The Burnside Ring and Equivariant Cohomotopy for Infinite Groups

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April 4, 2005

Abstract

After we have given a survey on the Burnside ring of a finite group, we discuss and analyze various extensions of this notion to infinite (discrete) groups. The first three are the finite-G-set-version, the inverse-limit-version and the covariant Burnside group. The most sophisticated one is the fourth definition as the equivariant zero-th cohomotopy of the classifying space for proper actions. In order to make sense of this definition we define equivariant cohomotopy groups of finite proper equivariant CW-complexes in terms of maps between the sphere bundles associated to equivariant vector bundles. We show that this yields an equivariant cohomology theory with a multiplicative structure. We formulate a version of the Segal Conjecture for infinite groups. All this is analogous and related to the question what are the possible extensions of the notion of the representation ring of a finite group to an infinite group. Here possible candidates are projective class groups, Swan groups and the equivariant topological K-theory of the classifying space for proper actions.

Key words: Burnside ring, equivariant cohomotopy, infinite groups.
Mathematics Subject Classification 2000: 55P91, 19A22.

0 Introduction

The basic notions of the Burnside ring and of stable equivariant cohomotopy have been defined and investigated in detail for finite groups. The purpose of this article is to discuss how these can be generalized to infinite (discrete) groups.

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The guideline will be the related notion of the representation ring which allows several generalizations to infinite groups, each of which reflects one aspect of the original notion for finite groups. Analogously we will present several possible generalizations of the Burnside ring for finite groups to infinite (discrete) groups. There seems to be no general answer to the question which generalization is the right one. The answer depends on the choice of the background problem such as universal additive properties, induction theory, equivariant stable homotopy theory, representation theory, completion theorems and so on. For finite groups the representation ring and the Burnside ring are related to all these topics simultaneously and for infinite groups the notion seems to split up into different ones which fall together for finite groups but not in general.

The following table summarizes in the first column the possible generalizations to infinite groups of the representation ring $R_F(G)$ with coefficients in a field F of characteristic zero. In the second column we list the analogous generalizations for the Burnside ring. In the third column we give key words for its main property, relevance or application. Explanations will follow in the main body of the text.

$R_F(G)$	$A(G)$	key words
$K_0(FG)$	$\underline{A}(G)$	universal additive invariant, equivariant Euler characteristic
$\mathrm{Sw}^f(G;F)$	$\overline{A}(G)$	induction theory, Green functors
$R_{\mathrm{cov},F}(G) :=$ $\mathrm{colim}_{H \in \mathrm{Sub}_{\mathcal{FIN}}(G)} R_F(H)$	$A_{\mathrm{cov}}(G) :=$ $\mathrm{colim}_{H \in \mathrm{Sub}_{\mathcal{FIN}}(G)} A(H)$	collecting all values for finite subgroups with respect to induction
$R_{\mathrm{inv},F}(G) :=$ $\mathrm{invlim}_{H \in \mathrm{Sub}_{\mathcal{FIN}}(G)} R_F(H)$	$A_{\mathrm{inv}}(G) :=$ $\mathrm{invlim}_{H \in \mathrm{Sub}_{\mathcal{FIN}}(G)} A(H)$	collecting all values for finite subgroups with respect to restriction
$K_G^0(\underline{E}G)$	$A_{\mathrm{ho}}(G) := \pi_G^0(\underline{E}G)$	completion theorems, equivariant vector bundles
$K_0^G(\underline{E}G)$	$\pi_0^G(\underline{E}G)$	representation theory, Baum-Connes Conjecture, equivariant homotopy theory,

The various rings are linked by the following diagram of ring homomorphisms

$$\begin{tikzcd}
A_{\mathrm{ho}}(G) = \pi^0_G(\underline{E}G) \arrow[r, "\mathrm{edge}^G"] \arrow[dd, "\psi^0_G(\underline{E}G)"] & A_{\mathrm{inv}}(G) \arrow[d, "P^G_{\mathrm{inv}}"] & \overline{A}(G) \arrow[l, "T^G"'] \arrow[d, "\overline{P}^G"] \\
 & R_{\mathrm{inv},\mathbb{Q}}(G) \arrow[d, "c"] & \mathrm{Sw}^f(G;\mathbb{Q}) \arrow[l, "S^{G,\mathbb{Q}}"] \arrow[d, "c"] \\
K^0_G(\underline{E}G) \arrow[r, "\mathrm{edge}^G"] & R_{\mathrm{inv},\mathbb{C}}(G) & \mathrm{Sw}^f(G;\mathbb{C}) \arrow[l, "S^{G,\mathbb{C}}"]
\end{tikzcd}$$

where c denotes the obvious change of coefficients homomorphisms and the other maps will be explained later.

We will also define various pairings which are summarized in the following diagram which reflects their compatibilities.

$$\begin{tikzcd}
\overline{A}(G) \times \underline{A}(G) \arrow[rr, "\mu^G_A"] \arrow[d, "\overline{P}^G \times \underline{P}^G"'] \arrow[drr, "T^G \times \mathrm{id}"] & & \underline{A}(G) \arrow[d, "\underline{P}^G"] \arrow[drr, "Q^G_A"] & & \\
\mathrm{Sw}^f(G;\mathbb{Q}) \times K_0(\mathbb{Q}G) \arrow[rr, "\mu^G_K"] \arrow[ddr, shift left, "S^{G,\mathbb{Q}}"] & & K_0(\mathbb{Q}G) \arrow[ddrr, "Q^G_K"] & & \\
 & & A_{\mathrm{inv}}(G) \times \underline{A}(G) \arrow[rr, "\nu^G_A"] \arrow[d, "P^G_{\mathrm{inv}} \times (P^G_{\mathrm{cov}} \circ (V^G)^{-1})"] & & \mathbb{Z} \arrow[d, "\mathrm{id}"] \\
 & R_{\mathrm{inv},\mathbb{Q}}(G) \times R_{\mathrm{cov},\mathbb{Q}}(G) \arrow[uul, shift left, "W^{G,\mathbb{Q}}"] \arrow[rrr, "\nu^G_R"] & & & \mathbb{Z}
\end{tikzcd} \tag{0.1}$$

In Section 1 we give a brief survey about the Burnside ring $A(G)$ of a finite group G in order to motivate the generalizations. In Sections 2, 3 and 4 we treat the finite-G-set-version of the Burnside Ring $\overline{A}(G)$, the inverse-limit-version of the Burnside ring $A_{\mathrm{inv}}(G)$ and the covariant Burnside group $\underline{A}(G)$. These definitions are rather straightforward. The most sophisticated version of the Burnside ring for infinite groups is the equivariant zero-th cohomotopy $\pi^0_G(\underline{E}G)$ of the classifying space $\underline{E}G$ for proper G-actions. It will be constructed in Section 6 after we have explained the notion of an equivariant cohomology theory with multiplicative structure in Section 5. One of the main result of this paper is

Theorem 6.5 *Equivariant Cohomotopy $\pi^*_?$ defines an equivariant cohomology theory with multiplicative structure for finite proper equivariant CW-complexes. For every finite subgroup H of the group G the abelian groups $\pi^n_G(G/H)$ and π^n_H are isomorphic for every $n \in \mathbb{Z}$ and the rings $\pi^0_G(G/H)$ and $\pi^0_H = A(H)$ are isomorphic.*

An important test in the future will be whether the version of the Segal Conjecture for infinite groups discussed in Section 8 is true.

The papers is organized as follows:

1 Review of the Burnside Ring for Finite Groups

In this section we give a brief review of the definition, properties and applications of the Burnside ring for finite groups in order to motivate our definitions for infinite groups.

Definition 1.1. (Burnside ring of a finite group). *The isomorphism classes of finite G-sets form a commutative associative semi-ring with unit under the disjoint union and the cartesian product. The* Burnside ring $A(G)$ *is the Grothendieck ring associated to this semi-ring.*

As abelian group the Burnside ring $A(G)$ is the free abelian group with the set $\{G/H \mid (H) \in \operatorname{ccs}(G)\}$ as basis, where $\operatorname{ccs}(G)$ denotes the set of conjugacy classes of subgroups of G. The zero element is represented by the empty set, the unit is represented by G/G. The interesting feature of the Burnside ring is its multiplicative structure.

Given a group homomorphism $f\colon G_0 \to G_1$ of finite groups, restriction with f defines a ring homomorphism $f^*\colon A(G_1) \to A(G_0)$. Thus $A(G)$ becomes a contravariant functor from the category of finite groups to the category of commutative rings. Induction defines a homomorphism of abelian groups $f_*\colon A(G_0) \to A(G_1)$, $[S] \mapsto [G_1 \times_f S]$, which is not compatible with the multiplication. Thus $A(G)$ becomes a becomes a covariant functor from the category of finite groups to the category of abelian groups.

1.1 The Character Map and the Burnside Ring Congruences

Let G be a finite group. Let $\operatorname{ccs}(G)$ be *the set of conjugacy classes (H) of subgroups $H \subseteq G$.* Define the character map

$$\operatorname{char}^G\colon A(G) \quad \to \quad \prod_{(H)\in\operatorname{ccs}(G)} \mathbb{Z} \tag{1.2}$$

by sending the class of a finite G-set S to the numbers $\{|S^H| \mid (H) \in \operatorname{ccs}(G)\}$. This is an injective ring homomorphism whose image can be described by the so called *Burnside ring congruences* which we explain next.

In the sequel we denote for a subgroup $H \subseteq G$ by $N_G H$ its *normalizer* $\{g \in G \mid g^{-1}Hg = H\}$, by $C_G H = \{g \in G \mid gh = hg \text{ for } h \in H\}$ its *centralizer*, by $W_G H$ its *Weyl group* $N_G H/H$ and by $[G : H]$ its *index*. Let $p_H : N_G H \to W_G H$ be the canonical projection. Denote for a cyclic group C by $\mathrm{Gen}(C)$ the set of its generators. We conclude from [46, Proposition 1.3.5]

Lemma 1.3. *An element $\{x(H)\} \in \prod_{(H)\in \mathrm{ccs}(G)} \mathbb{Z}$ lies in the image of the injective character map char^G defined in* (1.2) *if and only if we have for every $(H) \in \mathrm{ccs}(G)$*

$$\sum_{\substack{(C)\in \mathrm{ccs}(W_G H) \\ C \text{ cyclic}}} |\operatorname{Gen}(C)| \cdot [W_G H : N_{W_G H} C] \cdot x(p_H^{-1}(C)) \;\equiv\; 0 \qquad \mod |W_G H|.$$

Example 1.4 ($A(\mathbb{Z}/p)$)**.** Let p be a prime and let G be the cyclic group $\mathbb{Z}/p$ of order p. Then $A(G)$ is the free abelian group generated by $[G]$ and $[G/G]$. The multiplication is determined by the fact that $[G/G]$ is the unit and $[G] \cdot [G] = p \cdot [G]$. There is exactly one non-trivial Burnside ring congruence, namely the one for $H = \{1\}$ which is in the notation of Lemma 1.3

$$x(1) \;\equiv\; x(G) \mod p.$$

1.2 The Equivariant Euler Characteristic

Next we recall the notion of a G-CW-complex.

Definition 1.5 (G-CW-complex)**.** *Let G be a group. A G-CW-*complex *X is a G-space together with a G-invariant filtration*

$$\emptyset = X_{-1} \subseteq X_0 \subset X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots \subseteq \bigcup_{n\geq 0} X_n = X$$

such that X carries the colimit topology with respect to this filtration (i.e. a set $C \subseteq X$ is closed if and only if $C \cap X_n$ is closed in X_n for all $n \geq 0$) and X_n is obtained from X_{n-1} for each $n \geq 0$ by attaching equivariant n-dimensional cells, i.e. there exists a G-pushout

$$\begin{array}{ccc} \coprod_{i\in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i\in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i\in I_n} G/H_i \times D^n & \xrightarrow[\coprod_{i\in I_n} Q_i^n]{} & X_n \end{array}$$

A G-CW-complex X is called *finite* if it is built by finitely many equivariant cells $G/H \times D^n$ and is called *cocompact* if $G\backslash X$ is compact. The conditions finite and cocompact are equivalent for a G-CW-complex. Provided that G is finite, X is compact if and only if X is cocompact A G-map $f\colon X \to Y$ of G-CW-complexes is called *cellular* if $f(X_n) \subseteq Y_n$ holds for all n.

Definition 1.6 (Equivariant Euler Characteristic). *Let G be a finite group and X be a finite G-CW-complex. Define its* equivariant Euler characteristic

$$\chi^G(X) \ \in A(G)$$

by

$$\chi^G(X) := \sum_{n=0}^{\infty}(-1)^n \cdot \sum_{i \in I_n}[G/H_i]$$

after choices of the G-pushouts as in Definition 1.5.

This definition is independent of the choice of the G-pushouts by the next result. The elementary proofs of the next two results are left to the reader. We denote by X^H and $X^{>H}$ respectively the subspace of X consisting of elements $x \in X$ whose isotropy group G_x satisfies $H \subseteq G_x$ and $H \subsetneq G_x$ respectively.

Lemma 1.7. *Let G be a finite group.*

(i) Let X be a finite G-CW-complex. Then

$$\chi^G(X) \ = \sum_{(H) \in \operatorname{ccs}(G)} \chi\left(W_H G \backslash (X^H, X^{>H})\right) \cdot [G/H],$$

where χ denotes the classical (non-equivariant) Euler characteristic;

(ii) If X and Y are G-homotopy equivalent finite G-CW-complexes, then

$$\chi^G(X) = \chi^G(Y);$$

(iii) If

$$\begin{array}{ccc} X_0 & \xrightarrow{i_0} & X_1 \\ {\scriptstyle i_2}\big\downarrow & & \big\downarrow \\ X_2 & \longrightarrow & X \end{array}$$

is a G-pushout of finite G-CW-complexes such that i_1 is an inclusion of finite G-CW-complexes and i_2 is cellular, then

$$\chi^G(X) = \chi^G(X_1) + \chi^G(X_1) - \chi^G(X_0);$$

(iv) If X and Y are finite G-CW-complexes, then $X \times Y$ with the diagonal G-action is a finite G-CW-complex and

$$\chi^G(X \times Y) \ = \ \chi^G(X) \cdot \chi^G(Y);$$

(v) The image of $\chi^G(X)$ under the character map char^G of (1.2) is given by the collection of classical (non-equivariant) Euler characteristics $\{\chi(X^H) \mid (H) \in \operatorname{ccs}(G)\}$.

An *equivariant additive invariant* for finite G-CW-complexes is a pair (A,a) consisting of an abelian group and an assignment a which associates to every finite G-CW-complex X an element $a(X) \in A$ such that $a(\emptyset) = 0$, G-homotopy invariance and Additivity holds, i.e. the obvious versions of assertions (ii) and (iii) appearing in Lemma 1.7 are true. An equivariant additive invariant (U,u) is called *universal* if for every equivariant additive invariant (A,a) there is precisely one homomorphism of abelian groups $\phi\colon U \to A$ such that $\phi(u(X)) = a(X)$ holds for every finite G-CW-complex X. Obviously (U,u) is (up to unique isomorphism) unique if it exists.

Theorem 1.8 (The universal equivariant additive invariant). *Let G be a finite group. The pair $(A(G),\chi^G)$ is the universal equivariant additive invariant for finite G-CW-complexes.*

1.3 The Equivariant Lefschetz Class

The notion of an equivariant Euler characteristic can be extended to the notion of an equivariant Lefschetz class as follows.

Definition 1.9. *Let G be a finite group and X be a finite G-CW-complex. We define the* equivariant Lefschetz class *of a cellular G-selfmap $f\colon X \to X$*

$$\Lambda^G(f) \quad \in \quad A(G)$$

by

$$\Lambda^G(f) \;=\; \sum_{(H)\in \mathrm{ccs}(G)} \Lambda\left(W_GH\backslash(f^H, f^{>H})\right)\cdot [G/H],$$

where $\Lambda(W_GH\backslash(f^H,f^{>H})) \in \mathbb{Z}$ is the classical Lefschetz number of the endomorphism $W_GH\backslash(f^H,f^{>H})$ of the pair of finite CW-complexes $W_GH\backslash(X^H,X^{>H})$ induced by f.

Obviously $\Lambda^G(\mathrm{id}\colon X\to X)$ agrees with $\chi^G(X)$. The elementary proof of the next result is left to the reader.

Lemma 1.10. *Let G be a finite group.*

(i) If f and g are G-homotopic G-selfmaps of a finite GW-CW-complex X, then

$$\Lambda^G(f) = \Lambda^G(g);$$

(ii) Let

$$\begin{array}{ccc} X_0 & \xrightarrow{i_0} & X_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow \\ X_2 & \longrightarrow & X \end{array}$$

is a G-pushout of finite G-CW-complexes such that i_1 is an inclusion of finite G-CW-complexes and i_2 is cellular. Let $f_i\colon X_i \to X_i$ for $i = 0,1,2$ and $f\colon X\to X$ be the G-selfmaps compatible with this G-pushout. Then

$$\Lambda^G(f) = \Lambda^G(f_1) + \Lambda^G(f_2) - \Lambda^G(f_0);$$

(iii) *Let X and Y be finite G-CW-complexes and $f\colon X \to X$ and $g\colon Y \to Y$ be G-selfmaps. Then*

$$\Lambda^G(f \times g) \;=\; \chi^G(X) \cdot \Lambda^G(g) + \chi^G(Y) \cdot \Lambda^G(f);$$

(iv) *Let $f\colon X \to Y$ and $g\colon Y \to X$ be G-maps of finite G-CW-complexes. Then*

$$\Lambda^G(f \circ g) \;=\; \Lambda^G(g \circ f);$$

(v) *The image of $\Lambda^G(f)$ under the character map* char^G *of* (1.2) *is given by the collection of classical (non-equivariant) Lefschetz numbers $\{\Lambda(f^H) \mid (H) \in \mathrm{ccs}(G)\}$.*

One can also give a universal property characterizing the equivariant Lefschetz class (see [22]).

The equivariant Lefschetz class has also the following homotopy theoretic meaning.

Definition 1.11. *A G-*homotopy representation *X is a finite-dimensional G-CW-complex such that for each subgroup $H \subseteq G$ the fixed point set X^H is homotopy equivalent to a sphere $S^{n(H)}$ for $n(H)$ the dimension of the CW-complex X^H.*

An example is the unit sphere SV in an orthogonal representation V of G. Denote by $[X,X]^G$ the set of G-homotopy classes of G-maps $X \to X$. The proof of the next theorem can be found in [24, Theorem 3.4 on page 139] and is a consequence of the equivariant Hopf Theorem (see for instance [46, page 213],[49, II.4], [21]).

Theorem 1.12. *Let X be a G-homotopy representation of the finite group G. Suppose that*

(i) *Every subgroup $H \subseteq G$ occurs as isotropy group of X;*

(ii) $\dim(X^G) \geq 1$*;*

(iii) *The group G is nilpotent or for every subgroup $H \subseteq G$ we have $\dim(X^{>H}) + 2 \leq \dim(X^H)$.*

Then the following map is an bijection of monoids, where the monoid structure on the source comes from the composition and the one on the target from the multiplication

$$\deg^G\colon [X,X]^G \xrightarrow{\cong} A(G), \quad [f] \;\mapsto (\Lambda^G(f) - 1) \cdot (\chi^G(X) - 1).$$

We mention that the image of the element $\deg^G(f)$ for a self-G-map of a G-homotopy representation under the character map char^G of (1.2) is given by the collection of (non-equivariant) degrees $\{\deg(f^H) \mid (H) \in \mathrm{ccs}(G)\}$.

1.4 The Burnside Ring and Stable Cohomotopy

Let X and Y be two finite pointed G-CW-complexes. Pointed means that we have specified an element in its 0-skeleton which is fixed under the G-action. If V is a real G-representation, let S^V be its one-point compactification. We will use the point at infinity as base point for S^V. If V is an orthogonal representation, i.e. comes with an G-invariant scalar product, then S^V is G-homeomorphic to the unit sphere $S(V \oplus \mathbb{R})$. Given two pointed G-CW-complexes X and Y with base points x and y, define their *one-point-union* $X \vee Y$ to be the pointed G-CW-complex $X \times \{y\} \cup \{x\} \times Y \subseteq X \times Y$ and their *smash product* $X \wedge Y$ to be the pointed G-CW-complex $X \times Y/X \vee Y$.

We briefly introduce equivariant stable homotopy groups following the approach due to tom Dieck [49, II.6].

If V and W are two complex G-representations, we write $V \leq W$ if there exists a complex G-representation U and a linear G-isomorphism $\phi\colon U \oplus V \to W$. If $\phi\colon U \oplus V \to W$ is a linear G-isomorphism, define a map

$$b_{V,W}\colon [S^V \wedge X, S^V \wedge Y]^G \to [S^W \wedge X, S^W \wedge Y]^G$$

by the composition

$$\begin{aligned}
[S^V \wedge X, S^V \wedge Y]^G &\xrightarrow{u_1} [S^U \wedge S^V \wedge X, S^U \wedge S^V \wedge Y]^G \\
&\xrightarrow{u_2} [S^{U \oplus V} \wedge X, S^{U \oplus V} \wedge Y]^G \xrightarrow{u_3} [S^W \wedge X, S^W \wedge Y]^G,
\end{aligned}$$

where the map u_1 is given by $[f] \mapsto [\mathrm{id}_{S^U} \wedge f]$, the map u_2 comes from the obvious G-homeomorphism $S^{U \oplus V} \xrightarrow{\cong} S^U \wedge S^V$ induced by the inclusion $V \oplus W \to S^V \wedge S^W$ and the map u_3 from the G-homeomorphism $S^\phi\colon S^{U \oplus V} \xrightarrow{\cong} S^W$. Any two linear G-isomorphisms $\phi_0, \phi_1\colon V_1 \to V_2$ between to complex G-representations are isotopic as linear G-isomorphisms. (This is not true for real G-representations.) This implies that the map $b_{V,W}$ is indeed independent of the choice of U and ϕ. One easily checks that $b_{V_2,V_1} \circ b_{V_0,V_1} = b_{V_0,V_2}$ holds for complex G-representations V_0, V_1 and V_2 satisfying $V_0 \leq V_1$ and $V_1 \leq V_2$.

Let I be the set of complex G-representations with underlying complex vector space $\mathbb{C}^n$ for some n. (Notice that the collections of all complex G-representations does not form a set.) Define on the disjoint union

$$\coprod_{V \in I} [S^V \wedge X, S^V \wedge Y]^G$$

an equivalence relation by calling $f \in [S^V \wedge X, S^V \wedge Y]^G$ and $g \in [S^W \wedge X, S^W \wedge Y]^G$ equivalent if there exists a representation $U \in I$ with $V \leq U$ and $W \leq U$ such that $b_{V,U}([f]) = b_{W,U}([g]$ holds. Let $\omega_0^G(X,Y)$ for two pointed G-CW-complexes X and Y be the set of equivalence classes.

If V is any complex G-representation (not necessarily in I) and $f\colon S^V \wedge X \to S^V \wedge Y$ is any G-map, there exists an element $W \in I$ with $V \leq W$ and we get an element in $\omega_n^G(X,Y)$ by $b_{V,W}([f])$. This element is independent of the choice of W and also denoted by $[f] \in \omega_n^G(X,Y)$.

One can define the structure of an abelian group on the set $\omega_0^G(X,Y)$ as follows. Consider elements $x, y \in \omega_0^G(X,Y)$. We can choose an element of the shape $\mathbb{C} \oplus U$ in I for $\mathbb{C}$ equipped with the trivial G-action and G-maps $f, g\colon S^{\mathbb{C}\oplus U} \wedge X \to S^{\mathbb{C}\oplus U} \wedge X$ representing x and y. Now using the standard pinching map $\nabla\colon S^{\mathbb{C}} \to S^{\mathbb{C}} \wedge S^{\mathbb{C}}$ one defines $x + y$ as the class of the G-map

$$S^{\mathbb{C}\oplus U} \wedge X \xrightarrow{\cong} S^{\mathbb{C}} \wedge S^U \wedge X \xrightarrow{\nabla \wedge \mathrm{id}_{S^U} \wedge \mathrm{id}_X} (S^{\mathbb{C}} \vee S^{\mathbb{C}}) \wedge S^U \wedge X$$
$$\xrightarrow{\cong} (S^{\mathbb{C}} \wedge S^U \wedge X) \vee (S^{\mathbb{C}} \wedge S^U \wedge X) \xrightarrow{\cong} (S^{\mathbb{C}\oplus U} \wedge X) \vee (S^{\mathbb{C}\oplus U} \wedge X) \xrightarrow{f \vee g} S^{\mathbb{C}\oplus U} \wedge X.$$

The inverse of x is defined by the class of

$$S^{\mathbb{C}\oplus U} \wedge X \xrightarrow{\cong} S^{\mathbb{C}} \wedge S^U \wedge X \xrightarrow{d \wedge f} S^{\mathbb{C}} \wedge S^U \wedge X \xrightarrow{\cong} S^{\mathbb{C}\oplus U} \wedge X$$

where $d\colon S^{\mathbb{C}} \to S^{\mathbb{C}}$ is any pointed map of degree -1. This is indeed independent of the choices of U, f and g.

We define the abelian groups

$$\begin{array}{rcll} \omega_n^G(X,Y) & = & \omega_0^G(S^n \wedge X, Y) & n \geq 0; \\ \omega_n^G(X,Y) & = & \omega_0^G(X, S^{-n} Y) & n \leq 0; \\ \omega_G^n(X,Y) & = & \omega_{-n}^G(X,Y) & n \in \mathbb{Z}; \end{array}$$

Obviously $\omega_n^G(X,Y)$ is functorial, namely contravariant in X and covariant in Y.

Let X and Y be (unpointed) G-CW-complexes. Let X_+ and Y_+ be the pointed G-CW-complexes obtained from X and Y by adjoining a disjoint base point. Denote by $\{\bullet\}$ the one-point-space. Define abelian groups

$$\begin{array}{rcll} \pi_n^G(Y) & = & \omega_n^G(\{\bullet\}_+, Y_+) & n \in \mathbb{Z}; \\ \pi_G^n(X) & = & \omega_G^n(X_+, \{\bullet\}_+) & n \in \mathbb{Z}; \\ \pi_n^G & = & \pi_n^G(\{\bullet\}) & n \in \mathbb{Z}; \\ \pi_G^n & = & \pi_{-n}^G & n \in \mathbb{Z}. \end{array}$$

The abelian group $\pi_0^G = \pi_G^0$ becomes a ring by the composition of maps. The abelian groups $\pi_n^G(Y)$ define covariant functors in Y and are called the *equivariant stable homotopy groups* of Y. The abelian groups $\pi_G^n(X)$ define contravariant functors in X and are called the *equivariant stable cohomotopy groups* of X.

We emphasize that our input in π_G^n and π_n^G are unpointed G-CW-complexes. This is later consistent with our constructions for infinite groups, where all G-CW-spaces must be proper and therefore have empty G-fixed point sets and cannot have base points.

Theorem 1.12 implies the following result due to Segal [41].

Theorem 1.13. *The isomorphism* $\deg^G$ *appearing in Theorem 1.12 induces an isomorphism of rings*

$$\deg^G\colon \pi_G^0 \xrightarrow{\cong} A(G).$$

For a more sophisticated and detailed construction of and more information about the equivariant stable homotopy category we refer for instance to [17], [23].

1.5 The Segal Conjecture for Finite Groups

The equivariant cohomotopy groups $\pi_G^n(X)$ are modules over the ring $\pi_G^0 = A(G)$, the module structure is given by composition of maps. The *augmentation homomorphism* $\epsilon^G\colon A(G) \to \mathbb{Z}$ is the ring homomorphism sending the class of a finite set S to $|S|$ which is just the component belonging to the trivial subgroup of the character map defined in (1.2). The *augmentation ideal* $\mathbf{I}_G \subseteq A(G)$ is the kernel of the augmentation homomorphism ϵ^G.

For an (unpointed) CW-complex X we denote by $\pi_s^n(X)$ the (non-equivariant) stable cohomotopy group of X_+. This is in the previous notation for equivariant stable cohomotopy the same as $\pi_{\{1\}}^n(X)$ for $\{1\}$ the trivial group. If X is a finite G-CW-complex, we can consider $\pi_s^n(EG \times_G X)$. Since $\pi_G^n(X)$ is a $A(G)$-module, we can also consider its $\mathbf{I}_G$-adic completion denoted by $\pi_G^n(X)\widehat{_{\mathbf{I}_G}}$. The following result is due to Carlsson [8].

Theorem 1.14 (Segal Conjecture for finite groups). *The* Segal Conjecture for finite groups G *is true, i.e. for every finite group G and finite G-CW-complex X there is an isomorphism*

$$\pi_G^n(X)\widehat{_{\mathbf{I}_G}} \xrightarrow{\cong} \pi_s^n(EG \times_G X).$$

In particular we get in the case $X = \{\bullet\}$ and $n = 0$ an isomorphism

$$A(G)\widehat{_{\mathbf{I}_G}} \xrightarrow{\cong} \pi_s^0(BG). \tag{1.15}$$

Thus the Burnside ring is linked via its $\mathbf{I}_G$-adic completion to the stable cohomotopy of the classifying space BG of a finite group G.

Example 1.16 (Segal Conjecture for $\mathbb{Z}/p$). Let G be the cyclic group $\mathbb{Z}/p$ of order p. We have computed $A(G)$ in Example 1.4. If we put $x = [G] - p\cdot[G/G]$, then the augmentation ideal is generated by x. Since

$$x^2 \;=\; ([G]-p)^2 \;=\; [G]^2 - 2p\cdot[G] + p^2 = (-p)\cdot x,$$

we get $x^n = (-p)^{n-1}x$ and hence $\mathbf{I}_G^n = p^{n-1}\cdot \mathbf{I}_G$ for $n \in \mathbb{Z}, n \geq 1$. This implies

$$A(G)\widehat{_{\mathbf{I}_G}} \;=\; \operatorname{invlim}_{n\to\infty} \mathbb{Z} \oplus \mathbf{I}_G/\mathbf{I}_G^n \;=\; \mathbb{Z} \times \mathbb{Z}\widehat{_p},$$

where $\mathbb{Z}\widehat{_p}$ denotes the ring of p-adic integers.

1.6 The Burnside Ring as a Green Functor

Let R be an associative commutative ring with unit. Let FGINJ be the category of finite groups with injective group homomorphisms as morphisms. Let $M\colon \mathsf{FGINJ} \to R\text{-}\mathsf{MODULES}$ be a bifunctor, i.e. a pair (M_*, M^*) consisting of a covariant functor M_* and a contravariant functor M^* from FGINJ to R-$\mathsf{MODULES}$ which agree on objects. We will often denote for an injective group homomorphism $f\colon H \to G$ the map $M_*(f)\colon M(H) \to M(G)$ by ind_f and the map $M^*(f)\colon M(G) \to M(H)$ by res_f and write $\operatorname{ind}_H^G = \operatorname{ind}_f$ and $\operatorname{res}_G^H = \operatorname{res}_f$ if f is an inclusion of groups. We call such a bifunctor M a *Mackey functor* with values in R-modules if

(i) For an inner automorphism $c(g)\colon G \to G$ we have $M_*(c(g)) = \mathrm{id}\colon M(G) \to M(G)$;

(ii) For an isomorphism of groups $f\colon G \xrightarrow{\cong} H$ the composites $\mathrm{res}_f \circ \mathrm{ind}_f$ and $\mathrm{ind}_f \circ \mathrm{res}_f$ are the identity;

(iii) Double coset formula

We have for two subgroups $H, K \subset G$

$$\mathrm{res}_G^K \circ \mathrm{ind}_H^G \quad = \sum_{KgH \in K\backslash G/H} \mathrm{ind}_{c(g)\colon H\cap g^{-1}Kg \to K} \circ \mathrm{res}_H^{H\cap g^{-1}Kg},$$

where $c(g)$ is conjugation with g, i.e. $c(g)(h) = ghg^{-1}$.

Let $\phi\colon R \to S$ be a homomorphism of associative commutative rings with unit. Let M be a Mackey functor with values in R-modules and let N and P be Mackey functors with values in S-modules. A *pairing* with respect to ϕ is a family of maps

$$m(H)\colon M(H) \times N(H) \to P(H), \quad (x, y) \mapsto m(H)(x, y) =: x \cdot y,$$

where H runs through the finite groups and we require the following properties for all injective group homomorphisms $f\colon H \to K$ of finite groups:

$$\begin{array}{ll}
(x_1 + x_2) \cdot y = x_1 \cdot y + x_2 \cdot y & \text{for } x_1, x_2 \in M(H), y \in N(H);\\
x \cdot (y_1 + y_2) = x \cdot y_1 + x \cdot y_2 & \text{for } x \in M(H), y_1, y_2 \in N(H);\\
(rx) \cdot y = \phi(r)(x \cdot y) & \text{for } r \in R, x \in M(H), y \in N(H);\\
x \cdot sy = s(x \cdot y) & \text{for } s \in S, x \in M(H), y \in N(H);\\
\mathrm{res}_f(x \cdot y) = \mathrm{res}_f(x) \cdot \mathrm{res}_f(y) & \text{for } x \in M(K), y \in N(K);\\
\mathrm{ind}_f(x) \cdot y = \mathrm{ind}_f\,(x \cdot \mathrm{res}_f(y)) & \text{for } x \in M(H), y \in N(K);\\
x \cdot \mathrm{ind}_f(y) = \mathrm{ind}_f\,(\mathrm{res}_f(x) \cdot y) & \text{for } x \in M(K), y \in N(H).
\end{array}$$

A *Green functor* with values in R-modules is a Mackey functor U together with a pairing with respect to $\mathrm{id}\colon R \to R$ and elements $1_H \in U(H)$ for each finite group H such that for each finite group H the pairing $U(H) \times U(H) \to U(H)$ induces the structure of an R-algebra on $U(H)$ with unit 1_H and for any morphism $f\colon H \to K$ in FGINJ the map $U^*(f)\colon U(K) \to U(H)$ is a homomorphism of R-algebras with unit. Let U be a Green functor with values in R-modules and M be a Mackey functor with values in S-modules. A (left) U-module structure on M with respect to the ring homomorphism $\phi\colon R \to S$ is a pairing such that any of the maps $U(H) \times M(H) \to M(H)$ induces the structure of a (left) module over the R-algebra $U(H)$ on the R-module $\phi^*M(H)$ which is obtained from the S-module $M(H)$ by $rx := \phi(r)x$ for $r \in R$ and $x \in M(H)$.

Theorem 1.17. *(i) The Burnside ring defines a Green functor with values in $\mathbb{Z}$-modules;*

(ii) If M is a Mackey functor with values in R-modules, then M is in a canonical way a module over the Green functor given by the Burnside ring with respect to the canonical ring homomorphism $\phi\colon \mathbb{Z} \to R$.

Proof. (i) Let $f\colon H \to G$ be an injective homomorphism of groups. Define $\mathrm{ind}_f\colon A(H) \to A(G)$ by sending the class of a finite H-set S to the class of the finite G-set $G \times_f S$. Define $\mathrm{res}_f\colon A(G) \to A(H)$ by considering a finite G-set as an H-set by restriction with f. One easily verifies that the axioms of a Green functor with values in $\mathbb{Z}$-modules are satisfied.

(ii) We have to specify for any finite group G a pairing $m(G)\colon A(G) \times M(G) \to M(G)$. This is done by the formula

$$m(G)\left(\sum_i n_i \cdot [G/H_i], x\right) \quad := \quad \sum_i n_i \cdot \mathrm{ind}_{H_i}^G \circ \mathrm{res}_G^{H_i}(x).$$

One easily verifies that the axioms of a module over the Green functor given by the Burnside ring are satisfied. □

Theorem 1.17 is the main reason, why the Burnside ring plays an important role in induction theory. Induction theory addresses the question whether one can compute the values of a Mackey functor on a finite group by its values on a certain class of subgroups such as the family of cyclic or hyperelementary groups. Typical examples of such Mackey functors are the representation ring $R_F(G)$ or algebraic K and L-groups $K_n(RG)$ and $L_n(RG)$ of groups rings. The applications require among other things a good understanding of the prime ideals of the Burnside ring. For more information about induction theory for finite groups we refer to the fundamental papers by Dress [12], [13] and for instance to [46, Chapter 6]. Induction theory for infinite groups is developed in [5].

As an illustration we give an example how the Green-functor mechanism works.

Example 1.18 (Artin's Theorem). Let $R_{\mathbb{Q}}(G)$ be the rational representation ring of the finite group G. For any finite cyclic group C one can construct an element

$$\theta_C \in R_{\mathbb{Q}}(C)$$

which is uniquely determined by the property that its character function sends a generator of C to $|C|$ and every other element of C to zero.

Let G be a finite group. Let $\mathbb{Q}$ be the trivial 1-dimensional rational G-representation. It is not hard to check by a calculation with characters that

$$|G| \cdot [\mathbb{Q}] \quad = \quad \sum_{\substack{C \subset G \\ C \text{ cyclic}}} \mathrm{ind}_C^H \theta_C. \tag{1.19}$$

Assigning to a finite group G the rational representation ring $R_{\mathbb{Q}}(G)$ inherits the structure of a Green functor with values in $\mathbb{Z}$-modules by induction and restriction. Suppose that M is a Mackey functor with values in $\mathbb{Z}$-modules which is a module over the Green functor $R_{\mathbb{Q}}$. Then for every finite group G

the cokernel of the map

$$\bigoplus_{\substack{C\subset G\\ C\text{ cyclic}}} \operatorname{ind}_C^G\colon \bigoplus_{\substack{C\subset G\\ C\text{ cyclic}}} M(C) \;\to\; M(G)$$

is annihilated by multiplication with $|G|$. This follows from the following calculation for $x \in M(G)$ based on (1.19) and the axioms of a Green functor and a module over it

$$|G|\cdot x \;=\; (|G|\cdot[\mathbb{Q}])\cdot x \;=\; \sum_{\substack{C\subset G\\ C\text{ cyclic}}} \operatorname{ind}_C^H(\theta_C)\cdot x \;=\; \sum_{\substack{C\subset G\\ C\text{ cyclic}}} \operatorname{ind}_C^H(\theta_C\cdot \operatorname{res}_H^C x).$$

Examples for M are algebraic K- and L-groups $K_n(RG)$ and $L_n(RG)$ for any ring R with $\mathbb{Q}\subseteq R$. We may also take M to be R_F for any field F of characteristic zero and then the statement above is Artin's Theorem (see [43, Theorem 26 on page 97].

1.7 The Burnside Ring and Rational Representations

Let $R_{\mathbb{Q}}(G)$ be the representation ring of finite-dimensional rational G-representation. Given a finite G-set S, let $\mathbb{Q}[S]$ be the rational G-representation given by the $\mathbb{Q}$-vector space with the set S as basis. The next result is due to Segal [40].

Theorem 1.20. (The Burnside ring and the rational representation ring for finite groups). *Let G be a finite group. We obtain a ring homomorphism*

$$P^G\colon A(G)\to R_{\mathbb{Q}}(G),\quad [S]\mapsto[\mathbb{Q}[S]].$$

It is rationally surjective. If G is a p-group for some prime p, it is surjective. It is bijective if and only if G is cyclic.

1.8 The Burnside Ring and Homotopy Representations

We have introduced the notion of a G-homotopy representation in Definition 1.11. The join of two G-homotopy representations is again a G-homotopy representation. We call two G-homotopy representations X and Y stably G-homotopy equivalent if for some G-homotopy representation Z the joins $X*Z$ and $Y*Z$ are G-homotopy equivalent. The stable G-homotopy classes of G-homotopy representations together with the join define an abelian semi-group. The *G-homotopy representation group* $V(G)$ is the associated Grothendieck group. It may be viewed as the homotopy version of the representation ring. Taking the unit sphere yields a group homomorphism $R_{\mathbb{R}}(G)\to V(G)$.

The *dimension function* of a G-homotopy representation X

$$\dim(X)\in\prod_{(H)}\mathbb{Z}$$

associates to the conjugacy class (H) of a subgroup $H \subseteq G$ the dimension of X^H. The question which elements in $\prod_{(H)} \mathbb{Z}$ occur as $\dim(X)$ is studied for instance in [47], [48], [49, III.5] and [50]. Define $V(G, \dim)$ by the exact sequence

$$0 \to V(G, \dim) \to V(G) \xrightarrow{\dim} \prod_{(H)} \mathbb{Z}.$$

Let $\mathrm{Pic}(A(G))$ be the Picard group of the Burnside ring, i.e. the abelian group of projective $A(G)$-module of rank one with respect to the tensor product. The next result is taken from [50, 6.5].

Theorem 1.21 ($V(G, \dim)$ and the Picard group of $A(G)$). *There is an isomorphism*

$$V(G, \dim) \xrightarrow{\cong} \mathrm{Pic}(A(G)).$$

Further references about the Burnside ring of finite groups are [7], [10],[11], [16], [19], [20], [36], [45], [52].

2 The Finite-G-Set-Version of the Burnside Ring

From now on G can be any (discrete) group and needs not to be finite anymore. Next we give a first definition of the Burnside ring for infinite groups.

Definition 2.1. (The finite-G-set-version of the Burnside ring). *The isomorphism classes of finite G-sets form a commutative associative semi-ring with unit under the disjoint union and the cartesian product. The* finite-G-set-version of the Burnside ring $\overline{A}(G)$ *is the Grothendieck ring associated to this semi-ring.*

To avoid any confusion, we emphasize that finite G-set means a finite set with a G-action. This definition is word by word the same as given for a finite group in Definition 1.1.

Given a group homomorphism $f\colon G_0 \to G_1$ of groups, restriction with f defines a ring homomorphism $f^*\colon A(G_1) \to A(G_0)$. Thus $A(G)$ becomes a contravariant functor from the category of groups to the category of commutative rings. Provided that the image of f has finite index, induction defines a homomorphism of abelian groups $f_*\colon A(G_0) \to A(G_1)$, $[S] \mapsto [G_1 \times_f S]$, which is not compatible with the multiplication.

2.1 Character Theory and Burnside Congruences for the Finite-G-Set-Version

The definition of the character map (1.2) makes also sense for infinite groups and we denote it by

$$\overline{\mathrm{char}}^G\colon \overline{A}(G) \quad \to \prod_{(H)\in \mathrm{ccs}(G)} \mathbb{Z}, \quad [S] \mapsto [|(S^H|)_{(H)}. \tag{2.2}$$

Given a group homomorphism $f\colon G_0 \to G_1$, define a ring homomorphism

$$f^*\colon \prod_{(H_1)\in \mathrm{ccs}(G_1)} \to \prod_{(H_0)\in \mathrm{ccs}(G_0)} \mathbb{Z} \tag{2.3}$$

by sending $\{x(H_1)\}$ to $\{x(f(H_1))\}$. One easily checks

Lemma 2.4. *The following diagram of commutative rings with unit commutes for every group homomorphism $f\colon G_0 \to G_1$*

$$\begin{array}{ccc} \overline{A}(G_1) & \xrightarrow{\overline{\mathrm{char}}^{G_1}} & \prod_{(H_1)\in \mathrm{ccs}(G_1)} \mathbb{Z} \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \overline{A}(G_0) & \xrightarrow[\overline{\mathrm{char}}^{G_0}]{} & \prod_{(H_0)\in \mathrm{ccs}(G_0)} \mathbb{Z} \end{array}$$

Theorem 2.5 (Burnside ring congruences for $\overline{A}(G)$). *The character map $\overline{\mathrm{char}}^G$ is an injective ring homomorphism.*

Already the composition

$$\overline{A}(G) \xrightarrow{\overline{\mathrm{char}}^G} \prod_{(H)\in \mathrm{ccs}(G)} \mathbb{Z} \xrightarrow{\mathrm{pr}} \prod_{\substack{(H)\in \mathrm{ccs}(G)\\ [G:H]<\infty}} \mathbb{Z}$$

for pr *the obvious projection is injective.*

An element $x = \{x(H)\} \in \prod_{(H)\in \mathrm{ccs}(G)} \mathbb{Z}$ lies in the image of the character map $\overline{\mathrm{char}}^G$ defined in (2.2) *if and only if it satisfies the following two conditions:*

(i) There exists a normal subgroup $K_x \subseteq G$ of finite index such that $x(H) = x(H \cdot K_x)$ holds for all $H \subseteq G$, where $H \cdot K_x$ is the subgroup $\{hk \mid h \in H, k \in K_x\}$;

(ii) We have for every $(H) \in \mathrm{ccs}(G)$ with $[G : H] < \infty$:

$$\sum_{\substack{(C)\in \mathrm{ccs}(W_GH)\\ C \text{ cyclic}}} |\operatorname{Gen}(C)|\cdot[W_GH : N_{W_GH}C]\cdot x(p_H^{-1}(C)) \equiv 0 \mod |W_GH|,$$

where $p_H\colon N_GH \to W_GH$ is the obvious projection.

Proof. Obviously $\overline{\mathrm{char}}^G$ is a ring homomorphism.

Suppose that $x \in \overline{A}(G)$ lies in the kernel of $\overline{\mathrm{char}}^G$. For any finite G-set the intersection of all its isotropy groups is a normal subgroup of finite index in G. Hence can find an epimorphism $p_x\colon G \to Q_x$ onto a finite group Q_x and $\overline{x} \in A(Q_x)$ such that x lies in the image of $p_x^*\colon \overline{A}(Q_x) \to \overline{A}(G)$. Since the map

$$p_x^*\colon \prod_{(H)\in \mathrm{ccs}(Q_x)} \mathbb{Z} \to \prod_{(K)\in \mathrm{ccs}(G)} \mathbb{Z}$$

is obviously injective and the character map $\overline{\operatorname{char}}^{Q_x}$ is injective by Lemma 1.3, we conclude $x = 0$. Hence $\overline{\operatorname{char}}^{G}$ is injective.

Suppose that y lies in the image of $\overline{\operatorname{char}}^{G}$. Choose $x \in \overline{A}(G)$ with $\overline{\operatorname{char}}^{G}(x) = y$. As explained above we can find an epimorphism $p_x \colon G \to Q_x$ onto a finite group Q_x and $\overline{x} \in A(Q_x)$ such that $p_x^* \colon \overline{A}(Q_x) \to \overline{A}(G)$ maps $\overline{x}$ to x. Then Condition (i) is satisfied by Lemma 2.4 if we take K_x to be the kernel of p_x. Condition (ii) holds for x since the proof of Lemma 1.3 carries though word by word to the case, where G is possibly infinite but $H \subseteq G$ is required to have finite index in G and hence $W_G H$ is finite.

We conclude that $\overline{\operatorname{char}}^{G}(x) = 0$ if and only if $\operatorname{pr} \circ \overline{\operatorname{char}}^{G}(x) = 0$ holds. Hence $\operatorname{pr} \circ \overline{\operatorname{char}}^{G}$ is injective.

Now suppose that $x = \{y(H)\} \in \prod_{(H) \in \operatorname{ccs}(G)} \mathbb{Z}$ satisfies Condition (i) and Condition (ii). Let $Q_x = G/K_x$ and let $p_x \colon G \to Q_x$ be the projection. In the sequel we abbreviate $Q = Q_x$ and $p = p_x$. Then Condition (i) ensures that x lies in the image of the injective map

$$p^* \colon \prod_{(H) \in \operatorname{ccs}(Q)} \mathbb{Z} \to \prod_{(K) \in \operatorname{ccs}(G)} \mathbb{Z}.$$

Let $y \in \prod_{(H) \in \operatorname{ccs}(Q)} \mathbb{Z}$ be such a preimage. Because of Lemma 2.4 it suffices to prove that y lies in the image of the character map

$$\operatorname{char}^{Q} \colon A(Q) \to \prod_{(H) \in \operatorname{ccs}(Q)} \mathbb{Z}.$$

By Lemma 1.3 this is true if and only if for every subgroup $K \subseteq Q$ the congruence

$$\sum_{\substack{(C) \in \operatorname{ccs}(W_Q K) \\ C \text{ cyclic}}} |\operatorname{Gen}(C)| \cdot [W_Q K : N_{W_Q K} C] \cdot y\left((p_K^Q)^{-1}(C)\right) \equiv 0 \mod |W_Q K|$$

holds, where $p_K^Q \colon N_Q K \to W_G K$ is the projection. Fix a subgroup $K \subseteq Q$. Put $H = p^{-1}(K) \subseteq G$. The epimorphism $p \colon G \to Q$ induces an isomorphism $\overline{p} \colon W_G H \xrightarrow{\cong} W_Q K$. Condition (ii) applied to x and H yields

$$\sum_{\substack{(C) \in \operatorname{ccs}(W_G H) \\ C \text{ cyclic}}} |\operatorname{Gen}(C)| \cdot [W_G H : N_{W_G H} C] \cdot x(p_H^{-1}(C)) \equiv 0 \mod |W_G H|.$$

For any cyclic subgroup $C \subseteq W_G H$ we obtain a cyclic subgroup $\overline{p}(C) \subseteq W_Q K$ and we have

$$\begin{aligned} |\operatorname{Gen}(C)| &= |\operatorname{Gen}(\overline{p}(C))|; \\ [W_G H : N_{W_G H} C] &= [W_Q K : N_{W_Q K} \overline{p}(C)]; \\ x\left(p_H^{-1}(C)\right) &= y\left((p_K^Q)^{-1}(\overline{p}(C))\right). \end{aligned}$$

Now the desired congruence for y follows. This finishes the proof of Theorem 2.5. $\square$

Example 2.6 ($\overline{A}$ of the integers). Consider the infinite cyclic group $\mathbb{Z}$. Any subgroup of finite index is of the form $n\mathbb{Z}$ for some $n \in \mathbb{Z}$, $n \geq 1$. As an abelian group $\overline{A}(Z)$ is generated by the classes $[\mathbb{Z}/n\mathbb{Z}]$ for $n \in \mathbb{Z}$, $n \geq 1$. The condition (ii) appearing in Theorem 2.5 reduces to the condition that for every subgroup $n\mathbb{Z}$ for $n \in \mathbb{Z}$, $n \geq 1$ the congruence

$$\sum_{m \in \mathbb{Z}, m \geq 1, m | n} \phi\left(\frac{n}{m}\right) \cdot x(m) \equiv 0 \qquad \bmod n$$

holds, where ϕ is the Euler function, whose value $\phi(k)$ is $|\operatorname{Gen}(\mathbb{Z}/k\mathbb{Z})|$. The condition (i) reduces to the condition that there exists $n_x \in \mathbb{Z}$, $n_x \geq 1$ such that for all $m \in \mathbb{Z}$, $m \geq 1$ we have $x(m\mathbb{Z}) = x(gcd(m, n_x)\mathbb{Z})$, where $gcd(m, n_x)$ is the greatest common divisor of m and n_x.

Remark 2.7 (The completion $\widehat{A}(G)$ of $\overline{A}(G)$). We call a G-set *almost finite* if each isotropy group has finite index and for every positive integer n the number of orbits G/H in S with $[G : H] \leq n$ is finite. A G-set S is almost finite if and only if for every subgroup $H \subseteq G$ of finite index the H-fixed point set S^H is finite and S is the union $\bigcup_{\substack{(H) \in \operatorname{ccs}(G) \\ [G:H] < \infty}} S^H$. Of course every finite G-set S is almost finite. The disjoint union and the cartesian product with the diagonal G-action of two almost finite G-sets is again almost finite. Define $\widehat{A}(G)$ as the Grothendieck ring of the semi-ring of almost finite G-sets under the disjoint union and the cartesian product. There is an obvious inclusion of rings $\overline{A}(G) \to \widehat{A}(G)$. We can define as before a character map

$$\widehat{\operatorname{char}}^G : \widehat{A}(G) \quad \to \quad \prod_{\substack{(H) \in \operatorname{ccs}(G) \\ [G:H] < \infty}} \mathbb{Z}, \quad [S] \ \mapsto (|(S^H|)_{(H)}. \tag{2.8}$$

We leave it to the reader to check that $\widehat{\operatorname{char}}^G$ is injective and that an element x in $\prod_{\substack{(H) \in \operatorname{ccs}(G) \\ [G:H] < \infty}} \mathbb{Z}$ lies in its image, if and only if x satisfies condition (ii) appearing in Theorem 2.5.

Dress and Siebeneicher [14] analyze $\widehat{A}(G)$ for profinite groups G and put it into relation with the Witt vector construction. They also explain that $\widehat{A}(G)$ is a completion of $\overline{A}(G)$. The ring $\widehat{A}(\mathbb{Z})$ is studied and put in relation to the necklace algebra, λ-rings and the universal ring of Witt vectors in [15].

2.2 The Finite-G-Set-Version and the Equivariant Euler Characteristic and the Equivariant Lefschetz Class

The results of Sections 1.2 and 1.3 carry over to $\overline{A}(G)$ if one considers only finite G-CW-complexes X whose isotropy group all have finite index in G. But this is not really new since for any such G-CW-complex X there is a subgroup $H \subseteq G$, namely the intersection of all isotropy groups, such that H is normal, has finite index in G and acts trivially on X. Thus X is a finite Q-CW-complex for the finite group $Q = G/H$ and all these invariant are obtained from the one over Q by the applying the obvious ring homomorphism $A(Q) = \overline{A}(Q) \to \overline{A}(G)$ to the invariants already defined over the finite group Q.

2.3 The Finite-G-Set-Version as a Green Functor

The notions and results of Subsection 1.6 carry over to the finite-G-set-version $\overline{A}(G)$ for an infinite group G, we replace the category FGINJ by the category GRIFI whose objects are groups and whose morphisms are injective group homomorphisms whose image has finite index in the target. However, for infinite groups this does not seem to be the right approach to induction theory. The approach presented in Bartels-Lück [5] is more useful. It is based on classifying spaces for families and aims at reducing the family of subgroups, for instance from all finite subgroups to all hyperelementary finite subgroups or from all virtually cyclic subgroups to the family of subgroups which admit an epimorphism to a hyperelementary group and whose kernel is trivial or infinite cyclic.

2.4 The Finite-G-Set-Version and the Swan Ring

Let R be a commutative ring. Let $\mathrm{Sw}^f(G;R)$ be the abelian group which is generated by the RG-isomorphisms classes of RG-modules which are finitely generated free over R with the relations $[M_0] - [M_1] - [M_2] = 0$ for any short exact RG-sequence $0 \to M_0 \to M_1 \to M_2 \to 0$ of such RG-modules. It becomes a commutative ring, the so called *Swan ring* $\mathrm{Sw}^f(G;R)$), by the tensor product $\otimes_R$. If G is finite and F is a field, then $\mathrm{Sw}^f(G;F)$ is the same as the representation ring $R_F(G)$ of (finite-dimensional) G-representations over K.

Let $G_0(RG)$ be the abelian group which is generated by the RG-isomorphism classes of finitely generated RG-modules with the relations $[M_0]-[M_1]-[M_2] = 0$ for any short exact RG-sequence $0 \to M_0 \to M_1 \to M_2 \to 0$ of such RG-modules. There is an obvious map

$$\phi\colon \mathrm{Sw}^f(G;R) \to G_0(RG)$$

of abelian groups. It is an isomorphism if G is finite and R is a principle ideal domain. This follows from [9, Theorem 38.42 on page 22].

We obtain a ring homomorphism

$$\overline{P}^G\colon \overline{A}(G) \;\to\; \mathrm{Sw}^f(G;\mathbb{Q}), \quad [S] \mapsto [R[S]], \tag{2.9}$$

where $R[S]$ is the finitely generated free R-module with the finite set S as basis and becomes a RG-module by the G-action on S.

Theorem 1.20 does not carry over to $\overline{A}(G)$ for infinite groups. For instance, the determinant induces a surjective homomorphism

$$\det\colon \mathrm{Sw}(\mathbb{Z};\mathbb{Q}) \to \mathbb{Q}^*, \quad [V] \;\mapsto\; \det(l_t\colon V \to V),$$

where l_t is left multiplication with a fixed generator $t \in \mathbb{Z}$. Given a finite $\mathbb{Z}$-set S, the map $l_t\colon \mathbb{Q}[S] \to \mathbb{Q}[S]$ satisfies $(l_t)^n = \mathrm{id}$ for some $n \geq 1$ and hence

$\overline{P}^{\mathbb{Z}}(\mathbb{Q}[S]) = \pm - 1$. Hence the image of the composition $\det \circ \overline{P}_{\mathbb{Z}}$ is contained in $\{\pm 1\}$. Therefore the map $\overline{P}^{\mathbb{Z}}$ of (2.9) is not rationally surjective.

2.5 Maximal Residually Finite Quotients

Let G be a group. Denote by G_0 the intersection of all normal subgroups of finite index. This is a normal subgroup. Let $p\colon G \to G/G_0$ be the projection. Recall that G is called *residually finite* if for every element $g \in G$ with $g \neq 1$ there exists a homomorphism onto a finite group which sends g to an element different from 1. If G is countable, then G is residually finite if and only if G_0 is trivial. The projection $p\colon G \to G_{\mathrm{mrf}} := G/G_0$ is the projection onto the maximal residually finite quotient of G, i.e. G_{mrf} is residually finite and every epimorphism $f\colon G \to Q$ onto a residually finite group Q factorizes through p into a composition $G \xrightarrow{p} G_{\mathrm{mrf}} \xrightarrow{\overline{f}} Q$. If G is a finitely generated subgroup of $GL_n(F)$ for some field F, then G is residually finite (see [35], [51, Theorem 4.2]). Hence for every finitely generated group G each G-representation V with coefficients in a field F is obtained by restriction with $p\colon G \to G_{\mathrm{mrf}}$ from a G_{mrf}-representation. In particular every G-representation with coefficient in a field F is trivial if G is finitely generated and G_{mrf} is trivial.

One easily checks that

$$p^* : \overline{A}(G_{\mathrm{mrf}}) \xrightarrow{\cong} \overline{A}(G)$$

is an isomorphism. In particular we have $\overline{A}(G) = \mathbb{Z}$ if G_{mrf} is trivial. If G is finitely generated, then

$$p^* : \mathrm{Sw}^f(G_{\mathrm{mrf}}; F) \xrightarrow{\cong} \mathrm{Sw}^f(G; F)$$

is an isomorphism. In particular we have $\mathrm{Sw}^f(G; F) = \mathbb{Z}$ if G is finitely generated and G_{mrf} is trivial.

Example 2.10 ($\overline{A}(\mathbb{Z}/p^\infty)$ and $\mathrm{Sw}^f(\mathbb{Z}/p^\infty; \mathbb{Q})$). Let $\mathbb{Z}/p^\infty$ be the *Prüfer group*, i.e. the colimit of the directed system of injections of abelian groups $\mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p^3 \to \cdots$. It can be identified with $\mathbb{Q}/\mathbb{Z}_{(p)}$ or $\mathbb{Z}[1/p]/\mathbb{Z}$. We want to show that the following diagram is commutative and consists of isomorphisms

$$\begin{array}{ccc} \overline{A}(\{1\}) = \mathbb{Z} & \xrightarrow[\cong]{p^*} & \overline{A}(\mathbb{Z}/p^\infty) \\ {\scriptstyle \overline{P}^{\{1\}}} \downarrow \cong & & {\scriptstyle \overline{P}^{\mathbb{Z}/p^\infty}} \downarrow \cong \\ \mathrm{Sw}^f(\{1\}; \mathbb{Q}) = \mathbb{Z} & \xrightarrow[\cong]{p^*} & \mathrm{Sw}^f(\mathbb{Z}/p^\infty; \mathbb{Q}) \end{array}$$

where $p\colon \mathbb{Z}/p^\infty \to \{1\}$ is the projection. Obviously the diagram commutes and the left vertical arrow is bijective. Hence it remains to show that the horizontal arrows are bijective.

Let $f\colon \mathbb{Z}/p^\infty \to Q$ be any epimorphism onto a finite group. Since $\mathbb{Z}/p^\infty$ is abelian, Q is a finite abelian group. Since any element in $\mathbb{Z}/p^n$ has p-power order, we conclude from the definition of $\mathbb{Z}/p^\infty$ as a colimit that Q is a finite abelian p-group. Since $\mathbb{Q}$ is p-divisible, the quotient Q must be p-divisible. Therefore Q must be trivial. Hence $(\mathbb{Z}/p^\infty)_{\mathrm{mrf}}$ is trivial and the upper horizontal arrow is bijective.

In order to show that the lower horizontal arrow is bijective, it suffices to show that every (finite-dimensional) rational $\mathbb{Z}/p^\infty$-representation V is trivial. It is enough to show that for every subgroup $\mathbb{Z}/p^m$ its restriction $\operatorname{res}_{i_m} V$ for the inclusion $i_m\colon \mathbb{Z}/p^m \to \mathbb{Z}/p^\infty$ is trivial. For this purpose choose a positive integer n such that $\dim_{\mathbb{Q}}(V) < (p-1)\cdot p^n$. Consider the rational $\mathbb{Z}/p^{m+n}$-representation $\operatorname{res}_{i_{m+n}} V$. Let $p_k^{m+n}\colon \mathbb{Z}/p^{m+n} \to \mathbb{Z}/p^k$ be the canonical projection. Let $\mathbb{Q}(p^k)$ be the rational $\mathbb{Z}/p^k$-representation given by adjoining a primitive p^k-th root of unity to $\mathbb{Q}$. Then the dimension of $\mathbb{Q}(p^k)$ is $(p-1)\cdot p^{k-1}$. A complete system of representatives for the isomorphism classes of irreducible rational $\mathbb{Z}/p^{m+n}$-representations is $\{\operatorname{res}_{p_k^{m+n}} \mathbb{Q}(p^k) \mid k = 0,1,2,\ldots m+n\}$. Since $\dim_{\mathbb{Q}}(V) < (p-1)\cdot p^n$, there exists a rational $\mathbb{Z}/p^n$-representation with W with $\operatorname{res}_{i_{m+n}} V \cong \operatorname{res}_{p_n^{m+n}} W$. Hence we get an isomorphism of rational $\mathbb{Z}/p^m$-representations

$$\operatorname{res}_{i_m} V \cong \operatorname{res}_{i_{m,m+n}} \circ \operatorname{res}_{p_n^{m+n}} W$$

where $i_{m,m+n}\colon \mathbb{Z}/p^m \to \mathbb{Z}/p^{m+n}$ is the inclusion. Since the composition $p_n^{m+n} \circ i_{m,m+n}$ is trivial, the rational $\mathbb{Z}/p^m$-representation $\operatorname{res}_{i_m} V$ is trivial.

It is not true that

$$\mathrm{Sw}^f(\mathbb{Z}/p^\infty;\mathbb{C}) \to \mathrm{Sw}^f(\{1\};\mathbb{C}) = \mathbb{Z}$$

is bijective because $\mathrm{Sw}^f(\mathbb{Z}/p^\infty;\mathbb{C})$ has as abelian group infinite rank (see Example 3.16).

3 The Inverse-Limit-Version of the Burnside Ring

In this section we present the inverse-limit-definition of the Burnside ring for infinite groups.

The *orbit category* $\mathsf{Or}(G)$ has as objects homogeneous spaces G/H and as morphisms G-maps. Let $\mathsf{Sub}(G)$ be the category whose objects are subgroups H of G. For two subgroups H and K of G denote by $\operatorname{conhom}_G(H,K)$ the set of group homomorphisms $f\colon H \to K$, for which there exists an element $g \in G$ with $gHg^{-1} \subset K$ such that f is given by conjugation with g, i.e. $f = c(g)\colon H \to K,\ h \mapsto ghg^{-1}$. Notice that $c(g) = c(g')$ holds for two elements $g, g' \in G$ with $gHg^{-1} \subset K$ and $g'H(g')^{-1} \subset K$ if and only if $g^{-1}g'$ lies in the centralizer $C_GH = \{g \in G \mid gh = hg \text{ for all } h \in H\}$ of H in G. The group of inner automorphisms of K acts on $\operatorname{conhom}_G(H,K)$ from the left by composition. Define the set of morphisms

$$\operatorname{mor}_{\mathsf{Sub}(G)}(H,K) \;:=\; \operatorname{inn}(K)\backslash \operatorname{conhom}_G(H,K).$$

There is a natural projection $\operatorname{pr}\colon \mathsf{Or}(G) \to \mathsf{Sub}(G)$ which sends a homogeneous space G/H to H. Given a G-map $f\colon G/H \to G/K$, we can choose an element $g \in G$ with $gHg^{-1} \subset K$ and $f(g'H) = g'g^{-1}K$. Then $\operatorname{pr}(f)$ is represented by $c(g)\colon H \to K$. Notice that $\operatorname{mor}_{\mathsf{Sub}(G)}(H,K)$ can be identified with the quotient $\operatorname{mor}_{\mathsf{Or}(G)}(G/H, G/K)/C_GH$, where $g \in C_GH$ acts on $\operatorname{mor}_{\mathsf{Or}(G)}(G/H, G/K)$ by composition with $R_{g^{-1}}\colon G/H \to G/H, \quad g'H \mapsto g'g^{-1}H$. We mention as illustration that for abelian G the set of morphisms $\operatorname{mor}_{\mathsf{Sub}(G)}(H,K)$ is empty if H is not a subgroup of K, and consists of precisely one element given by the inclusion $H \to K$ if H is a subgroup in K.

Denote by $\mathsf{Or}_{\mathcal{FIN}}(G) \subset \mathsf{Or}(G)$ and $\mathsf{Sub}_{\mathcal{FIN}}(G) \subset \mathsf{Sub}(G)$ the full subcategories, whose objects G/H and H are given by finite subgroups $H \subset G$.

Definition 3.1. (The inverse-limit-version of the Burnside ring). *The* inverse-limit-version of the Burnside ring $A_{\mathrm{inv}}(G)$ *is defined to be the commutative ring with unit given by the inverse limit of the contravariant functor*

$$A(?)\colon \mathsf{Sub}_{\mathcal{FIN}}(G) \to \mathsf{RINGS}, \quad H \ \mapsto \ A(H).$$

Since inner automorphisms induce the identity on $A(H)$, the contravariant functor appearing in the definition above is well-defined.

Consider a group homomorphism $f\colon G_0 \to G_1$. We obtain a covariant functor $\mathsf{Sub}_{\mathcal{FIN}}(f)\colon \mathsf{Sub}_{\mathcal{FIN}}(G_0) \to \mathsf{Sub}_{\mathcal{FIN}}(G_1)$ sending an object H to $f(H)$. A morphism $u\colon H \to K$ given by $c(g)\colon H \to K$ for some $g \in G$ with $gHg^{-1} \subseteq K$ is sent to the morphism given by $c(f(g))\colon f(H) \to f(K)$. There is an obvious transformation from the composite of the functor $A(?)\colon \mathsf{Sub}_{\mathcal{FIN}}(G_1) \to \mathsf{RINGS}$ with $\mathsf{Sub}_{\mathcal{FIN}}(f)$ to the functor $A_{\mathrm{inv}}(?)\colon \mathsf{Sub}_{\mathcal{FIN}}(G_0) \to \mathsf{RINGS}$. It is given for an object $H \in \mathsf{Sub}_{\mathcal{FIN}}(G_0)$ by the ring homomorphism $A(f(H)) \to A(H)$ induced by the group homomorphism $f|_H\colon H \to f(H)$. Thus we obtain a ring homomorphism $A_{\mathrm{inv}}(f)\colon A_{\mathrm{inv}}(G_1) \to A_{\mathrm{inv}}(G_0)$. So A_{inv} becomes a contravariant functor $\mathsf{GROUPS} \to \mathsf{RINGS}$.

Definition 3.1 reduces to the one for finite groups presented in Subsection 1 since for a finite group G the object $G \in \mathsf{Sub}_{\mathcal{FIN}}(G)$ is a terminal object.

There is an obvious ring homomorphism, natural in G,

$$T^G\colon \overline{A}(G) \quad \to \quad A_{\mathrm{inv}}(G) \tag{3.2}$$

which is induced from the various ring homomorphisms $\overline{A}(i_H)\colon \overline{A}(G) \to \overline{A}(H) = A(H)$ for the inclusions $i_H\colon H \to G$ for each finite subgroup $H \subseteq G$. The following examples show that it is neither injective nor surjective in general.

3.1 Some Computations of the Inverse-Limit-Version

Example 3.3 ($A_{\mathrm{inv}}(G)$ for torsionfree G). Suppose that G is torsionfree. Then $\mathsf{Sub}_{\mathcal{FIN}}(G)$ is the trivial category with precisely one object and one morphisms. This implies that the projection $\operatorname{pr}\colon G \to \{1\}$ induces a ring isomorphism

$$A_{\mathrm{inv}}(\operatorname{pr})\colon A_{\mathrm{inv}}(\{1\}) = \mathbb{Z} \ \xrightarrow{\cong} \ A_{\mathrm{inv}}(G).$$

In particular we conclude from Example 2.6 that the canonical ring homomorphism

$$T^{\mathbb{Z}}\colon \overline{A}(\mathbb{Z}) \xrightarrow{\cong} A_{\mathrm{inv}}(\mathbb{Z})$$

of (3.2) is not injective.

Example 3.4 (Groups with appropriate maximal finite subgroups). Let G be a discrete group which is not torsionfree. Consider the following assertions concerning G:

(M) Every non-trivial finite subgroup of G is contained in a unique maximal finite subgroup;

(NM) If $M \subseteq G$ is maximal finite, then $N_G M = M$.

The conditions (M) and (NM) imply the following: Let H be a non-trivial finite subgroup of G. Then there is a unique maximal finite subgroup M_H with $H \subseteq M_H$ and the set of morphisms in $\mathrm{Sub}_{\mathcal{FIN}}(G)$ from H to M_H consists of precisely one element which is represented by the inclusion $H \to M_H$. Let $\{M_i \mid i \in I\}$ be a complete set of representatives of the conjugacy classes of maximal finite subgroups of G. Denote by $j_i \colon M_i \to G$, $k_i \colon \{1\} \to M_i$ and $k \colon \{1\} \to G$ the inclusions. Then we obtain a short exact sequence

$$0 \to A_{\mathrm{inv}}(G) \xrightarrow{A_{\mathrm{inv}}(j_{\{1\}}) \times \prod_{i \in I} A_{\mathrm{inv}}(j_i)} A_{\mathrm{inv}}(\{1\}) \times \prod_{i \in I} A_{\mathrm{inv}}(M_i) \xrightarrow{\Delta - \prod_{i \in I} A_{\mathrm{inv}}(k_i)} \prod_{i \in I} A_{\mathrm{inv}}(\{1\} \to 0,$$

where $\Delta \colon A_{\mathrm{inv}}(\{1\} \to \prod_{i \in I} A_{\mathrm{inv}}(\{1\}$ is the diagonal embedding. If we define $\widetilde{A_{\mathrm{inv}}}(G)$ as the kernel of $A_{\mathrm{inv}}(G) \to A_{\mathrm{inv}}(\{1\})$, this gives an isomorphism

$$\widetilde{A_{\mathrm{inv}}}(G) \xrightarrow{\prod_{i \in I} \widetilde{A_{\mathrm{inv}}}(j_i)} \prod_{i \in I} \widetilde{A_{\mathrm{inv}}}(M_i).$$

Here are some examples of groups Q which satisfy conditions (M) and (NM):

- Extensions $1 \to \mathbb{Z}^n \to G \to F \to 1$ for finite F such that the conjugation action of F on $\mathbb{Z}^n$ is free outside $0 \in \mathbb{Z}^n$.

  The conditions (M) and (NM) are satisfied by [33, Lemma 6.3].

- Fuchsian groups F

  See for instance [33, Lemma 4.5]). In [33] the larger class of cocompact planar groups (sometimes also called cocompact NEC-groups) is treated.

- Finitely generated one-relator groups G

  Let $G = \langle (q_i)_{i \in I} \mid r \rangle$ be a presentation with one relation. Let F be the free group with basis $\{q_i \mid i \in I\}$. Then r is an element in F. There exists an element $s \in F$ and an integer $m \geq 1$ such that $r = s^m$, the cyclic subgroup C generated by the class $\overline{s} \in Q$ represented by s has order m, any finite subgroup of G is subconjugated to C and for any $q \in Q$ the implication $q^{-1}Cq \cap C \neq 1 \Rightarrow q \in C$ holds. These claims follows from [34, Propositions 5.17, 5.18 and 5.19 in II.5 on pages 107 and 108]. Hence Q satisfies (M) and (NM) and the inclusion $i\colon C \to G$ induces an isomorphism

$$A_{\mathrm{inv}}(i)\colon A_{\mathrm{inv}}(G) \xrightarrow{\cong} A_{\mathrm{inv}}(C).$$

Example 3.5 (Olshanskii's group). There is for any prime number $p > 10^{75}$ an infinite finitely generated group G all of whose proper subgroups are finite of order p [37]. Obviously G contains no subgroup of finite index. Hence the inclusion $i\colon \{1\} \to G$ induces an isomorphism

$$\overline{A}(i)\colon \overline{A}(G) \to A(\{1\}) = \mathbb{Z}$$

(see Subsection 2.5). Let H be a finite non-trivial subgroup of G. Then H is isomorphic to $\mathbb{Z}/p$ and agrees with its normalizer. So the conditions appearing in Example 3.4 are satisfied. Hence we obtain an isomorphism

$$\widetilde{A_{\mathrm{inv}}}(G) \xrightarrow{\cong} \prod_{\substack{(H)\in \mathrm{ccs}_f(G)\\ H\neq\{1\}}} \widetilde{A_{\mathrm{inv}}}(\mathbb{Z}/p),$$

where $\mathrm{ccs}_f(G)$ is the set of conjugacy classes of finite subgroups. This implies that the natural map

$$T^G\colon \overline{A}(G) \xrightarrow{\cong} A_{\mathrm{inv}}(G)$$

of (3.2) is not surjective.

Example 3.6 (Extensions of $\mathbb{Z}^n$ with $\mathbb{Z}/p$ as quotient). Suppose that G can be written as an extension $1 \to A \to G \to \mathbb{Z}/p \to 1$ for some fixed prime number p and for $A = \mathbb{Z}^n$ for some integer $n \geq 0$ and that G is not torsionfree. The conjugation action of G on the normal subgroup A yields the structure of a $\mathbb{Z}[\mathbb{Z}/p]$-module on A. Every non-trivial element $g \in G$ of finite order G has order p and satisfies

$$N_G\langle g\rangle = C_G\langle g\rangle = A^{\mathbb{Z}/p} \times \langle g\rangle.$$

In particular the conjugation action of $N_G\langle g\rangle$ on $\langle g\rangle$ is trivial. There is a bijection

$$\mu\colon H^1(\mathbb{Z}/p; A) \xrightarrow{\cong} \mathrm{ccs}_f(G),$$

where $H^1(\mathbb{Z}/p; A)$ is the first cohomology of $\mathbb{Z}/p$ with coefficients in the $\mathbb{Z}[\mathbb{Z}/p]$-module A. If we fix an element $g \in G$ of order p and a generator $s \in \mathbb{Z}/p$, the bijection μ sends $[u] \in H^1(\mathbb{Z}/p; A)$ to $(\langle ug\rangle)$ of the cyclic group $\langle ug\rangle$ of order p if $[u] \in H^1(\mathbb{Z}/p; A)$ is represented by the element u in the kernel of the second differential $A \to A,\ a \mapsto \sum_{i=0}^{p-1} s^i \cdot a$. Hence we obtain an exact sequence

$$0 \to A_{\mathrm{inv}}(G) \to A_{\mathrm{inv}}(\{1\}) \times \prod_{H^1(\mathbb{Z}/p;A)} A_{\mathrm{inv}}(\mathbb{Z}/p) \to \prod_{H^1(\mathbb{Z}/p;A)} A_{\mathrm{inv}}(\{1\}) \to 0$$

This gives an isomorphism

$$\widetilde{A_{\mathrm{inv}}}(G) \xrightarrow{\cong} \prod_{H^1(\mathbb{Z}/p;A)} \widetilde{A}(\mathbb{Z}/p).$$

3.2 Character Theory and Burnside Congruences for the Inverse-Limit-Version

Next we define a character map for infinite groups G and determine its image generalizing Lemma 1.3.

Let $\operatorname{ccs}_f(G)$ be the set of conjugacy classes (H) of finite subgroups $H \subseteq G$. Given a group homomorphism $f\colon G_0 \to G_1$, let $\operatorname{ccs}_f(G_0) \to \operatorname{ccs}_f(G_1)$ be the map sending the G_0-conjugacy class of a finite subgroup $H \subseteq G_0$ to the G_1-conjugacy class of $f(H) \subseteq G_1$. We obtain a covariant functor

$$\operatorname{ccs}\colon \mathsf{Sub}_{\mathcal{FIN}}(G) \to \mathsf{SETS}, \quad H \mapsto \operatorname{ccs}(H).$$

For each finite subgroup $H \subseteq G$ the inclusion $H \to G$ induces a map $\operatorname{ccs}(H) \to \operatorname{ccs}_f(G)$ sending (K) to (K). These fit together to a bijection of sets

$$\phi^G\colon \operatorname{colim}_{H \in \mathsf{Sub}_{\mathcal{FIN}}(G)} \operatorname{ccs}(H) \xrightarrow{\cong} \operatorname{ccs}_f(G). \tag{3.7}$$

One easily checks that ϕ^G is well-defined and surjective. Next we show injectivity. Consider two elements x_0 and x_1 in the source of ϕ^G with $\phi^G(x_0) = \phi^G(x_1)$. For $i = 0, 1$ we can choose an object $H_i \in \mathsf{Sub}_{\mathcal{FIN}}(G)$ and an element $(K_i) \in \operatorname{ccs}_f(H_i)$ such that the structure map of the colimit for the object (H_i) sends (K_i) to x_i. Then $\phi^G(x_0) = \phi^G(x_1)$ means that the subgroups K_0 and K_1 of G are conjugated in G. Hence we can find $g \in G$ with $gK_0g^{-1} = K_1$. The morphism $K_0 \to H_0$ induced by the inclusion yields a map $\operatorname{ccs}(K_0) \to \operatorname{ccs}(H_0)$ sending (K_0) to (K_0). The morphism $K_0 \to H_1$ induced by the conjugation homomorphism $c(g)\colon K_0 \to H_1$ yields a map $\operatorname{ccs}(K_0) \to \operatorname{ccs}(H_1)$ sending (K_0) to (K_1). This implies $x_0 = x_1$.

By the universal property of the colimit we obtain an isomorphism of abelian groups

$$\begin{aligned}\psi^G\colon \operatorname{map}\left(\operatorname{colim}_{H \in \mathsf{Sub}_{\mathcal{FIN}}(G)} \operatorname{ccs}(H), \mathbb{Z}\right) & \\ \xrightarrow{\cong} \operatorname{invlim}_{(H) \in \mathsf{Sub}_{\mathcal{FIN}}(G)} & \operatorname{map}(\operatorname{ccs}(H); \mathbb{Z}).\end{aligned} \tag{3.8}$$

Define the character map

$$\operatorname{char}^G_{\mathrm{inv}}\colon A_{\mathrm{inv}}(G) \to \prod_{(H) \in \operatorname{ccs}_f(G)} \mathbb{Z} \tag{3.9}$$

to be the map for which the composition with the isomorphism

$$\begin{aligned}\prod_{(H) \in \operatorname{ccs}_f(G)} \mathbb{Z} = \operatorname{map}(\operatorname{ccs}_f(G), \mathbb{Z}) & \xrightarrow{\operatorname{map}(\phi^G, \mathrm{id})} \operatorname{map}\left(\operatorname{colim}_{H \in \mathsf{Sub}_{\mathcal{FIN}}(G)} \operatorname{ccs}(H), \mathbb{Z}\right) \\ & \xrightarrow{\psi^G} \operatorname{invlim}_{H \in \mathsf{Sub}_{\mathcal{FIN}}(G)} \operatorname{map}(\operatorname{ccs}(H), \mathbb{Z})\end{aligned}$$

is the map

$$A_{\mathrm{inv}}(G) \;=\; \mathrm{invlim}_{H\in \mathrm{Sub}_{\mathcal{FIN}}(G)} \mathrm{map}(\mathrm{ccs}(H),\mathbb{Z}) \xrightarrow{\mathrm{invlim}_{H\in \mathrm{Sub}_{\mathcal{FIN}}(G)} \mathrm{char}^H} \mathrm{invlim}_{H\in \mathrm{Sub}_{\mathcal{FIN}}(G)} \mathrm{map}(\mathrm{ccs}(H);\mathbb{Z}),$$

where $\mathrm{char}^H\colon A(H) \to \mathrm{map}(\mathrm{ccs}(H),\mathbb{Z})$ is the map defined in (1.2).

Theorem 3.10 (Burnside ring congruences for $A_{\mathrm{inv}}(G)$). *Let x be an element in $\prod_{(H)\in\mathrm{ccs}_f(G)}\mathbb{Z}$. Then:*

(i) The character map

$$\mathrm{char}^G_{\mathrm{inv}}\colon A_{\mathrm{inv}}(G) \to \prod_{(H)\in\mathrm{ccs}_f(G)}\mathbb{Z}$$

of (3.9) is injective;

(ii) The element x lies in the image of the character map

$$\mathrm{char}^G_{\mathrm{inv}}\colon A_{\mathrm{inv}}(G) \to \prod_{(H)\in\mathrm{ccs}_f(G)}\mathbb{Z}$$

of (3.9) if and only if for every finite subgroup $K \subseteq G$ the following condition $C(K)$ is satisfied: The image of x under the map induced by the inclusion $i_K\colon K\to G$

$$(i_K)^* \prod_{(H)\in\mathrm{ccs}_f(G)}\mathbb{Z} \;=\; \mathrm{map}(\mathrm{ccs}(H),\mathbb{Z}) \;\to\; \prod_{(L)\in\mathrm{ccs}_f(K)}\mathbb{Z} \;=\; \mathrm{map}(\mathrm{ccs}(K),\mathbb{Z})$$

satisfies the Burnside ring congruences for the finite group K appearing in Lemma 1.3;

(iii) If $K_0 \subseteq K_1 \subseteq G$ are two subgroups, then condition $C(K_1)$ implies condition $C(K_0)$.

Proof. This follows from Lemma 1.3 and the fact that the inverse limit is left exact. □

Example 3.11. (Finitely many conjugacy classes of finite subgroups). Suppose that G has only finitely many conjugacy classes of finite subgroups. Then we conclude from Theorem 3.10 that the cokernel of the injective character map $\mathrm{char}^G_{\mathrm{inv}}\colon A_{\mathrm{inv}}(G) \to \prod_{(H)\in\mathrm{ccs}_f(G)}\mathbb{Z}$ is finite. Hence $A_{\mathrm{inv}}(G)$ is a finitely generated free abelian group of rank $|\mathrm{ccs}_f(G)|$.

Example 3.12 ($A_{\mathrm{inv}}(\mathbb{Z}/p^\infty)$). We have introduced in Example 2.10 the Prüfer group $\mathbb{Z}/p^\infty$ as $\mathrm{colim}_{n\to\infty}\mathbb{Z}/p^n$. Each $\mathbb{Z}/p^n$ represents a finite subgroup and each finite subgroup arises in this way. Hence $\mathrm{ccs}(\mathbb{Z}/p^\infty)$ is on one-to-one-correspondence with $\mathbb{Z}^{\geq 0} = \{n\in\mathbb{Z} \mid n\geq 0\}$. Thus $x\in\prod_{(H)\in\mathrm{ccs}(\mathbb{Z}/p^\infty)}\mathbb{Z}$ can be written as a sequence $\{x(n)\} = \{x(n) \mid n \in \mathbb{Z}^{\geq 0}\}$, where $x(n)$ corresponds to the value of x at $\mathbb{Z}/p^n$.

Consider the finite subgroup $\mathbb{Z}/p^m$. Its subgroups are given by $\mathbb{Z}/p^k$ for $k = 0,1,2\ldots m$. Then condition $C(\mathbb{Z}/p^m)$ reduces to the set of congruences for each $k = 0,1,2,\ldots,m-1$

$$\sum_{C\subset(\mathbb{Z}/p^m)/(\mathbb{Z}/p^k)} \operatorname{Gen}(C)\cdot x(p_k^{-1}(C)) \;\equiv\; 0 \qquad \bmod p^{m-k},$$

where $p_k\colon \mathbb{Z}/p^m \to (\mathbb{Z}/p^m)/(\mathbb{Z}/p^k)$ is the projection. More explicitly, the condition $C(\mathbb{Z}/p^m)$ reduces to the set of congruences for each $k = 0,1,2,\ldots,m-1$

$$x(k) + \sum_{i=1}^{m-k} p^{i-1}\cdot(p-1)\cdot x(k+i) \;\equiv\; 0 \qquad \bmod p^{m-k},$$

which can be rewritten as

$$\sum_{i=0}^{m-k-1} p^i\cdot(x(k+i)-x(k+i+1)) \;\equiv\; 0 \qquad \bmod p^{m-k}.$$

One can see that $C(\mathbb{Z}/p^{m_1})$ implies $C(\mathbb{Z}/p^{m_0})$ for $m_0 \leq m_1$ as predicted by Theorem 3.10 (iii).

Suppose that x satisfies $C(\mathbb{Z}/p^m)$ for $m = 0,1,2,\ldots$. We want to show inductively for $l = 0,1,2\ldots$ that $x(j) \equiv x(j+1) \mod p^l$ holds for $j = 0,1,2,\ldots$. The induction begin $l=0$ is trivial, the induction step from $l-1$ to $l \geq 1$ done as follows. The m-th equation appearing in condition $C(l+m)$ yields

$$\sum_{i=0}^{l-1} p^i\cdot(x(m+i)-x(m+i+1)) \;\equiv\; 0 \qquad \bmod p^l.$$

Since by induction hypothesis $x(k+i)-x(k+i+1) \equiv 0 \mod p^{l-1}$ holds, this reduces to

$$x(m)-x(m+1) \;\equiv\; 0 \qquad \bmod p^l.$$

This finishes the induction step.

Since $x(j) \equiv x(j+1) \mod p^l$ holds for $l = 0,1,2,\ldots$, we conclude $x(j) = x(j+1)$ for $j = 0,1,2,\ldots$. On the other hand, if $x(j) = x(j+1)$ holds for $j = 0,1,2,\ldots$, then x obviously satisfies the conditions $C(\mathbb{Z}/p^m)$ for $m = 0,1,2,\ldots$. Theorem 3.10 (i) and (ii) shows that the character map

$$\operatorname{char}_{\mathrm{inv}}^{\mathbb{Z}/p^\infty}\colon A_{\mathrm{inv}}(\mathbb{Z}/p^\infty) \;\to\; \prod_{(H)\in\operatorname{ccs}_f(\mathbb{Z}/p^\infty)} \mathbb{Z}$$

is injective and its image consists of the copy of the integers given by the constant series. This implies that the projection $\mathrm{pr}\colon \mathbb{Z}/p^\infty \to \{1\}$ induces a ring isomorphism

$$A_{\mathrm{inv}}(\mathrm{pr})\colon A_{\mathrm{inv}}(\{1\}) = \mathbb{Z} \;\xrightarrow{\cong}\; A_{\mathrm{inv}}(\mathbb{Z}/p^\infty).$$

In particular we conclude from Example 2.10 that the canonical ring homomorphism

$$T^{\mathbb{Z}/p^\infty}\colon \overline{A}(\mathbb{Z}/p^\infty) \xrightarrow{\cong} A_{\mathrm{inv}}(\mathbb{Z}/p^\infty)$$

of (3.2) is bijective.

3.3 The Inverse Limit Version of the Burnside Ring and Rational Representations

Analogously to $A_{\mathrm{inv}}(G)$ one defines $R_{\mathrm{inv},F}(G)$ for a field F to be the commutative ring with unit given by the inverse limit of the contravariant functor

$$R_{\mathrm{inv},F}(?)\colon \mathsf{Sub}_{\mathcal{FIN}}(G) \to \mathsf{RINGS}, \quad H \mapsto R_F(H).$$

This functor has been studied for $F = \mathbb{C}$ for instance in [1], [2]. The system of maps $P^H\colon A(H) \to R_{\mathbb{Q}}(H)$ for the finite subgroups $H \subseteq G$ appearing in Theorem 1.20 defines a ring homomorphism

$$P^G_{\mathrm{inv}}\colon A_{\mathrm{inv}}(G) \quad \to \quad R_{\mathrm{inv},\mathbb{Q}}(G). \tag{3.13}$$

The system of the restriction maps for every finite subgroup $H \subseteq G$ induces a homomorphism

$$S^{G,F}\colon \mathrm{Sw}^f(G;F) \quad \to \quad R_{\mathrm{inv},F}(G). \tag{3.14}$$

Although each of the maps P^H for the finite subgroups $H \subseteq G$ are rationally surjective by Theorem 1.20, the map P_{inv} need not to be rationally surjective in general, since inverse limits do not respects surjectivity or rationally surjectivity in general.

Example 3.15 ($R_{\mathrm{inv};\mathbb{Q}}(\mathbb{Z}/^\infty)$). Since every finite subgroup of $\mathbb{Z}/p^\infty$ is cyclic, we conclude from Theorem 1.20 that the map

$$P^{\mathbb{Z}/p^\infty}_{\mathrm{inv}}\colon A_{\mathrm{inv}}(\mathbb{Z}/p^\infty) \xrightarrow{\cong} R_{\mathrm{inv},\mathbb{Q}}(\mathbb{Z}/p^\infty)$$

is bijective. We have already seen in Example 3.12 that $p^*\colon A_{\mathrm{inv}}(\{1\}) \to A_{\mathrm{inv}}(\mathbb{Z}/p^\infty)$ is bijective. We conclude from Example 2.10 that the following diagram is commutative and consists of isomorphisms

$$\begin{tikzcd}
\overline{A}(\mathbb{Z}/p^\infty) \arrow[r, "T^{\mathbb{Z}/p^\infty}", "\cong"'] \arrow[d, "\overline{P}^{\mathbb{Z}/p^\infty}"', "\cong"] & A_{\mathrm{inv}}(\mathbb{Z}/p^\infty) \arrow[d, "P^{\mathbb{Z}/p^\infty}_{\mathrm{inv}}"', "\cong"] \\
\mathrm{Sw}^f(\mathbb{Z}/p^\infty;\mathbb{Q}) \arrow[r, "S^{\mathbb{Z}/p^\infty;\mathbb{Q}}_{\mathrm{inv}}", "\cong"'] & R_{\mathrm{inv},\mathbb{Q}}(\mathbb{Z}/p^\infty)
\end{tikzcd}$$

and is isomorphic by the maps induced by the projection $p\colon \mathbb{Z}/p^\infty \to \{1\}$ to the following commutative diagram whose corners are all isomorphic to $\mathbb{Z}$ and whose arrows are all the identity under this identification.

$$\begin{array}{ccc} \overline{A}(\{1\}) & \xrightarrow[\cong]{T^{\{1\}}} & A_{\mathrm{inv}}(\{1\}) \\ {\scriptstyle \overline{P}^{\{1\}}}\downarrow\cong & & {\scriptstyle P^{\{1\}}_{\mathrm{inv}}}\downarrow\cong \\ \mathrm{Sw}^f(\{1\};\mathbb{Q}) & \xrightarrow[\cong]{S^{\{1\};\mathbb{Q}}_{\mathrm{inv}}} & R_{\mathrm{inv},\mathbb{Q}}(\{1\}) \end{array}$$

Example 3.16 ($\mathrm{Sw}^f(\mathbb{Z}/p^\infty;\mathbb{C})$ and $R_{\mathrm{inv};\mathbb{C}}(\mathbb{Z}/^\infty)$). On the other hand let us consider $\mathbb{C}$ as coefficients. Consider the canonical map

$$S^{\mathbb{Z}/p^\infty,\mathbb{C}}\colon \mathrm{Sw}^f(\mathbb{Z}/p^\infty;\mathbb{C}) \to R_{\mathrm{inv},\mathbb{C}}(\mathbb{Z}/p^\infty)$$

which is induced by the restriction maps for all inclusions $H \to G$ of finite subgroups. If $\phi_H\colon R_{\mathrm{inv},\mathbb{C}}(\mathbb{Z}/p^\infty) \to R_{\mathbb{C}}(H)$ is the structure map of the inverse limit defining $R_{\mathrm{inv},\mathbb{C}}(\mathbb{Z}/p^\infty)$ for the finite subgroup $H \subseteq \mathbb{Z}/p^\infty$, then the composition

$$\mathrm{Sw}^f(\mathbb{Z}/p^\infty;\mathbb{C}) \xrightarrow{S^{\mathbb{Z}/p^\infty,\mathbb{C}}} R_{\mathrm{inv},\mathbb{C}}(\mathbb{Z}/p^\infty) \xrightarrow{\psi_H} R_{\mathbb{C}}(H)$$

is the map given by restriction with the inclusion of the finite subgroup H in $\mathbb{Z}/p^\infty$. We claim that this composition is surjective. Choose n with $H = \mathbb{Z}/p^n$. We have to find for every 1-dimensional complex $\mathbb{Z}/p^n$-representation V a 1-dimensional complex $\mathbb{Z}/p^\infty$-representation W such that V is the restriction of V. If V is given by the homomorphism $\mathbb{Z}/p^n \to S^1, \overline{k} \mapsto \exp(2\pi i k/p^n)$, then the desired W is given by the homomorphism

$$\mathbb{Z}/p^\infty = \mathbb{Z}[1/p]/\mathbb{Z} \to S^1, \quad k \mapsto \exp(2\pi i k).$$

This implies that both $\mathrm{Sw}^f(\mathbb{Z}/p^\infty;\mathbb{C})$ and $R_{\mathrm{inv},\mathbb{C}}(\mathbb{Z}/p^\infty)$ have infinite rank as abelian groups.

4 The Covariant Burnside Group

Next we give a third version for infinite groups which however will only be an abelian group, not necessarily a ring.

Definition 4.1 (Covariant Burnside group). *Define the* covariant Burnside group $\underline{A}(G)$ *of a group G to be the Grothendieck group which is associated to the abelian monoid under disjoint union of G-isomorphism classes of proper cofinite G-sets S, i.e. G-sets S for which the isotropy group of each element in S and the quotient $G\backslash S$ are finite.*

The cartesian product of two proper cofinite G-sets with the diagonal action is proper but not cofinite unless G is finite. So for infinite group G we do not get a ring structure on the Burnside group $\underline{A}(G)$. If G is finite the underlying abelian group of the Burnside ring $A(G)$ is just $\underline{A}(G)$. Given a group homomorphism

$f\colon G_0 \to G_1$, induction yields a homomorphism of abelian group $\underline{A}(G_0) \to \underline{A}(G_1)$ sending $[S]$ to $[G_1 \times_f S]$. Thus $\underline{A}$ becomes a covariant functor from GROUPS to $\mathbb{Z}$ – MODULES.

In the sequel we denote by $R[S]$ for a commutative ring R and a set S the free R-module with the set S as R-basis. We obtain an isomorphism of abelian groups

$$\beta^G\colon \mathbb{Z}[\mathrm{ccs}_f(G)] \xrightarrow{\cong} \underline{A}(G), \quad (H) \mapsto [G/H]. \tag{4.2}$$

The elementary proof of the following lemma is left to the reader.

Lemma 4.3. *Let H and K be subgroups of G. Then*

(i) $G/H^K = \{gH \mid g^{-1}Kg \subset H\}$;

(ii) The map
$$\phi\colon G/H^K \to \mathrm{ccs}(H), \qquad gH \mapsto g^{-1}Kg$$
induces an injection
$$W_GK\backslash(G/H^K) \to \mathrm{ccs}(H);$$

(iii) The W_GK-isotropy group of $gH \in G/H^K$ is $(gHg^{-1} \cap N_GK)/K$;

(iv) If H is finite, then G/H^K is a finite union of W_GK-orbits of the shape W_GK/L for finite subgroups $L \subset W_GK$. □

The next definition make sense because of the Lemma 4.3 above.

Definition 4.4 (L^2-character map). *Define for a finite subgroup $K \subset G$ the* L^2-character map at (K)

$$\underline{\mathrm{char}}_K^G\colon \underline{A}(G) \to \mathbb{Q}, \quad [S] \mapsto \sum_{i=1}^{r} |L_i|^{-1}$$

if W_GK/L_1, $W_GK/L_2,\ldots$, W_GK/L_r are the W_GK-orbits of S^K. Define the global L^2-character map *by*

$$\underline{\mathrm{char}}^G\colon \underline{A}(G) \to \mathbb{Q}[\mathrm{ccs}_f(G)], \quad [S] \mapsto \sum_{(K)\in \mathrm{ccs}_f(G)} \underline{\mathrm{char}}_K^G([S]) \cdot (K).$$

Notice that one gets from Lemma 4.3 the following explicit formula for the value of $\underline{\mathrm{char}}_K^G(G/H)$. Namely, define

$$\mathcal{L}_K(H) \quad := \quad \{(L) \in \mathrm{ccs}(H) \mid L \text{ conjugate to } K \text{ in } G\}.$$

For $(L) \in \mathcal{L}_K(H)$ choose $L \in (L)$ and $g \in G$ with $g^{-1}Kg = L$. Then

$$\begin{aligned} g(H \cap N_GL)g^{-1} &= gHg^{-1} \cap N_GK; \\ |(gHg^{-1} \cap N_GK)/K|^{-1} &= \frac{|K|}{|H \cap N_GL|}. \end{aligned}$$

This implies

$$\underline{\mathrm{char}}_K^G(G/H) \quad = \sum_{(L)\in\mathcal{L}_K(H)} \frac{|K|}{|H\cap N_GL|}. \tag{4.5}$$

Remark 4.6 (Burnside integrality relations). Let $T \subseteq \mathrm{ccs}_f(G)$ be a finite subset closed under taking subgroups, i.e. if $(H) \in T$, then $(K) \in T$ for every subgroup $K \subseteq H$. Since a finite subgroup contains only finitely many subgroups, one can write $\mathrm{ccs}_f(G)$ as the union of such subsets T. The union of two such subsets is again such a subset. So $R[\mathrm{ccs}_f(G)]$ is the colimit of the finitely generated free R-modules $R[T]$ if T runs to the finite subsets of $\mathrm{ccs}_f(G)$ closed under taking subgroups.

Fix a subset T of $\mathrm{ccs}_f(G)$ closed under taking subgroups. One easily checks using Lemma 4.3 that the composition

$$\mathbb{Z}[\mathrm{ccs}_f(G)] \xrightarrow{\beta^G} \underline{A}(G) \xrightarrow{\underline{\mathrm{char}}^G} \mathbb{Q}[\mathrm{ccs}_f(G)]$$

maps $\mathbb{Z}[T]$ to $\mathbb{Q}[T]$. We numerate the elements in T by (H_1), (H_2), ..., (H_r) such that H_i is subconjugated to (H_j) only if $i \leq j$ holds. Then the composition

$$\mathbb{Q}[\mathrm{ccs}_f(G)] \xrightarrow{\beta^G\otimes_{\mathbb{Z}}\mathbb{Q}} \underline{A}(G)\otimes_{\mathbb{Z}}\mathbb{Q} \xrightarrow{\underline{\mathrm{char}}^G} \mathbb{Q}[\mathrm{ccs}_f(G)]$$

induces a $\mathbb{Q}$-homomorphism

$$\mathbb{Q}[T] \xrightarrow{A_T} \mathbb{Q}[T]$$

given with respect to the basis $\{(H_i) \mid i = 1,2,\ldots r\}$ by a matrix A which is triangular and all whose diagonal entries are equal to 1. The explicit values of the entries in A_T are given by (4.5). The matrix A_T is invertible and one can actually write down an explicit formula for its inverse matrix B_T in terms of Möbius inversion [3, Chaper IV]. The matrix B_T yields an isomorphism

$$B_T\colon \mathbb{Q}[T] \xrightarrow{\cong} \mathbb{Q}[T].$$

Given an element $x \in \mathbb{Q}[\mathrm{ccs}_f(G)]$, we can find a finite subset $T \subseteq \mathrm{ccs}_f(G)$ closed under taking subgroups such that x lies already in $\mathbb{Q}[T]$. Then x lies in the image of the injective L^2-character map

$$\underline{\mathrm{char}}^G\colon \underline{A}(G) \to \mathbb{Q}[\mathrm{ccs}_f(G)]$$

of Definition 4.4 if and only if

$$B_T\colon \mathbb{Q}[T] \xrightarrow{\cong} \mathbb{Q}[T]$$

maps x to an element in $\mathbb{Z}[T]$. This means that the following rational numbers

$$\sum_{j=1}^{r} B_T(i,j)\cdot x(j)$$

for $i = 1, 2 \ldots, r$ are integers, where $B_T(i,j)$ and $x(j)$ are the components of B_T and x belonging to (i,j) and j. We call the condition that these rational numbers are integral numbers the *Burnside integrality relations*.

Now suppose that G is finite. Then the global L^2-character of Definition 4.4 is related to the classical character map (1.2) by the factors $|WK|^{-1}$, i.e. we have for each subgroup K of G and any finite G-set S

$$\mathrm{ch}_K^G(S) \quad = \quad |WK|^{-1} \cdot |S^K|. \tag{4.7}$$

One easily checks that for finite G under the identification (4.7) the Burnside integrality relations can be reformulated as a set of congruences, which consists of one congruence modulo $|W_G H|$ for every subgroup $H \subseteq G$ (compare Subsection 1.1).

4.1 Relation to L^2-Euler characteristic and Universal Property of the Covariant Burnside Group

The Burnside group $\underline{A}(G)$ can be characterized as the universal additive invariant for finite proper G-CW-complexes and the universal equivariant Euler characteristic of a finite proper G-CW-complex is mapped to the L^2-Euler characteristics of the $W_G H$-CW-complexes X^H by the character map at (H) for every finite subgroup $H \subseteq G$. In particular it is interesting to investigate the universal equivariant Euler characteristic of the classifying space for proper G-actions $\underline{E}G$ provided that there is a finite G-CW-model for $\underline{E}G$. All this is explained in [26, Section 6.6.2].

The relation of the universal equivariant Euler characteristic to the equivariant Euler class which is by definition the class of the Euler operator on a cocompact proper smooth G-manifold with G-invariant Riemannian metric in equivariant K-homology defined by Kasparov is analyzed in [31]. Equivariant Lefschetz classes for G-maps of finite proper G-CW-complexes are studied in [32].

4.2 The Covariant Burnside Group and the Colimit Version of the Burnside Ring Agree

Instead of the inverse-limit-version one may also consider the colimit-version

$$A_{\mathrm{cov}}(G) \ := \mathrm{colim}_{H \in \mathsf{Sub}_{\mathcal{FIN}}(G)} A(H)$$

where we consider A as a covariant functor from $\mathsf{Sub}_{\mathcal{FIN}}(G)$ to the category of $\mathbb{Z}$-modules by induction.

Theorem 4.8 ($A_{\mathrm{cov}}(G)$ and $\underline{A}(G)$ agree). *There obvious map induced by the various inclusions of a finite subgroup $H \subseteq G$*

$$V^G \colon A_{\mathrm{cov}}(G) \xrightarrow{\cong} \underline{A}(G)$$

is an bijection of abelian groups.

Proof. Recall that $\underline{A}(G)$ is the free abelian group with the set $\mathrm{ccs}_f(G)$ of conjugacy classes of finite subgroups as basis. Now the claim follows from the bijection (3.7). $\square$

The analogue for the representation ring is an open conjecture. Namely if we define for a field F of characteristic zero

$$R_{\mathrm{cov},F}(G) \ := \mathrm{colim}_{H \in \mathrm{Sub}_{\mathcal{FIN}}(G)} R_F(H)$$

we can consider

Conjecture 4.9. *The obvious map*

$$W^{G,F} \colon R_{\mathrm{cov},F}(G) \to K_0(F[G])$$

is an bijection of abelian groups.

This conjecture follows from the Farrell-Jones Conjecture for algebraic K-theory for $F[G]$ as explained in [30, Conjecture 3.3]. No counterexamples are known at the time of writing. For a status report about the Farrell-Jones Conjecture we refer for instance to [30, Section 5].

Let

$$P^G_{\mathrm{cov}} \colon A_{\mathrm{cov}}(G) \quad \to \quad R_{\mathrm{cov},F}(G). \tag{4.10}$$

be the map induced by the maps $P^H \colon A(H) \to R_F(H), [S] \mapsto [F[S]]$ for the various finite subgroups $H \subseteq G$.

4.3 The Covariant Burnside Group and the Projective Class Group

Given a finite proper G-set, the $\mathbb{Q}$-module $\mathbb{Q}[S]$ with the set S as basis becomes a finitely generated projective $\mathbb{Q}G$-module by the G-action on S. Thus we obtain a homomorphism

$$\underline{P}^G \colon \underline{A}(G) \quad \to \quad K_0(\mathbb{Q}G). \tag{4.11}$$

Conjecture 4.12. *The map $\underline{P}^G \colon \underline{A}(G) \to K_0(\mathbb{Q}G)$ is rationally surjective.*

This conjecture is motivated by the fact that it is implied by Theorem 1.20 and Theorem 4.8 together with Conjecture 4.9.

4.4 The Covariant Burnside Group as Module over the Finite-Set-Version

If S is a finite G-set and T is a cofinite proper G-set, then their cartesian product with the diagonal G-action is a cofinite proper G-set. Thus we obtain a pairing

$$\mu^G_A \colon \overline{A}(G) \times \underline{A}(G) \to \underline{A}(G), \quad ([S],[T]) \ \mapsto \ [S \times T] \tag{4.13}$$

Analogously one defines a pairing

$$\mu_K^G\colon \operatorname{Sw}^f(G;\mathbb{Q}) \times K_0(\mathbb{Q}G) \to K_0(\mathbb{Q}G), \quad ([M],[P]) \;\mapsto\; [M \otimes_{\mathbb{Q}} P] \tag{4.14}$$

which turns $K_0(\mathbb{Q}G)$ into a $\operatorname{Sw}^f(G;\mathbb{Q})$-module. These two pairings are compatible in the obvious sense (see (0.1)).

4.5 A Pairing between the Inverse-Limit-Version and the Covariant Burnside Group

Given a finite group H, we obtain a homomorphism of abelian groups

$$\nu^H\colon A(H) \to \hom_{\mathbb{Z}}(A(H),\mathbb{Z}), \quad [S] \mapsto \nu^H(S),$$

where $\nu^H(S)\colon A(H) \to \mathbb{Z}$ maps $[T]$ to $|G\backslash(S\times T)|$ for the diagonal G-operation on $S \times T$. A group homomorphism $f\colon H \to K$ induces a homomorphism of abelian groups $\operatorname{res}_\phi\colon A(K) \to A(H)$ by restriction and a homomorphism of abelian groups $\operatorname{ind}_\phi\colon A(H) \to A(K)$ by induction. The latter induces a homomorphism of abelian groups $(\operatorname{ind}_\phi)^*\colon A(K) \to A(H)$. One easily checks that the collection of the homomorphisms ν^H for the subgroups $H \subseteq G$ induces a natural transformation of the contravariant functors from $\mathtt{Sub}_{\mathcal{FIN}}(G)$ to $\mathbb{Z}$-modules given by $A(?)$ and $\hom_{\mathbb{Z}}(A(?),\mathbb{Z})$. Passing to the inverse limit, the canonical isomorphism of abelian groups

$$\hom_{\mathbb{Z}}\left(\operatorname{colim}_{H\in\mathtt{Sub}_{\mathcal{FIN}}(G)} A(H),\mathbb{Z}\right) \xrightarrow{\cong} \operatorname{invlim}_{H\in\mathtt{Sub}_{\mathcal{FIN}}(G)} \hom_{\mathbb{Z}}(A(H),\mathbb{Z})$$

and the isomorphism appearing in Theorem 4.8 yield a homomorphism of abelian groups

$$\nu_A^G\colon A_{\mathrm{inv}}(G) \to \hom_{\mathbb{Z}}(\underline{A}(G);\mathbb{Z})$$

which we can also write a bilinear pairing

$$\nu_A^G\colon A_{\mathrm{inv}}(G) \times \underline{A}(G) \quad\to\quad \mathbb{Z}. \tag{4.15}$$

For a field F of characteristic zero, there is an analogous pairing

$$\nu_R^G\colon R_{\mathrm{inv},F}(G) \times R_{\mathrm{cov},F}(G) \quad\to\quad \mathbb{Z} \tag{4.16}$$

which comes from the various homomorphisms of abelian groups for each finite subgroup $H \subseteq G$

$$R_F(H) \to \hom_{\mathbb{Z}}(R_F(H);\mathbb{Z}), [V] \;\mapsto\; ([W] \mapsto \dim_F(F \otimes_{FG} (V \otimes_F W)))\,.$$

The pairings ν_A^G and ν_R^G are compatible in the obvious sense (see (0.1)).

The homomorphism of abelian groups $\overline{\nu_R^G}\colon R_{\mathrm{inv},F}(G) \to \hom_{\mathbb{Z}}(R_{\mathrm{cov},F}(G),\mathbb{Z})$ associated to the pairing ν_R^G is injective. Its cokernel is finite if G has only finitely many conjugacy classes of finite subgroups. It is rationally surjective if there is an upper bound on the orders of finite subgroups of G.

Define the homomorphism $Q_A^G\colon \underline{A}(G) \to \mathbb{Z}$ and $Q_K^G\colon K_0(\mathbb{Q}G) \to \mathbb{Z}$ respectively by sending $[S]$ to $|G\backslash S|$ and $[P]$ to $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Q}G} P)$ respectively. Then the pairings μ_A^G, ν_A^G, μ_K^G and ν_R^G are compatible in the obvious sense (see (0.1)).

4.6 Some Computations of the Covariant Burnside Group

Example 4.17 ($\underline{A}(G)$ for torsionfree G). Suppose that G is torsionfree. Then the inclusion $i\colon \{1\} \to G$ induces a $\mathbb{Z}$-isomorphism

$$\underline{A}(i)\colon \underline{A}(\{1\}) = \mathbb{Z} \xrightarrow{\cong} \underline{A}(G).$$

Example 4.18 (Extensions of $\mathbb{Z}^n$ with $\mathbb{Z}/p$ as quotient). Suppose that G can be written as an extension $1 \to A \to G \to \mathbb{Z}/p \to 1$ for some fixed prime number p and for $A = \mathbb{Z}^n$ for some integer $n \geq 0$ and that G is not torsionfree. We use the notation of Example 3.6 in the sequel. We obtain an exact sequence

$$0 \to \bigoplus_{H^1(\mathbb{Z}/p;A)} \underline{A}(\{1\}) \to \underline{A}(\{1\}) \oplus \bigoplus_{H^1(\mathbb{Z}/p;A)} \underline{A}(\mathbb{Z}/p) \to \underline{A}(G) \to 0$$

If we define $\widetilde{\underline{A}}(G)$ as the kernel of $\underline{A}(G) \to \underline{A}(\{1\})$, we obtain an isomorphism

$$\bigoplus_{H^1(\mathbb{Z}/p;A)} \widetilde{\underline{A}}(\mathbb{Z}/p) \xrightarrow{\cong} \widetilde{\underline{A}}(G).$$

Let H_0 be the trivial subgroup and H_1, H_2, ..., H_r be a complete set of representatives of the conjugacy classes of finite subgroups. Then $r = |H^1(\mathbb{Z}/p;A)|$ and $\underline{A}(G)$ is the free abelian group of rank $r+1$ with $\{[G/H_0], [G/H_1], \ldots [G/H_r]\}$ as basis. Each H_i is isomorphic to $\mathbb{Z}/p$. We compute using (4.5)

$$\begin{array}{lcll}
\mathrm{ch}^G_{H_0}(G/H_0) & = & 1; & \\
\mathrm{ch}^G_{H_0}(G/H_j) & = & \frac{1}{p} & j = 1, 2, \ldots, r; \\
\mathrm{ch}^G_{H_i}(G/H_j) & = & 1 & i = j,\ i,j = 1, 2, \ldots, r; \\
\mathrm{ch}^G_{H_i}(G/H_j) & = & 0 & i \neq j,\ i,j = 1, 2, \ldots, r.
\end{array}$$

The Burnside integrality conditions (see Remark 4.6) become in this case for $x = (x(i)) \in \bigoplus_{i=0}^r \mathbb{Q}$.

$$\begin{array}{rcll}
x(0) - \frac{1}{p} \cdot \sum_{i=1}^r x(i) & \in & \mathbb{Z}; & \\
x(i) & \in & \mathbb{Z} & i = 1, 2, \ldots, r.
\end{array}$$

Example 4.19 (Groups with appropriate maximal finite subgroups). Consider the groups appearing in Example 3.4. In the notation of Example 3.4 we get an isomorphism of $\mathbb{Z}$-modules

$$\bigoplus_{i \in I} \widetilde{A}(M_i) \xrightarrow{\bigoplus_{i \in I} \widetilde{\underline{A}}(j_i)} \widetilde{\underline{A}}(G).$$

5 Equivariant Cohomology Theories

In this section we recall the axioms of a (proper) equivariant cohomology theory of [27]. They are dual to the ones of a (proper) equivariant homology theory as described in [25, Section 1].

5.1 Axiomatic Description of a G-Cohomology Theory

Fix a group G and an commutative ring R. A G-CW-pair (X, A) is a pair of G-CW-complexes. Recall that a G-CW-complex X is proper if and only if all isotropy groups of X are finite, and is finite if X is obtained from A by attaching finitely many equivariant cells, or, equivalently, if $G\backslash X$ is compact. A *G-cohomology theory $\mathcal{H}_G^*$ with values in R-modules* is a collection of covariant functors $\mathcal{H}_G^n$ from the category of G-CW-pairs to the category of R-modules indexed by $n \in \mathbb{Z}$ together with natural transformations $\delta_G^n(X, A)\colon \mathcal{H}_G^n(X, A) \to \mathcal{H}_G^{n+1}(A) := \mathcal{H}_G^{n+1}(A, \emptyset)$ for $n \in \mathbb{Z}$ such that the following axioms are satisfied:

- G-homotopy invariance

  If f_0 and f_1 are G-homotopic maps $(X, A) \to (Y, B)$ of G-CW-pairs, then $\mathcal{H}_G^n(f_0) = \mathcal{H}_G^n(f_1)$ for $n \in \mathbb{Z}$;

- Long exact sequence of a pair

  Given a pair (X, A) of G-CW-complexes, there is a long exact sequence

  $$\cdots \xrightarrow{\delta_G^{n-1}} \mathcal{H}_G^n(X, A) \xrightarrow{\mathcal{H}_G^n(j)} \mathcal{H}_G^n(X) \xrightarrow{\mathcal{H}_G^n(i)} \mathcal{H}_G^n(A) \xrightarrow{\delta_G^n} \cdots,$$

  where $i\colon A \to X$ and $j\colon X \to (X, A)$ are the inclusions;

- Excision

  Let (X, A) be a G-CW-pair and let $f\colon A \to B$ be a cellular G-map of G-CW-complexes. Equip $(X \cup_f B, B)$ with the induced structure of a G-CW-pair. Then the canonical map $(F, f)\colon (X, A) \to (X \cup_f B, B)$ induces an isomorphism

  $$\mathcal{H}_G^n(F, f)\colon \mathcal{H}_G^n(X, A) \xrightarrow{\cong} \mathcal{H}_G^n(X \cup_f B, B);$$

- Disjoint union axiom

  Let $\{X_i \mid i \in I\}$ be a family of G-CW-complexes. Denote by $j_i\colon X_i \to \coprod_{i \in I} X_i$ the canonical inclusion. Then the map

  $$\prod_{i \in I} \mathcal{H}_G^n(j_i)\colon \mathcal{H}_G^n\left(\coprod_{i \in I} X_i\right) \xrightarrow{\cong} \prod_{i \in I} \mathcal{H}_G^n(X_i)$$

  is bijective.

If $\mathcal{H}_G^*$ is defined or considered only for proper G-CW-pairs (X, A), we call it a *proper G-cohomology theory $\mathcal{H}_G^*$ with values in R-modules.*

5.2 Axiomatic Description of an Equivariant Cohomology Theory

Let $\alpha\colon H \to G$ be a group homomorphism. Given an H-space X, define the *induction of X with α* to be the G-space $\operatorname{ind}_\alpha X$ which is the quotient of $G \times X$

by the right H-action $(g,x)\cdot h := (g\alpha(h), h^{-1}x)$ for $h \in H$ and $(g,x) \in G \times X$. If $\alpha\colon H \to G$ is an inclusion, we also write ind_H^G instead of $\operatorname{ind}_\alpha$.

A *(proper) equivariant cohomology theory $\mathcal{H}^*_?$ with values in R-modules* consists of a collection of (proper) G-cohomology theories $\mathcal{H}^*_G$ with values in R-modules for each group G together with the following so called *induction structure*: given a group homomorphism $\alpha\colon H \to G$ and a (proper) H-CW-pair (X,A) there are for each $n \in \mathbb{Z}$ natural homomorphisms

$$\operatorname{ind}_\alpha\colon \mathcal{H}^n_G(\operatorname{ind}_\alpha(X,A)) \quad\to\quad \mathcal{H}^n_H(X,A) \tag{5.1}$$

satisfying

(i) Bijectivity

If $\ker(\alpha)$ acts freely on X, then $\operatorname{ind}_\alpha\colon \mathcal{H}^n_G(\operatorname{ind}_\alpha(X,A)) \to \mathcal{H}^n_H(X,A)$ is bijective for all $n \in \mathbb{Z}$;

(ii) Compatibility with the boundary homomorphisms

$\delta^n_H \circ \operatorname{ind}_\alpha = \operatorname{ind}_\alpha \circ \delta^n_G$;

(iii) Functoriality

Let $\beta\colon G \to K$ be another group homomorphism. Then we have for $n \in \mathbb{Z}$

$$\operatorname{ind}_{\beta\circ\alpha} \quad=\quad \operatorname{ind}_\alpha \circ \operatorname{ind}_\beta \circ \mathcal{H}^n_K(f_1)\colon \mathcal{H}^n_H(\operatorname{ind}_{\beta\circ\alpha}(X,A)) \to \mathcal{H}^n_K(X,A),$$

where $f_1\colon \operatorname{ind}_\beta \operatorname{ind}_\alpha(X,A) \xrightarrow{\cong} \operatorname{ind}_{\beta\circ\alpha}(X,A)$, $(k,g,x) \mapsto (k\beta(g),x)$ is the natural K-homeomorphism;

(iv) Compatibility with conjugation

For $n \in \mathbb{Z}$, $g \in G$ and a (proper) G-CW-pair (X,A) the homomorphism $\operatorname{ind}_{c(g)\colon G\to G}\colon \mathcal{H}^n_G(\operatorname{ind}_{c(g)\colon G\to G}(X,A)) \to \mathcal{H}^n_G(X,A)$ agrees with $\mathcal{H}^n_G(f_2)$, where f_2 is the G-homeomorphism $f_2\colon (X,A) \to \operatorname{ind}_{c(g)\colon G\to G}(X,A)$, $x \mapsto (1, g^{-1}x)$ and $c(g)(g') = gg'g^{-1}$.

This induction structure links the various G-cohomology theories for different groups G.

Sometimes we will need the following lemma whose elementary proof is analogous to the one in [25, Lemma 1.2].

Lemma 5.2. *Consider finite subgroups $H, K \subseteq G$ and an element $g \in G$ with $gHg^{-1} \subseteq K$. Let $R_{g^{-1}}\colon G/H \to G/K$ be the G-map sending $g'H$ to $g'g^{-1}K$ and $c(g)\colon H \to K$ be the homomorphism sending h to ghg^{-1}. Let $\operatorname{pr}\colon (\operatorname{ind}_{c(g)\colon H\to K}\{\bullet\}) \to \{\bullet\}$ be the projection to the one-point space $\{\bullet\}$. Then the following diagram commutes*

$$\begin{array}{ccc}
\mathcal{H}^n_G(G/K) & \xrightarrow{\mathcal{H}^n_G(R_{g^{-1}})} & \mathcal{H}^n_G(G/H) \\
{\scriptstyle \operatorname{ind}_K^G}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle \operatorname{ind}_H^G}\big\downarrow{\scriptstyle\cong} \\
\mathcal{H}^n_K(\{\bullet\}) & \xrightarrow{\operatorname{ind}_{c(g)} \circ \mathcal{H}^n_K(\operatorname{pr})} & \mathcal{H}^n_H(\{\bullet\})
\end{array}$$

5.3 Multiplicative Structures

Let $\mathcal{H}_G^*$ be a (proper) G-cohomology theory. A *multiplicative structure* assigns to a (proper) G-CW-complex X with G-CW-subcomplexes $A, B \subseteq X$ natural R-homomorphisms

$$\cup\colon \mathcal{H}_G^m(X,A) \otimes_R \mathcal{H}_G^n(X,B) \quad \to \quad \mathcal{H}_G^{m+n}(X, A \cup B). \tag{5.3}$$

This product is required to be compatible with the boundary homomorphism of the long exact sequence of a G-CW-pair, namely, for $u \in \mathcal{H}_G^m(A)$ and $v \in \mathcal{H}^nG(X)$ and $i\colon A \to X$ the inclusion we have $\delta(u \cup v) = \delta(u \cup \mathcal{H}^n(i)(v)$. Moreover, it is required to be graded commutative, to be associative and to have a unit $1 \in \mathcal{H}_G^0(X)$ for every (proper) G-CW-complex X.

Let $\mathcal{H}_?^*$ be a (proper) equivariant cohomology theory. A *multiplicative structure* on it assigns a multiplicative structure to the associated (proper) G-cohomology theory $\mathcal{H}_G^*$ for every group G such that for each group homomorphism $\alpha\colon H \to G$ the maps given by the induction structure of (5.1)

$$\operatorname{ind}_\alpha\colon \mathcal{H}_G^n(\operatorname{ind}_\alpha(X,A)) \quad \xrightarrow{\cong} \quad \mathcal{H}_H^n(X,A)$$

are in the obvious way compatible with the multiplicative structures on $\mathcal{H}_G^*$ and $\mathcal{H}_H^*$.

Example 5.4. Equivariant cohomology theories coming from non-equivariant ones). Let $\mathcal{K}^*$ be a (non-equivariant) cohomology theory with multiplicative structure, for instance singular cohomology or topological K-theory. We can assign to it an equivariant cohomology theory with multiplicative structure $\mathcal{H}_?^*$ in two ways. Namely, for a group G and a pair of G-CW-complexes (X,A) we define $\mathcal{H}_G^n(X,A)$ by $\mathcal{K}^n(G\backslash(X,A))$ or by $\mathcal{K}^n(EG \times_G (X,A))$.

5.4 Equivariant Topological K-Theory

In [29] equivariant topological K-theory is defined for finite proper equivariant CW-complexes in terms of equivariant vector bundles. For finite G it reduces to the classical notion which appears for instance in [4]. Its relation to equivariant KK-theory is explained in [38]. This definition is extended to (not necessarily finite) proper equivariant CW-complexes in [29] in terms of equivariant spectra using Γ-spaces. This equivariant cohomology theory $K_?^*$ has the property that for any finite subgroup H of a group G we have

$$K_G^n(G/H) \;=\; K_H^n(\{\bullet\}) \;=\; \begin{cases} R_{\mathbb{C}}(H) & n \text{ even;} \\ \{0\} & n \text{ odd.} \end{cases}$$

6 Equivariant Stable Cohomotopy in Terms of Real Vector Bundles

In this section we give a construction of equivariant cohomotopy for finite proper G-CW-complexes for infinite groups G in terms of real vector bundles and maps between the associated sphere bundles. The result will be an equivariant cohomology theory with multiplicative structure for finite proper equivariant CW-complexes. It generalizes the well-known approach for finite groups in terms of representations. We will first give the construction, show why it reduces to the classical construction for a finite group and explain why we need to consider equivariant vector bundles and not only representations in the case of an infinite group.

6.1 Preliminaries about Equivariant Vector Bundles

We will need the following notation. Given a finite-dimensional (real) vector space V, we denote by S^V its *one-point compactification*. We will use the point at infinity as the base point of S^V in the sequel. Given two finite-dimensional vector spaces V and W, the obvious inclusion $V \oplus W \to S^V \wedge S^W$ induces a natural homeomorphism

$$\phi(V,W)\colon S^{V\oplus W} \xrightarrow{\cong} S^V \wedge S^W. \tag{6.1}$$

Let

$$\nabla\colon S^{\mathbb{R}} \to S^{\mathbb{R}} \vee S^{\mathbb{R}}. \tag{6.2}$$

be the pinching map, which sends $x > 0$ to $\ln(x) \in \mathbb{R} \subseteq S^{\mathbb{R}}$ in the first summand, $x < 0$ to $-\ln(-x) \in \mathbb{R} \subseteq S^{\mathbb{R}}$ in the second summand and 0 and ∞ to the base point in $S^{\mathbb{R}} \vee S^{\mathbb{R}}$. This is under the identification $S^{\mathbb{R}} = S^1$ the standard pinching map $S^1 \to S^1/S^0 \cong S^1 \vee S^1$, at least up to pointed homotopy.

We need some basics about G-vector bundles over proper G-CW-complexes. More details can be found for instance in [29, Section 1]. Recall that a G-CW-complex is proper if and only if all its isotropy groups are finite. A *G-vector bundle* $\xi\colon E \to X$ over X is a real vector bundle with a G-action on E such that ξ is G-equivariant and for each $g \in G$ the map $l_g\colon E \to E$ is fiberwise a linear isomorphism. Such a G-vector bundle is automatically trivial in the equivariant sense that for each $x \in X$ there is a G-neighborhood U, a G-map $f\colon U \to G/H$ and a H-representation V such that $\xi|_U$ is isomorphic as G-vector bundle to the pullback of the G-vector bundle $G \times_H V \to G/H$ by f. Denote the fiber $\xi^{-1}(x)$ over a point $x \in X$ by E_x. This is a representation of the finite isotropy group G_x of $x \in X$. A *map of G-vector bundles* $(\overline{f}, f)$ from $\xi_0\colon E_0 \to X_0$ to $\xi_1\colon E_1 \to X_1$ consists of G-maps $\overline{f}\colon E_0 \to E_1$ and $f\colon X_0 \to X_1$ with $\xi_1 \circ \overline{f} = f \circ \xi_0$ such that $\overline{f}$ is fiberwise a (not necessarily injective or surjective) linear map.

Given a G-vector bundle $\xi\colon E \to X$, let $S^\xi\colon S^E \to X$ be the locally trivial G-bundle whose fiber over $x \in X$ is S^{E_x}. Consider two G-vector bundles $\xi\colon E \to X$ and $\mu\colon F \to X$. Let $S^\xi \wedge_X S^\mu\colon S^E \wedge_X S^F \to X$ be the G-bundle whose fiber of $x \in X$ is $S^{E_x} \wedge S^{F_x}$, in other word it is obtained from $S^\xi\colon S^E \to X$ and $S^\mu\colon S^F \to X$ by the fiberwise smash product. Define $S^\xi \vee_X S^\mu$ analogously. From (6.1) we obtain a natural G-bundle isomorphism

$$\phi(\xi,\mu)\colon S^{\xi\oplus\mu} \xrightarrow{\cong} S^\xi \wedge_X S^\mu. \tag{6.3}$$

The next basic lemma is proved in [29, Lemma 3.7].

Lemma 6.4. *Let $f\colon X \to Y$ be a G-map between finite proper G-CW-complexes and ξ a G-vector bundle over X. Then there is a G-vector bundle μ over Y such that ξ is a direct summand in $f^*\mu$.*

6.2 The Definition of Equivariant Cohomotopy Groups

Fix a proper G-CW-complex X. Let $\mathsf{SPHB}^G(X)$ be the following category. Objects are G-CW-vector bundles $\xi\colon E \to X$ over X. A morphism from $\xi\colon E \to X$ to $\mu\colon F \to X$ is a bundle map $u\colon S^\xi \to S^\mu$ covering the identity $\mathrm{id}\colon X \to X$ which fiberwise preserve the base points. (We do not require that u is fiberwise a homotopy equivalence.)

A *homotopy h between two morphisms* u_0, u_1 from $\xi\colon E \to X$ to $\mu\colon F \to X$ is a bundle map $h\colon S^\xi \times [0,1] \to S^\mu$ from the bundle $S^\xi \times \mathrm{id}_{[0,1]}\colon S^E \times [0,1] \to X \times [0,1]$ to S^μ which covers the projection $X \times [0,1] \to X$ and fiberwise preserve the base points such that its restriction to $X \times \{i\}$ is u_i for $i = 0,1$.

Let $\underline{\mathbb{R}^k}$ be the trivial vector bundle $X \times \mathbb{R}^k \to X$. We consider it as a G-vector bundle using the trivial G-action on $\mathbb{R}^k$. Fix an integer $n \in \mathbb{Z}$. Given two objects ξ_i, two non-negative integers k_i with $k_i + n \geq 0$ and two morphisms

$$u_i\colon S^{\xi_i \oplus \underline{\mathbb{R}^{k_i}}} \to S^{\xi_i \oplus \underline{\mathbb{R}^{k_i+n}}}$$

for $i = 0,1$, we call u_0 and u_1 *equivalent*, if there are objects μ_i in $\mathsf{SPHB}^G(X)$ for $i = 0,1$ and an isomorphism of G-vector bundles $v\colon \mu_0 \oplus \xi_0 \xrightarrow{\cong} \mu_1 \oplus \xi_1$ such that the following diagram in $\mathsf{SPHB}^G(X)$ commutes up to homotopy

$$\begin{array}{ccc}
S^{\mu_0 \oplus \underline{\mathbb{R}^{k_1}}} \wedge_X S^{\xi_0 \oplus \underline{\mathbb{R}^{k_0}}} & \xrightarrow{\mathrm{id} \wedge_X u_0} & S^{\mu_0 \oplus \underline{\mathbb{R}^{k_1}}} \wedge_X S^{\xi_0 \oplus \underline{\mathbb{R}^{k_0+n}}} \\
\sigma_1 \downarrow & & \sigma_2 \downarrow \\
S^{\mu_0 \oplus \xi_0 \oplus \underline{\mathbb{R}^{k_0+k_1}}} & & S^{\mu_0 \oplus \xi_0 \oplus \underline{\mathbb{R}^{k_0+k_1+n}}} \\
S^{v \oplus \mathrm{id}} \downarrow & & S^{v \oplus \mathrm{id}} \downarrow \\
S^{\mu_1 \oplus \xi_1 \oplus \underline{\mathbb{R}^{k_0+k_1}}} & & S^{\mu_1 \oplus \xi_1 \oplus \underline{\mathbb{R}^{k_0+k_1+n}}} \\
\sigma_3 \downarrow & & \sigma_4 \downarrow \\
S^{\mu_1 \oplus \underline{\mathbb{R}^{k_0}}} \wedge_X S^{\xi_1 \oplus \underline{\mathbb{R}^{k_1}}} & \xrightarrow{\mathrm{id} \wedge_X u_1} & S^{\mu_1 \oplus \underline{\mathbb{R}^{k_0}}} \wedge_X S^{\xi_1 \oplus \underline{\mathbb{R}^{k_1+n}}}
\end{array}$$

where σ_i stands for the obvious isomorphism coming from (6.3) and permutation.

We define $\pi^n_G(X)$ to be the set of equivalence classes of such morphisms $u\colon S^{\xi \oplus \underline{\mathbb{R}^k}} \to S^{\xi \oplus \underline{\mathbb{R}^{k+n}}}$ under the equivalence relation mentioned above. It becomes an abelian group as follows.

The zero element is represented by the class of any morphism $c\colon S^{\xi \oplus \underline{\mathbb{R}^k}} \to S^{\xi \oplus \underline{\mathbb{R}^{k+n}}}$ which is fiberwise the constant map onto the base point.

Consider classes $[u_0]$ and $[u_1]$ represented by two morphisms of the shape $u_i\colon S^{\xi_i \oplus \underline{\mathbb{R}^{k_i}}} \to S^{\xi_i \oplus \underline{\mathbb{R}^{k_i+n}}}$ for $i = 0, 1$. Define their sum by the class represented by the morphism

$$
\begin{aligned}
S^{\xi_0 \oplus \xi_1 \oplus \underline{\mathbb{R}^{k_0+k_1+1}}} &\xrightarrow{\sigma_1} S^{\xi_0 \oplus \underline{\mathbb{R}^{k_0}}} \wedge_X S^{\xi_1 \oplus \underline{\mathbb{R}^{k_1}}} \wedge_X S^{\underline{\mathbb{R}}} \\
&\xrightarrow{\mathrm{id} \wedge_X \mathrm{id} \wedge_X \underline{\nabla}} S^{\xi_0 \oplus \underline{\mathbb{R}^{k_0}}} \wedge_X S^{\xi_1 \oplus \underline{\mathbb{R}^{k_1}}} \wedge_X \left(S^{\underline{\mathbb{R}}} \vee_X S^{\underline{\mathbb{R}}}\right) \\
&\xrightarrow{\tau} \left(S^{\xi_0 \oplus \underline{\mathbb{R}^{k_0}}} \wedge_X S^{\xi_1 \oplus \underline{\mathbb{R}^{k_1}}} \wedge_X S^{\underline{\mathbb{R}}}\right) \vee_X \left(S^{\xi_0 \oplus \underline{\mathbb{R}^{k_0}}} \wedge_X S^{\xi_1 \oplus \underline{\mathbb{R}^{k_1}}} \wedge_X S^{\underline{\mathbb{R}}}\right) \\
&\xrightarrow{(u_0 \wedge_X \mathrm{id} \wedge_X \mathrm{id}) \vee_X (\mathrm{id} \wedge_X u_1 \wedge_X \mathrm{id})} \\
&\left(S^{\xi_0 \oplus \underline{\mathbb{R}^{k_0+n}}} \wedge_X S^{\xi_1 \oplus \underline{\mathbb{R}^{k_1}}} \wedge_X S^{\underline{\mathbb{R}}}\right) \vee_X \left(S^{\xi_0 \oplus \underline{\mathbb{R}^{k_0}}} \wedge_X S^{\xi_1 \oplus \underline{\mathbb{R}^{k_1+n}}} \wedge_X S^{\underline{\mathbb{R}}}\right) \\
&\xrightarrow{\sigma_5 \vee_X \sigma_6} \left(S^{\xi_0 \oplus \xi_1 \oplus \underline{\mathbb{R}^{k_0+k_1+1+n}}}\right) \vee_X \left(S^{\xi_0 \oplus \xi_1 \oplus \underline{\mathbb{R}^{k_0+k_1+1+n}}}\right) \\
&\xrightarrow{\mathrm{id} \vee_X \mathrm{id}} S^{\xi_0 \oplus \xi_1 \oplus \underline{\mathbb{R}^{k_0+k_1+1+n}}}
\end{aligned}
$$

where the isomorphisms σ_i are given by permutation and the isomorphisms (6.3) τ is given by the distributivity law for smash and wedge-products and $\underline{\nabla}$ is defined fiberwise by the map ∇ of (6.2).

Consider a class $[u]$ represented by the morphisms of the shape $u\colon S^{\xi \oplus \underline{\mathbb{R}^k}} \to S^{\xi \oplus \underline{\mathbb{R}^{k+n}}}$. Define its inverse as the class represented by the composition

$$
S^{\xi \oplus \underline{\mathbb{R}^{k+1}}} \xrightarrow{\sigma_1} S^{\xi \oplus \underline{\mathbb{R}^k}} \wedge_X S^{\underline{\mathbb{R}}} \xrightarrow{u \wedge_X \underline{-\mathrm{id}}} S^{\xi \oplus \underline{\mathbb{R}^{k+n}}} \wedge_X S^{\underline{\mathbb{R}}} \xrightarrow{\sigma_2} S^{\xi \oplus \underline{\mathbb{R}^{k+1+n}}},
$$

where $\underline{-\mathrm{id}}$ is fiberwise the map $-\mathrm{id}\colon \mathbb{R} \to \mathbb{R}$. The proof that this defines the structure of an abelian group is essentially the same as the one that the abelian group structure on the stable homotopy groups of a space is well-defined.

Next consider a pair (X, A) of proper G-CW-complexes. In order to define the abelian group $\pi_G^n(X, A)$ we consider morphisms $u\colon S^{\xi \oplus \underline{\mathbb{R}^k}} \to S^{\xi \oplus \underline{\mathbb{R}^{k+n}}}$ with $k + n \geq 0$ in $\mathsf{SPHB}^G(X)$ such that u is trivial over A, i.e. for every point $a \in A$ the map $u_a\colon S^{\xi_a \oplus \mathbb{R}^k} \to S^{\xi_a \oplus \mathbb{R}^{k+n}}$ is the constant map onto the base point. In the definition of the equivalence relation for such pairs we require that the homotopies of two morphisms are stationary over A. Then define $\pi_G^n(X, A)$ as the set of equivalence classes of morphism $u\colon S^{\xi \oplus \underline{\mathbb{R}^k}} \to S^{\xi \oplus \underline{\mathbb{R}^{k+n}}}$ in $\mathsf{SPHB}^G(X)$ with $k + n \geq 0$ which are trivial over A using this equivalence relation. The definition of the abelian group structure goes through word by word.

Notice that in the definition of $\pi_G^n(X, A)$ we cannot use as in the classical settings cones or suspensions since these contain G-fixed points and hence are not proper unless G is finite. The properness is needed to ensure that certain basic facts about bundles carry over to the equivariant setting.

6.3 The Proof of the Axioms of an Equivariant Cohomology Theory with Multiplicative Structure

In this subsection we want to prove

Theorem 6.5 (Equivariant cohomotopy in terms of equivariant vector bundles). *Equivariant Cohomotopy $\pi_?^*$ defines an equivariant cohomology theory with multiplicative structure for finite proper equivariant CW-complexes. For every finite subgroup H of the group G the abelian groups $\pi_G^n(G/H)$ and π_H^n are isomorphic for every $n \in \mathbb{Z}$ and the rings $\pi_G^0(G/H)$ and $\pi_H^0 = A(H)$ are isomorphic.*

Consider a G-map $f\colon (X,A) \to (Y,B)$ of pairs of proper G-CW-complexes. Using the pullback construction one defines a homomorphism of abelian groups

$$\pi_G^n(f)\colon \pi_G^n(Y,B) \to \pi_n^G(X,A).$$

Thus π_G^n becomes a contravariant functor from the category of proper G-CW-pairs to the category of abelian groups.

Lemma 6.6. *Let $f_0, f_1\colon (X,A) \to (Y,B)$ be two G-maps of pairs of proper G-CW-complexes. If they are G-homotopic, then $\pi_G^n(f_0) = \pi_G^n(f_1)$.*

Proof. By the naturality of π_G^n it suffices to prove that $\pi_G^n(h) = \mathrm{id}$ holds for the G-map

$$h\colon (X,A) \times [0,1] \to (X,A) \times [0,1], \quad (x,t) \ \mapsto \ (x,0).$$

Let the element $[u] \in \pi_G^n((X,A) \times [0,1])$ be given by the morphism $u\colon S^{\xi \oplus \underline{\mathbb{R}^k}} \to S^{\xi \oplus \underline{\mathbb{R}^{k+n}}}$ in $\mathsf{SPHB}^G(X \times [0,1])$ with $k+n \geq 0$ which is trivial over $X \times \{0,1\} \cup A \times [0,1]$. By [29, Theorem 1.2] there is an isomorphism of G-vector bundles $v\colon \xi \xrightarrow{\cong} h^*\xi$ covering the identity $\mathrm{id}\colon X \times [0,1] \to X \times [0,1]$ such that v restricted to $X \times \{0\}$ is the identity $\mathrm{id}\colon \xi|_{X \times \{0\}} \to \xi|_{X \times \{0\}}$. The composition of morphisms in $\mathsf{SPHB}^G(X \times [0,1])$

$$u'\colon S^{h^*\xi \oplus \underline{\mathbb{R}^k}} \xrightarrow{S^{v^{-1}} \oplus \mathrm{id}} S^{\xi \oplus \underline{\mathbb{R}^k}} \xrightarrow{u} S^{\xi \oplus \underline{\mathbb{R}^{k+n}}} \xrightarrow{S^v \oplus \mathrm{id}} S^{h^*\xi \oplus \underline{\mathbb{R}^{k+n}}}$$

has the property that its restriction to $X \times \{0\}$ agrees with the restriction of h^*u to $X \times \{0\}$. Hence this composite u' is homotopic to the morphism h^*u itself. Namely, if we write $h^*\xi = i_0^*\xi \times [0,1]$ for $i_0\colon X \to X \times [0,1], x \mapsto (x,0)$, then the homotopy is given at time $s \in [0,1]$ by the morphism

$$S^{i_0^*\xi \oplus \underline{\mathbb{R}^k}} \times [0,1] \to S^{i_0^*\xi \oplus \underline{\mathbb{R}^{k+n}}} \times [0,1], \quad (z,t) \ \mapsto \ (\mathrm{pr} \circ u'(z,st), t)$$

for $\mathrm{pr}\colon S^{i_0^*\xi \oplus \underline{\mathbb{R}^{k+n}}} \times [0,1] \to S^{i_0^*\xi \oplus \underline{\mathbb{R}^{k+n}}}$ the projection. Obviously this homotopy is stationary over A. We conclude from the equivalence relation appearing in the definition of π_G^n and the definition of $\pi_G^n(h)$ that $\pi_G^n(h)([u]) = [u'] = [u]$ holds. $\square$

Next we define the suspension homomorphism

$$\sigma_G^n(X,A)\colon \pi_G^n(X,A) \quad \to \quad \pi_G^{n+1}((X,A) \times ([0,1],\{0,1\})). \tag{6.7}$$

For a G-vector bundle ξ over X let $\xi \times [0,1]$ be the obvious G-vector bundle over $X \times [0,1]$, which is the same as the pullback of ξ for the projection $X \times$

$[0,1] \to X$. Consider a morphism $u\colon S^{\xi} \to S^{\mu}$ in $\mathsf{SPHB}^G(X)$ which is trivial over A. Let

$$\sigma(u)\colon S^{\xi\times[0,1]} = S^{\xi} \times [0,1] \;\to\; S^{(\mu\oplus\underline{\mathbb{R}})\times[0,1]} = \left(S^{\mu} \wedge_X S^{\underline{\mathbb{R}}}\right) \times [0,1]$$

be the morphism in $\mathsf{SPHB}^G(X \times [0,1])$ given by

$$(z,t) \in S^{\xi} \times [0,1] \;\mapsto\; \left((u(z) \wedge e(t))\,, t\right) \in \left(S^{\mu} \wedge_X S^{\underline{\mathbb{R}}}\right) \times [0,1],$$

where $e\colon [0,1] \to S^{\underline{\mathbb{R}}}$ comes from the homeomorphism $(0,1) \to \mathbb{R}, t \;\mapsto \ln(x) - \ln(1-x)$. The morphism $\sigma(u)$ is trivial over $X \times \{0,1\} \cup A \times [0,1]$. We define the in (X,A) natural homomorphism of abelian groups $\sigma_G^n(X,A)$ by sending the class of u to the class of $\sigma(u)$.

Lemma 6.8. *The homomorphism $\sigma_G^n(X,A)$ of* (6.7) *is bijective for all pairs of proper G-CW-complexes (X,A).*

Proof. We want to construct an inverse

$$\tau_G^{n+1}\colon \pi_G^{n+1}((X,A) \times ([0,1],\{0,1\})) \;\to\; \pi_G^n(X,A)$$

of $\sigma_G^n(X,A)$. Consider two G-vector bundles ξ and μ over X and a morphism $v\colon S^{\xi\times[0,1]} \to S^{\mu\times[0,1]}$ in $\mathsf{SPHB}^G(X \times [0,1])$ which is trivial over $A \times \{0,1\}$. Define a morphism in $\mathsf{SPHB}^G(X)$ which is trivial over A

$$\tau(v)\colon S^{\xi\oplus\underline{\mathbb{R}}} = S^{\xi} \wedge_X S^{\underline{\mathbb{R}}} \to S^{\mu}$$

by sending $(z,(e(t))) \in S^{\xi} \wedge_X S^{\underline{\mathbb{R}}}$ to $\mathrm{pr} \circ v(z,t)$ for $\mathrm{pr}\colon S^{\mu\times[0,1]} = S^{\mu} \times [0,1] \to S^{\mu}$ the projection, $e\colon [0,1] \to S^{\underline{\mathbb{R}}}$ the map defined above and $t \in [0,1]$. Next consider an element $[u] \in \pi_G^{n+1}(X,A)$ represented by a morphism $u\colon S^{\xi\oplus\underline{\mathbb{R}^k}} \to S^{\xi\oplus\underline{\mathbb{R}^{k+n+1}}}$ in $\mathsf{SPHB}^G(X \times [0,1])$ for $k+n \geq 0$ which is trivial over $X \times \{0,1\} \cup A \times [0,1]$. Choose an isomorphism of G-vector bundles $v\colon \xi_0 \times [0,1] \xrightarrow{\cong} \xi$ which covers the identity on $X \times [0,1]$ and is the identity over $X \times \{0\}$, where ξ_0 is the restriction of ξ to $X = X \times \{0\}$ (see [29, Theorem 1.2]). Let $u'\colon S^{(\xi_0 \oplus \underline{\mathbb{R}^k})\times[0,1]} \to S^{(\xi_0 \oplus \underline{\mathbb{R}^n})\times[0,1]}$ be the composition

$$S^{(\xi_0 \oplus \underline{\mathbb{R}^k})\times[0,1]} \xrightarrow{S^{v\oplus\mathrm{id}}} S^{(\xi \oplus \underline{\mathbb{R}^k})} \xrightarrow{u} S^{\xi\oplus\underline{\mathbb{R}^{k+n+1}}} \xrightarrow{S^{v^{-1}\oplus\mathrm{id}}} S^{(\xi_0 \oplus \underline{\mathbb{R}^{k+n+1}})\times[0,1]}.$$

Notice that $[u] = [u']$ holds in $\pi_G^{n+1}((X,A) \times ([0,1],\{0,1\}))$. Define $\tau_G^{n+1}([u])$ by the class of $\tau(u')\colon S^{\xi_0 \oplus \underline{\mathbb{R}^{k+1}}} \to S^{\xi_0 \oplus \underline{\mathbb{R}^{k+n+1}}}$.

Consider $[u] \in \pi_G^n(X,A)$ represented by the morphism $u\colon S^{\xi\oplus\underline{\mathbb{R}^k}} \to S^{\xi\oplus\underline{\mathbb{R}^{k+n}}}$ in $\mathsf{SPHB}^G(X)$ which is trivial over A. Then $\tau_G^{n+1} \circ \sigma_G^n([u])$ is represented by the morphism $\tau \circ \sigma(u)$ which can be identified with

$$S^{\xi\oplus\underline{\mathbb{R}^{k+1}}} \xrightarrow{\sigma_1} S^{\xi\oplus\underline{\mathbb{R}^k}} \wedge_X S^{\underline{\mathbb{R}}} \xrightarrow{u\wedge_X \mathrm{id}} S^{\xi \oplus \underline{\mathbb{R}^{k+n}}} \wedge_X S^{\underline{\mathbb{R}}} \xrightarrow{\sigma_2} S^{\xi\oplus\underline{\mathbb{R}^{k+1+n}}\oplus\underline{\mathbb{R}}}$$

But the latter morphism represents the same class as u. This shows $\tau_G^{n+1} \circ \sigma_G^n([u]) = [u]$ and hence $\tau_G^{n+1} \circ \sigma_G^n = \mathrm{id}$. The proof of $\sigma_G^n([u]) \circ \tau_G^{n+1} = \mathrm{id}$ is analogous. $\square$

So far we have only assumed that the G-CW-complex X is proper. In the sequel we will need additionally that it is finite since this condition appears in Lemma 6.4.

Lemma 6.9. *Let (X_1, X_0) be a pair of finite proper G-CW-complexes and $f\colon X_0 \to X_2$ be a cellular G-map of finite proper G-CW-complexes. Define the pair of finite proper G-CW-complexes (X, X_2) by the cellular G-pushout*

$$\begin{array}{ccc} X_0 & \xrightarrow{f} & X_2 \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow[F]{} & X \end{array}$$

Then the homomorphism

$$\pi_G^n(F,f)\colon \pi_G^n(X,X_2) \xrightarrow{\cong} \pi_G^n(X_1,X_0)$$

is bijective for all $n \in \mathbb{Z}$.

Proof. We begin with surjectivity. Consider an element $a \in \pi_G^n(X_1, X_0)$ represented by a morphism

$$u\colon S^{\xi \oplus \underline{\mathbb{R}^k}} \to S^{\xi \oplus \underline{\mathbb{R}^{k+n}}}$$

in $\mathsf{SPHB}^G(X_1)$ which is trivial over X_0. By Lemma 6.4 there is a G-vector bundle μ over X, a G-vector bundle ξ' over X_1 and an isomorphism of G-vector bundles $v\colon \xi \oplus \xi' \xrightarrow{\cong} F^*\mu$. Consider the morphism u' in $\mathsf{SPHB}^G(X_1)$ which is given by the composition

$$S^{F^*\mu \oplus \underline{\mathbb{R}^k}} \xrightarrow{S^{v^{-1}\oplus \mathrm{id}}} S^{\xi\oplus\xi'\oplus\underline{\mathbb{R}^k}} \xrightarrow{\sigma_1} S^{\xi'} \wedge_{X_1} S^{\xi\oplus\underline{\mathbb{R}^k}} \xrightarrow{\mathrm{id}\wedge_{X_1} u} S^{\xi'} \wedge_{X_1} S^{\xi\oplus\underline{\mathbb{R}^{k+n}}}$$
$$\xrightarrow{\sigma_2} S^{\xi\oplus\xi'\oplus\underline{\mathbb{R}^{k+n}}} \xrightarrow{S^{v\oplus\mathrm{id}}} S^{F^*\mu\oplus\underline{\mathbb{R}^{k+n}}}.$$

By definition it represents the same element in $\pi_G^n(X_1, X_0)$ as u. Hence we can assume without loss of generality for the representative u of a that the bundle ξ is of the form $F^*\mu$ for some G-vector bundle μ over X. Since the morphism u is trivial over X_0, we can find a morphism $\mathsf{SPHB}^G(X)$

$$\overline{u}\colon S^{\mu\oplus\underline{\mathbb{R}^k}} \to S^{\mu\oplus\underline{\mathbb{R}^{k+n}}}$$

which is trivial over X_2 and satisfies $F^*(\overline{u}) = u$. Hence the morphism $\overline{u}$ defines an element in $\pi_G^n(X, X_1)$ such that $\pi_G^n(F,f)([\overline{u}]) = [u] = a$ holds.

It remains to prove injectivity of $\pi_G^n(F,f)$. Consider an element b in the kernel of $\pi_G^n(F,f)$. Choose a morphism

$$u\colon S^{\xi\oplus\underline{\mathbb{R}^k}} \to S^{\xi\oplus\underline{\mathbb{R}^{k+n}}}$$

in $\mathsf{SPHB}^G(X)$ which is trivial over X_2 and represents b. Then $F^*u\colon S^{F^*\xi\oplus\underline{\mathbb{R}^k}} \to S^{F^*\xi\oplus\underline{\mathbb{R}^{k+n}}}$ represents zero in $\pi_G^n(X_1, X_0)$. Hence we can find a bundle μ over X_1 such that the composition

$$S^{\mu\oplus F^*\xi\oplus\underline{\mathbb{R}^k}} \xrightarrow{\sigma_1} S^\mu\wedge_{X_1} S^{F^*\xi\oplus\underline{\mathbb{R}^k}} \xrightarrow{\mathrm{id}\wedge_{X_1} u} S^\mu\wedge_{X_1} S^{F^*\xi\oplus\underline{\mathbb{R}^{k+n}}} \xrightarrow{\sigma_2} S^{\mu\oplus F^*\xi\oplus\underline{\mathbb{R}^{k+n}}}$$

is homotopic relative X_0 to the trivial morphism. As in the proof of the surjectivity we can arrange using Lemma 6.4 that μ is of the shape $F^*\xi'$ for some G-vector bundle ξ' over X. By replacing u by $\mathrm{id}_{S^{\xi'}}\wedge_X u$ we can achieve that $b = [u]$ still holds and additionally the morphism in $\mathsf{SPHB}^G(X_1)$

$$F^*u\colon S^{F^*(\xi\oplus\underline{\mathbb{R}^k})} \to S^{F^*(\xi\oplus\underline{\mathbb{R}^{k+n}})}$$

is homotopic relative X_0 to the trivial map. Since u is trivial over X_2, we can extend this homotopy trivially from X_1 to X to show that u itself is homotopic relative X_2 to the trivial map. But this means $b = [u] = 0$ in $\pi_G^n(X, X_2)$. This finishes the proof of Lemma 6.9. □

Lemma 6.10. *Let $A \subseteq Y \subset X$ be inclusions of finite proper G-CW-complexes. Then the sequence*

$$\pi_G^n(X,Y) \xrightarrow{\pi_G^n(j)} \pi_G^n(X,A) \xrightarrow{\pi_G^n(i)} \pi_G^n(Y,A)$$

is exact at $\pi_G^n(X,A)$ for all $n \in \mathbb{Z}$, where i and j denote the obvious inclusions.

Proof. The inclusion $j\circ i\colon (Y,A) \to (X,Y)$ induces the zero map $\pi_G^n(X,Y) \to \pi_n^G(Y,A)$ since an element in $a \in \pi_G^n(X,Y)$ is represented by a morphism $u\colon S^{\xi\oplus\underline{\mathbb{R}^k}} \to S^{\xi\oplus\underline{\mathbb{R}^{k+n}}}$ in $\mathsf{SPHB}^G(X)$ which is trivial over Y and $\pi_G^n(j\circ i)(a)$ is represented by the restriction of u to Y.

Consider an element $a \in \pi_G^n(X,A)$ which is mapped to zero under $\pi_G^n(i)$. Choose a morphism $u\colon S^{\xi\oplus\underline{\mathbb{R}^k}} \to S^{\xi\oplus\underline{\mathbb{R}^{k+n}}}$ in $\mathsf{SPHB}^G(X)$ which is trivial over A and represents a. Hence we can find a G-vector bundle ξ' over Y such that the morphism in $\mathsf{SPHB}^G(Y)$ given by the composition

$$S^{\xi'\oplus i^*\xi\oplus\underline{\mathbb{R}^k}} \xrightarrow{\sigma_1} S^{\xi'}\wedge_Y S^{i^*\xi\oplus\underline{\mathbb{R}^k}} \xrightarrow{\mathrm{id}\wedge_Y i^*u} S^{\xi'}\wedge_Y S^{i^*\xi\oplus\underline{\mathbb{R}^{k+n}}} \xrightarrow{\sigma_2} S^{\xi'\oplus i^*\xi\oplus\underline{\mathbb{R}^{k+n}}}$$

is homotopic to the trivial map relative A.

As in the proof of Lemma 6.9 we can arrange using Lemma 6.4 that ξ' is the of the shape $i^*\mu$ for some G-vector bundle μ over X. Hence we can achieve by replacing u by $\mathrm{id}_{S^\mu}\wedge_X u$ that $a = [u]$ still holds in $\pi_G^n(X,A)$ and additionally the morphisms in $\mathsf{SPHB}^G(Y)$

$$i^*u\colon S^{j^*\xi\oplus\underline{\mathbb{R}^k}} \to S^{i^*\xi\oplus\underline{\mathbb{R}^{k+n}}}$$

is homotopic relative A to the trivial map. One proves inductively over the number of equivariant cells in $X - Y$ and [29, Theorem 1.2] that this homotopy can be extended to a homotopy relative A of the morphism u to another morphism $v\colon S^{\xi} \to S^{\xi \oplus \underline{\mathbb{R}^n}}$ in $\mathsf{SPHB}^G(X)$ which is trivial over Y. But this implies that the element $[v] \in \pi^n_G(X,Y)$ represented by v is mapped to $a = [u]$ under $\pi^n_G(i)$. $\square$

In order to define a G-cohomology theory we must construct a connecting homomorphism for pairs. In the sequel maps denoted by i_k are the obvious inclusions and maps denoted by pr_k are the obvious projections. Consider a pair of finite proper G-CW-complexes (X,A) and $n \in \mathbb{Z}$, $n \geq 0$. We define the homomorphism of abelian groups

$$\delta^n_G(X,A)\colon \pi^n_G(A) \quad \to \quad \pi^{n+1}_G(X,A) \tag{6.11}$$

to be the composition

$$\begin{aligned}
\pi^n_G(A) &\xrightarrow{\sigma^n_G(A)} \pi^{n+1}_G(A \times [0,1], A \times \{0,1\}) \\
&\xrightarrow{\left(\pi^{n+1}_G(i_1)\right)^{-1}} \pi^{n+1}_G(X \cup_{A \times \{0\}} A \times [0,1], X \amalg A \times \{1\}) \\
&\xrightarrow{\pi^{n+1}_G(i_2)} \pi^{n+1}_G(X \cup_{A \times \{0\}} A \times [0,1], A \times \{1\}) \\
&\xrightarrow{\left(\pi^{n+1}_G(\mathrm{pr}_1)\right)^{-1}} \pi^{n+1}_G(X,A),
\end{aligned}$$

where $\sigma^n_G(A)$ is the suspension isomorphism (see Lemma 6.8), the map $\pi^{n+1}_G(i_1)$ is bijective by excision (see Lemma 6.9) and $\pi^{n+1}_G(\mathrm{pr}_1)$ is bijective by homotopy invariance (see Lemma 6.6) since pr_1 is a G-homotopy equivalence of pairs.

Lemma 6.12. *Let (X,A) be a pair for proper finite G-CW-complexes. Let $i\colon A \to X$ and $j\colon X \to (X,A)$ be the inclusions. Then the following long sequence is exact and natural in (X,A):*

$$\begin{aligned}
\cdots \xrightarrow{\delta^{n-1}_G} \pi^n_G(X,A) \xrightarrow{\pi^n_G(j)} \pi^n_G(X) \xrightarrow{\pi^n_G(i)} \pi^n_G(A) \\
\xrightarrow{\delta^n_G} \pi^{n+1}_G(X,A) \xrightarrow{\pi^{n+1}_G(j)} \pi^{n+1}_G(X) \xrightarrow{\pi^{n+1}_G(i)} \pi^{n+1}_G(A) \xrightarrow{\delta^{n-1}_G} \cdots .
\end{aligned}$$

Proof. It is obviously natural. It remains to prove exactness.

Exactness at $\pi^n_G(X)$ follows from Lemma 6.10.

Exactness at $\pi^n_G(X,A)$ follows from the following commutative diagram

$$\begin{tikzcd}
\pi^{n+1}_G\left(X \cup_{A \times \{0\}} A \times [0,1], X \amalg A \times \{1\}\right) \arrow[r, "(\sigma^n_G(A))^{-1} \circ \pi^{n+1}_G(i_1)", "\cong"'] \arrow[d, "\pi^{n+1}_G(i_2)"'] & \pi^n_G(A) \arrow[d, "\delta^{n+1}_G(X,A)"'] \\
\pi^{n+1}_G\left(X \cup_{A \times \{0\}} A \times [0,1], A \times \{1\}\right) \arrow[r, "\left(\pi^{n+1}_G(\mathrm{pr}_1)\right)^{-1}", "\cong"'] \arrow[d, "\pi^{n+1}_G(i_3)"'] & \pi^{n+1}_G(X,A) \arrow[d, "\pi^{n+1}_G(i)"'] \\
\pi^{n+1}_G\left(X \amalg A \times \{1\}, A \times \{1\}\right) \arrow[r, "\pi^{n+1}_G(i_4)", "\cong"'] & \pi^{n+1}_G(X)
\end{tikzcd}$$

whose left column is exact at $\pi_G^{n+1}\left(X \cup_{A\times\{0\}} A \times [0,1], A \times \{1\}\right)$ by Lemma 6.10.

Exactness at $\pi_G^n(A)$ is proved analogously by applying Lemma 6.10 to the inclusions

$$\begin{aligned}
&(X \times \{0,1\}, X \times \{0\} \amalg A \times \{1\}) \\
&\qquad\qquad \subseteq (X \times [0,1], X \times \{0\} \amalg A \times \{1\}) \\
&\qquad\qquad\qquad\qquad \subseteq (X \times [0,1], X \times \{0,1\}).
\end{aligned}$$

$\square$

We conclude from Lemma 6.6, Lemma 6.9 and Lemma 6.12 that π_G^* defines a G-cohomology theory on the category of finite proper G-CW-complexes.

Consider a finite proper G-CW-complex X with two subcomplexes $A, B \subseteq X$. We want to define a multiplicative structure, i.e. a cup-product

$$\cup\colon \pi_G^m(X,A) \times \pi_G^n(X,B) \quad \to \quad \pi_G^{m+n}(X, A \cup B). \tag{6.13}$$

Given elements $a \in \pi_G^m(X,A)$ and $b \in \pi_G^n(X,B)$, choose appropriate morphisms $u\colon S^{\xi\oplus\underline{\mathbb{R}^k}} \to S^{\xi\oplus\underline{\mathbb{R}^{m+k}}}$ and $v\colon S^{\eta\oplus\underline{\mathbb{R}^l}} \to S^{\eta\oplus\underline{\mathbb{R}^{n+l}}}$ in $\mathsf{SPHB}^G(X)$ representing a and b. Define $a \cup b$ by the class of the composition of morphisms in $\mathsf{SPHB}^G(X)$ which are trivial over $A \cup B$.

$$\begin{aligned}
&S^{\xi\oplus\eta\oplus\underline{\mathbb{R}^{k+l}}} \xrightarrow{\sigma_1} S^{\xi\oplus\underline{\mathbb{R}^k}} \wedge_X S^{\eta\oplus\underline{\mathbb{R}^l}} \\
&\qquad\qquad\qquad \xrightarrow{u\wedge_X v} S^{\xi\oplus\underline{\mathbb{R}^{k+m}}} \wedge_X S^{\eta\oplus\underline{\mathbb{R}^{l+n}}} \xrightarrow{\sigma_2} S^{\xi\oplus\eta\oplus\underline{\mathbb{R}^{k+l+m+n}}}.
\end{aligned}$$

Next we deal with the induction structure. Consider a group homomorphism $\alpha\colon H \to G$. The pullback construction for the $\alpha\colon H \to G$-equivariant map $X \to \operatorname{ind}_\alpha X = G \times_\alpha X$, $x \mapsto 1G \times_\alpha x$ defines a functor $\mathsf{SPHB}^G(\operatorname{ind}_\alpha X) \to \mathsf{SPHB}^H(X)$ which yields a homomorphism of abelian groups

$$\operatorname{ind}_\alpha\colon \pi_G^n\left(\operatorname{ind}_\alpha(X,A)\right) \quad \to \quad \pi_H^n(X). \tag{6.14}$$

Now suppose that the kernel of H acts trivially on X. Let $\xi\colon E \to X$ be a H-vector bundle. Then $G \times_\alpha X$ is a proper H-CW-complex. Since H acts freely on X, we obtain a well-defined G-vector bundle $G \times_\alpha \xi\colon G \times_\alpha E \to G \times_\alpha X$. Thus we obtain a functor $\operatorname{ind}_\alpha\colon \mathsf{SPHB}^H(X) \to \mathsf{SPHB}^G(G \times_\alpha X)$. It defines a homomorphism of abelian groups

$$\operatorname{ind}_\alpha\colon \pi_H^n(X) \quad \to \quad \pi_G^n\left(\operatorname{ind}_\alpha(X,A)\right). \tag{6.15}$$

which turns out to be an inverse of the induction homomorphism (6.14).

Now we have all ingredients of an equivariant cohomology theory with a multiplicative structure. We leave it to the reader to verify all the axioms. This finishes the proof of Theorem 6.5.

6.4 Comparison with the Classical Construction for Finite Groups

Next we want to show that for a finite group G our construction reduces to the classical one. We first explain why the finite group case is easier.

Remark 6.16 (Advantages in the case of finite groups). The finite group case is easier because for finite groups the following facts are true. The first fact is that every G-CW-complex is proper. Hence one can view pointed G-CW-complexes, where the base point is fixed under the G-action and one can carry out constructions like mapping cones without loosing the property proper. We need proper to ensure that certain basic facts about G-vector bundles are true (see [29, Section 1]). The second fact is that every G-vector bundle over a finite G-CW-complex ξ is a direct summand in a trivial G-vector bundle, i.e. a G-vector bundle given by the projection $V \times X \to X$ for some G-representation V. This makes for instance Lemma 6.4 superfluous whose proof is non-trivial in the infinite group case (see [29, Lemma 3.7]).

Next we identify $\pi_G^n(X)$ defined in Subsection 6.2 with the classical definitions which we have explained in Subsection 1.4 provided that G is a finite group.

Consider an element $a \in \pi_G^n(X)$ with respect to the definition given in Subsection 1.4. Obviously we can find a positive integer $k \in \mathbb{Z}$ with $k+n \geq 0$ such that a is represented for some complex G-representation V by a G-map $f\colon S^V \wedge S^k \wedge X_+ \to S^V \wedge S^{k+n}$. Let ξ be the trivial G-vector bundle $V \times X \to X$. Define a map

$$\overline{f}\colon S^{\xi \oplus \underline{\mathbb{R}^k}} \xrightarrow{\sigma_1} (S^V \wedge S^k) \times X \xrightarrow{f \times \mathrm{pr}_X} (S^V \wedge S^{k+n}) \times X \xrightarrow{\sigma_2} S^{\xi \oplus \underline{\mathbb{R}^{k+n}}}.$$

It is a morphism in $\mathsf{SPHB}^G(X)$ and hence defines an element $a' \in \pi_G^n(X)$ with respect to the definition of Subsection 6.2. Thus we get a homomorphism of abelian groups $a \mapsto a'$ from the definition of Subsection 1.4 to the one of Subsection 6.2.

Consider a morphism $u\colon S^{\xi \oplus \underline{\mathbb{R}^k}} \to S^{\xi \oplus \underline{\mathbb{R}^{k+n}}}$ in $\mathsf{SPHB}^G(X)$ representing an element in $b = [u]$ in $\pi_G^n(X)$ as defined in Subsection 6.2. Choose a G-vector bundle μ, a complex G-representation V and an isomorphism of (real) G-vector bundles $\phi\colon \mu \oplus \xi \xrightarrow{\cong} V \times X$. Then the morphism

$$\begin{aligned} v\colon S^{V \oplus \mathbb{R}^k} \times X &\xrightarrow{(\phi \oplus \mathrm{id})^{-1}} S^{\mu \oplus \xi \oplus \underline{\mathbb{R}^k}} \xrightarrow{\sigma_1} S^{\mu} \wedge_X S^{\xi \oplus \underline{\mathbb{R}^k}} \\ &\xrightarrow{\mathrm{id} \wedge u} S^{\mu} \wedge_X S^{\xi \oplus \underline{\mathbb{R}^{k+n}}} \xrightarrow{\sigma_2} S^{\mu \oplus \xi \oplus \underline{\mathbb{R}^{k+n}}} \xrightarrow{\phi \oplus \mathrm{id}} S^{V \oplus \mathbb{R}^{k+n}} \times X \end{aligned}$$

is equivalent to u and hence $b = [v]$. Since v covers the identity and is fiberwise a pointed map, its composition with the projection $S^{V \oplus \mathbb{R}^{k+n}} \times X \to S^{V \oplus \mathbb{R}^{k+n}}$ yields a map

$$\overline{v}\colon S^{V \oplus \mathbb{R}^k} \wedge X_+ \to S^{V \oplus \mathbb{R}^{k+n}}.$$

It defines an element $b' := [\overline{v}]$ in $\pi^n_G(X)$ with respect to the definition given in Subsection 1.4. The map $b \mapsto b'$ is the inverse of the map $a \to a'$ before.

Remark 6.17 (Why consider G-vector bundles?). One may ask why we consider G-vector bundles ξ in Section 6. It would be much easier if we would only consider trivial G-vector bundles $V \times X$ for G-representations V. Then we would not need Lemma 6.4. The proof that π^*_G is a G-cohomology theory with a multiplicative structure would go through and for finite groups we would get the classical notion. The problem is that the induction structure does not exists anymore as the following example shows.

Consider a finitely generated group such that G_{mrf} is trivial. Then any G-representation V is trivial (see Subsection 2.5). This implies that a morphism $u\colon S^{\xi \oplus \underline{\mathbb{R}^k}} \to S^{\xi \oplus \underline{\mathbb{R}^{k+n}}}$ in $\mathsf{SPHB}^G(X)$ for $\xi = V \times X$ is the same as a (non-equivariant) map

$$S^{V \oplus \mathbb{R}^k} \wedge (G\backslash X_+) \to S^{V \oplus \mathbb{R}^{k+n}}.$$

This yields an identification of $\pi^n_G(X)$ with respect to the definition, where all G-vector bundles are of the shape $X \times V$, with the (non-equivariant) stable cohomotopy group $\pi^n_s(G\backslash X)$. If G contains a non-trivial subgroup $H \subseteq G$, then the existence of an induction structure would predict for $X = G/H$ that $\pi^n_G(G/H)$ is isomorphic to π^n_H, which is in general different from $\pi^n_s(G\backslash(G/H)) = \pi^n_{\{1\}}$.

So we need to consider G-vector bundles in order to get induction structures and hence an equivariant cohomology theory. In particular our definition guarantees $\pi^n_G(G/H) = \pi^n_H$ for every group G with a finite subgroup $H \subseteq G$.

Remark 6.18 (The coefficients of equivariant stable cohomotopy). It is important to have information about the values $\pi^n_G(G/H)$ for a finite subgroup $H \subseteq G$ of a group G. By the induction structure and the identification above $\pi^n_G(G/H)$ agrees with the abelian groups $\pi^n_H = \pi^H_{-n}$ defined in Subsection 1.4. The equivariant homotopy groups π^H_{-n} are computed in terms of the splitting due to Segal and tom Dieck (see [49, Theorem 7.7 in Chapter II on page 154], [41, Proposition 2]) by

$$\pi^n_G(G/H) \;=\; \pi^H_{-n} \;=\; \bigoplus_{(K) \in \mathrm{ccs}(H)} \pi^s_{-n}(BW_HK).$$

The abelian group π^s_q is finite for $q \geq 1$ by a result of Serre [42] (see also [18]), is $\mathbb{Z}$ for $q = 0$ and is trivial for $q \leq -1$. Since W_HK is finite, $\widetilde{H}_p(BW_HK;\mathbb{Z})$ is finite for all $p \in \mathbb{Z}$. We conclude from the Atiyah-Hirzebruch spectral sequence that $\pi^s_{-n}(BW_HK)$ is finite for $n \leq -1$. This implies $|\pi^n_G(G/H)| < \infty$ for $n \leq -1$ and that $\pi^n_G(G/H) = 0$ for $n \geq 1$. We know already $\pi^0_H = A(H)$ from Theorem 1.13. Thus we get

$$\begin{array}{lcll} |\pi^n_G(G/H)| & < & \infty & n \leq -1; \\ \pi^0_G(G/H) & = & A(H); & \\ \pi^n_G(G/H) & = & \{0\} & n \geq 1. \end{array}$$

Remark 6.19. (Equivariant Cohomotopy for arbitrary G-CW-complexes). In order to construct an equivariant cohomology theory or an (equivariant homology theory) for arbitrary G-CW-complexes it suffices to construct a contravariant (covariant) functor from the category of small groupoids to the category of spectra (see [39], [30, Proposition 6.8]). In a different paper we will carry out such a construction yielding equivariant cohomotopy and homotopy for arbitrary equivariant CW-complexes and will identify the result with the one presented here for finite proper G-CW-complexes.

6.5 Rational Computation of Equivariant Cohomotopy

The cohomotopy theoretic Hurewicz homomorphism yields a transformation of cohomology theories

$$\pi_s^*(X) \xrightarrow{\cong} H^*(X;\mathbb{Z})$$

from the (non-equivariant) stable cohomotopy to singular cohomology with $\mathbb{Z}$-coefficients. It is rationally an isomorphism provided that X is a finite CW-complex. It is compatible with the multiplicative structures. The analogue for equivariant cohomotopy is described next.

Let G be a group and $H \subseteq G$ be a finite subgroup. Consider a pair of finite proper G-CW-complexes (X, A). Lemma 4.3 implies that (X^H, A^H) is a pair of finite proper W_GH-CW-complexes and $W_GH\backslash(X^H, A^H)$ is a pair of finite CW-complexes. Taking the H-fixed point set yields a homomorphism

$$\alpha^n_{(H)}(X, A)\colon \pi^n_G(X, A) \;\to\; \pi^n_{W_GH}(X^H, A^H).$$

This map is natural and compatible with long exact sequences of pairs and Mayer-Vietoris sequences.

The induction structure with respect to the homomorphism $W_GH \to \{1\}$ yields a homomorphism

$$\beta^n_{W_GH}(X^H, A^H)\colon \pi^n_s\left(W_GH\backslash(X^H, A^H)\right) \to \pi^n_{W_GH}(X^H, A^H).$$

We claim that $\beta^n_{W_GH}(Z, B)$ is a rational isomorphism for any pair of finite proper W_GH-CW-complexes (Z, B). Since $\beta^*_{W_GH}$ is natural and compatible with the long exact sequences of pairs and Mayer-Vietoris sequences, it suffices to prove the claim for $Z = W_GH/L$ and $B = \emptyset$ for any finite subgroup $L \subset W_GH$. But then $\beta^n_{W_GH}$ reduces to the obvious map $\pi^n(\{\bullet\}) \to \pi^n_L$ which is a rational isomorphism by Remark 6.18.

Let

$$h^n(X^H, A^H)\colon \pi^n_s\left(W_GH\backslash(X^H, A^H)\right) \;\to\; H^n\left(W_GH\backslash(X^H, A^H);\mathbb{Z}\right)$$

be the cohomotopy theoretic Hurewicz homomorphism. Let

$$\begin{aligned}\gamma^n\left(W_GH\backslash(X^H, A^H)\right) : H^n\left(W_GH\backslash(X^H, A^H);\mathbb{Z}\right) \otimes_{\mathbb{Z}} \mathbb{Q}\\ \to H^n\left(W_GH\backslash(X^H, A^H);\mathbb{Q}\right)\end{aligned}$$

be the natural map. Define a $\mathbb{Q}$-homomorphism by the composition

$$\begin{aligned}\zeta_G^n(X,A)_{(H)}: \pi_G^n(X,A)\otimes_{\mathbb{Z}}\mathbb{Q} &\xrightarrow{\alpha^n_{(H)}(X,A)\otimes_{\mathbb{Z}}\mathrm{id}_{\mathbb{Q}}} \pi^n_{W_GH}(X^H,A^H)\otimes_{\mathbb{Z}}\mathbb{Q}\\ &\xrightarrow{\left(\beta^n_{W_GH}(X^H,A^H)\otimes_{\mathbb{Z}}\mathrm{id}_{\mathbb{Q}}\right)^{-1}} \pi_s^n\left(W_GH\backslash(X^H,A^H)\right)\otimes_{\mathbb{Z}}\mathbb{Q}\\ &\xrightarrow{h^n(X^H,A^H)\otimes_{\mathbb{Z}}\mathrm{id}_{\mathbb{Q}}} H^n\left(W_GH\backslash(X^H,A^H);\mathbb{Z}\right)\otimes_{\mathbb{Z}}\mathbb{Q}\\ &\xrightarrow{\gamma^n\left(W_GH\backslash(X^H,A^H)\right)} H^n\left(W_GH\backslash(X^H,A^H);\mathbb{Q}\right).\end{aligned}$$

Define

$$\begin{aligned}\zeta_G^n(X,A) = \prod_{(H)\in\mathrm{ccs}(G)} \zeta_G^n(X,A)_{(H)}: \pi_G^n(X,A)\otimes_{\mathbb{Z}}\mathbb{Q}&\\ \to \prod_{(H)\in\mathrm{ccs}(G)} H^n\left(W_GH\backslash(X^H,A^H);\mathbb{Q}\right).&\end{aligned} \tag{6.20}$$

Theorem 6.21. *The maps*

$$\zeta_G^n(X,A)\colon \pi_G^n(X,A)\otimes_{\mathbb{Z}}\mathbb{Q} \xrightarrow{\cong} \prod_{(H)\in\mathrm{ccs}(G)} H^n\left(W_GH\backslash(X^H,A^H);\mathbb{Q}\right)$$

are bijective for all $n\in\mathbb{Z}$ and all pairs of finite proper G-CW-complexes (X,A). They are compatible with the obvious multiplicative structures.

Proof. One easily checks that ζ_G^* defines a transformation of G-homology theories, i.e. is natural in (X,A) and compatible with long exact sequences of pairs and Mayer-Vietoris sequences. Hence it suffices to show that $\zeta_G^n(G/K)$ is bijective for all $n\in\mathbb{Z}$ and finite subgroups $K\subset G$. The source and target of $\zeta_G^n(G/K)$ are trivial for $n\neq 0$ (see Remark 6.18). The map $\zeta_G^0(G/K)$ can be identified using Lemma 4.3 (ii) with the rationalization of the character map

$$\mathrm{char}^H\colon A(H)\to \prod_{(H)\in\mathrm{ccs}(G)}\mathbb{Z}$$

defined in (1.2) which is bijective by Lemma 1.3. ☐

6.6 Relating Equivariant Cohomotopy and Equivariant Topological K-Theory

We have introduced two equivariant cohomology theories with multiplicative structure, namely equivariant cohomotopy (see Theorem 6.5) and equivariant topological K-theory (see Subsection 5.4).

Let X be a finite proper G-CW-complex with G-CW-subcomplexes A and B and let $a\in\pi_G^m(X,A)$ be an element. We want to assign to it for every $m\in\mathbb{Z}$ a homomorphism of abelian groups

$$\phi_G^{m,n}(X,A)(a)\colon K_G^n(X,B)\to K_G^{m+n}(X,A\cup B). \tag{6.22}$$

Choose an integer $k \in \mathbb{Z}$ with $k \geq 0, k+m \geq 0$ and a morphism $u \colon S^{\xi \oplus \underline{\mathbb{R}^k}} \to S^{\xi \oplus \underline{\mathbb{R}^{k+m}}}$ in $\mathsf{SPHB}^G(X)$ which is trivial over A and represents a. Let v be the morphism in $\mathsf{SPHB}^G(X)$ which is given by the composite

$$v \colon S^{\xi \oplus \xi \oplus \underline{\mathbb{R}^k}} \xrightarrow{\sigma} S^{\xi} \wedge_X S^{\xi \oplus \underline{\mathbb{R}^k}} \xrightarrow{\mathrm{id} \wedge_X u} S^{\xi} \wedge_X S^{\xi \oplus \underline{\mathbb{R}^{k+m}}} \xrightarrow{\sigma^{-1}} S^{\xi \oplus \xi \oplus \underline{\mathbb{R}^{k+m}}}.$$

Then v is another representative of a. The bundle $\xi \oplus \xi$ carries a canonical structure of a complex vector bundle and we denote this complex vector bundle by $\xi_{\mathbb{C}}$.

Let $\sigma^k(X, A \cup B) \colon K_G^{m+n}(X, A \cup B) \xrightarrow{\cong} K_G^{m+n+k}\left((X, A \cup B) \times (D^k, S^{k-1})\right)$ be the suspension isomorphism. Let $\mathrm{pr}_k \colon X \times D^k \to X$ be the projection and $\mathrm{pr}_k^* \xi_{\mathbb{C}}$ be the complex vector bundle obtained from $\xi_{\mathbb{C}}$ by the pull back construction. Associated to it is a Thom isomorphism

$$\begin{aligned} T^{m+n+k}_{\mathrm{pr}_k^* \xi_{\mathbb{C}}} \colon & K_G^{m+n+k}\left((X, A \cup B) \times (D^k, S^{k-1})\right) \\ & \xrightarrow{\cong} K_G^{m+n+k+2\cdot\dim(\xi)}\left(S^{\mathrm{pr}_k^* \xi_{\mathbb{C}}}, S^{\mathrm{pr}_k^* \xi_{\mathbb{C}}|_{X \times S^{k-1} \cup (A \cup B) \times D^k}} \cup (X \times D^k)_\infty\right), \end{aligned}$$

where $(X \times D^k)_\infty$ is the copy of $X \times D^k$ given by the various points at infinity in the fibers $S^{\mathrm{pr}_k^* \xi_{\mathbb{C}}}$ and $\mathrm{pr}_k^* \xi_{\mathbb{C}}|_{X \times S^{k-1} \cup (A \cup B) \times D^k}$ is the restriction of $\mathrm{pr}_k^* \xi_{\mathbb{C}}$ to $X \times S^{k-1} \cup (A \cup B) \times D^k$ (see [29, Theorem 3.14]). Let

$$\begin{aligned} p_k \colon & \left(S^{\mathrm{pr}_k^* \xi_{\mathbb{C}}}, S^{\mathrm{pr}_k^* \xi_{\mathbb{C}}|_{X \times S^{k-1} \cup (A \cup B) \times D^k}} \cup (X \times D^k)_\infty\right) \\ & \qquad \left(S^{\xi \oplus \xi \oplus \underline{\mathbb{R}^k}}, S^{\xi \oplus \xi \oplus \underline{\mathbb{R}^k}|_{A \cup B}} \cup X_\infty\right) \end{aligned}$$

be the obvious projection which induces by excision an isomorphism on K_G^*. Define an isomorphism

$$\begin{aligned} \mu^{m+n, m+n+k+2\cdot\dim(\xi)} \colon & K_G^{m+n}(X, A \cup B) \\ & \to K_G^{m+n+k+2\cdot\dim(\xi)}\left(S^{\xi \oplus \xi \oplus \underline{\mathbb{R}^k}}, S^{\xi \oplus \xi \oplus \underline{\mathbb{R}^k}|_{A \cup B}} \cup X_\infty\right) \end{aligned}$$

by the composite $K_G^{m+n+k+2\cdot\dim(\xi)}(p_k)^{-1} \circ T^{n+m+k}_{\mathrm{pr}_k^* \xi_{\mathbb{C}}} \circ \sigma^k(X, A \cup B)$. Define

$$\begin{aligned} \mu^{n, m+n+k+2\cdot\dim(\xi)} \colon & K_G^{n}(X, B) \\ & \to K_G^{m+n+k+2\cdot\dim(\xi)}\left(S^{\xi \oplus \xi \oplus \underline{\mathbb{R}^{k+m}}}, S^{\xi \oplus \xi \oplus \underline{\mathbb{R}^{k+m}}|_{B}} \cup X_\infty\right) \end{aligned}$$

analogously. Let the desired map $\phi_G^{m,n}(X, A)(a)$ be the composite

$$\begin{aligned} \phi_G^{m,n}(X; A, B)(a) \colon & K_G^n(X, B) \\ & \xrightarrow{\mu^{n, m+n+k+2\cdot\dim(\xi)}} K_G^{m+n+k+2\cdot\dim(\xi)}\left(S^{\xi \oplus \xi \oplus \underline{\mathbb{R}^{k+m}}}, S^{\xi \oplus \xi \oplus \underline{\mathbb{R}^{k+m}}|_{B}} \cup X_\infty\right) \\ & \xrightarrow{K_G^{m+n+k+2\cdot\dim(\xi)}(v)} K_G^{m+n+k+2\cdot\dim(\xi)}\left(S^{\xi \oplus \xi \oplus \underline{\mathbb{R}^{k}}}, S^{\xi \oplus \xi \oplus \underline{\mathbb{R}^{k}}|_{A \cup B}} \cup X_\infty\right) \\ & \xrightarrow{\left(\mu^{m+n, m+n+k+2\cdot\dim(\xi)}\right)^{-1}} K_G^{m+n}(X, A \cup B). \end{aligned}$$

We leave it to the reader to check that the definition of $\phi_G^{m,n}(X;A,B)(a)$ is independent of the choices of k and u. The maps $\phi_G^{m,n}(X;A,B)(a)$ for the various elements $a \in \pi_G^m(X,A)$ define pairings

$$\phi_G^{m,n}(X;A,B)\colon \pi_G^m(X,A) \times K_G^n(X,B) \quad \to \quad K_G^{m+n}(X,A\cup B). \quad (6.23)$$

The verification of the next theorem is left to the reader.

Theorem 6.24. (Equivariant topological K-theory as graded algebra over equivariant cohomotopy).

(i) Naturality
The pairings $\phi_G^{m,n}(X;A.B)$ are natural in $(X;A,B)$;

(ii) Algebra structure
The collection of the pairings $\phi_G^{m,n}(X;\emptyset,A)$ defines the structure of a graded algebra over the graded ring $\pi_G^*(X)$ on $K_G^*(X,A)$;

(iii) Compatibility with induction
Let $\phi\colon H \to G$ be a group homomorphism and (X,A) be a pair of proper finite G-CW-complexes. Then the following diagram commutes

$$\begin{array}{ccc}
\pi_G^m\left(\operatorname{ind}_\alpha(X,A)\right) \times K_G^n\left(\operatorname{ind}_\alpha(X,B)\right) & \xrightarrow{\phi_G^{m,n}(\operatorname{ind}_\alpha(X;A,B))} & K_G^{m+n}\left(\operatorname{ind}_\alpha(X,A\cup B)\right) \\
{\scriptstyle \operatorname{ind}_\alpha \times \operatorname{ind}_\alpha}\downarrow & & {\scriptstyle \operatorname{ind}_\alpha}\downarrow \\
\pi_H^m(X,A) \times K_H^n(X,B) & \xrightarrow{\phi_H^{m,n}(X;A,B)} & K_H^{m+n}(X,A\cup B)
\end{array}$$

(iv) For $a \in \pi_G^{m-1}(A)$ and $b \in K_G^n(X)$ we have

$$\phi_G^{m,n}(X;\emptyset,\emptyset)(\delta(a),b) \; = \delta\left(\phi_G^{m-1,n}(A;\emptyset,\emptyset)(a,K_G^n(j)(b))\right),$$

where $\delta\colon \pi^{m-1}(A) \to \pi_G^m(X)$ and $\delta\colon K_G^{m+n-1}(A) \to K_G^{m+n}(X)$ are boundary operators for the pair (X,A) and $j\colon A \to X$ is the inclusion.

For every pair (X,A) of finite proper G-CW-complexes define a homomorphism

$$\psi_G^n(X,A)\colon \pi_G^n(X,A) \to K_G^n(X,A), \qquad a \;\mapsto \phi_G^{n,0}(X,A,\emptyset)(a,1_X), \; (6.25)$$

where $1_X \in K_G^0(X)$ is the unit element. Then Theorem 6.24 implies

Theorem 6.26 (Transformation from equivariant cohomotopy to equivariant topological K-theory). *We obtain a natural transformation of equivariant cohomology theories with multiplicative structure for pairs of equivariant proper finite CW-complexes by the maps*

$$\psi_?^*\colon \pi_?^* \to K_?^*.$$

If $H \subseteq G$ is a finite subgroup of the group G, then the map

$$\psi_G^n(G/H) \colon \pi_G^n(G/H) \to K_G^0(G/H)$$

is trivial for $n \geq 1$ and agrees for $n = 0$ under the identifications $\pi_G^0(G/H) = \pi_H^0 = A(H)$ and $K_G^0(G/H) = K_H^0(\{\bullet\}) = R_{\mathbb{C}}(H)$ with the ring homomorphism

$$A(H) \to R_{\mathbb{C}}(H), \quad [S] \mapsto [\mathbb{C}[S]]$$

which assigns to a finite H-set the associated complex permutation representation.

7 The Homotopy Theoretic Burnside Ring

In this section we introduce another version of the Burnside ring which is of homotopy theoretic nature and probably the most sophisticated and interesting one.

7.1 Classifying Space for Proper G-Actions

We need the following notion due to tom Dieck [44].

Definition 7.1 (Classifying space for proper G-actions). *A model for the* classifying space for proper G-actions *is a proper G-CW-complex $\underline{E}G$ such that $\underline{E}G^H$ is contractible for every finite subgroup $H \subseteq G$.*

Recall that a G-CW-complex is proper if and only if all its isotropy groups are finite. If $\underline{E}G$ is a model for the classifying space for proper G-actions, then for every proper G-CW-complex X there is up to G-homotopy precisely one G-map $X \to \underline{E}G$. In particular two models are G-homotopy equivalent and the G-homotopy equivalence between two models is unique up to G-homotopy. If G is finite, a model for $\underline{E}G$ is G/G. If G is torsionfree, $\underline{E}G$ is the same as EG which is by definition the total space of the universal principal G-bundle $G \to EG \to BG$.

Here is a list of groups G together with specific models for $\underline{E}G$ with the property that the model is a finite G-CW-complex.

G	$\underline{E}G$
word hyperbolic groups	Rips complex
discrete cocompact subgroup $G \subseteq L$ of a Lie group L with finite $\pi_0(L)$	L/K for a maximal compact subgroup $K \subseteq L$
G acts by isometries properly and cocompactly on a CAT(0)-space X, for instance on a tree or a simply-connected complete Riemannian manifold with non-positive sectional curvature	X
arithmetic groups	Borel-Serre completion
mapping class groups	Teichmüller space
outer automorphisms of finitely generated free groups	outer space

More information and more references about $\underline{E}G$ can be found for instance in [6] and [28].

7.2 The Definition of the Homotopy Theoretic Burnside Ring

We have introduced the equivariant cohomology theory with multiplicative structure for proper finite equivariant CW-complexes $\pi_?^*$ in Section 6 and the classifying space $\underline{E}G$ for proper G-actions in Subsection 7.1.

Definition 7.2 (Homotopy theoretic Burnside ring). *Let G be a (discrete) group such that there exists a finite model $\underline{E}G$ for the universal space for proper G-actions. Define the* homotopy theoretic Burnside ring *to be*

$$A_{\mathrm{ho}}(G) \ := \ \pi_G^0(\underline{E}G).$$

If G is finite, $\pi_G^0(\underline{E}G)$ agrees with π_G^0 which is isomorphic to the Burnside ring $A(G)$ by Theorem 1.13. So the homotopy theoretic definition $A_{\mathrm{ho}}(G)$ reflects this aspect of the Burnside ring which has not been addressed by the other definitions before.

After the program described in Remark 6.19 has been carried out, the assumption in Definition 7.2 that there exists a finite model for $\underline{E}G$ can be dropped and thus the Homotopy Theoretic Burnside ring $A_{\mathrm{ho}}(G)$ can be defined by $\pi_G^0(\underline{E}G)$ and analyzed for all discrete groups G.

If G is torsionfree, $A_{\mathrm{ho}}(G)$ agrees with $\pi_s^0(BG)$.

Theorem 6.26 implies that the map (see (6.25))

$$\psi_G^0(\underline{E}G) \colon A_{\mathrm{ho}}(G) = \pi_G^0(\underline{E}G) \ \to \ K_G^0(\underline{E}G)$$

is a ring homomorphism. It reduces for finite G to the ring homomorphism $A(G) \to R_{\mathbb{C}}(G)$ sending the class of a finite G-set to the class of the associated complex permutation representation.

7.3 Relation between the Homotopy Theoretic and the Inverse-Limit-Version

Suppose there is a finite model for $\underline{E}G$. Then there is an equivariant Atiyah-Hirzebruch spectral sequence which converges to $\pi_G^n(\underline{E}G)$ and whose E^2-term is given in terms of *Bredon cohomomology*

$$E_2^{p,q} = H^p_{\mathbb{Z}\mathsf{Sub}_{\mathcal{FIN}}(G)}(\underline{E}G; \pi_?^q).$$

Here $\pi_?^q$ is the contravariant functor

$$\pi_?^q \colon \mathsf{Sub}_{\mathcal{FIN}}(G) \to \mathbb{Z}-\mathsf{MODULES}, \quad H \mapsto \pi_H^q$$

and naturality comes from restriction with a group homomorphism $H \to K$ representing a morphism in $\mathsf{Sub}_{\mathcal{FIN}}(G)$. Usually the Bredon cohomology is defined over the orbit category, but in our case we can pass to the category $\mathsf{Sub}_{\mathcal{FIN}}(G)$ because of Lemma 5.2. Details of the construction of $H^p_{\mathbb{Z}\mathsf{Sub}_{\mathcal{FIN}}(G)}(\underline{E}G; \pi_q^?)$ can be found for instance in [27, Section 3]. We will only need the following elementary facts. There is a canonical identification

$$H^0_{\mathbb{Z}\mathsf{Sub}_{\mathcal{FIN}}(G)}(\underline{E}G; \pi_?^q) \quad \cong \quad \operatorname{invlim}_{H \in \mathsf{Sub}_{\mathcal{FIN}}(G)} \pi_H^q. \tag{7.3}$$

If we combine (7.3) with Theorem 1.13 we get an identification

$$H^0_{\mathbb{Z}\mathsf{Sub}_{\mathcal{FIN}}(G)}(\underline{E}G; \pi_?^0) \quad \cong \quad A_{\mathrm{inv}}(G). \tag{7.4}$$

The assumption that $\underline{E}G$ is finite implies together with Remark 6.18

$$|H^p_{\mathbb{Z}\mathsf{Sub}_{\mathcal{FIN}}(G)}(\underline{E}G; \pi_?^q)| < \infty \qquad \text{if } q \leq -1; \tag{7.5}$$

$$H^p_{\mathbb{Z}\mathsf{Sub}_{\mathcal{FIN}}(G)}(\underline{E}G; \pi_?^q) = \{0\} \qquad \text{if } p > \dim(\underline{E}G) \text{ or } p \leq -1 \text{ or } q \geq 1. \tag{7.6}$$

The equivariant Atiyah-Hirzebruch spectral sequence together with (7.4), (7.5) and (7.6) implies

Theorem 7.7 (Rationally $A_{\mathrm{ho}}(G)$ and $A_{\mathrm{inv}}(G)$ agree). *Suppose that there is a finite model for $\underline{E}G$. Then the edge homomorphism*

$$\operatorname{edge}^G \colon A_{\mathrm{ho}}(G) = \pi_G^0(\underline{E}G) \;\to\; A_{\mathrm{inv}}(G)$$

is a ring homomorphism whose kernel and the cokernel are finite.

The edge homomorphism appearing in Theorem 7.7 can be made explicit. Consider a morphism $u \colon S^{\xi \oplus \underline{\mathbb{R}^k}} \to S^{\xi \oplus \underline{\mathbb{R}^k}}$ in $\mathsf{SPHB}^G(\underline{E}G)$ representing the element $a \in \pi_G^0(\underline{E}G)$. In order to specify $\operatorname{edge}^G(a)$ we must define for every finite subgroup $H \subseteq G$ an element $\operatorname{edge}^G(a)_H \in A(H)$. Choose a point $x \in \underline{E}G^H$. Then u induces a pointed H-map $S^{\xi_x \oplus \mathbb{R}^k} \to S^{\xi_x \oplus \mathbb{R}^k}$. It defines an element in π_H^0. Let $\operatorname{edge}^G(a)_H$ be the image of this element under the ring isomorphism $\deg^H \colon \pi_H^0 \xrightarrow{\cong} A(H)$ appearing in Theorem 1.13. One easily checks that the collection of these elements $\operatorname{edge}_H^G(a)$ does define an element in the inverse limit $A_{\mathrm{inv}}(G)$. So essentially edge^G is the map which remembers just the system of the maps of the various fibers.

Remark 7.8. (Rank of the abelian group $A_{\mathrm{ho}}(G)$). A kind of character map for the homotopy theoretic version would be the composition of edge and the character map $\mathrm{char}^G_{\mathrm{inv}}$ of (3.9). Since we assume that $\underline{E}G$ has a finite model, there are only finitely many conjugacy classes of finite subgroups and the Burnside ring congruences appearing in Theorem 3.10 becomes easier to handle. In particular we conclude from Example 3.11 and Theorem 7.7 that $A_{\mathrm{ho}}(G)$ is a finitely generated abelian group whose rank is the number $|\operatorname{ccs}_f(G)|$ of conjugacy classes of finite subgroups of G.

7.4 Some Computations of the Homotopy Theoretic Burnside Ring

Example 7.9 (Groups with appropriate maximal finite subgroups). Suppose that the group G satisfies the conditions appearing in Example 3.4 and admits a finite model for $\underline{E}G$. In the sequel we use the notation introduced in Example 3.4. Then one can construct a G-pushout (see [28, Section 4.11])

$$\begin{array}{ccc} \coprod_{i\in I} G\times_{M_i} EM_i & \xrightarrow{i} & EG \\ \downarrow & & \downarrow \\ \coprod_{i\in I} G/M_i & \longrightarrow & \underline{E}G \end{array} \tag{7.10}$$

Taking the G-quotient, yields a non-equivariant pushout. There are long exact Mayer-Vietoris sequence associated to (7.10) and to the G-quotient. (We ignore the problem that $G\times_E M_i$ and EG may not be finite. It does not really matter since both are free or because we will in a different paper extend the definition of equivariant cohomotopy to all proper equivariant CW-complexes). These are linked by the induction maps with respect to the projections $G\to\{1\}$. Splicing these two long exact sequences together, yields the long exact sequence

$$\cdots\to\prod_{i\in I}\ker\left(\mathrm{res}^{\{1\}}_{M_i}\colon \pi^{-1}_{M_i}\to\pi^{-1}_s\right)\to\pi^0_s(G\backslash\underline{E}G)\to A_{\mathrm{ho}}(G)$$
$$\to\prod_{i\in I}\widetilde{A}(M_i)\to\pi^1_s(G\backslash\underline{E}G)\to\cdots$$

Example 7.11 (Extensions of $\mathbb{Z}^n$ with $\mathbb{Z}/p$ as quotient). Suppose that G satisfies the assumptions appearing in Example 3.6. Then G admits a finite model for $\underline{E}G$. In the sequel we use the notation introduced in Example 3.6. Then variation of the argument above yields a long exact sequence

$$\cdots\to\prod_{H^1(\mathbb{Z}/p;A)}\ker\left(\mathrm{res}^{\{1\}}_{\mathbb{Z}/p}\colon \pi^{-1}_{\mathbb{Z}/p}(BA^{\mathbb{Z}/p})\to\pi^{-1}_s(BA^{\mathbb{Z}/p})\right)\to\pi^0_s(G\backslash\underline{E}G)$$
$$\to A_{\mathrm{ho}}(G)\to\prod_{H^1(\mathbb{Z}/p;A)}\ker\left(\mathrm{res}^{\{1\}}_{\mathbb{Z}/p}\colon \pi^{0}_{\mathbb{Z}/p}(BA^{\mathbb{Z}/p})\to\pi^{0}_s(BA^{\mathbb{Z}/p})\right)\to\pi^1_s(G\backslash\underline{E}G)\to\cdots$$

where $\mathbb{Z}/p$ acts trivially on $BA^{\mathbb{Z}/p}$. If r is the rank of the finitely generated free abelian group $A^{\mathbb{Z}/p}$, then

$$\ker\left(\operatorname{res}_{\mathbb{Z}/p}^{\{1\}}\colon \pi_{\mathbb{Z}/p}^n(BA^{\mathbb{Z}/p}) \to \pi_s^n(BA^{\mathbb{Z}/p})\right) = \bigoplus_{k=0}^{r} \ker\left(\operatorname{res}_{\mathbb{Z}/p}^{\{1\}}\colon \pi_{\mathbb{Z}/p}^{n-k} \to \pi_s^{n-k}\right)^{\binom{r}{k}}.$$

8 The Segal Conjecture for Infinite Groups

We can now formulate a version of the Segal Conjecture for infinite groups. Let $\epsilon^G\colon A_{\mathrm{ho}}(G) \to \mathbb{Z}$ be the ring homomorphism which sends an element represented by a morphism $u\colon S^{\xi\oplus\underline{\mathbb{R}^k}} \to S^{\xi\oplus\underline{\mathbb{R}^k}}$ in $\mathsf{SPHB}^G(\underline{E}G)$ to the mapping degree of the map induced on the fiber $u_x\colon S^{\xi_x\oplus\mathbb{R}^k} \to S^{\xi_x\oplus\mathbb{R}^k}$ for some $x \in \underline{E}G$. This is the same as the composition

$$A_{\mathrm{ho}}(G) \xrightarrow{\mathrm{edge}^G} A_{\mathrm{inv}}(G) \xrightarrow{\mathrm{char}_{\mathrm{inv}}^G} \prod_{(H)\in \mathrm{ccs}_f(G)} \mathbb{Z} \xrightarrow{\mathrm{pr}_{\{1\}}} \mathbb{Z},$$

where $\mathrm{char}_{\mathrm{inv}}^G$ is the ring homomorphism defined in (3.9) and $\mathrm{pr}_{\{1\}}$ the projection onto the factor belonging to the trivial group. We define the *augmentation ideal* $\mathbf{I}_G$ of $A_{\mathrm{ho}}(G)$ to be the kernel of the ring homomorphism ϵ^G. Recall that for a finite proper G-CW-complex X the abelian group $\pi_G^n(X)$ is a $\pi_G^0(X)$-module. The classifying map $f\colon X \to \underline{E}G$ is unique up to G-homotopy. Suppose that $\underline{E}G$ is finite. Then f induces a uniquely defined ring homomorphism $\pi_G^0(f)\colon A_{\mathrm{ho}}(G) = \pi_G^0(\underline{E}G) \to \pi_G^0(X)$ and we can consider $\pi_G^n(X)$ is a $A_{\mathrm{ho}}(G)$-module.

Conjecture 8.1 (Segal Conjecture for infinite groups). *Let G be a group such that there is a finite model for the classifying space of proper G-actions $\underline{E}G$. Then for every finite proper G-CW-complex there is an isomorphism*

$$\pi_s^n(EG \times_G X) \xrightarrow{\cong} \pi_G^n(X)\widehat{_{\mathbf{I}_G}},$$

where $\pi_G^n(X)\widehat{_{\mathbf{I}_G}}$ is the $\mathbf{I}_G$-adic completion of the $A_{\mathrm{ho}}(G)$-module $\pi_G^n(X)$.

In particular we get for all $n \in \mathbb{Z}$ an isomorphism

$$\pi_s^n(BG) \xrightarrow{\cong} \pi_G^n(\underline{E}G)\widehat{_{\mathbf{I}_G}}$$

and especially for $n = 0$

$$\pi_s^0(BG) \xrightarrow{\cong} A_{\mathrm{ho}}(G)\widehat{_{\mathbf{I}_G}}.$$

If G is finite, Conjecture 8.1 reduces to the classical Segal Conjecture (see Theorem 1.14).

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