\wedge S^{\beta'}\xrightarrow{f^{-1}_{\alpha',\beta'}} S^{\alpha'+\beta'}.$$

Because of our restriction that $f_{0,\tau}$ and $f_{\tau,0}$ are the inverses of unit isomorphisms, the unit isomorphisms in $\mathfrak{C}$ are the appropriate identity maps. The associativity isomorphisms in $\mathfrak{C}$ are precisely the maps $\mathfrak{a}_{\alpha,\beta,\gamma}\cdot\mathrm{id}_{S^{\alpha+\beta+\gamma}}$. Since $\underline{B}(G/G)$ is commutative and composition in $\mathfrak{C}$ is bilinear over $\underline{B}(G/G)$, the pentagon and triangle laws [10, VII§1] for the monoidal structure on $\mathfrak{C}$ translate into the following assertion about the elements $\mathfrak{a}_{\alpha,\beta,\gamma}$:

Proposition A.8. *The elements $\mathfrak{a}_{\alpha,\beta,\gamma}$ satisfy the cocycle condition*

$$\mathfrak{a}_{\alpha,\beta,\gamma+\delta}\cdot\mathfrak{a}_{\alpha+\beta,\gamma,\delta} = \mathfrak{a}_{\beta,\gamma,\delta}\cdot\mathfrak{a}_{\alpha,\beta+\gamma,\delta}\cdot\mathfrak{a}_{\alpha,\beta,\gamma},$$

in the group of units of the Burnside ring and are normalized in the sense that $\mathfrak{a}_{\alpha,\beta,\gamma} = 1$ *when any of* α*,* β*, or* γ *are* 0*.*

This may be rephrased as the assertion that the collection $\{\mathfrak{a}_{\alpha,\beta,\gamma} \mid \alpha,\beta,\gamma \in RO(G)\}$ specifies a normalized 3-cocycle for the group cohomology of $RO(G)$ with coefficients in group A^* of units of the Burnside ring.

For the proof of Theorem A.1, we must consider how this cocycle changes when a different collection of isomorphisms $f_{\alpha,\beta}$ is selected. Let $f'_{\alpha,\beta}\colon S^{\alpha+\beta} \to S^\alpha \wedge S^\beta$ be another such collection satisfying our restriction on the maps $f'_{0,\tau}$ and $f'_{\tau,0}$. The composite $f'_{\alpha,\beta} \circ f^{-1}_{\alpha,\beta}\colon S^{\alpha+\beta} \to S^{\alpha+\beta}$ is $\mathfrak{b}_{\alpha,\beta} \cdot \mathrm{id}_{S^{\alpha+\beta}}$ for a well defined unit $\mathfrak{b}_{\alpha,\beta}$ in the Burnside ring. It is easy to verify that the cocycle $(\mathfrak{a}'_{\alpha,\beta,\gamma})$ associated to the isomorphisms $f'_{\alpha,\beta}$ is given by

$$\mathfrak{a}'_{\alpha,\beta,\gamma} = \mathfrak{b}_{\alpha,\beta} \cdot \mathfrak{b}_{\alpha+\beta,\gamma} \cdot \mathfrak{b}^{-1}_{\alpha,\beta+\gamma} \cdot \mathfrak{b}^{-1}_{\beta,\gamma} \cdot \mathfrak{a}_{\alpha,\beta,\gamma}.$$

When the collection $(\mathfrak{b}_{\alpha,\beta})$ is regarded as a 2-cochain for group cohomology, its boundary satisfies

$$(d\mathfrak{b})_{\alpha,\beta,\gamma} = \mathfrak{b}_{\alpha,\beta}\mathfrak{b}^{-1}_{\alpha,\beta+\gamma}\mathfrak{b}_{\alpha+\beta,\gamma}\mathfrak{b}^{-1}_{\beta,\gamma}.$$

Changing the choice of the maps $f_{\alpha,\beta}$ therefore changes the class $(\mathfrak{a}_{\alpha,\beta,\gamma})$ by a coboundary, from which it follows that $\mathfrak{a}$ determines a well-defined element of $H^3(RO(G);A^*)$.

Conversely, given any normalized 2-cochain $(\mathfrak{b}_{\alpha,\beta})$ for $RO(G)$ with coefficients in A^*, the rule $f'_{\alpha,\beta} = \mathfrak{b}_{\alpha,\beta} \cdot f_{\alpha,\beta}$ gives a collection of isomorphisms $f'_{\alpha,\beta}\colon S^{\alpha+\beta} \to S^\alpha \wedge S^\beta$ with $f'_{0,\tau}$ and $f'_{\tau,0}$ the inverses of the unit isomorphisms, and with associated cocycle $\mathfrak{a}' = d\mathfrak{b} \cdot \mathfrak{a}$. An easy computation indicates that changing the models S^τ does not alter the cohomology class represented by the collection $(\mathfrak{a}_{\alpha,\beta,\gamma})$. These observations are summarized in the following result:

Proposition A.9. *There exists a cohomology class* $\bar{\mathfrak{a}}$ *in* $H^3(RO(G);A^*)$ *such that, for any fixed choice of models* S^τ*, the association (A.6) defines a surjection from the set of collections of isomorphisms* $f_{\alpha,\beta}\colon S^{\alpha+\beta} \to S^\alpha \wedge S^\beta$ *(with* $f_{0,\tau}$ *and* $f_{\tau,0}$ *the inverses of the unit isomorphisms) onto the set of normalized* 3*-cocycles in the cohomology class* $\bar{\mathfrak{a}}$*.*

This result indicates, if the cohomology class $\bar{\mathfrak{a}}$ vanishes, then any collection of isomorphisms $f'_{\alpha,\beta}$ can be adjusted via some normalized 2-cochain into a collection $f_{\alpha,\beta}$ making diagram (A.5) commute. This reduction of the proof of Theorem A.1 to a question in cohomology opens the way for us to restrict our attention to actual, rather than virtual, real representations. Let $RO^+(G)$ be the commutative monoid of isomorphism classes of actual real representations of G. It follows from [13, 4.1] that the inclusion of $RO^+(G)$ into $RO(G)$ induces a homotopy equivalence from $B(RO^+(G))$ to $B(RO(G))$. The associated restriction map

$$H^3(RO(G);A^*) \longrightarrow H^3(RO^+(G);A^*)$$

is therefore an isomorphism. Thus, it suffices to show that the image $\bar{\mathfrak{a}}^+$ of the cohomology class $\bar{\mathfrak{a}}$ of Proposition A.9 is trivial in $H^3(RO^+(G);A^*)$.

The process just described for associating a normalized 3-cocycle in the cohomology of $RO(G)$ to any collection of maps $f_{\alpha,\beta}$ indexed on $\alpha,\beta \in RO(G)$ works equally well to associate a normalized 3-cocycle in the cohomology $RO^+(G)$ to any collection of maps $f_{\alpha,\beta}$ indexed on $\alpha,\beta \in RO^+(G)$. Moreover the resulting class in the cohomology of $RO^+(G)$ is clearly the restriction of the class in the cohomology of $RO(G)$. This allows us to turn the whole argument backwards: we prove that the class $\bar{\mathfrak{a}}^+$ is trivial by showing that we can choose model spheres S^τ for $\tau \in RO^+(G)$ and isomorphisms $f_{\alpha,\beta}$ for $\alpha, \beta \in RO^+(G)$ such that the diagrams (A.5) commute.

Let ρ_1, ρ_2, ..., ρ_r be an enumeration of the irreducible real representations of G. For each ρ_i, select an associated sphere S^{ρ_i}. Any nonzero $\tau \in RO^+(G)$ can be written as a sum

$$\tau = n_1\rho_1 + \cdots + n_r\rho_r$$

in which at least one of the n_i is nonzero. Let S^τ be $S \wedge (S^{\rho_1})^{(n_1)} \wedge \cdots \wedge (S^{\rho_r})^{(n_r)}$. Here, $(S^{\rho_i})^{(n_i)}$ denotes the n_i-fold smash product of copies of S^{ρ_i}. For each pair α, β in $RO^+(G)$, select the map $\bar{f}_{\alpha,\beta}\colon S^{\alpha+\beta} \to S^\alpha \wedge S^\beta$ to be that which uses the associativity and commutativity isomorphisms of the equivariant stable category to rearrange the spheres $(S^{\rho_i})^{(n_i)}$ appearing in $S^{\alpha+\beta}$ into the proper order for $S^\alpha \wedge S^\beta$ with as few transpositions as possible. Clearly this yields a collection of maps which are appropriately unital and associative. The associated cocycle is then identically 1, and the cohomology class $\bar{\mathfrak{a}}^+$ is therefore trivial. We have therefore proven the following result.

Proposition A.10. *The cohomology class $\bar{\mathfrak{a}}$ of Proposition A.9 is trivial. Thus, we can choose the isomorphisms $f_{\alpha,\beta}$ for $\alpha, \beta \in RO(G)$ so that ϕ satisfies (A.2) and (A.3).*

We still need to show that the natural transformation ϕ is symmetric, that is, that it satisfies (A.4). Let $u(\alpha, \beta)$ be the unique unit in the Burnside ring such that the composite map

$$S^{\alpha+\beta} \xrightarrow{f_{\alpha,\beta}} S^\alpha \wedge S^\beta \cong S^\beta \wedge S^\alpha \xrightarrow{f^{-1}_{\beta,\alpha}} S^{\alpha+\beta}$$

is $u(\alpha,\beta) \cdot \mathrm{id}_{S^{\alpha+\beta}}$. By examining the definition of ϕ and the symmetry isomorphism for the box product, it is easy to see that ϕ is symmetric if and only if $u(\alpha,\beta) = \sigma(\alpha,\beta)$. Clearly u is antisymmetric; that is, $u(\alpha,\beta)u(\beta,\alpha) = 1$. The commutativity of the diagram (A.5) implies that u is bilinear; that is, that $u(\alpha+\beta,\gamma) = u(\alpha,\gamma)u(\beta,\gamma)$. Thus, to see that our choice of signs σ in Section 2 is consistent with our choice of the maps $f_{\alpha,\beta}$, it suffices to check that $u(\rho,\rho) = \sigma(\rho,\rho)$ for each irreducible representation ρ and that $u(\rho,\rho') = 1$ whenever ρ and ρ' are distinct irreducible representations. This follows from the discussion above and the observation that the twist map on $S^\rho \wedge S^\rho$ is homotopic to the map

$$-1 \wedge \mathrm{id} : S^\rho \wedge S^\rho \longrightarrow S^\rho \wedge S^\rho.$$

Remark A.11. We hinted in Section 2 that there is some flexibility in the units $u(\alpha,\beta)$ determined by the symmetry isomorphism $S^{\alpha+\beta} \to S^{\beta+\alpha}$ and therefore in the choice of $\sigma(\alpha,\beta)$. The analysis leading to Proposition A.9 shows that, for a fixed choice of model spheres S^τ, the set of collections of isomorphisms $f_{\alpha,\beta}$ (with $f_{0,\tau}$ and $f_{\tau,0}$ the inverses of the unit isomorphisms) is a torsor for the normalized 2-cochains of $RO(G)$ with coefficients in A^*. Thus, the set of collections that satisfy (A.5) is a torsor for the normalized 2-cocycles. If we adjust the maps $f_{\alpha,\beta}$ by a 2-cocycle $\mathfrak{b}$, then the element $u(\alpha,\beta)$ changes by a factor of $\mathfrak{b}^{-1}_{\beta,\alpha} \cdot \mathfrak{b}_{\alpha,\beta}$. It follows that, for any collection of isomorphisms satisfying (A.5), $u(\rho,\rho)$ must be the unit described above. However, we can find a collection that takes on any desired set of values for $u(\rho, \rho')$ for distinct irreducibles ρ, ρ', subject only to the restriction $u(\rho, \rho')u(\rho', \rho) = 1$.

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