WHEN PROJECTIVE DOES NOT IMPLY FLAT, AND OTHER HOMOLOGICAL ANOMALIES

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ABSTRACT. The category $\mathfrak{M}_G$ of Mackey functors for a group G carries a symmetric monoidal closed structure. The $\Box$-product providing this structure encodes the Frobenius axiom, which describes the interaction of induction and multiplication in Mackey functor rings. Mackey functors are of interest in equivariant homotopy theory since good equivariant cohomology theories are Mackey functor valued. In this context, the $\Box$-product is useful not only because it encodes the interaction between induction and the cup product, but also because of the role it plays in the not yet fully understood universal coefficient and Künneth formulae. This role makes it important to know whether projective objects in $\mathfrak{M}_G$ are flat, and whether the $\Box$-product of projective objects in $\mathfrak{M}_G$ is projective. In the most familiar symmetric monoidal abelian categories, the tensor product obviously interacts appropriately with projective objects. However, the $\Box$-product for $\mathfrak{M}_G$ need not be so well behaved. For example, if G is $O(n)$, projectives need not be flat in $\mathfrak{M}_G$ and the $\Box$-product of projective objects need not be projective. This misbehavior complicates the search for full strength equivariant universal coefficient and Künneth formulae.

These questions about the interaction of the tensor product with projective objects can be regarded as compatibility conditions which may be satisfied by a symmetric monoidal closed category $\mathfrak{M}$. The primary purpose of this article is to investigate these, and related, conditions. Our focus is on functor categories whose monoidal structures arise in a fashion described by Day. Conditions are given under which such a structure interacts appropriately with projective objects. Further, examples are given to show that, when these conditions aren't met, this interaction can be quite bad. These examples were not fabricated to illustrate the abstract possibility of misbehavior. Rather, they are drawn from the literature. In particular, $\mathfrak{M}_G$ is badly behaved not only for the groups $O(n)$, but also for the groups $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$, and $Spin(n)$. Similar misbehavior occurs in two categories of global Mackey functors which are widely used in the study of classifying spaces of finite groups.

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Contents

Introduction

Let $\mathfrak{M}$ be a symmetric monoidal closed abelian category. The tensor product of two objects M and N in $\mathfrak{M}$ is denoted $M \,\Box\, N$. The closed structure of $\mathfrak{M}$ is provided by an internal hom functor adjoint to the tensor product; the result of applying this functor to M and N is an object of $\mathfrak{M}$ denoted $\langle M, N\rangle$. In contrast, the abelian group of morphisms from M to N in $\mathfrak{M}$ is denoted $\mathfrak{M}(M, N)$.

It is often important to know how well the symmetric monoidal closed and abelian category structures on $\mathfrak{M}$ interact. Two obvious questions about this interaction are whether projective objects are flat and whether the tensor product of two projectives is projective. In the classical examples, these questions are answered by the dicta "projective implies flat" and "the tensor product of projectives is projective". Unfortunately, these familiar dicta fail wildly in some less well known examples of symmetric monoidal closed abelian categories.

One of the primary purposes of this article is to describe the homological misbehavior of these examples, which arise in the study of Mackey functors. This context yields three families of symmetric monoidal closed abelian categories in which projectivity does not imply flatness. In one of these families, the tensor product of two projectives need not be projective. These examples were not contrived merely to demostrate the theoretical possibility of this sort of misbehavior. One of these families has appeared repeatedly in the literature in both representation theory and stable homotopy theory [1, 2, 4, 5, 7, 9, 10, 12–14, 24–26, 28, 29]. A second family, the one in which both homological anomalies occur, provides one of the very few reasonable definitions of a Mackey functor for a compact Lie group [21]. It is unlikely that the other definitions are better behaved in this regard. The misbehavior of projectivity with respect to tensor products is especially serious in this category because of the important role these two notions play in universal coefficient and Künneth theorems. The third family of categories arises naturally in the study of the equivariant Hurewicz and suspension homomorphisms in equivariant homotopy theory [18, 19].

Our two questions about the behavior of projective objects in symmetric monoidal closed abelian categories fit naturally into a larger group of six interrelated questions about the interaction of the monoidal closed and abelian structures on $\mathfrak{M}$. The other four questions in this group are less frequently discussed. Like our question about the relation between projectivity and flatness, each of these four deals with a pair of functors from the list

$$\langle M, ?\rangle : \mathfrak{M} \longrightarrow \mathfrak{M} \qquad \langle ?, M\rangle : \mathfrak{M}^{\mathrm{op}} \longrightarrow \mathfrak{M}$$
$$\mathfrak{M}(M, ?) : \mathfrak{M} \longrightarrow Ab \qquad \mathfrak{M}(?, M) : \mathfrak{M}^{\mathrm{op}} \longrightarrow Ab$$
$$M\Box ? : \mathfrak{M} \longrightarrow \mathfrak{M},$$

and asks whether the exactness of one of the functors in the pair implies the exactness of the other.

Some sense of the importance of one of the four is easily conveyed. This question asks whether the exactness of the functor $\langle M, ?\rangle$ implies the exactness of the functor $\mathfrak{M}(M, ?)$. Assume that $\mathcal{R}$ is a commutative ring, G is a finite group, and $\mathcal{R}[G]$ is the group ring of G over $\mathcal{R}$. The categories of $\mathcal{R}$-modules and of $\mathcal{R}[G]$-modules are the classic examples of symmetric monoidal closed abelian categories. In the category of $\mathcal{R}$-modules, the exactness of $\langle M, ?\rangle$ implies the exactness of $\mathfrak{M}(M, ?)$ because the two functors differ only in that $\mathfrak{M}(M, N)$ is the abelian group underlying the

$\mathcal{R}$-module $\langle M, N\rangle$. However, if M is taken to be $\mathcal{R}$ with trivial G-action in the category of $\mathcal{R}[G]$-modules, then the functor $\langle M, ?\rangle$ is exact for formal reasons, whereas the failure of the exactness of $\mathfrak{M}(M, ?)$ is the source for group cohomology. Thus, the relation between the exactness of $\langle M, ?\rangle$ and $\mathfrak{M}(M, ?)$ is not obvious and can have far-reaching consequences. A second purpose of this article is to call attention to these four infrequently discussed questions, and to describe their relation to the two more familiar questions in our group of six.

All of the symmetric monoidal closed abelian categories considered here are functor categories, and their monoidal structures are defined using a general procedure introduced by Day [6]. Roughly speaking, Day provides a symmetric monoidal closed structure on a functor category whose domain category $\mathcal{O}$ is a full subcategory of a symmetric monoidal category $\mathcal{S}$. The behavior of the projective and injective objects in these functor categories can be related to various properties of $\mathcal{O}$. In particular, there are obvious closure properties on the subcategory $\mathcal{O}$ which imply that the associated functor category is homologically well-behaved. A third purpose of this article is to describe these sufficient conditions for the good behavior of functor categories.

When the domain category $\mathcal{O}$ does not satisfy these simple closure conditions, the interaction between the monoidal closed and abelian structures on the associated functor category can be extremely bad, as our examples indicate. These badly behaved examples appear so ubiquitously in the context of Mackey functors that it is only prudent to assume that the interaction is bad in any case where our closure conditions fail. One is almost tempted to conjecture that our sufficient conditions are also necessary. The behavior of functor categories is not, however, this predictable. We give one example in which the closure conditions fail, but the functor category is perfectly well behaved.

The tensor product and internal hom functors which provide Day's symmetric monoidal closed structures are, to say the least, usually not easily computed. The final purpose of this article is to describe techniques which can be used to manipulate these functors.

In the first section, we give the precise statements of our six homological questions, and make a few observations about relations among them. The next section contains a review of Day's monoidal closed structures, and an introduction to the relevant examples of categories carrying these structures. The focus of the third section is on our positive results guaranteeing that a functor category carrying one of Day's monoidal structures satisfies our various compatibility conditions. The fourth and fifth sections contain more detailed descriptions of our families of examples. Section six is devoted to the statements of our negative results indicating how badly behaved the tensor product can be in a functor category to which the results of section three are inapplicable. The remaining sections provide the proofs of various results stated in sections three and six. In particular, our negative results about the category of Mackey functors associated to an incomplete indexing universe are proven in section seven. Sections eight and nine contain the proofs of our results about the homological misbehavior of the category of Mackey functors for a compact Lie group. Section ten provides the proofs of our positive results on the category of S^1-Mackey functors. Our results about the homological misbehavior of globally defined Mackey functors are proven in the last section.

1. Compatibility questions for symmetric monoidal closed abelian categories

Here, the compatibility questions of interest to us are phrased in terms of six axioms which a symmetric monoidal closed abelian category $\mathfrak{M}$ may satisfy. Certain fundamental connections between these axioms are also explored. Four of these six axioms are clearly satisfied by both the category $\mathcal{R}$-Mod of modules over a commutative ring $\mathcal{R}$ and the category $\mathcal{R}[G]$-Mod of modules over the group ring $\mathcal{R}[G]$ of a finite group G over the ring $\mathcal{R}$. The other two axioms are obviously satisfied by $\mathcal{R}$-Mod, but are typically not satisfied by $\mathcal{R}[G]$-Mod. The nature, and significance, of the misbehavior of $\mathcal{R}[G]$-Mod with respect to these two axioms is therefore discussed here.

Definition 1.1. **(a)** Since an object M of $\mathfrak{M}$ is said to be projective if the functor $\mathfrak{M}(M, ?) : \mathfrak{M} \longrightarrow Ab$ is exact, an object M is said to be internally projective if the functor $\langle M, ?\rangle : \mathfrak{M} \longrightarrow \mathfrak{M}$ is exact.

(b) Since M is injective if the functor $\mathfrak{M}(?, M) : \mathfrak{M}^{\mathrm{op}} \longrightarrow Ab$ is exact, M is said to be internally injective if the functor $\langle ?, M\rangle : \mathfrak{M}^{\mathrm{op}} \longrightarrow \mathfrak{M}$ is exact.

Our 6 axioms which may be satisfied by a symmetric monoidal closed abelian category $\mathfrak{M}$ are:

PiF If an object M of $\mathfrak{M}$ is projective, then it is also flat.

PiIP If an object M of $\mathfrak{M}$ is projective, then it is internally projective.

IiII If an object M of $\mathfrak{M}$ is injective, then it is internally injective.

IPiP If an object M of $\mathfrak{M}$ is internally projective, then it is projective.

IIiI If an object M of $\mathfrak{M}$ is internally injective, then it is injective.

TPPP If the objects M and N of $\mathfrak{M}$ are projective, then so is $M \,\square\, N$.

In the introduction, it was noted that **IPiP** holds in the category $\mathcal{R}$-Mod of modules over a commutative ring $\mathcal{R}$, but fails in the category $\mathcal{R}[G]$-Mod of modules over the group ring $\mathcal{R}[G]$ of a finite group G over $\mathcal{R}$. The argument for the failure of **IPiP** in $\mathcal{R}[G]$-Mod made use of $\mathcal{R}$ considered as an $\mathcal{R}[G]$-module with trivial G-action. The significance of the $\mathcal{R}[G]$-module $\mathcal{R}$ in this context is that it is the unit for the tensor product on $\mathcal{R}[G]$-Mod. The question of whether $\mathfrak{M}$ satisfies the axioms **IPiP** and **IIiI** is closely tied to the behavior of the unit for the tensor product on $\mathfrak{M}$, which we denote by $\bullet$ (or $\bullet_{\mathfrak{M}}$). In particular, the canonical unit isomorphism $\langle \bullet\, , M\rangle \cong M$ of $\mathfrak{M}$ implies that $\bullet$ is internally projective. Thus, **IPiP** can't be satisfied unless $\bullet$ is projective.

Lemma 1.2. **(a)** *The category $\mathfrak{M}$ satisfies* **IPiP** *if and only if the unit $\bullet$ for the tensor product on $\mathfrak{M}$ is projective.*

(b) *The category $\mathfrak{M}$ satisfies* **IIiI** *if the unit $\bullet$ for the tensor product on $\mathfrak{M}$ is projective.*

Proof. We have already noted that $\bullet$ must be projective if $\mathfrak{M}$ satisfies **IPiP**. The other half of part (a) and part (b) both follow easily from the natural isomorphism $\mathfrak{M}(\bullet\,, \langle M, N\rangle) \cong \mathfrak{M}(M, N)$. □

The question of whether the unit $\bullet$ for $\mathfrak{M}$ is projective seems to have an significance far beyond this lemma. There is a theory of localization for well-behaved symmetric monoidal closed abelian categories which generalizes the classical theory of localization for a commutative ring and its modules. This theory can be developed using techniques drawn from the theory of localization for noncommutative rings [27]. The various localizations of a category $\mathfrak{M}$ provided by this theory are all symmetric monoidal closed abelian categories. In the most favorable cases, such a localization of $\mathfrak{M}$ can can be identified as the category of modules over the localization of the unit $\bullet$ of $\mathfrak{M}$. However, as one might expect from the theory of localization of noncommutative rings, the typical localization of $\mathfrak{M}$ is a more subtle construction. The simplest indicator of whether the localized category is of this more subtle kind is that the more subtle construction occurs precisely when the unit of the localized category fails to be projective in that category.

The hom/tensor adjunction which provides the closed structure on $\mathfrak{M}$ implies that the remaining 4 axioms fall into two pairs of roughly equivalent conditions. In categories of functors into Ab, which always have enough projectives and injectives, the conditions in each pair are, in fact, equivalent.

Proposition 1.3. (a) *If $\mathfrak{M}$ satisfies* **PiIP***, then it also satisfies* **TPPP***. Moreover, if $\mathfrak{M}$ has enough projectives, then* **TPPP** *implies* **PiIP***.*

(b) *If $\mathfrak{M}$ has enough projectives, then* **PiF** *implies* **IiII***. If $\mathfrak{M}$ has enough injectives, then* **IiII** *implies* **PiF***.*

Proof. Let $f : L \longrightarrow M$ and $g : M \longrightarrow N$ be a monomorphism and an epimorphism in $\mathfrak{M}$, respectively. Both implications in part (a) follow from the equivalence of the lifting problems represented by the two diagrams

$$\begin{tikzcd} & P \,\Box\, Q \arrow[dl, dashed, "H"'] \arrow[d, "h"] \\ M \arrow[r, "g"'] & N \end{tikzcd} \qquad \begin{tikzcd} & P \arrow[dl, dashed, "\tilde{H}"'] \arrow[d, "\tilde{h}"] \\ \langle Q, M\rangle \arrow[r, "g_*"'] & \langle Q, N\rangle, \end{tikzcd}$$

in which P and Q are projective objects in $\mathfrak{M}$. Both implications in part (b) follow from the equivalence of the lifting problems represented by the two diagrams

$$\begin{tikzcd} I & \\ P \,\Box\, L \arrow[u, "k"] \arrow[r, "1\Box f"'] & P \,\Box\, M \arrow[ul, dashed, "K"'] \end{tikzcd} \qquad \begin{tikzcd} & P \arrow[dl, dashed, "\tilde{K}"'] \arrow[d, "\tilde{k}"] \\ \langle M, I\rangle \arrow[r, "f^*"'] & \langle L, I\rangle, \end{tikzcd}$$

in which P and I are projective and injective in $\mathfrak{M}$, respectively. □

2. Day's symmetric monoidal closed structures

The categories of greatest interest in this article are functor categories with symmetric monoidal closed structures of the sort described by Day in [6]. However, only a rather simple case of Day's general approach is needed for our examples. In this section, that case is reviewed, and our basic examples of functor categories carrying Day's structures are introduced.

Let $\mathcal{S}$ be a symmetric monoidal Ab-category whose tensor product $\wedge$ is bilinear, and $\mathcal{O}$ be a skeletally small full subcategory of $\mathcal{S}$. The relevant case of Day's approach applies to any reasonably well-behaved subcategory $\mathcal{O}$, and provides a symmetric monoidal closed structure on the category $\mathfrak{M}_{\mathcal{O}}$ of additive functors from $\mathcal{O}$ into Ab. In fact, Day's machinery applies equally well to functors into the category $\mathcal{R}$-Mod of modules over a commutative ring $\mathcal{R}$. However, restricting attention to functors into Ab somewhat simplifies our notation.

Our entire discussion of $\mathfrak{M}_{\mathcal{O}}$ is plagued by an annoying problem with the variance of our functors. The general observations in the next section about the relation between Day's structures and our axioms are most easily presented in terms of the category of covariant functors out of $\mathcal{O}$. However, some of our examples are drawn from geometric sources, like cohomology theories, which naturally yield contravariant functors. Most of the time, this difference in the preferred variance is nothing more than a notational nuisance. The preference for covariant functors in the next section arises from Proposition 3.3, which necessarily applies only to covariant functors. All of the other results in that section apply equally well to covariant and contravariant functors since they depend only on self-dual properties of $\mathcal{S}$ and $\mathcal{O}$. In many of our examples, the question of a preferred variance is moot, either because $\mathcal{O}$ and $\mathcal{O}^{\mathrm{op}}$ are isomorphic or because functors of both variances are of interest. Nevertheless, in a few key cases, variance is very significant. The preferred variance in these cases is perversely split about evenly. Since there is no clearly preferred variance and the choice of variance is often irrelevant, $\mathfrak{M}_{\mathcal{O}}$ is a deliberately ambiguous symbol denoting either of the two categories of additive functors from $\mathcal{O}$ to Ab. In the sections where functors of both variance are considered, this notation is used only in remarks applicable to both categories. When variance really matters, the symbols $\mathfrak{M}_{\mathcal{O}}^{\mathrm{cov}}$ and $\mathfrak{M}_{\mathcal{O}}^{\mathrm{cont}}$ are used to denote the categories of covariant and contravariant additive functors from $\mathcal{O}$ into Ab, respectively.

The case of Day's approach considered here is quite similar to that described in section 4 of [21]. However, two differences are worth noting. In [21], it is assumed that $\mathcal{O}$ contains the unit for the tensor product on $\mathcal{S}$. Here, that constraint is replaced by a weaker condition on $\mathcal{O}$ which fits more naturally into our discussion of the connection between the properties of $\mathcal{O}$ and our axioms. The second difference is that all the functors in [21] are contravariant. Since the contravariant case is discussed in detail in [21], the introductory discussion here is focused on the covariant case. Readers interested in the contravariant case may either make the necessary notational adjustments for themselves or look them up in [21].

The definition below provides the basic components of Day's symmetric monoidal closed structure on the category $\mathfrak{M}_{\mathcal{O}}^{\mathrm{cov}}$. In this definition, and throughout the rest of the article, objects of $\mathcal{O}$ are denoted by A, B, C, etc., and the objects of $\mathcal{S}$ that need not be in $\mathcal{O}$ are denoted by Z, Y, X, etc. The unit for the tensor product of $\mathcal{S}$ is denoted $\bullet$ (or $\bullet_{\mathcal{S}}$).

Definition 2.1. **(a)** For X in $\mathcal{S}$, let $\mathcal{H}_X^{\mathrm{cov}} : \mathcal{O} \longrightarrow Ab$ and $\mathcal{H}_X^{\mathrm{cont}} : \mathcal{O}^{\mathrm{op}} \longrightarrow Ab$ be the functors given by $\mathcal{H}_X^{\mathrm{cov}}(A) = \mathcal{S}(X, A)$ and $\mathcal{H}_X^{\mathrm{cont}}(A) = \mathcal{S}(A, X)$. Note that $\mathcal{H}_X^{\mathrm{cov}}$ is in $\mathfrak{M}_{\mathcal{O}}^{\mathrm{cov}}$ and $\mathcal{H}_X^{\mathrm{cont}}$ is in $\mathfrak{M}_{\mathcal{O}}^{\mathrm{cont}}$. In remarks applicable to the functor category of either variance and in contexts where the desired variance should be obvious, the notation $\mathcal{H}_X$ is used for the functor of the appropriate variance associated to X.

(b) If M and N are objects of $\mathfrak{M}_{\mathcal{O}}^{\mathrm{cov}}$, then the functor $M \,\Box\, N$ in $\mathfrak{M}_{\mathcal{O}}^{\mathrm{cov}}$ is given on an object A of $\mathcal{O}$ by

$$(M \,\Box\, N)(A) = \int^{B,C \in \mathcal{O}} M(B) \otimes N(C) \otimes \mathcal{S}(B \wedge C, A).$$

The naturality of coends provides the definition of $M \,\Box\, N$ on the morphisms of $\mathcal{O}$.

(c) If M and N are objects of $\mathfrak{M}_{\mathcal{O}}^{\mathrm{cov}}$, then the functor $\langle M, N\rangle$ in $\mathfrak{M}_{\mathcal{O}}^{\mathrm{cov}}$ is given on an object A of $\mathcal{O}$ by

$$\langle M, N\rangle(A) = \int_{B,C \in \mathcal{O}} \hom\big(M(B) \otimes \mathcal{S}(A \wedge B, C), N(C)\big).$$

The naturality of ends provides the definition of $\langle M, N\rangle$ on the morphisms of $\mathcal{O}$.

(d) Let A, C be in $\mathcal{O}$ and X, Y be in $\mathcal{S}$. Then the evaluation map

$$\mathcal{S}(X, C) \otimes \mathcal{S}(Y \wedge C, A) \longrightarrow \mathcal{S}(Y \wedge X, A)$$

induces a homomorphism

$$\rho^{\mathrm{cov}}(X, Y; A) : \int^{C} \mathcal{S}(X, C) \otimes \mathcal{S}(Y \wedge C, A) \longrightarrow \mathcal{S}(Y \wedge X, A)$$

of abelian groups. Note that, by Lemma 4.3(b) of [21], this map is an isomorphism if X is in $\mathcal{O}$. There is an an analogous map

$$\rho^{\mathrm{cont}}(X, Y; A) : \int^{C} \mathcal{S}(C, X) \otimes \mathcal{S}(A, Y \wedge C) \longrightarrow \mathcal{S}(A, Y \wedge X)$$

which must be used instead of $\rho^{\mathrm{cov}}(X, Y; A)$ in the context of categories of contravariant functors. This map is denoted θ in [21] and is discussed there in the proof of Proposition 5.2. In remarks applicable to functors of either variance, $\rho(X, Y; A)$ is used to denote the appropriate one of these two maps.

It is easy to see that the $\Box$ operation is symmetric, and a simple end calculation gives the desired adjunction relating $\Box$ and $\langle\ ,\ \rangle$. Thus, to show that these constructions provide $\mathfrak{M}_{\mathcal{O}}$ with a symmetric monoidal closed structure, it suffices to exhibit a unit for $\Box$, construct an associativity isomorphism, and prove that the appropriate diagrams commute. The unit for $\mathfrak{M}_{\mathcal{O}}$ should be $\mathcal{H}_{\bullet_{\mathcal{S}}}$. The associativity isomorphism for $\mathfrak{M}_{\mathcal{O}}$ should be easily derived from the associativity isomorphism for $\mathcal{S}$, and the commutativity of the required diagrams for $\mathfrak{M}_{\mathcal{O}}$ should follow from the commutativity of the analogous diagrams in $\mathcal{S}$. Nevertheless, these last three pieces of the monoidal structure need not fit properly into place unless the map $\rho(X, Y; A)$ is an isomorphism under the appropriate conditions.

Theorem 2.2. *Let $\mathcal{S}$ be an Ab-category with a symmetric monoidal structure derived from a bilinear tensor product $\wedge$, and let $\mathcal{O}$ be a skeletally small full subcategory of $\mathcal{S}$. If the map $\rho(X, Y; A)$ is an isomorphism whenever $X = \bullet_{\mathcal{S}}$ and $Y \in \mathcal{O}$ and whenever X and Y are both finite $\wedge$-products of objects in $\mathcal{O}$, then $\mathfrak{M}_{\mathcal{O}}$ is a symmetric monoidal closed category.*

Proof. We prove this result for $\mathfrak{M}^{\mathrm{cov}}_{\mathcal{O}}$; the proof for $\mathfrak{M}^{\mathrm{cont}}_{\mathcal{O}}$ is analogous. In this and several other proofs, we make use of a variety of folklore results about ends and coends. The contravariant analogs of these results are discussed in section 4 of [21]. The proofs of these results are formal, and obviously translate to the covariant context. Note, however, that the results in section 5 of [21] typically apply only in the contravariant case since their proofs make use of more geometric arguments. Let M, N, and P be in $\mathfrak{M}^{\mathrm{cov}}_{\mathcal{O}}$. The unit isomorphism $M \,\Box\, \mathcal{H}^{\mathrm{cov}}_{\bullet_{\mathcal{S}}} \cong M$ of $\mathfrak{M}^{\mathrm{cov}}_{\mathcal{O}}$ is given at $A \in \mathcal{O}$ by the composite

$$\begin{aligned}
(M \,\Box\, \mathcal{H}_{\bullet_{\mathcal{S}}})(A) \quad &= \quad \int^{B,C\in\mathcal{O}} M(B) \otimes \mathcal{S}(\bullet_{\mathcal{S}}, C) \otimes \mathcal{S}(B \wedge C, A) \\
&\xrightarrow{\int^B \rho} \quad \int^{B\in\mathcal{O}} M(B) \otimes \mathcal{S}(B \wedge \bullet_{\mathcal{S}}, A) \\
&\cong \quad \int^{B\in\mathcal{O}} M(B) \otimes \mathcal{S}(B, A) \\
&\cong \quad M(A),
\end{aligned}$$

in which the last isomorphism is given by Lemma 4.3(c) of [21]. The associativity isomorphism for $\mathfrak{M}^{\mathrm{cov}}_{\mathcal{O}}$ is obtained by using the maps $\rho^{\mathrm{cov}}(X, Y; A)$ to identify both $((M \,\Box\, N) \,\Box\, P)(A)$ and $(M \,\Box\, (N \,\Box\, P))(A)$ with

$$\int^{B,C,D\in\mathcal{O}} M(B) \otimes N(C) \otimes P(D) \otimes \mathcal{S}(B \wedge C \wedge D, A)$$

for each $A \in \mathcal{O}$. The maps $\rho^{\mathrm{cov}}(X, Y; A)$ are used in much the same fashion to reduce the question of the commutativity of each of the necessary diagrams in $\mathfrak{M}^{\mathrm{cov}}_{\mathcal{O}}$ to the commutativity of a corresponding diagram in $\mathcal{S}$. □

Scholium 2.3. The requirement in this theorem that $\rho(X, Y; A)$ be an isomorphism whenever both X and Y are finite $\wedge$-products of objects in $\mathcal{O}$ was unfortunately overlooked in [21]. However, the map θ introduced in the proof of Proposition 5.2 of [21] is our map $\rho^{\mathrm{cont}}(X, Y; A)$. In the proof of that proposition, θ is shown to be an isomorphism under far broader conditions than those needed to ensure that the categories considered in [21] are symmetric monoidal.

Remark 2.4. (a) If $\bullet_{\mathcal{S}}$ is in $\mathcal{O}$, then the map $\rho(\bullet_{\mathcal{S}}, Y; A)$ is an isomorphism by Lemma 4.3(b) of [21]. Thus, the restriction imposed on $\bullet_{\mathcal{S}}$ in the theorem above is weaker than that imposed in [21].

(b) If $\mathcal{O}$ contains $\bullet_{\mathcal{S}}$ and is closed under finite $\wedge$-products, so that it is a symmetric monoidal subcategory of $\mathcal{S}$, then both conditions on the maps $\rho(X, Y; A)$ are satisfied, again by Lemma 4.3(b) of [21].

We conclude this section with a list of examples of pairs $(\mathcal{S}, \mathcal{O})$ satisfying the hypotheses of Theorem 2.2. These examples serve a three-fold purpose. The next section is devoted to positive results giving conditions on a pair $(\mathcal{S}, \mathcal{O})$ which ensure that the associated functor category $\mathfrak{M}_{\mathcal{O}}$ satisfies one or more of our axioms. The power of those positive results is illustrated by the fact that they apply to several of the most important special cases of the examples below, and thus assure us that the associated functor categories are well-behaved. The limitation of our positive results is that they give sufficient, but not necessary, conditions for the good behavior of $\mathfrak{M}_{\mathcal{O}}$. Some special cases of the examples below are used to provide a measure of this lack of necessity. Section 6, and most of the sections following it, describe functor categories which fail to satisfy our compatibility axioms. These badly behaved categories are also special cases of the examples below.

Example 2.5. **(a)** Let $\mathcal{R}$ be a commutative ring, $\mathcal{S}$ be the category $\mathcal{R}$-Mod of $\mathcal{R}$-modules, and $\mathcal{O}$ be the full subcategory of $\mathcal{S}$ containing $\mathcal{R}$ as its only object. In this case, $\mathcal{S}$ is a symmetric monoidal closed category, and $\mathcal{O}$ is a symmetric monoidal closed subcategory. Of course, $\mathfrak{M}_{\mathcal{O}}$ is just $\mathcal{R}$-Mod, and Day's symmetric monoidal closed structure is identical to the standard one. In some sense, this example is frivolous in that we have applied a vast machine to $\mathcal{R}$-Mod with its symmetric monoidal closed structure only to recover that category with the same structure. However, in our discussion of the connections between the structure of $\mathcal{O}$ and our compatibility axioms, this example nicely illustrates the benefits of a very well-behaved category $\mathcal{O}$.

(b) Let $\mathcal{R}$ be a commutative ring, G be a finite group, and $\mathcal{S}$ be the category $\mathcal{R}[G]$-Mod of $\mathcal{R}[G]$-modules. In this context, there are two reasonable choices for $\mathcal{O}$. The smallest, which we denote $\mathcal{O}_1$, is the full subcategory of $\mathcal{S}$ containing $\mathcal{R}[G]$ as its only object. The other category, $\mathcal{O}_2$, is the full subcategory of $\mathcal{S}$ containing the n-fold direct sum of copies of $\mathcal{R}[G]$ for all $n \geq 1$. Note that neither of these subcategories contains the unit for the tensor product on $\mathcal{S}$, which is $\mathcal{R}$ with trivial G-action. The advantage of $\mathcal{O}_2$ over $\mathcal{O}_1$ is that $\mathcal{O}_2$ is closed under both the tensor product and the internal hom operations on $\mathcal{S}$. Of course, $\mathfrak{M}_{\mathcal{O}_1}$ and $\mathfrak{M}_{\mathcal{O}_2}$ are just $\mathcal{R}[G]$-Mod with its usual symmetric monoidal closed structure, so this example is just as frivolous as the previous one. However, it too serves to illustrate the relation between the structure of $\mathcal{O}$ and the compatibility axioms satisfied by $\mathfrak{M}_{\mathcal{O}}$.

(c) Let G be a compact Lie group, U be a G-universe, and $\mathcal{S}$ be the equivariant stable category $\bar{h}G\mathcal{SU}$ of G-spectra indexed on U (see chapter I of [23]). The obvious choice for the associated category $\mathcal{O}$ is the stable orbit category $\mathcal{O}_G(U)$. This is the full subcategory whose objects are suspension spectra $\Sigma_U^\infty G/H_+$ associated to the orbits G/H derived from the closed subgroups H of G. The category $\mathcal{O}_G(U)$ contains the unit for $\mathcal{S}$. However, if G is nontrivial, $\mathcal{O}_G(U)$ is closed under neither the tensor product nor the internal hom on $\mathcal{S}$. A contravariant functor out of $\mathcal{O}_G(U)$ is called a (G,U)-Mackey functor, and the category of such functors is denoted $\mathfrak{M}_G(U)$. If U is a complete G-universe, then $\mathcal{O}_G(U)$ and $\mathfrak{M}_G(U)$ are abbreviated to $\mathcal{O}_G$ and $\mathfrak{M}_G$, respectively. If G is finite and U is complete, then (G,U)-Mackey functors are the classical Mackey functors introduced by representation theorists for the study of induction theorems (see [8, 11, 16, 22] and Proposition V.9.9 of [23]). The case in which U is incomplete plays a role in the equivariant Hurewicz and suspension theorems [18, 19] and in the study of change of universe functors in equivariant stable homotopy theory [20]. The category $\mathfrak{M}_G(U)$ is discussed in greater detail in sections 4, 6, 7, 8, 9, and 10.

(d) If the group G in the previous example is finite, then the Burnside category $\mathcal{B}_G(U)$ is another choice for $\mathcal{O}$. This is the full subcategory of $\mathcal{S}$ containing the suspension spectra $\Sigma_U^\infty X_+$ of the finite G-sets X. Note that $\mathcal{O}_G(U)$ is a subcategory of $\mathcal{B}_G(U)$. The advantage of $\mathcal{B}_G(U)$ over $\mathcal{O}_G(U)$ is that it is closed under the tensor product operation on $\mathcal{S}$. If the universe U is complete, then $\mathcal{B}_G(U)$ is a symmetric monoidal closed subcategory of $\mathcal{S}$. Its closed structure is even nicer than that of the category of finite dimensional vector spaces over a field in that objects in $\mathcal{B}_G(U)$ are canonically self-dual. However, if U is incomplete, then $\mathcal{B}_G(U)$ is not closed under the internal hom operation on $\mathcal{S}$. As with $\mathcal{O}_G(U)$, one is usually interested in contravariant functors out of $\mathcal{B}_G(U)$. For either variance, the categories of functors out of $\mathcal{O}_G(U)$ and $\mathcal{B}_G(U)$ are equivalent via the restriction functor induced by the inclusion of $\mathcal{O}_G(U)$ into $\mathcal{B}_G(U)$. Thus, we abuse notation and employ $\mathfrak{M}_G(U)$ to denote the category of contravariant functors from $\mathcal{B}_G(U)$ to Ab.

(e) The objects of the global Burnside category $\mathcal{B}_*$ are the finite groups. If G and H are finite groups, then the set of morphisms from G to H in $\mathcal{B}_*$ is the Grothendieck group of isomorphism classes of finite $(G \times H)$-sets. The composition of a $(G \times H)$-set X, regarded as a morphism from G to H, with an $(H \times K)$-set Y, regarded as a morphism from H to a finite group K, is $(X \times Y)/H$, where the passage to orbits is over the diagonal action of H on $X \times Y$. The cartesian product of groups makes $\mathcal{B}_*$ into a symmetric monoidal Ab-category. There is an obvious duality functor $\mathfrak{D} : \mathcal{B}_* \longrightarrow \mathcal{B}_*^{\mathrm{op}}$ which is the identity on objects and which sends the $(G \times H)$-set X to itself regarded as an $(H \times G)$-set. This duality functor provides $\mathcal{B}_*$ with a closed structure like that for finite dimensional vector spaces; the internal hom object associated to groups G and H is $\mathfrak{D}(G) \times H$. Covariant additive functors from $\mathcal{B}_*$ to Ab are the most structured kind of globally defined Mackey functors. The category of such functors is denoted $\mathfrak{M}_*$.

Most globally defined Mackey functors carry a much less rich structure than that carried by the functors in $\mathfrak{M}_*$. These less structured Mackey functors are additive functors from some subcategory of $\mathcal{B}_*$ into Ab. The subcategories of $\mathcal{B}_*$ of interest to us here can be described in terms of pairs $(\mathfrak{P}, \mathfrak{Q})$ of sets of integer primes. A $(G \times H)$-set X is said to be a $(\mathfrak{P}, \mathfrak{Q})$-set if, for each $x \in X$, the G-isotropy subgroup G_x of x has order divisible only by the primes in $\mathfrak{P}$ and the H-isotropy subgroup H_x of x has order divisible only by the primes in $\mathfrak{Q}$. The subcategory $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ of $\mathcal{B}_*$ has the same objects as $\mathcal{B}_*$, but the set of morphisms from G to H in $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ is the Grothendieck group of isomorphism classes of finite $(G \times H)$-sets which are also $(\mathfrak{P}, \mathfrak{Q})$-sets. Observe that $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})^{\mathrm{op}}$ is just $\mathcal{B}_*(\mathfrak{Q}, \mathfrak{P})$. The category $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ inherits a symmetric monoidal structure from $\mathcal{B}_*$, but the internal hom on $\mathcal{B}_*$ does not restrict to give $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ a closed structure. In fact, $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ is typically not a closed category. Covariant additive functors from $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ to abelian groups are called global $(\mathfrak{P},\mathfrak{Q})$-Mackey functors. The category of such is denoted $\mathfrak{M}_*(\mathfrak{P}, \mathfrak{Q})$.

The category $\mathcal{B}_*$, and each of its subcategories $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$, can serve as both $\mathcal{S}$ and $\mathcal{O}$ in a pair $(\mathcal{S}, \mathcal{O})$ satisfying the hypotheses of Theorem 2.2. Note that this is our only example in which $\mathcal{S}$ need not be a closed category. The categories $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ appear, under various names and in various guises, in [1, 2, 4, 5, 7, 9, 10, 12–14, 24–26, 28, 29], and are discussed in greater detail in sections 5, 6 and 11.

Remark 2.6. The pairs $(\mathcal{S}, \mathcal{O})$ from Examples 2.5(a), 2.5(d), and 2.5(e) satisfy the hypotheses of Theorem 2.2 by Remark 2.4 since, in each of these cases, $\mathcal{O}$ is a symmetric monoidal subcategory of $\mathcal{S}$. Simple direct computations indicate that the pairs $(\mathcal{S}, \mathcal{O})$ from Example 2.5(b) satisfy the hypotheses of Theorem 2.2. If the group G is finite in Example 2.5(c), then the pairs introduced in that example must satisfy the hypotheses of Theorem 2.2 since the resulting functor categories $\mathfrak{M}_G(U)$ can be identified with the functor categories introduced in Example 2.5(d). If G is a nonfinite compact Lie group, then the argument needed to show that the pairs $(\mathcal{S}, \mathcal{O})$ of Example 2.5(c) satisfy the hypotheses of Theorem 2.2 is described in Scholium 2.3.

3. Positive results on functor categories

Throughout this section, $(\mathcal{S}, \mathcal{O})$ is assumed to be a pair of categories satisfying the hypotheses of Theorem 2.2. The focus of most of this article is on functor categories which do not satisfy our compatibility axioms. However, this section is devoted to positive results giving conditions on a pair $(\mathcal{S}, \mathcal{O})$ which ensure that the associated functor category $\mathfrak{M}_{\mathcal{O}}$ satisfies our various compatibility axioms. Our compatibility axioms naturally fall into three pairs. Associated to each of these pairs of axioms there is a fairly natural closure property on the subcategory $\mathcal{O}$ which ensures that the associated functor category $\mathfrak{M}_{\mathcal{O}}$ satisfies that pair of axioms. Unfortunately, as various pairs drawn from Example 2.5 illustrate, these sufficient conditions are far from necessary.

Proposition 3.1. *Let $\mathcal{S}$ be an Ab-category with a symmetric monoidal structure derived from a bilinear tensor product $\wedge$, and let $\mathcal{O}$ be a skeletally small full subcategory of $\mathcal{S}$ such that the pair $(\mathcal{S}, \mathcal{O})$ satisfies the hypotheses of Theorem 2.2. If the unit $\bullet_{\mathcal{S}}$ of $\mathcal{S}$ is in $\mathcal{O}$, then the unit of $\mathfrak{M}_{\mathcal{O}}$ is projective, and $\mathfrak{M}_{\mathcal{O}}$ satisfies the axioms* **IPiP** *and* **IIiI**.

Proof. The unit for $\mathfrak{M}_{\mathcal{O}}$ is the functor $\mathcal{H}_{\bullet_{\mathcal{S}}}$. Since $\bullet_{\mathcal{S}}$ is in $\mathcal{O}$, this functor is representable and therefore projective. The rest of the proposition follows immediately from Lemma 1.2. □

In all the pairs $(\mathcal{S}, \mathcal{O})$ from Example 2.5 except those from Example 2.5(b), $\mathcal{O}$ contains the unit of $\mathcal{S}$ so that the proposition above applies. However, in the two pairs introduced in Example 2.5(b), the unit of $\mathcal{S}$ is definitely not in $\mathcal{O}$. In this case, both $\mathcal{S}$ and $\mathfrak{M}_{\mathcal{O}}$ are the category $\mathcal{R}[G]$-Mod of modules over the group ring $\mathcal{R}[G]$ of a finite group G. The unit for this category is the ring $\mathcal{R}$ with trivial G-action, which is typically neither projective nor injective in $\mathcal{R}[G]$-Mod. However, being the unit, $\mathcal{R}$ is necessarily internally projective in $\mathcal{R}[G]$-Mod. Moreover, if $\mathcal{R}$ is a field, then, regarded as a trivial $\mathcal{R}[G]$-module, it is internally injective in $\mathcal{R}[G]$-Mod. Thus, $\mathcal{R}[G]$-Mod illustrates how badly behaved the category $\mathfrak{M}_{\mathcal{O}}$ can be when the hypotheses of the proposition don't hold. On the other hand, if $\mathcal{R}$ is a field of characteristic prime to the order of the group G, then $\mathcal{R}$ with trivial G-action is projective in $\mathcal{R}[G]$-Mod so that $\mathcal{R}[G]$-Mod satisfies the axioms **IPiP** and **IIiI**. Thus, the sufficient condition in the proposition is far from necessary.

Proposition 3.2. *Let $\mathcal{S}$ be an Ab-category with a symmetric monoidal structure derived from a bilinear tensor product $\wedge$, and let $\mathcal{O}$ be a skeletally small full subcategory of $\mathcal{S}$ such that the pair $(\mathcal{S}, \mathcal{O})$ satisfies the hypotheses of Theorem 2.2. If $\mathcal{O}$ is closed under the tensor product operation on $\mathcal{S}$, then the category $\mathfrak{M}_{\mathcal{O}}$ satisfies the axioms* **TPPP** *and* **PiIP**.

Proof. The representable functors form a set of projective generators for $\mathfrak{M}_{\mathcal{O}}$. Thus, to show that $\mathfrak{M}_{\mathcal{O}}$ satisfies **TPPP**, it suffices to show that, if A and B are in $\mathcal{O}$, then $\mathcal{H}_A \,\square\, \mathcal{H}_B$ is projective. Lemma 4.4(b) of [21] gives that $\mathcal{H}_A \,\square\, \mathcal{H}_B \cong \mathcal{H}_{A \wedge B}$. Since $A \wedge B$ is in $\mathcal{O}$, $\mathcal{H}_{A\wedge B}$ is a representable functor in $\mathfrak{M}_{\mathcal{O}}$ and so projective. The rest of the proposition follows from Proposition 1.3(a). □

The sufficient condition given by this proposition is not strictly necessary since the categories $\mathcal{R}[G]$-Mod and $\mathfrak{M}_G(U)$ associated to a finite group G in Examples 2.5(b) and 2.5(c) both satisfy **TPPP** and **PiIP**, but are of the form $\mathfrak{M}_{\mathcal{O}}$ for a category $\mathcal{O}$ that is not closed under the tensor product operation on $\mathcal{S}$. However, these examples are misleading in the sense that, in both cases, $\mathcal{O}$ is a proper subcategory of a larger subcategory $\mathcal{O}'$ of $\mathcal{S}$ which is closed under the tensor product and whose associated functor category $\mathfrak{M}_{\mathcal{O}'}$ is equivalent to $\mathfrak{M}_{\mathcal{O}}$ under the obvious restriction functor. Proposition 3.8 below provides a more convincing example of non-necessity.

Proposition 3.3. *Let $\mathcal{S}$ be a symmetric monoidal closed Ab-category, and let $\mathcal{O}$ be a skeletally small full subcategory of $\mathcal{S}$ such that the pair $(\mathcal{S}, \mathcal{O})$ satisfies the hypotheses of Theorem 2.2. If $\mathcal{O}$ is closed under the internal hom operation on $\mathcal{S}$, then the category $\mathfrak{M}^{\mathrm{cov}}_{\mathcal{O}}$ satisfies the axioms* **PiF** *and* **IiII**.

As with the previous proposition, the sufficient condition given by this proposition is not strictly necessary since the category $\mathcal{R}[G]$-Mod of Examples 2.5(b) satisfies **PiF** and **IiII**, but is a category of the form $\mathfrak{M}^{\mathrm{cov}}_{\mathcal{O}}$ for a category $\mathcal{O}$ that is not closed under the internal hom operation on $\mathcal{S}$. Another such example is provided by the category $\mathcal{O}_G(U)$ of Example 2.5(c) in the special case where G is finite and U is complete. Again, however, these examples are misleading in the sense that, in each of them, $\mathcal{O}$ is a proper subcategory of a larger subcategory $\mathcal{O}'$ of $\mathcal{S}$ which is closed under the internal hom operation and whose associated functor category $\mathfrak{M}_{\mathcal{O}'}$ is equivalent to $\mathfrak{M}_{\mathcal{O}}$ under the obvious restriction functor. Proposition 3.8 gives a more convincing example of non-necessity.

Remark 3.4. In all of our examples derived from the equivariant stable category $\bar{h}G\mathcal{SU}$ of G-spectra indexed on a G-universe U, the category $\mathcal{S}$, which was assumed to be $\bar{h}G\mathcal{SU}$, can be replaced with its full subcategory $\mathcal{S}'$ consisting of the G-spectra with the G-homotopy type of finite G-CW spectra. Equivariant Spanier-Whitehead duality provides a functor $\mathfrak{D} : \mathcal{S}' \longrightarrow (\mathcal{S}')^{\mathrm{op}}$ which is an equivalence of symmetric monoidal closed categories between $\mathcal{S}'$ and its opposite category. The internal hom in $\mathcal{S}'$ associated to objects X and Y is just $\mathfrak{D}(X) \wedge Y$. The category $\mathcal{B}_*$ of Example 2.5(e) has a similar closed structure. Categories with this sort of symmetric monoidal closed structure are sometimes called *-autonomous categories (see, for example, [3]). Whenever $\mathcal{S}$, or some full subcategory $\mathcal{S}'$ of $\mathcal{S}$ containing $\mathcal{O}$, has a closed structure of this sort, the category $\mathcal{O}$ is closed under the internal hom operation on $\mathcal{S}$ if and only if $\mathcal{O}^{\mathrm{op}}$ is analogously closed. Thus, the restriction of Proposition 3.3 to covariant functors is unnecessary in this special case.

To prove the proposition above, we need to introduce an important endofunctor on $\mathfrak{M}^{\mathrm{cov}}_{\mathcal{O}}$.

Definition 3.5. Let $\mathcal{S}$ be a symmetric monoidal closed Ab-category, and let $\mathcal{O}$ be a skeletally small full subcategory of $\mathcal{S}$ such that the pair $(\mathcal{S}, \mathcal{O})$ satisfies the hypotheses of Theorem 2.2. Assume also that $\mathcal{O}$ is closed under the internal hom operation on $\mathcal{S}$. If X and Y are in $\mathcal{S}$, then denote the internal hom of this pair in $\mathcal{S}$ by $\langle\!\langle X, Y \rangle\!\rangle$. For N in $\mathfrak{M}^{\mathrm{cov}}_{\mathcal{O}}$ and D in $\mathcal{O}$, let N^D be the functor in $\mathfrak{M}^{\mathrm{cov}}_{\mathcal{O}}$ given by $N^D(A) = N(\langle\!\langle D, A \rangle\!\rangle)$.

Lemma 3.6. *Let D be in $\mathcal{O}$.*

(a) *The assignment of N^D to $N \in \mathfrak{M}^{\mathrm{cov}}_{\mathcal{O}}$ is an exact additive functor on $\mathfrak{M}^{\mathrm{cov}}_{\mathcal{O}}$.*

(b) *There is an isomorphism*

$$\mathcal{H}^{\mathrm{cov}}_D \,\Box\, N \cong N^D$$

which is natural in both $N \in \mathfrak{M}^{\mathrm{cov}}_{\mathcal{O}}$ and $D \in \mathcal{O}$.

Proof. Clearly the assignment of N^D to N and D is functorial and is additive in each of N and D separately. It is exact in N because the exactness of sequences in the functor category $\mathfrak{M}^{\mathrm{cov}}_{\mathcal{O}}$ is determined pointwise. On an object A of $\mathcal{O}$, the isomorphism between $\mathcal{H}_D \,\square\, N$ and N^D is given by the composite

$$\begin{aligned}(\mathcal{H}_D \,\square\, N)(A) &= \int^{B,C\in\mathcal{O}} \mathcal{H}_D(B)\otimes N(C)\otimes \mathcal{S}(B\wedge C, A)\\ &= \int^{B,C\in\mathcal{O}} \mathcal{S}(D,B)\otimes N(C)\otimes \mathcal{S}(B\wedge C, A)\\ &\cong \int^{C\in\mathcal{O}} N(C)\otimes \mathcal{S}(D\wedge C, A)\\ &\cong \int^{C\in\mathcal{O}} N(C)\otimes \mathcal{S}(C, \langle\!\langle D,A\rangle\!\rangle)\\ &\cong N(\langle\!\langle D,A\rangle\!\rangle) = N^D(A).\end{aligned}$$

Here, the first and third isomorphisms are given by Lemmas 4.3(b) and 4.3(c) of [21], respectively. The second isomorphism comes from the adjunction isomorphism making $\mathcal{S}$ a closed category. Clearly the composite above is natural in A, D, and N. □

Remark 3.7. The construction N^D is very closely related to the construction N_D introduced in the context of Example 2.5(d) by Dress [8]. In fact, if the G-universe U of that example is complete and $\mathcal{O}$ is taken to be $\mathcal{B}_G(U)$, then the very simple nature of the symmetric monoidal closed structure carried by $\mathcal{B}_G(U)$ implies that the two constructions are isomorphic.

Proof of Proposition 3.3. Note first that the adjunction making $\mathcal{S}$ a closed category forces the tensor product on $\mathcal{S}$ to be bilinear. Since the representable functors form a set of projective generators for $\mathfrak{M}^{\mathrm{cov}}_{\mathcal{O}}$, showing that $\mathfrak{M}^{\mathrm{cov}}_{\mathcal{O}}$ satisfies **PiF** is easily reduced to showing that the functor $\mathcal{H}^{\mathrm{cov}}_D\square ?$ is exact for any $D \in \mathcal{O}$. This is established in Lemma 3.6. Since $\mathfrak{M}^{\mathrm{cov}}_{\mathcal{O}}$ has enough projectives, it must also satisfy **IiII** by Proposition 1.3(b). □

The following positive result about the category of Mackey functors for the circle group S^1 stands in sharp contrast to the host of negative results contained in section 6. It also provides clear evidence that the rather formal sufficient conditions contained in Propositions 3.2 and 3.3 above are very far from necessary. This result is proven in section 10.

Proposition 3.8. (a) *The category $\mathfrak{M}_{S^1}$ of S^1-Mackey functors satisfies the six axioms* **PiF**, **PiIP**, **IiII**, **IPiP**, **IIiI**, *and* **TPPP**.

(b) *Let $\mathcal{O}'$ be a full subcategory of the complete S^1-stable category which contains the stable orbit category $\mathcal{O}_{S^1}$. If $\mathcal{O}'$ is closed under either $\wedge$-products or function objects, then the restriction functor*

$$\mathfrak{M}^{\mathrm{cont}}_{\mathcal{O}'} \longrightarrow \mathfrak{M}^{\mathrm{cont}}_{\mathcal{O}_{S^1}} = \mathfrak{M}_{S^1}$$

is not an equivalence of categories.

4. An introduction to (G, U)-Mackey functors

Let G be a compact Lie group, and U be a possibly incomplete G-universe. This section is intended to provide a basic introduction to the categories $\mathcal{O}_G(U)$ and $\mathfrak{M}_G(U)$. The results presented here provide both some sense of why these categories have something to do with the classical notion of a Mackey functor, and some intuition about why, for various choices of G and U, the category $\mathfrak{M}_G(U)$ fails to satisfy our various axioms. These results also form the foundation for the proofs, given in later sections, of Theorems 3.8(b), 6.1, 6.5, and 6.9.

The structure of the morphism sets of the stable orbit category $\mathcal{O}_G(U)$ of Example 2.5(c) is described in Corollary 5.3(b) of [15] and Corollary 3.2 of [21]. An object of $\mathcal{O}_G(U)$ is the suspension spectrum $\Sigma^\infty_U G/H_+$ of an orbit G/H of G; however, to avoid unnecessary notational complexity, we hereafter denote this object by G/H. The set of morphisms in $\mathcal{O}_G(U)$ from G/H to G/K is a free abelian group whose generators are certain allowed equivalence classes of diagrams of the form

$$G/H \xleftarrow{\alpha} G/J \xrightarrow{\beta} G/K,$$

in which $\alpha : G/J \longrightarrow G/H$ and $\beta : G/J \longrightarrow G/K$ are space-level G-maps. Two such diagrams are equivalent if there is a G-homeomorphism $\gamma : G/J \longrightarrow G/J'$ making the space-level diagram

$$\begin{tikzcd} & G/J \arrow[dl, "\alpha"'] \arrow[dd, "\gamma"] \arrow[dr, "\beta"] & \\ G/H & & G/K \\ & G/J' \arrow[ul, "\alpha'"] \arrow[ur, "\beta'"'] & \end{tikzcd}$$

commute up to G-homotopy.

The morphism in $\mathcal{O}_G(U)$ represented by the diagram

$$G/H \xleftarrow{\alpha} G/J \xrightarrow{\beta} G/K$$

is the composite of the map, denoted $\tau(\alpha)$, which is represented by the diagram

$$G/H \xleftarrow{\alpha} G/J \xrightarrow{1_{G/J}} G/J$$

and the map, denoted $\rho(\beta)$, which is represented by the diagram

$$G/J \xleftarrow{1_{G/J}} G/J \xrightarrow{\beta} G/K.$$

If J is a subgroup of H, hereafter denoted $J \leq H$, then there is a canonical G-map $\pi^J_H : G/J \longrightarrow G/H$ which takes the identity coset eJ of G/J to the identity coset eH of G/H. The associated maps $\tau(\pi^J_H) : G/H \longrightarrow G/J$ and $\rho(\pi^J_H) : G/J \longrightarrow G/H$ in $\mathcal{O}_G(U)$ are denoted τ^J_H and ρ^J_H, respectively.

The connection between the category $\mathcal{O}_G(U)$ and the classical notion of a Mackey functor introduced by representation theorists can be seen from this sketch of the structure of the morphism sets of $\mathcal{O}_G(U)$. A contravariant functor M from $\mathcal{O}_G(U)$ to Ab assigns an abelian group $M(G/H)$ to each orbit G/H of G. This abelian group is the value of the Mackey functor M at the subgroup H of G and is often denoted $M(H)$ rather than $M(G/H)$. The map $\tau(\alpha)$ associated to a G-map $\alpha : G/J \longrightarrow G/H$ induces an induction (or transfer) map from $M(G/J)$ to $M(G/H)$, and the map $\rho(\alpha)$ induces a restriction map from $M(G/H)$ to $M(G/J)$.

Not every equivalence class of diagrams represents a generator of a morphism group in $\mathcal{O}_G(U)$. Each equivalence class of diagrams contains at least one diagram of the form

$$G/H \xleftarrow{\pi_H^J} G/J \xrightarrow{\beta} G/K.$$

The allowed equivalence classes are those having a representative of the above special type in which the subgroup J of H satisfies both the condition that H/J embeds in the universe U as an H-space and the condition that the index of J in its H-normalizer $N_H J$ is finite. For our purposes, the essential property of a complete G-universe U is that, for such a universe, the embedding condition on H/J is always satisfied.

If the universe U is contained in a larger universe U', then $\mathcal{O}_G(U)$ can be identified with a subcategory of $\mathcal{O}_G(U')$. In particular, since any G-universe is isomorphic to a subuniverse of a complete G-universe, $\mathcal{O}_G(U)$ is always a subcategory of $\mathcal{O}_G$. Viewing $\mathcal{O}_G(U)$ as a subcategory of $\mathcal{O}_G$ reveals a difference between the two restrictions imposed on the equivalence classes which index the generators of the morphism sets of $\mathcal{O}_G(U)$. If H/J does not embed in U, but the finiteness condition holds, then the diagram

$$G/H \xleftarrow{\pi_H^J} G/J \xrightarrow{\beta} G/K,$$

represents a morphism of $\mathcal{O}_G$ which has been omitted from $\mathcal{O}_G(U)$ by our choice of U. However, if the index of J in $N_H J$ is not finite, then this diagram does not represent a generator in $\mathcal{O}_G(U)$ for any U.

If the index of J in $N_H J$ is not finite, then for certain geometric reasons it is best to think of the diagram

$$G/H \xleftarrow{\pi_H^J} G/J \xrightarrow{\beta} G/K,$$

as representing a morphism in $\mathcal{O}_G(U)$ which, in some vague sense, should have been a generator, but has instead been identified with the zero map. This distinction between a morphism that has been omitted from $\mathcal{O}_G(U)$ and one that has been identified with zero is important for understanding composition of morphisms in $\mathcal{O}_G(U)$. It is also important for understanding the homological anomalies described in this paper. If the group G is finite and the G-universe U is incomplete, then the morphisms missing from $\mathcal{O}_G(U)$ may cause the axioms **PiF** and **IiII** to fail in the functor category $\mathfrak{M}_G(U)$. If the group G is a nonfinite compact Lie group, then the "unexpected" zero morphisms in $\mathcal{O}_G$ may cause the axioms **TPPP**, **PiIP**, **TPPP**, and **IiII** to fail in the functor category $\mathfrak{M}_G$. The arguments establishing these two types of failures are rather different since the failures happen for quite different reasons.

5. An introduction to globally defined Mackey functors

Here, we examine the structure of the category $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ associated to the category of global $(\mathfrak{P}, \mathfrak{Q})$-Mackey functors. Thus, throughout this section, all groups are assumed to be finite, and $\mathfrak{P}$, $\mathfrak{Q}$, and $\mathfrak{R}$ are assumed to be sets of primes. The observations presented here should provide both a sense of why the functors out of $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ should be regarded as globally defined Mackey functors and an understanding of the way in which the choice of the sets $\mathfrak{P}$ and $\mathfrak{Q}$ controls the level of structure carried by those functors. These remarks also lay the foundation for the proof of Theorem 6.10 in section 11.

Recall from Example 2.5(e) that the set of morphisms from G to H in $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ is the Grothendieck group of isomorphism classes of $(G \times H)$-sets which are also $(\mathfrak{P}, \mathfrak{Q})$-sets. This is a free abelian group whose generators are the isomorphism classes of $(G \times H)$-orbits which are $(\mathfrak{P}, \mathfrak{Q})$-sets. For any $(G \times H)$-orbit $(G \times H)/J$, the inclusion of J into $G \times H$ can be composed with the projections from $G \times H$ to G and H to produce group homomorphisms $\alpha : J \longrightarrow G$ and $\beta : J \longrightarrow H$. If M is a covariant functor out of $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$, then the appropriate intuitive understanding of the map from $M(G)$ to $M(H)$ induced by the $(G \times H)$-orbit $(G \times H)/J$ is that this map is the composite of a restriction map from $M(G)$ to $M(J)$ associated to the homomorphism α and an induction map from $M(J)$ to $M(H)$ associated to the homomorphism β. The imposed constraint that the $(G \times H)$-set $(G \times H)/J$ must be a $(\mathfrak{P}, \mathfrak{Q})$-set translates easily into the restriction that the kernel of α must be a $\mathfrak{Q}$-group (that is, have order divisible only by the primes in $\mathfrak{Q}$) and the kernel of β must be a $\mathfrak{P}$-group. Hereafter, we refer to a homomorphism whose kernel is a $\mathfrak{P}$-group as a homomorphism with $\mathfrak{P}$-kernel.

The most common choices for $\mathfrak{P}$ and $\mathfrak{Q}$ are the empty set $\emptyset$ of primes and the set of all primes, which we denote by ∞. If $\mathfrak{P} = \emptyset$, then the trivial group is the only $\mathfrak{P}$-group. On the other hand, if $\mathfrak{P} = \infty$, then all finite groups are $\mathfrak{P}$-groups. Thus, a homomorphism with $\emptyset$-kernel is a monomorphism, and every group homomorphism has ∞-kernel. At the level of global Mackey functors, this means, for example, that a global $(\emptyset, \infty)$-Mackey functor has induction maps only for injective group homomorphisms, but restriction maps for all homomorphisms. Note that $\mathcal{B}_*(\infty, \infty) = \mathcal{B}_*$.

In order to justify the intuitive description of the maps in $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ presented above, we must first identify the morphisms in $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ which are derived directly from ordinary group homomorphisms.

Definition 5.1. Let $\alpha : H \longrightarrow G$ be a group homomorphism. Then $\tau(\alpha)$ is the set G considered as a $(H \times G)$-set with action given by

$$(h, g)x = \alpha(h)xg^{-1},$$

for $(h, g) \in H \times G$ and $x \in G$. Also, $\rho(\alpha)$ is the set G considered as an $(G \times H)$-set with action given by

$$(g, h)x = gx\alpha(h^{-1}),$$

for $(g, h) \in G \times H$ and $x \in G$. Note that, if α has $\mathfrak{R}$-kernel, then $\tau(\alpha)$ is an $(\mathfrak{R}, \emptyset)$-set, and $\rho(\alpha)$ is an $(\emptyset, \mathfrak{R})$-set. If $\mathfrak{R} \subset \mathfrak{P}$, then $\tau(\alpha)$ is a generator of the free abelian group of morphisms from H to G in $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$. It should be thought of as the induction, or transfer, map associated to α. Similarly, if $\mathfrak{R} \subset \mathfrak{Q}$, then $\rho(\alpha)$ is a generator of the free abelian group of morphisms from G to H in $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$. It should be thought of as the restriction map associated to α. If α is the inclusion of a subgroup H into the group G, then $\tau(\alpha)$ and $\rho(\alpha)$ are denoted τ_G^H and ρ_G^H, respectively. If $G = H$ and α is the identity map, then $\tau(\alpha)$ and $\rho(\alpha)$ are equal and serve as the identity map 1_G of G in $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$.

If $\alpha : J \longrightarrow G$ and $\beta : J \longrightarrow H$ are the group homomorphisms associated in our introductory remarks to the $(G \times H)$-orbit $(G \times H)/J$, then it is fairly easy to see that, in $\mathcal{B}_*$, $(G \times H)/J$ is the composite $\tau(\beta) \circ \rho(\alpha)$. Further, the orbit $(G \times H)/J$ is a $(\mathfrak{P}, \mathfrak{Q})$-set precisely when α has $\mathfrak{Q}$-kernel and β has $\mathfrak{P}$-kernel. These observations allows us to think of the generators of the free abelian group $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})(G, H)$ as equivalence classes of diagrams of the form

$$G \xleftarrow{\ \alpha\ } J \xrightarrow{\ \beta\ } H,$$

in which α is a group homomorphism with $\mathfrak{Q}$-kernel, β is a group homomorphism with $\mathfrak{P}$-kernel, and the induced homomorphism $(\alpha, \beta) : J \longrightarrow G \times H$ is a monomorphism.

This description of the morphism sets of $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ plays a central role in the proof of Theorem 6.10. Three notes of caution are, however, necessary. First, since $\mathfrak{M}_*(\mathfrak{P}, \mathfrak{Q})$ is the category of covariant functors out of $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$, the Mackey functor interpretation of this diagram is the reverse of the interpretation associated to the analogous diagrams for $\mathcal{O}_G(U)$. Here, α is plays the role of the restriction map, and β plays the role of the induction map. Second, the equivalence relation which must be imposed on these diagrams cannot be described as neatly as the corresponding relation for $\mathcal{O}_G(U)$. For our purposes, it suffices to say that the diagram

$$G \xleftarrow{\ \alpha'\ } J' \xrightarrow{\ \beta'\ } H,$$

is equivalent to the unprimed diagram above if and only if the two $(G \times H)$-sets $\tau(\beta) \circ \rho(\alpha)$ and $\tau(\beta') \circ \rho(\alpha')$ are isomorphic. The third note is that composition in $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ cannot be computed using pullback diagrams like those used to compute composition in $\mathcal{O}_G$.

Remark 5.2. There is an alternative approach to globally defined Mackey functors in which the category $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ is replaced by a somewhat larger category $\mathcal{B}'_*(\mathfrak{P}, \mathfrak{Q})$ whose objects are finite groupoids rather than finite groups. In many ways, the relation between $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ and $\mathcal{B}'_*(\mathfrak{P}, \mathfrak{Q})$ is similar to that between $\mathcal{O}_G(U)$ and $\mathcal{B}_G(U)$. In particular, the categories $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ and $\mathcal{B}'_*(\mathfrak{P}, \mathfrak{Q})$ are similar enough that the associated categories of covariant functors into Ab are equivalent under the restriction functor derived from the inclusion of $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ into $\mathcal{B}'_*(\mathfrak{P}, \mathfrak{Q})$. The larger category $\mathcal{B}'_*(\mathfrak{P}, \mathfrak{Q})$ is somewhat more difficult to define because one has to deal with functors from finite groupoids into finite sets rather than sets carrying actions by finite groups. However, it has the advantages of an easily described equivalence relation on the generators of its morphism sets and a simple pullback formula for its composition. Further advantages of $\mathcal{B}'_*(\mathfrak{P}, \mathfrak{Q})$ over $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ are noted at the end of this section and in Remark 6.11(b).

One technical result describing the behavior of composition in $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ in the context of cartesian products of groups is needed for the proof of Theorem 6.10. The information this result provides about $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ is similar to that provided about $\mathcal{O}_G(U)$ by Lemma 3.3 of [21].

Lemma 5.3. *Let $\psi : J \longrightarrow P$, $\xi : J \longrightarrow Q$ and $\zeta : Q' \longrightarrow Q$ be homomorphisms between finite groups such that ξ has $\mathfrak{Q}$-kernel and ζ has $\mathfrak{P}$-kernel. Assume that the integers $\{n_i\}_{1 \leq i \leq m}$ and the diagrams*

$$Q' \xleftarrow{\alpha_i} K_i \xrightarrow{\beta_i} J,$$

for $1 \leq i \leq m$, are chosen so that the composite $\rho(\xi) \circ \tau(\zeta)$ in $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ is the sum over i of n_i times the generator of $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})(Q', J)$ represented by the i^{th} diagram. Then the composite $\rho((\psi, \xi)) \circ \tau(1 \times \zeta)$ of the restriction map associated to the homomorphism $(\psi, \xi) : J \longrightarrow P \times Q$ and the transfer map associated to the homomorphism $1 \times \zeta : P \times Q' \longrightarrow P \times Q$ is the sum over i of n_i times the generator of $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})(P \times Q', J)$ represented by the diagram

$$P \times Q' \xleftarrow{(\psi \circ \beta_i, \alpha_i)} K_i \xrightarrow{\beta_i} J.$$

This result can be proven by brute force computations with the obvious finite sets carrying the appropriate group actions. A further advantage of the category $\mathcal{B}'_*(\mathfrak{P}, \mathfrak{Q})$ described in Remark 5.2 is that the simple pullback formula for composition in $\mathcal{B}'_*(\mathfrak{P}, \mathfrak{Q})$ trivializes the proof of this lemma.

6. A bestiary of symmetric monoidal closed abelian categories

This section contains a catalog of functor categories which fail to satisfy the hypotheses of Propositions 3.2 and 3.3 and also fail to satisfy various of our compatibility axioms. These badly behaved categories come from the families of categories $\mathfrak{M}_G(U)$ of Example 2.5(c) and $\mathfrak{M}_*(\mathfrak{P},\mathfrak{Q})$ of Example 2.5(e). The results stated in this section about the misbehavior of our functor categories are proven in the subsequent sections.

Let G be a compact Lie group, and U be a possibly incomplete G-universe. The hypotheses of our theorems about the homological misbehavior of the category $\mathfrak{M}_G(U)$ of (G,U)-Mackey functors are certainly not as weak as they could be. Nevertheless, they are weak enough to produce an almost overwhelming supply of badly behaved categories.

Theorem 6.1. *Let G be a finite group, U be a G-universe, and $C \leq D \leq H$ be subgroups of G such that*

(i) *C is normal in D and $D/C \cong \mathbb{Z}/p$ for some prime p*
(ii) *H/C embeds as an H-space in U*
(iii) *H/D does not embed as an H-space in U.*

Then $\mathcal{H}_{G/D}$ is not flat in $\mathfrak{M}_G(U)$, and $\mathfrak{M}_G(U)$ satisfies neither **PiF** *nor* **IiII***.*

Remark 6.2. One way of understanding the misbehavior of $\mathfrak{M}_G(U)$ implied by Theorem 6.1 is that, if the subgroups $C \leq D \leq H$ satisfy the hypotheses of the theorem, then the induction maps τ_D^C and τ_H^C are contained in the subcategory $\mathcal{O}_G(U)$ of $\mathcal{O}_G$, but the induction map τ_H^D is not. Thus, even though the map τ_H^C can be factored as the composite $\tau_D^C \circ \tau_H^D$ in $\mathcal{O}_G$, it cannot be so factored in $\mathcal{O}_G(U)$. This failure of τ_H^C to factor properly in $\mathcal{O}_G(U)$ seems to be the source of the homological misbehavior of the category $\mathfrak{M}_G(U)$.

Example 6.3. Let p be a prime, $G = H = \mathbb{Z}/p^2$, $D = \mathbb{Z}/p \subset \mathbb{Z}/p^2$, and C be the trivial subgroup of G. Let U be a G-universe whose only irreducible summands are the trivial irreducible G-representation and a free irreducible G-representation. Then G/C embeds in U as a G-space, but G/D does not. By the theorem, the projective Mackey functor $\mathcal{H}_{G/D}$ is not flat in $\mathfrak{M}_G(U)$. Beyond the context of this paper, the special significance of this universe is that it has natural connections to the study of semi-free actions of G.

Corollary 6.4. *Let p be a prime, G be a finite p-group, and U be a G-universe. Then the category $\mathfrak{M}_G(U)$ cannot satisfy either* **PiF** *or* **IiII** *unless, for each subgroup H of G, the set*

$$\{K : K \leq H \text{ and } H/K \text{ does not embed as an } H\text{-space in } U\}$$

is closed under passage to subgroups. In particular, if the free orbit G/e embeds in U as a G-space, then $\mathfrak{M}_G(U)$ cannot satisfy either **PiF** *or* **IiII** *unless every G-orbit G/K embeds in U.*

The statement of our second main theorem requires a bit of additional notation. The Weyl group N_GH/H of a subgroup H of G is denoted W_GH, and the set of G-conjugates of $H \leq G$ is denoted $(H)_G$. If H and K are subgroups of G, then the notation $(K)_G \leq (H)_G$ indicates that K is subconjugate to H in G.

Theorem 6.5. *Let G be a compact Lie group, and U be a complete G-universe. Also, let $C \leq D \leq H$ be subgroups of G such that*

(i) *C is normal in D and D/C is a finite p-group for some prime p*
(ii) *$W_H D$ is finite*
(iii) *for every $K < D$ such that $(C)_G \leq (K)_G$, $W_H K$ is not finite.*

Then $\mathcal{H}_{G/D}$ is not flat in $\mathfrak{M}_G$, and $\mathfrak{M}_G$ satisfies neither **PiF** *nor* **IiII**.

Remark 6.6. One way of understanding the misbehavior of $\mathfrak{M}_G$ implied by Theorem 6.5 is that, if the subgroups $C \leq D \leq H$ satisfy the hypotheses of the theorem, then the induction maps τ_D^C and τ_H^D are generators of the morphism sets of $\mathcal{O}_G$. Their composite is the induction map τ_H^C associated to the containment $C \leq H$. This map τ_H^C should also be a generator of the appropriate morphism set. However, since $W_H C$ is not finite, τ_H^C is actually the zero map. This vanishing of the composite of two generators of the morphism sets of $\mathcal{O}_G$ seems to be the source of the homological misbehavior of the category $\mathfrak{M}_G(U)$.

Example 6.7. (a) Let G and H both be the orthogonal group $O(2)$, D be the dihedral group of order $2n$ (for $n \geq 3$) regarded as a subgroup of G, and C be the cyclic group of order n regarded as a subgroup of D. Then C is normal in G, and $D/C \cong \mathbb{Z}/2$. Moreover, $N_G D$ is the dihedral group of order $4n$, so $W_G D$ is finite. If K is a proper subgroup of D such that $(C)_G \leq (K)_G$, then $K = C$, and $W_G K$ is G/C, which is not finite. Thus, the projective Mackey functor $\mathcal{H}_{G/D}$ is not flat in $\mathfrak{M}_G$, and $\mathfrak{M}_G$ satisfies neither **PiF** nor **IiII**.

(b) If $C \leq D \leq H$ are subgroups of G which satisfy the hypotheses of the theorem and $\epsilon : G' \longrightarrow G$ is a surjective group homomorphism, then the subgroups $C' = \epsilon^{-1}(C)$, $D' = \epsilon^{-1}(D)$, and $H' = \epsilon^{-1}(H)$ are subgroups of G' which also satisfy the hypotheses of the theorem. Therefore, $\mathfrak{M}_{G'}$ does not satisfy the axioms **PiF** and **IiII**.

(c) We would like to argue that, if $C \leq D \leq H$ are subgroups of G which satisfy the hypotheses of the theorem and $G \leq G''$, then C, D, and H, regarded as subgroups of G'', satisfy the hypotheses of the theorem. However, condition (iii) in the hypotheses of the theorem might fail since there might be a subgroup K of D such that C was G''-subconjugate to K, but not G-subconjugate to K. Nevertheless, if $D/C \cong \mathbb{Z}/p$ and C is the unique subgroup of D of index p, as in part (a) of this example, then the triple $C \leq D \leq H$, regarded as subgroups of G'', must still satisfy the hypotheses of the theorem. In fact, it suffices to assume that $D/C \cong \mathbb{Z}/p$ and that every subgroup C' of index p in D is H-conjugate to C. Combining this observation with part (a) of this example, we see that, if a compact Lie group G contains the orthogonal group $O(2)$, then $\mathfrak{M}_G$ satisfies neither **PiF** nor **IiII**.

Combining the various parts of the example above yields the following corollary of Theorem 6.5.

Corollary 6.8. *If G is any one of*

$$O(m),\, SU(m),\, U(m), \quad \text{for } m \geq 2;$$
$$SO(n),\, Spin(n), \quad \text{for } n \geq 3;$$
$$Sp(q), \quad \text{for } q \geq 1;$$

then $\mathfrak{M}_G$ satisfies neither **PiF** *nor* **IiII**.

The family of categories $\mathfrak{M}_G$, for G a compact Lie group, also provides us with some examples of symmetric monoidal closed categories which fail to satisfy **TPPP** and **PiIP**.

Theorem 6.9. *Let G be the orthogonal group $O(m)$, for $m \geq 2$, or the special orthogonal group $SO(n)$, for $n \geq 3$. Then $\mathfrak{M}_G$ satisfies neither* **TPPP** *nor* **PiIP**.

Now we turn to the context of global Mackey functors. Recall that, in Example 2.5(e), we associated a category $\mathfrak{M}_*(\mathfrak{P}, \mathfrak{Q})$ of global $(\mathfrak{P}, \mathfrak{Q})$-Mackey functors to each pair $(\mathfrak{P}, \mathfrak{Q})$ of sets of integer primes.

Theorem 6.10. *Let $\mathfrak{P}$ and $\mathfrak{Q}$ be sets of integer primes. If the prime p is not in $\mathfrak{P}$, then the representable functor in $\mathfrak{M}_*(\mathfrak{P}, \mathfrak{Q})$ associated to the cyclic group $\mathbb{Z}/p$ is not flat in $\mathfrak{M}_*(\mathfrak{P}, \mathfrak{Q})$. Thus, $\mathfrak{M}_*(\mathfrak{P}, \mathfrak{Q})$ satisfies neither* **PiF** *nor* **IiII**.

Remark 6.11. (a) Recall that the empty set of primes and the set of all primes are denoted $\emptyset$ and ∞, respectively. The three types of global Mackey functors that appear most often in the literature seem to be $(\emptyset, \emptyset)$-, $(\emptyset, \infty)$- and $(\infty, \emptyset)$-Mackey functors (see, for example, [1, 2, 4, 5, 7, 9, 10, 12–14, 24–26, 28, 29]). Theorem 6.10 indicates that the first two of these categories of global Mackey functors are badly behaved.

(b) As noted in Remark 5.2, the category $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ used to define global $(\mathfrak{P}, \mathfrak{Q})$-Mackey functors can be replaced by a somewhat larger category $\mathcal{B}'_*(\mathfrak{P}, \mathfrak{Q})$ whose objects are finite groupoids. One advantage of this replacement is that, for any set of primes $\mathfrak{Q}$, the category $\mathcal{B}'_*(\infty, \mathfrak{Q})$ is a symmetric monoidal closed category. Proposition 3.3 therefore implies that $\mathfrak{M}_*(\infty, \mathfrak{Q})$ satisfies **PiF** and **IiII**. Thus, the theorem above gives sharp conditions under which $\mathfrak{M}_*(\mathfrak{P}, \mathfrak{Q})$ fails to satisfy **PiF** and **IiII**.

7. Mackey functors for incomplete universes and the proof of Theorem 6.1

Each of Theorems 6.1, 6.5, and 6.10 asserts that a projective object P in a certain functor category $\mathfrak{M}_{\mathcal{O}}$ is not flat. The proofs of these results follow the same pattern. It suffices to show that, for some object A in $\mathcal{O}$, the functor sending M in $\mathfrak{M}_{\mathcal{O}}$ to $(P \,\square\, M)(A)$ does not preserve monomorphisms. To show this, we restrict the domain of this functor to a nice full subcategory $\mathfrak{M}'$ of $\mathfrak{M}_{\mathcal{O}}$. We then identify a natural direct summand $\mathfrak{Z} : \mathfrak{M}' \longrightarrow Ab$ of the restricted functor, and show that the summand fails to preserve monomorphisms. In this section, we prove Theorem 6.1 by carrying out the appropriate special case of this general program. Thus, throughout the section, G is a finite group, U is a G-universe, and $C \le D \le H$ are subgroups of G. The appropriate projective object P in this context is the representable functor $\mathcal{H}_{G/D}$ in the category $\mathfrak{M}_G(U)$, and the appropriate object A of $\mathcal{O} = \mathcal{O}_G(U)$ is the orbit G/H.

We begin the proof of Theorem 6.1 by introducing the appropriate subcategory $\mathfrak{M}'$ of $\mathfrak{M}_G(U)$, and the appropriate direct summand functor $\mathfrak{Z}$.

Definition 7.1. (a) A Mackey functor M in $\mathfrak{M}_G(U)$ is concentrated over C if, for $K \le G$, $M(G/K) = 0$ unless $(C)_G \le (K)_G$. The subcategory $\mathfrak{M}'$ appropriate for the proof of Theorem 6.1 is the full subcategory of $\mathfrak{M}_G(U)$ consisting of Mackey functors which are concentrated over C; this subcategory is denoted $M^C_G(U)$.

(b) The Weyl group $W_G C$ is contained in the morphism set $\mathcal{O}_G(U)(G/C, G/C)$ as the set of maps $\rho(\beta)$ associated to the endomorphisms β of the G-set G/C. The group $W_G C$ therefore acts on the value $M(G/C)$ of a Mackey functor M at G/C. Let $\mathfrak{Z}_C : \mathfrak{M}_G(U) \longrightarrow Ab$ be the functor sending M in $\mathfrak{M}_G(U)$ to the quotient group $M(G/C)/W_D C$ of $M(G/C)$.

The technical foundation of the proof of Theorem 6.1 is the following result:

Proposition 7.2. *Let G be a finite group, U be a G-universe, and $C \le D \le H$ be subgroups of G such that*

(i) *C is normal in D, and $D/C \cong \mathbb{Z}/p$ for some prime p*
(ii) *H/C embeds as an H-space in U*
(iii) *H/D does not embed as an H-space in U.*

Then, for M in $M^C_G(U)$, $\mathfrak{Z}_C(M)$ splits off from $(\mathcal{H}_{G/D} \,\square\, M)(G/H)$ as a natural direct summand.

Before proving this result, we show how it can be used to complete the proof of our theorem.

Proof of Theorem 6.1. Assume that C is normal in D, and that $D/C \cong \mathbb{Z}/p$ for some prime p. Also assume that H/C embeds as an H-space in U, but that H/D does not so embed. We must construct a monomorphism $\iota : A \longrightarrow B$ which is not preserved by the functor $\mathfrak{Z}_C$. Define the object B of $\mathfrak{M}_G(U)$ by the exact sequence

$$\bigoplus_{K<C} \mathcal{H}_{G/K} \otimes \mathbb{Z}/p \xrightarrow{\oplus\, \tilde{\rho}^K_C \otimes 1} \mathcal{H}_{G/C} \otimes \mathbb{Z}/p \longrightarrow B \longrightarrow 0.$$

Here, the map $\tilde{\rho}^K_C : \mathcal{H}_{G/K} \longrightarrow \mathcal{H}_{G/C}$ is derived from the restriction map ρ^K_C in $\mathcal{O}_G(U)$. By construction, B is concentrated over C. Let A in $\mathfrak{M}_G(U)$ be the image of the composite map

$$\mathcal{H}_{G/D} \otimes \mathbb{Z}/p \xrightarrow{\tilde{\tau}_D^C \otimes 1} \mathcal{H}_{G/C} \otimes \mathbb{Z}/p \longrightarrow B,$$

in which $\tilde{\tau}_D^C$ is the map derived from the induction map τ_D^C in $\mathcal{O}_G(U)$. Note that there are an obvious monomorphism $\iota : A \longrightarrow B$ and an obvious epimorphism $\mu : \mathcal{H}_{G/D} \otimes \mathbb{Z}/p \longrightarrow A$. Since A embeds in B, it must also be concentrated over C.

The composition of morphisms in $\mathcal{O}_G(U)$ is discussed in section 3 of [21]. Using the results presented there, it is easy to see that $B(G/C)$ is just the group ring $\mathbb{Z}/p[W_GC]$. Let $t \in \mathbb{Z}/p[W_GC]$ be the sum $\sum_{d \in W_DC} d$. A straightforward double coset computation indicates that $A(G/C)$ is the image of the endomorphism of $\mathbb{Z}/p[W_GC]$ given by multiplication on the right by t. Thus, $A(G/C)$ can be identified with the $\mathbb{Z}/p$ vector space $\mathbb{Z}/p[W_GC/W_DC]$ whose basis set is the orbit W_GC/W_DC. Under these identifications, the actions of W_GC on $A(G/C)$ and $B(G/C)$ are the obvious ones coming from the left action of W_GC on itself. Moreover, the map

$$\iota(G/C) : A(G/C) \longrightarrow B(G/C)$$

is just the trace map

$$\mathbb{Z}/p[W_GC/W_DC] \longrightarrow \mathbb{Z}/p[W_GC]$$

which sends a coset in W_GC/W_DC, regarded as a basis element of $\mathbb{Z}/p[W_GC/W_DC]$, to the sum of its elements in $\mathbb{Z}/p[W_GC]$. These identifications allow us to analyse the map

$$\mathfrak{Z}_C(\iota) : \mathfrak{Z}_C(A) \longrightarrow \mathfrak{Z}_C(B).$$

The objects $\mathfrak{Z}_C(A)$ and $\mathfrak{Z}_C(B)$ are the $\mathbb{Z}/p$ vector spaces $\mathbb{Z}/p[W_DC\backslash W_GC/W_DC]$ and $\mathbb{Z}/p[W_DC\backslash W_GC]$ whose bases are the sets $W_DC\backslash W_GC/W_DC$ and $W_DC\backslash W_GC$, respectively. The map $\mathfrak{Z}_C(\iota)$ is the obvious one derived from the trace map $\iota(G/C)$. It follows that the double coset W_DCeW_DC is a basis element in $\mathfrak{Z}_C(A)$, but maps to zero in $\mathfrak{Z}_C(B)$. Thus, the map $\mathfrak{Z}_C(\iota)$ is not a monomorphism. $\square$

Note that Corollary 6.4 follows directly from Theorem 6.1 and the fact that p-groups are solvable. The remainder of this section is devoted to the postponed proof of our splitting result.

Proof of Proposition 7.2. By Lemma 4.4(a) of [21],

$$(\mathcal{H}_{G/D} \,\square\, M)(G/H) = \int^{G/Q} M(G/Q) \otimes [G/H, G/D \times G/Q].$$

Here, $[?,?]$ is used to denote the morphism set in $\mathcal{S} = \bar{h}G\mathcal{SU}$, and $G/D \times G/Q$ is used to denote the spectrum $\Sigma_U^\infty G/D_+ \wedge \Sigma_U^\infty G/Q_+ = \Sigma_U^\infty (G/D \times G/Q)_+$. By definition, this coend is a quotient of the abelian group $\bigoplus_{G/Q} M(G/Q) \otimes [G/H, G/D \times G/Q]$. We prove the proposition by constructing a natural map

$$\lambda : (\mathcal{H}_{G/D} \,\square\, M)(G/H) \longrightarrow \mathfrak{Z}_C(M),$$

and then showing that λ is a naturally split epimorphism. This map λ is derived from a map

$$\tilde{\lambda} : \bigoplus_{G/Q} M(G/Q) \otimes [G/H, G/D \times G/Q] \longrightarrow \mathfrak{Z}_C(M)$$

by factoring $\tilde{\lambda}$ through the quotient group $(\mathcal{H}_{G/D} \,\square\, M)(G/H)$.

The map $\tilde{\lambda}$ may be defined by specifying its restriction $\tilde{\lambda}_Q$ to each summand $M(G/Q) \otimes [G/H, G/D \times G/Q]$ of its domain. Proposition 3.1 of [21] indicates that $[G/H, G/D \times G/Q]$ is a free abelian group. Thus, $M(G/Q) \otimes [G/H, G/D \times G/Q]$ is a direct sum of one copy of $M(G/Q)$ for each generator of $[G/H, G/D \times G/Q]$. The generators of $[G/H, G/D \times G/Q]$ are certain equivalence classes of diagrams of the form

$$G/H \xleftarrow{\delta} G/J \xrightarrow{(\alpha,\beta)} G/D \times G/Q,$$

in which $\alpha : G/J \longrightarrow G/D$, $\beta : G/J \longrightarrow G/Q$, and $\delta : G/J \longrightarrow G/H$ are maps of G-spaces. Denote the equivalence class of this diagram by $(\delta; \alpha, \beta)$. The map $\tilde{\lambda}_Q$ may be defined by specifying its restriction $\tilde{\lambda}_{(\delta;\alpha,\beta)}$ to the copy of $M(G/Q)$ associated to each equivalence class $(\delta; \alpha, \beta)$.

If the equivalence class $(\delta; \alpha, \beta)$ does not contain a diagram of the form

$$G/H \xleftarrow{\pi_H^C} G/C \xrightarrow{(\pi_D^C,\beta)} G/D \times G/Q,$$

then $\tilde{\lambda}_{(\delta;\alpha,\beta)}$ is defined to be the zero map. On the other hand, the map $\tilde{\lambda}_{(\pi_H^C;\pi_D^C,\beta)}$ is defined to be the composite

$$M(G/Q) \xrightarrow{M(\rho(\beta))} M(G/C) \longrightarrow M(G/C)/W_D C = \mathfrak{z}_C(M).$$

There is, in fact, more than one choice for the map β in the diagram chosen to represent the equivalence class $(\pi_H^C; \pi_D^C, \beta)$. However, the passage to orbits in the codomain of $\tilde{\lambda}_{(\pi_H^C;\pi_D^C,\beta)}$ kills off the uncertainity that might have arisen in the definition of $\tilde{\lambda}_{(\pi_H^C;\pi_D^C,\beta)}$ from the possible choices for β.

To show that the map $\tilde{\lambda}$ induces the desired map λ, it suffices to show that, for each morphism $f : G/Q \longrightarrow G/Q'$ in $\mathcal{O}_G(U)$, the diagram

$$\begin{array}{ccc}
M(G/Q') \otimes [G/H, G/D \times G/Q] & \xrightarrow{M(f)\otimes 1} & M(G/Q) \otimes [G/H, G/D \times G/Q] \\
{\scriptstyle 1\otimes(1\times f)_*}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\lambda}_Q} \\
M(G/Q') \otimes [G/H, G/D \times G/Q'] & \xrightarrow{\tilde{\lambda}_{Q'}} & \mathfrak{z}_C(M)
\end{array}$$

commutes. In fact, the commutativity of this diagram needs to verified only for the morphisms f which are generators of the free abelian group $\mathcal{O}_G(U)(G/Q, G/Q')$. Moreover, since each morphism in $\mathcal{O}_G(U)$ is the composite of a restriction map and an induction map, it suffices to check the commutativity of the diagram for the cases in which f is either the restriction map $\rho(\beta)$ associated to a G-map $\beta : G/J \longrightarrow G/K$ or the induction map $\tau_Q^{Q'}$ associated to a containment $Q' \leq Q$.

Lemma 3.3 of [21] is the key to verifying the commutativity of the required diagram for both of these types of maps. Verifying the commutativity of the diagram for a restriction map $\rho(\beta)$ is entirely elementary, and uses none of the conditions imposed on C, D, and H other than the containments $C \leq D \leq H$. Thus, we can assume that f is the induction map $\tau_Q^{Q'}$ associated to the containment $Q' \leq Q$.

Note that $M(G/Q') \otimes [G/H, G/D \times G/Q]$ is a direct sum of one copy of $M(G/Q')$ for each generator of $[G/H, G/D \times G/Q]$. The commutativity of the diagram can therefore be verified by checking it on each of the copies of $M(G/Q')$. If the diagram

$$G/H \xleftarrow{\delta} G/J \xrightarrow{(\alpha,\beta)} G/D \times G/Q,$$

represents a generator of $[G/H, G/D \times G/Q]$, then $(J)_G \le (D)_G$. Using the equivalence relation, we may adjust the diagram so that $J \le D$ and $\alpha = \pi^J_D$. It is easy to verify that, if $(C)_G \not\le (J)_G$, then, on the copy of $M(G/Q')$ indexed on the equivalence class $(\delta; \pi^J_D, \beta)$, both of the composites in the diagram are zero. Thus, we may assume that $(C)_G \le (J)_G$. Since D/C is $\mathbb{Z}/p$, our assumptions now give that either $J = D$ or $(J)_G = (C)_G$.

First, let us assume that $J = D$ so that we are interested in the generator of $[G/H, G/D \times G/Q]$ associated to the equivalence class $(\delta; 1_{G/D}, \beta)$. We wish to show that both of the composites in the diagram are zero on the copy of $M(G/Q')$ indexed on this class. Clearly, the composite along the top and right of the diagram must be zero on this copy by the definition of $\tilde{\lambda}_Q$. Assume that δ takes the identity coset eD of G/D to the coset gH of G/H. Then $g \notin H$ since H/D doesn't embed in U and the equivalence class $(\delta; 1_{G/D}, \beta)$ is assumed to give a generator of $[G/H, G/D \times G/Q]$. The image of this generator of $[G/H, G/D \times G/Q]$ under the map

$$(1 \times f)_* : [G/H, G/D \times G/Q] \longrightarrow [G/H, G/D \times G/Q']$$

is given by the double coset formula, and is a sum of generators represented by diagrams of the form

$$G/H \xleftarrow{\delta \circ \pi^J_H} G/J \xrightarrow{(\pi^J_H, \beta)} G/D \times G/Q',$$

in which J is a subgroup of D. The composite $\delta \circ \pi^J_H$ appearing in this diagram cannot be π^J_H since it must take eJ to gH. But then $\tilde{\lambda}_{Q'}$ kills the copies of $M(G/Q')$ in its domain indexed on these generators of $[G/H, G/D \times G/Q']$. It follows that, on the $(\delta; 1_{G/D}, \beta)$-indexed copy of $M(G/Q')$, the composite along the left and bottom of the diagram also vanishes.

Now let us assume that $(J)_G = (C)_G$. Since $J \le D$ and C is normal in D, either $J = C$ or $(J)_D \ne (C)_D$. If $(J)_D \ne (C)_D$, then the definitions of $\tilde{\lambda}_Q$ and $\tilde{\lambda}_{Q'}$ easily imply that both of the composites in the diagram vanish on the $(\delta; \pi^J_D, \beta)$-indexed copy of $M(G/Q')$. Thus, we may assume that $J = C$. If $\delta \ne \pi^C_D$, then the definitions of $\tilde{\lambda}_Q$ and $\tilde{\lambda}_{Q'}$ again imply that both of the composites in the diagram vanish on the copy of $M(G/Q')$ in question. Thus, we need only consider the copy of $M(G/Q')$ indexed on the generator $(\pi^C_H; \pi^C_D, \beta)$. On this copy, simple double coset arguments indicate that the two composites in the diagram are equal. In these arguments, the fact that M is concentrated over C is used to ensure the vanishing of certain contributions to the composite along the top and right which might otherwise not vanish.

We have now constructed the map $\lambda : (\mathcal{H}_{G/D} \Box M)(G/H) \longrightarrow \mathfrak{Z}_C(M)$ for an arbitrary M in $M^C_G(U)$. It is easy to see that λ must be natural in M. We must still show that λ is naturally split. There is an obvious map $\tilde{\sigma}$ of $M(G/C)$ into $(\mathcal{H}_{G/D} \Box M)(G/H)$ coming from the inclusion

$$M(G/C) \subset \bigoplus_{G/Q} M(G/Q) \otimes [G/H, G/D \times G/Q]$$

associated to the copy of $M(G/C)$ in $M(G/C)\otimes[G/H, G/D\times G/C]$ indexed on the generator represented by the diagram

$$G/H \xleftarrow{\pi_H^C} G/C \xrightarrow{(\pi_D^C, 1_{G/C})} G/D\times G/C.$$

It follows directly from the definition of λ that the composite $\lambda\circ\tilde{\sigma}$ is just the projection $M(G/C)\longrightarrow M(G/C)/W_DC = \mathfrak{Z}_C(M)$. Moreover, the map $\tilde{\sigma}$ is obviously natural in M. Thus, it suffices to show that the map $\tilde{\sigma}$ factors through the projection $M(G/C)\longrightarrow M(G/C)/W_DC = \mathfrak{Z}_C(M)$ to give a map

$$\sigma : \mathfrak{Z}_C(M)\longrightarrow (\mathcal{H}_{G/D}\,\square\, M)(G/H).$$

For each $Q\leq G$, let

$$\nu_Q : M(G/Q)\otimes[G/H, G/D\times G/Q]\longrightarrow (\mathcal{H}_{G/D}\,\square\, M)(G/H)$$

be the map obtained from our description of $(\mathcal{H}_{G/D}\square M)(G/H)$ as a quotient group of $\bigoplus_{G/Q} M(G/Q)\otimes[G/H, G/D\times G/Q]$. Then, for any morphism $f : G/Q\longrightarrow G/Q'$ in $\mathcal{O}_G(U)$, the diagram

$$\begin{tikzcd}
M(G/Q')\otimes[G/H, G/D\times G/Q] \arrow[r, "M(f)\otimes 1"] \arrow[d, "1\otimes(1\times f)_*"'] & M(G/Q)\otimes[G/H, G/D\times G/Q] \arrow[d, "\nu_Q"] \\
M(G/Q')\otimes[G/H, G/D\times G/Q'] \arrow[r, "\nu_{Q'}"] & (\mathcal{H}_{G/D}\,\square\, M)(G/H)
\end{tikzcd}$$

commutes. Take $Q = Q' = C$ in this diagram, and consider the generators f in $\mathcal{O}_G(U)(G/C, G/C)$ which give the action of W_DC on $M(G/C)$. It follows immediately that $\tilde{\sigma}$ does factor through the indicated projection to give the desired splitting map σ for λ. $\square$

8. Mackey functors for compact Lie groups and the proof of Theorem 6.5

Throughout this section, G is a compact Lie group, U is a complete G-universe, and $C \leq D \leq H$ are subgroups of G satisfying the conditions of Theorem 6.5. Following the general program outlined in section 7, we show that $\mathcal{H}_{G/D}$ is not flat in $\mathfrak{M}_G$ by showing that the functor sending M in $\mathfrak{M}_G$ to $(\mathcal{H}_{G/D} \,\square\, M)(G/H)$ does not preserve monomorphisms. In the proof, the domain of this functor is restricted to the full subcategory M_G^C of $\mathfrak{M}_G$ consisting of the objects which are concentrated over C. This restricted functor has a natural direct summand which is easily computed and can be shown not to preserve monomorphisms.

Recall from Definition 2.1(b) of [21] that the Brauer quotient $br_D M$ of an object M in $\mathfrak{M}_G$ at D is defined by the exact sequence

$$\bigoplus_K M(G/K) \xrightarrow{\oplus M(\tau_D^K)} M(G/D) \longrightarrow br_D M \longrightarrow 0.$$

Here, the direct sum is indexed on the proper subgroups K of D for which $W_D K$ is finite. This Brauer quotient is the natural direct summand that is to be split off from our functor.

Proposition 8.1. *Let G be a compact Lie group, U be a complete G-universe, and $C \leq D \leq H$ be subgroups of G such that*

(i) *$W_D C$ and $W_H D$ are finite*

(ii) *for every $K < D$ such that $(C)_G \leq (K)_G$, $W_H K$ is not finite.*

Then, for M in M_G^C, $br_D M$ splits off from $(\mathcal{H}_{G/D} \,\square\, M)(G/H)$ as a natural direct summand.

Before proving this result, we show how it can be used to complete the proof of our theorem.

Proof of Theorem 6.5. Assume that $W_H D$ is finite, C is normal in D, and D/C is a finite p-group for some prime p. Also assume that, for every $K < D$ such that $(C)_G \leq (K)_G$, $W_H K$ is not finite. We must construct a monomorphism $\iota : A \longrightarrow B$ which is not preserved by the functor br_D. Define the object B of $\mathfrak{M}_G$ by the exact sequence

$$\bigoplus_{K<C} \mathcal{H}_{G/K} \xrightarrow{\oplus \tilde{\rho}_C^K} \mathcal{H}_{G/C} \xrightarrow{\phi} B \longrightarrow 0.$$

Here, the map $\tilde{\rho}_C^K : \mathcal{H}_{G/K} \longrightarrow \mathcal{H}_{G/C}$ is derived from the restriction map ρ_C^K in $\mathcal{O}_G$. By construction, B is concentrated over C. Let A in $\mathfrak{M}_G$ be the image of the composite map

$$\mathcal{H}_{G/D} \xrightarrow{\tilde{\tau}_D^C} \mathcal{H}_{G/C} \longrightarrow B,$$

in which $\tilde{\tau}_D^C$ is the map derived from the induction map τ_D^C in $\mathcal{O}_G$. Note that there are an obvious monomorphism $\iota : A \longrightarrow B$ and an obvious epimorphism $\mu : \mathcal{H}_{G/D} \longrightarrow A$. Since it is a subobject of B, A must be concentrated over C.

Using the results in section 3 of [21] about the composition in $\mathcal{O}_G$, it is easy to see that $B(G/D)$ is a free abelian group whose generators are the images under $\phi : \mathcal{H}_{G/C} \longrightarrow B$ of the induction maps $\tau(\alpha)$ in $\mathcal{H}_{G/C}(G/D) = \mathcal{O}_G(G/D, G/C)$

associated to G-maps $\alpha : G/C \longrightarrow G/D$. It follows that $br_D B$ is zero. Thus, to show that the functor br_D does not preserve monomorphisms, it suffices to show that $br_D A$ is nonzero.

Let $x \in A(G/D)$ be the image of the identity map $1_{G/D} \in \mathcal{H}_{G/D}(G/D) = \mathcal{O}_G(G/D, G/D)$ under $\mu : \mathcal{H}_{G/D} \longrightarrow A$, and let $y \in B(G/D)$ be $\iota(G/D)(x)$. It is easy to see that y is the generator of $B(G/D)$ derived from the transfer map τ_D^C in $\mathcal{H}_{G/C}(G/D)$. To show that the image of x in $br_D A$ is nonzero, it suffices to show that x is not in the image of the map

$$\bigoplus_K A(G/K) \xrightarrow{\oplus A(\tau_D^K)} A(G/D)$$

whose cokernel is $br_D A$. In fact, px is the smallest nonzero multiple of x which can appear in this image. This can be proven by showing that py is the smallest nonzero multiple of y which can appear in the image of the composite

$$\bigoplus_K A(G/K) \xrightarrow{\oplus A(\tau_D^K)} A(G/D) \xrightarrow{\iota(G/D)} B(G/D).$$

This claim about y can be proven using the commuting diagram

$$\begin{tikzcd}
\mathcal{H}_{G/D}(G/K) \arrow[r, "\mathcal{H}_{G/D}(\tau_D^K)"] \arrow[d, two heads, "\mu(G/K)"'] & \mathcal{H}_{G/D}(G/D) \arrow[r, "\tilde{\tau}_D^C"] \arrow[d, two heads, "\mu(G/D)"] & \mathcal{H}_{G/C}(G/D) \arrow[d, two heads, "\phi(G/D)"] \\
A(G/K) \arrow[r, "A(\tau_D^K)"] & A(G/D) \arrow[r, hook, "\iota(G/D)"] & B(G/D).
\end{tikzcd}$$

Let w be a generator of $\mathcal{H}_{G/D}(G/K) = \mathcal{O}_G(G/K, G/D)$, and z be its image in $B(G/D)$ under either composite in the diagram above. It suffices to show that, if z is written in terms of our standard basis for $B(G/D)$, then the coefficient of y is a multiple of p. We analyze z by first computing the element $\tilde{\tau}_D^C(\mathcal{H}_{G/D}(\tau_D^K)(w)) = \tau_D^C \circ w \circ \tau_D^K$ of $\mathcal{H}_{G/C}(G/D) = \mathcal{O}_G(G/D, G/C)$. The generator w is represented by a diagram of the form

$$G/K \xleftarrow{\alpha} G/J \xrightarrow{\beta} G/D,$$

and we can select this diagram so that $J \leq D$ and $\beta = \pi_D^J$. The composite $w \circ \tau_D^K$ is then represented by the diagram

$$G/D \xleftarrow{\pi_D^K \circ \alpha} G/J \xrightarrow{\pi_D^J} G/D.$$

Since G is a nonfinite compact Lie group, computing the composite of this map with τ_D^C would normally be rather difficult since it requires an analysis of the pullback diagram

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & G/C \arrow[d, "\pi_D^C"] \\
G/J \arrow[r, "\pi_D^J"] & G/D
\end{tikzcd}$$

in the category of G-spaces. However, it is easy to see that, if the composite $\pi_D^K \circ \alpha$ is not π_D^J, then the coefficient of y in the expression for z in terms of our basis is zero. Further, if J does not contain C, then the coefficient of y also has to be zero. Thus, we may assume that $\pi_D^K \circ \alpha = \pi_D^J$ and $C \leq J$. Now we can use the fact that D/C is a finite group to complete the computation. The D-orbits D/C, D/J, and D/D may be regarded as orbits of the finite group D/C. The diagram

$$\begin{tikzcd} D/J \times D/C \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & D/C \arrow[d, "\tilde{\pi}_D^C"] \\ D/J \arrow[r, "\tilde{\pi}_D^J"] & D/D \end{tikzcd}$$

is obviously a pullback in the category of D/C-sets. The functor $G\times_D ?$ preserves pullbacks, and so takes this pullback diagram into the pullback diagram we must analyze in the category of G-spaces. Since $D/J \times D/C$ decomposes into a disjoint union of finitely many D/C-orbits, the pullback P in our diagram of G-spaces is just a disjoint union of the analogous G-orbits. The free D/C-orbit D/C appears $|D/J|$ times in $D/J \times D/C$, and $|D/J| = p^n$ for some $n \geq 1$. From this, it follows that, if $\tilde{\tau}_D^C(\mathcal{H}_{G/D}(\tau_D^K)(w))$ is written in terms of the standard generators of $\mathcal{H}_{G/C}(G/D)$, then the coefficient of the generator τ_D^C is $|D/J| = p^n$. The coefficient of y in the expression for z in terms of the standard basis must then also be $|D/J| = p^n$. $\square$

Corollary 6.8 follows from Theorem 6.5 and the remarks in Example 6.7 by observing that the listed groups either contain $O(2)$ or map onto a group containing $O(2)$. In particular, the groups $O(m)$, $U(m)$, and $Sp(m)$ contain $O(2)$ for $m \geq 2$. For $n \geq 3$, the groups $SO(n)$ and $SU(n)$ also contain $O(2)$. The groups $Spin(n)$ map onto the groups $SO(n)$. The special cases $SU(2)$ and $Sp(1)$ are handled by their surjections onto $SO(3)$.

The remainder of this section is occupied by the postponed proof of our splitting result.

Proof of Proposition 8.1. Let M be in M_G^C. As in the proof of Proposition 7.2, we first construct a natural map

$$\lambda : (\mathcal{H}_{G/D} \,\square\, M)(G/H) \longrightarrow br_D M,$$

and then show that it is a split epimorphism. The map λ is, as in the previous section, derived from an appropriately behaved map

$$\tilde{\lambda} : \bigoplus_{G/Q} M(G/Q) \otimes [G/H, G/D \times G/Q] \longrightarrow br_D M.$$

As before, $\tilde{\lambda}$ is defined by specifying its restriction $\tilde{\lambda}_Q$ to the summand associated to each orbit G/Q. Moreover, $\tilde{\lambda}_Q$ can be defined by giving its restriction to the copy of $M(G/Q)$ in $M(G/Q) \otimes [G/H, G/D \times G/Q]$ associated to each generator of $[G/H, G/D \times G/Q]$. These generators correspond to equivalence classes of diagrams of the form

$$G/H \xleftarrow{\delta} G/J \xrightarrow{(\alpha,\beta)} G/D \times G/Q.$$

If the equivalence class $(\delta; \alpha, \beta)$ of such a diagram does not contain a diagram of the form

$$G/H \xleftarrow{\pi_H^D} G/D \xrightarrow{(1_{G/D},\beta)} G/D \times G/Q,$$

then the restriction $\tilde{\lambda}_{(\delta;\alpha,\beta)}$ to the associated copy of $M(G/Q)$ is the zero map. On the other hand, the map $\tilde{\lambda}_{(\pi_H^D;1_{G/D},\beta)}$ is defined to be the composite

$$M(G/Q) \xrightarrow{M(\rho(\beta))} M(G/D) \longrightarrow br_D M.$$

To show that λ can be derived from $\tilde{\lambda}$, it suffices to show that, for each morphism $f : G/Q \longrightarrow G/Q'$ in $\mathcal{O}_G(U)$, the diagram

$$\begin{tikzcd}
M(G/Q')\otimes[G/H, G/D\times G/Q] \arrow[r, "M(f)\otimes 1"] \arrow[d, "1\otimes(1\times f)_*"'] & M(G/Q)\otimes[G/H, G/D\times G/Q] \arrow[d, "\tilde{\lambda}_Q"] \\
M(G/Q')\otimes[G/H, G/D\times G/Q'] \arrow[r, "\tilde{\lambda}_{Q'}"] & br_D M
\end{tikzcd}$$

commutes. Again, we need only check the cases in which f is either a restriction map $\rho(\beta)$ or an induction map $\tau_Q^{Q'}$.

As in the previous proof, the commutativity of the diagram is purely formal if f is a restriction map $\rho(\beta)$. Thus, we assume that f is an induction map $\tau_Q^{Q'}$. The commutativity of the diagram can be verified by checking it on each summand $M(G/Q')$ of $M(G/Q')\otimes[G/H, G/D\times G/Q]$. On the summand associated to an equivalence class that does not contain a diagram of the form

$$G/H \xleftarrow{\pi_H^D} G/D \xrightarrow{(1_{G/D},\beta)} G/D\times G/Q,$$

it is easy to see that both composites in the diagram are zero. Thus, we restrict our attention to the summands indexed on equivalence classes of the form $(\pi_H^D; 1_{G/D}, \beta)$.

To compute the composite along the top and right side of the rectangle on such a summand, it is necessary to commute the composite in $\mathcal{O}_G$ of f and the map $\rho(\beta) : G/D \longrightarrow G/Q$. This composite is a sum of generators of $\mathcal{O}_G(G/D, G/Q')$ represented by diagrams of the form

$$G/D \xleftarrow{\pi_D^{K_i}} G/K_i \xrightarrow{\zeta_i} G/Q',$$

in which $K_i \leq D$. Because the target of the composite along the top and right side is $br_D M$, the generators for which $K_i \neq D$ can be ignored. The implications of the remaining terms for the composite along the top and right are easily understood.

To understand the composite along the left side and bottom of the rectangle, it is necessary to compute the image of the generator associcated to the equivalence class $(\pi_H^D; 1_{G/D}, \beta)$ under the map

$$(1\times f)_* : [G/H, G/D\times G/Q] \longrightarrow [G/H, G/D\times G/Q'].$$

By Lemma 3.3 of [21], this image is the sum of generators represented by diagrams of the form

$$G/H \xleftarrow{\pi_H^{K_i}} G/K_i \xrightarrow{(\pi_D^{K_i},\zeta_i)} G/D\times G/Q',$$

in which exactly the same subgroups K_i and the same morphisms ζ_i appear as those encountered in our examination of the other composite. By the definition of the map $\tilde{\lambda}_{Q'}$, the generators for which $K_i \neq D$ can be ignored. The implications of the remaining terms for the composite along the right and bottom are easily understood, and it follows that two composites in the diagram are equal. Thus, the desired map λ can be obtained from the defined map $\tilde{\lambda}$. Clearly, λ is natural in $M \in M_G^C$.

It is still necessary to show that λ is naturally split. There is an obvious map

$$\tilde{\sigma} : M(G/D) \longrightarrow (\mathcal{H}_{G/D} \,\square\, M)(G/H)$$

derived from the inclusion

$$M(G/D) \subset \bigoplus_{G/Q} M(G/Q) \otimes [G/H, G/D \times G/Q]$$

associated to the copy of $M(G/D)$ in $M(G/D) \otimes [G/H, G/D \times G/D]$ indexed on the equivalence class $(\pi_H^D; 1_{G/D}, 1_{G/D})$. Clearly, the composite $\lambda \circ \tilde{\sigma}$ is just the canonical projection $M(G/D) \longrightarrow br_D M$. Thus, to show that λ is a split epimorphism, it suffices to show that the map $\tilde{\sigma}$ factors through the projection $M(G/D) \longrightarrow br_D M$. This is equivalent to showing that, for each $K < D$ such that $W_D K$ is finite, the composite

$$M(G/K) \xrightarrow{M(\tau_D^K)} M(G/D) \xrightarrow{\tilde{\sigma}} (\mathcal{H}_{G/D} \,\square\, M)(G/H)$$

is zero. Unless $(C)_G \leq (K)_G$, $M(G/K)$ is zero, and the composite is trivially zero. Thus, we assume that $(C)_G \leq (K)_G$.

For each $Q \leq G$, let

$$\nu_Q : M(G/Q) \otimes [G/H, G/D \times G/Q] \longrightarrow (\mathcal{H}_{G/D} \,\square\, M)(G/H)$$

be the map obtained from our description of $(\mathcal{H}_{G/D} \,\square\, M)(G/H)$ as a quotient group of $\bigoplus_{G/Q} M(G/Q) \otimes [G/H, G/D \times G/Q]$. The diagram

$$\begin{array}{ccc}
M(G/K) \otimes [G/H, G/D \times G/D] & \xrightarrow{M(\tau_D^K) \otimes 1} & M(G/D) \otimes [G/H, G/D \times G/D] \\
\Big\downarrow {\scriptstyle 1 \otimes (1 \times \tau_D^K)_*} & & \Big\downarrow {\scriptstyle \nu_D} \\
M(G/K) \otimes [G/H, G/D \times G/K] & \xrightarrow{\quad \nu_K \quad} & (\mathcal{H}_{G/D} \,\square\, M)(G/H)
\end{array}$$

commutes by the definition of $(\mathcal{H}_{G/D} \,\square\, M)(G/H)$. The composite $\tilde{\sigma} \circ M(\tau_D^K)$ can be obtained by restricting the composite along the top and right hand side of the rectangle to the copy of $M(G/K)$ indexed on the generator of $[G/H, G/D \times G/D]$ associated to the equivalence class $(\pi_H^D; 1_{G/D}, 1_{G/D})$. The image of this generator under the map

$$(1 \times f)_* : [G/H, G/D \times G/D] \longrightarrow [G/H, G/D \times G/K].$$

is the element of $[G/H, G/D \times G/K]$ that should be represented by the diagram

$$G/H \xleftarrow{\pi_H^K} G/K \xrightarrow{(\pi_D^K, 1_{G/K})} G/D \times G/K.$$

However, by assumption, $(C)_G \leq (K)_G$ and so $W_H K$ is not finite. Thus, this diagram represents not a generator of $[G/H, G/D \times G/K]$, but the zero element. The composite along the left and bottom of the rectangle must then restrict to zero on the appropriate copy of $M(G/K)$. Thus, the composite $\tilde{\sigma} \circ M(\tau_D^K)$ is zero. It follows that the map $\tilde{\sigma} : M(G/D) \longrightarrow (\mathcal{H}_{G/D} \,\square\, M)(G/H)$ induces a map

$$\sigma : br_D M \longrightarrow (\mathcal{H}_{G/D} \,\square\, M)(G/H)$$

which splits λ. $\square$

9. Mackey functors for compact Lie groups and the proof of Theorem 6.9

Throughout this section, G is a compact Lie group, and U is a G-universe. Here we convert the question of whether $\mathfrak{M}_G(U)$ satisfies **TPPP** and **PiIP** into a question in equivariant stable homotopy theory. This conversion is used here to prove Theorem 6.9 by showing that $\mathfrak{M}_G$ fails to satisfy **TPPP** and **PiIP** if G is either the orthogonal group $O(m)$, for $m \geq 2$, or the special orthogonal group $SO(n)$, for $n \geq 3$. It is used in the next section to show that, if $G = S^1$, then $\mathfrak{M}_G$ does satisfy **TPPP** and **PiIP**.

The category $\mathfrak{M}_G(U)$ satisfies **TPPP** and **PiIP** if and only if, for every subgroup H of G, the functor $\langle \mathcal{H}_{G/H}, ?\rangle : \mathfrak{M}_G(U) \longrightarrow \mathfrak{M}_G(U)$ preserve epimorphisms. Associated to any short exact sequence

$$0 \longrightarrow M' \xrightarrow{\iota} M \xrightarrow{\epsilon} M'' \longrightarrow 0$$

in $\mathfrak{M}_G(U)$, there is a fibre sequence

$$K(M',0) \xrightarrow{\tilde{\iota}} K(M,0) \xrightarrow{\tilde{\epsilon}} K(M'',0)$$

of zero-dimensional equivariant Eilenberg-Mac Lane spectra in the stable category of G-spectra indexed on U. Proposition 5.2 of [21] allows us to use the long exact homotopy sequence derived from this fibre sequence to investigate the exactness of the functor $\langle \mathcal{H}_{G/H}, ?\rangle$.

That proposition gives an isomorphism

$$\langle \mathcal{H}_{G/H}, M\rangle(G/J) \cong [G/J_+ \wedge G/H_+, K(M,0)]_G$$

which is natural in $M \in \mathfrak{M}_G(U)$ and G/H, $G/J \in \mathcal{O}_G(U)$. Here, $[\,,\,]_G$ is used to denote the morphism sets in the G-stable homotopy category. This isomorphism identifies the map

$$\langle \mathcal{H}_{G/H}, M\rangle(G/J) \xrightarrow{\epsilon_*(G/J)} \langle \mathcal{H}_{G/H}, M''\rangle(G/J),$$

which must be an epimorphism if $\mathfrak{M}_G$ is to satisfy **TPPP** and **PiIP**, with the map

$$[G/J_+ \wedge G/H_+, K(M,0)]_G \xrightarrow{\tilde{\epsilon}_*} [G/J_+ \wedge G/H_+, K(M'',0)]_G.$$

From the long exact homotopy sequence, it follows that $\tilde{\epsilon}_*$ is an epimorphism if and only if the map

$$[G/J_+ \wedge G/H_+, \Sigma K(M',0)]_G \xrightarrow{\tilde{\iota}_G} [G/J_+ \wedge G/H_+, \Sigma K(M,0)]_G$$

is a monomorphism. The change of group isomorphisms given in section II.4 of [23] identify this map with the map

$$[G/H_+, \Sigma K(M',0)]_J \xrightarrow{\tilde{\iota}_J} [G/H_+, \Sigma K(M,0)]_J.$$

We have now reformulated the question of whether $\mathfrak{M}_G(U)$ satisfies **TPPP** and **PiIP** into the form needed for our discussion of S^1 in section 10.

Proposition 9.1. *Let G be a compact Lie group, and U be a G-universe. Then the category $\mathfrak{M}_G(U)$ satisfies* **TPPP** *and* **PiIP** *if and only if, for every monomorphism $\iota : M' \longrightarrow M$ in $\mathfrak{M}_G(U)$ and every pair H, J of subgroups of G, the map*

$$[G/H_+, \Sigma K(M',0)]_J \xrightarrow{\tilde{\iota}_J} [G/H_+, \Sigma K(M,0)]_J.$$

induced by ι is a monomorphism.

For the proof of Theorem 6.9, it suffices to exhibit a monomorphism $\iota : M' \longrightarrow M$ in $\mathfrak{M}_G$ and two subgroups H and J of G for which the map $\tilde{\iota}_J$ is not a monomorphism. It is possible to select these subgroups H and J so that $H \geq J$. For such a pair of subgroups, the identity coset eH of G/H is J-invariant and so provides a basepoint for G/H regarded as a J-space. The existence of this basepoint implies that G/H_+ is equivalent to $G/H \vee S^0$ in the J-stable category. Thus,

$$\begin{aligned} [G/H_+, \Sigma K(M',0)]_J &\cong [G/H \vee S^0, \Sigma K(M',0)]_J \\ &\cong [G/H, \Sigma K(M',0)]_J \oplus [S^0, \Sigma K(M',0)]_J. \end{aligned}$$

Similar observations apply to $[G/H_+, \Sigma K(M,0)]_J$. The dimension axiom implies that $[S^0, \Sigma K(M',0)]_J$ and $[S^0, \Sigma K(M,0)]_J$ are both zero. The map which we wish to show is not a monomorphism can therefore be identified with the map

$$[G/H, \Sigma K(M',0)]_J \xrightarrow{\hat{\iota}_J} [G/H, \Sigma K(M,0)]_J.$$

The task of showing that $\mathfrak{M}_G$ fails to satisfy **TPPP** and **PiIP** is now reduced to the two problems of analyzing the G-orbit G/H as a J-space so that the map $\hat{\iota}_J$ can be understood and of selecting a short exact sequence in $\mathfrak{M}_G$ for which this map is not a monomorphism. We address these two problems only for the case in which G is either $O(n)$ or $SO(n)$, and the universe U is complete.

If G is either $O(n)$ or $SO(n)$, then the subgroup J in the analysis above can be taken to be H. For either choice of G, our argument employs an appropriately chosen proper subgroup K of H. If $G = O(n)$, then $H = O(n-1)$ and $K = O(n-2)$. If $G = SO(n)$, then $H = SO(n-1)$ and $K = SO(n-2)$. Here, $O(1) = \mathbb{Z}/2$, and $O(0) = SO(1) = e$. For both $O(n)$ and $SO(n)$, standard geometry gives nonequivariant homeomorphisms $G/H \cong S^{n-1}$ and $H/K \cong S^{n-2}$. Moreover, for either choice of G, the G-orbit G/H, considered as an H-space, may be identified with the unreduced suspension $S(H/K)$ of the H-orbit H/K. Under this identification, the map $\hat{\iota}_J$ becomes the map

$$[S(H/K), \Sigma K(M',0)]_H \xrightarrow{\hat{\iota}} [S(H/K), \Sigma K(M,0)]_H.$$

The unreduced suspension $S(H/K)$ fits into a cofibre sequence

$$S^0 \longrightarrow S(H/K) \longrightarrow \Sigma H/K_+ \longrightarrow S^1$$

of H-spaces. From this cofibre sequence, we obtain the diagram

$$\begin{array}{ccc}
[S^1, \Sigma K(M',0)]_H & \xrightarrow{\quad \iota'' \quad} & [S^1, \Sigma K(M,0)]_H \\
\downarrow & & \downarrow \theta \\
[\Sigma H/K_+, \Sigma K(M',0)]_H & \xrightarrow{\iota'} & [\Sigma H/K_+, \Sigma K(M,0)]_H \\
\downarrow & & \downarrow \\
[S(H/K), \Sigma K(M',0)]_H & \xrightarrow{\hat{\iota}} & [S(H/K), \Sigma K(M,0)]_H \\
\downarrow & & \downarrow \\
[S^0, \Sigma K(M',0)]_H & & [S^0, \Sigma K(M,0)]_H \\
\| & & \| \\
0 & & 0
\end{array}$$

in which the columns are exact sequences derived from our cofibre sequence. Consider the pullback diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & {[S^1, \Sigma K(M,0)]_H} \arrow[d, "\theta"] \\ {[\Sigma H/K_+, \Sigma K(M',0)]_H} \arrow[r, "\iota'"] & {[\Sigma H/K_+, \Sigma K(M,0)]_H} \end{tikzcd}$$

associated with the top rectangle of the larger diagram. The commutativity of the top rectangle in the larger diagram implies the existence of a map

$$\phi : [S^1, \Sigma K(M',0)]_H \longrightarrow P.$$

Assume that the map θ in these two diagrams is a monomorphism. A simple diagram chase then gives that the map $\hat{\iota}$ of interest to us fails to be a monomorphism if and only if the map ϕ fails to be an epimorphism.

The change of group isomorphisms from section II.4 of [23] allow us to identify the top rectangle of our larger diagram with the diagram

$$\begin{tikzcd} M'(G/H) \arrow[r, "\iota(G/H)"] \arrow[d, "M'(\rho_H^K)"'] & M(G/H) \arrow[d, "M(\rho_H^K)"] \\ M'(G/K) \arrow[r, "\iota(G/K)"] & M(G/K). \end{tikzcd}$$

We can think of P as the pullback associated with this rectangle and of ϕ as a map from $M'(G/H)$ into P. Our geometric argument has therefore reduced the question of whether the functor $\langle \mathcal{H}_{G/H}, ? \rangle : \mathfrak{M} \longrightarrow \mathfrak{M}$ preserves epimorphisms to the question of whether there exists a monomorphism $M' \longrightarrow M$ in $\mathfrak{M}_G$ for which the map $M(\rho_H^K) : M(G/H) \longrightarrow M(G/K)$ is a monomorphism and the map $\phi : M'(G/H) \longrightarrow P$ is not an epimorphism.

Let M be the Mackey functor $\underline{Z}$ given by Proposition V.9.10 of [23]. For any subgroup J of G, $M(G/J) = \mathbb{Z}$. Further, for $L \leq J$, then the restriction map $M(\rho_J^L) : M(G/J) \longrightarrow M(G/L)$ is the identity map, and the induction map $M(\tau_J^L) : M(G/L) \longrightarrow M(G/J)$ is multiplication by the nonequivariant Euler characteristic $\chi(J/L)$. Let M' be the image of the canonical map $\mathcal{H}_{G/K} \square M \longrightarrow M$. By Proposition 5.5 of [21], the map $(\mathcal{H}_{G/K} \square M)(G/K) \longrightarrow M(G/K)$ is a split epimorphism. Therefore, $M'(G/K) = M(G/K)$, and the map $\iota(G/K)$ is the identity map. The image of the map $(\mathcal{H}_{G/K} \square M)(G/H) \longrightarrow M(G/H)$ is contained in the image of the transfer map $M(\tau_H^K) : M(G/K) \longrightarrow M(G/H)$ by Proposition 5.7 of [21]. This transfer map is multiplication by the nonequivariant Euler characteristic of $H/K \cong S^{n-2}$, which is either 2 or 0. Thus, $M'(G/H)$ is either $\mathbb{Z}$ or 0. If $M'(G/H) = \mathbb{Z}$, then the map $\iota(G/H) : M'(G/H) \longrightarrow M(G/H)$ is multiplication by 2. In either case, the map $\phi : M'(G/H) \longrightarrow P$ is obviously not an epimorphism. Thus, the functor $\langle \mathcal{H}_{G/H}, ? \rangle : \mathfrak{M} \longrightarrow \mathfrak{M}$ does not preserve epimorphisms, and the category $\mathfrak{M}_G$ satisfies neither **TPPP** nor **PiIP**.

10. S^1-Mackey functors and the proof of Theorem 3.8

To prove Theorem 3.8, we must show that $\mathfrak{M}_{S^1}$ satisfies all six of the axioms **PiF**, **PiIP**, **IiII**, **IPiP**, **IIiI**, and **TPPP**. It follows immediately from Proposition 3.1 that it satisfies **IPiP** and **IIiI**. Tools from equivariant stable homotopy theory are used to show that it satisfies the other four axioms. Proposition 9.1 provides the means of applying these tools to the question of whether $\mathfrak{M}_{S^1}$ satisfies **TPPP** and **PiIP**. This section begins with an analogous proposition providing a method for applying these tools to the question of whether the category $\mathfrak{M}_G(U)$ associated to a compact Lie group G and a G-universe U satisfies **PiF** and **IiII**. After proving this proposition, we specialize to the case in which $G = S^1$ and U is a complete G-universe, and prove Theorem 3.8.

As in section 9, we assume here that $\iota : M' \longrightarrow M$ is a monomorphism in $\mathfrak{M}_G(U)$, and $\tilde{\iota} : K(M', 0) \longrightarrow K(M, 0)$ is the induced map between the equivariant Eilenberg-Mac Lane spectra associated to M' and M.

Proposition 10.1. *Let G be a compact Lie group, and U be a G-universe. Then the category $\mathfrak{M}_G(U)$ satisfies* **PiF** *and* **IiII** *if and only if, for every monomorphism $\iota : M' \longrightarrow M$ in $\mathfrak{M}_G(U)$ and every pair H, J of subgroups of G, the map*

$$[S^0, G/H_+ \wedge K(M', 0)]_J \xrightarrow{\check{\iota}_J} [S^0, G/H_+ \wedge K(M, 0)]_J$$

induced by ι is a monomorphism.

Proof. To prove that $\mathfrak{M}_G(U)$ satisfies **PiF** and **IiII**, it suffices to show that, for every monomorphism $\iota : M' \longrightarrow M$ in $\mathfrak{M}_G(U)$ and every pair H and J of subgroups of G, the map

$$(1 \,\square\, \iota)(G/J) : (\mathcal{H}_{G/H} \,\square\, M')(G/J) \longrightarrow (\mathcal{H}_{G/H} \,\square\, M)(G/J)$$

is a monomorphism. Proposition 5.2 of [21] provides an isomorphism

$$(\mathcal{H}_{G/H} \,\square\, M)(G/J) \cong [G/J_+, G/H_+ \wedge K(M, 0)]_G$$

which is natural in $M \in \mathfrak{M}_G(U)$ and G/H, $G/J \in \mathcal{O}_G(U)$. Under this isomorphism, the map $(1 \,\square\, \iota)(G/J)$ is identified with the map

$$[G/J_+, G/H_+ \wedge K(M', 0)]_G \xrightarrow{\check{\iota}_G} [G/J_+, G/H_+ \wedge K(M, 0)]_G$$

induced by the map $\tilde{\iota} : K(M', 0) \longrightarrow K(M, 0)$. The change of group isomorphisms given in section II.4 of [23] allow us to identify the map $\check{\iota}_G$ with the map $\check{\iota}_J$ of the proposition. $\square$

Henceforth, we assume that $G = S^1$ and that the universe U is complete. The remainder of this section is devoted to the proofs of the two parts of Theorem 3.8.

Proof of Theorem 3.8(a). Propositions 3.1, 9.1 and 10.1 reduce the proof of this part of the theorem to showing that, for every monomorphism $\iota : M' \longrightarrow M$ in $\mathfrak{M}_G$ and every pair H and J of subgroups of $G = S^1$, the two maps

$$[G/H_+, \Sigma K(M', 0)]_J \xrightarrow{\check{\iota}_J} [G/H_+, \Sigma K(M, 0)]_J.$$

and

$$[S^0, G/H_+ \wedge K(M', 0)]_J \xrightarrow{\check{\iota}_J} [S^0, G/H_+ \wedge K(M, 0)]_J$$

are monomorphisms. For $H = G$, the dimension axiom indicates that the morphism set $[G/G_+, \Sigma K(M', 0)]_J$, and the analogous set for M, are both zero. Thus, $\tilde{\iota}_J$ is trivially a monomorphism. Further, the isomorphism

$$[S^0, G/G_+ \wedge K(M', 0)]_J \cong M(G/J),$$

and the analogous isomorphism for M, identify the map $\breve{\iota}_J$ with the map

$$\iota(G/J) : M'(G/J) \longrightarrow M(G/J),$$

which is assumed to be a monomorphism. Thus, we may assume that $H \neq G$.

If $J = G$ and $H \neq G$, then both $\tilde{\iota}_J$ and $\breve{\iota}_J$ have vanishing range and domain, and so are monomorphisms. For $\tilde{\iota}_J$, this follows directly from the dimension axiom. To see this for $\breve{\iota}_J$, note that in the equivariant stable category $G/H_+ \wedge K(M, 0)$ can be identified with a G-CW spectrum whose zero skeleton is a wedge $\vee_i G/K^i_+$ of orbits G/K^i associated to certain subgroups K^i of H. A simple connectivity argument gives that the map

$$\bigoplus_i [S^0, G/K^i_+]_G \cong [S^0, \vee_i G/K^i_+]_G \longrightarrow [S^0, G/H_+ \wedge K(M, 0)]_G$$

induced by the inclusion of this zero skeleton is an epimorphism. Thus, to show that $[S^0, G/H_+ \wedge K(M, 0)]_G$ is zero, it suffices to argue that $[S^0, G/K^i_+]_G$ is zero. Corollary 3.2 of [21] gives that $[S^0, G/K^i_+]_G = \mathcal{O}_G(G/G, G/K^i)$ is a free abelian group whose generators are equivalence classes of diagrams of the form

$$G/G \xleftarrow{\alpha} G/L \xrightarrow{\beta} G/K^i$$

in which $W_G L$ is finite. However, in any such diagram, $L \leq K^i \leq H$. Since $H \neq G$, L must be a finite cyclic group, and so $W_G L$ cannot be finite. Thus, there are no generators, and $\mathcal{O}_G(G/G, G/K^i) = 0$. Analogously, the domain of the map $\breve{\iota}_J$ is zero.

We can now assume that neither H nor J is equal to G. If $J \leq H$, then the identity coset eH of G/H is J-invariant, and so provides G/H, considered as a J-space, with a basepoint. Thus, in the J-stable category, $G/H_+ \cong G/H \vee S^0 = S^1 \vee S^0$, where S^1 is assumed to have trivial J-action. This decomposition of G/H_+ and the dimension axiom provide isomorphisms

$$[G/H_+, \Sigma K(M, 0)]_J \cong [S^1, \Sigma K(M, 0)]_J = M(G/J)$$

and

$$[S^0, G/H_+ \wedge K(M, 0)]_J \cong [S^0, S^0 \wedge K(M, 0)]_J = M(G/J).$$

These isomorphisms, and the analogous ones for M', identify the maps $\tilde{\iota}_J$ and $\breve{\iota}_J$ with the map $\iota(G/J) : M'(G/J) \longrightarrow M(G/J)$, which is assumed to be a monomorphism.

We can now assume that $H, J \neq G$ and $J \not\leq H$. Thus, $J \cap H$ is a proper subgroup of J. In this case, G/H, considered as a J-space, can be identified with the unit sphere $S\xi$ of an irreducible complex representation ξ of J whose kernel is $J \cap H$. The inclusion of any J-orbit into $G/H = S\xi$ yields a cofibre sequence

$$J/(J \cap H)_+ \longrightarrow G/H_+ \longrightarrow \Sigma J/(J \cap H)_+.$$

The next map $\Sigma J/(J \cap H)_+ \longrightarrow \Sigma J/(J \cap H)_+$ in this sequence is the difference of the identity map of $\Sigma J/(J \cap H)_+$ and the map $\bar{j} : \Sigma J/(J \cap H)_+ \longrightarrow \Sigma J/(J \cap H)_+$ given by multiplication by an appropriate generator j of J (see, for example, the proof of Lemma A.1 in the appendix of [17]). To understand the map $\tilde{\iota}_J$ in this context, consider the exact sequence

$$\begin{array}{rcl} [\Sigma J/(J\cap H)_+, \Sigma K(M,0)]_J & \xrightarrow{(1-\bar{j})^*} & [\Sigma J/(J\cap H)_+, \Sigma K(M,0)]_J \\ & \longrightarrow & [G/H_+, \Sigma K(M,0)]_J \\ & \longrightarrow & [J/(J\cap H)_+, \Sigma K(M,0)]_J = 0 \end{array}$$

derived from our cofibre sequence. The last group in this sequence is zero by the dimension axiom. The first map in this sequence can be identified with the map

$$M(G/(J\cap H)) \xrightarrow{1-\tilde{j}} M(G/(J\cap H)),$$

in which the map $\tilde{j}$ is given by the action of j, considered as an element of the Weyl group $W_G(J\cap H)$, on $M(G/(J\cap H))$. Since $W_G(J\cap H) = S^1$ is connected, it acts trivially on $M(G/(J\cap H))$. Thus, the map $(1-\bar{j})^*$ in the exact sequence above is trivial, and there is an isomorphism

$$[G/H_+, \Sigma K(M,0)]_J \cong [\Sigma J/(J\cap H)_+, \Sigma K(M,0)]_J = M(G/(J\cap H)).$$

This isomorphism, and the analogous isomorphism for M', identify the map $\tilde{\iota}_J$ with the map $\iota(G/(J\cap H)) : M'(G/(J\cap H)) \longrightarrow M(G/(J\cap H))$, which is assumed to be a monomorphism.

The cofibre sequence of the inclusion $J/(J\cap H)_+ \longrightarrow G/H_+$ can also be used to analyze the map $\breve{\iota}_J$ whenever $H, J \neq G$ and $J \not\leq H$. For this analysis, consider the exact sequence

$$\begin{array}{rcl} [S^0, J/(J\cap H)_+ \wedge K(M,0)]_J & \xrightarrow{((1-\bar{j})^1)_*} & [S^0, J/(J\cap H)_+ \wedge K(M,0)]_J \\ & \longrightarrow & [S^0, G/H_+ \wedge K(M,0)]_J \\ & \longrightarrow & [S^0, \Sigma J/(J\cap H)_+ \wedge K(M,0)]_J = 0, \end{array}$$

in which the last term is zero by the dimension axiom. Since the group J is finite, equivariant Spanier-Whitehead duality provides an identification of the group $[S^0, J/(J\cap H)_+ \wedge K(M,0)]_J$ with the group $[J/(J\cap H)_+, K(M,0)]_J$. Using this identification, the first map in our exact sequence can be identified, as in the previous exact sequence, with the map

$$M(G/(J\cap H)) \xrightarrow{1-\tilde{j}} M(G/(J\cap H)),$$

which is known to be zero. Thus, there is an isomorphism

$$[S^0, G/H_+ \wedge K(M,0)]_J \cong [J/(J\cap H)_+, K(M,0)]_J = M(G/(J\cap H)).$$

This isomorphism, and the analogous isomorphism for M', identify the map $\breve{\iota}_J$ with the map $\iota(G/(J\cap H)) : M'(G/(J\cap H)) \longrightarrow M(G/(J\cap H))$, which is assumed to be a monomorphism. $\square$

Proof of Theorem 3.8(b). Recall that $G = S^1$. In this proof, we make use of the ideas presented in Remark 3.4. The stable orbit category $\mathcal{O}_G$ is a full subcategory of the category $\mathcal{S}'$ of G-spectra which have the G-homotopy type of finite G-CW complexes. The category $\mathcal{S}'$ is a symmetric monoidal closed category whose internal hom functor $\langle\!\langle ?, ? \rangle\!\rangle$ is derived from a duality functor. Thus, for objects X and Y in $\mathcal{S}'$,

$$X \wedge Y \cong \mathfrak{D}(\mathfrak{D}(X)) \wedge Y = \langle\!\langle \mathfrak{D}(X), Y \rangle\!\rangle \cong \langle\!\langle \langle\!\langle X, \bullet_{\mathcal{S}'} \rangle\!\rangle, Y \rangle\!\rangle.$$

It follows that, if $\mathcal{O}$ is a subcategory of $\mathcal{S}'$ which is closed under the internal hom operation on $\mathcal{S}'$, then it must also be closed under the $\wedge$-product operation on $\mathcal{S}'$.

To prove Theorem 3.8(b), it therefore suffices to show that, if $\mathcal{O}'$ is a full subcategory of the G-stable category which contains $\mathcal{O}_G$ and which is closed under smash products, then the restriction functor

$$\mathfrak{M}^{\mathrm{cont}}_{\mathcal{O}'} \longrightarrow \mathfrak{M}^{\mathrm{cont}}_{\mathcal{O}_G} = \mathfrak{M}_G$$

is not an equivalence of categories. We can regard the functors $\mathcal{H}_{G/e}$ and $\mathcal{H}_{G/e\times G/e}$ as contravariant functors out of $\mathcal{O}'$. The diagonal map $\Delta : G/e \longrightarrow G/e \times G/e$ induces a map $\tilde{\Delta} : \mathcal{H}_{G/e} \longrightarrow \mathcal{H}_{G/e\times G/e}$. Since G/e and $G/e\times G/e$ are path connected, Proposition 3.1 of [21] implies that the map

$$\tilde{\Delta}(G/H) : \mathcal{H}_{G/e}(G/H) \longrightarrow \mathcal{H}_{G/e\times G/e}(G/H)$$

is an isomorphism for $H \leq G$. However, by the change of group isomorphisms in section II.4 of [23],

$$\begin{aligned}
\mathcal{H}_{G/e}(G/e \times G/e) &= [G/e_+ \wedge G/e_+, G/e_+]_G \\
&\cong [G/e_+, G/e_+]_e \\
&\cong [S^1 \vee S^0, S^1 \vee S^0]_e \\
&\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{H}_{G/e\times G/e}(G/e \times G/e) &= [G/e_+ \wedge G/e_+, G/e_+ \wedge G/e_+]_G \\
&\cong [G/e_+, G/e_+ \wedge G/e_+]_e \\
&\cong [S^1 \vee S^0, (S^1 \vee S^0) \wedge (S^1 \vee S^0)]_e \\
&\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2.
\end{aligned}$$

Thus, $\tilde{\Delta}$ is not an isomorphism in $\mathfrak{M}^{\mathrm{cont}}_{\mathcal{O}'}$, but its restriction to $\mathfrak{M}^{\mathrm{cont}}_{\mathcal{O}_G} = \mathfrak{M}_G$ is an isomorphism. □

11. Globally defined Mackey functors and the proof of Theorem 6.10

In this section, all groups are assumed to be finite, and $\mathfrak{P}$ and $\mathfrak{Q}$ are sets of primes. In proving Theorem 6.10, we follow the general pattern described in section 7. First, we identify a natural direct summand $\mathfrak{Z}_p(M)$ of the functor sending a $(\mathfrak{P},\mathfrak{Q})$-Mackey functor M to the abelian group $(\mathcal{H}_{\mathbb{Z}/p} \,\square\, M)(\mathbb{Z}/p)$. Then we show that the functor $\mathfrak{Z}_p$ does not preserve monomorphisms. The argument here is somewhat simpler than the previous ones in that no restriction on M is needed to split off $\mathfrak{Z}_p(M)$.

Definition 11.1. Assume that the prime p is not in the set $\mathfrak{P}$ of primes, and that M is in $\mathfrak{M}_*(\mathfrak{P},\mathfrak{Q})$. Denote the trivial group by e, and the direct sum of p copies of an abelian group A by A^p. Then $\mathfrak{Z}_p(M)$ is the pushout in the diagram

$$\begin{tikzcd} M(e)^p \arrow[r, "M(\tau^e_{\mathbb{Z}/p})^p"] \arrow[d, "\nabla"'] & M(\mathbb{Z}/p)^p \arrow[d, "\kappa_M"] \\ M(e) \arrow[r, "\eta_e"] & \mathfrak{Z}_p(M). \end{tikzcd}$$

Here, ∇ is the folding map. Regard $M(\mathbb{Z}/p)^p$ as being the direct sum of a collection of copies of $M(\mathbb{Z}/p)$ indexed on the set of group endomorphisms of $\mathbb{Z}/p$. Given such an endomorphism $\gamma : \mathbb{Z}/p \longrightarrow \mathbb{Z}/p$, let $\eta_\gamma : M(\mathbb{Z}/p) \longrightarrow \mathfrak{Z}_p(M)$ be the composite

$$M(\mathbb{Z}/p) \subset M(\mathbb{Z}/p)^p \xrightarrow{\kappa_M} \mathfrak{Z}_p(M),$$

in which the first map is the inclusion of $M(\mathbb{Z}/p)$ into the direct sum as the copy indexed on γ. Analogously, the map η_e in the pushout diagram above should be thought of as the map from $M(e)$ into $\mathfrak{Z}_p(M)$ associated to the unique group homomorphism $e \longrightarrow \mathbb{Z}/p$.

Proposition 11.2. *Let p be a prime and $\mathfrak{P}$ and $\mathfrak{Q}$ be sets of primes such that $p \notin \mathfrak{P}$. Then, for all M in $\mathfrak{M}_*(\mathfrak{P},\mathfrak{Q})$, $\mathfrak{Z}_p(M)$ splits off from $(\mathcal{H}_{\mathbb{Z}/p} \,\square\, M)(\mathbb{Z}/p)$ as a natural direct summand.*

As in the previous sections, we postpone the proof of this result until after we have shown how it can be used to complete the proof of our theorem

Proof of Theorem 6.10. Let p be a prime, and let $\mathfrak{P}$ and $\mathfrak{Q}$ be sets of primes such that $p \notin \mathfrak{P}$. We must construct a monomorphism $\iota : A \longrightarrow B$ which is not preserved by the functor $\mathfrak{Z}_p$. Let B be the representable Mackey functor $\mathcal{H}_e$, and let A be the image of the map

$$\tilde{\tau}^e_{\mathbb{Z}/p} : \mathcal{H}_{\mathbb{Z}/p} \longrightarrow \mathcal{H}_e = B.$$

induced by the map $\tau^e_{\mathbb{Z}/p}$ in $\mathcal{B}_*(\mathfrak{P},\mathfrak{Q})$. Note that there are an obvious monomorphism $\iota : A \longrightarrow B$ and an obvious epimorphism $\mu : \mathcal{H}_{\mathbb{Z}/p} \longrightarrow A$.

Consider the diagram

$$\begin{tikzcd}
A(e) \arrow[dr, dashed, "\theta"'] \arrow[drr, "\iota(e)"] \arrow[ddr, "A(\tau^e_{\mathbb{Z}/p})"'] & & \\
& P \arrow[r, "\iota'"'] \arrow[d, "\tau'"] & B(e) \arrow[d, "B(\tau^e_{\mathbb{Z}/p})"] \\
& A(\mathbb{Z}/p) \arrow[r, "\iota(\mathbb{Z}/p)"] & B(\mathbb{Z}/p),
\end{tikzcd}$$

in which the rectangle is a pullback, and θ is the induced map into the pullback. We show first that the map θ is not surjective, and then use this to show that the map $\mathfrak{Z}_p(\iota) : \mathfrak{Z}_p(A) \longrightarrow \mathfrak{Z}_p(B)$ is not a monomorphism.

To show that θ is not surjective, it is necessary to compute several values of the representable functors used to define A and B. It is easy to see that the morphism sets $[\mathbb{Z}/p, e] = \mathcal{H}_{\mathbb{Z}/p}(e)$ and $[e, e] = \mathcal{H}_e(e)$ of $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ are infinite cyclic groups generated by the morphisms $\rho^e_{\mathbb{Z}/p}$ and 1_e, respectively. The map $\tilde{\tau}^e_{\mathbb{Z}/p}(e)$ takes $\rho^e_{\mathbb{Z}/p}$ to $p \cdot 1_e$.

The morphism set $[e, \mathbb{Z}/p] = \mathcal{H}_e(\mathbb{Z}/p)$ has either one or two generators, depending on whether or not p is an element of $\mathfrak{Q}$. The morphism $\tau^e_{\mathbb{Z}/p}$ is always a generator. If $p \in \mathfrak{Q}$, then there is a second generator represented by the diagram

$$e \longleftarrow \mathbb{Z}/p \xrightarrow{1_{\mathbb{Z}/p}} \mathbb{Z}/p.$$

The morphism set $[\mathbb{Z}/p, \mathbb{Z}/p] = \mathcal{H}_{\mathbb{Z}/p}(\mathbb{Z}/p)$ is generated by the composite $\tau^e_{\mathbb{Z}/p} \circ \rho^e_{\mathbb{Z}/p}$ and by the morphisms in $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$ of the form $\rho(\alpha)$, where α is an endomorphism of $\mathbb{Z}/p$. If $p \notin \mathfrak{Q}$, then α must be a nonzero endomorphism; otherwise, it can be any endomorphism. The map $\tilde{\tau}^e_{\mathbb{Z}/p}(\mathbb{Z}/p)$ takes $\tau^e_{\mathbb{Z}/p} \circ \rho^e_{\mathbb{Z}/p}$ to $p \cdot \tau^e_{\mathbb{Z}/p}$ and $\rho(\alpha)$ to $\tau^e_{\mathbb{Z}/p}$.

From these computations, it follows immediately that the diagram above has the form

$$\begin{tikzcd}
\mathbb{Z} \arrow[dr, dashed, "p"'] \arrow[drr, "p"] \arrow[ddr, "p"'] & & \\
& \mathbb{Z} \arrow[r, "1_{\mathbb{Z}}"'] \arrow[d, "1_{\mathbb{Z}}"] & \mathbb{Z} \arrow[d, "i_1"] \\
& \mathbb{Z} \arrow[r, "i_1"] & \mathbb{Z} \oplus C.
\end{tikzcd}$$

Here, C is $\mathbb{Z}$ if $p \in \mathfrak{Q}$; otherwise, $C = 0$. Also, the map i_1 is the inclusion of the first summand of a direct sum, and p denotes the map given by multiplication by p. Clearly, θ is not surjective.

Pick $x \in A(\mathbb{Z}/p)$ and $y \in B(e)$ illustrating the fact that θ isn't onto; that is, such that $\iota(\mathbb{Z}/p)(x) = B(\tau^e_{\mathbb{Z}/p})(y) = z \in B(\mathbb{Z}/p)$ but there is no element of $A(e)$ hitting both x and y under the obvious maps. The choices $x = \mu(1_{\mathbb{Z}/p})$ and $y = 1_e$ are, for example, acceptable. Consider the elements

$$(x, -x, 0, \dots, 0) \in A(\mathbb{Z}/p)^p$$
$$(y, -y, 0, \dots, 0) \in B(e)^p \qquad (z, -z, 0, \dots, 0) \in B(\mathbb{Z}/p)^p$$

in the context of the pushout diagrams defining $\mathfrak{Z}_p(A)$ and $\mathfrak{Z}_p(B)$. From the equations

$$\nabla(y, -y, 0, \dots, 0) = 0$$

and

$$B(\tau^e_{\mathbb{Z}/p})^p(y, -y, 0, \dots, 0) = (z, -z, 0, \dots, 0),$$

it follows that

$$\kappa_B(z, -z, 0, \dots, 0) = 0 \in \mathfrak{Z}_p(B).$$

However, $\kappa_A(x, -x, 0, \dots, 0)$ must be a nonzero element of $\mathfrak{Z}_p(A)$ since x and y illustrate the fact that θ isn't surjective. Since

$$\mathfrak{Z}_p(\iota)(\kappa_A(x, -x, 0, \dots, 0)) = \kappa_B(z, -z, 0, \dots, 0) = 0,$$

it follows that $\mathfrak{Z}_p$ does not preserve monomorphisms. $\square$

The rest of this section is devoted to the postponed proof of our splitting result.

Proof of Proposition 11.2. Let M be in $\mathfrak{M}_*(\mathfrak{P}, \mathfrak{Q})$. As in the proof of Proposition 7.2, we first construct a natural map

$$\lambda : (\mathcal{H}_{\mathbb{Z}/p} \,\square\, M)(\mathbb{Z}/p) \longrightarrow \mathfrak{Z}_p(M),$$

and then show that it is a split epimorphism. The map λ is, as before, derived from an appropriately behaved map

$$\tilde{\lambda} : \bigoplus_Q M(Q) \otimes [\mathbb{Z}/p \times Q, \mathbb{Z}/p] \longrightarrow \mathfrak{Z}_p(M).$$

Again it suffices to specify the restriction $\tilde{\lambda}_Q$ of $\tilde{\lambda}$ to the summand indexed on each group Q. That summand is itself a direct sum of copies of $M(Q)$ indexed on the generators of $[\mathbb{Z}/p \times Q, \mathbb{Z}/p]$. These generators correspond to equivalence classes $(\alpha, \beta; \delta)$ of diagrams of the form

$$\mathbb{Z}/p \times Q \xleftarrow{(\alpha,\beta)} J \xrightarrow{\delta} \mathbb{Z}/p,$$

in which the induced map $(\alpha, \beta, \delta) : J \longrightarrow \mathbb{Z}/p \times Q \times \mathbb{Z}/p$ is a monomorphism. Denote the restriction of $\tilde{\lambda}_Q$ to the copy of $M(Q)$ indexed on the generator associated to this diagram by $\tilde{\lambda}_{(\alpha,\beta;\delta)}$.

Some manipulation of the diagram above must be done to define $\tilde{\lambda}_{(\alpha,\beta;\delta)}$. Let $L = \delta(J)$ so that L is either e or $\mathbb{Z}/p$, and regard δ as a map into L. Since the order of the kernel of δ is not divisible by p, there is a unique map $\gamma : L \longrightarrow \mathbb{Z}/p$ making the diagram

$$\begin{tikzcd} & J \arrow[dl, "\alpha"'] \arrow[dr, "\delta"] & \\ \mathbb{Z}/p & & L \arrow[ll, "\gamma"] \end{tikzcd}$$

commute. This map is perhaps best understood by looking at a p-Sylow subgroup P of J. The map δ must restrict to an isomorphism from P to L. The map γ is the composite of the inverse of this isomorphism and the restriction of α to P.

If the kernel of β is not a $\mathfrak{Q}$-group, then define $\tilde{\lambda}_{(\alpha,\beta;\delta)}$ to be zero. Otherwise, define it to be the composite

$$M(Q) \xrightarrow{M(\rho(\beta))} M(J) \xrightarrow{M(\tau(\delta))} M(L) \xrightarrow{\eta_\gamma} \mathfrak{Z}_p(M),$$

where η_γ is the map from Definition 11.1 associated to a homomorphism γ from either e or $\mathbb{Z}/p$ into $\mathbb{Z}/p$.

To show that λ can be derived from $\tilde{\lambda}$, it suffices to show that, for each morphism $f : Q' \longrightarrow Q$ in $\mathcal{B}_*(\mathfrak{P}, \mathfrak{Q})$, the diagram

$$\begin{tikzcd}
M(Q') \otimes [\mathbb{Z}/p \times Q, \mathbb{Z}/p] \arrow[r, "M(f)\otimes 1"] \arrow[d, "1\otimes(1\times f)^*"'] & M(Q) \otimes [\mathbb{Z}/p \times Q, \mathbb{Z}/p] \arrow[d, "\tilde{\lambda}_Q"] \\
M(Q') \otimes [\mathbb{Z}/p \times Q', \mathbb{Z}/p] \arrow[r, "\tilde{\lambda}_{Q'}"] & \mathfrak{Z}_p(M)
\end{tikzcd}$$

commutes. Further, we need only check the cases in which f is a generator of the form $\rho(\xi)$ associated to a homomorphism $\xi : Q \longrightarrow Q'$ with $\mathfrak{Q}$-kernel, or of the form $\tau(\zeta)$ associated to a homomorphism $\zeta : Q' \longrightarrow Q$ with $\mathfrak{P}$-kernel. As in the previous proof, the commutativity of the diagram is purely formal if $f = \rho(\xi)$ for some homomorphism ξ. Thus, we assume that $f = \tau(\zeta)$ for some homomorphism $\zeta : Q' \longrightarrow Q$ with $\mathfrak{P}$-kernel.

We can verify the commutativity of the diagram by checking it on each summand $M(Q')$ of $M(Q') \otimes [\mathbb{Z}/p \times Q, \mathbb{Z}/p]$. On the summand indexed by a diagram

$$\mathbb{Z}/p \times Q \xleftarrow{(\alpha,\beta)} J \xrightarrow{\delta} \mathbb{Z}/p,$$

in which the map $\beta : J \longrightarrow Q$ does not have $\mathfrak{Q}$-kernel, it is easy to see that the two composites in the diagram are both zero. Thus, we restrict our attention to the summands indexed on diagrams in which β does have $\mathfrak{Q}$-kernel.

For this case, we must use Lemma 5.3 in much the same way that Lemma 3.3 of [21] is used in the proofs of Propositions 7.2 and 8.1. We also need to know that, for any group endomorphism $\gamma : \mathbb{Z}/p \longrightarrow \mathbb{Z}/p$, the diagram

$$\begin{tikzcd}
M(e) \arrow[r, "M(\tau^E_{\mathbb{Z}/p})"] \arrow[dr, "\eta_e"'] & M(\mathbb{Z}/p) \arrow[d, "\eta_\gamma"] \\
& \mathfrak{Z}_p(M)
\end{tikzcd}$$

commutes. Ensuring the commutativity of this diagram is, however, the whole point of the pushout diagram used to define $\mathfrak{Z}_p(M)$. The commutativity of our naturality diagram on the appropriate summand $M(Q')$ of $M(Q') \otimes [\mathbb{Z}/p \times Q, \mathbb{Z}/p]$ follows from these observations by an easy diagram chase. Thus, the desired map λ can be obtained from the map $\tilde{\lambda}$. Clearly, λ is natural in M.

We still must show that the map λ splits naturally. Let D be the summand of $[\mathbb{Z}/p \times \mathbb{Z}/p, \mathbb{Z}/p]$ whose generators are represented by the diagrams

$$\mathbb{Z}/p \times \mathbb{Z}/p \xleftarrow{(\gamma, 1_{\mathbb{Z}/p})} \mathbb{Z}/p \xrightarrow{1_{\mathbb{Z}/p}} \mathbb{Z}/p,$$

in which γ is an endomorphism of $\mathbb{Z}/p$. Then $D \cong \mathbb{Z}^p$, and $M(\mathbb{Z}/p) \otimes D \cong M(\mathbb{Z}/p)^p$. Let $\tilde{\sigma} : M(\mathbb{Z}/p)^p \longrightarrow (\mathcal{H}_{\mathbb{Z}/p} \,\Box\, M)(\mathbb{Z}/p)$ be the composite

$$\begin{aligned} M(\mathbb{Z}/p)^p \;\cong\; M(\mathbb{Z}/p) \otimes D \quad &\subset \quad M(\mathbb{Z}/p) \otimes [\mathbb{Z}/p \times \mathbb{Z}/p, \mathbb{Z}/p] \\ &\subset \quad \bigoplus_Q M(Q) \otimes [\mathbb{Z}/p \times Q, \mathbb{Z}/p] \\ &\longrightarrow \quad (\mathcal{H}_{\mathbb{Z}/p} \,\Box\, M)(\mathbb{Z}/p), \end{aligned}$$

in which the last map is the standard projection. It is easy to see that the composite $\lambda \circ \tilde{\sigma} : M(\mathbb{Z}/p)^p \longrightarrow \mathfrak{Z}_p(M)$ is just the projection $\kappa_M : M(\mathbb{Z}/p)^p \longrightarrow \mathfrak{Z}_p(M)$ of Definition 11.1. Thus, to show that λ is a split epimorphism, it suffices to show that the map $\tilde{\sigma}$ factors through the projection κ_M.

For each group Q, let $\nu_Q : M(Q) \otimes [\mathbb{Z}/p \times Q, \mathbb{Z}/p] \longrightarrow (\mathcal{H}_{\mathbb{Z}/p} \,\Box\, M)(\mathbb{Z}/p)$ be the map obtained from our description of $(\mathcal{H}_{\mathbb{Z}/p} \,\Box\, M)(\mathbb{Z}/p)$ as a quotient group of $\bigoplus_Q M(Q) \otimes [\mathbb{Z}/p \times Q, \mathbb{Z}/p]$. The diagram

$$\begin{tikzcd} M(e) \otimes [\mathbb{Z}/p \times \mathbb{Z}/p, \mathbb{Z}/p] \arrow[r, "M(\tau^e_{\mathbb{Z}/p}) \otimes 1"] \arrow[d, "1 \otimes (1 \times \tau^e_{\mathbb{Z}/p})^*"'] & M(\mathbb{Z}/p) \otimes [\mathbb{Z}/p \times \mathbb{Z}/p, \mathbb{Z}/p] \arrow[d, "\nu_{\mathbb{Z}/p}"] \\ M(e) \otimes [\mathbb{Z}/p \times e, \mathbb{Z}/p] \arrow[r, "\nu_e"] & (\mathcal{H}_{\mathbb{Z}/p} \,\Box\, M)(\mathbb{Z}/p) \end{tikzcd}$$

commutes by the definition of $(\mathcal{H}_{\mathbb{Z}/p} \,\Box\, M)(\mathbb{Z}/p)$. For any endomorphism γ of $\mathbb{Z}/p$, the image under $(1 \times \tau^e_{\mathbb{Z}/p})^*$ of the generator of $[\mathbb{Z}/p \times \mathbb{Z}/p, \mathbb{Z}/p]$ represented by the diagram

$$\mathbb{Z}/p \times \mathbb{Z}/p \xleftarrow{(\gamma, 1_{\mathbb{Z}/p})} \mathbb{Z}/p \xrightarrow{1_{\mathbb{Z}/p}} \mathbb{Z}/p$$

is just the morphism represented by the diagram

$$\mathbb{Z}/p \times e \longleftarrow e \longrightarrow \mathbb{Z}/p$$

in which all the maps are the obvious ones. This observation, together with the commutativity of the above naturality diagram for the maps ν_e and $\nu_{\mathbb{Z}/p}$, easily implies that $\tilde{\sigma}$ factors through the projection κ_M. It follows that $\tilde{\sigma}$ induces a map σ which splits λ. The map σ is obviously natural. $\square$

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