Closed model categories for presheaves of simplicial groupoids and presheaves of 2-groupoids

by

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Abstract

We prove that the category of presheaves of simplicial groupoids and the category of presheaves of 2-groupoids have Quillen closed model structures. We also show that the homotopy categories associated to the two categories are equivalent to the homotopy categories of simplicial presheaves and homotopy 2-types, respectively.

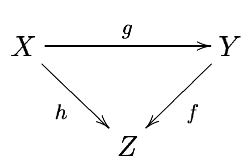
Key words: presheaves of simplicial groupoids, presheaves of 2-groupoids, Quillen closed model category

1 Introduction

A *Quillen closed model category* $\mathcal{D}$ is a category which is equipped with three classes of morphisms, called cofibrations, fibrations and weak equivalences which together satisfy the following axioms [9], [10], [3]:

CM1: The category $\mathcal{D}$ is closed under all finite limits and colimits.

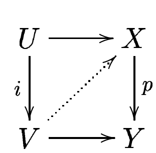
CM2: Suppose that the following diagram commutes in $\mathcal{D}$:



If any two of f, g and h are weak equivalences, then so is the third.

CM3: If f is a retract of g and g is a weak equivalence, fibration or cofibration, then so is f.

CM4: Suppose that we are given a commutative diagram



where i is a cofibration and p is a fibration. Then the lifting exists, making the diagram commute, if either i or p is also a weak equivalence.

CM5: Any map $f : X \to Y$ may be factored:

(a) $f = p \cdot i$ where p is a fibration and i is a trivial cofibration, and

(b) $f = q \cdot j$ where q is a trivial fibration and j is a cofibration.

The central foundational theorem of simplicial homotopy theory asserts that the category $\mathbf{S}$ of simplicial sets has a closed model structure [9]. Mathematicians have found a large quantity of categories enjoying the closed model structures. For example, the category of simplicial groupoids by Dwyer-Kan [2], [3] , the category of 2-groupoids by Moerdijk-Svensson [8], the category of simplicial presheaves by Jardine [5], the category of simplicial sheaves by Joyal [7] and so on. Crans [1] uses adjoint functors to prove that a kind of sheaves have closed model structures according to a well-known closed model category.

We use similar technique, basing on Jardine's paper [5], to prove that some presheaves have the closed model structures. One is the category of presheaves of simplicial groupoids in the section 2 and the other one is the category of presheaves of 2-groupoids in the section 3. We also show that the homotopy category associated to the first category is equivalent to the homotopy category of simplicial presheaves, the homotopy category associated to the latter category is equivalent to the homotopy category of homotopy 2-types.

2 Presheaves of simplicial groupoids

Let $\mathcal{C}$ be a fixed small Grothendieck site. $s\mathbf{Gd}\mathrm{Pre}(\mathcal{C})$ is the category of presheaves of simplicial groupoids on $\mathcal{C}$; its objects are the contravariant functors from $\mathcal{C}$ to the category $s\mathbf{Gd}$ of simplicial groupoids, and its morphisms are natural transformations.

Dwyer and Kan show that [2], [3], with the following definitions of weak equivalence, fibration and cofibration, the category $s\mathbf{Gd}$ of simplicial groupoids satisfies the axioms for a closed model category.

A map $f : G \to H$ of simplicial groupoids is said to be a *weak equivalence* of $s\mathbf{Gd}$ if

(1) the morphism f induces an isomorphism $\pi_0 G \cong \pi_0 H$, and

(2) each induced map $f : G(x, x) \to H(f(x), f(x))$, $x \in \mathrm{Ob}(G)$ is a weak equivalence of simplicial groups (or of simplicial sets).

A map $g : H \to K$ of simplicial groupoids is said to be a *fibration* of $s\mathbf{Gd}$ if

(1) the morphism g has path lifting property in the sense for every object x of H and morphism $\omega : g(x) \to y$ of the groupoids K_0, there is a morphism $\hat{\omega} : x \to z$ of H_0 such that $g(\hat{\omega}) = \omega$, and

(2) each induced map $g : H(x, x) \to K(g(x), g(x))$, $x \in \mathrm{Ob}(H)$ is a fibration of simplicial groups (or of simplicial sets).

A *cofibration* of simplicial groupoids is defined to be a map which has the left lifting property with respect to all morphisms of $s\mathbf{Gd}$ which are both fibrations and weak equivalences.

Recall the adjunction between the loop groupoid functor $G : \mathbf{S} \to s\mathbf{Gd}$ and the universal cocycle functor $\overline{W}$ [3, Lemma V.7.7]. By applying these functors pointwise to simplicial presheaves and presheaves of simplicial groupoids, one obtains functors

$$G : \mathbf{S}Pre(\mathcal{C}) \rightleftarrows s\mathbf{Gd}Pre(\mathcal{C}) : \overline{W}$$

So there is

Proposition 2.1. *The functor $G : \mathbf{S}Pre(\mathcal{C}) \to s\mathbf{Gd}Pre(\mathcal{C})$ is left adjoint to the functor $\overline{W}$.*

A map $f : X \to Y$ in the category $s\mathbf{Gd}\mathrm{Pre}(\mathcal{C})$ is said to be a *fibration* if the induced map $\overline{W}(f) : \overline{W}X \to \overline{W}Y$ is a global fibration in the category $\mathbf{S}\mathrm{Pre}(\mathcal{C})$ in the sense of [5].

A map $g : Z \to U$ in the category $s\mathbf{Gd}\mathrm{Pre}(\mathcal{C})$ is said to be a *weak equivalence* if the induced map $\overline{W}(g) : \overline{W}Z \to \overline{W}U$ is a topological weak equivalence in the category $\mathbf{S}\mathrm{Pre}(\mathcal{C})$ in the sense of [5].

A *cofibration* in the category $s\mathbf{Gd}\mathrm{Pre}(\mathcal{C})$ is a map of presheaves of simplicial groupoids which has the left lifting property with respect to all fibations and weak equivalences.

Say that a map of presheaves of simplicial groupoids f is a *trivial fibration* if it is both a fibration and a weak equivalence; a map g is a *trivial cofibration* if it is both a cofibration and a weak equivalence.

In [5], Jardine defines an important concept. The site $\mathcal{C}$ is "small", so that there is a cardinal number α such that α is larger than the cardinality of the set of subsets $\mathbf{P}\mathrm{Mor}(\mathcal{C})$ of the set of morphisms $\mathrm{Mor}(\mathcal{C})$ of $\mathcal{C}$. A simplicial presheaf X is said to be $\alpha - bounded$ if the cardinality of each $X_n(U), U \in \mathcal{C}, n \geq 0$, is smaller than α.

A map $p : X \to Y$ in the category $\mathbf{S}\mathrm{Pre}(\mathcal{C})$ is a global fibration if and only if it has the right lifting property with respect to all trivial cofibrations $i : U \to V$ such that V is α-bounded [5, Lemma 2.4]. Then a map $q : G \to H$ in the category $s\mathbf{Gd}\mathrm{Pre}(\mathcal{C})$ is a fibration if and only if it has the right lifting property with respect to all maps $G(i) : GU \to GV$ induced by those maps $i : U \to V$ since there exist the adjoint diagrams:

$$\begin{array}{ccc} GU & \longrightarrow & G \\ {\scriptstyle G(i)}\downarrow & \nearrow & \downarrow{\scriptstyle q} \\ GV & \longrightarrow & H \end{array} \qquad\qquad \begin{array}{ccc} U & \longrightarrow & \overline{W}G \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle \overline{W}(q)} \\ V & \longrightarrow & \overline{W}H \end{array} \qquad (\mathbf{D})$$

For each $W \in \mathcal{C}$, $GV(W)_n$ is the free groupoid on generators $x \in V(W)_{n+1}$ subject to some relations, and $\mathrm{Ob}(GV(W)) = V(W)_0$, so the cardinality of each $\mathrm{Mor}(GV(W)_n)$, $n \geq 0$ and $\mathrm{Ob}(GV(W))$ is smaller than $\beta = max(2^\alpha, \infty)$. We also call the presheaf of simplicial groupoids GV is $\beta - bounded$.

When G is a simplicial group there is a pullback diagram

$$\begin{array}{ccc} G & \xrightarrow{i} & WG \\ \downarrow & & \downarrow{\scriptstyle q} \\ * & \xrightarrow[*]{} & \overline{W}G \end{array}$$

where q is a fibration of simplicial sets [3, Lemma V.4.1], G is the fibre over the unique vertex $* \in \overline{W}G$. G is a simplicial group, so G is a Kan complex [3, Lemma I.3.4]. $\overline{W}G$ is a Kan complex [3, Corollary V.6.8], so is WG, then for any vertex $v \in G$ there exists a long exact sequence

$$... \to \pi_n(G, v) \xrightarrow{i_*} \pi_n(WG, v) \xrightarrow{q_*} \pi_n(\overline{W}G, *) \xrightarrow{\partial} \pi_{n-1}(G, v) \to ...$$

$$... \xrightarrow{q_*} \pi_1(\overline{W}G, *) \xrightarrow{\partial} \pi_0(G) \xrightarrow{i_*} \pi_0(WG) \xrightarrow{q_*} \pi_0(\overline{W}G)$$

by Lemma I.7.3 in [3]. WG is contractible [3, Lemma V.4.6], so $\pi_n(WG, v) = 0, n \geq 1$; and $\pi_0(WG) = 0$, since for any two vertices $a, b \in WG_0 = G_0$, there exists a 1-simplex $(s_0 b, b^{-1}a) \in WG_1 = G_1 \times G_0$, s.t., $d_1(s_0 b, b^{-1}a) = b, d_0(s_0 b, b^{-1}a) = a$. Then

$$\pi_n(G, v) = \pi_{n+1}(\overline{W}G, *), n \geq 1$$

$$\pi_0 G = \pi_1(\overline{W}G, *)$$

For an ordinary groupoid H, it's standard to write $\pi_0 H$ for the set of path components of H. By this one means that

$$\pi_0 H = Ob(H)/\sim$$

where there is a relation $x \sim y$ between two objects of H if and only if there is a morphism $x \to y$ in H.

If now A is a simplicial groupoid, all of the simplicial structure functors $\theta^* : A_n \to A_m$ induce isomorphisms $\pi_0 A_n \cong \pi_0 A_m$. We shall therefore refer to $\pi_0 A_0$ as the set of path components of the simplicial groupoid A, and denote it by $\pi_0 A$.

When A is a simplicial groupoid, $\mathrm{Ob}(A) = (\overline{W}A)_0$, $\mathrm{Mor}(A_0) = (\overline{W}A)_1$, so $\pi_0 A \cong \pi_0(\overline{W}A)$.

Choose a representative x for each $[x] \in \pi_0 A$, the inclusion

$$i : \bigsqcup_{[x]\in\pi_0 A} A(x, x) \to A$$

is a homotopy equivalence of simplicial groupoids, and the induced map

$$\overline{W}(i) : \overline{W}(\bigsqcup_{[x]\in\pi_0 A} A(x, x)) \to \overline{W}A$$

is a weak equivalence of simplicial sets. $\overline{W}$ preserves disjoint unions, $\overline{W}(\bigsqcup_{[x]\in\pi_0 A} A(x, x)) = \bigsqcup_{[x]\in\pi_0 A} \overline{W}(A(x, x))$ [3, p. 303,304].

$$\pi_n(\overline{W}(\bigsqcup_{[x]\in\pi_0 A} A(x, x)), x) = \pi_n(\overline{W}(A(x, x)), *) \cong \pi_{n-1}(A(x, x), v), n \geq 2, v \in A(x, x)_0$$

$$\pi_1(\overline{W}(\bigsqcup_{[x]\in\pi_0 A} A(x,x)),x) = \pi_1(\overline{W}(A(x,x)),*) \cong \pi_0(A(x,x)).$$

so one obtains

$$\pi_n(A(x,x),v) \cong \pi_{n+1}(\overline{W}A,x), n \geq 1, x \in \mathrm{Ob}(A), v \in A(x,x)_0,$$

$$\pi_0(A(x,x)) \cong \pi_1(\overline{W}A,x).$$

According to the definition of topological weak equivalence of simplicial presheaves in [5] and the relations between simplicial groupoids and simplicial sets, we can give an explicit description of weak equivalence of presheaves of simplicial groupoids.

For any presheaf of simplicial groupoids X and any object $U \in \mathcal{C}$ and $x \in \mathrm{Ob}(X(U))$, $X(U)(x,x)$ is a simplicial group. Associated to this presheaf of simplicial groupoids X on $\mathcal{C}$ and $* \in X(U)(x,x)_0$ is a presheaf $\pi_n^{simp}(X|_U,x,*)(n \geq 1)$ on the comma category $\mathcal{C} \downarrow U$, the presheaf of simplicial homtopy groups of $X|_U$, based at $*$, which is defined by

$$(\mathcal{C} \downarrow U)^{op} \to \mathbf{Grp}$$

$$\varphi : V \to U \mapsto \pi_n(X(V)(x_V,x_V),*_V)$$

where x_V and $*_V$ are the images of x and $*$ in $X(V)$ under the map $X(U) \to X(V)$ which is induced by $V \to U$, respectively; and the simplicial homotopy group $\pi_n(X(V)(x_V,x_V),*_V)$ exists since the simplicial group $X(V)(x_V,x_V)$ is a Kan complex [3, Lemma I.3.4].

Let $\pi_n(X|_U,x,*)$ be the associated sheaf of the presheaf $\pi_n^{simp}(X|_U,x,*)$, i.e., $\pi_n(X|_U,x,*) = L^2\pi_n^{simp}(X|_U,x,*)$. Then $\pi_n(X|_U,x,*)$ is a sheaf of groups which is abelian if $n \geq 2$. The sheaves $\pi_0(X|_U,x)$ and $\pi_0(X)$ of path components are defined similarly.

A map $f : X \to Y$ of presheaves of simplicial groupoids is said to be a *weak equivalence* if it induces isomorphisms of sheaves

$$f_* : \pi_n(X|_U,x,*) \cong \pi_n(Y|_U,fx,f*), n \geq 1, U \in \mathcal{C}, x \in \mathrm{Ob}(X(U)), * \in X(U)(x,x)_0$$

$$f_* : \pi_0(X|_U,x) \cong \pi_0(Y|_U,fx).$$

$$f_* : \pi_0(X) \cong \pi_0(Y).$$

In view of Proposition 1.18 in [5], the weak equivalence is just same as the combinatorial weak equivalence in [5]. Since the weak equivalence is defined by the isomorphisms between sheaves of groups, thus, Proposition 1.11 in [5] implies (or directly followed from the **CM2** of category $\mathbf{S}\mathrm{Pre}(\mathcal{C})$)

Lemma 2.1. *Given maps of presheaves of simplicial groupoids $f : X \to Y$ and $g : Y \to Z$, if any two of f, g, or $g \circ f$ are weak equivalences, then so is the third.*

Lemma 2.2. *The functor $X \mapsto \overline{W}G(X)$ preserves weak equivalences of simplicial presheaves.*

Proof. When T is a simplicial set, the natural simplicial map $\eta : T \to \overline{W}G(T)$ is a weak equivalence of simplicial sets [3, Theorem V.7.8]. So the map $X \to \overline{W}G(X)$ is a pointwise weak equivalence of simplicial presheaves, then it is a weak equivalence.

There exists a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \overline{W}G(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{W}G(f)} \\ Y & \xrightarrow[\eta_Y]{} & \overline{W}G(Y) \end{array}$$

where both η_X and η_Y are weak equivalences, if $f : X \to Y$ is a weak equivalences, so is $\overline{W}G(f)$ by the **CM2** of the closed model category $\mathbf{S}\mathrm{Pre}(\mathcal{C})$. □

Lemma 2.3. *The functor $G : \mathbf{S}Pre(\mathcal{C}) \to s\mathbf{Gd}Pre(\mathcal{C})$ preserves cofibrations and weak equivalences.*

Proof. The adjoint diagrams (**D**) imply that the functor G preserves cofibrations. Lemma 2.2 implies that G preserves weak equivalences. □

Lemma 2.4. *The category $s\mathbf{Gd}Pre(\mathcal{C})$ has all pushouts, and is hence cocomplete. The class of cofibrations in $s\mathbf{Gd}Pre(\mathcal{C})$ is closed under pushout.*

Proof. The category $s\mathbf{Gd}$ has all pushouts and is cocomplete, so is the category $s\mathbf{Gd}\mathrm{Pre}(\mathcal{C})$, since we can take the pushout and colimit pointwise. The second statement is obvious. □

There exists a Kan Ex^∞ functor from $\mathbf{S}\mathrm{Pre}(\mathcal{C})$ to $\mathbf{S}\mathrm{Pre}(\mathcal{C})$, such that $Ex^\infty X$ is locally fibrant for any simplicial presheaf X and the canonical map $\nu : X \to Ex^\infty X$ is a pointwise weak equivalence [6].

Fix a Boolean localization $\wp : Shv(\mathcal{B}) \to \mathcal{E}$, and consider the functors

$$\mathbf{S}Pre(\mathcal{C}) \xrightarrow{L^2} \mathbf{S}\mathcal{E} \xrightarrow{\wp^*} \mathbf{S}Shv(\mathcal{B})$$

relating the categories of simplicial presheaves on $\mathcal{C}$ and the categories of simplicial sheaves and the categories of simplicial objects in the categories of sheaves $Shv(\mathcal{B})$, where L^2 is the associated sheaf functor. In [6] Jardine proves that the topological weak equivalence between simplicial presheaves [5] coincides with the local weak equivalence [6], i.e., a map $f : X \to Y$ of simplicial presheaves on $\mathcal{C}$ is a topological weak equivalence if the induced map $\wp^* L^2 : \wp^* L^2 Ex^\infty X \to \wp^* L^2 Ex^\infty Y$ is a pointwise weak equivalence.

Notice that there is a commutative diagram

$$\begin{array}{ccccc} s\mathbf{Gd}Pre(\mathcal{C}) & \xrightarrow{L^2} & s\mathbf{Gd}\mathcal{E} & \xrightarrow{\wp^*} & s\mathbf{Gd}Shv(\mathcal{B}) \\ {\scriptstyle \overline{W}}\downarrow & & \downarrow{\scriptstyle \overline{W}} & & \downarrow{\scriptstyle \overline{W}} \\ \mathbf{S}Pre(\mathcal{C}) & \xrightarrow[L^2]{} & \mathbf{S}\mathcal{E} & \xrightarrow[\wp^*]{} & \mathbf{S}Shv(\mathcal{B}) \end{array}$$

$\overline{W}G$ is locally fibrant simplicial presheaf for any presheaf of simplicial groupoid G, so a map $f : G \to H$ of presheaves of simplicial groupoids on $\mathcal{C}$ is a weak equivalence if the induced map $\wp^* L^2 : \wp^* L^2 G \to \wp^* L^2 H$ is a pointwise weak equivalence.

Proposition 2.2. *Trivial cofibrations of presheaves of simplicial groupoids are closed under pushout.*

Proof. Suppose that

$$\begin{array}{ccc} G & \longrightarrow & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ H & \longrightarrow & D \end{array}$$

is a pushout in the category $s\mathbf{Gd}\mathrm{Pre}(\mathcal{C})$. i is a trivial cofibration, then i' is a cofibration by Lemma 2.4.

The heart of the matter for this proof is the weak equivalence. Both L^2 and $\wp^*$ are left adjoint functors, so the functor $\wp^* L^2$ preserves the pushout

$$\begin{array}{ccc} \wp^* L^2 G & \longrightarrow & \wp^* L^2 C \\ {\scriptstyle \wp^* L^2(i)}\downarrow & & \downarrow{\scriptstyle \wp^* L^2(i')} \\ \wp^* L^2 H & \longrightarrow & \wp^* L^2 D \end{array}$$

the map $\wp^* L^2(i)$ is a pointwise weak equivalence and pointwise cofibration, so for any $U \in \mathcal{B}$, the diagram

$$\begin{array}{ccc} \wp^* L^2 G(U) & \longrightarrow & \wp^* L^2 C(U) \\ {\scriptstyle \wp^* L^2(i)}\downarrow & & \downarrow{\scriptstyle \wp^* L^2(i')} \\ \wp^* L^2 H(U) & \longrightarrow & \wp^* L^2 D(U) \end{array}$$

is a pushout in the category $s\mathbf{Gd}$. The category $s\mathbf{Gd}$ is a closed model category, then the map $\wp^* L^2(i)$ is a trivial cofibration, so is $\wp^* L^2(i') : \wp^* L^2 C(U) \to \wp^* L^2 D(U)$. Then $\wp^* L^2(i') : \wp^* L^2 C \to \wp^* L^2 D$ is a pointwise weak equivalence, so $i' : C \to D$ is a weak equivalence in the category $s\mathbf{Gd}\mathrm{Pre}(\mathcal{C})$. □

Given a trivial cofibration $i : A \to B$ in the category $\mathbf{S}\mathrm{Pre}(\mathcal{C})$, suppose that

$$\begin{array}{ccc} GA & \longrightarrow & C \\ {\scriptstyle G(i)}\downarrow & & \downarrow{\scriptstyle i'} \\ GB & \longrightarrow & D \end{array}$$

is a pushout in the category $s\mathbf{Gd}\mathrm{Pre}(\mathcal{C})$. The map $G(i)$ is a trivial cofibration by Lemma 2.3, then the map i' is a trivial cofibration.

Lemma 2.5. *Every map $f : X \to Y$ of presheaves of simplicial groupoids may be factored*

$$\begin{array}{ccccc} X & & \xrightarrow{\quad f \quad} & & Y \\ & {\scriptstyle i}\searrow & & \nearrow{\scriptstyle p} & \\ & & Z & & \end{array}$$

where i is a trivial cofibration and p is a fibration.

Proof. We use transfinite small object argument. Choose a cardinal number $\gamma > 2^\beta$, and define a functor $F : \gamma \to s\mathbf{Gd}Pre(\mathcal{C}) \downarrow Y$ on the partially ordered set γ by setting $F(0) = f : X \to Y, F(s) : X(s) \to Y$ such that

(1) $X(0) = X$,

(2) $X(t) = \varinjlim_{s<t} X(s)$ for all limit ordinals $t < \gamma$, and

(3) the map $X(s) \to X(s+1)$ is defined by the pushout diagram

$$\begin{array}{ccc} \bigsqcup_D GU_D & \xrightarrow{(\alpha_D)} & X(s) \\ {\scriptstyle \bigsqcup_D Gi_D}\downarrow & & \downarrow \\ \bigsqcup_D GV_D & \longrightarrow & X(s+1) \end{array}$$

where the index D refers to a set of representatives for all diagrams

$$\begin{array}{ccc} GU_D & \xrightarrow{\alpha_D} & X(s) \\ {\scriptstyle Gi_D}\downarrow & & \downarrow{\scriptstyle F(s)} \\ GV_D & \longrightarrow & Y \end{array}$$

such that $Gi_D : GU_D \to GV_D$ is induced by $i_D : U_D \to V_D$, where i_D is a trivial cofibration in $\mathbf{S}\mathrm{Pre}(\mathcal{C})$ with V_D $\alpha-$bounded.

Then GV_D is $\beta-$bounded. Let

$$X(\gamma) = \varinjlim_{t<\gamma} X(t)$$

and consider the induced factorization of f

$$\begin{array}{ccc} X & \xrightarrow{i_\gamma} & X(\gamma) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle F(\gamma)} & \\ & Y & \end{array}$$

Then i_γ is a trivial cofibration, since it is a filtered colimit of such. Also, for any diagram

$$\begin{array}{ccc} GU & \longrightarrow & X(\gamma) \\ {\scriptstyle Gi}\downarrow & & \downarrow{\scriptstyle F(\gamma)} \\ GV & \longrightarrow & Y \end{array}$$

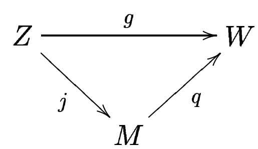
such that GV is $\beta-$bounded and Gi is a trivial cofibration, the map $GU \to X(\gamma)$ must factor through some $X(n) \to X(\gamma), n < \gamma$, for otherwise GU has too many subobjects. $\square$

For each object U of $\mathcal{C}$, the $U-$sections functor $X \to X(U)$ has a left adjoint $?_U : \mathbf{S} \to \mathbf{S}Pre(\mathcal{C})$ which sends the simplicial set Y to the simplicial presheaf Y_U, which is defined by

$$Y_U(V) = \coprod_{\varphi : V \to U} Y.$$

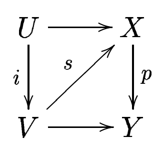
Then a simplicial presheaves map $q : Z \to X$ is a trivial fibration if and only if it has the right lifting property with respect to all inclusions $S \subset \triangle^n_U$ of subobjects of the $\triangle^n_U, U \in \mathcal{C}, n \geq 0$ [5, p. 68]. So a map $p : G \to H$ of presheaves of simplicial groupoids is a trivial fibration if and only if it has the right lifting property with respect to all inclusions $GS \subset G\triangle^n_U$ of subobjects of the $G\triangle^n_U, U \in \mathcal{C}, n \geq 0$. A transfinite small object argument, as in Lemma 2.5, shows that

Lemma 2.6. *Every map $g : Z \to W$ of presheaves of simplicial groupoids may be factored*



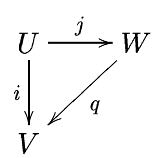
where j is a cofibration and q is a trivial fibration.

Lemma 2.7. *For the commutative diagram*

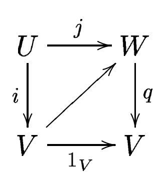


where i is a trivial cofibration and p is a fibration in the caregory $s\mathbf{Gd}Pre(\mathcal{C})$, there exists a lifting s.

Proof. Suppose that $i : U \to V$ is a trivial cofibration. Then i has a factorization



where q is a fibration and j is a trivial cofibration which has the left lifting property with respect to all fibrations by the construction in the proof of Lemma 2.5. But then q is a trivial fibration, and so the lifting exists in the diagram



It follows that i is a retract of j, so that i has the left lifting property with respect to all fibrations. □

Theorem 2.1. *The category $s\mathbf{Gd}Pre(\mathcal{C})$, with the classes of fibrations, weak equivalences and cofibrations as defined above, satisfies the axioms for a closed model category.*

Proof. The category $s\mathbf{Gd}$ is closed under all finite limits and colimits, we can take the limits and colimits pointwise, so the category $s\mathbf{Gd}\mathrm{Pre}(\mathcal{C})$ is also closed under all finite limits and colimits. This is **CM1**. **CM2** is the Lemma 2.1. **CM3** is trivial. The first part of **CM4** is the Lemma 2.7, the second part is the definition of cofibration. **CM5**(1) is the Lemma 2.5, **CM5**(2) is the Lemma 2.6. □

Remark 2.1. The fibration (trivial fibration) in the category $s\mathbf{Gd}\mathrm{Pre}(\mathcal{C})$ has the right lifting property with respect to all maps $G(i) : GU \to GV$ induced by the maps $i : U \to V$ where i is a trivial cofibration (cofibration) in the category $\mathbf{S}\mathrm{Pre}(\mathcal{C})$ and V is $\alpha-$bounded. So the category $s\mathbf{Gd}\mathrm{Pre}(\mathcal{C})$ is cofibrantly generated.

Lemma 2.8. *(1) The functor $\overline{W} : s\mathbf{Gd}Pre(\mathcal{C}) \to \mathbf{S}Pre(\mathcal{C})$ preserves fibrations and weak equivalences.*

(2) A map $K \to \overline{W}X \in \mathbf{S}Pre(\mathcal{C})$ is a weak equivalence if and only if its adjoint $GK \to X \in s\mathbf{Gd}Pre(\mathcal{C})$ is a weak equivalence.

Proof. (1) This is implied by the definitions of fibration and weak equivalence.

(2) There is a commutative diagram

$$\begin{array}{ccc} K & \longrightarrow & \overline{W}GK \\ & \searrow \quad \swarrow & \\ & \overline{W}X & \end{array}$$

where the map $K \to \overline{W}GK$ is a pointwise weak equivalence [3, Theorem V.7.8(3)]. So the map $K \to \overline{W}X$ is a weak equivalence if and only if the map $\overline{W}GK \to \overline{W}X$ is a weak equivalence, i.e., the map $GK \to X$ is a weak equivalence. □

Corollary 2.1. *The functor G and $\overline{W}$ induce an equivalence of homotopy categories*

$$Ho(s\mathbf{Gd}Pre(\mathcal{C})) \simeq Ho(\mathbf{S}Pre(\mathcal{C}))$$

Proof. Lemma 2.8 implies that the natural maps $\varepsilon : G\overline{W}K \to K$ and $\eta : X \to \overline{W}GX$ are weak equivalences for all presheaves of simplicial groupoids K and simplicial presheaves X. □

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are two closed model categories.

1. We call a functor $F : \mathcal{C} \to \mathcal{D}$ a *left Quillen functor* if F is a left adjoint and preserves cofibrations and trivial cofibrations.

2. We call a functor $U : \mathcal{D} \to \mathcal{C}$ a *right Quillen functor* if U is a right adjoint and preserves fibrations and trivial fibrations.

3. Suppose that (F, U, φ) is an adjunction from $\mathcal{C}$ to $\mathcal{D}$. That is, F is a functor $\mathcal{C} \to \mathcal{D}$, U is a functor $\mathcal{D} \to \mathcal{C}$, and φ is a natural isomorphism $\mathcal{D}(FC, D) \to \mathcal{C}(C, UD)$ expressing U as a right adjoint of F. We call (F, U, φ) a *Quillen adjunction* if F is a left Quillen functor (cf. [4]).

A Quillen adjunction $(F, U, \varphi) : \mathcal{C} \to \mathcal{D}$ is called a *Quillen equivalence* if and only if, for all cofibrant X in $\mathcal{C}$ and fibrant Y in $\mathcal{D}$, a map $f : FX \to Y$ is a weak equivalence in $\mathcal{D}$ if and only if $\varphi(f) : X \to UY$ is a weak equivalence in $\mathcal{C}$ (cf. [4]).

Corollary 2.2. *The adjunction* $G : \mathbf{S}Pre(\mathcal{C}) \rightleftarrows s\mathbf{Gd}Pre(\mathcal{C}) : \overline{W}$ *is a Quillen equivalence.*

Proof. It's obvious from Theorem 2.1, Proposition 2.1, Lemma 2.3, Lemma 2.8 and above definitions. □

3 Presheaves of 2-groupoids

2-Gpd Pre($\mathcal{C}$) is the category of presheaves of 2-groupoids on $\mathcal{C}$; its objects are the contravariant functors from $\mathcal{C}$ to the category **2-Gpd** of 2-groupoids, and its morphisms are natural transformations.

Moerdijk and Svensson show that [8], with the following definitions of weak equivalence, fibration and cofibration, the category **2-Gpd** of 2-groupoids satisfies the axioms for a closed model category.

A map $\varphi : \mathcal{A} \to \mathcal{B}$ of 2-groupoids is said to be a *weak equivalence* of **2-Gpd** if

(1) for every object b of $\mathcal{B}$ there exists an object a of $\mathcal{A}$ and an arrow $\varphi(a) \to b$;

(2) for any two objects a, a' in $\mathcal{A}$, φ induces an equivalence of categories (groupoids)

$$\varphi_{a,a'} : Hom_{\mathcal{A}}(a, a') \to Hom_{\mathcal{B}}(\varphi(a), \varphi(a')).$$

A map $\psi : \mathcal{B} \to \mathcal{A}$ of 2-groupoids is said to be a (*Grothendieck*) *fibration* of **2-Gpd** if for any arrow $f : b_1 \to b_2$ in $\mathcal{B}$ and any arrows $g : a_0 \to \psi(b_1)$ and $h : a_0 \to \psi(b_2)$, any deformation $\alpha : h \Rightarrow \psi(f) \circ g$ can be lifted to a deformation $\tilde{\alpha} : \tilde{h} \Rightarrow f \circ \tilde{g}$ in $\mathcal{B}$ (in the sense that $\psi(\tilde{\alpha}) = \alpha, \psi(\tilde{h}) = h$ and $\psi(\tilde{g}) = g$).

A *cofibration* of 2-groupoids is defined to be a map which has the left lifting property with respect to all morphisms of **2-Gpd** which are both fibrations and weak equivalences.

Recall the adjunction [8]:

$$G : \mathbf{S} \rightleftarrows \mathbf{2} - \mathbf{Gpd} : \overline{W}$$

where the functor $\overline{W}$ is the functor N in [8] and the functor G is the Whitehead 2-groupoid functor W in [8]: $W(X) = W(|X|, |X^{(1)}|, |X^{(0)}|)$. By applying these functors pointwise to simplicial presheaves and presheaves of 2-groupoids, one obtains functors

$$G : \mathbf{S}Pre(\mathbf{C}) \rightleftarrows \mathbf{2} - \mathbf{Gpd}Pre(\mathbf{C}) : \overline{W}$$

and there is

Proposition 3.1. *The functor $G : \mathbf{S}Pre(\mathcal{C}) \to \mathbf{2} - \mathbf{Gpd}Pre(\mathcal{C})$ is left adjoint to the functor $\overline{W}$.*

A map $f : X \to Y$ in the category **2-Gpd**Pre($\mathcal{C}$) is said to be a *fibration* if the induced map $\overline{W}(f) : \overline{W}X \to \overline{W}Y$ is a global fibration in the category **S**Pre($\mathcal{C}$).

A map $g : Z \to U$ in the category **2-Gpd**Pre($\mathcal{C}$) is said to be a *weak equivalence* if the induced map $\overline{W}(g) : \overline{W}Z \to \overline{W}U$ is a weak equivalence in the category **S**Pre($\mathcal{C}$).

A *cofibration* in the category **2-Gpd**Pre($\mathcal{C}$) is a map of presheaves of 2-groupoids which has the left lifting property with respect to all fibations and weak equivalences.

Say that a map of presheaves of 2-groupoids f is a *trivial fibration* if it is both a fibration and a weak equivalence; a map g is a *trivial cofibration* if it is both a cofibration and a weak equivalence.

Similarly, we can define the above concepts according to the category s**Gd**Pre($\mathcal{C}$).

A map $q : G \to H$ in the category **2-Gpd**Pre($\mathcal{C}$) is a fibration if and only if it has the right lifting property with respect to all maps $G(i) : GU \to GV$ induced by the maps $i : U \to V$ where i is a trivial cofibration in the category **S**Pre($\mathcal{C}$) and V is $\alpha-$bounded, since there exist two adjoint diagrams similar to the diagrams **D**.

For each $S \in \mathcal{C}$, $\mathrm{Ob}(GV(S)) = V(S)_0$, $\mathrm{Mor}(GV(S))$ and 2-cell($GV(S)$) are free generated by $V(S)_1$ and $V(S)_2$, subject to some relations, respectively. So the cardinality of objects, morphisms and 2-cells of 2-groupoid $GV(S)$ is smaller than $\beta = max(2^\alpha, \infty)$, where α is a boundary of the simplicial presheaf $V(S)$. We also call the presheaf of 2-groupoids GV is β *– bounded.*

For each 2-groupoid G and each object x of G, there are natural isomorphisms [8, Proposition 2.1(iii)]:

$$\pi_0(\overline{W}G) \cong \pi_0(G),$$

$$\pi_1(\overline{W}G, x) \cong \pi_1(G, x),$$

$$\pi_2(\overline{W}G, x) \cong \pi_2(G, x),$$

$$\pi_i(\overline{W}G, x) \cong 0 \ (i > 2).$$

According to the definition of topological weak equivalence of simplicial presheaves in [5] and the above relations, we can give an explicit description of weak equivalence of presheaves of 2-groupoids.

For any presheaf of 2-groupoids X and any object $U \in \mathcal{C}$ and $x \in \mathrm{Ob}(X(U))$, associated to this presheaf of 2-groupoids X on $\mathcal{C}$ and x is a presheaf on the comma category $\mathcal{C} \downarrow U$, the presheaf of homotopy groups of $X|_U$, based at x, which is defined by

$$(\mathcal{C} \downarrow U)^{op} \to \mathbf{Grp}$$

$$\varphi : V \to U \mapsto \pi_i(X(V), x_V), i = 1, 2$$

where x_V is the image of x in $X(V)$ under the map $X(U) \to X(V)$ which is induced by $V \to U$.

Let $\pi_i(X|_U, x), i = 1, 2$ be the associated sheaves of the above presheaves. The sheaf $\pi_0(X)$ of path components is defined similarly.

A map $f : X \to Y$ of presheaves of 2-groupoids is said to be a *weak equivalence* if it induces isomorphisms of sheaves

$$f_* : \pi_i(X|_U, x) \cong \pi_i(Y|_U, fx), i = 1, 2; U \in \mathcal{C}, x \in \mathrm{Ob}(X(U))$$

$$f_* : \pi_0(X) \cong \pi_0(Y).$$

In parallel with the corresponding arguments for presheaves of simplicial groupoids, we have

Lemma 3.1. *Given maps of presheaves of 2-groupoids $f : X \to Y$ and $g : Y \to Z$, if any two of f, g, or $g \circ f$ are weak equivalences, then so is the third.*

Lemma 3.2. *The functor $X \mapsto \overline{W}G(X)$ preserves weak equivalences of simplicial presheaves.*

Proof. When T is a simplicial set, there are isomorphisms [8]

$$\pi_0(\overline{W}GT) \cong \pi_0(GT) \cong \pi_0(T),$$

$$\pi_i(\overline{W}GT, t_0) \cong \pi_i(GT, t_0) \cong \pi_i(T, t_0)\ (i = 1, 2), t_0 \in T_0.$$

$$\pi_i(\overline{W}GT, t_0) = 0\ (i > 2).$$

so there exist isomorphisms of sheaves

$$\pi_0(\overline{W}GX) \cong \pi_0(GX) \cong \pi_0(X),$$

$$\pi_i(\overline{W}GX|_U, x) \cong \pi_i(GX|_U, x) \cong \pi_i(X|_U, x)\ (i = 1, 2)\ U \in \mathcal{C}\ x \in X(U)_0.$$

and $\pi_i(\overline{W}GX|_U, x) = 0\ (i > 2)$. □

Lemma 3.3. *The functor $G : \mathbf{S}Pre(\mathcal{C}) \to \mathbf{2\text{-}Gpd}Pre(\mathcal{C})$ preserves cofibrations and weak equivalences.*

Lemma 3.4. *The category $\mathbf{2\text{-}Gpd}Pre(\mathcal{C})$ has all pushouts, and is hence cocomplete. The class of cofibrations in $\mathbf{2\text{-}Gpd}Pre(\mathcal{C})$ is closed under pushout.*

Notice that there is a commutative diagram

$$\begin{array}{ccccc}
\mathbf{2} - \mathbf{Gpd}Pre(\mathcal{C}) & \xrightarrow{L^2} & \mathbf{2} - \mathbf{Gpd}\mathcal{E} & \xrightarrow{\wp^*} & \mathbf{2} - \mathbf{Gpd}Shv(\mathcal{B}) \\
\downarrow \overline{W} & & \downarrow \overline{W} & & \downarrow \overline{W} \\
\mathbf{S}Pre(\mathcal{C}) & \xrightarrow[L^2]{} & \mathbf{S}\mathcal{E} & \xrightarrow[\wp^*]{} & \mathbf{S}Shv(\mathcal{B})
\end{array}$$

$\overline{W}G$ is locally fibrant simplicial presheaf for any presheaf of 2-groupoids G, so a map $f : G \to H$ of presheaves of 2-groupoids on $\mathcal{C}$ is a weak equivalence if the induced map $\wp^* L^2 : \wp^* L^2 G \to \wp^* L^2 H$ is a pointwise weak equivalence.

Proposition 3.2. *Trivial cofibrations of presheaves of 2-groupoids are closed under pushout.*

Given a trivial cofibration $i : A \to B$ in the category $\mathbf{S}\mathrm{Pre}(\mathcal{C})$, suppose that

$$\begin{array}{ccc} GA & \longrightarrow & C \\ {\scriptstyle G(i)}\downarrow & & \downarrow{\scriptstyle i'} \\ GB & \longrightarrow & D \end{array}$$

is a pushout in the category $\mathbf{2\text{-}Gpd}\mathrm{Pre}(\mathcal{C})$. Then the map i' is a trivial cofibration.

Lemma 3.5. *Every map $f : X \to Y$ of presheaves of 2-groupoids may be factored*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle p} & \\ & Z & \end{array}$$

where i is a trivial cofibration and p is a fibration.

A map $p : G \to H$ of presheaves of 2-groupoids is a trivial fibration if and only if it has the right lifting property with respect to all inclusions $GS \subset G\triangle^n_U$ of subobjects of the $G\triangle^n_U, U \in \mathcal{C}, n \geq 0$. A transfinite small object argument, as in Lemma 2.5, shows that

Lemma 3.6. *Every map $g : Z \to W$ of presheaves of 2-groupoids may be factored*

$$\begin{array}{ccc} Z & \xrightarrow{\quad g \quad} & W \\ & {\scriptstyle j}\searrow \quad \nearrow{\scriptstyle q} & \\ & M & \end{array}$$

where j is a cofibration and q is a trivial fibration.

Lemma 3.7. *For the commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle p} \\ V & \longrightarrow & Y \end{array}$$

where i is a trivial cofibration and p is a fibration in the caregory $\mathbf{2\text{-}Gpd}Pre(\mathcal{C})$, there exists a lifting s.

Theorem 3.1. *The category $\mathbf{2\text{-}Gpd}Pre(\mathcal{C})$, with the classes of fibrations, weak equivalences and cofibrations as defined above, satisfies the axioms for a closed model category.*

Lemma 3.8. *(1) The functor $\overline{W} : \mathbf{2} - \mathbf{Gpd}Pre(\mathcal{C}) \to \mathbf{S}Pre(\mathcal{C})$ preserves fibrations and weak equivalences.*

(2) The functors G and $\overline{W}$ induce adjoint functors

$$G : Ho(\mathbf{S}Pre(\mathcal{C})) \rightleftarrows Ho(\mathbf{2} - \mathbf{Gpd}Pre(\mathcal{C})) : \overline{W}$$

at the level of homotopy categories.

Proof. (1) This is implied by the definitions of fibration and weak equivalence.

(2) The functors $\overline{W}$ and G both preserve weak equivalences ((1) of this Lemma and Lemma 3.3), they localize to functors of homotopy categories. The triangular identities for the unit and counit will still hold after localization. □

Corollary 3.1. *The adjunction* $G : \mathbf{S}Pre(\mathcal{C}) \rightleftarrows \mathbf{2} - \mathbf{Gpd}Pre(\mathcal{C}) : \overline{W}$ *is a Quillen adjunction.*

Proof. It's obvious from Theorem 3.1, Proposition 3.1, Lemma 3.3 and the definition of Quillen adjunction. □

Define the category $2 - types\mathbf{S}Pre(\mathcal{C})$ of homotopy 2-types to be the full subcategory of Ho($\mathbf{S}$Pre($\mathcal{C}$)) given by those simplicial presheaves with sheaves $\pi_i(X|_U, x) = 0$ for any integer $i > 2$, any object $U \in \mathcal{C}$ and any basepoint $x \in X(U)_0$.

Theorem 3.2. *The functors* G *and* $\overline{W}$ *induce an equivalence of homotopy categories*

$$Ho(\mathbf{2} - \mathbf{Gpd}Pre(\mathcal{C})) \simeq \textit{2-types}\mathbf{S}Pre(\mathcal{C})$$

Proof. For a simplicial presheaf X, the natural map $\eta : X \to \overline{W}G(X)$ is a weak equivalence if and only if $\pi_i(X|_U, x) = 0, i > 2, U \in \mathcal{C}, x \in X(U)_0$. For any presheaf of 2-groupoids K, $\pi_i(\overline{W}K|_U, *) = 0, i > 2, U \in \mathcal{C}, * \in Ob(K(U))$, and the natural map $\varphi : G\overline{W}(K) \to K$ is a weak equivalence. □

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References

[1] S.E. Crans, *Quillen closed model structures for sheaves*, J. Pure Appl. Algebra 101:35-57, 1995.

[2] W.G. Dwyer and D.M. Kan, *Homotopy theory and simplicial groupoids*, Indag. Math. **46**(1984), 379-385.

[3] P.G. Goerss and J.F. Jardine, *Simplicial Homotopy Theory*, Birkhäuser, PM174, 1999.

[4] M. Hovey, *Model Categories*, Mathematical Surveys and Monographs, V. 63, American Mathematics Society, 1999.

[5] J. F. Jardine, *Simplicial presheaves*, J. Pure and Appl. Algebra **47** (1987), 35-87.

[6] J. F. Jardine, *Boolean Localization, In Practice*, Doc. Math 1, 1996, 245-275.

[7] A. Joyal, *letter to A.Grothendieck*, (1984).

[8] I. Moerdijk and J. Svensson, *Algebraic classification of equivariant homotopy 2 − types I*, J. Pure Appl. Algebra 89:187-216, 1993.

[9] D. Quillen, *Homotopical Algebra*, Lecture Notes in Math., Vol.43, Springer, Berlin-Heidelberg-New York, 1967.

[10] D. Quillen, *Rational homotopy theory*, Ann. of Math. **90** (1969), 205-295.

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