Splitting theorems for certain equivariant spectra

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ABSTRACT. Let G be a compact Lie group, Π be a normal subgroup of G, $\mathcal{G} = G/\Pi$, X be a $\mathcal{G}$-space and Y be a G-space. There are a number of results in the literature giving a direct sum decomposition of the group $[\Sigma^\infty X, \Sigma^\infty Y]_G$ of equivariant stable homotopy classes of maps from X to Y. Here, these results are extended to a decomposition of the group $[B, C]_G$ of equivariant stable homotopy classes of maps from an arbitrary finite $\mathcal{G}$-CW spectrum B to any G-spectrum C carrying a geometric splitting (a new type of structure introduced here). Any naive G-spectrum, and any spectrum derived from such by a change of universe functor, carries a geometric splitting. Our decomposition of $[B, C]_G$ is a consequence of the fact that, if C is geometrically split and $(\mathfrak{F}', \mathfrak{F})$ is any reasonable pair of families of subgroups of G, then there is a splitting of the cofibre sequence

$$(E\mathfrak{F}_+ \wedge C)^\Pi \longrightarrow (E\mathfrak{F}'_+ \wedge C)^\Pi \longrightarrow (E(\mathfrak{F}', \mathfrak{F}) \wedge C)^\Pi$$

constructed from the universal spaces for the families. Both the decomposition of the group $[B, C]_G$ and the splitting of the cofibre sequence are proven here not just for complete G-universes, but for arbitrary G-universes.

Various technical results about incomplete G-universes that should be of independent interest are also included in this paper. These include versions of the Adams and Wirthmüller isomorphisms for incomplete universes. Also included is a vanishing theorem for the fixed-point spectrum $(E(\mathfrak{F}', \mathfrak{F}) \wedge C)^\Pi$ which gives computational force to the intuition that what really matters about a G-universe U is which orbits G/H embed as G-spaces in U.

Contents

Introduction

This is a memoir in three parts. The first two parts present two closely related collections of results on equivariant stable homotopy theory. The last part contains the details of some of the messier proofs of the results stated in the first two parts.

Roughly speaking, the central theme of the first part is that a family of well-known splitting results [**4–6, 10–13, 23, 24, 28**] about the fixed-point spectra of equivariant suspension spectra can be generalized to a significantly larger class of spectra. This larger class includes all spectra obtained by change of universe functors from spectra indexed on a trivial universe. The generalizations of these results, and the class of spectra to which they apply, are best understood by considering the relation between two types of equivariant fixed-point spectra. Assume that G is a compact Lie group, Λ is a (closed) subgroup of G, and C is a G-spectrum indexed on a G-universe U. The two different Λ-fixed-point spectra, C^Λ and $\Phi^\Lambda C$, associated to C are both $W_G\Lambda$-spectra indexed on the Λ-fixed universe U^Λ. The spectrum C^Λ is referred to here simply as the Λ-fixed-point spectrum of C. It could also be described as the categorical fixed-point spectrum because it is characterized by an adjunction, a special case of which identifies the equivariant stable homotopy group $[G/\Lambda \wedge S^n, C]_G$ with the non-equivariant stable homotopy group $[S^n, C^\Lambda]$. This identification is one of the primary reasons for the importance of this type of fixed-point spectra.

The other type $\Phi^\Lambda C$ of a fixed-point spectrum is referred to here as a geometric fixed-point spectrum. These spectra first became widely known in the context of the proof of the Segal conjecture. Frank Adams, in particular, discussed them extensively in [**1**]. They are best understood by considering the case in which C is the equivariant suspension spectrum $\Sigma^\infty_U Y$ associated to a based G-space Y. In this case, there is an obvious $W_G\Lambda$-spectrum, $\Sigma^\infty_{U^\Lambda}(Y^\Lambda)$, indexed on U^Λ which deserves to be considered as some sort of fixed-point spectrum of $\Sigma^\infty_U Y$. However, it is not immediately obvious that the assignment of $\Sigma^\infty_{U^\Lambda}(Y^\Lambda)$ to $\Sigma^\infty_U Y$ gives a well-defined functor from the full subcategory of the G-stable category generated by the suspension spectra to the $W_G\Lambda$-stable category. Establishing the functoriality of this construction in this context only raises the question of whether the construction can be extended to give a functor from the entire G-stable category to the $W_G\Lambda$-stable category. In fact, the geometric fixed-point functor $\Phi^\Lambda C$ provides such an extension. However, here we encounter a surprise. There is a canonical map

$$\zeta_\Lambda : \Sigma^\infty_{U^\Lambda}(Y^\Lambda) \longrightarrow (\Sigma^\infty_U Y)^\Lambda,$$

and it would be natural to expect that this map extends to a natural transformation from $\Phi^\Lambda C$ to C^Λ. In fact, the only known natural transformation relating these two functors is a map

$$\omega_\Lambda : C^\Lambda \longrightarrow \Phi^\Lambda(C)$$

going in the opposite direction. The connection between this map and the map ζ_Λ in the case where $C = \Sigma^\infty_U Y$ is that the composite $\omega_\Lambda \circ \zeta_\Lambda$ is a weak $W_G\Lambda$-equivalence. Thus, when $C = \Sigma^\infty_U Y$, $\Phi^\Lambda C$ splits off from C^Λ as a wedge summand.

The starting point for the research which led to this memoir was the observation that the proofs of the family of splitting results for suspension spectra mentioned earlier make use of nothing more than the fact that, for each subgroup Λ, $\Phi^\Lambda C$ splits off from C^Λ as a wedge summand. Thus, these splitting results extend to any G-spectrum C having a right inverse for the map ω_Λ for each subgroup Λ of G. This observation, together with a result indicating that the class of spectra having these right inverses is significantly larger than the class of suspension spectra, form the foundation for the work presented here.

It would be natural to call a spectrum having this collection of right inverses a split spectrum. However, that term has already been used to describe another class of spectra [**23**]. The spectra with this new structure are therefore called geometrically split spectra, emphasizing the role that the geometric fixed-point functors play in their definition. However, since the previously defined split spectra play no role here, geometrically split spectra are often referred to here simply as split spectra.

In the first part of this memoir, we introduce the precise definition of this notion of a geometrically split G-spectrum, provide a list of examples of such spectra, and describe a few spectra that are known not to be geometrically split. Also, we state two main splitting theorems for geometrically split spectra. One of these is our generalization of the family of splitting theorems for suspension spectra that have already appeared in the literature [**4–6, 10–13, 23, 24, 28**]. For a finite group G, the simplest case of this extension has roughly the form:

THEOREM. *Let C be a geometrically split G-spectrum indexed on a complete G-universe U. Then there is a weak equivalence*

$$C^G \simeq \bigvee_{(\Lambda)} EW\Lambda_+ \wedge_{W\Lambda} \Phi^\Lambda C$$

of nonequivariant spectra indexed on U^G. The wedge on the right is indexed over the G-conjugacy classes (Λ) of subgroups Λ of G. This weak equivalence is natural in C with respect to maps which preserve the geometric splittings.

The earlier forms of this splitting result have been used for a variety of tasks such as identifying the equivariant zeroth stable stem as the Burnside ring, studying Mackey functors, and proving the Segal conjecture. Our generalization can be used to identify the E^2-terms of some of the suspension and change of universe spectral sequences introduced in [**20**]. It also has applications to the study of Mackey functors [**21**].

Our other main splitting theorem seems to be essentially new. It applies to certain cofibre sequences and long exact sequences in homology derived from the cofibre sequence

$$E\mathfrak{F}_+ \longrightarrow E\mathfrak{F}'_+ \longrightarrow E(\mathfrak{F}', \mathfrak{F})$$

associated to a pair of G-families $(\mathfrak{F}', \mathfrak{F})$. In its simplest form, our splitting theorem asserts that, if C is a geometrically split G-spectrum, then the cofibre sequence

$$(E\mathfrak{F}_+ \wedge C)^G \longrightarrow (E\mathfrak{F}'_+ \wedge C)^G \longrightarrow (E(\mathfrak{F}', \mathfrak{F}) \wedge C)^G,$$

derived from the sequence above, is split — that is, that $(E\mathfrak{F}'_+ \wedge C)^G$ is the wedge of the other two spectra. Given the important role that cofibre sequences associated to pairs of G-families have played in equivariant homotopy theory, our splitting results for these cofibre sequences ought to have a wide variety of applications.

The impetus behind the second part of this memoir is the large number of connections between geometrically split spectra and change of universe functors. In the first place, change of universe functors are used in the construction of all the known geometrically split spectra. Further, one main application of our splitting results is to the study of change of universe functors. In particular, as we have already noted, these results can be used to compute the E^2-terms of some of the change of universe spectral sequences introduced in [**20**]. Moreover, some of our results, such as Corollaries 2.5 and 3.4 and Remark 3.9, provide explicit formulae for the homotopy groups of the spectra in the image of certain change of universe functors. Thus, this memoir provides tools for studying change of universe functors which should be thought of as complementary to the tools developed in [**18–20**].

The desire to use our splitting results for the study of change of universe functors made it imperative that all of them be proven for as wide a range of incomplete G-universes as possible. The effort to prove these splittings in this generality led to the discovery of a number of technical results on incomplete universes. These technical results are presented in the second part of this memoir. They should be of interest in their own right. Moreover, most of them are needed for the proofs of the results stated in the first part.

One of the most noteworthy of our results on incomplete universes gives a contractibility criterion for spectra of the form $(E(\mathfrak{F}', \mathfrak{F}) \wedge C)^\Lambda$. This result gives computational force to the suggestion offered by Theorem 1.2(b) of [**20**] that what really matters about a G-universe U is which orbits G/Λ embed in U as G-spaces. In particular, it is the key to showing that, if the orbit G/Λ does not embed in U as G-space, then it cannot have a properly behaved Spanier-Whitehead dual. Two other items of special interest in the second part are treatments of the Adams and Wirthmüller isomorphisms for incomplete universes.

One common thread in our generalizations of results to incomplete universes is the role played by new families of subgroups of the ambient group G. A typical extension associates a G-family $\mathfrak{F}(U)$ to each incomplete G-universe U and asserts that the conclusion applicable to all G-spectra indexed on a complete universe holds only for $\mathfrak{F}(U)$-spectra indexed on U. Providing insight into these new families, and tools for manipulating them, is therefore one of the goals of the second part.

The first part of this memoir is organized in a somewhat eccentric fashion because it is intended to serve two audiences. Some special cases of the splitting theorems mentioned above have a variety of applications and should be of interest to a broad audience. However, the most general versions of these splitting results are rather technical in form and of interest only to people looking closely at equivariant stable homotopy theory. Thus, after giving a precise definition of the notion of a geometrically split spectrum in section one, we present the special cases of broadest interest in section two and the first half of section three. In the rest of the third section, we state our splitting theorems in their full generality. This approach is somewhat inefficient, but it should make the results more accessible. In the fourth section, the simplifications of our results which occur when the group G is finite are discussed. The implications of our results for the stable orbit category associated to an incomplete universe are discussed in Section 5. This material is used in [**14, 21**] and in our discussion of the Adams and Wirthmüller isomorphisms. The proofs of some of the results stated in this part are long and technical; these proofs are given in Part 3.

Almost all of the second part of this memoir should be regarded as a supplement to [**24**] which extends to incomplete universes various results proven in [**24**] only for complete universes. The first four sections of Part 2 contain a discussion of families of subgroups, and pairs of families of subgroups, in the context of incomplete universes; these are essentially a supplement to Sections II.1, II.2, and V.7 of [**24**]. The fifth section is devoted to the behavior of the geometric fixed-point functor Φ^{Λ}, and extends Section II.9 of [**24**]. The sixth and seventh sections provide introductions to the Wirthmüller and Adams isomorphisms for incomplete universes, and so supplement Sections II.6 and II.7 of [**24**]. The proofs of most of the results in this part are rather short and are given immediately following the statements. However, a few longer proofs, such as the proof of the Adams isomorphism theorem for an incomplete universe, are delayed until the third part.

In addition to containing the proofs of various results from the first two parts, the third part also contains a construction of the Adams transfer in the context of an incomplete universe. This should be of interest in its own right because the procedures for constructing transfers introduced in Chapter 4 of [**24**] turn out to be inadequate for constructing many of the transfers that ought to exist in the context of an incomplete universe. The construction presented here is a small step toward supplying these missing transfers.

Notational conventions

Much of our notation, and many basic facts about equivariant stable homotopy theory, are drawn from the first two chapters of [**24**]. In particular, groups are assumed to be compact Lie groups, and subgroups are understood to be closed. Throughout this memoir, we work with an ambient compact Lie group G, a normal subgroup Π of G (which is G itself in one special case), and the associated quotient group G/Π. Our notation is complicated by the fact that, in both the statements of our results and their proofs, we must work with a collection of subgroups of G which have no particular relation to Π, a collection of subgroups of Π, and a collection of subgroups of G/Π. This large assortment of subgroups creates both a shortage of symbols and a certain amount of confusion about which group contains each subgroup. To resolve these problems, subgroups of G which have no particular relation to Π are denoted by the letters immediately following G, such as H and J. Subgroups of Π are denoted by capital Greek letters such as Λ and Ψ. The quotient group G/Π is denoted $\mathcal{G}$, and its various subgroups are denoted by other script capitals. Typically, whenever subgroups H of G and $\mathcal{H}$ of $\mathcal{G}$ appear in the same context, H is the inverse image of $\mathcal{H}$ under the projection map $G \longrightarrow \mathcal{G}$. The notation $K \leq H$ indicates that K is a subgroup of H. The set of subgroups of G which are G-conjugate to H is denoted $(H)_G$, and the notation $(K)_G \leq (H)_G$ indicates that K is subconjugate (in G) to H.

All topological spaces are assumed to be compactly generated, weak Hausdorff spaces [**15, 27, 30**]. All G-spaces are left G-spaces. Whenever possible, the prefix G is omitted from our notations, so that by spaces, subspaces, spectra, maps, etc., we mean G-spaces, sub-G-spaces, G-spectra, G-maps, etc. If H is a subgroup of G and Y is a G-space, then Y^H is the H-fixed subspace of Y. A based G-space is a G-space Y together with a specified basepoint, which is required to be in Y^G. If X is an unbased G-space, then X_+ denotes the disjoint union of X and a G-trivial basepoint. Our G-spectra are indexed on a G-universe U as defined in section I.2 of [**24**]. If C and D are G-spectra indexed on U, then the set of maps from C to D in the G-stable category of spectra indexed on U is denoted $[C, D]^U_G$.

All G-representations are assumed to have a G-invariant inner product. If W is a G-representation, then DW, SW, and S^W denote its unit disk, its unit sphere, and its one-point compactification, respectively. The basepoint of S^W is the point at infinity. If V is a subrepresentation of the G-representation W, then $W - V$ denotes the orthogonal complement of V in W. It is therefore an actual, and not a virtual, G-representation.

If Λ is a subgroup of G, then $N_G\Lambda$ and $W_G\Lambda = N_G\Lambda/\Lambda$ are the normalizer of Λ in G and the associated Weyl group of Λ. If U is a G-universe and Ψ is another subgroup of G such that $\Psi \leq \Lambda$, then $j^{\Psi,\Lambda} : U^\Lambda \longrightarrow U^\Psi$ is the inclusion of the Λ-fixed universe U^Λ into the Ψ-fixed universe U^Ψ. If Ψ is the trivial subgroup so that U^{Ψ} is just U, then we abbreviate $j^{\Psi,\Lambda}$ to $j^{\Lambda}: U^{\Lambda} \longrightarrow U$. Since the inclusion j^{Λ} is a linear $N_G\Lambda$-isometry between two $N_G\Lambda$-universes, there are two change of universe functors

$$j^{\Lambda}_{*}: N_G\Lambda\mathcal{S}U^{\Lambda} \longrightarrow N_G\Lambda\mathcal{S}U \text{ and } j^{*}_{\Lambda}: N_G\Lambda\mathcal{S}U \longrightarrow N_G\Lambda\mathcal{S}U^{\Lambda}$$

relating the categories $N_G\Lambda\mathcal{S}U^{\Lambda}$ and $N_G\Lambda\mathcal{S}U$ of $N_G\Lambda$-spectra indexed on the universes U^{Λ} and U respectively (see section II.1 of [**24**]). Recall that, if C is a G-spectrum indexed on U, then the Λ-fixed-point spectrum C^{Λ} of C is a $W_G\Lambda$-spectrum indexed on U^{Λ} (see p. 56, [**24**]). This spectrum is obtained by pulling C back to the Λ-trivial universe U^{Λ} via the functor j^{*}_{Λ} and then passing to fixed points in an obvious way. When it is necessary to emphasize the change of universe involved in the formation of C^{Λ}, the notation C^{Λ} is expanded to $(j^{*}_{\Lambda}C)^{\Lambda}$. We often regard C^{Λ} as a $N_G\Lambda$-spectrum via the projection $N_G\Lambda \longrightarrow W_G\Lambda$.

Throughout this paper, we work with families, and pairs of families, of subgroups of some group (usually G or $N_G\Lambda$). Whenever two families $\mathfrak{F}'$ and $\mathfrak{F}$ are referred to as a pair $(\mathfrak{F}', \mathfrak{F})$, it is assumed that $\mathfrak{F} \subset \mathfrak{F}'$. If $E\mathfrak{F}$ and $E\mathfrak{F}'$ are the universal spaces associated to these two families, then there is a canonical map $E\mathfrak{F} \longrightarrow E\mathfrak{F}'$. This map, and any map derived from it, such as the map $E\mathfrak{F}_+ \wedge Y \longrightarrow E\mathfrak{F}'_+ \wedge Y$ for some space or spectrum Y, is denoted by λ. The cofibre of the map $\lambda: E\mathfrak{F}_+ \longrightarrow E\mathfrak{F}'_+$ is the universal space $E(\mathfrak{F}', \mathfrak{F})$ of the pair $(\mathfrak{F}', \mathfrak{F})$. The canonical inclusion of $E\mathfrak{F}'$ into the cofibre $E(\mathfrak{F}', \mathfrak{F})$, and any map derived from this inclusion, is denoted μ. If $(\mathfrak{F}', \mathfrak{F})$ and $(\mathfrak{G}', \mathfrak{G})$ are two families of subgroups such that $\mathfrak{F}' \subset \mathfrak{G}'$ and $\mathfrak{F} \subset \mathfrak{G}$, then the canonical maps $\lambda: E\mathfrak{F}' \longrightarrow E\mathfrak{G}'$ and $\lambda: E\mathfrak{F} \longrightarrow E\mathfrak{G}$ induce a canonical comparison map $E(\mathfrak{F}', \mathfrak{F}) \longrightarrow E(\mathfrak{G}', \mathfrak{G})$ which is denoted κ. If $\mathfrak{F}'$ is the family of all subgroups of G, then $E\mathfrak{F}'_+$ is just S^0 and the universal space $E(\mathfrak{F}', \mathfrak{F})$ is usually denoted $\tilde{E}\mathfrak{F}$. In this context, the canonical maps λ and μ become the obvious collapse map $\lambda: E\mathfrak{F}_+ \longrightarrow S^0$ and a map $\mu: S^0 \longrightarrow \tilde{E}\mathfrak{F}$. For this memoir, one of the most important families of subgroups is the $N_G\Lambda$-family $\mathfrak{F}_{N_G\Lambda}[\Lambda]$ consisting of those subgroups H of $N_G\Lambda$ which do not contain Λ(see Definition II.2.3(ii) of [**24**]). When it is desirable to keep track of Λ in this setting, we denote the canonical $N_G\Lambda$-map $\mu: S^0 \longrightarrow \tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda]$ by μ_{Λ}.

If Λ is a subgroup not only of G but also of Π, as our notation is intended to suggest, then $N_{\Pi}\Lambda$ is equal to $\Pi \cap N_G\Lambda$ and is therefore a normal subgroup of $N_G\Lambda$. It follows that $W_{\Pi}\Lambda$ is a normal subgroup of $W_G\Lambda$ and that there are isomorphisms

$$W_G\Lambda/W_{\Pi}\Lambda \cong N_G\Lambda/N_{\Pi}\Lambda \cong (\Pi N_G\Lambda)/\Pi,$$

where $\Pi N_G\Lambda$ is the product of the subgroups Π and $N_G\Lambda$. These isomorphisms allow us to regard $W_G\Lambda/W_{\Pi}\Lambda$ as a subgroup of $\mathcal{G} = G/\Pi$. They also allow us to assign an action of $\Pi N_G\Lambda$ to any object which carries a $W_G\Lambda/W_{\Pi}\Lambda$ action. To emphasize the containment of $W_G\Lambda/W_{\Pi}\Lambda$ in $\mathcal{G}$ and to compactify our notation, the group $W_G\Lambda/W_{\Pi}\Lambda$ is sometimes denoted $\mathcal{W}\Lambda$.

Part 1

Geometrically Split Spectra

SECTION 1

The notion of a geometrically split G-spectrum

The general notion of a geometrically split G-spectrum C outlined in the introduction actually breaks up into a collection of different possible levels of structure on C. The level of structure required for any particular result about a spectrum C depends on the indexing universe U for C and the subgroup with respect to which the fixed points are taken. Here, we present the most general definition of our notion of a geometrically split spectrum and provide a collection of examples of such spectra.

DEFINITION 1.1. (a) A G-spectrum C indexed on a G-universe U is (geometrically) split at a subgroup Λ of G if there is a $W_G\Lambda$-spectrum $C[\Lambda]$ indexed on U^Λ and a $W_G\Lambda$-map $\zeta_\Lambda : C[\Lambda] \longrightarrow C^\Lambda$ such that the composite

$$C[\Lambda] \xrightarrow{\zeta_\Lambda} C^\Lambda \xrightarrow{\omega_\Lambda} \Phi^\Lambda(C)$$

is a weak $W_G\Lambda$-equivalence of $W_G\Lambda$-spectra indexed on U^Λ.

(b) Let $\mathfrak{D}$ be a collection of subgroups of G. A G-spectrum C indexed on a G-universe U is (geometrically) $\mathfrak{D}$-split if C is geometrically split at Λ for every subgroup Λ in $\mathfrak{D}$. Note that we might as well assume that the collection $\mathfrak{D}$ is closed under conjugation. We definitely do not assume that $\mathfrak{D}$ is a family of subgroups (that is, that it is closed under the passage to subgroups). A variety of different collections of subgroups of G are of interest to us. In particular, a G-spectrum C geometrically split with respect to the family of all subgroups of G is said to be fully (geometrically) split. For any G-universe U, a G-spectrum C is U-split if it is split with respect to the collection of subgroups Λ of G such that G/Λ embeds in U as a G-space. If Π is a normal subgroup of G, then a a G-spectrum C is Π-split if it is split with respect to the collection of subgroups of Π. Yet another collection of interest is that of those subgroups Λ of Π such that Π/Λ embeds in U as a Π-space. A spectrum split with respect to this collection is referred to as (U, Π)-split spectrum. In the context of studying pairs $(\mathfrak{F}', \mathfrak{F})$ of families of subgroups of G, spectra split with respect to the collection of subgroups of Π contained in $\mathfrak{F}' - \mathfrak{F}$ are often of interest. So are spectra split with respect to the collection of subgroups Λ of Π in $\mathfrak{F}' - \mathfrak{F}$ such that Π/Λ embeds in U as a Π-space.

(c) Let $\mathfrak{D}$ be a collection of subgroups of G. A G-map $f : C \longrightarrow C'$ between two $\mathfrak{D}$-split G-spectra is said to preserve the splittings if, for each subgroup Λ in $\mathfrak{D}$, there is a $W\Lambda$-map $f[\Lambda] : C[\Lambda] \longrightarrow C'[\Lambda]$ making the diagram

$$\begin{array}{ccc} C[\Lambda] & \xrightarrow{f[\Lambda]} & C'[\Lambda] \\ \downarrow{\scriptstyle \zeta_\Lambda} & & \downarrow{\scriptstyle \zeta'_\Lambda} \\ C^\Lambda & \xrightarrow{f^\Lambda} & (C')^\Lambda \end{array}$$

commute in the $W\Lambda$-stable category. Given a G-map $f : C \longrightarrow C'$, there is only one possible choice for the map $f[\Lambda]$. However, for an arbitrary G-map f, this choice need not make the diagram commute.

The following propositions and remarks provide a collection of examples of geometrically split spectra:

Proposition 1.2. *Let $\mathfrak{D}$ be a collection of subgroups of G, and let U be a G-universe.*

(a) *If Y is a G-space, then the U-indexed suspension spectrum $\Sigma_U^\infty Y$ is fully split. Moreover, any G-map $h : Y \longrightarrow Y'$ induces a map $\Sigma_U^\infty h : \Sigma_U^\infty Y \longrightarrow \Sigma_U^\infty Y'$ which preserves the splittings. The splitting map ζ_Λ which exhibits $\Sigma_U^\infty Y$ as a spectrum split at the subgroup Λ is just the canonical map $\zeta : \Sigma_{U^\Lambda}^\infty(Y^\Lambda) \longrightarrow (\Sigma_U^\infty Y)^\Lambda$ introduced in Remarks II.3.14(i) of* [**24**].

(b) *If every subgroup Λ of $\mathfrak{D}$ acts trivially on the indexing universe U, then every G-spectrum C indexed on U is $\mathfrak{D}$-split and every G-map between G-spectra indexed on U preserves these splittings. The $W_G\Lambda$-spectrum $C[\Lambda]$ indexed on $U^\Lambda = U$ which exhibits the splitting of C at Λ is just C^Λ, and the splitting map $\zeta_\Lambda : C[\Lambda] \longrightarrow C^\Lambda$ is just the identity map.*

(c) *The change of universe functor i_* associated to a G-isometry $i : U \longrightarrow U'$ takes $\mathfrak{D}$-split G-spectra to $\mathfrak{D}$-split G-spectra. Moreover, i_* takes maps which preserve splittings to maps which preserve splittings. Note in particular that, if U is a trivial universe, then every spectrum in the image of i_* is fully split and every map in the image of i_* preserves splittings. If $C[\Lambda]$ is the $W_G\Lambda$-spectrum indexed on U^Λ which exhibits the splitting of C at Λ, then $i_*^\Lambda C[\Lambda]$ is the $W_G\Lambda$-spectrum indexed on $(U')^\Lambda$ which exhibits the splitting of i_*C at Λ. The splitting map for i_*C at Λ is just the composite*

$$i_*^\Lambda C[\Lambda] \xrightarrow{i_*^\Lambda \zeta_\Lambda} i_*^\Lambda(C^\Lambda) \longrightarrow (i_*C)^\Lambda,$$

in which the second map is the canonical map. In this discussion, i_*^Λ is just the change of universe functor associated to the induced linear isometry $i : U^\Lambda \longrightarrow (U')^\Lambda$ between the fixed-point universes.

(d) *If Y is a G-space and C is a $\mathfrak{D}$-split G-spectrum, then $Y \wedge C$ is $\mathfrak{D}$-split. Moreover, if $h : Y \longrightarrow Y'$ is a G-map between G-spaces and $f : C \longrightarrow C'$ is a G-map between $\mathfrak{D}$-split G-spectra which preserves the splittings, then the induced map $h \wedge f : Y \wedge C \longrightarrow Y' \wedge C'$ preserves the splittings. If $C[\Lambda]$ is the $W_G\Lambda$-spectrum which exhibits the splitting of C at Λ, then $Y^\Lambda \wedge C[\Lambda]$ is the $W_G\Lambda$-spectrum exhibiting the splitting of $Y \wedge C$ at Λ. The splitting map is the composite*

$$Y^\Lambda \wedge C[\Lambda] \xrightarrow{1 \wedge \zeta_\Lambda} Y^\Lambda \wedge C^\Lambda \xrightarrow{\nu} (Y \wedge C)^\Lambda,$$

in which ν is the canonical map of Remarks II.3.14(ii) of [**24**].

(e) *If C and D are $\mathfrak{D}$-split G-spectra, then $C \wedge D$ is $\mathfrak{D}$-split. Moreover, any pair of G-maps $f : C \longrightarrow C'$ and $g : D \longrightarrow D'$ between $\mathfrak{D}$-split G-spectra which preserve the splittings induces a map $f \wedge g : C \wedge D \longrightarrow C' \wedge D'$ which preserves the splittings. If $C[\Lambda]$ and $D[\Lambda]$ are the $W_G\Lambda$-spectra which exhibit the splittings of C and D at Λ, then $C[\Lambda] \wedge D[\Lambda]$ is the $W_G\Lambda$-spectrum exhibiting the splitting of $C \wedge D$ at Λ. The splitting map is the composite*

$$C[\Lambda] \wedge D[\Lambda] \xrightarrow{\zeta_{\Lambda,C} \wedge \zeta_{\Lambda,D}} C^\Lambda \wedge D^\Lambda \xrightarrow{\omega} (C \wedge D)^\Lambda,$$

in which ω is the map of Remarks II.3.14(iii) of [**24**].

(f) *If C is a $\mathfrak{D}$-split G-spectrum, and E is a retract of C in the G-stable category, then E is also $\mathfrak{D}$-split. If the maps $i : E \longrightarrow C$ and $r : C \longrightarrow E$ exhibit E as a retract of C and the map $\zeta_\Lambda : C[\Lambda] \longrightarrow C^\Lambda$ provides the splitting of C at the subgroup Λ, then the composite*

$$\Phi^\Lambda E \xrightarrow{\Phi^\Lambda i} \Phi^\Lambda C \xrightarrow{(\omega_\Lambda \circ \zeta_\Lambda)^{-1}} C[\Lambda] \xrightarrow{\zeta_\Lambda} C^\Lambda \xrightarrow{r^\Lambda} E^\Lambda$$

provides the required splitting of E at Λ.

(g) *If T is a collection of integer primes and C is a $\mathfrak{D}$-split G-spectrum, then the localization C_T of C at T is also $\mathfrak{D}$-split.*

(h) *If C is bounded below and is a $\mathfrak{D}$-split G-spectrum, then its completion $C_p^\wedge$ at any integer prime p is also $\mathfrak{D}$-split.*

REMARK 1.3. (a) Every spectrum C is split at the trivial subgroup e since $C^e = \Phi^e C = C$.

(b) Part (d) of the proposition indicates that our splitting theorems for split G-spectra apply not only to these spectra, but also to the homology theories on G-spaces derived from them.

(c) Recall from [**18, 19, 22**] that an Eilenberg-Mac Lane G-spectrum indexed on a universe U is a G-CW spectrum C indexed on U such that, for every integer $n \neq 0$ and every subgroup K of G, $\pi_n^K C = 0$. If C is an Eilenberg-Mac Lane G-spectrum, then, for any subgroup Λ of G, C^Λ is an Eilenberg-Mac Lane $W_G\Lambda$-spectrum. In order for C to be split at Λ, the spectrum $\Phi^\Lambda C$, being a retract of C^Λ, would also have to be an Eilenberg-Mac Lane $W_G\Lambda$-spectrum. If the indexing universe U is nontrivial, then the spectrum $\Phi^\Lambda C$ typically has nonvanishing homotopy groups in positive dimensions (see Theorem II.9.8 and Proposition II.9.13 in [**24**]). Thus, equivariant Eilenberg-Mac Lane spectra indexed on non-trivial universes are typically not split at any subgroup Λ other than the trivial subgroup. However, the general result about geometric splittings stated below indicates certain special Eilenberg-Mac Lane spectra are geometrically split at some nontrivial subgroups.

PROPOSITION 1.4. *Let C be a G-spectrum, and Λ be a subgroup of G. If, for every integer n and every subgroup K of $N_G\Lambda$ which does not contain Λ, $\pi_n^K C = 0$, then the canonical map $\omega_\Lambda : C^\Lambda \longrightarrow \Phi^\Lambda C$ is a weak $W_G\Lambda$-equivalence, and so C is split at Λ.*

PROOF. Since the existence of a splitting for C at Λ depends only on the structure of C as a $N_G\Lambda$-spectrum, we may assume that Λ is normal in G. Proposition II.9.2 and Theorem II.9.8 of [**24**] then imply that the map ω_Λ is a weak $W_G\Lambda$-equivalence. Theorem II.9.5 of [**24**] indicates that this result is actually a special case of Proposition 1.2(c). □

SECTION 2

Geometrically split G-spectra and G-fixed-point spectra

In this section, we explore the implications of a geometric splitting of a G-spectrum C for the structure of various G-fixed-point spectra associated to C. The primary implications are two splitting theorems. One of these (Theorem 2.1) seems new, but the other (Theorem 2.4) generalizes a large number of splitting results that have already appeared in the literature [**4–6, 10–13, 23, 24, 28**]. The statement of each of these results consists of two parts: one part is a statement about G-fixed-point spectra, and the other is the represented form of the spectrum-level assertion.

It turns out that, since fixed-point spectra with respect to proper subgroups of G are not considered here, there is no real advantage to assuming here that the indexing G-universe U for our spectra is complete. Thus, in this section, no restrictions are imposed on the indexing universe U. Here, also, the Weyl group of a subgroup Λ of G is denoted $W\Lambda$, rather than $W_G\Lambda$.

THEOREM 2.1. *Let C be a geometrically U-split G-spectrum indexed on a G-universe U, and $(\mathfrak{F}', \mathfrak{F})$ be a pair of G-families. Then the cofibre sequence*

$$(E\mathfrak{F}_+ \wedge C)^G \longrightarrow (E\mathfrak{F}'_+ \wedge C)^G \longrightarrow (E(\mathfrak{F}', \mathfrak{F}) \wedge C)^G$$

is split. Thus, if B is a nonequivariant spectrum indexed on U^G and B is regarded as a G-spectrum with trivial action, then the portion

$$\left[j_*^G B, E\mathfrak{F}_+ \wedge C\right]_G^U \longrightarrow \left[j_*^G B, E\mathfrak{F}'_+ \wedge C\right]_G^U \longrightarrow \left[j_*^G B, E(\mathfrak{F}', \mathfrak{F}) \wedge C\right]_G^U$$

of the long exact sequence associated to the pair $(\mathfrak{F}', \mathfrak{F})$ is a split short exact sequence. These two splittings are natural in C with respect to maps of spectra which preserve the geometric splittings. The second splitting is also natural in B.

REMARK 2.2. In this theorem, one can weaken the requirement that C be geometrically U-split to the requirement that C is geometrically split with respect to the collection $\mathfrak{D}$ of subgroups Λ in $\mathfrak{F}' - \mathfrak{F}$ for which the orbit G/Λ embeds in U as a G-space.

The theorem above provides us with several examples of spectra that are not geometrically split.

EXAMPLE 2.3. (a) In [**2**], Costenoble describes the structure of the equivariant Thom spectrum MO_G and the spectrum mO_G representing unoriented geometric G-bordism for the group $G = \mathbb{Z}/2$. His results include the observation that, if $\mathfrak{F}'$ is the family of all subgroups of $\mathbb{Z}/2$ and $\mathfrak{F}$ is the family consisting only of the trivial subgroup, then the maps of equivariant stable homotopy groups induced by the maps

$$\mu : E\mathfrak{F}'_+ \wedge MO_G \longrightarrow E(\mathfrak{F}', \mathfrak{F}) \wedge MO_G$$

and

$$\mu : E\mathfrak{F}'_+ \wedge mO_G \longrightarrow E(\mathfrak{F}', \mathfrak{F}) \wedge mO_G$$

are monomorphisms, but not isomorphisms. If either MO_G or mO_G were geometrically fully split, then the corresponding map on homotopy groups would have to be a split epimorphism, which it clearly cannot be. It follows that MO_G and mO_G are not geometrically fully split for any compact Lie group G which contains $\mathbb{Z}/2$ as a subgroup.

(b) Even though completion at an integer prime typically preserves geometric splittings (see Proposition 1.2(h)), completion at ideals in the Burnside ring need not. For example, let G be $\mathbb{Z}/p$, for some prime p, and let I be the augmentation ideal of the Burnside ring of G. The completion $(S^0)^\wedge_I$ of the sphere G-spectrum at I is not geometrically split even though the sphere G-spectrum itself is geometrically split by Proposition 1.2(a). The nonexistence of a splitting for $(S^0)^\wedge_I$ can be seen by examining the cofibre seqence

$$EG_+ \wedge (S^0)^\wedge_I \longrightarrow (S^0)^\wedge_I \longrightarrow \tilde{E}G \wedge (S^0)^\wedge_I.$$

By Theorem 2.1, the associated exact sequence

$$\pi_0^G\left(EG_+ \wedge (S^0)^\wedge_I\right) \longrightarrow \pi_0^G\left((S^0)^\wedge_I\right) \longrightarrow \pi_0^G\left(\tilde{E}G \wedge (S^0)^\wedge_I\right).$$

of zeroth homotopy groups would have to be a split short exact sequence if $(S^0)^\wedge_I$ were split. The first two groups in this sequence, and the map between them, are easily computed. Proposition 3.1 of [**7**] and the observation that $(\tilde{E}G \wedge (S^0)^\wedge_I)^G$ is $\Phi^G((S^0)^\wedge_I)$ make it easy to compute the third group. These computations indicate that the sequence of homotopy groups has the form

$$\mathbb{Z} \xrightarrow{(p,\gamma)} \mathbb{Z} \oplus \mathbb{Z}_p^\wedge \longrightarrow \mathbb{Z}_p^\wedge,$$

where $\gamma : \mathbb{Z} \longrightarrow \mathbb{Z}_p^\wedge$ is the completion map. Note that the composite of the first map in this sequence with the projection $\mathbb{Z} \oplus \mathbb{Z}_p^\wedge \longrightarrow \mathbb{Z}$ is not surjective. This observation and the fact that there are no nontrivial maps from $\mathbb{Z}_p^\wedge$ to $\mathbb{Z}$ imply that the sequence of homotopy groups cannot be a split short exact sequence.

A bit of additional notation is needed for our second splitting theorem. Let Λ be a subgroup of G. Then the adjoint representation $Ad(W\Lambda)$ of the Weyl group $W\Lambda$ of Λ is the tangent space of $W\Lambda$ at the identity element e with the $W\Lambda$-action derived from the conjugation action of $W\Lambda$ on itself. Let C be a U-split G-spectrum, and assume that the orbit G/Λ embeds in U as a G-space. The U-splitting of C provides a $W\Lambda$-spectrum $C[\Lambda]$ indexed on U^Λ. The spectrum $EW\Lambda_+ \wedge \Sigma^{Ad(W\Lambda)} C[\Lambda]$ is a free $W\Lambda$-spectrum indexed on U^Λ. Thus, by Theorem II.2.6 of [**24**], there is a $W\Lambda$-spectrum $Z(\Lambda)$ indexed on U^G such that $j_*^{\Lambda,G} Z(\Lambda)$ is weakly $W\Lambda$-equivalent to $EW\Lambda_+ \wedge \Sigma^{Ad(W\Lambda)} C[\Lambda]$. Since $W\Lambda$ acts trivially on the universe U^G, there is an orbit spectrum $Z(\Lambda)/W\Lambda$ indexed on U^G. This spectrum is hereafter denoted $EW\Lambda_+ \wedge_{W\Lambda} \Sigma^{Ad(W\Lambda)} C[\Lambda]$. Note that the spectrum $C[\Lambda]$ in this construction could be replaced by any other $W\Lambda$-spectrum D indexed on U^Λ. The resulting nonequivariant spectrum $EW\Lambda_+ \wedge_{W\Lambda} \Sigma^{Ad(W\Lambda)} D$ depends functorially on D.

THEOREM 2.4. *Let C be a geometrically U-split G-spectrum indexed on a G-universe U. Then there is a weak equivalence*

$$C^G \simeq \bigvee_{(\Lambda)} EW\Lambda_+ \wedge_{W\Lambda} \Sigma^{Ad(W\Lambda)} C[\Lambda]$$

of nonequivariant spectra indexed on U^G. The indexing in the target of this map is over the G-conjugacy classes (Λ) of subgroups Λ of G such that G/Λ embeds as a G-space in U. Further, let B be a nonequivariant spectrum indexed on U^G which is regarded as a G-spectrum with trivial action. If either B is a finite CW spectrum or G is a finite group, then there is an induced isomorphism

$$\left[j_*^G B, C\right]_G^U \cong \bigoplus_{(\Lambda)} \left[B, EW\Lambda_+ \wedge_{W\Lambda} \Sigma^{Ad(W\Lambda)} C[\Lambda]\right]^{U^G},$$

in which the direct sum has the same indexing as the wedge in the weak equivalence. Both the weak equivalence and the isomorphism are natural in C with respect to maps which preserve splittings; the isomorphism is also natural in B.

In some important special cases, the spectra $EW\Lambda_+ \wedge_{W\Lambda} \Sigma^{Ad(W\Lambda)} C[\Lambda]$ appearing in Theorem 2.4 have particularly simple descriptions. The resulting simplifications of the formulae from the theorem are recorded below. The first of these simplifications is, essentially, a special case of Theorem V.11.1 of [**24**].

COROLLARY 2.5. *Let B be a nonequivariant spectrum indexed on U^G regarded as a G-spectrum with trivial action, and assume that either B is a finite CW spectrum or G is a finite group.*

(a) *For any G-space Y, there is a weak equivalence*

$$(\Sigma_U^\infty Y)^G \simeq \bigvee_{(\Lambda)} \Sigma_{U^G}^\infty (EW\Lambda_+ \wedge_{W\Lambda} \Sigma^{Ad(W\Lambda)} Y^\Lambda)$$

of nonequivariant spectra indexed on U^G. This map induces an isomorphism

$$\left[j_*^G B, \Sigma_U^\infty Y\right]_G^U \cong \bigoplus_{(\Lambda)} \left[B, \Sigma_{U^G}^\infty (EW\Lambda_+ \wedge_{W\Lambda} \Sigma^{Ad(W\Lambda)} Y^\Lambda)\right]^{U^G}.$$

The indexing in the targets of these maps is over the G-conjugacy classes (Λ) of subgroups Λ of G such that G/Λ embeds as a G-space in U. Both maps are natural in Y. The isomorphism is also natural in B.

(b) *Let X be a G-spectrum indexed on the trivial universe U^G. Then there is a weak equivalence*

$$(j_*^G X)^G \simeq \bigvee_{(\Lambda)} EW\Lambda_+ \wedge_{W\Lambda} \Sigma^{Ad(W\Lambda)} X^\Lambda$$

of nonequivariant spectra indexed on U^G. This map induces an isomorphism

$$\left[j_*^G B, j_*^G X\right]_G^U \cong \bigoplus_{(\Lambda)} \left[B, EW\Lambda_+ \wedge_{W\Lambda} \Sigma^{Ad(W\Lambda)} X^\Lambda\right]^{U^G}.$$

The indexing in the targets of these maps is over the G-conjugacy classes (Λ) of subgroups Λ of G such that G/Λ embeds as a G-space in U. Both maps are natural in X. The isomorphism is also natural in B.

REMARK 2.6. As noted in Proposition 1.16 of [**20**], the second part of the corollary above can be used to compute the E^2-terms of some of the change of universe spectral sequences introduced in Theorem 1.14(b) of [**20**]. If G is a finite group, X is an Eilenberg-Mac Lane G-spectrum indexed on the trivial universe U^G, and n is an integer, then $Ad(W\Lambda) = 0$ and $\left[S^n, EW\Lambda_+ \wedge_{W\Lambda} X^\Lambda\right]^{U^G}$ is just the ordinary group homology $H_n(W\Lambda; \pi_0 X^\Lambda)$ of the group $W\Lambda$. Here, the action of $W\Lambda$ on the coefficient group $\pi_0 X^\Lambda$ is just the natural one derived from the action of $W\Lambda$ on X^Λ. Thus, when G is finite and X is an Eilenberg-Mac Lane G-spectrum, the entire right-hand side of the isomorphism in Corollary 2.5(b) can be described in terms of ordinary group homology.

Our second splitting theorem provides some insight into the relation between change of universe functors and fixed-point spectra.

COROLLARY 2.7. *Let C be a G-spectrum indexed on a universe U, $i : U \longrightarrow U'$ be a linear G-isometry, and $i^G : U^G \longrightarrow (U')^G$ be the induced linear isometry on the fixed-point universes. If C is geometrically U'-split, then the natural map*

$$\delta : i_*^G(C^G) \longrightarrow (i_* C)^G$$

is a split monomorphism in the stable category of spectra indexed on $(U')^G$.

A key ingredient in the proofs of all the splitting results stated above is an observation about the fixed-point spectrum $(E(\mathfrak{F}', \mathfrak{F}) \wedge C)^G$ associated to any G-spectrum C and any adjacent pair $(\mathfrak{F}', \mathfrak{F})$ of families of subgroups of G. Being applicable to non-split spectra, this result ought to be of independent interest, and is therefore recorded here. Some precursor of this result lies at the heart of the proofs of most of the predecessors of our splitting theorems.

PROPOSITION 2.8. *Let C be a G-spectrum indexed on a G-universe U, $(\mathfrak{F}', \mathfrak{F})$ be an adjacent pair of families of subgroups of G, and B be a nonequivariant spectrum indexed on U^G. Also, let Λ be a subgroup of G such that $\mathfrak{F}' - \mathfrak{F} = (\Lambda)$. If G/Λ does not embed in U as a G-space, then $(E(\mathfrak{F}', \mathfrak{F}) \wedge C)^G$ is weakly contractible, and $\left[j_*^G B, E(\mathfrak{F}', \mathfrak{F}) \wedge C\right]_G^U = 0$. Otherwise, there is a natural isomorphism*

$$\left[j_*^G B, E(\mathfrak{F}', \mathfrak{F}) \wedge C\right]_G^U \cong \left[B, EW\Lambda_+ \wedge_{W\Lambda} \Sigma^{Ad(W\Lambda)} \Phi^\Lambda C\right]^{U^G}.$$

This isomorphism arises from a natural weak equivalence

$$(E(\mathfrak{F}', \mathfrak{F}) \wedge C)^G \simeq EW\Lambda_+ \wedge_{W\Lambda} \Sigma^{Ad(W\Lambda)} \Phi^\Lambda C$$

of nonequivariant spectra indexed on U^G.

SECTION 3

Geometrically split G-spectra and Π-fixed-point spectra

Here, the results on G-fixed-point spectra presented in the previous section are generalized to results on Π-fixed-point spectra, where Π is a proper subgroup of G. In this broader context, a variety of rather unpleasant complications arise when the indexing universe U is incomplete. Thus, in this section, some results are stated first for the special case in which the universe U is complete. This is the most interesting special case, and the results available for it are much cleaner and more satisfying than those available in the general case.

If C is a G-spectrum indexed on a G-universe U and Π is a proper subgroup of G, then the Π-fixed-point spectrum C^Π is a $W_G\Pi$-spectrum indexed on U^Π. Any elements of G not in $N_G\Pi$ have no influence on C^Π, so we lose nothing by assuming that Π is normal in G. In this case, $W_G\Pi$ is just G/Π, which is hereafter denoted $\mathcal{G}$.

The results of Theorem 2.1 on the standard cofibre sequences associated to a pair of families $(\mathfrak{F}', \mathfrak{F})$ extend to the context of Π-fixed-point spectra only if a restriction is placed on the pair. This restriction is expressed as a closure condition on the difference $\mathfrak{F}' - \mathfrak{F}$. The closure condition seems to be a natural one, and appears repeatedly in the study of Π-fixed-point spectra.

DEFINITION 3.1. A collection $\mathfrak{D}$ of subgroups of G (such as $\mathfrak{F}' - \mathfrak{F}$) is Π-closed if, whenever H and K are subgroups of G such that $H \cap \Pi = K \cap \Pi$ and $H \in \mathfrak{D}$, then K is also in $\mathfrak{D}$. Note that, if $\mathfrak{D}$ is a Π-closed collection of subgroups of G, then $H \in \mathfrak{D}$ if and only if $H \cap \Pi \in \mathfrak{D}$. Thus, a Π-closed collection of subgroups $\mathfrak{D}$ is completely determined by the set of subgroups Λ of Π such that Λ is in $\mathfrak{D}$. A pair $(\mathfrak{F}', \mathfrak{F})$ of families of subgroups of G is said to be a Π-closed pair if the difference $\mathfrak{F}' - \mathfrak{F}$ is Π-closed.

THEOREM 3.2. *Let Π be a normal subgroup of the compact Lie group G, $\mathcal{G} = G/\Pi$, C be a G-spectrum indexed on a G-universe U, and $(\mathfrak{F}', \mathfrak{F})$ be a Π-closed pair of G-families. If C is geometrically split at every subgroup Λ of Π such that Λ is in $\mathfrak{F}' - \mathfrak{F}$ and Π/Λ embeds in U as a Π-space, then the cofibre sequence*

$$(E\mathfrak{F}_+ \wedge C)^\Pi \longrightarrow (E\mathfrak{F}'_+ \wedge C)^\Pi \longrightarrow (E(\mathfrak{F}', \mathfrak{F}) \wedge C)^\Pi$$

is split in the $\mathcal{G}$-stable category. Thus, if B is a $\mathcal{G}$-spectrum indexed on U^Π, then the portion

$$\left[j_*^\Pi B, E\mathfrak{F}_+ \wedge C\right]_G^U \longrightarrow \left[j_*^\Pi B, E\mathfrak{F}'_+ \wedge C\right]_G^U \longrightarrow \left[j_*^\Pi B, E(\mathfrak{F}', \mathfrak{F}) \wedge C\right]_G^U$$

of the long exact sequence associated to the pair $(\mathfrak{F}', \mathfrak{F})$ is a split short exact sequence. These two splittings are natural in C with respect to maps of spectra which preserve the geometric splittings. Also, the second splitting is natural in B.

The generalization of Theorem 2.4 to Π-fixed-point spectra is a result in which rather unpleasant complications arise when the universe U is incomplete. Thus, before stating the full generalization, we consider first the complete universe case. In order to state even this generalization, we must introduce the appropriate replacement for the construction $EW\Lambda_+ \wedge_{W\Lambda} \Sigma^{Ad(W\Lambda)}C[\Lambda]$. In the context of studying Π-fixed-point spectra, the significant subgroups Λ of G are those which are also subgroups of Π. Recall that, if Λ is a subgroup of Π, then $W_\Pi\Lambda$ is a normal subgroup of $W_G\Lambda$, and the quotient group $W_G\Lambda/W_\Pi\Lambda$ may be regarded as a subgroup of $\mathcal{G} = G/\Pi$. To compactify our notation, we denote the group $W_G\Lambda/W_\Pi\Lambda$ by $\mathcal{W}\Lambda$. Also recall from Section V.10 of [**24**] that $E(W_\Pi\Lambda, W_G\Lambda)$ is the universal $W_\Pi\Lambda$-free $W_G\Lambda$-space. Observe that the adjoint representation $Ad(W_\Pi\Lambda)$ of $W_\Pi\Lambda$ actually carries a compatible $W_G\Lambda$-action since the conjugation action of $W_\Pi\Lambda$ on itself extends to a conjugation action of $W_G\Lambda$ on $W_\Pi\Lambda$. If C is a Π-split G-spectrum indexed on U and $C[\Lambda]$ is the associated $W_G\Lambda$-spectrum indexed on U^Λ, then the $W_G\Lambda$-spectrum $E(W_\Pi\Lambda, W_G\Lambda)_+ \wedge \Sigma^{Ad(W_\Pi\Lambda)}C[\Lambda]$ is a $W_\Pi\Lambda$-free $W_G\Lambda$-spectrum indexed on U^Λ. Thus, by Theorem II.2.6 of [**24**], there is a $W_G\Lambda$-spectrum $Z(\Lambda)$ indexed on U^Π such that $j_*^{\Lambda,\Pi}Z(\Lambda)$ is weakly $W_G\Lambda$-equivalent to $E(W_\Pi\Lambda, W_G\Lambda)_+ \wedge \Sigma^{Ad(W_\Pi\Lambda)}C[\Lambda]$. Since $W_\Pi\Lambda$ acts trivially on the universe U^Π, there is an orbit spectrum $Z(\Lambda)/W_\Pi\Lambda$ indexed on U^Π. This $\mathcal{W}\Lambda$-spectrum is denoted by $E(W_\Pi\Lambda, W_G\Lambda)_+ \wedge_{W_\Pi\Lambda} \Sigma^{Ad(W_\Pi\Lambda)}C[\Lambda]$. Note that the spectrum $C[\Lambda]$ in this construction could be replaced by any other $W_G\Lambda$-spectrum D indexed on U^Λ. The resulting $\mathcal{W}\Lambda$-spectrum $E(W_\Pi\Lambda, W_G\Lambda)_+ \wedge_{W_\Pi\Lambda} \Sigma^{Ad(W_\Pi\Lambda)}D$ depends functorially on D.

THEOREM 3.3. *Let Π be a normal subgroup of the compact Lie group G, $\mathcal{G} = G/\Pi$, and C be a geometrically Π-split G-spectrum indexed on a complete G-universe U. Then there is a weak equivalence*

$$C^\Pi \simeq \bigvee_{(\Lambda)_G} \mathcal{G} \ltimes_{\mathcal{W}\Lambda} \left(E(W_\Pi\Lambda, W_G\Lambda)_+ \wedge_{W_\Pi\Lambda} \Sigma^{Ad(W_\Pi\Lambda)}C[\Lambda]\right)$$

of $\mathcal{G}$-spectra indexed on U^Π. The indexing in the target of this map is over the G-conjugacy classes $(\Lambda)_G$ of subgroups Λ of Π. Further, let B be a $\mathcal{G}$-spectrum indexed on U^Π. If either B is a finite $\mathcal{G}$-CW spectrum or Π is a finite group, then there is an induced isomorphism

$$\left[j_*^\Pi B, C\right]_G^U \cong \bigoplus_{(\Lambda)_G} \left[B, E(W_\Pi\Lambda, W_G\Lambda)_+ \wedge_{W_\Pi\Lambda} \Sigma^{Ad(W_\Pi\Lambda)}C[\Lambda]\right]_{\mathcal{W}\Lambda}^{U^\Pi},$$

in which the direct sum has the same indexing as the wedge in the weak equivalence. Both the weak equivalence and the isomorphism are natural in C with respect to maps which preserve the splittings. The isomorphism is also natural in B.

The spectra $E(W_\Pi\Lambda, W_G\Lambda)_+ \wedge_{W_\Pi\Lambda} \Sigma^{Ad(W_\Pi\Lambda)}C[\Lambda]$, like the analogous spectra $EW_G\Lambda_+ \wedge_{W_G\Lambda} \Sigma^{Ad(W_G\Lambda)}C[\Lambda]$ of Section 2, have simpler descriptions in certain special cases. The analog of Corollary 2.5 stated below records the resulting simplifications of the formulae from Theorem 3.3. The first part of this corollary is a restatement of Theorem V.11.1 of [**24**], which is the most general precursor of Theorems 2.4 and 3.3 to be found in the literature.

COROLLARY 3.4. *Let U be a complete G-universe, and B be a $\mathcal{G}$-spectrum indexed on U^Π. Assume that either B is a finite $\mathcal{G}$-CW spectrum or Π is a finite group.*

(a) *For any G-space Y, there is a weak equivalence*

$$(\Sigma_U^\infty Y)^\Pi \simeq \bigvee_{(\Lambda)} \Sigma_{U^\Pi}^\infty \mathcal{G} \ltimes_{\mathcal{W}\Lambda} \left(E(W_\Pi\Lambda, W_G\Lambda)_+ \wedge_{W_\Pi\Lambda} \Sigma^{Ad(W_\Pi\Lambda)} Y^\Lambda \right)$$

of $\mathcal{G}$-spectra indexed on U^Π. This map induces an isomorphism

$$\left[j_*^\Pi B, \Sigma_U^\infty Y\right]_G^U \cong \bigoplus_{(\Lambda)} \left[B, \Sigma_{U^\Pi}^\infty \left(E(W_\Pi\Lambda, W_G\Lambda)_+ \wedge_{W_\Pi\Lambda} \Sigma^{Ad(W_\Pi\Lambda)} Y^\Lambda \right)\right]_{\mathcal{W}\Lambda}^{U^\Pi}.$$

The indexing in the targets of these maps is over the G-conjugacy classes $(\Lambda)_G$ of subgroups Λ of Π. Both maps are natural in Y. The second is also natural in B.

(b) *Let X be a G-spectrum indexed on the Π-trivial G-universe U^Π. Then there is a weak equivalence*

$$(j_*^\Pi X)^\Pi \simeq \bigvee_{(\Lambda)} \mathcal{G} \ltimes_{\mathcal{W}\Lambda} \left(E(W_\Pi\Lambda, W_G\Lambda)_+ \wedge_{W_\Pi\Lambda} \Sigma^{Ad(W_\Pi\Lambda)} X^\Lambda \right)$$

of $\mathcal{G}$-spectra indexed on U^Π. This map induces an isomorphism

$$\left[j_*^\Pi B, j_*^\Pi X\right]_G^U \cong \bigoplus_{(\Lambda)} \left[B, E(W_\Pi\Lambda, W_G\Lambda)_+ \wedge_{W_\Pi\Lambda} \Sigma^{Ad(W_\Pi\Lambda)} X^\Lambda \right]_{\mathcal{W}\Lambda}^{U^\Pi}.$$

The indexing in the targets of these maps is over the G-conjugacy classes $(\Lambda)_G$ of subgroups Λ of Π. Both maps are natural in X. The second is also natural in B.

Proposition 2.8, which describes certain G-fixed-point spectra even in the absence of a geometric splitting, is easily extended to Π-fixed-point spectra in the context of a complete G-universe U. For this extension, we need the analog of the notion of an adjacent family appropriate to the context of Π-fixed-point spectra.

Definition 3.5. Let Π be a normal subgroup of G, and $(\mathfrak{F}', \mathfrak{F})$ be a pair of families of subgroups of G. Then $(\mathfrak{F}', \mathfrak{F})$ is a Π-adjacent pair if there is a subgroup Λ of Π such that

$$\mathfrak{F}' - \mathfrak{F} = \{H \leq G \mid (H \cap \Pi)_G = (\Lambda)_G\}.$$

This Π-adjacent pair is said to be associated to the subgroup Λ. Observe that, if $\Pi = G$, then a Π-adjacent pair is just an adjacent pair in the usual sense.

Proposition 3.6. *Let C be a G-spectrum indexed on a complete G-universe U, Π be a normal subgroup of G, $\mathcal{G} = G/\Pi$, and B be a $\mathcal{G}$-spectrum indexed on U^Π. Also, let Λ be a subgroup of Π, and $(\mathfrak{F}', \mathfrak{F})$ be a Π-adjacent pair of families of subgroups of G associated to Λ. Then there is a natural isomorphism*

$$\left[j_*^\Pi B, E(\mathfrak{F}', \mathfrak{F}) \wedge C\right]_G^U \cong \left[B, E(W_\Pi\Lambda, W_G\Lambda)_+ \wedge_{W_\Pi\Lambda} \Sigma^{Ad(W_\Pi\Lambda)} \Phi^\Lambda C\right]_{\mathcal{W}\Lambda}^{U^\Pi}.$$

This isomorphism arises from a natural weak equivalence

$$(E(\mathfrak{F}', \mathfrak{F}) \wedge C)^\Pi \simeq \mathcal{G} \ltimes_{\mathcal{W}\Lambda} \left(E(W_\Pi\Lambda, W_G\Lambda)_+ \wedge_{W_\Pi\Lambda} \Sigma^{Ad(W_\Pi\Lambda)} \Phi^\Lambda C \right)$$

of $\mathcal{G}$-spectra indexed on U^Π.

The difference between Theorem 3.3 and the corresponding result applicable to incomplete universes arises from the fact that Theorem 3.3 is really a composite result consisting of a splitting result for geometrically split spectra coupled with instances of the Adams and the Wirthmüller isomorphisms. In an incomplete universe, neither of these two isomorphisms is commonly available. Thus, the generalization of Theorems 2.4 and 3.3 to a result about the Π-fixed-point spectra of geometrically split spectra indexed on an incomplete universe is a pure splitting results unadorned with either of the two isomorphisms. The conditions under which the Adams and Wirthmüller isomorphisms can be applied to improve these results are discussed in Sections 11 and 12. One unfortunate consequence of the lack of a Wirthmüller isomorphism in this context is that the change of group functor $\mathcal{G}\ltimes_{\mathcal{W}\Lambda}?$ which appears in Theorem 3.3 must be replaced by the less well-understood change of group functor $F_{\mathcal{W}\Lambda}[\mathcal{G},?)$ introduced in Section II.4 of [**24**].

In order to state the full generalization of Theorems 2.4 and 3.3, we must first introduce the appropriate replacement for the universal spaces $EW\Lambda$ and $E(W_\Pi\Lambda, W_G\Lambda)$.

Definition 3.7. Let $\rho : N_G\Lambda \longrightarrow W_G\Lambda$ be the standard projection. Then the $EW_G\Lambda$-space $E(\Lambda, \Pi, G; U)$ is the universal space associated to the family of subgroups W of $W_G\Lambda$ such that both $W \cap W_\Pi\Lambda = e$ and, if $H = \rho^{-1}(W)$, then $H\Pi/H$ embeds as a $H\Pi$-space into U. Note that the space $E(\Lambda, \Pi, G; U)$ is a $W_\Pi\Lambda$-free $W_G\Lambda$-space. It is the empty space unless the orbit Π/Λ embeds in U as a Π-space. If U is a complete G-universe, then $E(\Lambda, \Pi, G; U)$ is just $E(W_\Pi\Lambda, W_G\Lambda)$. If $\Pi = G$ and G/Λ embeds in U as a G-space, then $E(\Lambda, G, G; U)$ is just the universal free $W_G\Lambda$-space $EW_G\Lambda$.

One might expect the space $E(\Lambda, \Pi, G; U)$ to depend only on the groups $W_\Pi\Lambda$ and $W_G\Lambda$ and the $W_G\Lambda$-universe U^Λ. Unfortunately, this need not be the case. The requirement in the definition that $H\Pi/H$ embed as a $H\Pi$-space into U is too strong, and may cause the universal space to depend in a very subtle way on the embedding of Λ in G. The weaker condition that $HN_G\Lambda/H$ embeds as a HN_G-space into U is what might have been expected in this definition, but it isn't restrictive enough to allow a generalization of Theorems 2.4 and 3.3.

Theorem 3.8. *Let Π be a normal subgroup of the compact Lie group G, $\mathcal{G}$ be G/Π, C be a G-spectrum indexed on a G-universe U, and $(\mathfrak{F}', \mathfrak{F})$ be a Π-closed pair of G-families. If C is geometrically split at every subgroup Λ of Π such that Λ is in $\mathfrak{F}' - \mathfrak{F}$ and Π/Λ embeds in U as a Π-space, then there is a weak equivalence*

$$(E(\mathfrak{F}', \mathfrak{F}) \wedge C)^\Pi \simeq \bigvee_{(\Lambda)_G} F_{\mathcal{W}\Lambda}[\mathcal{G}, (j^*_{\Lambda,\Pi}(E(\Lambda, \Pi, G; U)_+ \wedge C[\Lambda]))^{W_\Pi\Lambda})$$

of $\mathcal{G}$-spectra indexed on U^Π. The indexing in the target of this map is over the G-conjugacy classes $(\Lambda)_G$ of subgroups Λ of Π such that Λ is in $\mathfrak{F}' - \mathfrak{F}$ and Π/Λ embeds as a Π-space in U. Further, let B be a $\mathcal{G}$-spectrum indexed on U^Π. If either B is a finite $\mathcal{G}$-CW spectrum or Π is a finite group, then there is an induced isomorphism

$$\left[j^\Pi_* B, E(\mathfrak{F}', \mathfrak{F}) \wedge C\right]^U_G \cong \bigoplus_{(\Lambda)_G} \left[j^{\Lambda,\Pi}_* B, E(\Lambda, \Pi, G; U)_+ \wedge C[\Lambda]\right]^{U^\Lambda}_{W_G\Lambda},$$

in which the direct sum has the same indexing as the wedge in the weak equivalence. Both the weak equivalence and the isomorphism are natural in C with respect to maps which preserve the splittings. The isomorphism is also natural in B.

Remark 3.9. (a) If $\mathfrak{F}'$ is the family of all subgroups of G and $\mathfrak{F}$ is the empty family of subgroups, then $E(\mathfrak{F}', \mathfrak{F}) = S^0$. Thus, applied to this pair of families, the theorem gives formulae for $\left[j^\Pi_* B, C\right]^U_G$ and C^Λ.

(b) If $C = \Sigma^\infty_U Y$ for some G-space Y, then the terms on the right-hand sides of the formulae in the theorem are given by

$$F_{\mathcal{W}\Lambda}[\mathcal{G}, (j^*_{\Lambda,\Pi}(E(\Lambda,\Pi,G;U)_+ \wedge C[\Lambda]))^{W_\Pi\Lambda}) = \\ F_{\mathcal{W}\Lambda}[\mathcal{G}, (j^*_{\Lambda,\Pi}(\Sigma^\infty_{U^\Lambda} E(\Lambda,\Pi,G;U)_+ \wedge Y^\Lambda))^{W_\Pi\Lambda})$$

and

$$\left[j^{\Lambda,\Pi}_* B, E(\Lambda,\Pi,G;U)_+ \wedge C[\Lambda]\right]^{U^\Lambda}_{W_G\Lambda} = \left[j^{\Lambda,\Pi}_* B, \Sigma^\infty_{U^\Lambda} E(\Lambda,\Pi,G;U)_+ \wedge Y^\Lambda\right]^{U^\Lambda}_{W_G\Lambda}.$$

(c) If $C = j^\Pi_* X$ for some G-spectrum X indexed on the Π-trivial G-universe U^Π, then the spectrum $C[\Lambda]$ appearing on the right-hand sides of the formulae in the theorem is just $j^{\Lambda,\Pi}_*(X^\Lambda)$.

By comparing Theorem 3.8 with Theorems 2.4 and 3.3, one can come to a fuller appreciation of the roles played by the Adams and Wirthmüller isomorphisms in the two earlier results. Essentially all that is known about the change of groups functor $F_{\mathcal{W}\Lambda}[\mathcal{G}, ?)$ comes directly from either its defining adjunction or the Wirthmüller isomorphism. In the context of an incomplete universe, the Wirthmüller isomorphism is available only for $\mathfrak{W}$-spectra, where $\mathfrak{W}$ is some poorly understood $\mathcal{W}\Lambda$-family. Thus, almost nothing is known about the behavior of the functor $F_{\mathcal{W}\Lambda}[\mathcal{G}, ?)$ when the indexing universe is incomplete. One might hope to argue that the spectra to which this functor is applied in our results happen to be $\mathfrak{W}$-spectra. However, they are all fixed-point spectra, and, even for a well-understood family $\mathfrak{F}$ of subgroups, it is hard to determine whether a particular fixed-point spectrum is an $\mathfrak{F}$-spectrum. If the Adams isomorphism were available, then it would convert these fixed-point spectra to orbit spectra. It is somewhat easier to settle the question of whether a particular orbit spectrum is an $\mathfrak{F}$-spectrum for any given family $\mathfrak{F}$. Unfortunately, for a typical incomplete universe U, the Adams isomorphism is unlikely to apply to the spectra appearing on the right-hand sides of the formulae in Theorem 3.8. Moreover, even when it does apply, considering the question of whether the Wirthmüller isomorphism also applies leads to the conclusion that the Wirthmüller isomorphism is typically unavailable for these spectra. One point of the discussion of the Adams and Wirthmüller isomorphisms for incomplete universes in Sections 11 and 12 is to provide information on when the formulae of Theorem 3.8 can be simplified using these isomorphisms. Various conditions under which these isomorphisms are available are described in Proposition 11.1 and Theorems 11.4, 11.6, 11.8, and 12.2.

One problem caused by the inapplicability of the Adams and Wirthmüller isomorphisms in Theorem 3.8 is that there seems to be no reasonable generalization of Corollary 2.7 to Π-fixed-point spectra. The difficulty here is that this corollary describes the interaction between of a change of universe functor i_*, which is a left adjoint, and the G-fixed-point functor, which is a right adjoint. The Adams isomorphism plays a critical role in the proof of the corollary because it allows us to replace certain instances of the G-fixed-point functor with a more easily handled G-orbit functor. Any generalization of the corollary to Π-fixed-point spectra would have to describe the relation of i_* to a composite of a fixed-point functor and a change of group functor, which is also a right adjoint. In the absence of the Adams and Wirthmüller isomorphisms, each of which would relate one of these right adjoints to a more tractable left adjoint, it is unreasonable to expect to be able to generalize the corollary.

Propositions 2.8 and 3.6, which describe certain G-fixed-point spectra even in the absence of a geometric splitting, are easily extended to Π-fixed-point spectra in the context of an arbitary G-universe U. For this extension, recall the notion of a Π-adjacent pair of G-families from Definition 3.5.

PROPOSITION 3.10. *Let C be a G-spectrum indexed on a G-universe U, Π be a normal subgroup of G, $\mathcal{G} = G/\Pi$, and B be a $\mathcal{G}$-spectrum indexed on U^Π. Also, let Λ be a subgroup of Π, and $(\mathfrak{F}', \mathfrak{F})$ be a Π-adjacent pair of families of subgroups of G associated to Λ. Then there is a natural isomorphism*

$$\gamma : \left[j_*^\Pi B, E(\mathfrak{F}', \mathfrak{F}) \wedge C\right]_G^U \xrightarrow{\cong} \left[j_*^{\Lambda,\Pi} B, E(\Lambda, \Pi, G; U)_+ \wedge \Phi^\Lambda C\right]_{W_G\Lambda}^{U^\Lambda}.$$

This isomorphism arises from a natural weak equivalence

$$\chi : (E(\mathfrak{F}', \mathfrak{F}) \wedge C)^\Pi \xrightarrow{\simeq} F_{\mathcal{W}\Lambda}[\mathcal{G}, (j_{\Lambda,\Pi}^*(E(\Lambda, \Pi, G; U)_+ \wedge \Phi^\Lambda C))^{W_\Pi\Lambda})$$

of $\mathcal{G}$-spectra indexed on U^Π.

SECTION 4

Geometrically split spectra and finite groups

If G is a finite group, then the splittings in Theorem 3.3 can be derived directly from Theorem 3.2 and Proposition 3.6 by using a sequence $\mathfrak{F}^*_{G,\Pi}$ of families of subgroups of G. This sequence provides a natural finite filtration for the entire G-stable category. Theorem 3.3 implies that, if C is a Π-split G-spectrum, then the induced filtration on C^Π is derived from a wedge decomposition of C^Π. In a similar fashion, Theorem 3.8 can be derived from Theorem 3.2 and Proposition 3.10 using the same sequence of families. Moreover, Theorem 2.4 can be derived from Theorem 2.1 and Proposition 2.8 using a related, simpler sequence $\mathfrak{E}^*_G$ of families of subgroups of G. In this section, these two sequence of families are defined, the filtration quotients of the associated natural filtrations on G-spectra are analyzed, and the relation between these filtrations and geometric splittings is described.

The sequence of G-families $\mathfrak{F}^*_{G,\Pi}$ is most easily defined in terms of the sequence $\mathfrak{E}^*_\Pi$ for the subgroup Π of G. The motivation behind the definition of the sequence of families $\mathfrak{F}^*_{G,\Pi}$ is best understood by recalling the notion of a Π-closed collection of subgroups of G from Definition 3.1.

Definition 4.1. (a) Let $\mathfrak{E}^0_G$ be the set consisting only of the trivial subgroup $\{e\}$ of G, and define the families $\mathfrak{E}^n_G$, for $n \geq 1$, inductively by

$$\mathfrak{E}^n_G = \{K \leq G \mid \text{if } J \text{ is a proper subgroup of } K, \text{ then } J \in \mathfrak{E}^{n-1}_G\}.$$

Thus, $\mathfrak{E}^1_G$ consists of the trivial subgroup and all the subgroups of G which are cyclic of prime order. Since G is finite, there are only finitely many distinct terms in this sequence. Observe that the filtration $\mathfrak{E}^*_G$ is natural in G in the sense that, if H is a subgroup of G, then, for any $n \geq 0$, the family $\mathfrak{E}^n_H$ of subgroups of H is just the intersection of $\mathfrak{E}^n_G$ with the set of subgroups of H. Thus, as a H-space, the universal $\mathfrak{E}^n_G$-space $E\mathfrak{E}^n_G$ is just $E\mathfrak{E}^n_H$.

(b) For each Λ in $\mathfrak{E}^n_G - \mathfrak{E}^{n-1}_G$, let

$$\mathfrak{E}^n_G(\Lambda) = \mathfrak{E}^{n-1}_G \cup (\Lambda)_G.$$

Then $\mathfrak{E}^n_G(\Lambda)$ is a family of subgroups of G such that $\mathfrak{E}^{n-1}_G \subset \mathfrak{E}^n_G(\Lambda) \subset \mathfrak{E}^n_G$. Moreover, the pair $(\mathfrak{E}^n_G(\Lambda), \mathfrak{E}^{n-1}_G)$ is adjacent.

(c) If Π is a normal subgroup of G, then define the families $\mathfrak{F}^n_{G,\Pi}$, for $n \geq 0$, by

$$\mathfrak{F}^n_{G,\Pi} = \{K \leq G \mid K \cap \Pi \in \mathfrak{E}^n_\Pi\}.$$

The filtration $\mathfrak{F}^n_{G,\Pi}$ is natural in G in the sense that, if $H \leq G$, then $\mathfrak{F}^n_{H,H\cap\Pi}$ is just the intersection of $\mathfrak{F}^n_{G,\Pi}$ with the set of subgroups of H. Observe that $\mathfrak{F}^n_{G,G} = \mathfrak{E}^n_G$. If C is a G-spectrum indexed on a G-universe U, then the spectra $(E\mathfrak{F}^n_{G,\Pi})_+ \wedge C$, for $n \geq 0$, form a finite natural filtration of C with filtration quotients the spectra $E(\mathfrak{F}^n_{G,\Pi}, \mathfrak{F}^{n-1}_{G,\Pi}) \wedge C$.

(d) For each Λ in $\mathfrak{E}^n_\Pi - \mathfrak{E}^{n-1}_\Pi$, let

$$\mathfrak{F}^n_{G,\Pi}(\Lambda) = \{K \leq G \mid K \cap \Pi \in \mathfrak{E}^n_\Pi(\Lambda)\}.$$

Then $\mathfrak{F}^n_{G,\Pi}(\Lambda)$ is a family of subgroups of G such that $\mathfrak{F}^{n-1}_{G,\Pi} \subset \mathfrak{F}^n_{G,\Pi}(\Lambda) \subset \mathfrak{F}^n_{G,\Pi}$. Moreover, the pair $(\mathfrak{F}^n_{G,\Pi}(\Lambda), \mathfrak{F}^{n-1}_{G,\Pi})$ is Π-adjacent.

We begin our discussion of the relation between our splitting results and the sequences of families of subgroups introduced above by considering the implications of these sequences of families for the Π-fixed-point spectra of spectra indexed on a complete G-universe.

THEOREM 4.2. *Let G be a finite group, $\Pi \leq G$ be a normal subgroup, $\mathcal{G} = G/\Pi$, U be a complete G-universe, C be a G-spectrum indexed on U, and B be a $\mathcal{G}$-spectrum indexed on U^Π. Then*

(a) *There is an isomorphism*

$$\left[j^\Pi_* B, E(\mathfrak{F}^n_{G,\Pi}, \mathfrak{F}^{n-1}_{G,\Pi}) \wedge C\right]^U_G \cong \bigoplus_{(\Lambda)_G} \left[B, E(W_\Pi\Lambda, W_G\Lambda)_+ \wedge_{W_\Pi\Lambda} \Phi^\Lambda C\right]^{U^\Pi}_{\mathcal{W}\Lambda}.$$

Moreover, there is a weak equivalence

$$(E(\mathfrak{F}^n_{G,\Pi}, \mathfrak{F}^{n-1}_{G,\Pi}) \wedge C)^\Pi \simeq \bigvee_{(\Lambda)_G} \mathcal{G} \ltimes_{\mathcal{W}\Lambda} \left(E(W_\Pi\Lambda, W_G\Lambda)_+ \wedge_{W_\Pi\Lambda} \Phi^\Lambda C\right)$$

of $\mathcal{G}$-spectra indexed on U^Π. The sum and wedge are indexed on the G-conjugacy classes $(\Lambda)_G$ of subgroups Λ of Π such that $\Lambda \in \mathfrak{E}^n_\Pi - \mathfrak{E}^{n-1}_\Pi$. Both maps are natural in C; the first is also natural in B.

(b) *If C is Π-split, then there is an isomorphism*

$$\left[j^\Pi_* B, C\right]^U_G \cong \bigoplus_n \left[j^\Pi_* B, E(\mathfrak{F}^n_{G,\Pi}, \mathfrak{F}^{n-1}_{G,\Pi}) \wedge C\right]^U_G$$

which is natural in B and in C with respect to maps which preserve the splittings. Moreover, there is a weak equivalence

$$C^\Pi \simeq \bigvee_n (E(\mathfrak{F}^n_{G,\Pi}, \mathfrak{F}^{n-1}_{G,\Pi}) \wedge C)^\Pi$$

of $\mathcal{G}$-spectra indexed on U^Π which is natural in C with respect to maps which preserve the splittings.

PROOF. By checking the fixed-point sets, it is easy to see that, as G-spaces,

$$E(\mathfrak{F}^n_{G,\Pi}, \mathfrak{F}^{n-1}_{G,\Pi}) \simeq \bigvee_{(\Lambda)_G} E(\mathfrak{F}^n_{G,\Pi}(\Lambda), \mathfrak{F}^{n-1}_{G,\Pi}),$$

where the wedge product is indexed on the set of G-conjugacy classes of subgroups Λ of Π in $\mathfrak{E}^n_\Pi - \mathfrak{E}^{n-1}_\Pi$. Part (a) follows directly from this decomposition and Proposition 3.6. Part (b) is proven by applying Theorem 3.2 inductively to the cofibre sequences associated to the pairs $(\mathfrak{F}^n_{G,\Pi}, \mathfrak{F}^{n-1}_{G,\Pi})$. □

REMARK 4.3. (a) If C is a Π-split G-spectrum, then the splittings of Theorem 3.3 are the result of combining the splittings provided in the two parts of this theorem and identifying the spectra $\Phi^\Lambda C$ with the spectra $C[\Lambda]$.

(b) If C is a G-spectrum indexed on U which is not (geometrically) split, then the description provided in part (a) of the theorem above for the filtration quotients of the filtration $((E\mathfrak{F}^n_{G,\Pi})_+ \wedge C)^\Pi$ should be thought of as a partial substitute for the splittings of Theorem 3.3.

The analog of Theorem 4.2 for the G-fixed-point spectrum of a G-spectrum indexed on any G-universe U is:

THEOREM 4.4. *Let G be a finite group, U be a G-universe, C be a G-spectrum indexed on U, and B be a nonequivariant spectrum indexed on U^G. Regard B as a G-spectrum with trivial action. Then*

(a) *There is an isomorphism*

$$\left[j_*^G B, E(\mathfrak{E}_G^n, \mathfrak{E}_G^{n-1}) \wedge C\right]_G^U \cong \bigoplus_{(\Lambda)_G} \left[B, EW_G\Lambda_+ \wedge_{W_G\Lambda} \Phi^\Lambda C\right]^{U^G}.$$

Moreover, there is a weak equivalence

$$(E(\mathfrak{E}_G^n, \mathfrak{E}_G^{n-1}) \wedge C)^G \simeq \bigvee_{(\Lambda)_G} EW_G\Lambda_+ \wedge_{W_G\Lambda} \Phi^\Lambda C$$

of nonequivariant spectra indexed on U^G. The sum and wedge are indexed on the G-conjugacy classes (Λ) of subgroups Λ of G such that $\Lambda \in \mathfrak{E}_G^n - \mathfrak{E}_G^{n-1}$ and G/Λ embeds as a G-space in U. Both maps are natural in C; the first is also natural in B.

(b) *If C is U-split, then there is an isomorphism*

$$\left[j_*^G B, C\right]_G^U \cong \bigoplus_n \left[j_*^G B, E(\mathfrak{E}_G^n, \mathfrak{E}_G^{n-1}) \wedge C\right]_G^U$$

which is natural in B and in C with respect to maps which preserve the splittings. Moreover, there is a weak equivalence

$$C^G \simeq \bigvee_n (E(\mathfrak{E}_G^n, \mathfrak{E}_G^{n-1}) \wedge C)^G$$

of nonequivariant spectra indexed on U^G which is natural in C with respect to maps which preserve the splittings.

In our general setting of looking at the Π-fixed-point spectra of spectra indexed on an incomplete universe, we lose the Adams and Wirthmüller isomorphisms that are implicitly contained in the first parts of Theorems 4.2 and 4.4. Thus, our most general result about the sequence of families $\mathfrak{F}_{G,\Pi}^*$ is:

THEOREM 4.5. *Let G be a finite group, $\Pi \le G$ be a normal subgroup, $\mathcal{G} = G/\Pi$, U be a G-universe, C be a G-spectrum indexed on U, and B be a $\mathcal{G}$-spectrum indexed on U^Π. Then*

(a) *There is an isomorphism*

$$\left[j_*^\Pi B, E(\mathfrak{F}_{G,\Pi}^n, \mathfrak{F}_{G,\Pi}^{n-1}) \wedge C\right]_G^U \cong \bigoplus_{(\Lambda)_G} \left[j_*^{\Lambda,\Pi} B, E(\Lambda, \Pi, G; U)_+ \wedge \Phi^\Lambda C\right]_{W_G\Lambda}^{U^\Lambda}.$$

Moreover, there is a weak $\mathcal{G}$-equivalence

$$(E(\mathfrak{F}_{G,\Pi}^n, \mathfrak{F}_{G,\Pi}^{n-1}) \wedge C)^\Pi \simeq \bigvee_{(\Lambda)_G} F_{\mathcal{W}\Lambda}[\mathcal{G}, (j_{\Lambda,\Pi}^*(E(\Lambda, \Pi, G; U)_+ \wedge \Phi^\Lambda C))^{W_\Pi\Lambda})$$

of $\mathcal{G}$-spectra indexed on U^Π. The sum and wedge are indexed on the G-conjugacy classes $(\Lambda)_G$ of subgroups Λ of Π such that $\Lambda \in \mathfrak{E}_\Pi^n - \mathfrak{E}_\Pi^{n-1}$ and Π/Λ embeds as a Π-space in U. Both maps are natural in C; the first is also natural in B.

(b) *If C is (U,Π)-split, then there is an isomorphism*

$$\left[j_*^{\Pi}B, C\right]_G^U \cong \bigoplus_n \left[j_*^{\Pi}B, E(\mathfrak{F}^n_{G,\Pi}, \mathfrak{F}^{n-1}_{G,\Pi}) \wedge C\right]_G^U$$

which is natural in B and in C with respect to maps which preserve the splittings. Moreover, there is a weak $\mathcal{G}$-equivalence

$$C^{\Pi} \simeq \bigvee_n (E(\mathfrak{F}^n_{G,\Pi}, \mathfrak{F}^{n-1}_{G,\Pi}) \wedge C)^{\Pi}$$

which is natural in C with respect to maps which preserve the splittings.

REMARK 4.6. (a) Theorems 4.4 and 4.5 follow from Theorems 2.1 and 3.2 and Propositions 2.8 and 3.10 just as Theorem 4.2 followed from Theorem 3.2 and Proposition 3.6. The one additional result needed for them is Theorem 6.1, which allows us to eliminate the summands in the formulae of part (a) of each theorem corresponding to the orbits which do not embed in U.

(b) If G is a finite group, then Theorems 2.4 and 3.8 follow from Theorems 4.4 and 4.5 just as Theorem 3.3 followed from Theorem 4.2. Moreover, if C is not geometrically split, then the descriptions provided in part (a) of Theorems 4.4 and 4.5 for the filtration quotients of the filtrations $((E\mathfrak{C}^n_G)_+ \wedge C)^G$ and $((E\mathfrak{F}^n_{G,\Pi})_+ \wedge C)^{\Pi}$ can be used as a partial substitutes for the splittings of Theorems 2.4 and 3.8.

SECTION 5

The stable orbit category for an incomplete universe

The precursors of Theorem 2.4 have been used to provide a useful description of the set of maps in the complete G-stable category between the suspension spectra of two orbits G/H and G/K (see Section V.9 of [**24**]). An extension of this description to the context of the G-stable category indexed on an incomplete universe has already been used in [**18–21**]. This extension is also used in our discussion of the Wirthmüller and Adams isomorphisms in Sections 11 and 12. Further, it plays a key role in [**14**], in which some homological misbehavior of the category of Mackey functors for a compact Lie group is described. There are several ways to prove this extension, but the easiest is to derive it from Theorem 2.4 in the same way that the result for a complete G-universe was derived from the precursors of this Theorem. This section is devoted to the statement and proof of this extension.

Although our primary concern is the set of stable maps between orbits, we begin with a generalization of Corollary V.9.3 of [**24**]. This generalization characterizes stable morphism sets in a somewhat broader context. For this characterization, recall that a space-level G-map $f : G/J \longrightarrow G/H$ between two orbits is determined by the image gH of the identity coset eJ of G/J and that the possible image cosets gH are those associated to the elements $g \in G$ such that $g^{-1}Jg \leq H$.

THEOREM 5.1. *Let G be a compact Lie group, U be a G-universe, H be a subgroup of G, and Y be a based G-space. Then the morphism set $[\Sigma^\infty_U G/H_+, \Sigma^\infty_U Y]^U_G$ is a free abelian group generated by the equivalence classes $[f,k]$ of certain diagrams (f,k) of the form*

$$G/H \xleftarrow{\;f\;} G/J \xrightarrow{\;k\;} Y,$$

in which $f : G/J \longrightarrow G/H$ and $k : G/J \longrightarrow Y$ are space-level G-maps. If $f(eJ) = gH$, then the diagram (f,k) represents a generator if the orbit $H/(g^{-1}Jg)$ embeds as an H-space in U and the map k is not null-G-homotopic. The diagram

$$G/H \xleftarrow{\;f'\;} G/J' \xrightarrow{\;k'\;} Y$$

is equivalent to the diagram (f,k) if there is a G-homeomorphism $\alpha : G/J \longrightarrow G/J'$ making the space-level diagram

$$\begin{array}{ccccc} & & G/J & & \\ & \swarrow^{f} & \big\downarrow{\scriptstyle\alpha} & \searrow^{k} & \\ G/H & & & & Y \\ & \nwarrow_{f'} & & \nearrow_{k'} & \\ & & G/J' & & \end{array}$$

commute up to G-homotopy.

PROOF. The change of group isomorphism

$$[\Sigma_U^\infty G/H_+, \Sigma_U^\infty Y]_G^U \cong \left[\Sigma_U^\infty S^0, \Sigma_U^\infty Y\right]_H^U$$

reduces the proof to the case in which $H = G$. This special case follows from Theorem 2.4 in exactly the same way that Corollary V.9.3 of [**24**] follows from Theorem V.9.1 of [**24**]. The embedding condition in the hypotheses of this corollary which is not in Corollary V.9.3 of [**24**] arises from the difference between the indexing sets for the direct sums which appear in Theorem 2.4 and Theorem V.9.1 of [**24**]. □

REMARK 5.2. (a) If $f(eJ) = gH$, then the diagram

$$G/H \xleftarrow{f} G/J \xrightarrow{k} Y,$$

represents the composite

$$\Sigma_U^\infty G/H_+ \xrightarrow{t} \Sigma_U^\infty G/(g^{-1}Jg)_+ \xrightarrow{c_g} \Sigma_U^\infty G/J_+ \xrightarrow{\Sigma_U^\infty k} \Sigma_U^\infty Y,$$

in which t is the transfer associated to the equivariant bundle $G/(g^{-1}Jg) \longrightarrow G/H$ and c_g is the usual G-homeomorphism between orbits associated to conjugate subgroups.

(b) In each equivalence class of diagrams, the representative (f, k) can be chosen so that $J \leq H$ and the map $f : G/J \longrightarrow G/H$ is the standard projection; that is, so that $f(eJ) = eH$.

(c) The equivalence relation on diagrams described in the theorem can also be obtained by requiring that the left triangle commute exactly and the right triangle G-homotopy commute.

If the target in the morphism set of the theorem is the suspension spectrum of an unbased G-space, then the morphism set can be described in a particularly simple way.

COROLLARY 5.3. *Let G be a compact Lie group, U be a G-universe, H be a subgroup of G, and Z be an unbased G-space.*

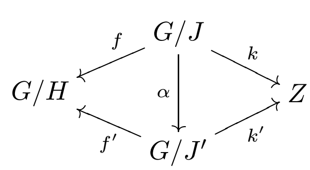
(a) *The morphism set $[\Sigma_U^\infty G/H_+, \Sigma_U^\infty Z_+]_G^U$ is a free abelian group whose generators are the equivalence classes of certain diagrams (f, k) of the form*

$$G/H \xleftarrow{f} G/J \xrightarrow{k} Z,$$

in which $f : G/J \longrightarrow G/H$ and $k : G/J \longrightarrow Z$ are space-level G-maps. If $f(eJ) = gH$, then the diagram (f, k) represents a generator if the orbit $H/(g^{-1}Jg)$ embeds as an H-space in U. The diagram

$$G/H \xleftarrow{f'} G/J' \xrightarrow{k'} Z$$

is equivalent to the diagram (f, k) if there is a G-homeomorphism $\alpha : G/J \longrightarrow G/J'$ making the space-level diagram



commute up to G-homotopy.

(b) *In particular, for* $K \leq G$*, the morphism set* $[\Sigma^\infty_U G/H_+, \Sigma^\infty_U G/K_+]^U_G$ *is a free abelian group whose generators are the equivalence classes of certain diagrams* (f,k) *of the form*

$$G/H \xleftarrow{f} G/J \xrightarrow{k} G/K,$$

in which $f : G/J \longrightarrow G/H$ *and* $k : G/J \longrightarrow G/K$ *are space-level* G*-maps. If* $f(eJ) = gH$*, then the diagram* (f,k) *represents a generator if the orbit* $H/(g^{-1}Jg)$ *embeds as an* H*-space in* U*.*

REMARK 5.4. (a) In both parts of the corollary, each equivalence class of diagrams has a representative (f,k) for which $J \leq H$ and $f(eJ) = eH$. In the second part of the corollary, there is also a representative for which $J \leq K$ and $k(eJ) = eK$. However, there may be equivalence classes for which one cannot arrange that J is in both H and K.

(b) In both parts of the corollary, the desired equivalence relation on diagrams may also be obtained by requiring that the left triangle commute exactly and the right triangle G-homotopy commute. In the second part, the desired relation may be obtained by requiring that the right triangle commute exactly and the left triangle commute up to G-homotopy.

(c) The composite of two stable maps represented by the diagrams

$$G/H \xleftarrow{f} G/J \xrightarrow{k} G/K$$

and

$$G/K \xleftarrow{m} G/J' \xrightarrow{n} G/Q$$

is obtained by considering the space-level pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{k'} & G/J' \\ {\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle m} \\ G/J & \xrightarrow{k} & G/K, \end{array}$$

and applying the standard result about the interaction between transfers and space-level maps in a pullback diagram (see, for example, axiom (A1) in Definition 1.1 of [**17**]). Unfortunately, if G is not finite, then the pullback P need not be a disjoint union of orbits. Thus, a decomposition formula like Theorem IV.6.1 of [**24**] (or a double coset formula like Theorem IV.6.3 of [**24**]) is needed to express the composite of the maps represented by (f,k) and (m,n) as a sum of the generators of the morphism set $[\Sigma^\infty_U G/H_+, \Sigma^\infty_U G/Q_+]^U_G$. It seems to be quite difficult to determine the multiplicities with which the various generators appear in this composite. However, it is clear that the only generators which can appear with a nonzero coefficient are those represented by diagrams

$$G/H \xleftarrow{p} G/L \xrightarrow{q} G/Q$$

in which L is an isotropy subgroup of the G-space P. In particular, L must be subconjugate to both J and J'. Moreover, if the representatives (f,k) and (m,n) are chosen so that $J, J' \leq K$, $k(eJ) = eK$, and $m(eJ') = eK$, then the allowed subgroups L are those which appear in the usual double coset decomposition of $J\backslash G/J'$.

Part 2

A Toolkit for Incomplete Universes

SECTION 6

A vanishing theorem for fixed-point spectra

Let U be an incomplete G-universe, Π be a normal subgroup of G, and $\mathcal{G} = G/\Pi$. Then there are certain pairs $(\mathfrak{F}', \mathfrak{F})$ of families of subgroups of G such that, for any G-spectrum D indexed on U, the fixed-point spectrum $(E(\mathfrak{F}', \mathfrak{F}) \wedge D)^{\Pi}$ is weakly $\mathcal{G}$-contractible. This vanishing result gives computational force to the suggestion offered by Theorem 1.2(b) of [**20**] that what really matters about a G-universe U is which orbits G/H embed in U as G-spaces. It is, moreover, one of the key tools in the proofs of our splitting theorems.

This section contains the statement and proof of this vanishing result. It also contains a simple lemma about pairs of families of subgroups which allows us to use the vanishing result to show that certain pairs of families may be replaced by somewhat smaller pairs of families. We begin with the statement of our vanishing theorem.

THEOREM 6.1. *Let Π be a normal subgroup of G, $\mathcal{G} = G/\Pi$, U be a G-universe, and $(\mathfrak{F}', \mathfrak{F})$ be a pair of families of subgroups of G such that, for every subgroup H in $\mathfrak{F}' - \mathfrak{F}$, the orbit $H\Pi/H$ does not embed in U as a $H\Pi$-space. Then, for any G-spectrum D indexed on U, the spectrum $(E(\mathfrak{F}', \mathfrak{F}) \wedge D)^{\Pi}$ is weakly $\mathcal{G}$-contractible.*

An argument like that used in the proof of Proposition V.7.5 of [**24**] reduces the proof of this proposition to the following lemma about adjacent families.

LEMMA 6.2. *Let H be a subgroup of G, Π be a normal subgroup of G, $\mathcal{G} = G/\Pi$, and $(\mathfrak{F}', \mathfrak{F})$ be an adjacent pair of families of subgroups of G such that $\mathfrak{F}' - \mathfrak{F} = (H)_G$. If U is a G-universe such that $H\Pi/H$ does not embed in U as a $H\Pi$-space and D is a G-spectrum indexed on U, then the spectrum $(E(\mathfrak{F}', \mathfrak{F}) \wedge D)^{\Pi}$ is weakly $\mathcal{G}$-contractible.*

PROOF. Let $\rho : G \longrightarrow \mathcal{G}$ be the projection, $\mathcal{K}$ be a subgroup of $\mathcal{G}$, $K = \rho^{-1}(\mathcal{K})$, and n be an integer. The homotopy group $\left[\mathcal{G}/\mathcal{K}_+ \wedge S^n, (E(\mathfrak{F}', \mathfrak{F}) \wedge D)^{\Pi}\right]_{\mathcal{G}}^{U^{\Pi}}$ must be shown to vanish. Note that G/K and $\mathcal{G}/\mathcal{K}$ are homeomorphic G-spaces. This observation, together with the change of group and change of universe adjunctions in chapter II of [**24**], yields the sequence of isomorphisms:

$$\begin{aligned}
\left[\mathcal{G}/\mathcal{K}_+ \wedge S^n, (E(\mathfrak{F}', \mathfrak{F}) \wedge D)^{\Pi}\right]_{\mathcal{G}}^{U^{\Pi}} &\cong \left[G/K_+ \wedge S^n, j_{\Pi}^*(E(\mathfrak{F}', \mathfrak{F}) \wedge D)\right]_G^{U^{\Pi}} \\
&\cong \left[G/K_+ \wedge S^n, E(\mathfrak{F}', \mathfrak{F}) \wedge D\right]_G^U \\
&\cong \left[S^n, E(\mathfrak{F}', \mathfrak{F}) \wedge D\right]_K^U .
\end{aligned}$$

Here, j_{Π}^* is the change of universe functor derived from the inclusion $j^{\Pi} : U^{\Pi} \longrightarrow U$. If $(H)_G \nleq (K)_G$, then $E(\mathfrak{F}', \mathfrak{F})$ is K-contractible and the group $[S^n, E(\mathfrak{F}', \mathfrak{F}) \wedge D]_K^U$ vanishes.

If $(H)_G \le (K)_G$, then we may assume that $H \le K$. Clearly, $\Pi \le K$, and therefore $H\Pi$ is a subgroup of K. Since $H\Pi/H$ does not embed in U as a $H\Pi$-space, K/H cannot embed in U as a K-space. Recall from section I.8 of [**24**] that we can approximate D up to a weak G-equivalence by a Σ-cofibrant G-prespectrum $\tilde{D}$ indexed on an indexing sequence $\{V_m\}_{m\ge 0}$ for the universe U. Then

$$[S^n, E(\mathfrak{F}', \mathfrak{F}) \wedge D]_K^U \cong \operatorname*{colim}_m \left[S^{n+V_m}, E(\mathfrak{F}', \mathfrak{F}) \wedge \tilde{D}V_m\right]_K.$$

If $n < 0$, then, for sufficiently large m, the representation V_m contains a trivial representation of dimension at least $|n|$ so that $n + V_m$ is a well defined representation. Thus, the colimit is well defined even when n is negative. By Lemma V.7.6 of [**24**], the homotopy group $\left[S^{n+V_m}, E(\mathfrak{F}', \mathfrak{F}) \wedge \tilde{D}V_m\right]_K$ always vanishes because K/H does not embed as a K-space in $V_m \subset U$. □

In the proofs of our main splitting theorems, Theorem 6.1 is frequently used to show that a pair of families of subgroups can be replaced by somewhat smaller pair. The following lemma, whose proof requires nothing more than an examination of fixed-point sets, facilitates this use of our vanishing theorem.

LEMMA 6.3. *Let* $(\mathfrak{F}_2, \mathfrak{F}_1)$ *and* $(\mathfrak{F}_4, \mathfrak{F}_3)$ *be pairs of families of subgroups of* G *such that* $\mathfrak{F}_2 \subset \mathfrak{F}_4$ *and* $\mathfrak{F}_1 = \mathfrak{F}_2 \cap \mathfrak{F}_3$*. Then the pair of canonical* G*-maps*

$$E(\mathfrak{F}_2, \mathfrak{F}_1) \longrightarrow E(\mathfrak{F}_4, \mathfrak{F}_3) \longrightarrow E(\mathfrak{F}_4, \mathfrak{F}_2 \cup \mathfrak{F}_3)$$

between the universal spaces associated to these pairs of families is an equivariant cofibre sequence.

SECTION 7

Spanier-Whitehead duality and incomplete universes

It is well known that, if the G-orbit G/H embeds in the G-universe U, then G/H has a Spanier-Whitehead dual in the equivariant stable category of G-spectra indexed on U (see, for example, Chapter 3 of [**24**]) . It has always been assumed that this sufficient condition was also necessary. Our vanishing theorem for fixed-point spectra (Theorem 6.1) provides a way of verifying this assumption.

PROPOSITION 7.1. *Let G be a compact Lie group, H be a closed subgroup of G, and U be a G-universe into which G/H does not embed as a G-space. Then $\Sigma_U^\infty G/H_+$ does not have a Spanier-Whitehead dual in the G-stable category of spectra indexed on U.*

PROOF. Let $(\mathfrak{F}', \mathfrak{F})$ be an adjacent pair of G-families such that $\mathfrak{F}' - \mathfrak{F} = (H)_G$. If there were a G-spectrum Z which was the Spanier-Whitehead dual of $\Sigma_U^\infty G/H_+$, then there would be an isomorphism

$$[\Sigma_U^\infty G/H_+, \Sigma_U^\infty E(\mathfrak{F}', \mathfrak{F})]_G^U \cong [S^0, \Sigma_U^\infty E(\mathfrak{F}', \mathfrak{F}) \wedge Z]_G^U.$$

Since G/H doesn't embed as a G-space in U and $\mathfrak{F}' - \mathfrak{F} = (H)_G$, the spectrum $(\Sigma_U^\infty E(\mathfrak{F}', \mathfrak{F}) \wedge Z)^G$ is weakly contractible by Theorem 6.1. This reduces the nonexistence proof to showing that the left-hand side of the putative equation above is nonzero. Various change of group isomorphisms can be applied to that side to obtain

$$[\Sigma_U^\infty G/H_+, \Sigma_U^\infty E(\mathfrak{F}', \mathfrak{F})]_G^U \cong [S^0, \Sigma_U^\infty E(\mathfrak{F}', \mathfrak{F})]_H^U \cong [S^0, (\Sigma_U^\infty E(\mathfrak{F}', \mathfrak{F}))^H]^{U^H}.$$

Let $\mathfrak{F}[H]$ be the H-family consisting of all of the proper subgroups of H. Then the natural projection

$$E(\mathfrak{F}', \mathfrak{F}) \longrightarrow E(\mathfrak{F}', \mathfrak{F}) \wedge \tilde{E}\mathfrak{F}[H]$$

is an H-equivalence, as is shown by a computation of the fixed-point sets. Corollary II.9.9 of [**24**] therefore indicates that the nonequivariant spectrum $(\Sigma_U^\infty E(\mathfrak{F}', \mathfrak{F}))^H$ is equivalent to $\Sigma_{U^H}^\infty (E(\mathfrak{F}', \mathfrak{F})^H)$. But $E(\mathfrak{F}', \mathfrak{F})^H \simeq S^0$. Thus, the morphism set on the left-hand side of the asserted equation can be identified with $[S^0, S^0]^{U^H} \cong \mathbb{Z}$. $\square$

SECTION 8

Change of group functors and families of subgroups

Here, we study certain pairs $(\mathfrak{F}', \mathfrak{F})$ of families of subgroups of G, such as Π-adjacent pairs, which are naturally tied to a single subgroup Λ of G. Our objective is to give conditions under which the universal space $E(\mathfrak{F}', \mathfrak{F})$ associated to such a pair $(\mathfrak{F}', \mathfrak{F})$ can be approximated by a space of the form $G \ltimes_{N_G\Lambda} E(\mathfrak{E}', \mathfrak{E})$, where $N_G\Lambda$ is the normalizer of Λ in G and $(\mathfrak{E}', \mathfrak{E})$ is a pair of families of subgroups of $N_G\Lambda$. Our main result is that, for any G-universe U, there is a minimal pair $(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda; U\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda; U\rangle)$ of families of subgroups of $N_G\Lambda$ such that, for any G-spectrum D indexed on U and any Π-adjacent pair $(\mathfrak{F}', \mathfrak{F})$ of families of subgroups of G associated to Λ, the Π-fixed-point spectrum $(E(\mathfrak{F}', \mathfrak{F}) \wedge D)^{\Pi}$ can be approximated, up to a weak $\mathcal{G}$-equivalence, by the Π-fixed-point spectrum $((G \ltimes_{N_G\Lambda} E(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda; U\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda; U\rangle)) \wedge D)^{\Pi}$. This approximation result generalizes some results contained in [**3**] and chapter 7 of [**5**]. It forms the heart of the proofs of Propositions 2.8, 3.6, and 3.10, and also plays a critical role in the proofs of our main splitting theorems.

Our approximation result describes the behavior of a particular member of a family of canonical comparison maps. We begin by introducing this family of comparison maps and certain key families of subgroups.

For any G-family $\mathfrak{F}$, let $\mathfrak{F}\,|_{N_G\Lambda}$ be the $N_G\Lambda$-family $\{H \le N_G\Lambda \mid H \in \mathfrak{F}\}$. The universal G-space $E\mathfrak{F}$ associated with $\mathfrak{F}$, when considered as an $N_G\Lambda$-space, is $N_G\Lambda$-equivalent to $E(\mathfrak{F}\,|_{N_G\Lambda})$. Similarly, the universal G-space $E(\mathfrak{F}', \mathfrak{F})$ associated to a pair of G-families $(\mathfrak{F}', \mathfrak{F})$ is $N_G\Lambda$-equivalent to $E(\mathfrak{F}'\,|_{N_G\Lambda}, \mathfrak{F}\,|_{N_G\Lambda})$. Thus, if $\mathfrak{E}$ is a family of subgroups of $N_G\Lambda$ and $\mathfrak{F}$ is a family of subgroups of G such that $\mathfrak{E} \subset \mathfrak{F}\,|_{N_G\Lambda}$, then the canonical $N_G\Lambda$-map $\lambda : E\mathfrak{E}_+ \longrightarrow E(\mathfrak{F}\,|_{N_G\Lambda})_+$ induces a G-map

$$\tilde{\lambda} : G \ltimes_{N_G\Lambda} E\mathfrak{E}_+ \longrightarrow E\mathfrak{F}_+.$$

Similarly, if $(\mathfrak{E}', \mathfrak{E})$ is a pair of families of subgroups of $N_G\Lambda$ and $(\mathfrak{F}', \mathfrak{F})$ is a pair of families of subgroups of G such that $\mathfrak{E}' \subset \mathfrak{F}'\,|_{N_G\Lambda}$ and $\mathfrak{E} \subset \mathfrak{F}\,|_{N_G\Lambda}$, then the canonical $N_G\Lambda$-map $\kappa : E(\mathfrak{E}', \mathfrak{E}) \longrightarrow E(\mathfrak{F}'\,|_{N_G\Lambda}, \mathfrak{F}\,|_{N_G\Lambda})$ induces a canonical G-map

$$\tilde{\kappa} : G \ltimes_{N_G\Lambda} E(\mathfrak{E}', \mathfrak{E}) \longrightarrow E(\mathfrak{F}', \mathfrak{F}).$$

If $\mathfrak{E}$ is a family of subgroups of $N_G\Lambda$, then, by closing the collection $\mathfrak{E}$ under conjugation by elements of G, we obtain a family of subgroups of G which is denoted $\overline{\mathfrak{E}}$. Note that, for any $N_G\Lambda$-family $\mathfrak{E}$, $\mathfrak{E} \subset \overline{\mathfrak{E}}\,|_{N_G\Lambda}$. Thus, there is a canonical G-map

$$\tilde{\lambda} : G \ltimes_{N_G\Lambda} E\mathfrak{E}_+ \longrightarrow E\overline{\mathfrak{E}}_+.$$

Also, for any pair $(\mathfrak{E}', \mathfrak{E})$ of families of subgroups of $N_G\Lambda$, there is a canonical G-map

$$\tilde{\kappa} : G \ltimes_{N_G\Lambda} E(\mathfrak{E}', \mathfrak{E}) \longrightarrow E(\overline{\mathfrak{E}}', \overline{\mathfrak{E}}).$$

There are two pairs of G-families of subgroups and two pairs of $N_G\Lambda$-families of subgroups which play a special role in the study of the Π-adjacent pairs $(\mathfrak{F}', \mathfrak{F})$ of subgroups of G associated to Λ.

Definition 8.1. (a) Let $(\mathfrak{F}'_G\langle\Lambda\rangle, \mathfrak{F}_G\langle\Lambda\rangle)$ be the pair of G-families of subgroups given by

$$\mathfrak{F}'_G\langle\Lambda\rangle = \{H \leq G \mid \exists g \in G, gHg^{-1} \leq N_G\Lambda \text{ and } gHg^{-1} \cap \Pi \leq \Lambda\}$$

and

$$\mathfrak{F}_G\langle\Lambda\rangle = \{H \leq G \mid \exists g \in G, gHg^{-1} \leq N_G\Lambda \text{ and } gHg^{-1} \cap \Pi < \Lambda\}.$$

If K is a subgroup of G such that $K \cap \Pi = \Lambda$, then $K \leq N_G\Lambda$, and so $K \in \mathfrak{F}'_G\langle\Lambda\rangle$. It follows that $\mathfrak{F}'_G\langle\Lambda\rangle - \mathfrak{F}_G\langle\Lambda\rangle = \{H \leq G \mid H \cap \Pi \in (\Lambda)_G\}$ so that the pair $(\mathfrak{F}'_G\langle\Lambda\rangle, \mathfrak{F}_G\langle\Lambda\rangle)$ is Π-adjacent. This pair is minimal among Π-adjacent pairs associated to the subgroup Λ of Π in the sense that, if $(\mathfrak{F}', \mathfrak{F})$ is another such pair, then $\mathfrak{F}'_G\langle\Lambda\rangle \subset \mathfrak{F}'$ and $\mathfrak{F}_G\langle\Lambda\rangle \subset \mathfrak{F}$. There is therefore a comparison G-map

$$\kappa : E(\mathfrak{F}'_G\langle\Lambda\rangle, \mathfrak{F}_G\langle\Lambda\rangle) \longrightarrow E(\mathfrak{F}', \mathfrak{F}),$$

which is unique up to G-homotopy. Also, the G-family $\mathfrak{F}'_G\langle\Lambda\rangle$ is minimal among the G-families containing the set $\{H \leq G \mid H \cap \Pi \in (\Lambda)_G\}$ in the sense that, if $\mathfrak{F}'$ is any other G-family containing this set, then $\mathfrak{F}'_G\langle\Lambda\rangle \subset \mathfrak{F}'$. Thus, there is a comparison G-map $\lambda : E\mathfrak{F}'_G\langle\Lambda\rangle \longrightarrow E\mathfrak{F}'$ which is unique up to G-homotopy.

(b) Let $(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda\rangle)$ be the pair of $N_G\Lambda$-families given by

$$\mathfrak{E}'_{N_G\Lambda}\langle\Lambda\rangle = \{H \leq N_G\Lambda \mid H \cap \Pi \leq \Lambda\}$$

and

$$\mathfrak{E}_{N_G\Lambda}\langle\Lambda\rangle = \{H \leq N_G\Lambda \mid H \cap \Pi < \Lambda\}.$$

The pair $(\overline{\mathfrak{E}'_{N_G\Lambda}\langle\Lambda\rangle}, \overline{\mathfrak{E}_{N_G\Lambda}\langle\Lambda\rangle})$ is just the minimal Π-adjacent pair $(\mathfrak{F}'_G\langle\Lambda\rangle, \mathfrak{F}_G\langle\Lambda\rangle)$ introduced above. Thus, there is a canonical G-map

$$\tilde{\kappa} : G \ltimes_{N_G\Lambda} E(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda\rangle) \longrightarrow E(\mathfrak{F}'_G\langle\Lambda\rangle, \mathfrak{F}_G\langle\Lambda\rangle).$$

Moreover, for any Π-adjacent pair $(\mathfrak{F}', \mathfrak{F})$ of subgroups of G associated to Λ, there is a canonical comparison map

$$\tilde{\kappa} : G \ltimes_{N_G\Lambda} E(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda\rangle) \longrightarrow E(\mathfrak{F}', \mathfrak{F}).$$

(c) If U is an incomplete G-universe, then the $N_G\Lambda$-pair $(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda\rangle)$ is not as well-behaved as one might like. A better-behaved replacement for this pair is given by

$$\mathfrak{E}'_{N_G\Lambda}\langle\Lambda; U\rangle = \{H \leq N_G\Lambda \mid H \cap \Pi \leq \Lambda \text{ and } H\Pi/H\Lambda \ H\Pi\text{-embeds in } U\}$$

and

$$\mathfrak{E}_{N_G\Lambda}\langle\Lambda; U\rangle = \{H \leq N_G\Lambda \mid H \cap \Pi < \Lambda \text{ and } H\Pi/H\Lambda \ H\Pi\text{-embeds in } U\}.$$

Define the G-pair $(\mathfrak{F}'_G\langle\Lambda; U\rangle, \mathfrak{F}_G\langle\Lambda; U\rangle)$ to be $(\overline{\mathfrak{E}'_{N_G\Lambda}\langle\Lambda; U\rangle}, \overline{\mathfrak{E}_{N_G\Lambda}\langle\Lambda; U\rangle})$. Note that there are canonical comparison maps

$$\kappa : E(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda; U\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda; U\rangle) \longrightarrow E(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda\rangle),$$

$$\tilde{\kappa} : G \ltimes_{N_G\Lambda} E(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda;U\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda;U\rangle) \longrightarrow E(\mathfrak{F}'_G\langle\Lambda;U\rangle, \mathfrak{F}_G\langle\Lambda;U\rangle),$$

and

$$\kappa : E(\mathfrak{F}'_G\langle\Lambda;U\rangle, \mathfrak{F}_G\langle\Lambda;U\rangle) \longrightarrow E(\mathfrak{F}'_G\langle\Lambda\rangle, \mathfrak{F}_G\langle\Lambda\rangle).$$

Moreover, for any Π-adjacent pair $(\mathfrak{F}', \mathfrak{F})$ of subgroups of G associated to Λ, there is a canonical comparison map

$$\tilde{\kappa} : G \ltimes_{N_G\Lambda} E(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda;U\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda;U\rangle) \longrightarrow E(\mathfrak{F}', \mathfrak{F}).$$

The embedding condition on $H\Pi/H\Lambda$ in Definition 8.1(c), like the embedding condition in Definition 3.7, is stronger than one might expect. Its significance can be understood by examining Theorem 11.8, which plays an essential role in the proofs of our main results.

The following lemma, which follows from an examination of fixed-point sets, describes an important connection between the universal spaces $E(\Lambda, \Pi, G; U)$ of Definition 3.7 and $E(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda;U\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda;U\rangle)$ of Definition 8.1(c).

LEMMA 8.2. *Let Λ be a subgroup of the normal subgroup Π of G, and let U be a G-universe. If the $W_G\Lambda$-space $E(\Lambda, \Pi, G; U)$ is regarded as a $N_G\Lambda$-space via the projection $N_G\Lambda \longrightarrow W_G\Lambda$, then the composite*

$$E(\Lambda, \Pi, G; U) \xrightarrow{\lambda} E\mathfrak{E}'_{N_G\Lambda}\langle\Lambda;U\rangle \xrightarrow{\mu} E(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda;U\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda;U\rangle)$$

is, up to an $N_G\Lambda$-equivalence, the inclusion into $E(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda;U\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda;U\rangle)$ of its Λ-fixed-point set.

Our main approximation result describes the behavior of the final comparison maps introduced in parts (b) and (c) of Definition 8.1.

PROPOSITION 8.3. *Let Λ be a subgroup of the normal subgroup Π of G, and let $(\mathfrak{F}', \mathfrak{F})$ be a Π-adjacent pair of families of subgroups of G associated to Λ. Then the canonical map*

$$\tilde{\kappa} : G \ltimes_{N_G\Lambda} E(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda\rangle) \longrightarrow E(\mathfrak{F}', \mathfrak{F})$$

is a weak G-equivalence. Moreover, if D is a G-spectrum indexed on a G-universe U, then the canonical map

$$(\tilde{\kappa} \wedge 1)^{\Pi} : ((G \ltimes_{N_G\Lambda} E(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda;U\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda;U\rangle)) \wedge D)^{\Pi} \longrightarrow (E(\mathfrak{F}', \mathfrak{F}) \wedge D)^{\Pi}$$

is a weak $\mathcal{G}$-equivalence.

The first part of the proposition above follows from a general result giving conditions under which the canonical comparison map $\tilde{\kappa} : G \ltimes_{N_G\Lambda} E(\mathfrak{E}', \mathfrak{E}) \longrightarrow E(\mathfrak{F}', \mathfrak{F})$ is a G-equivalence.

PROPOSITION 8.4. *Let Λ be a subgroup of the normal subgroup Π of G, and let $(\mathfrak{E}', \mathfrak{E})$ be a pair of $N_G\Lambda$-families of subgroups such that*

$$\mathfrak{E}' - \mathfrak{E} \subset \{H \leq N_G\Lambda \mid H \cap \Pi = \Lambda\}.$$

Let $\overline{\mathfrak{E}}'$ and $\overline{\mathfrak{E}}$ be the G-families of subgroups obtained by closing $\mathfrak{E}'$ and $\mathfrak{E}$ under G-conjugation. If

$$\overline{\mathfrak{E}} \cap \{H \leq G \mid (H \cap \Pi)_G = (\Lambda)_G\} = \emptyset,$$

then the canonical G-map

$$\tilde{\kappa} : G \ltimes_{N_G\Lambda} E(\mathfrak{E}', \mathfrak{E}) \longrightarrow E(\overline{\mathfrak{E}}', \overline{\mathfrak{E}})$$

is a G-equivalence.

PROOF. Consider the commuting diagram

$$\begin{array}{ccc}
G \ltimes_{N_G\Lambda} E(\mathfrak{C}', \mathfrak{C}) & \xrightarrow{\quad\tilde{\kappa}\quad} & E(\overline{\mathfrak{C}}', \overline{\mathfrak{C}}) \\
{\scriptstyle 1\ltimes\Delta}\big\downarrow & & \big\downarrow{\scriptstyle \Delta} \\
G \ltimes_{N_G\Lambda} (E(\mathfrak{C}', \mathfrak{C}) \wedge E(\mathfrak{C}', \mathfrak{C})) & & E(\overline{\mathfrak{C}}', \overline{\mathfrak{C}}) \wedge E(\overline{\mathfrak{C}}', \overline{\mathfrak{C}}) \\
{\scriptstyle 1\ltimes(\kappa\wedge 1)}\big\downarrow & & \big\uparrow{\scriptstyle 1\wedge\tilde{\kappa}} \\
G \ltimes_{N_G\Lambda} (E(\overline{\mathfrak{C}}', \overline{\mathfrak{C}}) \wedge E(\mathfrak{C}', \mathfrak{C})) & \xrightarrow{\ \cong\ } & E(\overline{\mathfrak{C}}', \overline{\mathfrak{C}}) \wedge G \ltimes_{N_G\Lambda} E(\mathfrak{C}', \mathfrak{C})
\end{array}$$

in which the bottom horizontal map is the inverse of G-homeomorphism ζ of Lemma II.4.9 of [**24**]. An examination of its behavior on fixed-point sets indicates that the diagonal map

$$\Delta : E(\mathfrak{C}', \mathfrak{C}) \longrightarrow E(\mathfrak{C}', \mathfrak{C}) \wedge E(\mathfrak{C}', \mathfrak{C})$$

appearing in the left column is an $N_G\Lambda$-equivalence. A similar argument indicates that the other diagonal map appearing in the diagram is a G-equivalence. Thus, to show that the map $\tilde{\kappa}$ is a G-equivalence, it suffices to show that the map $\kappa \wedge 1$ appearing in the left column is an $N_G\Lambda$-equivalence, and that the map $1 \wedge \tilde{\kappa}$ appearing in the right column is a G-equivalence.

The conditions imposed on $\mathfrak{C}' - \mathfrak{C}$ and $\overline{\mathfrak{C}}$ in the hypotheses of this proposition imply that $\mathfrak{C} = \overline{\mathfrak{C}} \cap \mathfrak{C}'$. It follows that, for each $H \in \mathfrak{C}' - \mathfrak{C}$, the map on fixed-point sets $\kappa^H : E(\mathfrak{C}', \mathfrak{C})^H \longrightarrow E(\overline{\mathfrak{C}}', \overline{\mathfrak{C}})^H$ is a nonequivariant equivalence between spaces homotopy equivalent to S^0. Thus, by Lemma V.7.4 of [**24**], the map $\kappa \wedge 1$ is an $N_G\Lambda$-equivalence.

To show that $1 \wedge \tilde{\kappa}$ is a G-equivalence, it suffices to show that the map $\tilde{\kappa}^H : (G \ltimes_{N_G\Lambda} E(\mathfrak{C}', \mathfrak{C}))^H \longrightarrow E(\overline{\mathfrak{C}}', \overline{\mathfrak{C}})^H$ is a nonequivariant equivalence for each subgroup H in $\overline{\mathfrak{C}}' - \overline{\mathfrak{C}}$. Every H in $\overline{\mathfrak{C}}'$ is G-conjugate to a subgroup in $\mathfrak{C}'$, so we may as well assume that H is in $\mathfrak{C}'$. Then, since $\mathfrak{C} = \overline{\mathfrak{C}} \cap \mathfrak{C}'$ and $H \in \overline{\mathfrak{C}}' - \overline{\mathfrak{C}}$, H must be in $\mathfrak{C}' - \mathfrak{C}$. Recall from Section II.4 of [**24**] that there is a natural inclusion $\eta : E(\mathfrak{C}', \mathfrak{C}) \longrightarrow G \ltimes_{N_G\Lambda} E(\mathfrak{C}', \mathfrak{C})$ such that $\tilde{\kappa} \circ \eta = \kappa$. Since we have already shown that κ^H is a nonequivariant equivalence, to show that $\tilde{\kappa}^H$ is a nonequivariant equivalence, it suffices to show that η^H is a nonequivariant equivalence.

In fact, we show that, for an appropriately chosen model of $E(\mathfrak{C}', \mathfrak{C})$, η^H is a homeomorphism. Let $g \in G$ and $x \in E(\mathfrak{C}', \mathfrak{C})$ be such that the equivalence class $[g, x]$ of the pair (g, x) in $G \ltimes_{N_G\Lambda} E(\mathfrak{C}', \mathfrak{C})$ is in $(G \ltimes_{N_G\Lambda} E(\mathfrak{C}', \mathfrak{C}))^H$ and is not the basepoint. Then, for each $h \in H$, there is an $n \in N_G\Lambda$ such that $(hg, x) = (gn, n^{-1}x)$. It follows that n must be in the isotropy subgroup $N_G\Lambda_x$ of x. There is a model for $E(\mathfrak{C}', \mathfrak{C})$ in which the isotropy groups of all of the points except the basepoint are in $\mathfrak{C}'$. By using this model, we can ensure that $N_G\Lambda_x$ is in $\mathfrak{C}'$. Since $hg = gn$, $g^{-1}Hg$ must be a subgroup of $N_G\Lambda_x$ and so must be in $\mathfrak{C}'$. By our hypothesis on $\mathfrak{C}' - \mathfrak{C}$, $H \cap \Pi = \Lambda$, and so $g^{-1}Hg \cap \Pi = g^{-1}\Lambda g$. But then $g^{-1}Hg$ cannot be in $\overline{\mathfrak{C}}$. Thus, $g^{-1}Hg \in \mathfrak{C}' - \mathfrak{C}$, and $g^{-1}Hg \cap \Pi = \Lambda$. It follows that $g^{-1}\Lambda g = \Lambda$ so that $g \in N_G\Lambda$. By the definition of $G \ltimes_{N_G\Lambda} E(\mathfrak{C}', \mathfrak{C})$, the equivalence classes $[g, x]$ and $[e, g^{-1}x]$ are equal. The map η^H is therefore onto. Since it is also an inclusion, it is a homeomorphism. □

REMARK 8.5. Let $(\mathfrak{E}', \mathfrak{E})$ be a pair of $N_G\Lambda$-families such that

$$\mathfrak{E}' - \mathfrak{E} \subset \{H \leq N_G\Lambda \mid H \cap \Pi = \Lambda\} \text{ and } \mathfrak{E} \subset \{H \leq N_G\Lambda \mid H \cap \Pi < \Lambda\}.$$

From these conditions, it follows easily that

$$\overline{\mathfrak{E}} \cap \{H \leq G \mid (H \cap \Pi)_G = (\Lambda)_G\} = \emptyset.$$

Thus, the pair $(\mathfrak{E}', \mathfrak{E})$ satisfies the hypotheses of the proposition. Observe that the pairs $(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda\rangle)$ and $(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda; U\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda; U\rangle)$ satisfy these stronger conditions, and so satisfy the hypotheses of the proposition.

We can now prove our main approximation result.

PROOF OF PROPOSITION 8.3. Observe that, by Proposition 8.4 and Remark 8.5, the canonical maps

$$\tilde{\kappa} : G \ltimes_{N_G\Lambda} E(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda\rangle) \longrightarrow E(\mathfrak{F}'_G\langle\Lambda\rangle, \mathfrak{F}_G\langle\Lambda\rangle)$$

and

$$\tilde{\kappa} : G \ltimes_{N_G\Lambda} E(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda; U\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda; U\rangle) \longrightarrow E(\mathfrak{F}'_G\langle\Lambda; U\rangle, \mathfrak{F}_G\langle\Lambda; U\rangle)$$

of Definition 8.1 are G-equivalences. Let $(\mathfrak{F}', \mathfrak{F})$ be a Π-adjacent pair of G-families associated to Λ. An examination of the fixed-point sets reveals that the map

$$\kappa : E(\mathfrak{F}'_G\langle\Lambda\rangle, \mathfrak{F}_G\langle\Lambda\rangle) \longrightarrow E(\mathfrak{F}', \mathfrak{F})$$

is a G-equivalence. The map $\tilde{\kappa} : G \ltimes_{N_G\Lambda} E(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda\rangle) \longrightarrow E(\mathfrak{F}', \mathfrak{F})$ of part (a) of the proposition is just the composite of this map κ and the map $\tilde{\kappa}$ of Definition 8.1(b), which was just shown to be a G-equivalence.

For part (b) of the proposition, note that the two pairs $(\mathfrak{F}'_G\langle\Lambda; U\rangle, \mathfrak{F}_G\langle\Lambda; U\rangle)$ and $(\mathfrak{F}'_G\langle\Lambda\rangle, \mathfrak{F}_G\langle\Lambda\rangle)$ satisfy the hypotheses of Lemma 6.3 so that the sequence

$$E(\mathfrak{F}'_G\langle\Lambda; U\rangle, \mathfrak{F}_G\langle\Lambda; U\rangle) \longrightarrow E(\mathfrak{F}'_G\langle\Lambda\rangle, \mathfrak{F}_G\langle\Lambda\rangle) \longrightarrow E(\mathfrak{F}'_G\langle\Lambda\rangle, \mathfrak{F}'_G\langle\Lambda; U\rangle \cup \mathfrak{F}_G\langle\Lambda\rangle)$$

is a G-equivariant cofibre sequence. If $H \in \mathfrak{E}'_{N_G\Lambda}\langle\Lambda; U\rangle - \mathfrak{E}_{N_G\Lambda}\langle\Lambda; U\rangle$, then $H \cap \Pi = \Lambda$ and $H\Lambda = H$. Thus, $H\Pi/H$ embeds as a $H\Pi$-space in U. It follows that

$$\mathfrak{E}'_{N_G\Lambda}\langle\Lambda; U\rangle - \mathfrak{E}_{N_G\Lambda}\langle\Lambda; U\rangle = \{H \leq N_G\Lambda \mid H \cap \Pi = \Lambda \text{ and } H\Pi/H\ H\Pi\text{-embeds in } U\}$$

and

$$\mathfrak{F}'_G\langle\Lambda; U\rangle - \mathfrak{F}_G\langle\Lambda; U\rangle = \{H \leq G \mid (H \cap \Pi)_G = (\Lambda)_G \text{ and } H\Pi/H\ H\Pi\text{-embeds in } U\}.$$

The second equation above implies that the G-pair $(\mathfrak{F}'_G\langle\Lambda\rangle, \mathfrak{F}'_G\langle\Lambda; U\rangle \cup \mathfrak{F}_G\langle\Lambda\rangle)$ satisfies the hypotheses of Proposition 6.1. Thus, for any G-spectrum D indexed on U, the canonical map

$$(\kappa \wedge 1)^\Pi : (E(\mathfrak{F}'_G\langle\Lambda; U\rangle, \mathfrak{F}_G\langle\Lambda; U\rangle) \wedge D)^\Pi \longrightarrow (E(\mathfrak{F}'_G\langle\Lambda\rangle, \mathfrak{F}_G\langle\Lambda\rangle) \wedge D)^\Pi$$

is a weak $\mathcal{G}$-equivalence. The map $(\tilde{\kappa} \wedge 1)^\Pi$ of part (b) is just the composite of this map and two $\mathcal{G}$-equivalences derived from the G-equivalences

$$\kappa : E(\mathfrak{F}'_G\langle\Lambda\rangle, \mathfrak{F}_G\langle\Lambda\rangle) \longrightarrow E(\mathfrak{F}', \mathfrak{F})$$

and

$$\tilde{\kappa} : G \ltimes_{N_G\Lambda} E(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda; U\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda; U\rangle) \longrightarrow E(\mathfrak{F}'_G\langle\Lambda; U\rangle, \mathfrak{F}_G\langle\Lambda; U\rangle). \qquad \square$$

SECTION 9

Change of universe functors and families of subgroups

In this section, we examine the question of when a G-spectrum C indexed on a G-universe U can be represented by a G-spectrum C' indexed on a smaller G-universe U'. This question always arises when the Adams isomorphism is invoked. In particular, the remarks preceding Theorems 2.4 and 3.3 deal with the special cases of this question which arise in those two theorems. Section II.2 of [**24**] addresses this issue. However, the results presented there are not as sharp as possible, and are inadequate for our discussion of the Adams isomorphism for incomplete universes. The essential difference between what is done here and what was done in Section II.2 of [**24**] is that here, when we compare two G-universes U and U', we look at the subgroups H of G for which U and U' are H-orbit equivalent rather than at the subgroups H for which U and U' are isomorphic. Thus, the foundation for the results here is Theorem 1.2(b) of [**20**] rather than the earlier, weaker Corollary II.1.8 of [**24**]. For convenience, we assemble here a collection of relevant definitions from [**20, 24**].

Definition 9.1. Let U and U' be G-universes, and let H be a subgroup of G.

(a) The universe U embeds in U' up to H-orbits if, for each pair $J \leq K$ of subgroups of H such that the orbit K/J K-embeds in U, this orbit also K-embeds in U'. Note that, if there is a linear H-isometry from U into U', then U embeds in U' up to H-orbits. However, it is possible for U to embed in U' up to H-orbits even if there is no linear H-isometry from U into U'. For example, if $G = \mathbb{Z}/p$ for some prime p, then any nontrivial G-universe U embeds in any other nontrivial G-universe U' up to G-orbits.

(b) The universes U' and U are H-orbit equivalent if, for each pair $J \leq K$ of subgroups of H, the orbit K/J K-embeds in U if and only if it K-embeds in U'. Thus, U' and U are H-orbit equivalent if and only if each embeds in the other up to H-orbits.

(c) Let $\tilde{\mathfrak{E}}(U, U') \subset \tilde{\mathfrak{F}}(U, U')$ be the families of subgroups of G consisting respectively of those $H \leq G$ such that U and U' are H-orbit equivalent and those such that U embeds in U' up to H-orbits. Similarly, let $\mathfrak{E}(U, U') \subset \mathfrak{F}(U, U')$ be the families of subgroups of G consisting respectively of those $H \leq G$ such that U and U' are H-isomorphic and those such that there is a linear H-isometry from U into U'. Note that $\mathfrak{E}(U, U') \subset \tilde{\mathfrak{E}}(U, U')$ and $\mathfrak{F}(U, U') \subset \tilde{\mathfrak{F}}(U, U')$. Also observe that, if there is a linear G-isometry $i : U' \longrightarrow U$, then $\tilde{\mathfrak{E}}(U, U') = \tilde{\mathfrak{F}}(U, U')$ and $\mathfrak{E}(U, U') = \mathfrak{F}(U, U')$. The families $\mathfrak{E}(U, U')$ and $\mathfrak{F}(U, U')$ were introduced in [**24**] before results like Theorem 1.2(b) of [**20**] were known. However, now that stronger results are available, the families $\mathfrak{E}(U, U')$ and $\mathfrak{F}(U, U')$ should be replaced by the families $\tilde{\mathfrak{E}}(U, U')$ and $\tilde{\mathfrak{F}}(U, U')$, respectively, in almost every context.

DEFINITION 9.2. Let $i : U' \longrightarrow U$ be a linear G-isometry. A U'-representation of a G-spectrum C indexed on U is a G-spectrum C' indexed on U' together with a weak G-equivalence $i_*C' \longrightarrow C$.

The main result in this section is a generalization of Theorem II.2.6 of [**24**].

THEOREM 9.3. *Let $i : U' \longrightarrow U$ be a linear G-isometry, E' and F' be G-spectra indexed on U', and C be a G-spectrum indexed on U.*

(a) *If E' is an $\tilde{\mathfrak{F}}(U, U')$-spectrum, then the map*

$$i_* : [E', F']_G^{U'} \longrightarrow [i_*E', i_*F']_G^U$$

is an isomorphism.

(b) *If C is an $\tilde{\mathfrak{F}}(U, U')$-spectrum, then C admits a U'-representation by an $\tilde{\mathfrak{F}}(U, U')$-spectrum C' indexed on U'. Moreover, C' is unique up to G-equivalence, and can be chosen to have cells in a canonical bijective correspondence with the cells of C.*

PROOF. This result follows from Theorem 1.2(b) of [**20**] in the same way that Theorem II.2.6 of [**24**] follows from Corollary II.1.8 of [**24**]. □

In the setting in which this theorem is most frequently applied, Π is a normal subgroup of G, $\mathcal{G} = G/\Pi$, U is a G-universe, and U' is a Π-trivial G-universe which embeds in U via a linear G-isometry $i : U' \longrightarrow U$. Typically, in fact, $U' = U^\Pi$. If $\mathfrak{F}$ is a family of subgroups of G contained in $\tilde{\mathfrak{F}}(U, U')$ and C is a G-spectrum indexed on U, then the spectrum $E\mathfrak{F}_+ \wedge C$ is an $\mathfrak{F}$-spectrum and so also an $\tilde{\mathfrak{F}}(U, U')$-spectrum. Thus, by the theorem, there is a G-spectrum $\widehat{C}$ indexed on U' and a weak G-equivalence $i_*\widehat{C} \longrightarrow E\mathfrak{F}_+ \wedge C$. Since Π acts trivially on the universe U', $\widehat{C}$ has an associated orbit spectrum $\widehat{C}/\Pi$ which carries a canonical $\mathcal{G}$-action. This orbit spectrum is usually denoted $E\mathfrak{F}_+ \wedge_\Pi C$. Theorem 9.3 guarantees both the existence and the naturality of this construction.

COROLLARY 9.4. *Let Π be a normal subgroup of the compact Lie group G, $\mathcal{G}$ be G/Π, U be a G-universe, and U' be a Π-trivial G-universe which G-embeds in U. Then, for any family $\mathfrak{F}$ of subgroups of G which is contained in $\tilde{\mathfrak{F}}(U, U')$, the assignment of the $\mathcal{G}$-spectrum $E\mathfrak{F}_+ \wedge_\Pi C$ indexed on U' to each G-spectrum C indexed on U gives a functor from the stable category of G-spectra indexed on U to the stable category of $\mathcal{G}$-spectra indexed on U'.*

SECTION 10

The geometric fixed-point functor Φ^Λ for incomplete universes

There are actually two definitions of the geometric fixed-point functor Φ^Λ. One is very straightforward and geometric; the other is homotopy theoretic and makes use of universal spaces for families of subgroups and the categorical fixed-point functor. The equivalence of these two rather disparate definitions is extremely important because it allows us to use two quite different sets of techniques when working with this functor. Most of the basic properties of the geometric fixed-point functor, including the equivalence of the two definitions, are discussed in Section II.9 of [**24**]. That section of [**24**], unlike some of the other technical sections, was explicitly written so that it applies to incomplete universes. However, there are some minor oversights in what is said there. Moreover, for our work here, we need a few properties of the geometric fixed-point functor that are not covered there. In this section, we correct the minor difficulties in Section II.9 of [**24**], and supply the necessary additional results. Once this is done, we have all that is needed for the proof of Proposition 1.2. The latter part of this section is devoted to that proof.

We begin with the definition of Φ^Λ in terms of universal spaces for families of subgroups and the categorical fixed-point functor since this definition is the one most directly connected to geometric splittings.

Definition 10.1. Let Λ be a subgroup of a compact Lie group G, U be a G-universe, and C be a G-spectrum indexed on U. Recall that $\mathfrak{F}_{N_G\Lambda}[\Lambda]$ is the $N_G\Lambda$-family consisting of those subgroups H of $N_G\Lambda$ which do not contain Λ. Also recall that the space $\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda]$ is the cofibre of the obvious collapse map $E\mathfrak{F}_{N_G\Lambda}[\Lambda]_+ \longrightarrow S^0$. The geometric fixed-point spectrum $\Phi^\Lambda C$ of C is the $W_G\Lambda$-spectrum $(\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda]\wedge C)^\Lambda$ indexed on the universe U^Λ. The natural transformation $\omega_\Lambda : C^\Lambda \longrightarrow \Phi^\Lambda C$ is the canonical map induced by the projection $S^0 \longrightarrow \tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda]$.

The oversights in Section II.9 of [**24**] must be addressed because of their impact on the proofs of various results such as Proposition 1.2. In correcting these mistakes, we switch from the perspective of [**24**], which describes the geometric fixed-point functor Φ^N associated to a normal subgroup N of a group G, to the perspective of the previous sections, in which the fixed-point functor Φ^Λ associated to an arbitrary subgroup Λ of G is used. Thus, the N and G of [**24**] are here replaced by Λ and $N_G\Lambda$, respectively.

Scholium 10.2. (a) Several results in Section II.9 of [**24**] assert the existence of an equivariant weak equivalence between two spectra without identifying the map which gives that equivalence. The identity of these maps is important for our work. In particular, the weak $W_G\Lambda$-equivalence

$$\Sigma^\infty(X^\Lambda) \simeq \Phi^\Lambda(\Sigma_U^\infty X)$$

of Corollary II.9.9 of [**24**] is the composite

$$\Sigma^\infty(X^\Lambda) \xrightarrow{\zeta} (\Sigma^\infty X)^\Lambda \xrightarrow{\mu} (\Sigma^\infty \tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge X)^\Lambda = \Phi^\Lambda(\Sigma_U^\infty X),$$

in which ζ is the natural transformation introduced in Remarks II.3.14(i) of [**24**] and μ is the map induced by the canonical projection $S^0 \longrightarrow \tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda]$. Moreover, the weak $W_G\Lambda$-equivalences

$$(\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge D \wedge X)^\Lambda \simeq (\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge D)^\Lambda \wedge X^\Lambda$$

and

$$(\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge D \wedge D')^\Lambda \simeq (\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge D)^\Lambda \wedge (\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge D')^\Lambda$$

of parts (i) and (ii) of Proposition II.9.12 of [**24**] are the inverses of the map

$$(\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge D)^\Lambda \wedge X^\Lambda \xrightarrow{\nu} (\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge D \wedge X)^\Lambda$$

and the composite

$$\begin{aligned} &(\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge D)^\Lambda \wedge (\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge D')^\Lambda \\ \xrightarrow{\omega}\quad &(\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge D \wedge \tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge D')^\Lambda \\ \simeq\quad &(\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge D \wedge D')^\Lambda. \end{aligned}$$

Here, ν and ω are the natural transformations of Remarks II.3.14 (ii) and (iii) of [**24**], and the weak equivalence in the composite is derived from the fact that the diagonal map

$$\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \longrightarrow \tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge \tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda]$$

is an equivariant homotopy equivalence. This identification of the maps in Corollary II.9.9 and Proposition II.9.12 of [**24**] follows from an inspection of the proofs of those results and the descriptions of the natural transformations ζ, ν and ω given in Remarks II.3.14 of [**24**].

(b) Proposition II.9.13 of [**24**] applies for incomplete universes in spite of the fact that the proof given for it implicitly assumes that the universe is complete. In the proof of the proposition, it is asserted that the colimit of a certain collection of spheres S^V forms a model for the universal space $\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda]$. In fact, if the $N_G\Lambda$-universe U is incomplete, then this colimit forms a model for the universal space $\tilde{E}\mathfrak{F}$, where $\mathfrak{F}$ is the somewhat smaller family consisting of those subgroups K of $N_G\Lambda$ such that K does not contain Λ and such that there is a finite-dimensional subrepresentation V of U for which $(V - V^\Lambda)^K \neq 0$. Proposition 6.1 indicates that, for any $N_G\Lambda$-spectrum D indexed on U, the Λ-fixed-point spectrum $(E(\mathfrak{F}_{N_G\Lambda}[\Lambda], \mathfrak{F}) \wedge D)^\Lambda$ is weakly $W_G\Lambda$-contractible. It follows by an inspection of the cofibre sequence

$$E(\mathfrak{F}_{N_G\Lambda}[\Lambda], \mathfrak{F}) \longrightarrow \tilde{E}\mathfrak{F} \longrightarrow \tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda]$$

that, for the purpose of proving Proposition II.9.13, $\tilde{E}\mathfrak{F}$ may be substituted for $\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda]$.

The first of our supplementary technical lemmas provides the basis for showing that, if a subgroup Λ acts trivially on an indexing universe U, then all G-spectra indexed on U are geometrically split at Λ.

LEMMA 10.3. *Let U be a Λ-trivial $N_G\Lambda$-universe and D be an $N_G\Lambda$-spectrum indexed on U. Then the canonical map*

$$\omega_\Lambda : D^\Lambda \longrightarrow \Phi^\Lambda D$$

is a weak $W_G\Lambda$-equivalence.

PROOF. Since there is a cofibre sequence

$$(E\mathfrak{F}_{N_G\Lambda}[\Lambda]_+ \wedge D)^\Lambda \longrightarrow D^\Lambda \xrightarrow{\omega_\Lambda} (\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge D)^\Lambda = \Phi^\Lambda D,$$

it suffices to show that the spectrum $(E\mathfrak{F}_{N_G\Lambda}[\Lambda]_+ \wedge D)^\Lambda$ is weakly $W_G\Lambda$-contractible. Let $\mathfrak{F}$ be the empty family of subgroups of $N_G\Lambda$. The space $E\mathfrak{F}_{N_G\Lambda}[\Lambda]_+$ may be regarded as the universal space $E(\mathfrak{F}_{N_G\Lambda}[\Lambda], \mathfrak{F})$ of the pair $(\mathfrak{F}_{N_G\Lambda}[\Lambda], \mathfrak{F})$. Thus, by Proposition 6.1, to show the desired weak contractibility, it suffices to show that, if $H \in \mathfrak{F}_{N_G\Lambda}[\Lambda]$, then $H\Lambda/H$ does not embed in U as a $H\Lambda$-space. This is obvious since the $H\Lambda$-isotropy subgroup of every point of the Λ-trivial universe U must contain Λ and so cannot be H. □

Our second supplementary technical lemma describes the interaction of change of universe functors and geometric fixed-point functors. It is the key to showing that change of universe functors preserve geometric splittings. Since geometric fixed-point functors for two different universes appear in this result, the notation Φ^Λ_U is used here to denote the geometric fixed-point functor associated to the universe U.

LEMMA 10.4. *Let $i : U \longrightarrow U'$ be a $N_G\Lambda$-isometry between two $N_G\Lambda$-universes U and U', and $i^\Lambda : U^\Lambda \longrightarrow (U')^\Lambda$ be the isometry obtained from i by passage to Λ-fixed-points. If D is an $N_G\Lambda$-spectrum indexed on U, then there are natural $W_G\Lambda$-maps*

$$\delta : i^\Lambda_*(D^\Lambda) \longrightarrow (i_*D)^\Lambda$$

and

$$\hat{\delta} : i^\Lambda_*(\Phi^\Lambda_U D) \longrightarrow \Phi^\Lambda_{U'}(i_*D)$$

making the diagram

$$\begin{array}{ccc} i^\Lambda_*(D^\Lambda) & \xrightarrow{\delta} & (i_*D)^\Lambda \\ {\scriptstyle i^\Lambda_*\omega_\Lambda}\downarrow & & \downarrow{\scriptstyle \omega_\Lambda} \\ i^\Lambda_*(\Phi^\Lambda_U D) & \xrightarrow{\hat{\delta}} & \Phi^\Lambda_{U'}(i_*D) \end{array}$$

commute. Moreover, the map $\hat{\delta}$ is a weak $W_G\Lambda$-equivalence.

PROOF. Let $j^\Lambda : U^\Lambda \longrightarrow U$ and $k^\Lambda : (U')^\Lambda \longrightarrow U'$ be the inclusions of the two Λ-fixed universes. To keep track of the change of universe functors appearing in this proof, we expand our usual notation for fixed-point spectra to include explicitly the change of universes involved in the passage to fixed points. Thus, the spectrum usually denoted D^Λ is here denoted $(j^*_\Lambda D)^\Lambda$.

The natural map δ is a composite of the form

$$i^\Lambda_*((j^*_\Lambda D)^\Lambda) \longrightarrow (i^\Lambda_* j^*_\Lambda D)^\Lambda \longrightarrow (k^*_\Lambda i_* D)^\Lambda.$$

The first map in this composite is an instance of a $W_G\Lambda$-map $i^\Lambda_*(E^\Lambda) \longrightarrow (i^\Lambda_* E)^\Lambda$ applicable to all $N_G\Lambda$-spectra E indexed on U^Λ. This natural map is the adjoint of the $N_G\Lambda$-map $i^\Lambda_*(E^\Lambda) \longrightarrow i^\Lambda_* E$ derived from the inclusion $E^\Lambda \longrightarrow E$. The second map in the composite is obtained by passage to Λ-fixed points from the natural $N_G\Lambda$-map $i_*^\Lambda j_\Lambda^* D \longrightarrow k_\Lambda^* i_* D$ which is the adjoint of the map

$$k_*^\Lambda i_*^\Lambda j_\Lambda^* D \cong i_* j_*^\Lambda j_\Lambda^* D \xrightarrow{i_*\epsilon} i_* D.$$

Here, ϵ is the counit of the $(j_*^\Lambda, j_\Lambda^*)$-adjunction and the isomorphism is derived from the equation $k^\Lambda \circ i^\Lambda = i \circ j^\Lambda$. The natural map $\hat{\delta}$ is just the composite

$$i_*^\Lambda \Phi_U^\Lambda D = i_*^\Lambda((j_\Lambda^*(\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge D))^\Lambda) \xrightarrow{\delta} (k_\Lambda^* i_*(\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge D))^\Lambda = \Phi_{U'}^\Lambda(i_* D).$$

The diagram asserted to commute is therefore just a naturality diagram for δ.

In order to show that $\hat{\delta}$ is a weak $W_G\Lambda$-equivalence, we must use the alternative geometric definition of the geometric fixed-point functor which is given in Definition II.9.7 of [**24**]. To distinguish the geometric version of the functor from the the homotopy-theoretic version, we denote the geometric version from Definition II.9.7 of [**24**] by $\tilde{\Phi}_U^\Lambda$. This functor is applicable to $N_G\Lambda$-spectra indexed on U. It is constructed in [**24**] by first introducing a prespectrum-level version and then passing to the spectrum-level version using an approximation by CW-prespectra. It is easy to see that, if we apply the prespectrum-level versions of the composite functors $i_*^\Lambda \circ \tilde{\Phi}_U^\Lambda$ and $\tilde{\Phi}_{U'}^\Lambda \circ i_*$ to a Σ-inclusion prespectrum $\tilde{D}$ indexed on U, then the resulting prespectra $i_*^\Lambda(\tilde{\Phi}_U^\Lambda \tilde{D})$ and $\tilde{\Phi}_{U'}^\Lambda(i_*\tilde{D})$ are naturally isomorphic. It follows that there is a natural weak $W_G\Lambda$-equivalence

$$\tilde{\delta} : i_*^\Lambda \tilde{\Phi}_U^\Lambda D \longrightarrow \tilde{\Phi}_{U'}^\Lambda(i_* D)$$

relating the spectrum-level functors. Under the natural weak $W_G\Lambda$-equivalence

$$\xi : \Phi_U^\Lambda D = (j_\Lambda^*(\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge D))^\Lambda \longrightarrow \tilde{\Phi}_U^\Lambda D$$

of Theorem II.9.8(ii) of [**24**] and the analogous map for the universe U', the natural map $\hat{\delta}$ is identified with $\tilde{\delta}$ and so is a weak $W_G\Lambda$-equivalence. □

We conclude this section with the proof of Proposition 1.2.

PROOF OF PROPOSITION 1.2. Part (a) of the proposition follows directly from Corollary II.9.9 of [**24**] and Scholium 10.2, which identifies the map asserted to be a weak equivalence in the corollory. Parts (b) and (c) follow directly from Lemmas 10.3 and 10.4, respectively. Scholium 10.2 and Proposition II.9.12 of [**24**] provide all that is needed for the proofs of parts (d) and (e). Part (f) follows from an obvious diagram chase. Part (g) follows from parts (b), (c), and (e), since the localization C_T of C at T can be identified with $C \wedge j_*^G S_T^0$, where S_T^0 is the localization at T of the sphere spectrum S^0 indexed on the trivial G-universe U^G. Part (h) follows from Proposition 3.1 of [**7**], which describes the interaction between completions and the geometric fixed-point functor. □

SECTION 11

The Wirthmüller isomorphism for incomplete universes

Let N be a subgroup of a compact Lie group G, L be the N-representation given by the tangent space of G/N at the identity coset eN, and D be a N-spectrum indexed on a complete G-universe. Then the Wirthmüller isomorphism (see [**29**] and Theorem II.6.2 of [**24**]) identifies the G-spectra $G \ltimes_N D$ and $F_N[G, \Sigma^L D)$. It is easy to see that the existence of such an isomorphism for every N-spectrum D implies that the G-spectrum $\Sigma^\infty G/N_+$ has a well-behaved Spanier-Whitehead dual. This observation severely restricts the extent to which the Wirthmüller isomorphism can be extended to an incomplete G-universe U since, by Proposition 7.1, the G-spectrum $\Sigma_U^\infty G/N_+$ cannot have a well-behaved Spanier-Whitehead dual unless the orbit G/N embeds as a G-space in U.

What can be expected for an incomplete universe is a pair of partial extensions. One result in this pair should assert that, if the orbit G/N does embed, then there is an Wirthmüller isomorphism for N-spectra indexed on U just as in the context of a complete universe. The other result should indicate that, even if the orbit does not embed, there is some reasonable N-family $\mathfrak{W}$ of subgroups such that a Wirthmüller isomorphism exists for all $\mathfrak{W}$-spectra indexed on U. The first result of this pair is contained implicitly in Section II.6 of [**24**], and is explicitly provided as the first result in this section. The present status of the second part of this pair is less satisfactory. For trivial reasons, there is a family $\mathfrak{W}(N, G; U)$ of subgroups of N such that a Wirthmüller isomorphism exists for all $\mathfrak{W}(N, G; U)$-spectra indexed on U. This family is maximal in the sense that, for any N-family $\mathfrak{F}$ not contained in $\mathfrak{W}(N, G; U)$, there is an $\mathfrak{F}$-spectrum D for which there is no Wirthmüller isomorphism. However, for an arbitrary compact Lie group G, very little is known about the family $\mathfrak{W}(N, G; U)$ beyond these formal properties. Thus, here we introduce two more tractable families of subgroups, $\mathfrak{W}'(N, G; U)$ and $\mathfrak{W}''(N, G; U)$, and state several results about them which are first approximations to the desired second half of our pair. In particular, we show that, if G is a finite group, then the mysterious family $\mathfrak{W}(N, G; U)$ is just the more accessibly defined family $\mathfrak{W}'(N, G; U)$. The family $\mathfrak{W}''(N, G; U)$ is the most easily understood of the three. Moreover, for any compact Lie group G, any subgroup N, and any G-universe U, there is a Wirthmüller isomorphism for all $\mathfrak{W}''(N, G; U)$-spectra indexed on U.

In the places where a Wirthmüller isomorphism is used in the proofs of our splitting theorems, the subgroup N is the normalizer of another subgroup Λ. In this situation, our results on the Wirthmüller isomorphism take a particularly simple form. In addition to playing a role in the proofs of our splitting theorems, the results presented here provide insight into the conditions under which Theorem 3.8 can be extended to a result more like Theorems 2.4 and 3.3.

The approach taken to the Wirthmüller isomorphism in Theorem II.6.2 of [**24**] is not suitable in the context of incomplete universes because the map

$$\omega : F_N[G, \Sigma^L D) \longrightarrow G \ltimes_N D,$$

of that theorem is not even defined if the orbit G/N does not embed in U as a G-space. There is, however, another comparison map

$$\psi : G \ltimes_N D \longrightarrow F_N[G, \Sigma^L D),$$

introduced in Definition II.6.8 of [**24**], which exists for any G-universe U. The proper approach to the Wirthmüller isomorphism for incomplete universes is to look for conditions under which this map ψ is an isomorphism. Hereafter, we refer to ψ as the Wirthmüller map.

If G/N embeds in U as a G-space so that the map ω exists, then results contained in [**24**] suffice to show that ψ and ω are inverse G-equivalences.

PROPOSITION 11.1. *Let N be a subgroup of a compact Lie group G, U be a G-universe into which G/N embeds as a G-space, and D be a N-CW spectrum indexed on U. Then the maps*

$$\psi : G \ltimes_N D \longrightarrow F_N[G, \Sigma^L D)$$

and

$$\omega : F_N[G, \Sigma^L D) \longrightarrow G \ltimes_N D$$

are inverse weak G-equivalences.

PROOF. Let V be a finite dimensional G-representation contained in U into which G/N embeds as a G-space. As a N-representation, V decomposes as a direct sum of L and some other N-representation W. Thus, L is contained in U as a N-representation. By Theorem I.6.1 of [**24**], suspension by L is an invertible functor on the equivariant stable category of N-spectra indexed on U. Also, suspension by V is an invertible functor on the equivariant stable category of G-spectra indexed on U. Since the invertibility of suspension by L and V are the only two properties of the equivariant stable category used in the proof of the Wirthmüller isomorphism in section II.6 of [**24**], that proof extends to a proof of this proposition. □

To begin our discussion of the availability of the Wirthmüller isomorphism when the orbit G/N does not embed in U, we define the three families of subgroups of N mentioned in the introduction. The kindest thing that can be said about the definitions of these families is that they are less enlightening than one would like. This section concludes with an investigation of a very special case of the Wirthmüller isomorphism. This special case offers some insight into the strange nature of our three families of subgroups.

DEFINITION 11.2. Let N be a subgroup of a compact Lie group G, and let U be a G-universe.

(a) Let $\mathfrak{W}(N, G; U)$ be the family of subgroups K of N such that, for every K-spectrum C indexed on U, the map

$$\psi : G \ltimes_N (N \ltimes_K C) \longrightarrow F_N[G, \Sigma^L N \ltimes_K C)$$

is a weak G-equivalence.

(b) Let $\mathfrak{W}'(N,G;U)$ be the family of subgroups K of N such that, for every pair of subgroups J of K and H of G with $J \leq H$, the orbit $(N \cap H)/J$ embeds as an $(N \cap H)$-space in U if and only if the orbit H/J embeds as a H-space in U.

(c) Let $\mathfrak{W}''(N,G;U)$ be the family of subgroups K of N for which there is a subgroup H of N such that $K \leq H$ and G/H embeds as a G-space in U.

Note that, for any family $\mathfrak{W}$ of subgroups of N, to show that the Wirthmüller map ψ is a weak G-equivalence for every $\mathfrak{W}$-spectrum D indexed on U, it suffices to show that the map

$$\psi : G \ltimes_N \Sigma_U^\infty N/K_+ \longrightarrow F_N[G, \Sigma^L \Sigma_U^\infty N/K_+)$$

is a weak G-equivalence for every $K \in \mathfrak{W}$. With this observation, it is easy to see that $\mathfrak{W}(N,G;U)$ is maximal among the families of subgroups of N for which there is a Wirthmüller isomorphism.

PROPOSITION 11.3. *Let N be a subgroup of a compact Lie group G, and let U be a G-universe. Then, for every $\mathfrak{W}(N,G;U)$-spectrum D indexed on U, the map*

$$\psi : G \ltimes_N D \longrightarrow F_N[G, \Sigma^L D),$$

is a weak G-equivalence. Moreover, if $\mathfrak{F}$ is any family of subgroups of N which is not contained in $\mathfrak{W}(N,G;U)$, then there is an $\mathfrak{F}$-spectrum D' indexed on U for which the Wirthmüller map ψ is not an isomorphism.

This result would be somewhat vacuous without the following theorem and conjecture.

THEOREM 11.4. *Let N be a subgroup of a finite group G, and let U be a G-universe. Then, $\mathfrak{W}(N,G;U) = \mathfrak{W}'(N,G;U)$. Thus, for any $\mathfrak{W}'(N,G;U)$-spectrum D indexed on U, the natural map*

$$\psi : G \ltimes_N D \longrightarrow F_N[G, \Sigma^L D),$$

is a weak G-equivalence.

CONJECTURE 11.5. *Let N be a subgroup of a compact Lie group G, and let U be a G-universe. Then, $\mathfrak{W}(N,G;U) = \mathfrak{W}'(N,G;U)$.*

The proof of the theorem above, which is given in Section 15, makes use of the fact that, if G is a finite group and N and H are subgroups, then the orbit G/N, regarded as a H-space, decomposes into a disjoint union of H-orbits. The obvious way to try to extend this proof to a proof of the conjecture would be to replace this decomposition by the skeletal filtration of G/N regarded as a H-CW-complex. This approach cannot work because the orbit G/N (or, more precisely, some closely related constructions) appears covariantly in the domain of the map ψ and contravariantly in the range. It seems likely that a rather delicate geometric argument will be needed to prove the conjecture.

The utility of Proposition 11.3 and Theorem 11.4 is limited by the fact that the families of subgroups associated to those results are not easily understood. The family $\mathfrak{W}''(N,G;U)$ is typically smaller than either of these, but is certainly much easier to work with. Thus, the following version of the Wirthmüller isomorphism theorem is likely to be the most general one which is widely applicable in the context of incomplete universes.

THEOREM 11.6. *Let N be a subgroup of a compact Lie group G, and let U be a G-universe. Then, for every $\mathfrak{W}''(N,G;U)$-spectrum D indexed on U, the map*

$$\psi : G \ltimes_N D \longrightarrow F_N[G, \Sigma^L D),$$

is a weak G-equivalence.

PROOF. As we have already noted, it suffices to consider the case in which $D = \Sigma_U^\infty N/K_+$, for $K \in \mathfrak{W}''(N,G;U)$. Throughout this proof, we work with the suspension spectra of orbits. Thus, to reduce notational clutter, we omit all instances of Σ_U^∞ from our notation. Select a subgroup H of G such that $K \leq H \leq N$ and G/H embeds in U as a G-space. Denote the N-representation given by the tangent space of G/N at the identity coset eN by $L(N)$ rather than just L. Also, denote the H-representations given by the tangent spaces of N/H and G/H at the identity cosets eH by $L(H,N)$ and $L(H)$, respectively. As a H-representation, $L(H)$ is the direct sum of $L(H,N)$ and $L(N)$.

By Lemma II.6.13 of [**24**], the diagram

$$\begin{array}{ccc}
G \ltimes_N N/K_+ & \xrightarrow{\quad\psi\quad} & F_N[G, \Sigma^{L(N)} N/K_+) \\
\cong\downarrow & & \downarrow\cong \\
G \ltimes_H H/K_+ & & F_N[G, \Sigma^{L(N)} N \ltimes_H H/K_+) \\
\psi\downarrow\cong & & \cong\downarrow F_N[G,\Sigma^{L(N)}\psi) \\
F_H[G, \Sigma^{L(H)} H/K_+) & \xrightarrow{\ \cong\ } & F_N[G, \Sigma^{L(N)} F_H[N, \Sigma^{L(H,N)} H/K_+))
\end{array}$$

commutes in the equivariant stable category of G-spectra indexed on U. The two unlabeled vertical isomorphisms are derived from the space-level isomorphism $N/K_+ \cong N \ltimes_H H/K_+$ and one of the isomorphisms given by Lemma II.4.10 of [**24**]. The horizontal isomorphism at the bottom of the diagram is also given by Lemma II.4.10 of [**24**]. The vertical maps labeled ψ and $F_N[G, \Sigma^L \psi)$ are isomorphisms by Proposition 11.1 since the orbit G/H embeds in U as a G-space. It now follows that the upper horizontal arrow

$$\psi : G \ltimes_N N/K_+ \longrightarrow F_N[G, \Sigma^{L(N)} N/K_+)$$

is an isomorphism in the equivariant stable category of G-spectra indexed on U. ☐

REMARK 11.7. Theorem 11.6 implies, of course, that the family $\mathfrak{W}''(N,G;U)$ is contained in the Wirthmüller family $\mathfrak{W}(N,G;U)$. It is fairly easy to see that, as the conjecture would lead us to expect, $\mathfrak{W}''(N,G;U)$ is also contained in $\mathfrak{W}'(N,G;U)$.

In the instances where the Wirthmüller isomorphism is needed for the proof of our splitting theorems, the subgroup which has so far been denoted N is actually the normalizer $N_G\Lambda$ of some other subgroup Λ. In this context, a connection between the family $\mathfrak{E}'_{N_G\Lambda}\langle\Lambda;U\rangle$ of Definition 8.1(c) and the family $\mathfrak{W}''(N_G\Lambda, G;U)$ makes it easier to verify that certain spectra satisfy the hypotheses of Theorem 11.6. As in our earlier notation, the $N_G\Lambda$-representation L is just that derived from the tangent space of $G/N_G\Lambda$ at the identity coset $eN_G\Lambda$.

THEOREM 11.8. *Let Π be a normal subgroup of a compact Lie group G, Λ be a subgroup of Π, and U be a G-universe. Then the $N_G\Lambda$-family $\mathfrak{E}'_{N_G\Lambda}\langle\Lambda;U\rangle$*

is contained in the $N_G\Lambda$-family $\mathfrak{W}''(N_G\Lambda, G; U)$. Thus, if D is a $\mathfrak{E}'_{N_G\Lambda}\langle\Lambda; U\rangle$-spectrum indexed on U, then the natural map

$$\psi : G \ltimes_{N_G\Lambda} D \longrightarrow F_{N_G\Lambda}[G, \Sigma^L D)$$

is a weak G-equivalence. In particular, for any $N_G\Lambda$-spectrum C indexed on U, the maps

$$\psi : G \ltimes_{N_G\Lambda} (E\mathfrak{E}'_{N_G\Lambda}\langle\Lambda; U\rangle_+ \wedge C) \longrightarrow F_{N_G\Lambda}[G, \Sigma^L E\mathfrak{E}'_{N_G\Lambda}\langle\Lambda; U\rangle_+ \wedge C)$$

and

$$\psi : G \ltimes_{N_G\Lambda} (E(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda; U\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda; U\rangle) \wedge C) \longrightarrow$$
$$F_{N_G\Lambda}[G, \Sigma^L E(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda; U\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda; U\rangle) \wedge C)$$

are weak G-equivalences.

This result follows from a simple technical lemma about the isotropy subgroups of arbitrary G-sets.

LEMMA 11.9. *Let Π be a normal subgroup of the group G, Q be a subgroup of G, X be a G-set, and x be a point of X such that the $Q\Pi$-isotropy subgroup $(Q\Pi)_x$ of x is Q. Then the G-isotropy subgroup G_x of x is contained in the G-normalizer of $Q \cap \Pi$.*

PROOF. Let $g \in G$ such that $gx = x$. Since $\Pi \leq Q\Pi$ and $(Q\Pi)_x = Q$, $\Pi_x = Q \cap \Pi$ and $(gQ\Pi g^{-1})_{gx} = gQg^{-1}$. The subgroup $gQ\Pi g^{-1}$ is equal to $gQg^{-1}\Pi$ since Π is normal. Thus,

$$\Pi_{gx} = (gQg^{-1}\Pi)_{gx} \cap \Pi = gQg^{-1} \cap \Pi.$$

The normality of Π also implies that $gQg^{-1} \cap \Pi = g(Q \cap \Pi)g^{-1}$. The equality $gx = x$ then implies the sequence of identifications

$$Q \cap \Pi = \Pi_x = \Pi_{gx} = gQg^{-1} \cap \Pi = g(Q \cap \Pi)g^{-1}.$$

It follows that g is in the G-normalizer of $Q \cap \Pi$. □

PROOF OF THEOREM 11.8. Let K be in $\mathfrak{E}'_{N_G\Lambda}\langle\Lambda; U\rangle$. Then $K \cap \Pi \leq \Lambda$ and $K\Pi/K\Lambda$ $K\Pi$-embeds in U. Clearly $K\Lambda \cap \Pi$ is Λ. Applying the lemma with the subgroup Q of the lemma taken to be $K\Lambda$ provides a subgroup H of G such that $K\Lambda \leq H \leq N_G\Lambda$ and G/H embeds as a G-space in U. Since $K \leq K\Lambda \leq H$, K is in $\mathfrak{W}''(N_G\Lambda, G; U)$. The remaining assertions of the theorem follow from Theorem 11.6 and the fact that the spectra $E\mathfrak{E}'_{N_G\Lambda}\langle\Lambda; U\rangle_+ \wedge C$ and $E(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda; U\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda; U\rangle) \wedge C$ are both $\mathfrak{E}'_{N_G\Lambda}\langle\Lambda; U\rangle$-spectra. □

The remainder of this section is devoted to an examination of a very special case of the Wirthmüller isomorphism. This investigation leads incidentally to a proof that, for G finite, the family $\mathfrak{W}(N, G; U)$ is contained in the family $\mathfrak{W}'(N, G; U)$. However, its deeper purpose is to provide the intuition behind the definition of $\mathfrak{W}'(N, G; U)$ and to give some insight into the conditions needed to obtain a Wirthmüller isomorphism. For this investigation, assume that G is a finite group and that K is a subgroup of N contained in $\mathfrak{W}(N, G; U)$. Since $K \in \mathfrak{W}(N, G; U)$, the composite

$$\Sigma^\infty_U G/K_+ \cong G \ltimes_N \Sigma^\infty_U N/K_+ \xrightarrow{\psi} F_N[G, \Sigma^\infty_U N/K_+)$$

is a weak G-equivalence. Thus, for any subgroup H of G, the composite

$$\begin{array}{rcl} [\Sigma_U^\infty G/H_+, \Sigma_U^\infty G/K_+]_G^U & \longrightarrow & [\Sigma_U^\infty G/H_+, F_N[G, \Sigma_U^\infty N/K_+)]_G^U \\ & \cong & [\Sigma_U^\infty G/H_+, \Sigma_U^\infty N/K_+)]_N^U \end{array}$$

is an isomorphism.

The domain and range of this isomorphism can be computed using Corollary 5.3(b), which asserts that each of them is a free abelian group. The generators of $[\Sigma_U^\infty G/H_+, \Sigma_U^\infty G/K_+]_G^U$ are equivalence classes of certain diagrams of the form

$$G/H \xleftarrow{\ f\ } G/J \xrightarrow{\ p\ } G/K,$$

where $J \leq K$, p is the obvious projection, and f is a G-map between G-sets. If the map f takes the coset eJ to the coset gH, then $g^{-1}Jg \leq H$. The allowed diagrams are those for which $H/(g^{-1}Jg)$ embeds in U as an H-space. Analogously, the generators of $[\Sigma_U^\infty G/H_+, \Sigma_U^\infty N/K_+)]_N^U$ are equivalence classes of certain diagrams of the form

$$G/H \xleftarrow{\ h\ } N/J \xrightarrow{\ q\ } N/K,$$

where $J \leq K$, q is the obvious projection, and h is an N-map between N-sets. If h takes the coset eJ to the coset gH, then $g^{-1}Jg \leq g^{-1}Ng \cap H$. The allowed diagrams are those for which $(g^{-1}Ng \cap H)/(g^{-1}Jg)$ embeds in U as an $(g^{-1}Ng \cap H)$-space.

The inverse of the Wirthmüller isomorphism displayed above takes the generator associated to the diagram

$$G/H \xleftarrow{\ h\ } N/J \xrightarrow{\ q\ } N/K$$

to the generator associated to the diagram

$$G/H \xleftarrow{\ \tilde{h}\ } G \ltimes_N N/J \xrightarrow{\ 1 \ltimes_N q\ } G \ltimes_N N/K \cong G/K,$$

where $\tilde{h}$ is the G-map induced by the N-map h. From the existence of a Wirthmüller isomorphism for the N-spectrum $\Sigma_U^\infty N/mgK_+$, we can therefore derive the condition that, for every pair of subgroups J of N and H of G and every element g of G such that $g^{-1}Jg \leq g^{-1}Ng \cap H$, the orbit $(g^{-1}Ng \cap H)/(g^{-1}Jg)$ embeds in U as an $(g^{-1}Ng \cap H)$-space if and only if the orbit $H/(g^{-1}Jg)$ embeds in U as an H-space. In fact, it is easy to check that this embedding condition for arbitrary $g \in G$ follows from the analogous condition in which g is restricted to being e. But this is precisely the condition that K be in $\mathfrak{W}'(N, G; U)$. Thus, we have shown that, if G is finite, then $\mathfrak{W}(N, G; U)$ is contained in $\mathfrak{W}'(N, G; U)$. If G is not finite, then the G-orbit G/H, considered as an N-space, need not decompose into a disjoint union of orbits. Thus, Corollary 5.3(b) cannot be applied, and the argument presented here does not suffice to show the containment.

SECTION 12

An introduction to the Adams isomorphism for incomplete universes

One of the ways in which Theorem 3.8 differs from Theorems 2.4 and 3.3 is that the summand of the splitting indexed on a subgroup Λ of Π is described in terms of the $W_G\Lambda$-spectrum $E(\Lambda,\Pi,G;U)_+ \wedge C[\Lambda]$ rather than in terms of a $W_G\Lambda/W_\Pi\Lambda$-spectrum of the form $E(\Lambda,\Pi,G;U)_+ \wedge_{W_\Pi\Lambda} \Sigma^{Ad(W_\Pi\Lambda)}C[\Lambda]$. This difference is necessary because the Adams isomorphism (see section 5 of [**1**] and section II.7 of [**24**]) need not be available if the universe U is incomplete. Here we discuss the conditions under which an Adams isomorphism is available for incomplete universes.

There are two easily seen obstacles to obtaining an Adams isomorphism of the desired sort. The first is that any sort of Adams isomorphism must use a representability theorem like Theorem II.2.6 of [**24**] to approximate the spectrum $E(\Lambda,\Pi,G;U)_+ \wedge C[\Lambda]$ indexed on the universe U^Λ by a spectrum indexed on some $W_\Pi\Lambda$-trivial universe so that an orbit spectrum can be constructed. If the G-universe U is complete, then such an approximation is certainly available. However, if U is incomplete, then the approximation need not exist, and, even if it does, Theorem II.2.6 of [**24**] may be inadequate to imply its existence. Moreover, the approximation problem is exacerbated by the fact that the natural indexing universe for such an approximating spectrum is $(U^\Lambda)^{W_\Pi\Lambda} = U^{N_\Pi\Lambda}$, whereas, for the purposes of our splitting theorem, the only reasonable indexing universe is U^Π. If U is complete, then U^Π and $U^{N_\Pi\Lambda}$ are equivalent $W_G\Lambda$-universes, so this apparent difference in indexing universe is immaterial. However, if U is not complete, then there is no reason to assume that U^Π and $U^{N_\Pi\Lambda}$ are equivalent as $W_G\Lambda$-universes. Thus, the kind of approximation needed for an incomplete universe is more far-reaching than the one needed for a complete universe. This approximation problem is, in fact, resolved by Theorem 9.3 and Lemma 12.3, which provide the desired approximation under any conditions where other, more serious, problems do not eliminate the possibility of an Adams isomorphism.

In the context of a complete universe, the $W_\Pi\Lambda$-freeness of a spectrum like $E(\Lambda,\Pi,G;U)_+ \wedge C[\Lambda]$ suffices to guarantee the existence of an Adams isomorphism. However, in the context of an incomplete universe, this condition need not be sufficient. Thus, the more serious obstacle to obtaining the desired isomorphism is the necessity of identifying conditions under which it can be expected to exist. These conditions could take the form of rather stringent constraints on the universe U or of some new restrictions on the spectra to which we wish to apply the isomorphism.

To describe our resolution of this more serious problem, it seems best to simplify our notation. In the context of our splitting theorems, we want an Adams isomorphism for a certain $W_\Pi\Lambda$-free $W_G\Lambda$-spectrum indexed on U^Λ. However, for our discussion of the existence of this isomorphism, what matters about these two groups is that $W_\Pi\Lambda$ is a normal subgroup of $W_G\Lambda$. Thus, for the rest of our discussion of the Adams isomorphism, we replace $W_G\Lambda$ by G, $W_\Pi\Lambda$ by Π, and the $W_G\Lambda$-universe U^Λ by the G-universe U. We are then looking for an Adams isomorphism for a Π-free G-spectrum D. As in [**24**], we denote the adjoint representation of G derived from Π by A. The differences between this notation and the notation in Section II.7 of [**24**] are that we are using Π instead of N for the normal subgroup and $\mathcal{G}$ rather than J for the quotient group G/Π.

To deal with the difficulty that we wish to index the orbit spectrum D/Π on a universe other than the natural one, we replace the universe U^Π that is the indexing universe of D in [**24**] with an arbitrary Π-trivial G-universe U'. We also replace the inclusion i of U^Π into U by an arbitrary linear G-isometry $i : U' \longrightarrow U$. Thus, in the remainder of this section, the spectrum D, all of our Π-fixed-point spectra, and all of our Π-orbit spectra are indexed on U' rather than U^Π.

With this notation in place, we can describe another problem with the Adams isomorphism that arises in incomplete universes. In [**24**], the Adams isomorphism is derived from a dimension-shifting transfer of the form $\tau : i_*(D/\Pi) \longrightarrow \Sigma^{-A} i_* D$. The desuspension by A appearing here may be undefined if A is not isomorphic to a subrepresentation of U. This difficulty can be avoided by working with a transfer of the form $\tau : i_*((\Sigma^A D)/\Pi) \longrightarrow i_* D$. Such a change amounts to nothing more than replacing D by $\Sigma^A D$ in Theorem II.7.1 of [**24**]. In the applications of the Adams isomorphism to our splitting theorems, this replacement is, in fact, always made. Thus, for our purposes, an Adams isomorphism derived from this second type of transfer is preferred. Hereafter, a dimension-shifting transfer of the form

$$\tau : i_*((\Sigma^A D)/\Pi) \longrightarrow i_* D$$

is referred to as an Adams transfer.

We deal with the problem that the Π-freeness of the action on D is not a sufficient condition for the existence of the desired Adams isomorphism by imposing on the universe U the minimal conditions under which an Adams isomorphism might possibly exist and then describing the additional conditions that must be imposed on D to obtain the isomorphism. The minimal condition on U is that the free orbit Π/e embeds in U as a Π-space. The conditions that must be imposed on D depend on both U and U', but not the choice of i. They are best described in terms of a family $\mathfrak{A}_G(\Pi; U, U')$ of subgroups of G. This family is one of two natural generalizations to incomplete universes of the family $\mathfrak{F}_G(\Pi)$ of Definitions II.2.3(ii) of [**24**].

Definition 12.1. If Π is a normal subgroup of a compact Lie group G, U is a G-universe into which Π/e embeds as a Π-space, and U' is a Π-trivial G-universe that embeds in U, then the family $\mathfrak{A}_G(\Pi; U, U')$ of subgroups of G consists of those $H \le G$ such that

(i) $H \cap \Pi = \{e\}$
(ii) $H\Pi/H$ embeds in U as an $H\Pi$-space
(iii) for every subgroup K of H and Q of G such that $K\Pi \le Q$, the orbit Q/K embeds in U as a Q-space if and only if the orbit $Q/K\Pi$ embeds in U' as a Q-space.

If U is a complete G-universe and $U' = U^\Pi$, then $\mathfrak{A}_G(\Pi; U, U') = \mathfrak{F}_G(\Pi)$. For any universe U, we denote $\mathfrak{A}_G(\Pi; U, U^\Pi)$ by $\mathfrak{A}_G(\Pi; U)$. If $\Pi = G$, then, for any trivial universe U', $\mathfrak{A}_G(\Pi; U, U')$ is just the family $\mathfrak{F}_G(G)$ consisting of the trivial subgroup of G.

Our generalization of Theorem II.7.1 of [**24**] to incomplete universes can now be stated. This result is proven in sections 16 and 17.

THEOREM 12.2. *Let Π be a normal subgroup of a compact Lie group G, $\mathcal{G}$ be G/Π, A be the adjoint representation of G derived from Π, U be a G-universe into which Π/e embeds as a Π-space, U' be a Π-trivial G-universe, and $i : U' \longrightarrow U$ be a linear G-isometry. If D is an $\mathfrak{A}_G(\Pi; U, U')$-spectrum indexed on U', then there is an Adams transfer*

$$\tau : i_*((\Sigma^A D)/\Pi) \longrightarrow i_* D$$

whose adjoint

$$\tilde{\tau} : (\Sigma^A D)/\Pi \longrightarrow (i^* i_* D)^\Pi$$

is a weak $\mathcal{G}$-equivalence of $\mathcal{G}$-spectra indexed on U'. Thus, for any $\mathcal{G}$-spectrum B indexed on U', there is a natural isomorphism

$$\left[B, (\Sigma^A D)/\Pi\right]_{\mathcal{G}}^{U'} \xrightarrow[\cong]{\tilde{\tau}_*} \left[B, (i^* i_* D)^\Pi\right]_{\mathcal{G}}^{U'} \cong [i_* B, i_* D]_G^U.$$

In order to apply this theorem to our splitting results, we must deal with the representability problem mentioned in the introduction to this section. The issue is that Theorem 12.2 applies to $\mathfrak{A}_G(\Pi; U, U')$-spectra indexed on U', but, in practice, one always begins with an $\mathfrak{A}_G(\Pi; U, U')$-spectrum C indexed on U. To apply the theorem, one must find an $\mathfrak{A}_G(\Pi; U, U')$-spectrum D indexed on U' such that $i_* D \simeq C$. As in the remarks preceding Theorems 2.4 and 3.3 and Corollary 9.4, we denote the spectrum $(\Sigma^A D)/\Pi$ indexed on U' by $(\Sigma^A C)/\Pi$. The Adams isomorphism then becomes a weak $\mathcal{G}$-equivalence $\tilde{\tau} : (\Sigma^A C)/\Pi \longrightarrow (i^* C)^\Pi$. The following result indicates that, if C is an $\mathfrak{A}_G(\Pi; U, U')$-spectrum, then the desired spectrum D can always be found. For this result, recall from Definition 9.1 the family $\tilde{\mathfrak{E}}(U, U')$ of subgroups associated to the universes U and U' and the notion of two universes being G-orbit equivalent.

LEMMA 12.3. *Let Π be a normal subgroup of a compact Lie group G, U be a G-universe into which Π/e embeds as a Π-space, U' be a Π-trivial G-universe, and $i : U' \longrightarrow U$ be a linear G-isometry. If $H \in \mathfrak{A}_G(\Pi; U, U')$, then the universes U and U' are H-orbit equivalent. Thus, the family $\mathfrak{A}_G(\Pi; U, U')$ is contained in the family $\tilde{\mathfrak{E}}(U, U')$.*

PROOF. Let $H \in \mathfrak{A}_G(\Pi; U, U')$. Since U' embeds in U as a G-universe, U' is clearly contained in U up to H-orbits. Assume that $J \leq K \leq H$ and that K/J embeds as a K-space in U. We must show that it also embeds in U'. Because $K \leq H$ and $H \in \mathfrak{A}_G(\Pi; U, U')$, $K\Pi/K$ embeds as a $K\Pi$-space in U. But then $K\Pi/J$ must also embed as a $K\Pi$-space in U because the set of $K\Pi$-isotropy subgroups of U is closed under intersection. By condition (iii) of the definition of $\mathfrak{A}_G(\Pi; U, U')$, $K\Pi/J\Pi$ must then embed as a $K\Pi$-space in U'. Since $K \cap \Pi = \{e\}$, K/J embeds as a K-space in U', and U is contained in U' up to H-orbits. $\square$

There are two ways in which Theorem 12.2 might seem unsatisfactory. Certainly, the family $\mathfrak{A}_G(\Pi; U, U')$ can be rather smaller than we would like. Also, the motivation behind its definition is hardly obvious. The remainder of this section is devoted to providing the intuition behind this definition; it contains a heuristic argument that the hypotheses of the theorem are necessary as well as sufficient. The purpose of condition (i) in the definition should be clear; it ensures that $\mathfrak{A}_G(\Pi; U, U')$-spectra are Π-free. Without such a freeness condition, one cannot hope to have the Adams transfer from which the Adams isomorphism is obtained. The other two conditions are more exotic looking. However, examining the Adams isomorphism in a very special case suffices to explain the need for conditions (ii) and (iii). For the remainder of this section, assume that G is a finite group and H is a subgroup of G such that $H \cap \Pi = \{e\}$. Then G/H is a Π-free G-space. Note that $(G/H)/\Pi$ is just $G/H\Pi$. If there were an Adams isomorphism theorem for the spectrum $D = \Sigma^\infty_{U'} G/H_+$, then its represented form would assert that, for each subgroup Q of G such that $\Pi \leq Q$, the composite

$$\begin{aligned}[\Sigma^\infty_{U'} G/Q_+, \Sigma^\infty_{U'} G/H\Pi_+]^{U'}_{\mathcal{G}} &\cong [\Sigma^\infty_{U'} G/Q_+, \Sigma^\infty_{U'} G/H\Pi_+]^{U'}_G \\ &\xrightarrow{i_*} [\Sigma^\infty_{U} G/Q_+, \Sigma^\infty_{U} G/H\Pi_+]^{U}_G \\ &\xrightarrow{\tau_*} [\Sigma^\infty_{U} G/Q_+, \Sigma^\infty_{U} G/H_+]^{U}_G\end{aligned}$$

is an isomorphism, where $\tau : \Sigma^\infty_U G/H\Pi_+ \longrightarrow \Sigma^\infty_U G/H_+$ is the standard transfer associated to the projection $G/H \longrightarrow G/H\Pi$. In order for the transfer τ to exist as a map between spectra indexed on U, the orbit $H\Pi/H$ must embed in U as an $H\Pi$-space. Thus, condition (ii) in the definition is necessary for the existence of the map which is supposed to give the Adams isomorphism.

In order to understand the third condition in the definition, let us assume that H satisfies the first two conditions so that the composite above is well-defined, and let us consider the question of whether this composite is an isomorphism. This question can be addressed by using Corollary 5.3(b) to take a closer look at the morphism sets $[\Sigma^\infty_{U'} G/Q_+, \Sigma^\infty_{U'} G/H\Pi_+]^{U'}_{\mathcal{G}}$ and $[\Sigma^\infty_{U} G/Q_+, \Sigma^\infty_{U} G/H_+]^{U}_G$. This theorem indicates that $[\Sigma^\infty_{U} G/Q_+, \Sigma^\infty_{U} G/H_+]^{U}_G$ is a free abelian group whose generators are equivalence classes of certain diagrams of the form

$$G/Q \xleftarrow{p} G/K \xrightarrow{f} G/H,$$

where $K \leq Q$, p is the obvious projection, and f is a G-map between G-sets. The allowed diagrams are those for which Q/K embeds in U as a Q-space. The morphism set $[\Sigma^\infty_{U'} G/Q_+, \Sigma^\infty_{U'} G/H\Pi_+]^{U'}_{\mathcal{G}}$ has an analogous description; each of its generators can be represented by a diagram of the form

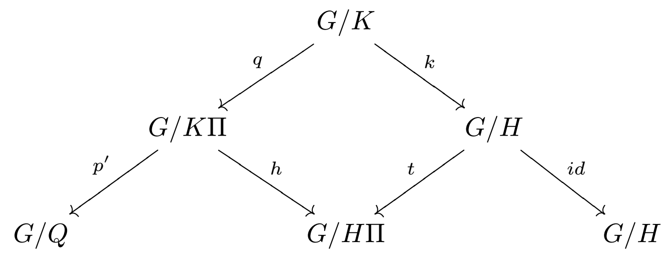
$$G/Q \xleftarrow{p'} G/K\Pi \xrightarrow{h} G/H\Pi$$

in which K is a subgroup of G subconjugate to H such that $K\Pi \leq Q$. Again, p' is the obvious projection and h is a G-map.

The image of the generator associated to the diagram (p', h) under the first two maps in the composite above is just the generator of $[\Sigma^\infty_{U} G/Q_+, \Sigma^\infty_{U} G/H\Pi_+]^{U}_G$ represented by the same diagram. The transfer τ which induces the third map in the composite is represented by the diagram

$$G/H\Pi \xleftarrow{t} G/H \xrightarrow{id} G/H,$$

where t is just the standard projection. The composite of τ with our generator in $[\Sigma^\infty_U G/Q_+, \Sigma^\infty_U G/H\Pi_+]^U_G$ can therefore be computed from the diagram



of G-sets and G-maps in which the quadrilateral is a pullback and q is the standard projection. Thus, the image of a generator

$$G/Q \xleftarrow{p'} G/K\Pi \xrightarrow{h} G/H\Pi$$

of $[\Sigma^\infty_{U'} G/Q_+, \Sigma^\infty_{U'} G/H\Pi_+]^{U'}_{\mathcal{G}}$ under our putative Adams isomorphism is the morphism represented by the diagram

$$G/Q \xleftarrow{p} G/K \xrightarrow{k} G/H.$$

Note that the diagram

$$G/Q \xleftarrow{p'} G/K\Pi \xrightarrow{h} G/H\Pi$$

represents an actual generator of $[\Sigma^\infty_{U'} G/Q_+, \Sigma^\infty_{U'} G/H\Pi_+]^{U'}_{\mathcal{G}}$ only when $Q/K\Pi$ embeds as a Q-space in U'. Similarly, the diagram

$$G/Q \xleftarrow{p} G/K \xrightarrow{k} G/H$$

represents an actual generator of $[\Sigma^\infty_U G/Q_+, \Sigma^\infty_U G/H_+]^U_G$ only when Q/K embeds in U as a Q-space. Thus, condition (iii) of the definition is essentially the condition H must satisfy to ensure that the composite above is an isomorphism. Our analysis suggests that condition (iii) should be imposed on subgroups K of G which are subconjugate to H. However, it is easy to check that it suffices to consider the actual subgroups K of H. This discussion indicates that, at least when G is finite, the family $\mathfrak{A}_G(\Pi; U, U')$ is the largest family for which we can hope to have an Adams isomorphism.

Part 3

The Longer Proofs

SECTION 13

The proof of Proposition 3.10 and its consequences

The first purpose of this section is to provide the proofs of Proposition 3.10 and two related results, Propositions 2.8 and 3.6. These proofs provide the motivation for the definitions of two maps which play a critical role in the proofs of our main splitting theorems. The second purpose of this section is to introduce these maps, and to use them to prove a special case of Theorem 3.2.

PROOF OF PROPOSITION 3.10. Assume that C is a G-spectrum indexed on a G-universe U, Π is a normal subgroup of G, $\mathcal{G} = G/\Pi$, and B is a $\mathcal{G}$-spectrum indexed on U^Π. Also, let Λ be a subgroup of Π, and $(\mathfrak{F}', \mathfrak{F})$ be a Π-adjacent pair of families of subgroups of G associated to Λ. Let L be the tangent $N_G\Lambda$-representation at the identity coset $eN_G\Lambda$ of $G/N_G\Lambda$. Note that, by Lemma V.10.3(ii) of [**24**], $L^\Lambda = 0$. The pair $(\mathfrak{E}'_{N_G\Lambda}\langle\Lambda; U\rangle, \mathfrak{E}_{N_G\Lambda}\langle\Lambda; U\rangle)$ of families of subgroups of $N_G\Lambda$ (see Definition 8.1(c)) appears throughout this proof, and, to reduce notational clutter, we denote this pair by ($\mathfrak{E}$', $\mathfrak{E}$) hereafter in the proof. For the same reason, the quotient group $W_G\Lambda/W_\Pi\Lambda$ is denoted here by $\mathcal{W}\Lambda$.

The isomorphism γ of Proposition 3.10 is the composite:

$$\begin{aligned}
&\big[j_*^\Pi B, E(\mathfrak{F}', \mathfrak{F}) \wedge C\big]_G^U \\
&\xrightarrow[\cong]{((\tilde{\kappa}\wedge 1)^{-1})_*} \big[j_*^\Pi B, G \ltimes_{N_G\Lambda} E(\mathfrak{E}', \mathfrak{E}) \wedge C\big]_G^U \\
&\xrightarrow[\cong]{(\zeta^{-1})_*} \big[j_*^\Pi B, G \ltimes_{N_G\Lambda} (E(\mathfrak{E}', \mathfrak{E}) \wedge C)\big]_G^U \\
&\xrightarrow[\cong]{\tilde{\psi}} \big[j_*^\Pi B, \Sigma^L E(\mathfrak{E}', \mathfrak{E}) \wedge C\big]_{N_G\Lambda}^U \\
&\xrightarrow{\cong} \big[j_*^{\Lambda,\Pi} B, (\Sigma^L E(\mathfrak{E}', \mathfrak{E}) \wedge C)^\Lambda\big]_{W_G\Lambda}^{U^\Lambda} \\
&\xrightarrow[\cong]{\mu_*^\Lambda} \Big[j_*^{\Lambda,\Pi} B, (\Sigma^L E(\mathfrak{E}', \mathfrak{E}) \wedge \tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge C)^\Lambda\Big]_{W_G\Lambda}^{U^\Lambda} \\
&\xrightarrow[\cong]{(\tilde{\nu}^{-1})_*} \Big[j_*^{\Lambda,\Pi} B, E(\Lambda, \Pi, G; U)_+ \wedge (\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge C)^\Lambda\Big]_{W_G\Lambda}^{U^\Lambda} \\
&= \big[j_*^{\Lambda,\Pi} B, E(\Lambda, \Pi, G; U)_+ \wedge \Phi^\Lambda C\big]_{W_G\Lambda}^{U^\Lambda}.
\end{aligned}$$

Here, $((\tilde{\kappa} \wedge 1)^{-1})_*$ is the inverse of the isomorphism given by Proposition 8.3. The isomorphism $(\zeta^{-1})_*$ is derived from the inverse of the isomorphism ζ of Lemma II.4.9 of [**24**], and the isomorphism $\tilde{\psi}$ is derived from the Wirthmüller isomorphism of Theorem 11.8. The unlabeled isomorphism is the adjunction isomorphism for the spectrum-level Λ-fixed-point functor. The projection

$$\mu_\Lambda : E(\mathfrak{E}', \mathfrak{E}) \longrightarrow E(\mathfrak{E}', \mathfrak{E}) \wedge \tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda]$$

is easily seen to be a weak $N_G\Lambda$-equivalence by checking fixed-point sets. Thus, the map μ_*^Λ in the composite above is an isomorphism. The isomorphism $(\tilde{\nu}^{-1})_*$ is derived from the map ν of Remarks II.3.14(ii) of [**24**] via the identification in Lemma 8.2 of the Λ-fixed-point set of $E(\mathfrak{E}', \mathfrak{E})$. Scholium 10.2(a) and Proposition II.9.12(i) of [**24**] indicate that the map ν is an isomorphism in this case.

The weak $\mathcal{G}$-equivalence χ of Proposition 3.10 involves several isomorphisms relating various change of group and change of universe functors. Thus, in describing this weak equivalence, we temporarily abandon the convention of suppressing from our notation the change of universe functor included in any spectrum-level fixed-point functor. The weak equivalence of the proposition is then the composite:

$$\begin{aligned}
&(j_\Pi^*(E(\mathfrak{F}', \mathfrak{F}) \wedge C))^\Pi \\
&\xrightarrow[\simeq]{((\tilde{\kappa}\wedge 1)^\Pi)^{-1}} (j_\Pi^*((G \ltimes_{N_G\Lambda} E(\mathfrak{E}', \mathfrak{E})) \wedge C))^\Pi \\
&\xrightarrow[\cong]{(\zeta^\Pi)^{-1}} (j_\Pi^* G \ltimes_{N_G\Lambda} (E(\mathfrak{E}', \mathfrak{E}) \wedge C))^\Pi \\
&\xrightarrow[\simeq]{\psi} (j_\Pi^* F_{N_G\Lambda}[G, \Sigma^L E(\mathfrak{E}', \mathfrak{E}) \wedge C))^\Pi \\
&\xrightarrow{\cong} (F_{N_G\Lambda}[G, j_\Pi^*(\Sigma^L E(\mathfrak{E}', \mathfrak{E}) \wedge C)))^\Pi \\
&\xrightarrow{\cong} F_{\mathcal{W}\Lambda}[\mathcal{G}, ((j_\Pi^*(\Sigma^L E(\mathfrak{E}', \mathfrak{E}) \wedge C))^\Lambda)^{W_\Pi\Lambda}) \\
&\xrightarrow{\cong} F_{\mathcal{W}\Lambda}[\mathcal{G}, (j_{\Lambda,\Pi}^*((j_\Lambda^*(\Sigma^L E(\mathfrak{E}', \mathfrak{E}) \wedge C))^\Lambda))^{W_\Pi\Lambda}) \\
&\xrightarrow[\simeq]{\mu_*^\Lambda} F_{\mathcal{W}\Lambda}[\mathcal{G}, (j_{\Lambda,\Pi}^*((j_\Lambda^*(\Sigma^L E(\mathfrak{E}', \mathfrak{E}) \wedge \tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge C))^\Lambda))^{W_\Pi\Lambda}) \\
&\xrightarrow[\simeq]{(\tilde{\nu}^{-1})_*} F_{\mathcal{W}\Lambda}[\mathcal{G}, (j_{\Lambda,\Pi}^*(E(\Lambda, \Pi, G; U)_+ \wedge (j_\Lambda^*(\tilde{E}\mathfrak{F}_{N_G\Lambda}[\Lambda] \wedge C))^\Lambda))^{W_\Pi\Lambda}) \\
&= F_{\mathcal{W}\Lambda}[\mathcal{G}, (j_{\Lambda,\Pi}^*(E(\Lambda, \Pi, G; U)_+ \wedge \Phi^\Lambda C))^{W_\Pi\Lambda}).
\end{aligned}$$

Here, the maps $(\tilde{\kappa} \wedge 1)^\Pi)^{-1}$, $(\zeta^\Pi)^{-1}$, ψ, μ_*^Λ, and $(\tilde{\nu}^{-1})_*$ are similar in origin to the analogously named maps appearing in the description of the isomorphism of the proposition. The first of the unlabeled isomorphisms is one of those given by Lemma II.4.14 of [**24**]. To understand the second unlabeled isomorphism, note that its domain and range are derived from the spectrum $j_\Pi^*(\Sigma^L E(\mathfrak{E}', \mathfrak{E}) \wedge C)$ by applying the sequences of change of group functors associated to the sequences of group homomorphisms

$$N_G\Lambda \subset G \longrightarrow \mathcal{G}$$

and

$$N_G\Lambda \longrightarrow W_G\Lambda \longrightarrow W_G\Lambda/W_\Pi\Lambda = \mathcal{W}\Lambda \subset \mathcal{G},$$

respectively. The composites of these two sequences of homomorphisms are equal, and so Lemma II.4.10 of [**24**] provides the desired natural isomorphism between the composites of the associated sequences of functors. The last unlabeled isomorphism is the composite of the natural isomorphism $j_{\Lambda,\Pi}^* j_\Lambda^* \cong j_\Pi^*$ arising from the fact that $j^\Pi = j^\Lambda j^{\Lambda,\Pi}$ and an isomorphism from Lemma II.4.14 of [**24**] describing the commutativity of certain change of group and change of universe functors. It is easy to verify that the isomorphism γ is, in fact, induced by the weak equivalence χ. □

Propositions 2.8 and 3.6 are, roughly, the special cases of Proposition 3.10 in which, respectively, $\Pi = G$ and the indexing universe U is complete. These two results are derived from from Proposition 3.10 by inserting the Adams and Wirthmüller isomorphisms in the appropriate places:

PROOFS OF PROPOSITIONS 2.8 AND 3.6. The isomorphisms in each of these two propositions follow from the isomorphism γ of Proposition 3.10 by inserting the appropriate Adams isomorphisms. The observation in Definition 12.1 that $\mathfrak{A}_G(G;U,U')$ is always just $\mathfrak{F}_G(G)$ ensures that the required Adams isomorphism exists in the context of Proposition 2.8. The weak equivalence of Proposition 3.6 follows from the weak equivalence χ of Proposition 3.10 by inserting both the appropriate Adams isomorphism and the appropriate Wirthmüller isomorphism. Lemma V.10.3(*iii*) of [**24**] indicates that $\mathcal{W}\Lambda$ has finite index in $\mathcal{G}$, so the usual suspension by a representation is not required in this instance of the Wirthmüller isomorphism. The weak equivalence in Proposition 2.8 requires only the appropriate Adams isomorphism since here $\mathcal{G} = e$ so that the change of group functor appearing in the weak equivalence χ of Proposition 3.10 is irrelevant in this special case. □

Our ultimate goal is to construct a well-behaved splitting of the map

$$(E\mathfrak{F}'_+ \wedge C)^{\Pi} \longrightarrow (E(\mathfrak{F}',\mathfrak{F}) \wedge C)^{\Pi}$$

for appropriately split G-spectra C and suitable pairs $(\mathfrak{F}',\mathfrak{F})$ of families of subgroups of G. Such a splitting induces a splitting of the map

$$\left[j_*^{\Pi}B, E\mathfrak{F}'_+ \wedge C\right]_G^U \longrightarrow \left[j_*^{\Pi}B, E(\mathfrak{F}',\mathfrak{F}) \wedge C\right]_G^U.$$

The descriptions of the maps χ and γ of Proposition 3.10 given above suggest definitions of these splitting maps for the special case in which the pair $(\mathfrak{F}',\mathfrak{F})$ is Π-adjacent. The remainder of this section is devoted to introducing the splitting maps for this special case and showing that they do provide the desired splitting. We introduce the more easily understood splitting at the morphism-set level before introducing the spectrum-level splitting.

DEFINITION 13.1. Let Π be a normal subgroup of G, $\mathcal{G} = G/\Pi$, Λ be a subgroup of Π, and $\mathfrak{F}'$ be a family of subgroups of G such that $\Lambda \in \mathfrak{F}'$. Also, let B be a $\mathcal{G}$-spectrum indexed on U^{Π}, and C be a G-spectrum indexed on U which is geometrically split at Λ. Recall that, if L is the tangent $N_G\Lambda$-representation at the identity coset $eN_G\Lambda$ of $G/N_G\Lambda$, then $L^{\Lambda} = 0$.

(a) The map

$$\alpha_{\Lambda}(\mathfrak{F}') : \left[j_*^{\Lambda,\Pi}B, E(\Lambda,\Pi,G;U)_+ \wedge C[\Lambda]\right]_{W_G\Lambda}^{U^{\Lambda}} \longrightarrow \left[j_*^{\Pi}B, E\mathfrak{F}'_+ \wedge C\right]_G^U$$

is the composite

$$
\begin{aligned}
&\big[j_*^{\Lambda,\Pi}B, E(\Lambda,\Pi,G;U)_+ \wedge C[\Lambda]\big]^{U^\Lambda}_{W_G\Lambda}\\
&\xrightarrow{\zeta_*^\Lambda} \big[j_*^{\Lambda,\Pi}B, E(\Lambda,\Pi,G;U)_+ \wedge C^\Lambda\big]^{U^\Lambda}_{W_G\Lambda}\\
&\xrightarrow{\nu_*} \big[j_*^{\Lambda,\Pi}B, (\Sigma^L E(\Lambda,\Pi,G;U)_+ \wedge C)^\Lambda\big]^{U^\Lambda}_{W_G\Lambda}\\
&\xrightarrow{\lambda_*} \big[j_*^{\Lambda,\Pi}B, (\Sigma^L E\mathfrak{E}'_{N_G\Lambda}\langle\Lambda;U\rangle_+ \wedge C)^\Lambda\big]^{U^\Lambda}_{W_G\Lambda}\\
&\xrightarrow{\cong} \big[j_*^{\Pi}B, \Sigma^L E\mathfrak{E}'_{N_G\Lambda}\langle\Lambda;U\rangle_+ \wedge C\big]^{U}_{N_G\Lambda}\\
&\underset{\cong}{\xrightarrow{\tilde\psi^{-1}}} \big[j_*^{\Pi}B, G\ltimes_{N_G\Lambda}(E\mathfrak{E}'_{N_G\Lambda}\langle\Lambda;U\rangle_+ \wedge C)\big]^{U}_{G}\\
&\underset{\cong}{\xrightarrow{\tilde\zeta_*}} \big[j_*^{\Pi}B, (G\ltimes_{N_G\Lambda}E\mathfrak{E}'_{N_G\Lambda}\langle\Lambda;U\rangle_+) \wedge C)\big]^{U}_{G}\\
&\xrightarrow{\tilde\lambda_*} \big[j_*^{\Pi}B, E\mathfrak{F}'_+ \wedge C)\big]^{U}_{G}.
\end{aligned}
$$

Here, ζ_*^Λ is derived from the map ζ_Λ splitting C at Λ. The maps ν_* and $\tilde\zeta_*$ are defined in Remarks II.3.14(*ii*) and Lemma II.4.9 of [**24**]. The maps λ_* and $\tilde\lambda_*$ are formed from canonical maps between universal spaces. The unlabeled isomorphism is the adjunction isomorphism for the spectrum-level fixed-point functor, and $\tilde\psi^{-1}$ is the inverse of the Wirthmüller isomorphism of Theorem 11.8. Observe that $\alpha_\Lambda(\mathfrak{F}')$ is natural in B and in C with respect to maps which preserve the splitting at Λ.

(b) The $\mathcal{G}$-map

$$\beta_\Lambda(\mathfrak{F}') : F_{\mathcal{W}\Lambda}[\mathcal{G}, (j^*_{\Lambda,\Pi}(E(\Lambda,\Pi,G;U)_+ \wedge C[\Lambda]))^{W_\Pi\Lambda}) \longrightarrow (E\mathfrak{F}'_+ \wedge C)^\Pi$$

is a composite containing several natural maps relating change of universe functors. Thus, in describing $\beta_\Lambda(\mathfrak{F}')$, we again explicitly note the change of universe functors included in our fixed-point functors. The appropriate composite is then:

$$
\begin{aligned}
&F_{\mathcal{W}\Lambda}[\mathcal{G}, (j^*_{\Lambda,\Pi}(E(\Lambda,\Pi,G;U)_+ \wedge C[\Lambda]))^{W_\Pi\Lambda})\\
&\xrightarrow{\zeta_*^\Lambda} F_{\mathcal{W}\Lambda}[\mathcal{G}, (j^*_{\Lambda,\Pi}(E(\Lambda,\Pi,G;U)_+ \wedge (j^*_\Lambda C)^\Lambda))^{W_\Pi\Lambda})\\
&\xrightarrow{\nu_*} F_{\mathcal{W}\Lambda}[\mathcal{G}, (j^*_{\Lambda,\Pi}((j^*_\Lambda(\Sigma^L E(\Lambda,\Pi,G;U)_+ \wedge C))^\Lambda))^{W_\Pi\Lambda})\\
&\xrightarrow{\lambda_*} F_{\mathcal{W}\Lambda}[\mathcal{G}, (j^*_{\Lambda,\Pi}((j^*_\Lambda(\Sigma^L E\mathfrak{E}'_{N_G\Lambda}\langle\Lambda;U\rangle_+ \wedge C))^\Lambda))^{W_\Pi\Lambda})\\
&\xrightarrow{\cong} F_{\mathcal{W}\Lambda}[\mathcal{G}, ((j^*_\Pi(\Sigma^L E\mathfrak{E}'_{N_G\Lambda}\langle\Lambda;U\rangle_+ \wedge C))^\Lambda)^{W_\Pi\Lambda})\\
&\xrightarrow{\cong} (F_{N_G\Lambda}[G, j^*_\Pi(\Sigma^L E\mathfrak{E}'_{N_G\Lambda}\langle\Lambda;U\rangle_+ \wedge C)))^\Pi\\
&\xrightarrow{\cong} (j^*_\Pi F_{N_G\Lambda}[G, \Sigma^L E\mathfrak{E}'_{N_G\Lambda}\langle\Lambda;U\rangle_+ \wedge C))^\Pi\\
&\underset{\cong}{\xrightarrow{\psi^{-1}}} (j^*_\Pi G\ltimes_{N_G\Lambda}(E\mathfrak{E}'_{N_G\Lambda}\langle\Lambda;U\rangle_+ \wedge C))^\Pi\\
&\underset{\cong}{\xrightarrow{\tilde\zeta}} (j^*_\Pi(G\ltimes_{N_G\Lambda}(E\mathfrak{E}'_{N_G\Lambda}\langle\Lambda;U\rangle_+) \wedge C))^\Pi\\
&\xrightarrow{\tilde\lambda} (j^*_\Pi(E\mathfrak{F}'_+ \wedge C))^\Pi.
\end{aligned}
$$

Here, the maps ζ_*^Λ, ν_*, λ_*, ψ^{-1}, $\tilde{\zeta}$, and $\tilde{\lambda}$ are similar in origin to the analogously named maps of part (a). The three unlabeled isomorphisms are, essentially, the inverses of the three unlabeled isomorphisms appearing in the definition of χ in the proof of Proposition 3.10. However, the isomorphisms here appear in the reverse order from those in the proof. Observe that the map $\beta_\Lambda(\mathfrak{F}')$ is natural in C with respect to maps which preserve the splitting at Λ.

The following result, which forms the basis for the proof of our main splitting theorems, indicates that the maps $\alpha_\Lambda(\mathfrak{F}')$ and $\beta_\Lambda(\mathfrak{F}')$ provide the desired splittings.

PROPOSITION 13.2. *Let Π be a normal subgroup of a compact Lie group G, $\mathcal{G} = G/\Pi$, Λ be a subgroup of Π, and $(\mathfrak{F}', \mathfrak{F})$ be a Π-adjacent pair of families of subgroups of G associated to Λ. Also, let B be a $\mathcal{G}$-spectrum indexed on U^Π, and C be a G-spectrum indexed on U which is geometrically split at Λ. Then the composite*

$$\begin{aligned}
&\left[j_*^{\Lambda,\Pi} B, E(\Lambda,\Pi,G;U)_+ \wedge C[\Lambda]\right]_{W_G\Lambda}^{U^\Lambda} \\
&\xrightarrow{\alpha_\Lambda(\mathfrak{F}')} \left[j_*^{\Pi} B, E\mathfrak{F}'_+ \wedge C\right]_G^U \\
&\xrightarrow{\mu_*} \left[j_*^{\Pi} B, E(\mathfrak{F}',\mathfrak{F}) \wedge C\right]_G^U \\
&\underset{\cong}{\xrightarrow{\gamma}} \left[j_*^{\Lambda,\Pi} B, E(\Lambda,\Pi,G;U)_+ \wedge \Phi^\Lambda C\right]_{W_G\Lambda}^U
\end{aligned}$$

is the isomorphism induced by the weak $W_G\Lambda$-equivalence

$$C[\Lambda] \xrightarrow{\zeta_\Lambda} C^\Lambda \xrightarrow{\omega_\Lambda} \Phi^\Lambda C.$$

Moreover, the composite

$$\begin{aligned}
&F_{\mathcal{W}\Lambda}[\mathcal{G}, (j_{\Lambda,\Pi}^*(E(\Lambda,\Pi,G;U)_+ \wedge C[\Lambda]))^{W_\Pi\Lambda})\\
&\xrightarrow{\beta_\Lambda(\mathfrak{F}')} (E\mathfrak{F}'_+ \wedge C)^\Pi \\
&\xrightarrow{(\mu\wedge 1)^\Pi} (E(\mathfrak{F}',\mathfrak{F}) \wedge C)^\Pi \\
&\underset{\simeq}{\xrightarrow{\chi}} F_{\mathcal{W}\Lambda}[\mathcal{G}, (j_{\Lambda,\Pi}^*(E(\Lambda,\Pi,G;U)_+ \wedge \Phi^\Lambda C))^{W_\Pi\Lambda})
\end{aligned}$$

is the weak $\mathcal{G}$-equivalence induced by the same weak $W_G\Lambda$-equivalence. Thus, the maps

$$\mu_* : \left[j_*^\Pi B, E\mathfrak{F}'_+ \wedge C\right]_G^U \longrightarrow \left[j_*^\Pi B, E(\mathfrak{F}',\mathfrak{F}) \wedge C\right]_G^U$$

and

$$(\mu \wedge 1)^\Pi : (E\mathfrak{F}'_+ \wedge C)^\Pi \longrightarrow (E(\mathfrak{F}',\mathfrak{F}) \wedge C)^\Pi$$

are split epimorphisms.

PROOF. Rewrite the definitions of $\alpha_\Lambda(\mathfrak{F}')$ and $\beta_\Lambda(\mathfrak{F}')$ replacing the spaces $E\mathfrak{C}'_{N_G\Lambda}\langle\Lambda;U\rangle_+$ and $E\mathfrak{F}'_+$ by $E(\mathfrak{C}'_{N_G\Lambda}\langle\Lambda;U\rangle, \mathfrak{C}_{N_G\Lambda}\langle\Lambda;U\rangle)$ and $E(\mathfrak{F}',\mathfrak{F})$, respectively. Observe that most of the maps in these definitions become isomorphisms when these replacements are made, and that γ and χ are constructed largely by going backwards along these isomorphisms. Thus, γ and χ undo most of what $\mu_* \circ \alpha_\Lambda(\mathfrak{F}')$ and $(\mu \wedge 1)^\Pi \circ \beta_\Lambda(\mathfrak{F}')$ have done. Once these cancellations are removed from the composites $\gamma \circ \mu_* \circ \alpha_\Lambda(\mathfrak{F}')$ and $\chi \circ (\mu \wedge 1)^\Pi \circ \beta_\Lambda(\mathfrak{F}')$, easy diagram chases confirm that these maps have the asserted form. $\square$

SECTION 14

The proofs of the main splitting theorems

This section begins with the proofs of our main splitting theorems (Theorems 2.1, 2.4, 3.2, 3.3 and 3.8) and concludes with the proof of Corollary 2.7. All of the main splitting theorems follow from a single result which is, essentially, an extension of Proposition 13.2 to pairs of families $(\mathfrak{F}', \mathfrak{F})$ which are not Π-adjacent. Thus, the first step in proving these splittings is introducing the maps which play the roles analogous to those played by the maps $\alpha_\Lambda(\mathfrak{F}')$ and $\beta_\Lambda(\mathfrak{F}')$ in Proposition 13.2.

DEFINITION 14.1. Let Π be a normal subgroup of G, $\mathcal{G} = G/\Pi$, and $(\mathfrak{F}', \mathfrak{F})$ be a Π-closed pair of families of subgroups of G. Also, let B be a $\mathcal{G}$-spectrum indexed on U^Π, and C be a G-spectrum indexed on U which is geometrically split at every subgroup Λ of Π such that $\Lambda \in \mathfrak{F}' - \mathfrak{F}$ and Π/Λ embeds as a Π-space in U.

(a) The homomorphism

$$\alpha(\mathfrak{F}', \mathfrak{F}) : \bigoplus_{(\Lambda)_G} \left[j_*^{\Lambda,\Pi} B, E(\Lambda, \Pi, G; U)_+ \wedge C[\Lambda]\right]_{W_G\Lambda}^{U^\Lambda} \longrightarrow \left[j_*^{\Pi} B, E\mathfrak{F}'_+ \wedge C\right]_G^U$$

is the sum over the G-conjugacy classes of subgroups Λ of Π such that $\Lambda \in \mathfrak{F}' - \mathfrak{F}$ and Π/Λ embeds as a Π-space in U of the maps

$$\alpha_\Lambda(\mathfrak{F}') : \left[j_*^{\Lambda,\Pi} B, E(\Lambda, \Pi, G; U)_+ \wedge C[\Lambda]\right]_{W_G\Lambda}^{U^\Lambda} \longrightarrow \left[j_*^{\Pi} B, E\mathfrak{F}'_+ \wedge C\right]_G^U$$

introduced in Definition 13.1(a). Note that $\alpha(\mathfrak{F}', \mathfrak{F})$ is natural in B and in C with respect to maps which preserve the splitting at all the specified subgroups Λ.

(b) The $\mathcal{G}$-map

$$\beta(\mathfrak{F}', \mathfrak{F}) : \bigvee_{(\Lambda)_G} F_{\mathcal{W}\Lambda}[\mathcal{G}, (j_{\Lambda,\Pi}^*(E(\Lambda, \Pi, G; U)_+ \wedge C[\Lambda]))^{W_\Pi\Lambda}) \longrightarrow (E\mathfrak{F}'_+ \wedge C)^\Pi$$

is the wedge sum over the G-conjugacy classes of subgroups Λ of Π such that $\Lambda \in \mathfrak{F}' - \mathfrak{F}$ and Π/Λ embeds as a Π-space in U of the maps

$$\beta_\Lambda(\mathfrak{F}') : F_{\mathcal{W}\Lambda}[\mathcal{G}, (j_{\Lambda,\Pi}^*(E(\Lambda, \Pi, G; U)_+ \wedge C[\Lambda]))^{W_\Pi\Lambda}) \longrightarrow (E\mathfrak{F}'_+ \wedge C)^\Pi$$

introduced in Definition 13.1(b). Note that $\beta(\mathfrak{F}', \mathfrak{F})$ is natural in C with respect to maps which preserve the splitting at all the specified subgroups Λ.

Our generalization of Proposition 13.2 to Π-closed pairs of families $(\mathfrak{F}', \mathfrak{F})$ which are not Π-adjacent is then:

THEOREM 14.2. *Let Π be a normal subgroup of a compact Lie group G, $\mathcal{G} = G/\Pi$, and $(\mathfrak{F}', \mathfrak{F})$ be a Π-closed pair of families of subgroups of G. Also, let B be a $\mathcal{G}$-spectrum indexed on U^Π, and C be a G-spectrum indexed on U which is geometrically split at every subgroup Λ of Π such that $\Lambda \in \mathfrak{F}' - \mathfrak{F}$ and Π/Λ embeds* *as a* Π*-space in* U*. Then the composite*

$$\bigvee_{(\Lambda)_G} F_{\mathcal{W}\Lambda}[\mathcal{G}, (j^*_{\Lambda,\Pi}(E(\Lambda,\Pi,G;U)_+ \wedge C[\Lambda]))^{W_\Pi\Lambda}) \xrightarrow{\beta(\mathfrak{F}',\mathfrak{F})} (E\mathfrak{F}'_+ \wedge C)^\Pi$$

$$\xrightarrow{(\mu\wedge 1)^\Pi} (E(\mathfrak{F}',\mathfrak{F}) \wedge C)^\Pi$$

is a weak $\mathcal{G}$*-equivalence. Moreover, if either* B *is a finite* $\mathcal{G}$*-CW spectrum or the indexing set for the direct sum in the domain of* $\alpha(\mathfrak{F}',\mathfrak{F})$ *is finite, then the composite*

$$\bigoplus_{(\Lambda)_G} \left[j^{\Lambda,\Pi}_* B, E(\Lambda,\Pi,G;U)_+ \wedge C[\Lambda]\right]^{U^\Lambda}_{W_G\Lambda} \xrightarrow{\alpha(\mathfrak{F}',\mathfrak{F})} \left[j^\Pi_* B, E\mathfrak{F}'_+ \wedge C\right]^U_G$$

$$\xrightarrow{\mu_*} \left[j^\Pi_* B, E(\mathfrak{F}',\mathfrak{F}) \wedge C\right]^U_G$$

is an isomorphism. Thus, the maps

$$\mu_* : \left[j^\Pi_* B, E\mathfrak{F}'_+ \wedge C\right]^U_G \longrightarrow \left[j^\Pi_* B, E(\mathfrak{F}',\mathfrak{F}) \wedge C\right]^U_G$$

and

$$(\mu \wedge 1)^\Pi : (E\mathfrak{F}'_+ \wedge C)^\Pi \longrightarrow (E(\mathfrak{F}',\mathfrak{F}) \wedge C)^\Pi$$

are split epimorphisms.

Note that Theorems 2.1, 3.2, and 3.8 are obvious consequences of this result. The other two main splitting theorems, Theorems 2.4 and 3.3, can be derived from the special case of this result in which $\mathfrak{F}'$ is the family of all subgroups and $\mathfrak{F}$ is the empty family of subgroups. These two remaining splitting theorems are obtained from this special case by inserting the Adams and Wirthmüller isomorphisms in the appropriate places (as in the proofs of Propositions 2.8 and 3.6 given in the previous section). Note that, by Lemma V.10.3(*iii*) of [**24**], the Wirthmüller isomorphism used in Theorem 3.3 does not require the usual suspension by a representation.

Proof of Theorem 14.2. To prove the claims about $\mu_* \circ \alpha(\mathfrak{F}',\mathfrak{F})$, it suffices to show that this composite is an isomorphism whenever B is a finite $\mathcal{G}$-CW spectrum. If the indexing set for the sum in the domain of $\alpha(\mathfrak{F}',\mathfrak{F})$ is finite, then the apparently stronger assertion about the composite being an isomorphism for all B follows because, in this case, both the domain and range of the composite are cohomology theories in B. If the indexing set is not finite, then the domain of the composite may fail to satisfy the wedge axiom for B.

Our proof employs a form of induction over the families of subgroups of G similar to that described in Proposition V.7.5(*iii*) of [**24**]. Assume first that the pair $(\mathfrak{F}',\mathfrak{F})$ of the theorem is a Π-adjacent pair associated to the subgroup Λ of Π. If Π/Λ embeds as a Π-space in U, then the indexing sets for the domains of $\alpha(\mathfrak{F}',\mathfrak{F})$ and $\beta(\mathfrak{F}',\mathfrak{F})$ consist of the single element $(\Lambda)_G$ so that $\alpha(\mathfrak{F}',\mathfrak{F}) = \alpha_\Lambda(\mathfrak{F}')$ and $\beta(\mathfrak{F}',\mathfrak{F}) = \beta_\Lambda(\mathfrak{F}')$. In this case, Theorem 14.2 is, essentially, just a restatement of Proposition 13.2, and so is proven. If, on the other hand, Π/Λ does not embed in U as a Π-space, then the domains of $\alpha(\mathfrak{F}',\mathfrak{F})$ and $\beta(\mathfrak{F}',\mathfrak{F})$ are trivial because they are sums indexed on the empty set. The failure of Π/Λ to embed implies that, if $H \in \mathfrak{F}' - \mathfrak{F}$, then $H\Pi/H$ cannot embed in U as an $H\Pi$-space. Theorem 6.1 therefore indicates that the spectrum $(E(\mathfrak{F}',\mathfrak{F}) \wedge C)^\Pi$ is weakly $\mathcal{G}$-contractible. The ranges of $\alpha(\mathfrak{F}',\mathfrak{F})$ and $\beta(\mathfrak{F}',\mathfrak{F})$ are then also trivial. Thus, the conclusion of the theorem holds whenever the pair $(\mathfrak{F}',\mathfrak{F})$ is Π-adjacent.

For the inductive step in the proof, let $\mathfrak{D}$ be the collection of G-families $\mathfrak{E}$ such that $\mathfrak{F} \subset \mathfrak{E} \subset \mathfrak{F}'$, the pair $(\mathfrak{E}, \mathfrak{F})$ is Π-closed, and the conclusion of the theorem holds for the pair $(\mathfrak{E}, \mathfrak{F})$. Order $\mathfrak{D}$ by inclusion. Clearly, $\mathfrak{D}$ is not empty since it contains $\mathfrak{F}$. We wish to show that $\mathfrak{F}'$ is in $\mathfrak{D}$. Since G has only countably many conjugacy classes of subgroups, any totally ordered subset of $\mathfrak{D}$ has a cofinal sequence $\{\mathfrak{E}_i\}$. If $\overline{\mathfrak{E}} = \cup \mathfrak{E}_i$, then $\mathfrak{F} \subset \overline{\mathfrak{E}} \subset \mathfrak{F}'$ and the pair $(\overline{\mathfrak{E}}, \mathfrak{F})$ is Π-closed. A straightforward argument using the fact that $\operatorname{Tel} E\mathfrak{E}_i$ is a model for $E\overline{\mathfrak{E}}$ indicates that the conclusion of the theorem holds for the pair $(\overline{\mathfrak{E}}, \mathfrak{F})$. Thus, $\overline{\mathfrak{E}}$ is in $\mathfrak{D}$, and $\mathfrak{D}$ must have a maximal element $\mathfrak{E}'$. If $\mathfrak{E}' \neq \mathfrak{F}'$, then there are subgroups of Π contained in $\mathfrak{F}' - \mathfrak{E}'$. Pick a minimal such subgroup Ψ of Π. Let $\mathfrak{E}'' = \mathfrak{E}' \cup \{H \leq G \mid (H \cap \Pi)_G = (\Psi)_G\}$. Then $\mathfrak{E}''$ is a family of subgroups of G, $\mathfrak{F} \subset \mathfrak{E}'' \subset \mathfrak{F}'$, the pair $(\mathfrak{E}'', \mathfrak{F})$ is Π-closed, and the pair $(\mathfrak{E}'', \mathfrak{E}')$ is Π-adjacent. To complete the proof of the theorem, it suffices to show that its conclusion holds for the pair $(\mathfrak{E}'', \mathfrak{F})$. This contradicts the assumed maximal nature of $\mathfrak{E}'$ and so implies that $\mathfrak{E}' = \mathfrak{F}'$.

The sequence

$$E(\mathfrak{E}', \mathfrak{F}) \xrightarrow{\kappa} E(\mathfrak{E}'', \mathfrak{F}) \xrightarrow{\kappa'} E(\mathfrak{E}'', \mathfrak{E}')$$

is a cofibre sequence. It therefore provides long exact sequences of the form

$$\cdots \xrightarrow{\partial} \left[j_*^{\Pi} B, E(\mathfrak{E}', \mathfrak{F}) \wedge C\right]_G^U \xrightarrow{\kappa_*} \left[j_*^{\Pi} B, E(\mathfrak{E}'', \mathfrak{F}) \wedge C\right]_G^U \xrightarrow{\kappa'_*} \left[j_*^{\Pi} B, E(\mathfrak{E}'', \mathfrak{E}') \wedge C\right]_G^U \xrightarrow{\partial} \cdots$$

and

$$\cdots \xrightarrow{\partial} \left[S^n \wedge \mathcal{G}/\mathcal{H}_+, (E(\mathfrak{E}', \mathfrak{F}) \wedge C)^{\Pi}\right]_G^U \xrightarrow{\kappa_*} \left[S^n \wedge \mathcal{G}/\mathcal{H}_+, (E(\mathfrak{E}'', \mathfrak{F}) \wedge C)^{\Pi}\right]_G^U \xrightarrow{\kappa'_*} \left[S^n \wedge \mathcal{G}/\mathcal{H}_+, (E(\mathfrak{E}'', \mathfrak{E}') \wedge C)^{\Pi}\right]_G^U \xrightarrow{\partial} \cdots,$$

where $n \in \mathbb{Z}$ and $\mathcal{H} \leq \mathcal{G}$.

For any Π-closed pair $(\mathfrak{G}', \mathfrak{G})$ of G-families, denote the direct sum which forms the domain of the map $\alpha(\mathfrak{G}', \mathfrak{G})$ by $D(\mathfrak{G}', \mathfrak{G})$. Then $D(\mathfrak{E}'', \mathfrak{F})$ is the direct sum of $D(\mathfrak{E}', \mathfrak{F})$ and $D(\mathfrak{E}'', \mathfrak{E}')$. The canonical inclusion ι of $D(\mathfrak{E}', \mathfrak{F})$ into $D(\mathfrak{E}'', \mathfrak{F})$ and the canonical projection π of $D(\mathfrak{E}'', \mathfrak{F})$ onto $D(\mathfrak{E}'', \mathfrak{E}')$ therefore form a short exact sequence which appears as the top row in the diagram:

$$\begin{array}{ccccc}
D(\mathfrak{E}', \mathfrak{F}) & \xrightarrow{\iota} & D(\mathfrak{E}'', \mathfrak{F}) & \xrightarrow{\pi} & D(\mathfrak{E}'', \mathfrak{E}') \\
\downarrow{\scriptstyle \alpha(\mathfrak{E}', \mathfrak{F})} & & \downarrow{\scriptstyle \alpha(\mathfrak{E}'', \mathfrak{F})} & & \downarrow{\scriptstyle \alpha(\mathfrak{E}'', \mathfrak{E}')} \\
\left[j_*^{\Pi} B, E\mathfrak{E}'_+ \wedge C\right]_G^U & \xrightarrow{\lambda_*} & \left[j_*^{\Pi} B, E\mathfrak{E}''_+ \wedge C\right]_G^U & & \left[j_*^{\Pi} B, E\mathfrak{E}''_+ \wedge C\right]_G^U \\
\downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \mu'_*} & & \downarrow{\scriptstyle \mu''_*} \\
\left[j_*^{\Pi} B, E(\mathfrak{E}', \mathfrak{F}) \wedge C\right]_G^U & \xrightarrow[\kappa_*]{} & \left[j_*^{\Pi} B, E(\mathfrak{E}'', \mathfrak{F}) \wedge C\right]_G^U & \xrightarrow[\kappa'_*]{} & \left[j_*^{\Pi} B, E(\mathfrak{E}'', \mathfrak{E}') \wedge C\right]_G^U.
\end{array}$$

It is easy to see that the two left-hand squares in this diagram commute. To see that the right-hand rectangle commutes, it suffices to check its commutativity when restricted to each of the summands of $D(\mathfrak{E}'', \mathfrak{F})$. Restricted to the summand indexed on $(\Psi)_G$, it clearly commutes. The commutativity of the restriction to each of the other summands follows easily from the commutativity of the two left-hand squares and the exactness of the bottom row. Since the maps π and $\mu''_* \circ \alpha(\mathfrak{E}'', \mathfrak{E}')$ are both surjective, the map κ'_* must also be surjective. Since this argument applies for any finite $\mathcal{G}$-CW spectrum B and the bottom row of the diagram is part of a long exact sequence, the map κ must be injective. The bottom row of the diagram is therefore a short exact sequence. The left and right vertical composites are isomorphisms, and so the center vertical composite $\mu'_* \circ \alpha(\mathfrak{E}'', \mathfrak{F})$ is also an isomorphism.

A similar argument applied to the $\mathcal{G}$-homotopy groups of the domains of the maps $\beta(\mathfrak{E}', \mathfrak{F})$, $\beta(\mathfrak{E}'', \mathfrak{F})$, and $\beta(\mathfrak{E}'', \mathfrak{E}')$ and the long exact sequence of $\mathcal{G}$-homotopy groups displayed above proves that the composite $(\mu \wedge 1)^{\Pi} \circ \beta(\mathfrak{E}'', \mathfrak{F})$ is a weak $\mathcal{G}$-equivalence. Thus, the conclusion of the theorem holds for the pair $(\mathfrak{E}'', \mathfrak{F})$, and we have the desired contradiction. □

The proof below is the last of the delayed proofs from Part I, and concludes this section.

Proof of Corollary 2.7. Let C be a G-spectrum indexed on a universe U, and $i : U \longrightarrow U'$ be a linear G-isometry. Assume that C is U'-split. We wish to show that the natural inclusion

$$\delta : i^G_*(C^G) \longrightarrow (i_*C)^G$$

is a split monomorphism in the stable category of spectra indexed on $(U')^G$. Theorem 2.4 provides weak equivalences:

$$C^G \simeq \bigvee_{(\Lambda)} EW\Lambda_+ \wedge_{W\Lambda} \Sigma^{Ad(W\Lambda)} C[\Lambda]$$

and

$$(i_*C)^G \simeq \bigvee_{(\Lambda)} EW\Lambda_+ \wedge_{W\Lambda} \Sigma^{Ad(W\Lambda)} i^\Lambda_* C[\Lambda],$$

in which the first wedge is indexed on the G-conjugacy classes (Λ) of subgroups Λ of G such that G/Λ embeds as a G-space in U and the second is indexed on those classes (Λ) for which G/Λ embeds as a G-space in U'. Thus, to show that $i^G_*(C^G)$ is a wedge summand of $(i_*C)^G$, it suffices to show that, if G/Λ embeds as a G-space in U, then $i^G_*(EW\Lambda_+ \wedge_{W\Lambda} \Sigma^{Ad(W\Lambda)} C[\Lambda])$ and $EW\Lambda_+ \wedge_{W\Lambda} \Sigma^{Ad(W\Lambda)} i^\Lambda_* C[\Lambda]$ are weakly equivalent spectra. Recall that the spectrum $EW\Lambda_+ \wedge_{W\Lambda} \Sigma^{Ad(W\Lambda)} C[\Lambda]$ is the orbit spectrum $Z(\Lambda)/W\Lambda$ of the $W\Lambda$-spectrum $Z(\Lambda)$ indexed on U^G which represents the U^Λ-indexed $W\Lambda$-spectrum $EW\Lambda_+ \wedge \Sigma^{Ad(W\Lambda)} C[\Lambda]$. It is easy to see that the spectrum $i^G_* Z(\Lambda)$ indexed on $(U')^G$ represents the spectrum $EW\Lambda_+ \wedge \Sigma^{Ad(W\Lambda)} i^\Lambda_* C[\Lambda]$. The desired weak equivalence follows from this coupled with the fact that passing to orbits over $W\Lambda$ commutes with the functor i^G_*. An easy diagram chase confirms that the resulting identification of $i^G_*(C^G)$ as a wedge summand of $(i_*C)^G$ is consistent with the canonical map δ. □

SECTION 15

The proof of the sharp Wirthmüller isomorphism theorem

This section contains the proof of Theorem 11.4. This result asserts that, if G is a finite group, U is a G-universe, and $N \leq G$, then the formally defined Wirthmüller family $\mathfrak{W}(N,G;U)$ of subgroups of N is equal to the more concretely defined family $\mathfrak{W}'(N,G;U)$. Recall that, in the discussion of the Wirthmüller isomorphism at the end of Section 11, we showed that $\mathfrak{W}(N,G;U) \subset \mathfrak{W}'(N,G;U)$. Thus, it suffices to show the other inclusion. This is easily reduced to showing that, if $H \leq G$, $n \in \mathbb{Z}$, $K \in \mathfrak{W}'(N,G;U)$, and $(\mathfrak{D}', \mathfrak{D})$ is an adjacent pair of families of subgroups of K, then the Wirthmüller map

$$\psi_* : [G/H_+ \wedge S^n, G \ltimes_N (N \ltimes_K E(\mathfrak{D}', \mathfrak{D}))]_G^U \longrightarrow \\ [G/H_+ \wedge S^n, F_N[G, N \ltimes_K E(\mathfrak{D}', \mathfrak{D}))]_G^U$$

is an isomorphism. Here, and throughout the remainder of this section, we are working entirely in the equivariant stable category, and so omit the Σ_U^∞ when referring to the suspension spectrum of a space. By a change of groups isomorphism, the map above can be identified with the map

$$\psi_* : [S^n, G \ltimes_N (N \ltimes_K E(\mathfrak{D}', \mathfrak{D}))]_H^U \longrightarrow [S^n, F_N[G, N \ltimes_K E(\mathfrak{D}', \mathfrak{D}))]_H^U.$$

Thus, the central issue in the proof of Theorem 11.4 is the structure of the G-spectra $G \ltimes_N (N \ltimes_K E(\mathfrak{D}', \mathfrak{D}))$ and $F_N[G, N \ltimes_K E(\mathfrak{D}', \mathfrak{D}))$ as H-spectra. This structure is well understood only when G is finite. For this reason, Theorem 11.4 is restricted to finite groups.

Our description of the structure of these two G-spectra considered as H-spectra is obtained from a double coset formula applicable to the G-spectra $G \ltimes_N C$ and $F_N[G,C)$ derived from any N-spectrum C. In this formula, we denote by $[g]$ the equivalence class of an element g of G in the double coset $H\backslash G/N$. As one might expect in a double coset formula, the N-spectrum C must be converted into an appropriate $(H \cap gNg^{-1})$-spectrum. This is accomplished using the change of universe functor g_* derived from the action map of g on U. The spectrum g_*C is a obviously a gNg^{-1}-spectrum, and thus an $(H \cap gNg^{-1})$-spectrum.

PROPOSITION 15.1. *Let H and N be subgroups of a finite group G, and let C be an N-spectrum indexed on a G-universe U. Then there are natural isomorphisms*

$$G \ltimes_N C \cong \bigvee_{[g] \in H\backslash G/N} H \ltimes_{(H \cap gNg^{-1})} g_*C$$

and

$$F_N[G,C) \cong \prod_{[g] \in H\backslash G/N} F_{(H \cap gNg^{-1})}[H, g_*C)$$

of H-spectra. Moreover, under these isomorphisms, the Wirthmüller map

$$\psi : G \ltimes_N C \longrightarrow F_N[G, C)$$

is identified with the composite H-map

$$\begin{aligned} \bigvee_{[g]\in H\backslash G/N} H \ltimes_{(H\cap gNg^{-1})} g_*C \quad & \xrightarrow{\vee\,\psi} \quad \bigvee_{[g]\in H\backslash G/N} F_{(H\cap gNg^{-1})}[H, g_*C) \\ & \xrightarrow{\simeq} \quad \prod_{[g]\in H\backslash G/N} F_{(H\cap gNg^{-1})}[H, g_*C) \end{aligned}$$

in which the second map is the natural inclusion of the wedge into the product.

PROOF. Let $\{g_i\}$ be a set of representatives for the cosets in G/N. Then the G-spectrum $G \ltimes_N C$ is isomorphic, before passage to the stable category, to the spectrum $\vee_i\, g^i_*C$. If $g \in G$ and $gg_iN = g_jN$, then the action map

$$G \ltimes (G \ltimes_N C) \longrightarrow G \ltimes_N C.$$

of G on $G \ltimes_N C$ takes the wedge summand of the domain indexed on g and g_i to the wedge summand of the range indexed on g_j via the map

$$g_*g^i_*C \cong g^j_*(g_j^{-1}gg_i)_*C \xrightarrow{g^j_*(g_j^{-1}gg_i)} g^j_*C$$

derived from the action of $g_j^{-1}gg_i \in N$ on C.

Analogously, the G-spectrum $F_N[G, C)$ can be identified, before passage to the stable category, with the spectrum $\prod_i (g_i^{-1})^*C$. However, $(g_i^{-1})^*$ is naturally isomorphic to g^i_*, so that $F_N[G, C)$ can be identified with the spectrum $\prod_i g^i_*C$. Under this identification, the action of G on $F_N[G, C)$ is the obvious variant of the action of G on $G \ltimes_N C$.

Under these identifications, the Wirthmüller map

$$\psi : G \ltimes_N C \longrightarrow F_N[G, C)$$

becomes the natural map

$$\bigvee_i g^i_*C \longrightarrow \prod_i g^i_*C$$

of the wedge into the product. The isomorphisms of the proposition and the description of the Wirthmüller map in the proposition follow from these identifications merely by grouping together the terms in $\vee_i\, g^i_*C$ and $\prod_i g^i_*C$ which lie in the same H-orbit. □

This proposition and Theorems 6.1 and 11.6 suffice for our proof of the sharp Wirthmüller isomorphism theorem.

PROOF OF THEOREM 11.4. The proposition allows us to identify the map

$$\begin{aligned} \psi_* : [G/H_+ \wedge S^n, G \ltimes_N (N \ltimes_K E(\mathfrak{D}', \mathfrak{D}))]^U_G \longrightarrow \\ [G/H_+ \wedge S^n, F_N[G, N \ltimes_K E(\mathfrak{D}', \mathfrak{D}))]^U_G \end{aligned}$$

with the map

$$\oplus\ \psi_*: \bigoplus_{[g]\in H\backslash G/N} [S^n, H \ltimes_{(H\cap gNg^{-1})} g_*(N \ltimes_K E(\mathfrak{D}',\mathfrak{D}))]^U_H \longrightarrow$$
$$\bigoplus_{[g]\in H\backslash G/N} [S^n, F_{(H\cap gNg^{-1})}[H, g_*(N \ltimes_K E(\mathfrak{D}',\mathfrak{D})))]^U_H.$$

Thus, it suffices to show that the composite

$$\begin{array}{ll} & [S^n, H \ltimes_{(H\cap gNg^{-1})} g_*(N \ltimes_K E(\mathfrak{D}',\mathfrak{D}))]^U_H \\ \xrightarrow{\psi_*} & [S^n, F_{(H\cap gNg^{-1})}[H, g_*(N \ltimes_K E(\mathfrak{D}',\mathfrak{D})))]^U_H \\ \cong & [S^n, g_*(N \ltimes_K E(\mathfrak{D}',\mathfrak{D}))]^U_{H\cap gNg^{-1}} \end{array}$$

is an isomorphism for some set $\{g\}$ of representatives for the double coset space $H\backslash G/N$. Since the pair $(\mathfrak{D}',\mathfrak{D})$ of families of subgroups of K is adjacent, there is a subgroup J of K such that $\mathfrak{D}'-\mathfrak{D}=(J)_K$. Select a pair $(\mathfrak{E}',\mathfrak{E})$ of families of subgroups of N such that $\mathfrak{E}'-\mathfrak{E}=(J)_N$, and observe that $N \ltimes_K E(\mathfrak{D}',\mathfrak{D})$ and $E(\mathfrak{E}',\mathfrak{E}) \wedge N \ltimes_K E(\mathfrak{D}',\mathfrak{D})$ are homotopy equivalent N-spaces.

Three cases must be considered. If $(J)_N \not\leq (g^{-1}Hg\cap N)_N$, then the space $E(\mathfrak{E}',\mathfrak{E})\wedge N \ltimes_K E(\mathfrak{D}',\mathfrak{D})$ is obviously $(g^{-1}Hg\cap N)$-contractible. In this case, both the domain and range of the above composite vanish. Thus, we may assume that $(J)_N \leq (g^{-1}Hg\cap N)_N$.

Now consider the case in which there is no subgroup $J' \in (J)_N$ such that $J' \leq g^{-1}Hg\cap N$ and $(g^{-1}Hg\cap N)/J'$ embeds in U as a $(g^{-1}Hg\cap N)$-space. In this case, the target of the composite vanishes by Theorem 6.1. To see that the domain of the composite also vanishes, select a pair $(\mathfrak{F}',\mathfrak{F})$ of families of subgroups of H such that

$$\mathfrak{F}'-\mathfrak{F} = \{Q \leq H \mid \exists J' \leq g^{-1}Hg\cap N,\ (J')_N=(J)_N \text{ and } (gJ'g^{-1})_H=(Q)_H\}.$$

Note that, if $Q \in \mathfrak{F}'-\mathfrak{F}$, then H/Q cannot embed in U as an H-space because of our assumption about the nonexistence of $(g^{-1}Hg\cap N)$-embeddings. It is easy to check that the two spectra $E(\mathfrak{F}',\mathfrak{F}) \wedge H \ltimes_{(H\cap gNg^{-1})} g_*(N \ltimes_K E(\mathfrak{D}',\mathfrak{D}))$ and $H \ltimes_{(H\cap gNg^{-1})} g_*(N \ltimes_K E(\mathfrak{D}',\mathfrak{D}))$ are H-equivalent. Thus, Theorem 6.1 implies that the domain of our composite vanishes.

The only remaining case is that in which $(J)_N \leq (g^{-1}Hg\cap N)_N$ and there is a subgroup $J' \in (J)_N$ such that $J' \leq g^{-1}Hg\cap N$ and $(g^{-1}Hg\cap N)/J'$ embeds in U as a $(g^{-1}Hg\cap N)$-space. By choosing g appropriately in its double coset, we may assume that $J'=J$. Since $J\leq K$, $K\in\mathfrak{W}'(N,G;U)$, and $(g^{-1}Hg\cap N)/J$ embeds in U as a $(g^{-1}Hg\cap N)$-space, $(g^{-1}Hg)/J$ must embed in U as a $g^{-1}Hg$-space. There is a model for the K-space $E(\mathfrak{D}',\mathfrak{D})$ in which every orbit is of the form K/L for some subgroup L of J. The existence of the embedding of $(g^{-1}Hg)/J$ in U then implies that $g_*(N \ltimes_K E(\mathfrak{D}',\mathfrak{D}))$, constructed using this model for $E(\mathfrak{D}',\mathfrak{D})$, is a $\mathfrak{W}''(H\cap gNg^{-1},H;U)$-spectrum. Thus, by Theorem 11.6, our composite is an isomorphism in this final case. □

SECTION 16

The proof of the Adams isomorphism theorem for incomplete universes

Theorem 12.2 is proven here. The first step in this proof ought to be showing that the Adams transfer $\tau : i_*((\Sigma^A D)/\Pi) \longrightarrow i_* D$ exists. However, the general process for constructing transfers developed in [**24**] does not seem adequate for constructing τ when the universe is incomplete. Rather than obscuring the proof with the gory details of an alternative construction, we begin here by summarizing, via axioms, the properties of τ needed for this proof. The theorem is then proven under the assumption that a transfer exists satisfying those axioms. Section 17 completes this proof by supplying the desired transfer. Throughout this section, we follow the variant of the notational conventions of section II.7 of [**24**] introduced in Section 12. Thus, Π is a normal subgroup of a compact Lie group G, $\mathcal{G} = G/\Pi$, $i : U' \longrightarrow U$ is a linear G-isometry from a Π-trivial G-universe U' to a G-universe U, D is a Π-free G-spectrum indexed on U', and A is the adjoint representation of G derived from Π.

It turns out that the Adams transfer exists for a somewhat larger class of spectra than the class $\mathfrak{A}_G(\Pi; U, U')$ for which it yields an isomorphism. We begin by introducing this larger class and stating our axioms in terms of it.

Definition 16.1. If U is a G-universe into which Π/e embeds as a Π-space, then the G-family $\mathfrak{F}_G(\Pi; U)$ of subgroups consists of those $H \leq G$ such that $H \cap \Pi = \{e\}$ and $H\Pi/H$ embeds in U as an $H\Pi$-space. Note that, for any Π-trivial G-universe U' which embeds in U, the Adams family $\mathfrak{A}_G(\Pi; U, U')$ is contained in $\mathfrak{F}_G(\Pi; U)$. Also, if P is a subgroup of G such that $\Pi \leq P$, then $\mathfrak{F}_P(\Pi; U)$ is just the intersection of $\mathfrak{F}_G(\Pi; U)$ with the set of subgroups of P. If U is a complete G-universe, then $\mathfrak{F}_G(\Pi; U) = \mathfrak{F}_G(\Pi)$. Moreover, if $\Pi = G$, then, for any universe U, $\mathfrak{F}_G(\Pi; U)$ is just the family $\mathfrak{F}_G(G)$ consisting of the trivial subgroup of G.

The first two axioms impose the obvious naturality and change of group restrictions.

> **(A1)** The Adams transfer τ is natural with respect to maps between $\mathfrak{F}_G(\Pi; U)$-spectra; that is, if $f : D \longrightarrow D'$ is a G-map between $\mathfrak{F}_G(\Pi; U)$-spectra indexed on U, then the diagram
>
> $$\begin{array}{ccc} i_*((\Sigma^A D)/\Pi) & \xrightarrow{i_*((\Sigma^A f)/\Pi)} & i_*((\Sigma^A D')/\Pi) \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau'} \\ i_* D & \xrightarrow{i_* f} & i_* D' \end{array}$$
>
> commutes in the stable category of G-spectra indexed on U.

> **(A2)** If $\Pi \le P \le G$ and D is an $\mathfrak{F}_G(\Pi; U)$-spectrum indexed on U', then the Adams transfer $\tau_G : i_*((\Sigma^A D)/\Pi) \longrightarrow i_* D$, considered as a P-map of P-spectra indexed on U, is P-homotopic to the Adams transfer $\tau_P : i_*((\Sigma^A D)/\Pi) \longrightarrow i_* D$ obtained by regarding D as an $\mathfrak{F}_P(\Pi; U)$-spectrum.

The third axiom describes the interaction between the transfer and smash products. A technical lemma about orbit spectra is needed for its statement.

LEMMA 16.2. *Let Π be a normal subgroup of a compact Lie group G, $\mathcal{G}$ be G/Π, D be a G-spectrum indexed on a Π-trivial G-universe U', and Y be a $\mathcal{G}$-space regarded as a G-space via the projection $G \longrightarrow \mathcal{G}$. Then there is a natural isomorphism of spectra*

$$Y \wedge (D/\Pi) \cong (Y \wedge D)/\Pi.$$

This lemma asserts that two functors, both of which are left adjoints, are naturally isomorphic. It is easy to see that the corresponding right adjoints are naturally isomorphic, and the lemma follows by the uniqueness of left adjoints. Note that, if D is an $\mathfrak{F}_G(\Pi; U)$-spectrum and Y is a $\mathcal{G}$-CW complex, then $Y \wedge D$ is an $\mathfrak{F}_G(\Pi; U)$-spectrum. Our third axiom relates the Adams transfers for D and $Y \wedge D$.

> **(A3)** If Y is a $\mathcal{G}$-CW complex regarded as a G-space via the projection $G \longrightarrow \mathcal{G}$ and D is an $\mathfrak{F}_G(\Pi; U)$-spectrum indexed on U', then the diagram
>
> $$\begin{array}{ccc} Y \wedge i_*((\Sigma^A D)/\Pi) & \xrightarrow{1 \wedge \tau} & Y \wedge i_* D \\ \cong \downarrow & & \downarrow \cong \\ i_*((\Sigma^A (Y \wedge D))/\Pi) & \xrightarrow{\tau} & i_*(Y \wedge D) \end{array}$$
>
> commutes in the stable category of G-spectra indexed on U.

The vertical isomorphisms in this axiom are given by Lemma 16.2 and Proposition II.1.4 of [**24**]. Observe that this axiom applied to the case $Y = S^1$ indicates that the Adams transfer is preserved under suspension by a trivial representation.

The final axiom is a normalization axiom indicating that τ is the obvious dimension-shifting transfer when D is just the suspension spectrum $\Sigma^\infty_{U'} G/H_+$ associated to some subgroup H in $\mathfrak{F}_G(\Pi; U)$. If G is finite, then this axiom is just the assertion made in the discussion at the end of Section 12 that τ is the ordinary transfer associated to the projection $G/H \longrightarrow G/H\Pi$. However, to state this axiom properly for nonfinite groups, we must first describe the structure of the orbit spectrum $(\Sigma^A \Sigma^\infty_{U'} G/H_+)/\Pi$. A difficulty with the action of the group $H\Pi$ on the vector space A comes up in this description. Let $\rho : H\Pi \longrightarrow H\Pi/\Pi \cong H$ be the obvious projection. The group $H\Pi$ can act on A via the inclusion $H\Pi \subset G$ or via the composite of ρ and the inclusion $H \subset G$. These two actions are usually not the same since $\Pi \le H\Pi$ acts trivially on A with the second action; whereas A, regarded as a Π-representation with the first action, is just the adjoint representation of Π. The second action must be used for the analysis of the orbit spectrum

$(\Sigma^A\Sigma^\infty_{U'}G/H_+)/\Pi$. As a reminder of this, we denote A with this somewhat nonstandard $H\Pi$-action as ρ^*A. Note that A and ρ^*A are isomorphic as H-representations since the composite of ρ and the inclusion $H \subset H\Pi$ is just the identity map on H.

LEMMA 16.3. *If $H \in \mathfrak{F}_G(\Pi; U)$, then there is an isomorphism*

$$i_*((\Sigma^A\Sigma^\infty_{U'}G/H_+)/\Pi) \cong G \ltimes_{H\Pi} \Sigma^\infty_U S^{\rho^*A}$$

of G-spectra indexed on U.

PROOF. By Propositions I.3.6 and I.3.8 of [**24**], $(\Sigma^A\Sigma^\infty_{U'}G/H_+)/\Pi$ is isomorphic to $\Sigma^\infty_{U'}((\Sigma^A G/H_+)/\Pi)$. Moreover, $\Sigma^A G/H_+ \cong G \ltimes_H S^A$. There is a natural projection from $G \ltimes_H S^A$ to $G \ltimes_{H\Pi} S^{\rho^*A}$. Since Π acts trivially on ρ^*A, $G \ltimes_{H\Pi} S^{\rho^*A}$ is G-isomorphic to $\mathcal{G} \ltimes_{H\Pi/\Pi} S^A$ considered as a G-space via the projection $G \longrightarrow \mathcal{G}$. Thus, Π acts trivially on $G \ltimes_{H\Pi} S^{\rho^*A}$ and the projection $G \ltimes_H S^A \longrightarrow G \ltimes_{H\Pi} S^{\rho^*A}$ factors through a map $(G \ltimes_H S^A)/\Pi \longrightarrow G \ltimes_{H\Pi} S^{\rho^*A}$. It is easy to see that this last map is an isomorphism of $\mathcal{G}$-spaces. Combining this space-level isomorphism with the isomorphism between $(\Sigma^A\Sigma^\infty_{U'}G/H_+)/\Pi$ and $\Sigma^\infty_{U'}((\Sigma^A G/H_+)/\Pi)$ yields an isomorphism

$$(\Sigma^A\Sigma^\infty_{U'}G/H_+)/\Pi \cong G \ltimes_{H\Pi} \Sigma^\infty_{U'} S^{\rho^*A}$$

in the category of G-spectra indexed on U'. Applying the functor i_* to this isomorphism and composing the result with two isomorphisms derived from Proposition II.1.4 and Lemma II.4.14 of [**24**] gives the isomorphism whose existence is asserted by the lemma. □

Now recall from the proof of Lemma II.7.6 of [**24**] that the H-representation derived from the tangent space of $H\Pi/H$ at eH is just A. Since $H\Pi/H$ embeds in U as an $H\Pi$-space, A must be contained in U as an H-represention. Thus, there is a pretransfer $t : S^0 \longrightarrow H\Pi \ltimes_H S^{-A}$ in the category of $H\Pi$-spectra indexed on U. Our final axiom describes the Adams transfer for $\Sigma^\infty_{U'}G/H_+$ in terms of this pretransfer.

(A4) If $H \in \mathfrak{F}_G(\Pi; U)$, then the Adams transfer

$$\tau : i_*((\Sigma^A\Sigma^\infty_{U'}G/H_+)/\Pi) \longrightarrow i_*\Sigma^\infty_{U'}G/H_+$$

for the $\mathfrak{F}_G(\Pi; U)$-spectrum $\Sigma^\infty_{U'}G/H_+$ is just the composite

$$\begin{aligned}
i_*((\Sigma^A\Sigma^\infty_{U'}G/H_+)/\Pi) &\cong G \ltimes_{H\Pi} \Sigma^\infty_U S^{\rho^*A} \\
&\cong G \ltimes_{H\Pi} (\Sigma^\infty_U S^{\rho^*A} \wedge S^0) \\
&\xrightarrow{1\ltimes_{H\Pi}(1\wedge t)} G \ltimes_{H\Pi} (\Sigma^\infty_U S^{\rho^*A} \wedge H\Pi \ltimes_H S^{-A}) \\
&\cong G \ltimes_{H\Pi} H\Pi \ltimes_H (\Sigma^\infty_U S^A \wedge S^{-A}) \\
&\cong G \ltimes_H S^0 \cong \Sigma^\infty_U G/H_+ \cong i_*\Sigma^\infty_{U'}G/H_+
\end{aligned}$$

The isomorphisms in the composite map of axiom (A4) are derived from Lemma 16.3, the fact that A and ρ^*A are isomorphic H-representations, and an assortment of results from chapter II of [**24**].

Assume now that there is an Adams transfer defined for all $\mathfrak{F}_G(\Pi; U)$-spectra indexed on U' and satisfying axioms **(A1)** through **(A4)**. We would like to complete the proof of Theorem 12.2 by simply copying the proof of Theorem II.7.1 of [**24**]. However, that proof employs the Wirthmüller isomorphism repeatedly.

This isomorphism is not necessarily available in an incomplete universe U unless an appropriate orbit embeds in U. Thus, our proof begins with a long sequence of inductive arguments whose purpose is to reduce the proof of the theorem to a case in which the appropriate orbits embed. Once this reduction is accomplished, the Wirthmüller isomorphism can be employed to finish the proof in the same way that the proof of Theorem II.7.1 was finished in [**24**].

PROOF OF THEOREM 12.2. Assume that D is an $\mathfrak{A}_G(\Pi;U,U')$-spectrum indexed on U'. We must show that, for all $\mathcal{P}\le\mathcal{G}$ and $m\in\mathbb{Z}$, the map

$$\big[\mathcal{G}/\mathcal{P}_+\wedge S^m,(\Sigma^A D)/\Pi\big]_{\mathcal{G}}^{U'}\xrightarrow{\tilde{\tau}_*}\big[\mathcal{G}/\mathcal{P}_+\wedge S^m,(i_*D)^{\Pi}\big]_{\mathcal{G}}^{U'}$$

is an isomorphism. Since the diagram

$$\begin{array}{ccc}
\big[\mathcal{G}/\mathcal{P}_+\wedge S^m,(\Sigma^A D)/\Pi\big]_{\mathcal{G}}^{U'} & \xrightarrow{\tilde{\tau}_*} & \big[\mathcal{G}/\mathcal{P}_+\wedge S^m,(i_*D)^{\Pi}\big]_{\mathcal{G}}^{U'}\\
\cong\downarrow & & \\
\big[\mathcal{G}/\mathcal{P}_+\wedge S^m,(\Sigma^A D)/\Pi\big]_{G}^{U'} & & \downarrow\cong\\
i_*\downarrow & & \\
\big[\mathcal{G}/\mathcal{P}_+\wedge S^m,i_*((\Sigma^A D)/\Pi)\big]_{G}^{U} & \xrightarrow{\tau_*} & \big[\mathcal{G}/\mathcal{P}_+\wedge S^m,i_*D\big]_{G}^{U}
\end{array}$$

commutes, it suffices to show that the composite along the left side and bottom of the diagram is an isomorphism. Let $\rho:G\longrightarrow\mathcal{G}$ be the canonical projection, and let $P=\rho^{-1}(\mathcal{P})$. Using change of group isomorphisms, we can identify this composite with the composite

$$\begin{aligned}
\big[S^m,(\Sigma^A D)/\Pi\big]_{\mathcal{P}}^{U'} &\cong \big[S^m,(\Sigma^A D)/\Pi\big]_{P}^{U'}\\
&\xrightarrow{i_*}\big[S^m,i_*((\Sigma^A D)/\Pi)\big]_{P}^{U}\\
&\xrightarrow{\tau_*}\big[S^m,i_*D\big]_{P}^{U}.
\end{aligned}$$

Here, axiom **(A2)** implies that we can take τ to be either the P-map coming from D regarded as an $\mathfrak{A}_P(\Pi;U,U')$-spectrum or the G-map coming from D regarded as an $\mathfrak{A}_G(\Pi;U,U')$-spectrum. Therefore, an induction over the subgroups $\mathcal{P}$ of $\mathcal{G}$ reduces our problem to showing that the composite above is an isomorphism in the special case where $\mathcal{P}=\mathcal{G}$ and $P=G$.

A further induction over the cells of D reduces the proof to showing that this composite is an isomorphism when $D=\Sigma^\infty_{U'}G/H_+$, for $H\in\mathfrak{A}_G(\Pi;U,U')$. This reduction uses the naturality axiom **(A1)**. It also uses the smash product axiom **(A3)** to shift the dimension of the cells of D over to the dimension of the sphere in the domain. We prove the apparently more general result that, for any $\mathcal{G}$-CW-complex Y, the composite

$$\begin{aligned}
\big[S^m,Y\wedge(\Sigma^A\Sigma^\infty_{U'}G/H_+)/\Pi\big]_{\mathcal{G}}^{U'} &\cong \big[S^m,Y\wedge(\Sigma^A\Sigma^\infty_{U'}G/H_+)/\Pi\big]_{G}^{U'}\\
&\xrightarrow{i_*}\big[S^m,Y\wedge i_*((\Sigma^A\Sigma^\infty_{U'}G/H_+)/\Pi)\big]_{G}^{U}\\
&\xrightarrow{(1\wedge\tau)_*}\big[S^m,Y\wedge i_*\Sigma^\infty_{U'}G/H_+\big]_{G}^{U}
\end{aligned}$$

is an isomorphism whenever $H \in \mathfrak{A}_G(\Pi; U, U')$. Hereafter, we refer to this composite as $\widehat{\tau}(Y, H)$. We carry out this proof using an induction over the subgroups H of G in $\mathfrak{A}_G(\Pi; U, U')$. Thus, in proving that $\widehat{\tau}(Y, H)$ is an isomorphism, we can assume that $\widehat{\tau}(Y, K)$ is an isomorphism for every proper subgroup K of H.

For each $H \in \mathfrak{A}_G(\Pi; U, U')$, we show that $\widehat{\tau}(Y, H)$ is an isomorphism using an induction over the families of subgroups of $\mathcal{G}$. This induction reduces the problem to showing that, for each pair $(\mathfrak{F}', \mathfrak{F})$ of adjacent families of subgroups of $\mathcal{G}$, the map $\widehat{\tau}(E(\mathfrak{F}', \mathfrak{F}) \wedge Y, H)$ is an isomorphism. Since the pair $(\mathfrak{F}', \mathfrak{F})$ is adjacent, there is a subgroup $\mathcal{P}$ of $\mathcal{G}$ such that $\mathfrak{F}' - \mathfrak{F} = (\mathcal{P})_{\mathcal{G}}$. If $(\mathcal{P})_{\mathcal{G}} \not\leq (H\Pi/\Pi)_{\mathcal{G}}$, then both the domain and range of the map $\widehat{\tau}(E(\mathfrak{F}', \mathfrak{F}) \wedge Y, H)$ vanish. To see that the domain vanishes, note that the $\mathcal{G}$-space $E(\mathfrak{F}', \mathfrak{F}) \wedge Y \wedge (\Sigma^A G/H_+)/\Pi$ is weakly contractible since $((\Sigma^A G/H_+)/\Pi)^{\mathcal{P}}$ is a point if $(\mathcal{P})_{\mathcal{G}} \not\leq (H\Pi/\Pi)_{\mathcal{G}}$. To see that the range vanishes, note that the G-space $E(\mathfrak{F}', \mathfrak{F}) \wedge Y \wedge G/H_+$ is G-homeomorphic to $G \ltimes_H (E(\mathfrak{F}', \mathfrak{F}) \wedge Y)$ and that $E(\mathfrak{F}', \mathfrak{F})$ is H-contractible unless $(\mathcal{P})_{\mathcal{G}} \leq (H\Pi/\Pi)_{\mathcal{G}}$.

We may therefore assume that $(\mathcal{P})_{\mathcal{G}} \leq (H\Pi/\Pi)_{\mathcal{G}}$ and may then pick $\mathcal{P}$ so that $\mathcal{P} \leq H\Pi/\Pi$. If $\mathcal{G}/\mathcal{P}$ does not embed in U', then the domain and range of $\widehat{\tau}(E(\mathfrak{F}', \mathfrak{F}) \wedge Y, H)$ also vanish. The vanishing of the domain follows directly from Lemma 6.2. To see that the range vanishes, consider the two G-families

$$\mathfrak{E}_2 = \{K \leq G \mid (K)_G \leq (H)_G \text{ and } (K\Pi/\Pi)_{\mathcal{G}} \leq (\mathcal{P})_{\mathcal{G}}\}$$

and

$$\mathfrak{E}_1 = \{K \leq G \mid (K)_G \leq (H)_G \text{ and } (K\Pi/\Pi)_{\mathcal{G}} < (\mathcal{P})_{\mathcal{G}}\}.$$

A check of fixed-point sets reveals that the two canonical maps

$$(E\mathfrak{E}_2)_+ \wedge E(\mathfrak{F}', \mathfrak{F}) \wedge Y \wedge G/H_+ \longrightarrow E(\mathfrak{F}', \mathfrak{F}) \wedge Y \wedge G/H_+$$

and

$$(E\mathfrak{E}_2)_+ \wedge E(\mathfrak{F}', \mathfrak{F}) \wedge Y \wedge G/H_+ \xrightarrow{\mu\wedge 1\wedge 1\wedge 1} E(\mathfrak{E}_2, \mathfrak{E}_1) \wedge E(\mathfrak{F}', \mathfrak{F}) \wedge Y \wedge G/H_+$$

are G-equivalences. Thus, the range of $\widehat{\tau}(E(\mathfrak{F}', \mathfrak{F}) \wedge Y, H)$ is isomorphic to

$$[S^m, E(\mathfrak{E}_2, \mathfrak{E}_1) \wedge E(\mathfrak{F}', \mathfrak{F}) \wedge Y \wedge i_* \Sigma^\infty_{U'} G/H_+]^U_G.$$

Theorem 6.1 will imply that this morphism set vanishes if we can show that, for every K in $\mathfrak{E}_2 - \mathfrak{E}_1$, the orbit G/K does not embed in U as a G-space. A subgroup K in $\mathfrak{E}_2 - \mathfrak{E}_1$ is subconjugate to H, and it suffices to consider the case in which $K \leq H$. Since $K \in \mathfrak{E}_2 - \mathfrak{E}_1$, $(K\Pi/\Pi)_{\mathcal{G}} = (\mathcal{P})_{\mathcal{G}}$. But then, if $\mathcal{G}/\mathcal{P}$ doesn't embed in U' as a $\mathcal{G}$-space, $G/K\Pi$ cannot embed in U' as a G-space. Since $H \in \mathfrak{A}_G(\Pi; U, U')$ and $K \leq H$, this implies that G/K does not embed in U as a G-space.

We now need to consider only the case in which $\mathcal{G}/\mathcal{P}$ embeds as a $\mathcal{G}$-space in U'. Using axiom **(A3)** and the equivariant homeomorphisms

$$\begin{aligned} E(\mathfrak{F}', \mathfrak{F}) \wedge Y \wedge (\Sigma^A G/H_+)/\Pi &\cong (\Sigma^A G/H_+ \wedge E(\mathfrak{F}', \mathfrak{F}) \wedge Y)/\Pi \\ &\cong (\Sigma^A G \ltimes_H (E(\mathfrak{F}', \mathfrak{F}) \wedge Y))/\Pi \end{aligned}$$

and

$$E(\mathfrak{F}', \mathfrak{F}) \wedge Y \wedge G/H_+ \cong G \ltimes_H (E(\mathfrak{F}', \mathfrak{F}) \wedge Y),$$

the map $\widehat{\tau}(E(\mathfrak{F}', \mathfrak{F}) \wedge Y, H)$ can be identified with the composite

$$\begin{aligned}
&\left[S^m, (\Sigma^A \Sigma^\infty_{U'} G \ltimes_H (E(\mathfrak{F}', \mathfrak{F}) \wedge Y))/\Pi\right]^{U'}_{\mathcal{G}} \\
&\quad \cong \quad \left[S^m, (\Sigma^A \Sigma^\infty_{U'} G \ltimes_H (E(\mathfrak{F}', \mathfrak{F}) \wedge Y))/\Pi\right]^{U'}_{G} \\
&\quad \xrightarrow{i_*} \quad \left[S^m, i_*((\Sigma^A \Sigma^\infty_{U'} G \ltimes_H (E(\mathfrak{F}', \mathfrak{F}) \wedge Y))/\Pi)\right]^{U}_{G} \\
&\quad \xrightarrow{\tau_*} \quad [S^m, i_* \Sigma^\infty_{U'} G \ltimes_H (E(\mathfrak{F}', \mathfrak{F}) \wedge Y)]^U_G,
\end{aligned}$$

where τ is the Adams transfer associated to the spectrum $\Sigma^\infty_{U'} G \ltimes_H (E(\mathfrak{F}', \mathfrak{F}) \wedge Y)$. Consider the two H-families

$$\mathfrak{E}_4 = \{K \leq H \mid (K\Pi/\Pi)_{\mathcal{G}} \leq (\mathcal{P})_{\mathcal{G}}\}$$

and

$$\mathfrak{E}_3 = \{K \leq H \mid (K\Pi/\Pi)_{\mathcal{G}} < (\mathcal{P})_{\mathcal{G}}\}.$$

It is easy to show that $E(\mathfrak{F}', \mathfrak{F})$, considered as an H-space, is H-equivalent to $E(\mathfrak{E}_4, \mathfrak{E}_3)$. Thus, it suffices to show that the composite

$$\begin{aligned}
&\left[S^m, (\Sigma^A \Sigma^\infty_{U'} G \ltimes_H (E(\mathfrak{E}_4, \mathfrak{E}_3) \wedge Y))/\Pi\right]^{U'}_{\mathcal{G}} \\
&\quad \cong \quad \left[S^m, (\Sigma^A \Sigma^\infty_{U'} G \ltimes_H (E(\mathfrak{E}_4, \mathfrak{E}_3) \wedge Y))/\Pi\right]^{U'}_{G} \\
&\quad \xrightarrow{i_*} \quad \left[S^m, i_*((\Sigma^A \Sigma^\infty_{U'} G \ltimes_H (E(\mathfrak{E}_4, \mathfrak{E}_3) \wedge Y))/\Pi)\right]^{U}_{G} \\
&\quad \xrightarrow{\tau_*} \quad [S^m, i_* \Sigma^\infty_{U'} G \ltimes_H (E(\mathfrak{E}_4, \mathfrak{E}_3) \wedge Y)]^U_G
\end{aligned}$$

is an isomorphism. This we prove by induction over the H-cells of the H-CW-complex $E(\mathfrak{E}_4, \mathfrak{E}_3) \wedge Y$. All of the H-cells of this space are constructed from spheres of the form $H/K_+ \wedge S^n$, where $K \in \mathfrak{E}_4$. Thus, we have reduced the problem to showing that the map $\widehat{\tau}(S^0, K)$ is an isomorphism for $K \in \mathfrak{E}_4$. If K is a proper subgroup of H, then this map is an isomorphism by our induction over the subgroups of G in $\mathfrak{A}_G(\Pi; U, U')$.

The proof has now been reduced to showing that $\widehat{\tau}(S^0, H)$ is an isomorphism when $H \in \mathfrak{E}_4$. In this case, $\mathcal{P} = H\Pi/\Pi$ and $\mathcal{G}/\mathcal{P} \cong G/H\Pi$ embeds as a G-space in U'. Since $H \in \mathfrak{A}_G(\Pi; U, U')$, G/H must embed as a G-space in U. From this point on, the argument given in [**24**] for Theorem II.7.1 can be used with only minor adjustments to compensate for the replacement of U^Π by U' and the different treatment of the suspension by A.

To show that $\widehat{\tau}(S^0, H)$ is an isomorphism, it, of course, suffices to show that the adjoint

$$\tilde{\tau} : (\Sigma^A \Sigma^\infty_{U'} G/H_+)/\Pi \longrightarrow (i^* i_* \Sigma^\infty_{U'} G/H_+)^\Pi$$

of the Adams transfer

$$\tau : i_*((\Sigma^A \Sigma^\infty_{U'} G/H_+)/\Pi) \longrightarrow i_* \Sigma^\infty_{U'} G/H_+$$

is a weak $\mathcal{G}$-equivalence. Axiom **(A4)** allows us to identify τ as the composite

$$\begin{aligned}
i_*((\Sigma^A \Sigma^\infty_{U'} G/H_+)/\Pi) &\cong G \ltimes_{H\Pi} \Sigma^\infty_U S^{\rho^* A} \\
&\cong G \ltimes_{H\Pi} (\Sigma^\infty_U S^{\rho^* A} \wedge S^0) \\
&\xrightarrow{1 \ltimes_{H\Pi}(1 \wedge t)} G \ltimes_{H\Pi} (\Sigma^\infty_U S^{\rho^* A} \wedge H\Pi \ltimes_H S^{-A}) \\
&\cong G \ltimes_{H\Pi} H\Pi \ltimes_H (\Sigma^\infty_U S^A \wedge S^{-A}) \\
&\cong G \ltimes_{H\Pi} H\Pi \ltimes_H S^0 \\
&\cong G \ltimes_H S^0 \cong \Sigma^\infty_U G/H_+ \cong i_* \Sigma^\infty_{U'} G/H_+.
\end{aligned}$$

Since $H\Pi/H$ embeds as an $H\Pi$-space in U and A is the H-representation derived from the tangent space at eH of $H\Pi/H$, the Wirthmüller map

$$\psi : H\Pi \ltimes_H S^0 \longrightarrow F_H[H\Pi, \Sigma^\infty_U S^A)$$

is a weak $H\Pi$-equivalence in the category of $H\Pi$-spectra indexed on U. Thus, it suffices to show that the adjoint of the composite

$$\begin{aligned}
G \ltimes_{H\Pi} \Sigma^\infty_U S^{\rho^* A} &\cong G \ltimes_{H\Pi} (\Sigma^\infty_U S^{\rho^* A} \wedge S^0) \\
&\xrightarrow{1 \ltimes_{H\Pi}(1 \wedge t)} G \ltimes_{H\Pi} (\Sigma^\infty_U S^{\rho^* A} \wedge H\Pi \ltimes_H S^{-A}) \\
&\cong G \ltimes_{H\Pi} H\Pi \ltimes_H (\Sigma^\infty_U S^A \wedge S^{-A}) \\
&\cong G \ltimes_{H\Pi} H\Pi \ltimes_H S^0 \\
&\xrightarrow{1 \ltimes_{H\Pi} \psi} G \ltimes_{H\Pi} F_H[H\Pi, \Sigma^\infty_U S^A)
\end{aligned}$$

is a weak $\mathcal{G}$-equivalence. Just as in the proof of Theorem II.7.1 of [**24**], this composite may be identified with the map

$$1 \ltimes_{H\Pi} \nu : G \ltimes_{H\Pi} \Sigma^\infty_U S^{\rho^* A} \longrightarrow G \ltimes_{H\Pi} F_H[H\Pi, \Sigma^\infty_U S^A),$$

where $\nu : \Sigma^\infty_U S^{\rho^* A} \longrightarrow F_H[H\Pi, \Sigma^\infty_U S^A)$ is the coaction map of $H\Pi$ on $\Sigma^\infty_U S^{\rho^* A}$.

Let L be the $H\Pi$-representation derived from the tangent space of $G/H\Pi$ at $eH\Pi$. Since $G/H\Pi$ embeds as a G-space in U', it also embeds in U. Thus, the map

$$\psi : G \ltimes_{H\Pi} F_H[H\Pi, \Sigma^\infty_U S^A) \longrightarrow F_{H\Pi}[G, \Sigma^L F_H[H\Pi, \Sigma^\infty_U S^A))$$

is a weak G-equivalence in the category of G-spectra indexed on U. It therefore suffices to show that the adjoint of the composite

$$\begin{aligned}
G \ltimes_{H\Pi} \Sigma^\infty_U S^{\rho^* A} &\xrightarrow{1 \ltimes_{H\Pi} \nu} G \ltimes_{H\Pi} F_H[H\Pi, \Sigma^\infty_U S^A) \\
&\xrightarrow{\psi} F_{H\Pi}[G, \Sigma^L F_H[H\Pi, \Sigma^\infty_U S^A)) \\
&\cong F_H[G, \Sigma^\infty_U S^{A+L})
\end{aligned}$$

is an isomorphism. As in the proof of Theorem II.7.1 of [**24**], this adjoint is the composite

$$\mathcal{G} \ltimes_{H\Pi/\Pi} \Sigma^\infty_{U'} S^A \xrightarrow{\psi} F_{H\Pi/\Pi}[\mathcal{G}, \Sigma^\infty_{U'} S^{A+L}) \xrightarrow{F_{H\Pi/\Pi}[1,\eta)} F_{H\Pi/\Pi}[\mathcal{G}, i^* i_* \Sigma^\infty_{U'} S^{A+L}),$$

where η is the unit of the (i_*, i^*)-adjunction. Since $\mathcal{G}/(H\Pi/\Pi) \cong G/H\Pi$ embeds in U' as a $\mathcal{G}$-space, the map ψ is a weak $\mathcal{G}$-equivalence. Also, since $H\Pi/\Pi \cong H$ and the universes U' and U are H-orbit equivalent by Lemma 12.3, the map η is a weak $H\Pi/\Pi$-equivalence by Theorem 1.2(b) of [**20**]. It follows that the composite is a weak $\mathcal{G}$-equivalence, as required. □

SECTION 17

The Adams transfer for incomplete universes

Here, an ad hoc construction of the dimension-shifting Adams transfer is given. The notation of Section II.7 of [**24**], as modified in Section 12, is used throughout this section. Recall the family $\mathfrak{F}_G(\Pi; U)$ of subgroups of G introduced in Definition 16.1 and the axioms **(A1)** through **(A4)** for the Adams transfer presented in Section 16. This entire section is devoted to proving the following existence theorem for the Adams transfer.

THEOREM 17.1. *Let Π be a normal subgroup of a compact Lie group G, A be the adjoint representation of G derived from Π, U be a G-universe into which Π/e embeds as a Π-space, U' be a Π-trivial G-universe, and $i : U' \longrightarrow U$ be a linear G-isometry. Then, for each $\mathfrak{F}_G(\Pi; U)$-spectrum D indexed on U', there is an Adams transfer*

$$\tau : i_*((\Sigma^A D)/\Pi) \longrightarrow i_* D.$$

Moreover, this transfer satisfies axioms **(A1)** *through* **(A4)**.

Several auxiliary groups must be introduced for the proof of this theorem. In particular, let $\mathfrak{G} = G \times_c \Pi$ be the semidirect product of G and Π formed using the conjugation action of G on Π. Let $\mathfrak{N}$ be the subgroup $e \times_c \Pi$ of $\mathfrak{G}$ so that $\mathfrak{N}$ is a normal subgroup of $\mathfrak{G}$. Identify G with the subgroup $G \times_c e$ of $\mathfrak{G}$. Define the homomorphisms $\epsilon : \mathfrak{G} \longrightarrow G$ and $\theta : \mathfrak{G} \longrightarrow G$ by

$$\epsilon(g, u) = g \quad \text{and} \quad \theta(g, u) = gu.$$

This notation deviates from that in section II.7 of [**24**] in that the groups denoted here by $\mathfrak{G}$, $\mathfrak{N}$, and Π are denoted there by Γ, Π, and N, respectively. Recall from [**24**] that there is an action of $\mathfrak{G}$ on Π given by

$$(g, u)v = guvg^{-1} \text{ for } g \in G \text{ and } u, v \in \Pi,$$

and that, with this action, Π is $\mathfrak{G}$-homeomorphic to $\mathfrak{G}/G$. In [**24**], this orbit was consistently referred to as Π (or, rather N in the notation of [**24**]) to compactify notation. Here, however, we denote it by $\mathfrak{G}/G$ because viewing it as a coset space clarifies the definitions of several of our maps. If X is a G-space, G-spectrum, or some other kind of G-object, then $\epsilon^* X$ and $\theta^* X$ denote the $\mathfrak{G}$-objects derived from X via the homomorphisms ϵ and θ. We always regard the G-universe U as a $\mathfrak{G}$-universe via ϵ. Note that $\mathfrak{N}$ acts trivially on $\epsilon^* U$ so that we may form $\mathfrak{N}$-orbit spectra indexed on $\epsilon^* U$. Since Π acts trivially on U', $\epsilon^* U'$ and $\theta^* U'$ are the same $\mathfrak{G}$-universe, and the inclusion $i : \theta^* U' \longrightarrow \epsilon^* U$ is a $\mathfrak{G}$-map. Thus, if D is a G-spectrum indexed on U', then $i_* \theta^* D$ is a well-defined $\mathfrak{G}$-spectrum indexed on the $\mathfrak{G}$-universe $\epsilon^* U$. The importance of $i_* \theta^* D$ comes from the isomorphisms

$$(i_* \theta^* D \wedge \mathfrak{G}/G_+)/\mathfrak{N} \cong i_* D \quad \text{and} \quad (i_* \theta^* D)/\mathfrak{N} \cong i_*(D/\Pi),$$

provided by Lemma II.7.4 of [**24**]. These maps play a key role in the construction of the Adams transfer.

In [**24**], an essential step in constructing the Adams transfer for a complete G-universe U is embedding $\epsilon^* U$ into a complete $\mathfrak{G}$-universe U'' (called, unfortunately, U' in [**24**]) via a $\mathfrak{G}$-isometry $k : U \longrightarrow U''$. Since $\mathfrak{G}/G$ embeds in U'' as a $\mathfrak{G}$-space, there is a pretransfer $t : S^0 \longrightarrow \Sigma^{-A}\Sigma^\infty_{U''}\mathfrak{G}/G_+$ in the category of $\mathfrak{G}$-spectra indexed on U''. If D is a Π-free G-spectrum indexed on U', then the induced map

$$1 \wedge t : k_* i_* \theta^* D \wedge S^0 \longrightarrow k_* i_* \theta^* D \wedge \Sigma^{-A}\Sigma^\infty_{U''}\mathfrak{G}/G_+$$

can be pulled back to a $\mathfrak{G}$-map

$$\widehat{\tau} : i_* \theta^* D \longrightarrow \Sigma^{-A}(i_* \theta^* D \wedge \mathfrak{G}/G_+)$$

between $\mathfrak{G}$-spectra indexed on U. In [**24**], the transfer $\tau : i_*(D/\Pi) \longrightarrow \Sigma^{-A} i_* D$ is obtained from $\widehat{\tau}$ by passing to orbits over $\mathfrak{N}$ and invoking the isomorphisms recalled above from Lemma II.7.4 of [**24**].

If U is an incomplete G-universe, then two difficulties arise in this approach to forming the Adams transfer. The most obvious is that A need not be a subrepresentation of U so that desuspension by A may not be defined in the category of $\mathfrak{G}$-spectra (or G-spectra) indexed on U. As noted in Section 12, this difficulty is easily avoided by replacing the map $\widehat{\tau}$ of [**24**] by a map of the form

$$\widehat{\tau} : i_* \theta^* \Sigma^A D \longrightarrow i_* \theta^* D \wedge \mathfrak{G}/G_+$$

from which one derives an Adams transfer of the form

$$\tau : i_*((\Sigma^A D)/\Pi) \longrightarrow i_* D$$

rather than of the form

$$\tau : i_*(D/\Pi) \longrightarrow \Sigma^{-A} i_* D$$

described in [**24**].

The more serious difficulty is that there does not seem to be a general method for forming a $\mathfrak{G}$-universe U'' that is both complete enough to admit an embedding of $\mathfrak{G}/G$ as a $\mathfrak{G}$-space and, at the same time, incomplete enough to allow the map $1 \wedge t$ (or, rather, $\Sigma^A(1 \wedge t)$ since we wish to eliminate the desuspension problem) to be pulled back to obtain $\widehat{\tau}$. In general, we resolve this difficulty by introducing an entirely new approach to forming the map $\widehat{\tau}$. However, before introducing this alternative method, we record here the one situation in which the method of [**24**] suffices for forming the Adams transfer for an incomplete G-universe U.

CONSTRUCTION 17.2. Assume that $\Pi = G$, so that U' is a trivial G-universe and U is a G-universe into which the free orbit G/e embeds as a G-space. Let V be a finite-dimensional subrepresentation of U such that G/e embeds as a G-space in V and such that V^G is the one-dimensional trivial G-representation $\mathbb{R}$. Let U'' be the vector space $\hom(\theta^* V, \epsilon^* U)$ of all linear transformations from V to U, regarded as a $\mathfrak{G}$-space via the usual conjugation action on a function space. Topologize U'' with the usual colimit topology derived from its finite-dimensional subspaces. Clearly, U'' is a $\mathfrak{G}$-universe. The canonical projection of V onto its fixed-point subspace $\mathbb{R}$ induces a linear injection

$$k : \epsilon^* U \cong \hom(\mathbb{R}, \epsilon^* U) \longrightarrow U''$$

which is $\mathfrak{G}$-equivariant, and can be made into an isometry by the choice of an appropriate inner product on U''. Let $\chi : V \longrightarrow U$ be the inclusion of V into U.

The $\mathfrak{G}$-isotropy subgroup of χ, regarded as an element of U'', is G, so that $\mathfrak{G}/G$ embeds in U'' as a $\mathfrak{G}$-space. Thus, in the category of $\mathfrak{G}$-spectra indexed on U'', there is a pretransfer

$$t : \Sigma^\infty_{U''} S^A \longrightarrow \Sigma^\infty_{U''} \mathfrak{G}/G_+.$$

Since Π is G, $\mathfrak{F}_G(\Pi; U)$ is just $\mathfrak{F}_G(G)$ and $\mathfrak{F}_G(\Pi; U)$-spectra are just G-free G-spectra. If D is a G-free G-spectrum indexed on U', then Theorem 9.3 can be used to pull the $\mathfrak{G}$-map

$$1 \wedge t : k_* i_* \theta^* D \wedge \Sigma^\infty_{U''} S^A \longrightarrow k_* i_* \theta^* D \wedge \Sigma^\infty_{U''} \mathfrak{G}/G_+$$

between $\mathfrak{G}$-spectra indexed on U'' back to a $\mathfrak{G}$-map

$$\widehat{\tau} : i_* \theta^* \Sigma^A D \longrightarrow i_* \theta^* D \wedge \mathfrak{G}/G_+$$

between $\mathfrak{G}$-spectra indexed on U. The isomorphisms of Lemma II.7.4 of [**24**] may then be applied just as they were in [**24**] to obtain the Adams transfer

$$\tau : i_*((\Sigma^A D)/G) \longrightarrow i_* D$$

Since this transfer is formed using the general approach to constructing transfers presented in [**24**], the general results in [**24**] concerning the properties of transfers suffice to show that τ satisfies axioms **(A1)** through **(A4)**.

One might hope to generalize this process to obtain an Adams transfer for any normal subgroup Π of G simply by taking V to be a suitable G-representation into which Π/e embeds as a Π-space. Unfortunately, in this more general context, it seems very hard to decide which $\mathfrak{G}$-orbits embed in $U'' = \hom(\theta^* V, \epsilon^* U)$. It therefore seems impossible to ensure that Theorem 9.3 can be applied to pull back $1 \wedge t$ to obtain $\widehat{\tau}$. The key to our alternative procedure for obtaining the map $\widehat{\tau}$ is the observation that, if D is an $\mathfrak{F}_G(\Pi; U)$-spectrum, then the projection $D \wedge E\mathfrak{F}_G(\Pi; U)_+ \longrightarrow D$ is a G-equivalence. Thus, if we can form the $\mathfrak{G}$-map

$$\widehat{\tau} : i_* \theta^* \Sigma^A \Sigma^\infty_{U'} E\mathfrak{F}_G(\Pi; U)_+ \longrightarrow i_* \theta^* \Sigma^\infty_{U'} E\mathfrak{F}_G(\Pi; U)_+ \wedge \mathfrak{G}/G_+$$

for the single $\mathfrak{F}_G(\Pi; U)$-spectrum $\Sigma^\infty_{U'} E\mathfrak{F}_G(\Pi; U)_+$ indexed on U', then we can obtain the $\mathfrak{G}$-map $\widehat{\tau}$ for every other $\mathfrak{F}_G(\Pi; U)$-spectrum D indexed on U' as the composite

$$\begin{aligned} i_* \theta^* \Sigma^A D \quad &\simeq \quad i_* \theta^* D \wedge i_* \theta^* \Sigma^A \Sigma^\infty_{U'} E\mathfrak{F}_G(\Pi; U)_+ \\ &\xrightarrow{1 \wedge \widehat{\tau}} \quad i_* \theta^* D \wedge i_* \theta^* \Sigma^\infty_{U'} E\mathfrak{F}_G(\Pi; U)_+ \wedge \mathfrak{G}/G_+ \\ &\simeq \quad i_* \theta^* D \wedge \mathfrak{G}/G_+. \end{aligned}$$

Once we have the $\mathfrak{G}$-map $\widehat{\tau}$ for every $\mathfrak{F}_G(\Pi; U)$-spectrum D indexed on U', the Adams transfer

$$\tau : i_*((\Sigma^A D)/\Pi) \longrightarrow i_* D$$

can be formed as the composite

$$i_*((\Sigma^A D)/\Pi) \cong (i_* \theta^* \Sigma^A D)/\mathfrak{N} \xrightarrow{\widehat{\tau}/\mathfrak{N}} (i_* \theta^* D \wedge \mathfrak{G}/G_+)/\mathfrak{N} \cong i_* D,$$

where the isomorphisms are those given by Lemma II.7.4 of [**24**].

Defined in this way, the Adams transfer obviously satisfies axioms **(A1)** and **(A3)** of Section 16. Thus, to complete the proof of Theorem 17.1, we need only construct the map $\widehat{\tau}$ for the spectrum $\Sigma^\infty E\mathfrak{F}_G(\Pi; U)_+$ and verify that the resulting Adams transfer satisfies axioms **(A2)** and **(A4)**. To see what is required to verify axiom **(A2)**, assume that P is a subgroup of G such that $\Pi \leq P$, and let $\mathfrak{P}$ be the subgroup $P \times_c \Pi$ of $\mathfrak{G}$. Identify P with the subgroup $P \times_c e$ of $\mathfrak{P}$. The inclusion of $\mathfrak{P}$ into $\mathfrak{G}$ induces a $\mathfrak{P}$-homeomorphism $\mathfrak{P}/P \longrightarrow \mathfrak{G}/G$. The homomorphisms $\epsilon : \mathfrak{G} \longrightarrow G$ and $\theta : \mathfrak{G} \longrightarrow G$, restricted to $\mathfrak{P}$, provide maps into P for which we continue to use the names ϵ and θ. The space $E\mathfrak{F}_G(\Pi; U)$, regarded as a P-space, is P-equivalent to $E\mathfrak{F}_P(\Pi; U)$. Thus, to verify axiom **(A2)**, it suffices to show that, under these identifications of $\mathfrak{G}/G$ with $\mathfrak{P}/P$ and of $E\mathfrak{F}_G(\Pi; U)$ with $E\mathfrak{F}_P(\Pi; U)$, the $\mathfrak{G}$-map

$$\widehat{\tau} : i_*\theta^*\Sigma^A\Sigma^\infty_{U'}E\mathfrak{F}_G(\Pi; U)_+ \longrightarrow i_*\theta^*\Sigma^\infty_{U'}E\mathfrak{F}_G(\Pi; U)_+ \wedge \mathfrak{G}/G_+,$$

regarded as a $\mathfrak{P}$-map, is just the map

$$\widehat{\tau} : i_*\theta^*\Sigma^A\Sigma^\infty_{U'}E\mathfrak{F}_P(\Pi; U)_+ \longrightarrow i_*\theta^*\Sigma^\infty_{U'}E\mathfrak{F}_P(\Pi; U)_+ \wedge \mathfrak{P}/P_+$$

from which the Adams transfer for $\mathfrak{F}_P(\Pi; U)$-spectra is constructed.

The map $\widehat{\tau}$ for the spectrum $\Sigma^\infty_{U'}E\mathfrak{F}_G(\Pi; U)_+$ is constructed from a space-level $\mathfrak{G}$-map

$$\delta_G : \theta^*E\mathfrak{F}_G(\Pi; U)_+ \longrightarrow \Omega^{\theta^*A}\Omega^\infty_U\Sigma^\infty_U\mathfrak{G}/G_+.$$

There is a spectrum-level $\mathfrak{G}$-map

$$\tilde{\delta}_G : i_*\theta^*\Sigma^A\Sigma^\infty_{U'}E\mathfrak{F}_G(\Pi; U)_+ \longrightarrow \Sigma^\infty_U\mathfrak{G}/G_+$$

adjoint to δ_G, and $\widehat{\tau}$ is the composite

$$\begin{aligned} i_*\theta^*\Sigma^A\Sigma^\infty_{U'}E\mathfrak{F}_G(\Pi; U)_+ &\longrightarrow i_*\theta^*\Sigma^\infty_{U'}E\mathfrak{F}_G(\Pi; U)_+ \wedge i_*\theta^*\Sigma^A\Sigma^\infty_{U'}E\mathfrak{F}_G(\Pi; U)_+ \\ &\xrightarrow{1\wedge\tilde{\delta}_G} i_*\theta^*\Sigma^\infty_{U'}E\mathfrak{F}_G(\Pi; U)_+ \wedge \Sigma^\infty_U\mathfrak{G}/G_+ \\ &\cong i_*\theta^*\Sigma^\infty_{U'}E\mathfrak{F}_G(\Pi; U)_+ \wedge \mathfrak{G}/G_+ \end{aligned}$$

in which the first map is derived from the diagonal map for the space $E\mathfrak{F}_G(\Pi; U)$. This reduction of the construction of the Adams transfer to the construction of the $\mathfrak{G}$-map δ_G also reduces the verification of axiom **(A2)** to showing that δ_G, regarded as a $\mathfrak{P}$-map, can be identified with the $\mathfrak{P}$-map

$$\delta_P : \theta^*E\mathfrak{F}_P(\Pi; U)_+ \longrightarrow \Omega^{\theta^*A}\Omega^\infty_U\Sigma^\infty_U\mathfrak{P}/P_+$$

from which the Adams transfer for $P \leq G$ is constructed.

We form the $\mathfrak{G}$-map δ_G by employing a simplified version of methods used in [**26**] to produce approximation maps for loop spaces. The first step in defining δ_G is forming a very simple model for the space $E\mathfrak{F}_G(\Pi; U)$ out of the universe U itself.

DEFINITION 17.3. If V is a G-representation into which Π/e embeds as a Π-space, then let $V[\Pi]$ be the set $\{z \in V \mid G_z \cap \Pi = \{e\}\}$, where G_z is the isotropy subgroup of the point z. Note that $V[\Pi]$ is a G-space. If $H \leq G$, then the set $V[\Pi]^H$ is nonempty if and only if $H \cap \Pi = \{e\}$ and the orbit $H\Pi/H$ embeds in V as an $H\Pi$-space. Moreover, if $V[\Pi]^H$ is nonempty, then

$$V[\Pi]^H = \{z \in V \mid (H\Pi)_z = H\}.$$

PROPOSITION 17.4. *If Π is a normal subgroup of a compact Lie group G and U is a G-universe into which Π/e embeds as a Π-space, then the G-space $U[\Pi]$ is G-equivalent to $\mathfrak{F}_G(\Pi; U)$.*

PROOF. The space $U[\Pi]$ is the colimit of the collection of spaces $V[\Pi]$ indexed on the finite-dimensional subrepresentations V of U. For such a V, $V[\Pi]$ clearly has the G-homotopy type of a G-CW-complex. Moreover, if $V \subset W$, then the inclusion of $V[\Pi]$ into $W[\Pi]$ is a G-cofibration. Thus, $U[\Pi]$ has the G-homotopy type of a G-CW-complex. If $H \notin \mathfrak{F}_G(\Pi; U)$, then $U[\Pi]^H = \emptyset$. If $H \in \mathfrak{F}_G(\Pi; U)$, then $U[\Pi]^H = \{z \in U \mid (H\Pi)_z = H\}$, and so $U[\Pi]^H$ is contractible by the lemma below. It follows that $U[\Pi]$ is an $\mathfrak{F}_G(\Pi; U)$-space and that the canonical map $U[\Pi] \longrightarrow \mathfrak{F}_G(\Pi; U)$ is a G-equivalence. □

LEMMA 17.5. *Let K be a compact Lie group, J be a subgroup of K, and U be a K-universe. If the space $U_J = \{z \in U \mid K_z = J\}$ is nonempty, then it is contractible.*

PROOF. The space U_J clearly has the homotopy type of a CW complex, so it suffices to show that, for every $n \geq 0$, any map $f : S^n \longrightarrow U_J$ extends to a map $\tilde{f} : D^{n+1} \longrightarrow U_J$. The map f, regarded as a map into U, must factor through some finite-dimensional subrepresentation V of U. Since U is a universe, there is another subrepresentation V' of U such that V and V' are isomorphic and $V \perp V'$. Select $z \in V'_J$. Regard D^{n+1} as $S^n \times I/S^n \times \{0\}$, and define $\tilde{f} : D^{n+1} \longrightarrow V \oplus V'$ by $\tilde{f}(s,t) = (tf(s), (1-t)z)$, for $s \in S^n$, $t \in I$. Since $f(s) \in V_J$ for all $s \in S^n$ and $z \in V'_J$, $\tilde{f}(D^{n+1}) \subset (V \oplus V')_J \subset U_J$. □

If V is a finite-dimensional subrepresentation of U, then there is a reasonably obvious $\mathfrak{G}$-map

$$\delta_G(V) : \theta^* V[\Pi]_+ \longrightarrow \Omega^{\theta^* A} \Omega^{\epsilon^* V} \Sigma^{\epsilon^* V} \mathfrak{G}/G_+$$

which sends each $z \in V[\Pi]$ to a map $S^{\epsilon^* V + \theta^* A} \longrightarrow \Sigma^{\epsilon^* V} \mathfrak{G}/G_+$ derived from the map collapsing out the complement in S^V of a tubular neighborhood of the embedding of $G_z\Pi/G_z$ into V that sends eG_z to z. Since $U[\Pi]$ is the colimit of the spaces $V[\Pi]$, where V runs over the finite-dimensional subrepresentations of U, and $\Omega^{\theta^* A} \Omega^\infty_U \Sigma^\infty_U \mathfrak{G}/G_+$ is the colimit of the corresponding collection of spaces $\Omega^{\theta^* A} \Omega^{\epsilon^* V} \Sigma^{\epsilon^* V} \mathfrak{G}/G_+$, we would like to form the $\mathfrak{G}$-map

$$\delta_G : \theta^* E\mathfrak{F}_G(\Pi; U)_+ \simeq \theta^* U[\Pi]_+ \longrightarrow \Omega^{\theta^* A} \Omega^\infty_U \Sigma^\infty_U \mathfrak{G}/G_+$$

from the maps $\delta_G(V)$ by passage to colimits. Unfortunately, it does not seem possible to construct the maps $\delta_G(V)$ in such a way that they commute with the structure maps in these two colimit diagrams. Thus, δ_G must be constructed in a more indirect fashion.

For this indirect approach, we first form a collection of G-spaces $C_G(V)$, indexed on the finite-dimensional subrepresentations V of U, together with a collection of inclusions of G-spaces $\eta_G(V,W) : C_G(V) \longrightarrow C_G(W)$ indexed on the pairs of subrepresentations V, W of U such that $V \subset W$. Then we introduce a collection of G-equivalences

$$\alpha_G(V) : C_G(V) \longrightarrow V[\Pi],$$

and a collection of $\mathfrak{G}$-maps

$$\beta_G(V) : \theta^* C_G(V)_+ \longrightarrow \Omega^{\theta^* A} \Omega^{\epsilon^* V} \Sigma^{\epsilon^* V} \mathfrak{G}/G_+$$

such that the diagrams

$$\begin{array}{ccc} C_G(V) & \xrightarrow{\alpha_G(V)} & V[\Pi] \\ {\scriptstyle \eta_G(V,W)}\downarrow & & \cap \\ C_G(W) & \xrightarrow{\alpha_G(W)} & W[\Pi] \end{array}$$

and

$$\begin{array}{ccc} \theta^* C_G(V)_+ & \xrightarrow{\beta_G(V)} & \Omega^{\theta^* A}\Omega^{\epsilon^* V}\Sigma^{\epsilon^* V}\mathfrak{G}/G_+ \\ {\scriptstyle \theta^*\eta_G(V,W)}\downarrow & & \downarrow{\scriptstyle \overline{\eta}} \\ \theta^* C_G(W)_+ & \xrightarrow{\beta_G(W)} & \Omega^{\theta^* A}\Omega^{\epsilon^* W}\Sigma^{\epsilon^* W}\mathfrak{G}/G_+ \end{array}$$

commute for each pair V, W with $V \subset W$. Here, $\overline{\eta}$ is the usual inclusion. Let $C_G(U)$ be the colimit of the diagram formed from the spaces $C_G(V)$ and the inclusions $\eta_G(V, W)$. The maps $\alpha_G(V)$ and $\beta_G(V)$ induce a G-map

$$\alpha_G : C_G(U) \longrightarrow U[\Pi]$$

and a $\mathfrak{G}$-map

$$\beta_G : \theta^* C_G(U)_+ \longrightarrow \Omega^{\theta^* A}\Omega^\infty_U \Sigma^\infty_U \mathfrak{G}/G_+.$$

The map α_G is a weak G-equivalence because the maps $\alpha_G(V)$ are G-equivalences, and so the Whitehead theorem provides a G-map $\gamma_G : U[\Pi] \longrightarrow C_G(U)$ right inverse to α_G. Our $\mathfrak{G}$-map δ_G is the composite

$$\theta^* E\mathfrak{F}_G(\Pi; U)_+ \simeq \theta^* U[\Pi]_+ \xrightarrow{\theta^*\gamma_G} \theta^* C_G(U)_+ \xrightarrow{\beta_G} \Omega^{\theta^* A}\Omega^\infty_U \Sigma^\infty_U \mathfrak{G}/G_+.$$

If P is a subgroup of G such that $\Pi \le P$, then, for each finite-dimensional sub-G-representation V of U, there is a P-equivalence $\iota^P_G(V) : C_G(V) \longrightarrow C_P(V)$ such that the diagrams

$$\begin{array}{ccc} C_G(V) & & \\ {\scriptstyle \iota^P_G(V)}\downarrow & \searrow^{\alpha_G(V)} & V[\Pi] \\ C_P(V) & \nearrow_{\alpha_P(V)} & \end{array} \qquad \begin{array}{ccc} \theta^* C_G(V)_+ & \xrightarrow{\beta_G(V)} & \Omega^{\theta^* A}\Omega^{\epsilon^* V}\Sigma^{\epsilon^* V}\mathfrak{G}/G_+ \\ {\scriptstyle \theta^*\iota^P_G(V)}\downarrow & & \uparrow{\scriptstyle \cong} \\ \theta^* C_P(V)_+ & \xrightarrow{\beta_P(V)} & \Omega^{\theta^* A}\Omega^{\epsilon^* V}\Sigma^{\epsilon^* V}\mathfrak{p}/P_+ \end{array}$$

and

$$\begin{array}{ccc} C_G(V) & \xrightarrow{\iota^P_G(V)} & C_P(V) \\ {\scriptstyle \eta_G(V,W)}\downarrow & & \downarrow{\scriptstyle \eta_P(V,W)} \\ C_G(W) & \xrightarrow{\iota^P_G(W)} & C_P(W) \end{array}$$

commute. It follows immediately that the $\mathfrak{G}$-map δ_G, regarded as a $\mathfrak{p}$-map, can be identified with the map δ_P from which the Adams transfer for the subgroup P is formed. Thus, axiom **(A2)** holds for our Adams transfer.

The space $C_G(V)$ is a space of embeddings resembling the spaces of embeddings that form the little cubes and little disks operads used in loop space theory. Thus, the first step in constructing $C_G(V)$ is introducing the type of embeddings which it contains. If $b_1, b_2, \dots, b_n$ are nonnegative real numbers, then we call the graph of the equation $\sum_{i=1}^n b_i^2 x_i^2 = 1$ in $\mathbb{R}^n$ an ellipsoid. Note that we allow some of the b_i to be zero so that our ellipsoids may extend infinitely along some axes in $\mathbb{R}^n$. We also refer to the image of this graph under any matrix B in $SO(n)$ as an ellipsoid. Define the map

$$j(b_1, b_2, \dots, b_n) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

by

$$j(b_1, b_2, \dots, b_n)(x_1, x_2, \dots, x_n) = \frac{1}{1 + \sqrt{\sum b_i^2 x_i^2}}(x_1, x_2, \dots, x_n).$$

It is easy to check that $j(b_1, b_2, \dots, b_n)$ is an embedding of $\mathbb{R}^n$ into the interior of the ellipsoid $\sum_{i=1}^n b_i^2 x_i^2 = 1$. Moreover, $j(b_1, b_2, \dots, b_n)$ acts by moving each point of $\mathbb{R}^n$ toward the origin along the line joining that point to the origin. Note also that, if $b_i = 0$, then $j(b_1, b_2, \dots, b_n)$ fixes all the points on the x_i-axis. In particular, the embedding $j(b_1, b_2, \dots, b_n, 0) : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ is just $j(b_1, b_2, \dots, b_n) \times 1$. This property of these embeddings makes possible the passage to colimits that is an essential part of the definition of δ_G.

If $B \in SO(n)$, then let

$$j(b_1, b_2, \dots, b_n; B) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

be the composite $Bj(b_1, b_2, \dots, b_n)B^{-1}$. This map embeds $\mathbb{R}^n$ into the interior of the ellipsoid obtained by rotating the graph of $\sum_{i=1}^n b_i^2 x_i^2 = 1$ by B. Note that $j(b_1, b_2, \dots, b_n; B)$ also acts by moving each point of $\mathbb{R}^n$ toward the origin along the line joining that point to the origin. Hereafter, we refer to embeddings of the form $j(b_1, b_2, \dots, b_n; B)$ as ellipsoidal embeddings. Let $I(\mathbb{R}^n)$ be the set of all ellipsoidal embeddings of $\mathbb{R}^n$ into $\mathbb{R}^n$. Assigning the embedding $j(b_1, b_2, \dots, b_n; B)$ to the element $(b_1, b_2, \dots, b_n, B)$ of $[0, \infty)^n \times SO(n)$ gives a surjective map from $[0, \infty)^n \times SO(n)$ to $I(\mathbb{R}^n)$. We give $I(\mathbb{R}^n)$ the quotient topology derived from this map. It is fairly easy to see that this agrees with the subspace topology that $I(\mathbb{R}^n)$ inherits from the space of all continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. If V is a finite dimensional inner product space of dimension n, then we identify V with $\mathbb{R}^n$ by choosing an orthonormal basis. This identification allows us to define both an ellipsoidal embedding of V into V and the space $I(V)$ of all such embeddings.

The ellipsoidal embeddings that are, essentially, the points of the space $C_G(V)$ have as their domains the fibres of a vector bundle over $V[\Pi]$. If $z \in V[\Pi]$, then there is an embedding of the $G_z\Pi$-space $G_z\Pi/G_z$ into V which sends the identity coset eG_z to z. Let $E_z(V)$ be the vector subspace of V orthogonal to this embedding at z. Let $E(V) = \bigcup_{z \in V[\Pi]} E_z(V)$ and define $\xi(V) : E(V) \longrightarrow V[\Pi]$ to be the map sending the entire vector space $E_z(V)$ to z. It is easy to see that $\xi(V)$ is a G-vector bundle over $V[\Pi]$. For each z in $V[\Pi]$, the inclusion of Π into $G_z\Pi/G_z$ is a diffeomorphism. Thus, the bundle $\xi(V)$ depends on Π and V, but not on the ambient group G containing Π. This observation is critical for the proof that axiom **(A2)** is satisfied. The diffeomorphism between Π and $G_z\Pi/G_z$ also implies that the tangent space of $G_z\Pi/G_z$ at eG_z is isomorphic, as a G_z-representation, to A. Let $\phi(V) : A \times E(V) \longrightarrow V \times V[\Pi]$ be the map which sends a pair (a, v) consisting of $a \in A$ and $v \in E_z(V)$, for $z \in V[\Pi]$, to the pair $(v + d_z a, z)$, where d_z is the differential of the embedding of $G_z\Pi/G_z$ into V sending eG_z to z. Clearly, $\phi(V)$ is a G-homeomorphism which identifies the bundle $A \oplus \xi(V)$ with the trivial bundle $V \times V[\Pi] \longrightarrow V[\Pi]$. Denote the restriction of $\phi(V)$ to the fibre over $z \in V[\Pi]$ by $\phi_z(V) : A \oplus E_z(V) \longrightarrow V$. Note that $\phi_z(V)$ is a G_z-equivariant linear isomorphism. We can now describe the space $C_G(V)$ and the map $\alpha_G(V)$ precisely.

Definition 17.6. (a) Let V be a G-representation into which the free orbit Π/e embeds as a Π-space. The space $C_G(V)$ is the subset of the set $(A \times E(V))^V$ of continuous maps from V to $A \times E(V)$ consisting of those maps λ satisfying the three conditions:

(i) There is an $z \in V[\Pi]$ and an ellipsoidal embedding $j \in I(E_z(V))$ such that the diagram

$$\begin{array}{ccc} A \oplus E_z(V) & \xrightarrow{\phi_z(V)} & V \\ {\scriptstyle 1\oplus j}\downarrow & \swarrow{\scriptstyle \lambda} & \\ A \oplus E_z(V) & & \end{array}$$

commutes.

(ii) The map λ is G_z-equivariant, where z is the point of $V[\Pi]$ specified in condition (i).

(iii) The subset $z + j(E_z(V))$ of V is a $G_z\Pi$-slice at z; that is, the G_z-map from $E_z(V)$ to V which sends $v \in E_z(V)$ to $z + j(v)$ extends to a $G_z\Pi$-homeomorphism from $G_z\Pi \times_{G_z} E_z(V)$ to an open neighborhood of $G_z\Pi z$.

Of course, the map $\alpha_G(V) : C_G(V) \longrightarrow V[\Pi]$ sends each $\lambda \in C_G(V)$ to the $z \in V[\Pi]$ associated with λ in condition (i). Observe that the set $C_G(V)$ is a G-invariant subset of $(A \times E(V))^V$ and that $\alpha_G(V)$ is a G-map. If $\lambda \in C_G(V)$ and $z = \alpha_G(V)(\lambda)$, then λ can be factored as the composite

$$V \xrightarrow{j'} V \xrightarrow{\phi_z^{-1}(V)} A \oplus E_z(V)$$

for some ellipsoidal embedding j' of V into V. The assignment of the pair (z, j') to the map $\lambda \in C_G(V)$ embeds the set $C_G(V)$ into $V[\Pi] \times I(V)$. Give $C_G(V)$ the subspace topology derived from this embedding; this actually agrees with the subspace topology that $C_G(V)$ inherits from $(A \times E(V))^V$. The map $\alpha_G(V)$ is continuous since it is just the restriction of the projection of $V[\Pi] \times I(V)$ onto $V[\Pi]$.

(b) In order to show that the map $\alpha_G(V)$ is a G-equivalence, we must construct its homotopy inverse $s_G(V) : V[\Pi] \longrightarrow C_G(V)$. For each $z \in V[\Pi]$, there is a real number $r_z > 0$ such that the disk of radius r_z in $E_z(V)$ provides a slice at z. There is an ellipsoidal embedding $j_z(r_z) : E_z(V) \longrightarrow E_z(V)$ which maps $E_z(V)$ onto the interior of this disk. The element $s_G(V)(z)$ of $C_G(V)$ should be the composite

$$V \xrightarrow{\phi_z^{-1}(V)} A \oplus E_z(V) \xrightarrow{1\oplus j_z(r_z)} A \oplus E_z(V).$$

The only difficulty with this definition of $s_G(V)$ is that the radii r_z must be selected so that the assignment of r_z to z gives a continuous G-map from $V[\Pi]$ to the space $(0, \infty)$, which carries a trivial G-action. It is easy to see that a radius that works at $z \in V[\Pi]$ also works at each point in the G-orbit of z. Moreover, at each point z, a radius can be selected which works for every point in a neighborhood of z. A partition of unity argument can then be used to construct an appropriate equivariant continuous function $r : V[\Pi] \longrightarrow (0, \infty)$.

Since our ellipsoidal embeddings simply move points inward along lines through the origin, it is easy to see that, for each $z \in V[\Pi]$, every λ in $\alpha_G(V)^{-1}(z)$ can be deformed continuously to the standard embedding $s_G(V)(z)$ in $\alpha_G(V)^{-1}(z)$. This deformation can be performed continuously in z. We have thus proven the following result:

LEMMA 17.7. *Let V be a G-representation into which the free orbit Π/e embeds as a Π-space. Then the maps*

$$s_G(V) : V[\Pi] \longrightarrow C_G(V) \quad \textit{and} \quad \alpha_G(V) : C_G(V) \longrightarrow V[\Pi]$$

display $V[\Pi]$ as a G-equivariant strong deformation retract of $C_G(V)$.

The maps

$$\iota_G^P(V) : C_G(V) \longrightarrow C_P(V),$$
$$\eta_G(V,W) : C_G(V) \longrightarrow C_G(W),$$

and

$$\beta_G(V) : \theta^* C_G(V)_+ \longrightarrow \Omega^{\theta^* A}\Omega^{\epsilon^* V}\Sigma^{\epsilon^* V}\mathfrak{G}/G_+$$

must still be defined, and it must be shown that the various diagrams relating these maps to each other and to $\alpha_G(V)$ commute. Also, we must prove that axiom **(A4)** is satisfied.

DEFINITION 17.8. (a) Assume that P is a subgroup of G such that $\Pi \leq P$, and let V be a G-representation into which the free orbit Π/e embeds as a Π-space. The spaces $C_G(V)$ and $C_P(V)$ are both subsets of $(A \times E(V))^V$. It is easy to see that $C_P(V)$ contains $C_G(V)$, but is generally strictly larger. Take the P-map $\iota_G^P(V) : C_G(V) \longrightarrow C_P(V)$ to be just the inclusion of $C_G(V)$ as a subset. Clearly, $\alpha_G(V) = \alpha_P(V) \circ \iota_G^P(V)$. It follows from Lemma 17.7 that $\iota_G^P(V)$ is a P-equivalence.

(b) If W is a G-representation which contains V and $W - V$ is the orthogonal complement of V in W, then $E_z(W) = E_z(V) \oplus (W - V)$ for each $z \in V[\Pi]$. Thus, the restriction of $\xi(W)$ to $V[\Pi] \subset W[\Pi]$ is just $\xi(V) \oplus (W - V)$. Under these identifications, the map $\phi_z(W) : A \oplus E_z(W) \longrightarrow W$ becomes the map

$$\phi_z(V) \oplus 1 : A \oplus E_z(V) \oplus (W - V) \longrightarrow V \oplus (W - V)$$

whenever $z \in V[\Pi]$. If $\lambda \in C_G(V)$, $\alpha_G(V)(\lambda) = z$, and $j : E_z(V) \longrightarrow E_z(V)$ is the ellipsoidal embedding associated to λ by condition (i) of Definition 17.6(a), then the composite

$$\tilde{j} : E_z(W) = E_z(V) \oplus (W - V) \xrightarrow{j \oplus 1} E_z(V) \oplus (W - V) = E_z(W)$$

is also an ellipsoidal embedding. Moreover, since $j(E_z(V))$ provides a $G_z\Pi$-slice at $z \in V$, $\tilde{j}(E_z(W))$ provides a $G_z\Pi$-slice at z considered as an element of W. Thus, the map

$$\tilde{\lambda} : W \xrightarrow{\phi_z^{-1}(W)} A \oplus E_z(W) \xrightarrow{1 \oplus \tilde{j}} A \oplus E_z(W)$$

is an element of $C_G(W)$. Let $\eta_G(V,W) : C_G(V) \longrightarrow C_G(W)$ be the map given by $\eta_G(V,W)(\lambda) = \tilde{\lambda}$. Clearly, $\eta_G(V,W)$ is a G-inclusion, $\alpha_G(V) = \alpha_G(W) \circ \eta_G(V,W)$, and $\eta_P(V,W) \circ \iota_G^P(V) = \iota_G^P(W) \circ \eta_G(V,W)$.

(c) Let λ be an element of $C_G(V)$, $\alpha_G(V)(\lambda) = z$, and $j : E_z(V) \longrightarrow E_z(V)$ be the ellipsoidal embedding associated to λ by condition (i) of Definition 17.6(a). To describe the map

$$\beta_G(V)(\lambda) \in \Omega^{\theta^* A}\Omega^{\epsilon^* V}\Sigma^{\epsilon^* V}\mathfrak{G}/G_+,$$

we must first introduce the collection of auxiliary maps from which $\beta_G(V)(\lambda)$ is formed. The sphere S^V contains an open neighborhood of the orbit $G_z\Pi z$ which is $G_z\Pi$-homeomorphic to $G_z\Pi \times_{G_z} j(E_z(V))$. Collapsing out the complement of this neighborhood gives a $G_z\Pi$-map

$$p(\lambda) : S^V \longrightarrow G_z\Pi \ltimes_{G_z} S^{j(E_z(V))},$$

where $S^{j(E_z(V))}$ denotes the one-point compactification of $j(E_z(V))$. Let $\mathfrak{G}_z = \theta^{-1}(G_z)$. The homomorphism ϵ induces an isomorphism between $\mathfrak{G}_z$ and $G_z\Pi$. This isomorphism maps $G_z \times_c e \leq \mathfrak{G}_z$ onto $G_z \leq G_z\Pi$. Hereafter, we identify G_z and $G_z \times_c e$. From these observations about ϵ and $\mathfrak{G}_z$, we obtain an $\mathfrak{G}_z$-isomorphism

$$\widehat{\epsilon} : \mathfrak{G}_z \ltimes_{G_z} S^{j(E_z(V))} \longrightarrow \epsilon^*\Big(G_z\Pi \ltimes_{G_z} S^{j(E_z(V))}\Big).$$

By regarding A as a $\mathfrak{G}$-representation via θ, we obtain an isomorphism

$$\zeta : \mathfrak{G}_z \ltimes_{G_z} \Big(S^A \wedge S^{j(E_z(V))}\Big) \longrightarrow S^A \wedge \mathfrak{G}_z \ltimes_{G_z} S^{j(E_z(V))}$$

which is the space-level analog of the isomorphism ζ of Lemma II.4.9 of [**24**]. Also, let

$$\zeta' : \mathfrak{G}_z \ltimes_{G_z} S^V \longrightarrow (\mathfrak{G}_z/G_z)_+ \wedge S^V$$

be the analogous isomorphism obtained by regarding V as a $\mathfrak{G}$-space via ϵ. The map $\lambda : V \longrightarrow A \oplus E_z(V)$ induces a G_z-homeomorphism

$$\widehat{\lambda} : S^V \longrightarrow S^A \wedge S^{j_z(E(V))}.$$

The inclusion of $\mathfrak{G}_z$ into $\mathfrak{G}$ induces a homeomorphism from $\mathfrak{G}_z/G_z$ to $\mathfrak{G}/G$ which we denote by ν.

The composite

$$\begin{aligned} S^A \wedge S^V &\xrightarrow{1\wedge p(\lambda)} S^A \wedge G_z\Pi \ltimes_{G_z} S^{j(E_z(V))} \\ &\xrightarrow{1\wedge\widehat{\epsilon}^{-1}} S^A \wedge \mathfrak{G}_z \ltimes_{G_z} S^{j(E_z(V))} \\ &\xrightarrow{\zeta^{-1}} \mathfrak{G}_z \ltimes_{G_z} \Big(S^A \wedge S^{j(E_z(V))}\Big) \\ &\xrightarrow{1\ltimes\widehat{\lambda}^{-1}} \mathfrak{G}_z \ltimes_{G_z} S^V \\ &\xrightarrow{\zeta'} (\mathfrak{G}_z/G_z)_+ \wedge S^V \\ &\xrightarrow{\nu\wedge 1} \mathfrak{G}/G_+ \wedge S^V \end{aligned}$$

may be regarded as an element of $\Omega^{\theta^* A}\Omega^{\epsilon^* V}\Sigma^{\epsilon^* V}\mathfrak{G}/G_+$, and we define $\beta_G(V)$ by letting $\beta_G(V)(\lambda)$ be this element. The continuity of $\beta_G(V)$ is most easily checked by looking at its adjoint

$$\tilde{\beta}_G(V) : \theta^* C_G(V)_+ \wedge S^{\theta^* A} \wedge S^{\epsilon^* V} \longrightarrow \mathfrak{G}/G_+ \wedge S^{\epsilon^* V}.$$

A somewhat messy diagram chase indicates that $\beta_G(V)$ is a $\mathfrak{G}$-map. In chasing this diagram, it is essential to remember that the action of $\mathfrak{G}$ on $\theta^* C_G(V)_+$ is derived from the action of $\mathfrak{G}$ on V via θ, whereas the action of $\mathfrak{G}$ on $\Omega^{\theta^*A}\Omega^{\epsilon^*V}\Sigma^{\epsilon^*V}\mathfrak{G}/G_+$ is derived from the action of $\mathfrak{G}$ on V via ϵ.

To verify that the diagram relating $\beta_G(V)$, $\beta_P(V)$, and $\iota_G^P(V)$ commutes, note that replacing G_z by P_z in the composite specifying $\beta_G(V)(\lambda)$ does not alter the composite at all because, at every stage of the composite in which $G_z\Pi$ or $\mathfrak{G}_z$ appears, the inclusion $P_z\Pi \subset G_z\Pi$ induces isomorphisms between the spaces appearing in the diagram and the spaces obtained by replacing $G_z\Pi$ by $P_z\Pi$ or $\mathfrak{G}_z$ by $\theta^{-1}(P_z)$. This completes the proof that our transfer satisfies axiom **(A2)**.

To see that $\overline{\eta}\circ\beta_G(V) = \beta_G(W)\circ\theta^*\eta_G(V,W)$, consider the adjoints of these two maps. Observe that both of these adjoints are just the composite

$$\begin{array}{rcl} \theta^*C_G(V)_+\wedge S^{\theta^*A}\wedge S^{\epsilon^*W} & \cong & \theta^*C_G(V)_+\wedge S^{\theta^*A}\wedge S^{\epsilon^*V}\wedge S^{\epsilon^*(W-V)} \\ & \xrightarrow{\tilde{\beta}_G(V)\wedge 1} & \mathfrak{G}/G_+\wedge S^{\epsilon^*V}\wedge S^{\epsilon^*(W-V)} \\ & \cong & \mathfrak{G}/G_+\wedge S^{\epsilon^*W}, \end{array}$$

where $\tilde{\beta}_G(V)$ is the adjoint of $\beta_G(V)$ that was used to check the continuity of $\beta_G(V)$.

The following result completes the last step in the proof of Theorem 17.1.

LEMMA 17.9. *The transfer derived from the map δ_G satisfies axiom* **(A4)**.

PROOF. Let $H\in\mathfrak{F}_G(\Pi;U)$. We must show that the Adams transfer

$$\tau : i_*((\Sigma^A\Sigma^\infty_{U'}G/H_+)/\Pi)\longrightarrow i_*\Sigma^\infty_{U'}G/H_+$$

for the $\mathfrak{F}_G(\Pi;U)$-spectrum $\Sigma^\infty_{U'}G/H_+$ is the composite specified by the axiom. The projection

$$G/H_+\wedge U[\Pi]_+\simeq G/H_+\wedge\mathfrak{F}_G(\Pi;U)_+\longrightarrow G/H_+$$

from which the transfer for $\Sigma^\infty_{U'}G/H_+$ is derived has an obvious homotopy inverse derived from the G-map $f : G/H\longrightarrow U[\Pi]$ whose existence is guaranteed by the universal properties of $U[\Pi]$. Since $C_G(U)^H$ is connected and contains points with isotropy subgroup H, we can select f so that the composite $\gamma_G\circ f : G/H\longrightarrow C_G(U)$ takes the identity coset eH to an element λ of $C_G(U)$ with isotropy subgroup H. Let $z=\alpha_G(\lambda)$. Note that z also has isotropy subgroup H. Since G/H is compact, there is a finite-dimensional subrepresentation V of U such that $f(G/H))\subset V[\Pi]$ and $\gamma_G(f(G/H))\subset C_G(V)$. Then λ is in $C_G(V)$, and z is in $V[\Pi]$.

Let

$$\widehat{\delta} : \theta^*G/H_+\wedge S^{\theta^*A}\wedge S^{\epsilon^*V}\longrightarrow\mathfrak{G}/G_+\wedge S^{\epsilon^*V}$$

be the adjoint of the composite

$$\theta^*G/H_+\longrightarrow\theta^*C_G(V)_+\xrightarrow{\beta_G(V)}\Omega^{\theta^*A}\Omega^{\epsilon^*V}\Sigma^{\epsilon^*V}\mathfrak{G}/G_+$$

in which the first map is the unique $\mathfrak{G}$-map sending eH to λ. The $\mathfrak{G}$-map

$$\widehat{\tau} : i_*\theta^*\Sigma^A\Sigma^\infty_{U'}G/H_+\longrightarrow i_*\theta^*\Sigma^\infty_{U'}G/H_+\wedge\mathfrak{G}/G_+$$

from which we construct the Adams transfer is just the stabilization of the composite

$$\begin{array}{rcl} \theta^*G/H_+\wedge S^{\theta^*A}\wedge S^{\epsilon^*V} & \xrightarrow{\Delta\wedge 1\wedge 1} & \theta^*G/H_+\wedge\theta^*G/H_+\wedge S^{\theta^*A}\wedge S^{\epsilon^*V} \\ & \xrightarrow{1\wedge\widehat{\delta}} & \theta^*G/H_+\wedge\mathfrak{G}/G_+\wedge S^{\epsilon^*V}, \end{array}$$

which we denote by $\overline{\tau}$. There are isomorphisms

$$(\theta^* G/H_+ \wedge \mathfrak{G}/G_+ \wedge S^{\epsilon^* V})/\mathfrak{N} \cong G/H_+ \wedge S^V$$

and

$$(\theta^* G/H_+ \wedge S^{\theta^* A} \wedge S^{\epsilon^* V})/\mathfrak{N} \cong ((\Sigma^A G/H_+)/\Pi) \wedge S^V$$

which are the space-level analogs of the isomorphisms of Lemma II.7.4 of [**24**]. Composing these isomorphisms with the map obtained from $\overline{\tau}$ by passage to $\mathfrak{N}$-orbits, we obtain the map

$$\tau_1 : ((\Sigma^A G/H_+)/\Pi) \wedge S^V \longrightarrow G/H_+ \wedge S^V$$

whose stabilization is the Adams transfer

$$\tau : i_*((\Sigma^A \Sigma^\infty_{U'} G/H_+)/\Pi) \longrightarrow i_* \Sigma^\infty_{U'} G/H_+.$$

The map which axiom **(A4)** asserts must be equal to the Adams transfer for $\Sigma^\infty_{U'} G/H_+$ is easily seen to be the stabilization of the composite

$$\begin{array}{rcl}
((\Sigma^A G/H_+)/\Pi) \wedge S^V & \cong & (G \ltimes_{H\Pi} S^{\rho^* A}) \wedge S^V \\
& \cong & G \ltimes_{H\Pi} (S^{\rho^* A} \wedge S^V) \\
& \xrightarrow{1 \ltimes (1 \wedge p(\lambda))} & G \ltimes_{H\Pi} (S^{\rho^* A} \wedge (H\Pi \ltimes_H S^{j(E_z(V))})) \\
& \cong & G \ltimes_{H\Pi} H\Pi \ltimes_H (S^A \wedge S^{j(E_z(V))}) \\
& \xrightarrow{1 \ltimes 1 \ltimes \widehat{\lambda}^{-1}} & G \ltimes_{H\Pi} H\Pi \ltimes_H S^V \\
& \cong & G/H_+ \wedge S^V,
\end{array}$$

which we denote τ_2. Here, the first isomorphism is one described in the proof of Lemma 16.3. Also, as defined just above that lemma, $\rho^* A$ is just A with the $H\Pi$-action derived from the projection $\rho : H\Pi \longrightarrow H\Pi/\Pi \cong H$ rather than the usual action derived from the inclusion of $H\Pi$ into G. The other unlabeled isomorphisms in this composite are space-level versions of the isomorphism ζ of Lemma II.4.9 of [**24**]. The remaining maps in the composite are all described in Definition 17.8(c).

Since the Adams transfer for $\Sigma^\infty_{U'} G/H_+$ and the map to which it is compared in axiom **(A4)** are the stabilizations of τ_1 and τ_2, axiom **(A4)** can be verified by showing that τ_1 and τ_2 are the same G-map. This follows from a rather tedious diagram chase which can be simplified by identifying the domain of both maps with $G \ltimes_{H\Pi} (S^{\rho^* A} \wedge S^V)$. The universal property of this space reduces the proof to establishing the equality of the two maps derived from τ_1 and τ_2 by precomposition with the canonical inclusion $S^{\rho^* A} \wedge S^V \longrightarrow G \ltimes_{H\Pi} (S^{\rho^* A} \wedge S^V)$. □

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