TOWER TECHNIQUES FOR COFACIAL RESOLUTIONS

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Abstract. Let J be a continuous coaugmented functor on spaces. For every space X one constructs a cofacial resolution $X \to J^\bullet X$ (namely a cosimplicial resolution without its codegeneracy maps) in the usual way. Following Bousfield and Kan, one defines $J_s X = \mathrm{tot}_s J^\bullet X$.

Suppose D is a small category and that X is a D-diagram of J-injective spaces, namely $X(d) \to JX(d)$ admits a left inverse for every object d in D, but in a way which need not be compatible, namely a map $JX \to X$ cannot be constructed out of this data. We show that for many free diagrams F, the spaces $\hom_D(F, X)$ are J_s-injective for $s < \infty$. Thus, the functors $\mathbb{Z}_s$ of Bousfield and Kan capture a large class of polyGEMs as their injective spaces. This generalises earlier results by the author. Our methods use pro-object arguments, which are originally due to Farjoun.

Keywords: Homotopy limits, Towers, Coaugmented functors

1. Introduction

1.1. Recall that a cofacial space $X^\bullet$ is a cosimplicial space without its codegeneracy maps. A cofacial resolution (or simply a resolution) is an augmented cofacial space $X^{-1} \to X^\bullet$ (compare [2, pp. 271]). Such a resolution is called trivial if it admits a left contraction, namely maps $X^n \xrightarrow{s} X^{n-1}$ which function as a codegeneracy map s^{-1}, namely $sd^0 = \mathrm{id}$ and $sd^i = d^{i-1}s$ for all $i > 0$. Trivial or not, such a resolution admits maps

$$X^{-1} \to \mathrm{tot}_n X^\bullet \qquad n \le \infty$$

where tot_n is the cofacial analogue of the classical cosimplicial tot_n in [2]. It is, in fact, the homotopy limit of the truncation of $X^\bullet$ in codimension n, denoted $(X^\bullet)^{\le n}$.

If D is a small category then the notion of resolution of D-diagrams is the obvious extension of the above. Of interest are such resolutions which are termwise trivial, namely $X^{-1}(d) \to X^\bullet(d)$ is a trivial resolution of spaces for all objects d in D, but they do not combine to give a map $X^\bullet \to X^{-1}$.

Theorem A. [15, Theorem 3.2] *Suppose that D is a small category such that $\dim\ N(D) < \infty$. Let $X^{-1} \to X^\bullet$ be a termwise trivial resolution of (fibrant) spaces. Then*

$$\mathrm{holim}_D X^{-1} \to \{tot_s \mathrm{holim}_D X^\bullet\}_{s \ge 0}$$

is a pro-equivalence in the homotopy category of spaces. In particular $\mathrm{holim}_D X^{-1}$ *is a retract of* $tot_s \mathrm{holim}_D X^\bullet$.

Suppose that J is a coaugmented functor on spaces which satisfies the mild assumption of being simplicial (alternatively, continuous [3, pp. 20], [14, 3.2],[13]).

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The functor J can be used to construct cofacial resolutions $X \to J^\bullet X$ and then, following Bousfield and Kan, to define

$$J_s X := \mathrm{tot}_s J^\bullet X = \mathrm{holim}\ (J^\bullet X)^{\leq s}.$$

A space X is called J-injective if $X \to JX$ admits a left inverse. For such spaces, the resolution $X \to J^\bullet X$ are easily seen to be trivial. Examples of this situation are described bellow. Theorem A was exploited to prove

Theorem B. [14, Theorem 4.7]. *Suppose that D is a small category whose nerve is finite dimensional, and that $X : D \to \mathcal{S}\mathrm{pc}$ is a diagram of J-injective spaces. Then there exists $s < \infty$ such that $\mathrm{holim}_D X$ is up to homotopy a retract of $J_s \mathrm{holim}_D X$. In fact $s = \dim(D)$.*

1.2. The motivation for this discussion lies in the study of polyGEMs. Recall from [3, pp. 87,101] that a space is called a GEM (Generalised Eilenberg MacLane space) if it is homotopy equivalent to a product of abelian Eilenberg MacLane spaces. It is a well known fact (due to J. Moore) that a space X is a GEM if and only if it is SP^∞-injective, namely if and only if $X \to \mathrm{SP}^\infty X$ admits a left homotopy inverse. It is also known that a space is a GEM if and only if it is equivalent to an abelian topological group.

A space X is called a polyGEM if it is homotopy equivalent to a space in a tower $\{\cdots \to X_2 \to X_1 \to X_0 = *\}$ of *principal* fibrations, all of whose fibres are abelian topological groups. Thus, the construction of a polyGEM is an iterative process, in which one starts with a GEM and repeatedly obtain the next space by taking the total space in a principal fibration in which the fibre is abelian and the base is the given space.

When D is a finite dimensional category and X is a D-diagram of GEMs, it is not hard to see that $\mathrm{holim}_D X$ is a polyGEM by appropriately filtering the category D (see [14, 5.4]). We call such spaces thin-polyGEMs. The simplicial analogue of SP^∞ is Bousfield Kan's functor $\mathbb{Z}$, and Theorem B implies

Theorem C. *If X is a thin polyGEM then X is up to homotopy a retract of $\mathbb{Z}_s X$ for some $s < \infty$.*

1.3. The purpose of this note is to extend these results. The main tool that we use is the notion of strong pro-equivalence which is introduced and explored in Section 2. It is essentially due to Farjoun [5]. The main technical result of this paper is 4.11

Theorem D. *Let $X^{-1} \to X^\bullet$ be a trivial resolution of (fibrant) spaces. Then the tower map*

$$X^{-1} \to \{tot_s X^\bullet\}_{s \geq 0}$$

is a strong pro-homotopy-equivalence.

The concept of strong pro-homotopy equivalence is much stronger than the more familiar weak-pro-equivalence ,see [2, pp. 76]. Its power is in enabling inductive arguments. This enables to transform the painfully complicated proof of Theorem A given in [15], into a much softer and conceptual one. It also enables us to generalise it in 5.11 and 5.6 as follows.

Theorem E. *Let D be a small category and that either*

(a) *D is finite dimensional and $F : D \to \mathcal{S}$ is a free diagram* [4], *or*

(b) *$F : D \to \mathcal{S}$ is a poly-free diagram (5.5).*

Then if $X^{-1} \to X^\bullet$ is a termwise trivial resolution of fibrant D-diagrams, then

$$\hom_D(F, X^{-1}) \to \{tot_s \hom_D(F, X^\bullet)\}_{s \geq 0}$$

is a strong pro-homotopy equivalence. In particular $\hom_D(F, X^{-1})$ is up to homotopy, a retract of $tot_s \hom_D(F, X^\bullet)$ for some $s < \infty$.

Theorem A is recovered by plugging in a weakly contractible F. Consequently, we deduce (6.4)

Corollary F. *Suppose that D and F are as in Theorem E, and J is a simplicial coaugmented functor. If $X : D \to \mathcal{S}\mathrm{pc}$ is a diagram of fibrant J-injective spaces, then there exists $s < \infty$ such that $\hom_D(F, X)$ is a retract of $J_s \hom_D(F, X)$.*

1.4. Applying Corollary F to the case when J is the functor $\mathbb{Z}$, we see that we have extended the class of polyGEMs which are detected by $\mathbb{Z}_s$ in the sense that a polyGEM X in this class is a retract of $\mathbb{Z}_s X$ for some $s < \infty$.

Firstly, if D is finite dimensional, F is *any* free D-diagram, and X a diagram of GEMs, then $\hom_D(F, X)$ is a retract of $\mathbb{Z}_s \hom_D(F, X)$ for some $s < \infty$. The older Theorem C is recovered by choosing F to be weakly contractible. Better, when D is *any* small category (even as not homologically finite as a finite group) and F is poly-free (5.5), then $\hom_D(F, X)$ is a retract of its $\mathbb{Z}_s$ for some $s < \infty$.

The class of poly-free diagrams is the smallest class of diagram which contains the diagrams $\coprod_{i \in I} A_i \times D(d_i, -)$ for (cofibrant) spaces A_i, and is closed under pushouts along cofibrations. It is, in some sense, a dual construction to polyGEMs which consist the smallest class of spaces which contain GEMs and is closed under pullbacks along fibrations between GEMs.

We still don't know if all polyGEMs are retracts of their $\mathbb{Z}_s$.

2. Towers

2.1. Throughout, $\mathcal{S}$ denotes the category of simplicial sets. The standard n-simplices are denoted Δ^n. These spaces are the objects in the standard cosimplicial space $\Delta^\bullet$. We let $d^i : \Delta^{n-1} \to \Delta^n$ and $s^i : \Delta^{n+1} \to \Delta^n$ be the standard cosimplicial coface and codegeneracy maps $(0 \leq i \leq n)$.

2.2. Let $\mathcal{C}$ be a simplicial closed model category (cf. [9, Ch.II.2]). This, in particular means that there is a pairing bifunctor

$$\otimes : \mathcal{C} \times \mathcal{S} \to \mathcal{C}$$

which has partial right adjoints

$$\mathrm{map} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S} \qquad \text{and} \qquad \hom : \mathcal{S}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}.$$

Explicitly, the first functor is given by

$$\mathrm{map}(X, Y)_n = \mathcal{C}(X \otimes \Delta^n, Y).$$

It is worthwhile noting that

(a) $X \otimes \Delta^0 = X$ for all $X \in \mathcal{C}$.

(b) If $f : A \to B$ is a (trivial) cofibration in $\mathcal{S}$ and if $X \in \mathcal{C}$ is cofibrant, then $X \otimes f$ is a (trivial) cofibration in $\mathcal{C}$ (see [9, II.3.4]).

(c) For every $X, Y \in \mathcal{C}$ and $A \in \mathcal{S}$, there is a map, natural in these objects

$$\mathrm{map}(X,Y) \times A \to \mathrm{map}(X, Y \otimes A)$$

which is the adjoint of

$$X \otimes (\mathrm{map}(X,Y) \times A) = (X \otimes \mathrm{map}(X,Y)) \otimes A \xrightarrow{\mathrm{ev}\otimes A} Y \otimes A$$

2.3. Recall that a tower of objects in $\mathcal{C}$ is a functor $X_\bullet : \mathbb{N}^{\mathrm{op}} \to \mathcal{C}$ where $\mathbb{N}$ is the category $\{0 \to 1 \to 2 \to \dots\}$. These objects form the usual functor category $\mathcal{C}^{\mathbb{N}^{\mathrm{op}}}$ which we denote tow–$\mathcal{C}$. It is, however, useful to treat towers as pro objects (cf. [1, Appendix]). Morphism sets of towers as pro-objects are by definition

$$\mathrm{pro}-\mathcal{C}(X_\bullet, Y_\bullet) = \lim_s \operatorname*{colim}_t \mathcal{C}(X_t, Y_s).$$

The convention will be that towers are denoted by the symbols $X_\bullet, Y_\bullet, \dots$ etc. Morphisms of towers as diagrams over $\mathbb{N}^{\mathrm{op}}$ are denoted by $\alpha, \beta, \dots$ or $f, g, \dots$. Morphisms of towers though of as pro objects will be denoted by $\bar{\alpha}, \bar{\beta}, \dots$ or $\bar{f}, \bar{g}, \dots$ etc.

Evidently, every pro-morphism of towers can be represented by a sequence of morphisms

$$\{f_s \colon X_{t(s)} \to Y_s\}_{s \geq 0}, \qquad (t(s+1) > t(s))$$

which render the following squares commutative ($s \geq 0$).

$$\begin{array}{ccc} X_{t(s+1)} & \xrightarrow{f_{s+1}} & Y_{s+1} \\ \downarrow & & \downarrow \\ X_{t(s)} & \xrightarrow{f_s} & Y_s \end{array}$$

Evidently, a level map $\{f_s\}_s : X_\bullet \to Y_\bullet$, namely a morphism in tow–$\mathcal{C}$, represents a pro-map $\bar{f}$.

2.4. Given a tower $X_\bullet$ and a simplicial set K, we denote by $X_\bullet \otimes K$ the obvious tower which at level s is $X_s \otimes K$.

2.5. Given towers $X_\bullet$ and $Y_\bullet$, let

$$\mathrm{map}(X_\bullet, Y_\bullet) \ := \ \lim_s \ \operatorname*{colim}_t \ \mathrm{map}(X_t, Y_s)$$

be the function complex (an object in $\mathcal{S}$). Evidently,

$$\mathrm{map}(X_\bullet, Y_\bullet)_n = \mathrm{pro}-\mathcal{C}(X_\bullet \otimes \Delta^n, Y_\bullet).$$

Moreover, this construction is natural in the sense that a morphism (of pro-objects) $\bar{f} : A_\bullet \to B_\bullet$ defines simplicial maps

$$\mathrm{map}(B_\bullet, X_\bullet) \xrightarrow{\bar{f}^*} \mathrm{map}(A_\bullet, X_\bullet) \qquad \text{and}$$

$$\mathrm{map}(X_\bullet, A_\bullet) \xrightarrow{\bar{f}_*} \mathrm{map}(X_\bullet, B_\bullet).$$

This is done in an obvious way by choosing a representative for the morphism $\bar{f}$ (2.3). We leave it to the reader to check that this is independent of the representative, and remark that when $\bar{f}$ is represented by a level map $\{f_s\}$ then

$$\mathrm{map}(\bar{f}, X_\bullet) = \lim_s \operatorname*{colim}_t \mathrm{map}(f_t, X_s) \qquad \text{and}$$

$$\mathrm{map}(X_\bullet, \bar{f}) = \lim_s \operatorname*{colim}_t \mathrm{map}(X_t, f_s).$$

2.6. Definition. (cf. [5].) A tower $X_\bullet$ is called *fibrant* if all the objects X_s are fibrant and cofibrant and the maps $X_{s+1} \to X_s$ are fibrations for all $s \geq 0$.

2.7. Remark. This is different than the terminology in [10] or [11]. The terminology and results that will follow in this section, are very reminiscent of the ones in (simplicial) model categories. However, we do not know how to endow the category tow–C with a model structure that correspond to our notion of fibrations etc.

2.8. Proposition. *Let $X_\bullet$ be a fibrant tower. Then for every tower $W_\bullet$ of cofibrant objects, the mapping space* $\mathrm{map}(W_\bullet, X_\bullet)$ *is a fibrant space (i.e. a Kan complex).*

Proof. Observe that $\mathrm{map}(W_t, X_s)$ is a fibrant space for every s and t, and that sequential colimits preserve fibrations in $\mathcal{S}$. Hence $\{\mathrm{colim}_t \mathrm{map}(W_t, X_s)\}_s$ if a fibrant tower in $\mathcal{S}$, and its inverse limit is fibrant. □

2.9. Definition. Tower maps $\bar{f}_0, \bar{f}_1 : X_\bullet \to Y_\bullet$ are called *homotopic*, if as vertices, they belong to the same component of $\mathrm{map}(X_\bullet, Y_\bullet)$. We call them *simplicially homotopic* if there exists a 1-simplex, namely a pro-map

$$\bar{h}\colon X_\bullet \otimes \Delta^1 \to Y_\bullet$$

such that $\partial_i(\bar{h}) = \bar{f}_i$.

By 2.8, when $Y_\bullet$ is fibrant and $X_\bullet$ is levelwise cofibrant, then both notions coincide.

2.10. Proposition. *Let $\bar{\alpha}_0, \bar{\alpha}_1 : X_\bullet \to Y_\bullet$ be simplicially homotopic pro-maps (2.9) between fibrant towers. Then*

(a) *For every fibrant tower $T_\bullet$, the induced maps $(\bar{\alpha}_i)^* = \mathrm{map}(\bar{\alpha}_i, T_\bullet)$ are simplicially homotopic (as maps between Kan complexes).*

(b) *For every tower $W_\bullet$ of cofibrant objects, the induced maps*

$$(\bar{\alpha}_i)_* = \mathrm{map}(W_\bullet, \bar{\alpha}_i)$$

are simplicially homotopic (as maps between Kan complexes).

Proof. Let $\bar{h}$ be a simplicial homotopy (2.9). That is, the compositions

$$X_\bullet \underset{d^1}{\overset{d^0}{\rightrightarrows}} X_\bullet \otimes \Delta^1 \xrightarrow{\bar{h}} Y_\bullet$$

are precisely $\bar{\alpha}_0$ and $\bar{\alpha}_1$.

To prove (a) it suffices to show that $\mathrm{map}(d^0, T_\bullet) \simeq \mathrm{map}(d^1, T_\bullet)$. Consider the projection map $\pi : X_\bullet \otimes \Delta^1 \to X_\bullet$. Since $\pi \cdot d^i = \mathrm{id}$, it suffices to show that $\mathrm{map}(\pi, T_\bullet)$ is a homotopy equivalence. This will follow if we show that, for instance, $\mathrm{map}(d^0, T_\bullet)$ is one. Observe that d^0 is a levelwise trivial cofibration, and since $T_\bullet$ is fibrant, it follows that the maps

$$\mathrm{map}(X_t \otimes \Delta^1, T_s) \xrightarrow{(d^0)^*} \mathrm{map}(X_t, T_s)$$

are all trivial fibrations. Using the fact that sequential colimits preserve trivial fibrations and fibrations in $\mathcal{S}$, we see that the induced map

$$\{\mathrm{colim}_t\, \mathrm{map}(X_t \otimes \Delta^1, T_s)\}_s \to \{\mathrm{colim}_t\, \mathrm{map}(X_t, T_s)\}_s$$

is a levelwise homotopy equivalence of fibrant towers in $\mathcal{S}$. Since the inverse limit of a tower of fibrations is equivalent to its homotopy limit, it follows that $\mathrm{map}(d^0, T_\bullet)$ is an equivalence of Kan complexes, and the result follows.

Point (b) is even easier since the natural map (2.2(c))

$$\mathrm{map}(W_\bullet, X_\bullet) \times \Delta^1 \to \mathrm{map}(W_\bullet, X_\bullet \otimes \Delta^1),$$

composed with $\mathrm{map}(W_\bullet, \bar{h})$ gives the desired simplicial homotopy. □

2.11. Definition. Let $\bar{\alpha}: X_\bullet \to Y_\bullet$ be a morphism of fibrant towers. We call $\bar{\alpha}$ a *strong pro-homotopy equivalence*, and write s.p.h.e. for short, if one of the following equivalent conditions hold.

(a) There exists a pro-map $\bar{\beta}: Y_\bullet \to X_\bullet$ such that $\bar{\alpha} \cdot \bar{\beta}$ and $\bar{\beta} \cdot \bar{\alpha}$ are simplicially homotopic to the identity maps (2.9).

(b) For every fibrant tower $T_\bullet$, the induced map

$$\mathrm{map}(Y_\bullet, T_\bullet) \xrightarrow{\bar{\alpha}^*} \mathrm{map}(X_\bullet, T_\bullet)$$

is a homotopy equivalence of fibrant spaces.

(c) For every tower $W_\bullet$ of cofibrant objects, the induced map

$$\mathrm{map}(W_\bullet, X_\bullet) \xrightarrow{\bar{\alpha}_*} \mathrm{map}(W_\bullet, Y_\bullet)$$

is a homotopy equivalence of fibrant spaces.

Proof. Evidently (a) implies both (b) and (c) using Proposition 2.10. For the converse we use the commutative diagram of fibrant objects in $\mathcal{S}$.

$$\begin{array}{ccc} \mathrm{map}(Y_\bullet, X_\bullet) & \xrightarrow{\bar{\alpha}^*} & \mathrm{map}(X_\bullet, X_\bullet) \\ {\scriptstyle \bar{\alpha}_*}\downarrow & & \downarrow{\scriptstyle \bar{\alpha}_*} \\ \mathrm{map}(Y_\bullet, Y_\bullet) & \xrightarrow[\bar{\alpha}^*]{} & \mathrm{map}(X_\bullet, Y_\bullet). \end{array}$$

If one assumes (b) then the horizontal arrows are homotopy equivalences between Kan complexes, hence one can choose $\bar{\beta} : Y_\bullet \to X_\bullet$ such that $\bar{\alpha}^*(\bar{\beta}) = \bar{\beta}\bar{\alpha}$ is simplicially homotopic to the identity on $X_\bullet$. Applying $\bar{\alpha}^*$ to these vertices, and using the fact that the bottom arrow is a homotopy equivalence, it follows that $\bar{\alpha} \cdot \bar{\beta} \simeq \mathrm{id}_{Y_\bullet}$, so (a) holds.

If one assumes (c), then the vertical arrows are homotopy equivalences, and the argument is similar. □

2.12. Lemma. *Let $f : X_\bullet \to Y_\bullet$ be a levelwise homotopy equivalence between fibrant towers. Then f is a s.p.h.e.*

Proof. If $W_\bullet$ is a tower of cofibrant objects, then for every $t, s \geq 0$, the map

$$\mathrm{map}(W_t, X_s) \xrightarrow{f_s} \mathrm{map}(W_t, Y_s)$$

is a homotopy equivalence of Kan complexes because X_s and Y_s are fibrant and W_t cofibrant. Since sequential colimits (indexed by t) carry equivalences between fibrant spaces (resp. fibrations) to an equivalence of fibrant spaces (resp. fibrations), it follows that

$$\{\operatorname*{colim}_t \mathrm{map}(W_t, X_s)\}_s \to \{\operatorname*{colim}_t \mathrm{map}(W_t, Y_s)\}_s$$

is a levelwise equivalence of fibrant towers in $\mathcal{S}$. The result follows because in this case the inverse limit (indexed by s) is the same as the homotopy inverse limit. □

2.13. Lemma. *Consider the following pullback diagram in* tow–$\mathcal{C}$ *of fibrant towers (and level maps).*

$$\begin{array}{ccc} U_\bullet & \xrightarrow{\{g'_s\}_s} & Y_\bullet \\ {\scriptstyle \{f'_s\}_s}\downarrow & & \downarrow{\scriptstyle \{f_s\}_s} \\ Z_\bullet & \xrightarrow[\{g_s\}_s]{} & X_\bullet . \end{array} \tag{2.13.1}$$

Suppose that g_s are fibrations in $\mathcal{C}$. Then for every tower $W_\bullet$ of cofibrant objects, applying $\operatorname{map}(W_\bullet, -)$ *to* (2.13.1) *yields a homotopy cartesian square of fibrant objects in $\mathcal{S}$.*

Proof. Let P_s denote the pullback square of (2.13.1) at level s. Since $\operatorname{map}(W_t, -)$ and sequential colimits carry pullback squares (resp. fibration) to pullback squares (resp. fibrations), it follows that

(i) For every s, the diagram $\operatorname{colim}_t \operatorname{map}(W_t, P_s)$ is a pullback square of fibrant spaces, and $\operatorname{colim}_t \operatorname{map}(W_t, g_s)$ is a fibration.

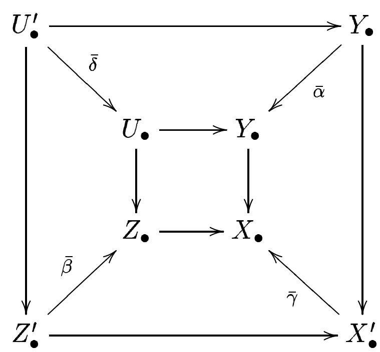
(ii) The induced maps between these pullback squares

$$\operatorname{colim}_t \operatorname{map}(W_t, P_{s+1}) \to \operatorname{colim}_t \operatorname{map}(W_t, P_s)$$

are objectwise fibrations.

Since sequential inverse limits (indexed by s) of fibrations are equivalent to homotopy limits, the result follows. □

2.14. Corollary. *Consider the diagram of fibrant towers commuting in* pro$-\mathcal{C}$



Assume that both squares are cartesian in tow–$\mathcal{C}$ *(so all vertical and horizontal arrows are level map), and that the horizontal arrows are levelwise fibrations. If $\bar{\alpha}, \bar{\beta}$ and $\bar{\gamma}$ are s.p.h.e. then so is $\bar{\delta}$.*

Proof. Immediate from 2.13 and 2.11(c). □

2.15. *(Loop objects and fibres.)* Suppose now that $\mathcal{C}$ is also pointed, namely it has an object $*$ which is both initial and terminal.

Given a fibration $f\colon X \to B$ in $\mathcal{C}$, we define its *fibre* F as the pullback of

$$X \xrightarrow{f} B \longleftarrow *.$$

Clearly, when B is fibrant, then so is the fibre of f. We call the resulting sequence

$$F \xrightarrow{i} X \xrightarrow{f} B$$

a *fibre sequence* in $\mathcal{C}$.

The *loop object* of every $X \in \mathcal{C}$, denoted ΩX, is the pullback of

$$\hom(\Delta^1, X) \longrightarrow \hom(\partial\Delta^1, X) \longleftarrow \hom(\partial\Delta^1, *) = *.$$

Observe that given objects W and X in $\mathcal{C}$, the function complex $\mathrm{map}(W, X)$ is naturally a pointed simplicial set. Using this basepoint, there is a natural isomorphism

$$\Omega \mathrm{map}(W, X) \cong \mathrm{map}(W, \Omega X) \tag{2.15.1}$$

which follows from the fact that $\mathrm{map}(W, -)$ preserve pullback squares and from the isomorphism (cf. [9, II.2.3])

$$\mathrm{map}_{\mathcal{C}}(W, \hom_{\mathcal{C}}(A, X)) \cong \mathrm{map}_{\mathcal{S}}(A, \mathrm{map}_{\mathcal{C}}(W, X)) = \hom_{\mathcal{S}}(A, \mathrm{map}_{\mathcal{C}}(W, X)).$$

2.16. Lemma. *Suppose that $F_\bullet \xrightarrow{i} X_\bullet \xrightarrow{p} B_\bullet$ is a level map of fibrant towers in a pointed simplicial closed model category $\mathcal{C}$. Assume further that at every level s the maps form a fibre sequence. Then for every levelwise cofibrant tower $W_\bullet$, the sequence*

$$\mathrm{map}(W_\bullet, F_\bullet) \to \mathrm{map}(W_\bullet, X_\bullet) \to \mathrm{map}(W_\bullet, B_\bullet)$$

is a homotopy fibre sequence of Kan complexes in $\mathcal{S}_*$.

Proof. Follows immediately from Lemma 2.13. □

2.17. Corollary. *Let $\mathcal{C}$ be a pointed simplicial closed model category in which all objects are cofibrant. Consider the following diagram of fibrant towers which commutes in* tow–$\mathcal{C}$.

$$\begin{array}{ccccc} F_\bullet & \longrightarrow & E_\bullet & \longrightarrow & B_\bullet \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ F'_\bullet & \longrightarrow & E'_\bullet & \longrightarrow & B'_\bullet . \end{array}$$

Suppose that the vertical arrows form levelwise fibre sequences. Then

(a) *If g, h are s.p.h.e. , then so is f.*
(b) *If f, g are s.p.h.e. then so is Ωh.*
(c) *If $f, \Omega h$ are s.p.h.e. then so is Ωg.*

Proof. (a) is immediate from 2.14. Point (b) follows by looping the resulting homotopy fibre sequences in 2.16, and using (2.15.1), and the fact that Ωh is a level map of fibrant towers because $B_\bullet$ and $B'_\bullet$ are. Point (c) follows by looping this sequence once more. □

3. Coherent functors

3.1. The concept of "coherent functors" is devised to give a uniform proof for our main results in both the pointed and unpointed categories. The unpointed case is more fundamental, and the reader who is happy with this case only may safely ignore the term "coherent" throughout the paper.

In this section we restrict attention to the categories of unpointed and pointed simplicial sets, denoted $\mathcal{S}$ and $\mathcal{S}_*$ denoted by $\mathcal{S}$pc.

These categories are related by the pair of adjoint functors u – the forgetful functor, and $(-)_+$ – adjoining a disjoint basepoint functor.

3.2. Recall that $\mathcal{S}$pc is a closed symmetric monoidal category (cf. [12]), and is, thus, enriched over itself. We lat "Map" denote the internal hom-spaces. It follows that if D is a small category, then the functor category $\mathcal{S}\mathrm{pc}^D$ is also an $\mathcal{S}$pc-category with hom-spaces given by the end formula

$$\mathrm{Map}_D(X,Y) = \int_{d\in D} \mathrm{Map}(X(d), Y(d)).$$

Similarly, the pairing $\otimes$ on $\mathcal{S}$pc (see 2.2)has an obvious prolongation to a pairing $\otimes\colon \mathcal{S}\mathrm{pc} \times \mathcal{S}^D \to \mathcal{S}\mathrm{pc}^D$, and the latter has an adjoint $\mathrm{hom}\colon (\mathcal{S}^D)^{\mathrm{op}} \times \mathcal{S}\mathrm{pc}^D \to \mathcal{S}\mathrm{pc}$ given by

$$\mathrm{hom}_D(K,X) = \int_{d\in D} \mathrm{hom}(K(d), X(d)).$$

It is easy to see that

$$\mathrm{hom}_D(K,X) = \begin{cases} \mathrm{Map}_D(K,X) & \mathcal{S}\mathrm{pc} = \mathcal{S} \\ \mathrm{Map}_D(K_+,X) & \mathcal{S}\mathrm{pc} = \mathcal{S}_* \end{cases} \tag{3.2.1}$$

3.3. Given a small functor $f : C \to D$, there is an induced restriction functor $f^* : \mathcal{S}\mathrm{pc}^D \to \mathcal{S}\mathrm{pc}^C$. By [16, pp. 229], this functor has a left adjoint, the left Kan extension of f, denoted Lan_f. It is not hard to verify that this adjunction prolongs to a natural isomorphism of spaces

$$\mathrm{Map}_C(X, f^*Y) \cong \mathrm{Map}_D(\mathrm{Lan}_f X, Y).$$

Moreover, since $\mathrm{Lan}_f(K_+) = (\mathrm{Lan}_f K)_+$ for every $K \in \mathcal{S}^C$, then

$$\mathrm{hom}_C(X, f^*Y) \cong \mathrm{hom}_D(\mathrm{Lan}_f X, Y).$$

3.4. Definition. A *coherent functor* $F : \mathcal{S}\mathrm{pc}^D \to \mathcal{S}\mathrm{pc}$ is comprised of a pair of functors – one for each of the categories $\mathcal{S}$ and $\mathcal{S}_*$, which are related by

$$uF = Fu$$

3.5. Example. Our prime examples of coherent functors are given by

$$F_A(-) := \mathrm{hom}_D(A,-)$$

where $A \in \mathcal{S}^D$. Indeed, if $X \in \mathcal{S}_*^D$, then

$$uF_A(X) = u\mathrm{Map}_D^*(A_+, X) = \mathrm{Map}_D(A, uX) = F_A(u(X)).$$

3.6. Definition. Let $F, G : \mathcal{S}\mathrm{pc}^D \to \mathcal{S}\mathrm{pc}$ be coherent functors. A coherent natural transformation $t : F \to G$ is a pair of natural transformations between the unpointed and pointed functors, which are subject to the condition

$$tu = ut.$$

The set of coherent natural maps is denoted $\mathrm{CNat}(F,G)$.

3.7. Example. If A and B are D-diagrams in $\mathcal{S}$, then any map $\varphi : B \to A$ induces a coherent map (3.5)

$$\mathrm{hom}_D(\varphi, -)\colon F_A(-) \to F_B(-).$$

Indeed, for all pointed diagrams of spaces,

$$u\mathrm{hom}_D(\varphi,-) = u\mathrm{Map}_D(\varphi_+,-) = \mathrm{Map}_D(\varphi, u(-)) = \mathrm{hom}_D(\varphi, u(-)).$$

This is, in fact, the only example:

3.8. Proposition. *Consider the functors F_A and F_B above (3.5). Then $t: F_A \to F_B$ is a coherent map if and only if $t = \hom_D(\varphi, -)$ for some $\varphi: B \to A$. That is, there is a natural bijection*

$$\mathcal{S}^D(B, A) \xrightarrow{\cong} \mathrm{CNat}(F_A, F_B).$$

Proof. The "if" part was proved in 3.7. Conversely, assume that t is coherent. Consider the unpointed t. Since $\mathcal{S}^D$ is an $\mathcal{S}$-category, and hom = Map (3.2.1), Yoneda's Lemma in enriched categories (see [12, pp. 45]) implies that the unpointed t is given by

$$t = \mathrm{Map}_D(\varphi, -) = \hom_D(\varphi, -)$$

for some unique $\varphi : B \to A$. Similarly, when $\mathcal{S}\mathrm{pc} = \mathcal{S}_*$, then by (3.2.1), and by Yoneda's Lemma again, the pointed t, denoted t_* is induced by a map

$$\psi: B_+ \to A_+.$$

By assumption t is coherent, so for all objects in $\mathcal{S}_*^D$,

$$u\mathrm{Map}_D(\varphi_+, -) = \mathrm{Map}_D(\varphi, u(-)) = tu(-) = ut_*(-) = u\mathrm{Map}_D(\psi, -).$$

But since the forgetful functor u is faithful, it follows that, φ_+ and ψ induce the same representable functor on $\mathcal{S}_*^D$, hence $\psi = \varphi_+$, and we have established that, as a coherent natural map, $t = \hom_D(\varphi, -)$. □

3.9. The set $\mathrm{CNat}(F, G)$ is naturally the vertex set of a simplicial set $\mathbf{CNat}(F, G)$, called the *space* of coherent maps. By definition

$$\mathbf{CNat}(F, G)_n = \mathrm{CNat}(F \otimes \Delta^n, G)$$

Observe that 1-simplices of $\mathbf{CNat}(F, G)$ correspond to simplicial homotopy of coherent natural maps $F \to G$.

3.10. Proposition. *Given $A, B \in \mathcal{S}^D$, there is a natural isomorphism of simplicial sets (3.5,3.9)*

$$\mathrm{Map}_D(B, A) \xrightarrow{\cong} \mathbf{CNat}(F_A, F_B).$$

Proof. Immediate from the following natural isomorphisms and 3.8

$$\begin{aligned} &\mathcal{S}\mathrm{pc}(\hom(A, -) \otimes \Delta^n, \hom(B, -)) \cong \\ &\qquad \mathcal{S}\mathrm{pc}(\hom(A, -), \hom(\Delta^n, \hom(B, -)) \cong \\ &\qquad \mathcal{S}\mathrm{pc}(\hom(A, -), \hom(B \times \Delta^n, -)). \end{aligned}$$

□

4. The tower associated to a resolution of spaces

4.1. We let $\mathcal{S}\mathrm{pc}$ denote either the unpointed or pointed categories of simplicial sets. See 3.1. Objects in these categories are called spaces.

4.2. For every $n \geq -1$ we let $[n] = \{0, \dots, n\}$ be linearly ordered in the usual way. By convention $[-1]$ is the empty set.

Let Λ denote the category whose objects are the sets $[0], [1], [2], \dots$ and whose morphisms are the *strictly* monotone maps between these sets. Obviously, Λ is a subcategory of the usual cosimplicial category Δ, and is generated by the morphisms $d^i: [n] \to [n+1]$ where $0 \leq i \leq n+1$, subject to the relation $d^j d^i = d^{i-1} d^j$ if $i > j$.

We further let Λ_0 denote the category Λ adjoined with the initial object $[-1]$.

Obviously, Λ is the category underlying "reduced cosimplicial objects" namely cosimplicial objects without their codegeneracy maps. Likewise, Λ_0 is the category underlying augmented such objects (cf. [2, pp. 271]). See bellow.

4.3. Definition. A *cofacial space* is a Λ-diagram of spaces. We shall denote cofacial spaces by $X^\bullet, Y^\bullet$ etc. An extremely important cofacial space is $\Lambda^\bullet$, which is the restriction of the standard cosimplicial space $\Delta^\bullet$ to Λ. Other useful variations are $\mathrm{sk}_n\Lambda^\bullet$ which is the simplicial n-skeleton of $\Lambda^\bullet$.

4.4. We note that the maps $(-1 \leq n \leq m \leq \infty)$

$$\mathrm{sk}_n\Lambda^\bullet \to \mathrm{sk}_m\Lambda^\bullet$$

are free, and hence cofibrations in $\mathcal{S}^\Lambda$ equipped with the usual model category structure (see [7]). In particular, $\mathrm{sk}_n\Lambda^\bullet$ are cofibrant.

4.5. Definition. A *cofacial resolution* (or merely "a resolution") of a space X, is a cofacial space $X^\bullet$ together with a map $X \xrightarrow{d^0} X^0$ such that $d^0d^0 = d^1d^0$. We usually write X^{-1} for X, and a resolution is thus noting but a functor $\mathbf{X}\colon\Lambda_0 \to \mathcal{S}\mathrm{pc}$. We shall use boldface capitals $\mathbf{X}, \mathbf{Y}$ etc. to denote resolutions. Given such $\mathbf{X}$, we let $X^\bullet$ be the cofacial part, and X^{-1} its augmentation.

4.6. Definition. Given a cofacial space $X^\bullet$, we let

$$\mathrm{tot}_s X^\bullet = \hom_\Lambda(\mathrm{sk}_s\Lambda^\bullet, X^\bullet).$$

Clearly, if $\mathbf{X}$ is a resolution, then there is a natural map

$$X^{-1} \xrightarrow{d^0} \mathrm{tot}_s X^\bullet.$$

Furthermore, from 4.4 it follows that if $X^\bullet$ is a fibrant cofacial space, namely objectwise fibrant, then $\mathrm{tot}_s X^\bullet$ is a fibrant space, and when $s = \infty$, it is homotopy equivalent to $\mathrm{holim}\, X^\bullet$.

4.7. Definition. A *left contraction* for a resolution $X^{-1} \xrightarrow{d^0} X^\bullet$ is a collection of maps $s\colon X^n \to X^{n-1}$ for all $n \geq 0$, such that $sd^0 = \mathrm{id}$ and $sd^i = d^{i-1}s$ for all $i > 0$. Compare this with [8].

A resolution which admits a left contraction is called *trivial*. Notice that a resolution may be trivialised in more than one way.

4.8. It is useful to define a category of trivial resolutions. More precisely, we let Λ_+ be the category whose objects are $\mathrm{Obj}(\Lambda_0)$, and whose morphism sets are generated by the (abstract) morphisms

$$d^i\colon[n-1] \to [n] \qquad 0 \leq n, \quad 0 \leq i \leq n$$
$$s\colon[n] \to [n-1] \qquad n \geq 0$$

subject to the relations

$$\begin{aligned} &d^jd^i = d^{i-1}d^j \qquad i > j \\ &sd^0 = \mathrm{id} \\ &sd^i = d^{i-1}s \qquad i > 0. \end{aligned} \tag{4.8.1}$$

One can, in fact, embed Λ_+ as the subcategory of Δ which consists of the objects $[1], [2], [3], \ldots$ and morphisms are those order preserving maps $\varphi : [n] \to [k]$ such that $0 \in \varphi^{-1}(0)$ and φ is strictly monotone outside $\varphi^{-1}(0)$. We shall, however, not need this description.

It is evident that trivial resolution are precisely diagrams over Λ_+.

4.9. Consider the small functors

$$\Lambda \xrightarrow{k} \Lambda_+ \xleftarrow{j} *$$

where $*$ denotes the trivial category,

$$j(*) = [-1]$$

and k is the inclusion functor, namely (4.8)

$$k([n]) = [n], \qquad k(d^i) = d^i.$$

Given a trivial resolution (a Λ_+-diagram) $\mathbf{X}$, then $X^\bullet = k^*(\mathbf{X})$ and $X^{-1} = j^*(\mathbf{X})$.

4.10. For every space Y we let $c^\bullet Y$ denote the constant cofacial space.

Suppose that $\mathbf{X}$ is a trivial resolution. Then the augmentation map $X^{-1} \xrightarrow{d^0} X^0$ and the contractions s induce natural cofacial space maps

$$d^0\colon c^\bullet X^{-1} \to X^\bullet \qquad \text{and} \qquad \sigma\colon X^\bullet \to c^\bullet X^{-1}$$

given by

$$(d^0)^{n+1}\colon X^{-1} \to X^n \qquad \text{and} \qquad s^{n+1}\colon X^n \to X^{-1}.$$

These maps give rise to natural maps (see 4.9)

$$d^0 \colon j^*(\mathbf{X}) \to \mathrm{tot}_n k^*(\mathbf{X}) \qquad \text{and} \qquad \sigma \colon \mathrm{tot}_n k^*(\mathbf{X}) \to j^*(\mathbf{X}).$$

Explicitly,

$$d^0\colon X^{-1} = \mathrm{hom}(*, c^\bullet X^{-1}) \to \mathrm{tot}_n c^\bullet X^{-1} \xrightarrow{d^0} \mathrm{tot}_n X^\bullet$$
$$\sigma\colon \mathrm{tot}_n X^\bullet \xrightarrow{s} \mathrm{tot}_n c^\bullet X^{-1} \to \mathrm{tot}_0 c^\bullet X^{-1} = X^{-1}.$$

We make the observation that these maps are coherent maps between coherent functors $\mathcal{S}\mathrm{pc}^{\Lambda_+} \to \mathcal{S}\mathrm{pc}$ (3.4,3.6). The rest of this section is dedicated to the proof of

4.11. Theorem. *Let $\mathbf{X}$ be a fibrant trivial resolution, i.e. a Λ_+-diagram of fibrant spaces. Then the natural map of fibrant towers*

$$d^0\colon \{X^{-1}\}_{s\geq 0} \to \{tot_s X^\bullet\}_{s \geq 0}$$

is a s.p.h.e. (2.11). In fact, σ defined in 4.10, is a pro-homotopy inverse.

4.12. Remark. The point of this theorem is in the strength of the relation between the towers involved. It is immediate that X^{-1} is a retract of $\mathrm{tot}_n X^\bullet$ for all n, and that $X^{-1} \simeq \mathrm{tot}_\infty X^\bullet$. But these facts are very weak statements compared to 4.11. It is also much stronger than the pro-equivalence in [15, Theorem 3.2].

4.13. The strategy of the proof is to show that both composition σd^0 and $d^0\sigma$ are s.p.h.e. Clearly, $\sigma d^0 = \mathrm{id}$, and so we are only left with $d^0\sigma$. The naturality in $\mathbf{X}$ of these maps, enables us to show that id and $d^0\sigma$ belong to the same component of the function complex $\mathrm{map}(\{\mathrm{tot}_s X^\bullet\}_s, \{\mathrm{tot}_s X^\bullet\}_s)$.

4.14. Proposition. *For every fibrant trivial resolution $\mathbf{X}$, $\sigma d^0 = id$.*

Proof. This is true on the level of cofacial maps (see 4.10). □

To establish the fact that $d^0\sigma$ is a s.p.h.e. we need some preparation.

4.15. Definition. For every $n \geq 0$, let v_n denote the collapsing map of Δ^n onto the vertex 0 of Δ^{n+1}. Specifically, v_n is the composition

$$v_n\colon \Delta^n \xrightarrow{(s^0)^n} \Delta^0 \xrightarrow{d^{n+1}\dots d^2 d^1} \Delta^{n+1}.$$

4.16. Proposition. *There exist maps* $(n \geq 0)$

$$\pi_n\colon \Delta^n \times \Delta^1 \to \Delta^{n+1}$$

such that

(a) $\pi_n \cdot (1 \times d^0) = d^0\colon \Delta^n \to \Delta^{n+1}$.
(b) $\pi_n \cdot (1 \times d^1) = v_n$ *(see 4.15)*
(c) *The following diagram commutes for every $n \geq 0$ and $0 \leq i \leq n+1$.*

$$\begin{array}{ccc} \Delta^n \times \Delta^1 & \xrightarrow{\pi_n} & \Delta^{n+1} \\ {\scriptstyle d^i \times 1}\big\downarrow & & \big\downarrow{\scriptstyle d^{i+1}} \\ \Delta^{n+1} \times \Delta^1 & \xrightarrow[\pi_{n+1}]{} & \Delta^{n+2}. \end{array}$$

Proof. Consider the linearly ordered sets $[n] = \{0, 1, \dots, n\}$ (see 4.2) as categories. Let $c_n\colon [n] \to [n+1]$ be the functor $(s^0)^n$, and let $j_n\colon [n] \to [n+1]$ be the functor d^0. Upon taking nerves, c_n induces v_n and j_n induces d^0. There is a unique natural transformation $\varphi_n\colon c_n \to j_n$ which induces the map π_n. We leave the rest to the reader. □

4.17. Given $s, q \geq 0$ we let $\Phi_{s,q}$ denote the composition of natural maps (4.10,4.9)

$$\Phi_{s,q}\colon \mathrm{tot}_s k^*(-) \xrightarrow{\sigma_s} j^*(-) \xrightarrow{d^0} \mathrm{tot}_q k^*(-).$$

Observe that the tower map

$$d^0 \cdot \sigma\colon \{\mathrm{tot}_s k^*(\mathbf{X})\}_s \to \{\mathrm{tot}_s k^*(\mathbf{X})\}_s$$

is in fact represented by the level map $\{\Phi_{s,s}(\mathbf{X})\}$.

We remark that $\Phi_{s,q}$ is natural in $\mathbf{X}$ as well as in s and q and that it is also coherent 3.6 (as a composition of coherent maps).

4.18. Proposition. *Let $\mathbf{X}$ be a trivial resolution, namely a Λ_+-diagram of spaces. There is a map of simplicial sets, natural in $\mathbf{X}$*

$$\mathrm{Map}(\Lambda^\bullet, k^*\mathrm{Lan}_k\Lambda^\bullet) \to \mathrm{map}(\{tot_s k^*\mathbf{X}\}_s, \{tot_s k^*\mathbf{X}\}_s)$$

whose image contain the vertices $\bar{id}$ and $d^0\sigma$ (see 4.10). In fact

(a) *The unit of adjunction $\eta\colon \Lambda^\bullet \to k^*\mathrm{Lan}_k\Lambda^\bullet$ is a preimage of $\bar{id}$ and*
(b) *A preimage Ψ for $d^0\sigma$ is given by*

$$\Psi^n\colon \Delta^n \xrightarrow{(s^0)^n} \Delta^0 \xrightarrow{\eta(0)} (k^*\mathrm{Lan}_k\Lambda^\bullet)(0) = (\mathrm{Lan}_k\Lambda^\bullet)(0) \xrightarrow{(d^0)^{n+1}s} (\mathrm{Lan}_k\Lambda^\bullet)(n).$$

Proof. Observe that (3.3)

$$\mathrm{tot}_n k^*(-) = \mathrm{hom}(\mathrm{sk}_n\Lambda^\bullet, k^*(-)) \cong \mathrm{hom}(\mathrm{Lan}_k\mathrm{sk}_n\Lambda^\bullet, -) \tag{4.18.1}$$

is a coherent functor (3.4,3.5). If $\mathbf{X}$ is a trivial resolution, then there is an obvious map

$$\mathbf{CNat}(\mathrm{tot}_p k^*(-), \mathrm{tot}_q k^*(-)) \to \mathrm{map}(\mathrm{tot}_p k^*(\mathbf{X}), \mathrm{tot}_q k^*(\mathbf{X})).$$

From 3.10, 3.3 and (4.18.1)

$$\text{(4.18.2)} \qquad \mathbf{CNat}(\mathrm{tot}_p k^*(-), \mathrm{tot}_q k^*(-)) \cong \mathrm{Map}(\mathrm{Lan}_k\, \mathrm{sk}_q\Lambda^\bullet, \mathrm{Lan}_k\, \mathrm{sk}_p\Lambda^\bullet) \cong \mathrm{Map}(\mathrm{sk}_q\Lambda^\bullet, k^*\mathrm{Lan}_k\, \mathrm{sk}_p\Lambda^\bullet).$$

We thus obtain natural maps (in $\mathcal{S}$)

$$\mathrm{Map}(\mathrm{sk}_q\Lambda^\bullet, k^*\mathrm{Lan}_k\, \mathrm{sk}_p\Lambda^\bullet) \to \mathrm{map}(\mathrm{tot}_p k^*(\mathbf{X}), \mathrm{tot}_q k^*(\mathbf{X})).$$

Now, sequential colimits (indexed by p) commute with $\mathrm{map}(\mathrm{sk}_q\Lambda^\bullet, -)$ (because $\mathrm{sk}_q\Lambda^\bullet$ is finite) and with k^* and Lan_k. Therefore, we obtain the desired map

$$\text{(4.18.3)} \qquad \nu\colon \mathrm{Map}(\Lambda^\bullet, k^*\mathrm{Lan}_k\Lambda^\bullet) \to \mathrm{map}(\{\mathrm{tot}_p k^*\mathbf{X}\}_p, \{\mathrm{tot}_q k^*\mathbf{X}\}_q).$$

Since id and $d^0\sigma = \{\Phi_{s,s}\}$ (4.17) are levelwise coherent maps, they correspond to vertices in the LHS of (4.18.3). In fact there is an effective way to compute these vertices by setting $p = q$, plugging $\mathrm{Lan}_k\mathrm{sk}_p\Lambda^\bullet$ into (4.18.2), computing the image under id and $d^0\sigma$ of the vertex

$$\eta \in \mathrm{tot}_p\, k^*\mathrm{Lan}_k\mathrm{sk}_p\Lambda^\bullet = \mathrm{Map}(\mathrm{sk}_p\Lambda^\bullet, k^*\mathrm{Lan}_k\, \mathrm{sk}_p\Lambda^\bullet)$$

where η is the unit of adjunction, and letting $p \to \infty$.

Evidently, id corresponds to the unit map $\eta : \mathrm{sk}_p\Lambda^\bullet \to k^*\mathrm{Lan}_k(\mathrm{sk}_p\Lambda^\bullet)$, and when p tends to ∞, we obtain the unit η of $\Lambda^\bullet$, which is point (a).

By the same method, the vertex $\Psi := d^0\sigma(\eta_{\mathrm{sk}_p\Lambda^\bullet})$ in the L.H.S of (4.18.3) corresponds to $d^0\sigma$. Using the explicit description of these maps (4.10), we see that Ψ^n is given by the composition

$$\Delta^n \xrightarrow{(s^0)^n} \Delta^0 \xrightarrow{\eta(0)} (k^*\mathrm{Lan}_k\mathrm{sk}_p\Lambda^\bullet)(0) = (\mathrm{Lan}_k\mathrm{sk}_p\Lambda^\bullet)(0) \xrightarrow{(d^0)^{n+1}s} (\mathrm{Lan}_k\mathrm{sk}_p\Lambda^\bullet)(n)$$

Letting $p \to \infty$ we obtain point (b). □

4.19. Proposition. *The vertices η and Ψ defined in 4.18 belong to the same component of* $\mathrm{Map}(\Lambda^\bullet, k^*\mathrm{Lan}_k\Lambda^\bullet)$. *In fact, they are the vertices of a* 1*-simplex in this space.*

Proof. We shall construct a 1-simplex h such that $\partial_0(h) = \eta$ and $\eta = \partial_1(h) = \Psi$. Curiously, the data for this homotopy is contained in η.

We first define maps τ_n for $n \geq 0$,

$$\text{(4.19.1)} \qquad \tau_n\colon \Delta^{n+1} \to (\mathrm{Lan}_k\Lambda^\bullet)(n)$$

by the composition

$$\tau_n\colon \Delta^{n+1} \xrightarrow{\eta(n+1)} (k^*\mathrm{Lan}_k\Lambda^\bullet)(n+1) = (\mathrm{Lan}_k\Lambda^\bullet)(n+1) \xrightarrow{(\mathrm{Lan}_k\Lambda^\bullet)(s)} (\mathrm{Lan}_k\Lambda^\bullet)(n).$$

Observe that

$$\text{(4.19.2)} \qquad \tau_n \cdot d^0 = \eta(n)$$

by the commutativity of the following diagram (use 4.8; the bottom row is τ_n)

$$\begin{array}{ccccccc}
\Delta^n & \xrightarrow{\eta(n)} & (k^*\mathrm{Lan}_k\Lambda^\bullet)(n) & = & (\mathrm{Lan}_k\Lambda^\bullet)(n) & \xrightarrow{\mathrm{id}} & \\
\downarrow{\scriptstyle d^0} & & \downarrow{\scriptstyle (k^*\mathrm{Lan}_k\Lambda^\bullet)(d^0)} & & \downarrow{\scriptstyle (\mathrm{Lan}_k\Lambda^\bullet)(d^0)} & & \\
\Delta^{n+1} & \xrightarrow[\eta(n+1)]{} & (k^*\mathrm{Lan}_k\Lambda^\bullet)(n+1) & = & (\mathrm{Lan}_k\Lambda^\bullet)(n+1) & \xrightarrow[(\mathrm{Lan}_k\Lambda^\bullet)(s)]{} & (\mathrm{Lan}_k\Lambda^\bullet)(n).
\end{array}$$

Furthermore,

(4.19.3)
$$\tau_n \cdot v_n = \Psi^n$$

due to the commutativity of (again use 4.8)

$$\begin{array}{ccccccc}
\Delta^0 & \xrightarrow{\eta(0)} & (k^*\mathrm{Lan}_k\Lambda^\bullet)(0) & = & (\mathrm{Lan}_k\Lambda^\bullet)(0) & \searrow^{(\mathrm{Lan}_k\Lambda^\bullet)(d^n\ldots d^0 s)} & \\
{\scriptstyle d^{n+1}\ldots d^1}\downarrow & & \downarrow{\scriptstyle d^{n+1}\ldots d^1} & & \downarrow{\scriptstyle (\mathrm{Lan}_k\Lambda^\bullet)(d^{n+1}\ldots d^1)} & & \\
\Delta^{n+1} & \xrightarrow[\eta(n+1)]{} & (k^*\mathrm{Lan}_k\Lambda^\bullet)(n+1) & = & (\mathrm{Lan}_k\Lambda^\bullet)(n+1) & \xrightarrow[(\mathrm{Lan}_k\Lambda^\bullet)(s)]{} & (\mathrm{Lan}_k\Lambda^\bullet)(n).
\end{array}$$

We now define the 1-simplex $h = \{h^n\}$ in $\mathrm{Map}(\Lambda^\bullet, k^*\mathrm{Lan}_k\,\Lambda^\bullet)$ by the composition

$$h^n \colon \Delta^n \times \Delta^1 \xrightarrow{\pi_n} \Delta^{n+1} \xrightarrow{\tau_n} (\mathrm{Lan}_k\Lambda^\bullet)(n).$$

First, we claim that $\{h^n\}$ is indeed a cofacial space map (namely, an 1-simplex). This follows from the commutativity of the following diagram for every $n \geq 0$ and every $0 \leq i \leq n+1$.

$$\begin{CD}
\Delta^n \times \Delta^1 @>{\pi_n}>> \Delta^{n+1} @>{\tau_n}>> (\mathrm{Lan}_k\Lambda^\bullet)(n) \\
@V{d^i \times 1}VV @VV{d^{i+1}}V @VV{(\mathrm{Lan}_k\Lambda^\bullet)(d^i)}V \\
\Delta^{n+1} \times \Delta^1 @>>{\pi_{n+1}}> \Delta^{n+2} @>>{\tau_{n+1}}> (\mathrm{Lan}_k\Lambda^\bullet)(n+1).
\end{CD}$$

The left square commutes by 4.16 while the right one by

$$\begin{CD}
\Delta^{n+1} @>{\eta(n+1)}>> (\mathrm{Lan}_k\Lambda^\bullet)(n+1) @>{(\mathrm{Lan}_k\Lambda^\bullet)(s)}>> (\mathrm{Lan}_k\Lambda^\bullet)(n) \\
@V{d^{i+1}}VV @V{(\mathrm{Lan}_k\Lambda^\bullet)(d^{i+1})}VV @VV{(\mathrm{Lan}_k\Lambda^\bullet)(d^0)}V \\
\Delta^{n+2} @>>{\eta(n+2)}> (\mathrm{Lan}_k\Lambda^\bullet)(n+2) @>{(\mathrm{Lan}_k\Lambda^\bullet)(s)}>> (\mathrm{Lan}_k\Lambda^\bullet)(n+1)
\end{CD}$$

which commutes by the identities in (4.8.1).

It follows that $\{h^n\}$ is a 1-simplex in $\mathrm{Map}(\Lambda^\bullet, k^*\mathrm{Lan}_k\Lambda^\bullet)$ and moreover, using 4.16 and (4.19.2)

$$\partial_0(h^n) = h^n \cdot (1 \times d^0) = \tau_n\pi_n(1 \times d^0) = \tau_n d^0 = \eta(n).$$

Using (4.19.3)

$$\partial_1(h^n) = h^n \cdot (1 \times d^1) = \tau_n\pi_n(1 \times d^1) = \tau_n v_n = \Psi^n.$$

□

Proof of Theorem 4.11. First, notice that, since $\mathbf{X}$ is fibrant and $\mathrm{sk}_p\Lambda^\bullet$ are all free diagrams then the tower $\{\mathrm{tot}_s\mathbf{X}\}_s$ is fibrant (2.6).

Since $\sigma d^0 = \mathrm{id}$ (4.14), it suffices to show, by 2.11, that id and $d^0\sigma$ belong to the same component of $\mathrm{map}(\mathrm{tot}_\bullet k^*\mathbf{X}, \mathrm{tot}_\bullet k^*\mathbf{X})$. Using 4.18 it suffices to show that η and Ψ belong to the same component of $\mathrm{Map}(\Lambda^\bullet, k^*\mathrm{Lan}_k\Lambda^\bullet)$, which is precisely the assertion of 4.19. □

5. Resolutions of diagrams

5.1. Definition. Let D be a small category. A *cofacial D-diagram* $X^\bullet$ is a cofacial object in the functor category $\mathcal{S}\text{pc}^D$, namely a functor $X^\bullet\colon\Lambda \to \mathcal{S}\text{pc}^D$. A *cofacial resolution*, or simply a *resolution*, of D-diagrams is a functor $\mathbf{X} : \Lambda_0 \to \mathcal{S}\text{pc}^D$. Compare this with 4.3,4.5. This data is the same as a functor $\mathbf{X}\colon D \times \Lambda_0 \to \mathcal{S}\text{pc}$, and we will usually prefer this description.

As before $X^\bullet, Y^\bullet, \dots$ denote cofacial D-diagrams (D should be understood from the context) and boldface letters $\mathbf{X}, \mathbf{Y}, \dots$ denote resolutions of D-diagrams.

5.2. Definition. We call a resolution $\mathbf{X}$ of D-diagrams *termwise trivial*, if for every object d in D, the resolution of spaces $\mathbf{X}(d)$ is trivial (4.7).

Observe that the trivialisations for every $\mathbf{X}(d)$ are *not* assumed to be compatible, namely, a termwise trivial resolution $\mathbf{X}$ is *not* a diagram $\mathbf{X}\colon \Lambda_+ \to \mathcal{S}\text{pc}^D$ and, in general, cannot be turned into one by a different choice of the trivialisations.

5.3. Given a cofacial resolution $\mathbf{X}$ as in 5.1, and a diagram $F : D \to \mathcal{S}$, we will consider the obvious cofacial resolution $\hom_D(F, \mathbf{X})$ which in codimension $i \geq -1$ is $\hom_D(F, X^i)$.

5.4. Definition. Let D be a small category. A diagram $Y : D \to \mathcal{S}$ is called a *relative free extension* of a diagram X if there is a family $\{d_\alpha\}_{\alpha\in\mathcal{A}}$ of objects in D and a family $\{A_\alpha \to B_\alpha\}_{\alpha\in\mathcal{A}}$ of cofibration in $\mathcal{S}$ which fit into the following pushout square

$$\begin{array}{ccc} \coprod_{\alpha\in\mathcal{A}} A_\alpha \times D(d_\alpha, -) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \coprod_{\alpha\in\mathcal{A}} B_\alpha \times D(d_\alpha, -) & \longrightarrow & Y \end{array}$$

Here $D(d_\alpha, -)$ is the free diagram at d_α (cf. [4]) considered as a diagram of discrete spaces.

Observe that the vertical arrows are cofibrations.

5.5. Definition. Let D be a small category. A diagram $X : D \to \mathcal{S}$ is called *poly-free* if

$$X = X_n \supseteq X_{n-1} \supseteq \dots \supseteq X_0 = \emptyset$$

where $X_i \subseteq X_{i+1}$ is a relative free extension (5.4).

Clearly X is cofibrant.

5.6. Theorem. *Let D be a small category and $F : D \to \mathcal{S}$ a poly-free diagram. Then for every termwise trivial (objectwise) fibrant resolution $\mathbf{X}$ of D-diagrams, the tower map*

$$\hom_D(F, X^{-1}) \to \{tot_s \hom_D(F, X^\bullet)\}_{s\geq 0}$$

is a s.p.h.e. In particular there exists $s < \infty$ such that $\hom_D(F, X^{-1})$ is up to homotopy a retract of $tot_s \hom_D(F, X^\bullet)$.

Proof. By definition there exists a filtration

$$F = F_n \supseteq F_{n-1} \supseteq \dots \supseteq F_0 = \emptyset$$

where $F_i \subseteq F_{i+1}$ is a free relative extension (5.4). We prove the theorem inductively for the F_i's.

To start with, the case $i = 0$ is trivial because $\hom_D(\emptyset, -) = *$. By definition (5.4) of relative free extension we obtain a pullback square of fibrant resolutions

$$\begin{array}{ccc} \hom_D(F_{i+1}, \mathbf{X}) & \longrightarrow & \prod_{\alpha\in\mathcal{A}} \hom_D(B_\alpha, X(d_\alpha)) \\ \downarrow & & \downarrow \\ \hom_D(F_i, \mathbf{X}) & \longrightarrow & \prod_{\alpha\in\mathcal{A}} \hom_D(A_\alpha, X(d_\alpha)) \end{array}$$

where the vertical arrows are fibrations. Since $\mathbf{X}$ is termwise trivial, then the resolutions of spaces in the right corners of the square are also trivial, so by Theorem 4.11 they induce a s.p.h.e. upon taking the $\mathrm{tot}_\bullet$ tower. By induction hypothesis the same happens with the resolution in the bottom left corner of the square. By 2.14 the induction step is complete. □

5.7. Remark. The same proof works for diagrams F which are elements in the smallest class of diagrams which contain the diagrams $\coprod_{i\in I} A_i \times D(d_i, -)$ and is closed under pushouts along cofibrations.

5.8. Corollary. *Suppose that $F \to F'$ is a cofibration between poly-free diagrams over a small category D. Let $\mathbf{X}$ be a termwise trivial fibrant resolution of* pointed *spaces. Then*

$$\mathrm{Map}_D(F'/F, X^{-1}) \to \{tot_s \mathrm{Map}_D(F'/F, X^\bullet)\}_{s\geq 0}$$

is a s.p.h.e. where F'/F is pointed in the obvious way.

Proof. Use the Theorem 5.6, 2.17(a) and the fibre sequence

$$\mathrm{Map}(F'/F, \mathbf{X}) \to \hom_D(F', \mathbf{X}) \to \hom_D(F, X).$$

□

5.9. Examples of poly-free diagrams include

(a) Classifying spaces $ED = D/-$ (see [2, pp. 292]) of finite dimensional categories D, i.e. ones whose nerve is finite dimensional.

(b) $\mathbb{R}$ with the usual $\mathbb{Z}$ action. More generally, if G is a group which has a finite dimensional classifying space, then its universal cover is an example of a poly-free diagram.

For (a) we can do even better:

5.10. Suppose that D is a finite dimensional small category, namely the nerve of D has dimension $n < \infty$. Then D is obviously a *direct* category. More explicitly, define a height function $|\cdot|\colon \mathrm{Obj}(D) \to \mathbb{N}$ by the assignment (cf. [2, pp. 292])

$$|d| = \dim N(D/d).$$

Then obviously, if $d \xrightarrow{\varphi} e$ is a morphism in D, then $h(d) \leq h(e)$, and equality holds if and only if $\varphi = \mathrm{id}$.

5.11. Theorem. *Let D be a finite dimensional small category. If $\mathbf{X}$ is a termwise trivial fibrant resolution of D-diagrams, then for every cofibrant diagram $F\colon D \to \mathcal{S}$, the tower map*

$$\hom_D(F, X^{-1}) \to \{tot_s \hom_D(F, X^\bullet)\}_s$$

is a s.p.h.e. In particular, there exists $s < \infty$ such that the natural map

$$\hom_D(F, X^{-1}) \to tot_s \hom_D(F, X^\bullet)$$

admits a left homotopy inverse, and $\mathrm{holim}_D X^{-1}$ *is, up to homotopy, a retract of* $tot_s \mathrm{holim}_D X^\bullet$ *for some* $s < \infty$.

Proof. Let D^k denote the full subcategory of D consisting of the objects d, such that $|d| \leq k$, and let $j: D^k \to D$ denote the inclusion. By inspection, the natural map $\mathrm{Lan}_j j^*(F) \to F$ is injective (for this use the description of left Kan extensions in [16, pp. 236], and the definition of free diagrams in [4] or [7]). Let F^k denote its image. It is easy to see that F^k is the subdiagram of F generated by the simplices of $F(d)$ for all objects d such that $|d| \leq k$.

Our proof proceeds by induction to show that for all $k \geq -1$

$$\mathrm{hom}_D(F^k, X^{-1}) \to \{\mathrm{tot}_s \mathrm{hom}_D(F^k, X^\bullet))\}_s \tag{5.11.1}$$

is a s.p.h.e. The conclusion of the theorem is obtained from $k = \dim(D)$.

The case $k = -1$ is trivial as $F^k = \emptyset$. For the induction step we consider the following commutative diagram in $\mathcal{S}\mathrm{pc}^{\Lambda_0}$.

$$\begin{array}{ccc} \mathrm{hom}_D(F^k, \mathbf{X}) & \longrightarrow & \mathrm{hom}_D(F^{k-1}, \mathbf{X}) \\ \downarrow & & \downarrow \\ \prod_{|d|=k} \mathrm{hom}(F(d), \mathbf{X}(d)) & \longrightarrow & \prod_{|d|=k} \mathrm{hom}(F^{k-1}(d), \mathbf{X}(d)) \end{array} \tag{5.11.2}$$

This square is cartesian because D is direct and $\mathrm{hom}_D(F^k, \mathbf{X}) \cong \mathrm{hom}_{D^k}(j^*F, j^*X)$. All the resolutions in the vertices of the squares are fibrant because $\mathbf{X}$ is fibrant and F^j are cofibrant for all j. Moreover, the horizontal maps are clearly (termwise) fibrations because $F^{k-1}(d) \to F^k(d)$ are cofibrations for all $d \in D$. Now use the induction hypothesis for the top right vertex of the square, and Theorem 4.11 for the resolutions at the bottom vertices, together with 2.14. □

6. Applications to simplicial functors and polyGEMs

6.1. Recall that a coaugmented functor J on spaces (pointed or not) is one with a natural map $X \to JX$ for every space X. Coaugmented functors J give rise to cofacial resolutions

$$X \to J^\bullet X$$

for every space X, in the usual way (cf. [14, Section 2]).

We say that a space X is J-injective if the map $X \to JX$ admits a left homotopy inverse. Our main example is when J is the Bousfield Kan [2] functor $R: \mathcal{S} \to \mathcal{S}$ where R is a unitary ring. Then the R-injective spaces are those which are homotopy equivalent to the product of Eilenberg-MacLane spaces of R-modules, known as GEMs (see [3, p. 87]).

For a J-injective space, the resolution $X \to J^\bullet X$ is easily seen to be trivial.

6.2. We remark that by a simple manipulation which does not alter the homotopy types, J can be assumed to satisfy
(a) the coaugmentation $X \to JX$ is a cofibration
(b) JX is fibrant (Kan complex) for all X. This will be assumed throughout. In that case, the coaugmentation $X \to JX$ admits a left homotopy inverse if and only if it admits a strict left inverse.

6.3. We say that J is simplicial if for every $A \in \mathcal{S}$ there is a natural map

$$(JX) \otimes A \to J(X \otimes A).$$

It is easy to show that a continuous coaugmented functor J, namely one which induces a continuous map for all spaces X and Y

$$\mathrm{Map}(X, Y) \to \mathrm{Map}(JX, JY)$$

is simplicial.

The force of simplicial functors is in the existence of a natural map of resolutions (compare [15])

$$J^\bullet \mathrm{hom}_D(F, X) \xrightarrow{c} \mathrm{hom}_D(F, J^\bullet X).$$

So in particular the composition

$$\mathrm{hom}_D(F, X) \to J^\bullet \mathrm{hom}_D(F, X) \xrightarrow{c} \mathrm{hom}_D(F, J^\bullet X)$$

is induced by $X \to J^\bullet X$.

6.4. Theorem. *Let D be a small category and assume that either*

(a) *$F : D \to \mathcal{S}$ is poly-free (5.5) or*

(b) *D is finite dimensional and $F : D \to \mathcal{S}$ is cofibrant.*

If $X : D \to \mathcal{S}\mathrm{pc}$ is a diagram of fibrant J-injective spaces, then

$$\mathrm{hom}_D(F, X) \to J_s \mathrm{hom}_D(F, X)$$

admits a left homotopy inverse for some $s < \infty$. In particular, when F is chosen weakly contractible (always possible in case (b)), then it follows that $\mathrm{holim}_D X$ is up to homotopy a retract of $J_s \mathrm{holim}_D X$.

Proof. The resolution $X \to J^\bullet X$ is fibrant and termwise trivial (6.1). The result follows from Theorems 5.11, 5.6 using the composition (see 6.3)

$$\mathrm{hom}_D(F, X) \to J_s \mathrm{hom}_D(F, X) = \mathrm{tot}_s J^\bullet \mathrm{hom}_D(F, X) \xrightarrow{c} \mathrm{tot}_s \mathrm{hom}_D(F, J^\bullet X).$$

□

Recall from the introduction the concepts of GEMs and polyGEMs. Recall also that X is a GEM if and only if it is up to homotopy $\mathbb{Z}$-injective where $\mathbb{Z}$ is the Bousfield Kan functor [2].

6.5. Corollary. *Assume that D and F are as in 6.4. If $X : D \to \mathcal{S}\mathrm{pc}$ is diagram of fibrant GEMs, then $\mathrm{hom}_D(F, X)$ is a polyGEM and is up to homotopy a retract of $\mathbb{Z}_s \mathrm{hom}_D(F, X)$ for some $s < \infty$.*

Proof. The first assertion is implicit in the proofs of 5.6 and 5.11. The second follows from 6.4 and the remarks in 6.2. □

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