

RATIONALIZED EVALUATION SUBGROUPS OF A MAP II: QUILLEN MODELS AND ADJOINT MAPS

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ABSTRACT. Let $\omega: \text{map}(X, Y; f) \rightarrow Y$ denote a general evaluation fibration. Working in the setting of rational homotopy theory via differential graded Lie algebras, we identify the long exact sequence induced on rational homotopy groups by ω in terms of (generalized) derivation spaces and adjoint maps. As a consequence, we obtain a unified description of the rational homotopy theory of function spaces, at the level of rational homotopy groups, in terms of derivations of Quillen models and adjoints. In particular, as a natural extension of a result of Tanré, we identify the rationalization of the evaluation subgroups of a map $f: X \rightarrow Y$ in this setting. As applications, we consider a generalization of a question of Gottlieb, within the context of rational homotopy theory. We also identify the rationalization of the G -sequence of f and make explicit computations of the homology of this sequence. In a separate result of independent interest, we give an explicit Quillen minimal model of a product $A \times X$, in the case in which A is a rational co- H -space.

1. INTRODUCTION

Let $f: X \rightarrow Y$ be a based map of connected spaces. Let $\text{map}(X, Y)$ denote the space of unbased maps from X to Y and $\text{map}(X, Y; f)$ the path component containing f . Evaluation at the basepoint of X determines a fibration $\omega: \text{map}(X, Y; f) \rightarrow Y$. The n th *evaluation subgroup of f* is then defined to be the subgroup

$$G_n(Y, X; f) = \text{Image}\{\omega_{\#}: \pi_n(\text{map}(X, Y; f)) \rightarrow \pi_n(Y)\}$$

of $\pi_n(Y)$. The n th Gottlieb group $G_n(X)$ of a space X occurs as the special case $X = Y$ and $f = 1$. Because $\omega: \text{map}(X, X; 1) \rightarrow X$ can be identified with the connecting map of the universal fibration for fibrations with fibre X , the Gottlieb group is an important universal object for fibrations with fibre X . The structure of the Gottlieb group and its role in homotopy theory represents a broad area of research having its beginnings, of course, with the papers of Gottlieb [Got65, Got68, Got69] (see [Opr95] for a recent survey and further references). A fundamental obstacle to computing Gottlieb groups is their lack of functorality, in that generally $f_{\#}(G_*(X)) \not\subseteq G_*(Y)$ for a map $f: X \rightarrow Y$. By widening our perspective so as to include the evaluation subgroups of a map, we remedy this situation somewhat, in that $f: X \rightarrow Y$ always induces a map $f_{\#}: G_n(X) \rightarrow G_n(Y, X; f)$.

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In a previous paper [LS03], we have developed a basic framework within which rational homotopy groups of function spaces and related topics—including rationalized evaluation subgroups— may be studied. All results in that paper were developed from the Sullivan minimal model point of view, that is, using DG algebras. The current paper is intended both as a complement to and a continuation of the earlier work. It is complementary, in that all results here, and the basic framework that we establish, are developed from the Quillen minimal model point of view, that is, using DG Lie algebras. It continues the earlier work in that we present some different developments of the basic results. We refer to [LS03] for a general introduction to the themes of this paper, and for a survey of the existing results in the area.

In [LS03, Th.2.1], we identified the rationalization of the induced homomorphism $\omega_{\#}: \pi_n(\text{map}(X, Y; f)) \rightarrow \pi_n(Y)$ in terms of the homology of derivation spaces of the Sullivan models of the spaces X and Y . Our result extends, to a connected component of a general function space, the isomorphism

$$\pi_n(\text{map}(X, X; 1)) \otimes \mathbb{Q} \cong H_n(\text{Der}_*(\mathcal{M}_X)),$$

which was first observed by Sullivan in [Sul78]. Here $\text{Der}_*(\mathcal{M}_X)$ is the differential graded vector space of degree-lowering derivations of the Sullivan model of X . Our results in [LS03] provide a general framework for studying certain long exact sequences of homotopy groups for function space components within the context of Sullivan minimal models. In particular, they allow for an extension of the Félix-Halperin description of the rational Gottlieb groups (see [FH82, Th.3]) to the more general setting of rational evaluation subgroups of a map.

The developments of this paper proceed in an analogous way to those of [LS03]. As a special case of Theorem 4.1, we obtain an isomorphism

$$\pi_n(\text{map}_*(X, X; 1)) \otimes \mathbb{Q} \cong H_n(\text{Der}_*(\mathcal{L}_X)),$$

where $(\text{Der}_*(\mathcal{L}_X))$ is the differential graded vector space of degree-raising derivations of the Quillen model of X . More generally, the results of Section 4 lead to a complete picture, within the framework of Quillen models, of various long exact sequences of rational homotopy groups of function space components. In particular, in Theorem 5.2 we extend Tanré’s description of the rationalized Gottlieb group [Tan83, VII.4.(10)] to a description of the rationalized evaluation subgroup of a map.

We pursue two applications of these results here. We first consider a generalization of a question of Gottlieb, in the context of rational homotopy theory. It is well-known that Gottlieb elements in $G_*(X)$ have vanishing Whitehead product with all elements of $\pi_*(X)$. Let $[\ , \]_w$ denote the Whitehead bracket in $\pi_*(X)$ and let $P_*(X)$ denote the subgroup of $\pi_*(X)$ consisting of homotopy elements with vanishing Whitehead product with all elements of $\pi_*(X)$ —the so-called *Whitehead center* of $\pi_*(X)$. Gottlieb’s question asks about the difference between $P_*(X)$ and its subgroup $G_*(X)$. More generally, given a map $f: X \rightarrow Y$ set

$$P_n(Y, X; f) = \{\alpha \in \pi_n(Y) \mid [\alpha, f_{\#}(\beta)]_w = 0 \text{ for all } \beta \in \pi_*(X)\}.$$

Then $G_n(Y, X; f)$ is a subgroup of $P_n(Y, X; f)$, and $P_*(X)$ occurs as the special case $X = Y$ and $f = 1$. In Theorem 5.4, we identify the quotient $P_*(Y, X; f)/G_*(Y, X; f)$ for general maps $f: X \rightarrow Y$ between rational spaces via a particular commutative diagram of adjoints and derivation spaces. On the other hand, in Theorem 5.9 we

prove $G_*(Y, X; f) = P_*(Y, X; f)$ when $f: X \rightarrow Y$ is a coformal map of rational spaces.

As a second application, we describe and study the rationalization of the so-called *G-sequence of a map* as constructed by Lee and Woo [WL88b]. This is a chain complex of the form

$$\cdots \longrightarrow G_n(X) \xrightarrow{f\#} G_n(Y, X; f) \longrightarrow G_n^{rel}(Y, X; f) \longrightarrow G_{n-1}(X) \longrightarrow \cdots$$

which arises naturally in the context of function spaces and evaluation maps described above. We identify the rationalization of the G -sequence in the framework of derivation spaces and adjoint maps of Quillen models (Corollary 6.4). We give, in particular, a complete calculation of the homology of this sequence at a particular term, in the case of a single cell-attachment (Theorem 6.7). We also prove the rationalized G -sequence is exact at the same particular term for coformal maps (Theorem 6.8), as an extension of the equality above.

The paper is organized as follows. Section 2 contains purely algebraic definitions and constructions used throughout the paper. Here, we describe the basic framework of generalized derivation spaces and generalized adjoint maps, and the exact sequences in homology which arise in this context. In Section 3 we obtain a first connection between generalized derivations and topology. In a self-contained argument, we describe an explicit Quillen minimal model for a product $A \times X$ in terms of the Quillen models of the factors, in the case in which A is a rational co- H -space. The description depends on a particular class of generalized derivations. While we believe the general result is of independent interest, we require it here mainly for the case in which A is a single sphere S^n and briefly for the case in which $A = S^n \vee S^n$. In Section 4 we connect the algebraic framework of Section 2 to the homotopy data of function spaces and evaluation maps. The main results here are Theorem 4.1 and Theorem 4.4, in which we identify the rational homotopy groups of function spaces, and more generally the long exact rational homotopy sequence of the evaluation fibration $\omega: \text{map}(X, Y; f) \rightarrow Y$, in terms of homology of derivation spaces of Quillen models. Sections 5 and 6 contain our examples and applications. In a technical appendix, we review some basic material concerning DG Lie algebra homotopy theory and provide details of some results from this area that we use for our proofs.

Throughout this paper, spaces X and Y will be based, simply connected CW complexes of finite type. Given a map $f: A \rightarrow B$, either topological or algebraic, f^* denotes pre-composition by f and f_* post-composition by f . We use “ $*$ ” to denote the constant map of spaces. We write ω generically for an evaluation map $\omega: \text{map}(X, Y; f) \rightarrow Y$, since it will be clear from the context which component is intended. We use $H(f)$ to denote the map induced on homology by f , and, when f is a map of spaces, $f_\#$ for the map induced on homotopy groups. We write $f \sim g$ to denote that based maps f and g are homotopic via a based homotopy. In some instances, we will consider homotopies that are either unbased— such as those in the unbased function space component $\text{map}(X, Y; f)$, or are relative to some subspace. In such instances, we will specify the nature of the homotopy. We will continue to introduce notation and set conventions throughout the paper, on an as-needed basis and usually at the start of each section.

2. LIE DERIVATION SPACES

In this section, we consider a class of chain complexes over \mathbb{Q} obtained from generalized derivations of differential graded Lie algebras. Geometric content will be obtained when we apply these basic constructions to Quillen models in Section 4. We first establish our notational conventions for this material.

By *vector space*, we mean a graded vector space of finite type over the rationals. Furthermore, all our vector spaces will be *connected*, that is, they will be positively graded. The degree of an element $x \in V$ is written $|x|$. The space of degree n elements of V will be denoted V_n . A vector space generated by a single element v will be written $\mathbb{Q}v$. That is, if $|v| = n$, then $(\mathbb{Q}v)_i$ is isomorphic to \mathbb{Q} if $i = n$ and is zero otherwise. The k th *suspension* of V , denoted by $s^k(V)$, is the vector space defined as $(s^k(V))_n = V_{n-k}$. In particular, the *desuspension* of V denoted $s^{-1}V$ is the vector space with $(s^{-1}V)_n = V_{n+1}$. A typical example of desuspension that we consider is that of the reduced rational homology of a simply connected space of finite type, denoted by $s^{-1}\tilde{H}_*(X; \mathbb{Q})$. A vector space V equipped with a *differential*, that is, a linear map $d: V \rightarrow V$ of degree -1 that satisfies $d^2 = 0$, will be called a *DG (differential graded) vector space*. Given a cycle $x \in V_n$, a DG vector space, we write $\langle x \rangle$ for the homology class in $H_n(V, d)$ represented by x .

A *Lie algebra* L is a (graded) vector space L equipped with a bilinear, degree zero map (the “bracket” multiplication) $[\ , \]: L \times L \rightarrow L$ that satisfies

- (i) Anti-symmetry: $[\alpha, \beta] = -(-1)^{|\alpha||\beta|}[\beta, \alpha]$
- (ii) Jacobi identity: $[\alpha, [\beta, \gamma]] = [[\alpha, \beta], \gamma] + (-1)^{|\alpha||\beta|}[\beta, [\alpha, \gamma]]$.

The motivating example here is $\pi_*(\Omega X) \otimes \mathbb{Q}$, the rational homotopy of ΩX with the Samelson bracket $[\ , \]$.

A *DG Lie algebra* is a pair (L, d) where L is a Lie algebra and d is a vector space differential that satisfies the derivation law $d([x, y]) = [d(x), y] + (-1)^{|x|}[x, d(y)]$. A map of DG Lie algebras $\phi: (L, d) \rightarrow (L', d')$ is a homomorphism that respects differentials, that is, satisfies $\phi d = d' \phi$. Given a map of DG Lie algebras, we may pass to homology in the usual way. If the induced homomorphism $H(\phi)$ is an isomorphism, then we say ϕ is a *quasi-isomorphism*. We will frequently use the symbol “ \simeq ” to denote the fact that a map is a quasi-isomorphism, especially in diagrams. We write $\mathbb{L}(V)$ for the free Lie algebra generated by the vector space V . The coproduct (or “free product”) of (DG) Lie algebras L and L' is written $L \sqcup L'$. We usually abuse notation somewhat and write $\mathbb{L}(V, W)$ for the coproduct $\mathbb{L}(V) \sqcup \mathbb{L}(W) = \mathbb{L}(V \oplus W)$. Similarly, we will also write $\mathbb{L}(v)$ for $\mathbb{L}(\mathbb{Q}v)$, $\mathbb{L}(v, w)$ for $\mathbb{L}(\mathbb{Q}v, \mathbb{Q}w) = \mathbb{L}(\mathbb{Q}v \oplus \mathbb{Q}w) = \mathbb{L}(\mathbb{Q}v) \sqcup \mathbb{L}(\mathbb{Q}w)$, $\mathbb{L}(v, W)$ for $\mathbb{L}(\mathbb{Q}v, W) = \mathbb{L}(\mathbb{Q}v \oplus W) = \mathbb{L}(\mathbb{Q}v) \sqcup \mathbb{L}(W)$, and so-forth. We commit another abuse of notation by saying that a DG Lie algebra is free if the underlying Lie algebra is a free Lie algebra. To reduce parentheses, we usually write a free DG Lie algebra $(\mathbb{L}(V), d)$ as $\mathbb{L}(V; d)$. Finally, a DG Lie algebra (L, d) is *minimal* if L is free and the differential is decomposable, that is, $d(L) \subseteq [L, L]$.

We assume the reader is familiar with the basic facts of rational homotopy theory from the Quillen point of view, that is, using DG Lie algebra minimal models. Good references for this material include [Tan83] and [FHT01, Part IV]. Specifically, we recall that each space X has a *Quillen minimal model* which is a minimal DG Lie algebra (\mathcal{L}_X, d_X) whose isomorphism type is a complete invariant of the rational homotopy type of X . As a Lie algebra, we have $\mathcal{L}_X = \mathbb{L}(s^{-1}\tilde{H}_*(X; \mathbb{Q}))$.

The differential d_X is determined by the topology of X in a more arcane way. The Quillen model of X recovers the rational homotopy Lie algebra of X via a Lie algebra isomorphism $H_*(\mathcal{L}_X, d_X) \cong \pi_*(\Omega X) \otimes \mathbb{Q}$. Furthermore, a map of spaces $f: X \rightarrow Y$ induces a map of Quillen models $\mathcal{L}_f: \mathcal{L}_X \rightarrow \mathcal{L}_Y$ with $H(\mathcal{L}_f)$ corresponding, via the above isomorphism, to the rationalization of the map induced on homotopy groups by $\Omega f: \Omega X \rightarrow \Omega Y$. We refer to this map of Quillen models as the *Quillen minimal model* of the map f . Our basic reference for rational homotopy theory is [FHT01].

Let (L, d) be a DG Lie algebra. A *derivation of degree n* of L is a linear map $\theta: L \rightarrow L$ that raises degree by n and satisfies the rule

$$\theta([\alpha, \beta]) = [\theta(\alpha), \beta] + (-1)^{|\theta||\alpha|}[\alpha, \theta(\beta)].$$

When $n = 1$ we also require $d \circ \theta = -\theta \circ d$. We write $\text{Der}_n(L)$ for the space of derivations of degree n of L . The space $\text{Der}_*(L)$ has the structure of a DG Lie algebra, with (commutator) bracket $[\theta_1, \theta_2] = \theta_1 \circ \theta_2 - (-1)^{|\theta_1||\theta_2|}\theta_2 \circ \theta_1$, and differential defined as $D(\theta) = [d, \theta]$. The *adjoint map* $\text{ad}: L \rightarrow \text{Der}_*(L)$, given by $\text{ad}(x)(y) = [x, y]$, is now a map of DG Lie algebras.

At the expense of the Lie bracket, we can extend the notion of a derivation of a DG Lie algebra to a derivation with respect to a map of Lie algebras, as follows. Given a map $\psi: (L, d_L) \rightarrow (K, d_K)$ of DG Lie algebras, define a *derivation of degree n with respect to ψ* , or simply a *ψ -derivation of degree n* , to be a linear map $\theta: L \rightarrow K$ that increases degree by n and satisfies

$$\theta([\alpha, \beta]) = [\theta(\alpha), \psi(\beta)] + (-1)^{n|\alpha|}[\psi(\alpha), \theta(\beta)]$$

for $\alpha, \beta \in L$. Let $\text{Der}_n(L, K; \psi)$ denote the space of all ψ -derivations of degree n from L to K . Next, define $D: \text{Der}_n(L, K; \psi) \rightarrow \text{Der}_{n-1}(L, K; \psi)$ by $D(\theta) = d_K \circ \theta - (-1)^{|\theta|}\theta \circ d_L$. The pair $(\text{Der}_*(L, K; \psi), D)$ is then a DG vector space. The *adjoint map associated to ψ* is $\text{ad}_\psi: K \rightarrow \text{Der}_*(L, K; \psi)$ where

$$\text{ad}_\psi(\alpha)(\beta) = [\alpha, \psi(\beta)].$$

It is easy to check that ad_ψ is a map of DG vector spaces. We continue to write ad for the adjoint map associated to the identity $1: L \rightarrow L$.

We now begin to consider homology of derivation spaces. At this point, we introduce only what is needed for Section 4. Further notions concerning derivation spaces and their homology will be introduced as needed in subsequent sections. To ease notation in what follows, we adopt the following conventions. If (V, d) is a DG vector space we suppress the differential when we write its n th homology group, writing just $H_n(V)$. Furthermore, we indicate only the outermost degree. For example, $H_n(\text{Der}(L, K; \psi))$ denotes the n th homology group of the DG vector space $(\text{Der}_*(L, K; \psi), D)$.

Next, we construct a long exact homology sequence that we will show corresponds to the long exact rational homotopy sequence of the evaluation fibration (Theorem 4.4). Given a map $\phi: V \rightarrow W$ of DG vector spaces we will need the *relativization of ϕ* . This is the DG vector space $(\text{Rel}_*(\phi), \delta)$ given, in degree n , by

$$\text{Rel}_n(\phi) = W_n \oplus V_{n-1}$$

with differential of degree -1 defined as

$$\delta(w, v) = (\phi(v) - d_W(w), d_V(v)).$$

The inclusion $J : W_n \rightarrow \text{Rel}_n(\phi)$ with $J(w_n) = (w_n, 0)$, and the projection $P : \text{Rel}_n(\phi) \rightarrow V_{n-1}$ with $P(w_n, v_{n-1}) = v_{n-1}$, give a short exact sequence of chain complexes

$$0 \longrightarrow W_* \xrightarrow{J} \text{Rel}_*(\phi) \xrightarrow{P} V_{*-1} \longrightarrow 0 .$$

This leads to a long exact sequence whose connecting homomorphism is $H(\phi)$. Actually, J is really an ‘‘anti-chain’’ map, in that $\delta J = -Jd_W$. But J still induces a homomorphism $H(J)$ on homology, and gives rise to a long exact sequence in the usual way. Thus we have a long exact homology sequence

$$\cdots \rightarrow H_{n+1}(\text{Rel}(\phi)) \xrightarrow{H(P)} H_n(V) \xrightarrow{H(\phi)} H_n(W) \xrightarrow{H(J)} H_n(\text{Rel}(\phi)) \rightarrow \cdots .$$

We refer to this sequence as the *long exact homology sequence of ϕ* .

Now apply this construction to the adjoint map $\text{ad}_\psi : K \rightarrow \text{Der}_*(L, K; \psi)$ from above. We obtain a long exact homology sequence

$$\begin{array}{ccc} & \cdots \xrightarrow{H(J)} & H_{n+1}(\text{Rel}(\text{ad}_\psi)) \\ & \searrow^{H(P)} & \uparrow \\ H_n(K) & \xrightarrow{H(\text{ad}_\psi)} & H_n(\text{Der}(L, K; \psi)) \xrightarrow{H(J)} H_n(\text{Rel}(\text{ad}_\psi)) \\ & \searrow^{H(P)} & \uparrow \\ H_{n-1}(K) & \xrightarrow{H(\text{ad}_\psi)} & \cdots \end{array}$$

We call this sequence the *long exact derivation homology sequence of ψ* .

3. QUILLEN MODELS FOR CERTAIN PRODUCTS

The Sullivan minimal model of a product of spaces is easily expressed as the tensor product of the Sullivan minimal models of the spaces. For Quillen minimal models, the situation is more complicated due to the fact that the direct sum of minimal DG Lie algebras—which is the categorical product in this setting—is not minimal. Tanré has described how to construct the Quillen minimal model of a product of spaces $A \times X$ in terms of the Quillen minimal models of A and X (see [Tan83, Prop.VII.1(2)]). For our purposes, we need explicit minimal models for the cases in which $A = S^n$ and $A = S^n \vee S^n$. In this section, we describe a Quillen minimal model of $A \times X$ for the more general case in which A is any rational co- H -space. Our description essentially makes Tanré’s construction explicit for these cases, although our treatment here is self-contained.

Suppose X has Quillen minimal model $\mathbb{L}(W; d_X)$, and that $\mathbb{L}(V; d = 0)$ is the Quillen minimal model of a simply connected, finite-type rational co- H -space A . Suppose $\{v_i\}_{i \in J}$ is a (connected, finite type) basis for V , and $|v_i| = n_i - 1$ for each $i \in J$. Topologically, this corresponds to A being of the rational homotopy type of the wedge of spheres $\bigvee_{i \in J} S^{n_i}$. We construct a new minimal DG Lie algebra from these data as follows: For each $i \in J$, let W_i denote the n_i -fold suspension of W , that is, set $W_i = s^{n_i}(W)$. Let $\lambda : \mathbb{L}(W) \rightarrow \mathbb{L}(W, V, \bigoplus_{i \in J} W_i)$ denote the inclusion of graded Lie algebras. For each $i \in J$, let $S_i : W \rightarrow W_i$ denote the n_i -fold suspension isomorphism and define a λ -derivation $S_i : \mathbb{L}(W) \rightarrow \mathbb{L}(W, V, \bigoplus_{i \in J} W_i)$ by extending

S_i using the λ -derivation rule. Now define a differential ∂ on $\mathbb{L}(W, V, \oplus_{i \in J} W_i)$ that extends the differentials on $\mathbb{L}(W)$ and $\mathbb{L}(V)$, by setting

$$\partial(S_i(w)) = [v_i, w] + (-1)^n S_i(d_X(w))$$

for each generator $w \in W$ (and thus each generator of W_i). This definition is equivalent to the identity $\text{ad}(v_i) = D(S_i)$ in $\text{Der}_{n_i-1}(\mathbb{L}(W), \mathbb{L}(W, V, \oplus_{i \in J} W_i); \lambda)$.

We will show that $\mathbb{L}(W, V, \oplus_{i \in J} W_i; \partial)$ is the Quillen minimal model of $A \times X$. First we check that the preceding formula defines a differential.

Lemma 3.1. *The derivation ∂ of $\mathbb{L}(W, V, \oplus_{i \in J} W_i)$ satisfies $\partial \circ \partial = 0$.*

Proof. It is sufficient to check on generators. Further, since ∂ extends the differentials on $\mathbb{L}(W)$ and $\mathbb{L}(V)$, it is sufficient to check that $(\partial)^2(S_i(w)) = 0$ for each $w \in W$ and $i \in J$. We compute as follows:

$$\begin{aligned} (\partial)^2(S_i(w)) &= \partial([v_i, w] + (-1)^{n_i} S_i(d_X(w))) \\ &= (-1)^{n_i-1} [v, d_X(w)] + (-1)^{n_i} \partial S_i(d_X(w)) \\ &= (-1)^{n_i-1} \text{ad}(v_i)(d_X(w)) + (-1)^{n_i} \partial S_i(d_X(w)) \\ &= S_i d_X(d_X(w)) = 0. \end{aligned}$$

The penultimate step follows from the identity $\text{ad}(v_i) = D(S_i)$ observed above, which expands to give $[v_i, \chi] = \partial S_i(\chi) - (-1)^{n_i} S_i d_X(\chi)$ for any $\chi \in \mathbb{L}(W)$, and which here is applied to $d_X(w) \in \mathbb{L}(W)$. \square

We will need the following technical point in our argument. The discussion here is lifted from [FHT01, Sec.22(f)]. Suppose that $\mathbb{L}(V; d)$ is a free, but not necessarily minimal, connected DG Lie algebra. That is, suppose that the differential d may have a non-trivial linear part. Since $\mathbb{L}(V)$ is free, we can write d as the sum of two derivations $d = d_0 + d_+$, where $d_0: V \rightarrow V$ is the linear part of the differential d and d_+ is the decomposable part of d that increases bracket length. Indeed, d_0 itself is a differential—although d_+ is generally not—and so we obtain a DG vector space (V, d_0) . Next, suppose $\phi: \mathbb{L}(V; d') \rightarrow \mathbb{L}(W; d)$ is a morphism of free, but not necessarily minimal, connected DG Lie algebras. Again, because the Lie algebras are free, we may write the linear map $\phi: V \rightarrow \mathbb{L}(W)$ as a sum $\phi = \phi_0 + \phi_+$, where $\phi_0: V \rightarrow W$ is the linear part of ϕ and ϕ_+ is the decomposable part of ϕ that increases bracket length. In this way, we obtain a morphism of DG vector spaces

$$\phi_0: (V, (d')_0) \rightarrow (W, d_0),$$

which we refer to as the *linearization* of ϕ . The following result can be interpreted as a version of Whitehead's theorem in our context.

Lemma 3.2. *Let $\phi: \mathbb{L}(V; d') \rightarrow \mathbb{L}(W; d)$ be a morphism of connected free DG Lie algebras. Let $\phi_0: (V, (d')_0) \rightarrow (W, d_0)$ be the linearization of ϕ . Then ϕ is a quasi-isomorphism of DG Lie algebras if, and only if, ϕ_0 is a quasi-isomorphism of DG vector spaces.*

Proof. This is proved as [FHT01, Prop.22.12]. \square

We now come to the main point of the section.

Theorem 3.3. *Let X be a simply connected space of finite type with Quillen minimal model $\mathbb{L}(W; d_X)$. Let A be a rational co- H -space of the rational homotopy type*

of the wedge of spheres $\bigvee_{i \in J} S^{n_i}$. Then $\mathbb{L}(W, V, \bigoplus_{i \in J} W_i; \partial)$, as described above, is the Quillen minimal model of $A \times X$.

Proof. Our starting point is the well-known fact that the direct sum of Quillen minimal models gives a (non-minimal) DG Lie algebra model for the product [FHT01, p.332, Ex.3]. In our case, this gives $\mathbb{L}(V) \oplus \mathbb{L}(W; d_X)$ as a non-minimal model for $A \times X$. We will show that the obvious projection

$$p: \mathbb{L}(W, V, \bigoplus_{i \in J} W_i; \partial) \rightarrow \mathbb{L}(V) \oplus \mathbb{L}(W; \partial)$$

is a quasi-isomorphism. Since the domain is a minimal DG Lie algebra, this is sufficient to show that it is the Quillen minimal model of the product.

So consider the following commutative diagram of DG Lie algebra morphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{i} & \mathbb{L}(W, V, \bigoplus_{i \in J} W_i; \partial) & \xrightarrow{q} & \mathbb{L}(V) & \longrightarrow & 0 \\ & & \downarrow p' & & \downarrow p & & \downarrow 1 & & \\ 0 & \longrightarrow & \mathbb{L}(W; d_X) & \xrightarrow{i'} & \mathbb{L}(V) \oplus \mathbb{L}(W; d_X) & \xrightarrow{q'} & \mathbb{L}(V) & \longrightarrow & 0 \end{array}$$

Here, q and q' are the obvious (quotient) projections onto $\mathbb{L}(V)$, and i and i' are the inclusions of the kernels, so that the rows are short exact sequences of DG Lie algebras. We will argue that $p': K \rightarrow \mathbb{L}(W; d_X)$ is a quasi-isomorphism. First note that, as a sub-DG Lie algebra of a connected, free DG Lie algebra, K is itself a connected, free DG Lie algebra. Indeed, as a Lie algebra, we may write

$$K = \mathbb{L}(W, \bigoplus_i W_i, [V, W], \bigoplus_i [V, W_i], [V, [V, W]], \bigoplus_i [V, [V, W_i]], \dots; \partial_K)$$

or more succinctly $K = \mathbb{L}(\{\text{ad}^j(V)(W)\}_{j \geq 0}, \{\bigoplus_i \text{ad}^j(V)(W_i)\}_{j \geq 0}; \partial_K)$. In these expressions, $[V, W]$ denotes the vector space spanned by brackets $[v, w]$ with $v \in V$ and $w \in W$, and so-forth, and $\text{ad}^0(V)(W)$ denotes W , $\text{ad}^1(V)(W)$ denotes $[V, W]$, $\text{ad}^2(V)(W)$ denotes $[V, [V, W]]$, and so-forth. We now claim that $(\partial_K)_0$, the linear part of the differential in K , induces isomorphisms

$$(\partial_K)_0: \bigoplus_{i \in J} \text{ad}^j(V)(W_i) \rightarrow \text{ad}^{j+1}(V)(W)$$

for each $j \geq 0$. First recall that ∂_K is simply the restriction of the differential ∂ to the kernel of q , and that $\partial(V) = 0$. Extending our notation a little further, we can denote a typical spanning element of $\text{ad}^j(V)(W)$ as follows. Suppose $(v_{r_1}, v_{r_2}, \dots, v_{r_j}) \in V^j$ is a j -tuple. Then write $\text{ad}(v_{r_1}, v_{r_2}, \dots, v_{r_j})(w)$ for $[v_{r_1}, [v_{r_2}, [\dots, [v_{r_{j-1}}, v_{r_j}] \dots]](w)$. Likewise for elements of $\text{ad}^j(V)(W_i)$. Now let $w \in W$ be a typical element. From the definition of ∂ above, we have

$$\begin{aligned} \partial(\text{ad}(v_{r_1}, v_{r_2}, \dots, v_{r_j})(S_i(w))) &= \pm \text{ad}(v_{r_1}, v_{r_2}, \dots, v_{r_j})(\partial(S_i(w))) \\ &= \pm \text{ad}(v_{r_1}, v_{r_2}, \dots, v_{r_j}, v_i)(w) \\ &\quad \pm \text{ad}(v_{r_1}, v_{r_2}, \dots, v_{r_j})(S_i d_X(w)). \end{aligned}$$

Since $\mathbb{L}(W; d_X)$ is the Quillen minimal model of X , $d_X(w)$ is decomposable in $\mathbb{L}(W)$ and thus $S_i d_X(w)$ is decomposable in $\mathbb{L}(W, W_i)$. From this, it follows that the last term displayed above, namely $\text{ad}(v_{r_1}, v_{r_2}, \dots, v_{r_j})(S_i d_X(w))$, is decomposable in K . We prove this assertion in Lemma 3.4 below. Assuming for the time being its validity, it follows that the linear part of the differential in K induces isomorphisms $(\partial_K)_0: \text{ad}^j(V)(W_i) \cong \text{ad}^j(V) \text{ad}(v_i)(W)$ for each $i \in J$ and each $j \geq 0$, and hence isomorphisms $(\partial_K)_0: \bigoplus_i \text{ad}^j(V)(W_i) \cong \text{ad}^{j+1}(V)(W)$ for each $j \geq 0$, as claimed. Notice that as a consequence of this, we must have $(\partial_K)_0 = 0$ on each vector space

of generators $\mathrm{ad}^{j+1}(V)(W)$ in K , for $j \geq 0$, since the linear part of a differential is itself a differential. In any case, this latter fact also follows from Lemma 3.4. Finally, notice that $(\partial_K)_0 = 0$ on the vector space of generators W in K , since $\partial = d_X$ is decomposable on W . It now follows that the DG vector space $(Q(K), (\partial_K)_0)$ obtained by linearizing K may be written as a direct sum

$$(Q(K), (\partial_K)_0) \cong (W, (\partial_K)_0 = 0) \oplus_{j \geq 0} \left((\oplus_i \mathrm{ad}^j(V)(W_i)) \oplus (\mathrm{ad}^{j+1}(V)(W)), (\partial_K)_0 \right),$$

in which each summand $\left((\oplus_i \mathrm{ad}^j(V)(W_i)) \oplus (\mathrm{ad}^{j+1}(V)(W)), (\partial_K)_0 \right)$ is an acyclic DG vector space. It is now evident that $H(Q(K), (\partial_K)_0) \cong W$ and that the linearization of p' , that is, $(p')_0: (Q(K), (\partial_K)_0) \rightarrow (W, \partial_0 = 0)$, is a quasi-isomorphism of DG vector spaces. It follows from Lemma 3.2 that p' is a quasi-isomorphism of DG Lie algebras.

Returning to the diagram of short exact sequences, we now have left and right vertical arrows that are quasi-isomorphisms. Therefore, by passing to the induced diagram of long exact homology sequences and applying the five-lemma, we obtain that p is a quasi-isomorphism. Hence $\mathbb{L}(W, V, \oplus_{i \in J} W_i; \partial)$ is the Quillen minimal model of $A \times X$. \square

The proof of Theorem 3.3 will be completed when we establish the following lemma:

Lemma 3.4. *With notation as in the proof above, suppose χ is a decomposable element in K . Then $[v, \chi]$ is also decomposable in K , for any $v \in V$. In particular, if χ is decomposable in $\mathbb{L}(W, W_i)$ for some $i \in J$, and hence decomposable in K , then $\mathrm{ad}(v_{r_1}, v_{r_2}, \dots, v_{r_j})(\chi)$ is decomposable in K for any j -tuple $(v_{r_1}, v_{r_2}, \dots, v_{r_j}) \in V^j$.*

Proof. Recall that $K = \mathbb{L}(\{\mathrm{ad}^j(V)(W)\}_{j \geq 0}, \oplus_i \{\mathrm{ad}^j(V)(W_i)\}_{j \geq 0})$ is a sub-Lie algebra of $\mathbb{L}(W, V, \oplus_i W_i)$, and observe that $v \in V$ is not a generator—is not even an element—of K , so the statement is not entirely trivial. Without loss of generality, we may assume that χ is a monomial term. We argue by induction on the bracket length in K of χ . When χ has length 2 in K , we have $\chi = [\chi_1, \chi_2]$ with χ_1 and χ_2 indecomposable monomials in K . From the Jacobi identity in $\mathbb{L}(W, V, \oplus_i W_i)$, we may write $[v, \chi] = \pm[\chi_1, [v, \chi_2]] \pm [\chi_2, [v, \chi_1]]$. Now χ_1 is an element from either $\mathrm{ad}^j(V)(W)$ or $\mathrm{ad}^j(V)(W_i)$ for some $i \in J$ and $j \geq 0$, and likewise for χ_2 . So $[v, \chi_1]$ is from $\mathrm{ad}^{j+1}(V)(W) \subseteq K$ or $\mathrm{ad}^{j+1}(V)(W_i) \subseteq K$, and likewise for $[v, \chi_2]$. That is, if χ_1 and χ_2 are indecomposable in K , then $[v, \chi] = \pm[\chi_1, [v, \chi_2]] \pm [\chi_2, [v, \chi_1]]$ displays $[v, \chi]$ as decomposable (and again of length 2) in K . Now assume inductively that the assertion is true for χ of bracket length $\leq r$ in K . Let χ be a monomial of bracket length $r + 1$ in K . By judicious use of the Jacobi identity in K , we may assume that $\chi = [\chi_1, \chi_2]$ with χ_1 an indecomposable in K and χ_2 a decomposable monomial of bracket length r in K . Once again, the Jacobi identity in $\mathbb{L}(W, V, \oplus_i W_i)$ yields $[v, \chi] = \pm[\chi_1, [v, \chi_2]] \pm [\chi_2, [v, \chi_1]]$. Our inductive hypothesis implies that $[v, \chi_2]$ is (decomposable) in K , and the same observations as were used to start the induction show that $[v, \chi_1]$ is an (indecomposable) element in K . Hence $[v, \chi]$ is decomposable in K , and the induction is complete. The result follows. \square

Since it is the main case we require here, we write out explicitly what this gives for the model of $S^n \times X$, with a slight easing of notation.

Corollary 3.5. *Suppose X has Quillen minimal model $\mathbb{L}(W; d_X)$, let $\mathbb{L}(v)$ with $|v| = n - 1$ and zero differential be the Quillen model of S^n , and set $W' = s^n(W)$. Let $\lambda: \mathbb{L}(W) \rightarrow \mathbb{L}(W, v, W')$ be the inclusion, and $S: \mathbb{L}(W) \rightarrow \mathbb{L}(W, v, W')$ be the λ -derivation that extends the linear map $S(w) = w'$. Define a differential ∂ on $\mathbb{L}(W, v, W')$ by $\partial(w) = d_X(w)$, $\partial(v) = 0$, and*

$$\partial(w') = [v, w] + (-1)^n S(d_X(w)),$$

for each $w \in W$. Then $\mathbb{L}(W, v, W'; \partial)$ is the Quillen minimal model of $S^n \times X$.

4. LIE DERIVATIONS AND HOMOTOPY GROUPS OF FUNCTION SPACES

Say two maps of vector spaces $f: U \rightarrow V$ and $g: U' \rightarrow V'$ are *equivalent* if there exist isomorphisms α and β which make the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \alpha \downarrow \cong & & \beta \downarrow \cong \\ U' & \xrightarrow{g} & V' \end{array}$$

commutative. We will extend this notion of equivalence in the obvious way to exact sequences of vector spaces, and any other diagram of vector space maps. Given any map $f: X \rightarrow Y$, we have the homomorphism

$$j_{\#} \otimes 1: \pi_n(\text{map}_*(X, Y; f)) \otimes \mathbb{Q} \rightarrow \pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q}$$

induced on rational homotopy groups by the fibre inclusion of the general evaluation fibration $\text{map}_*(X, Y; f) \xrightarrow{j} \text{map}(X, Y; f) \xrightarrow{\omega} Y$. On the other hand, we have the homomorphism

$$H(J): H_n(\text{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)) \rightarrow H_n(\text{Rel}(\text{ad}_{\mathcal{L}_f}))$$

that forms part of the long exact homology sequence of the adjoint map $\text{ad}_{\mathcal{L}_f}: \mathcal{L}_Y \rightarrow \text{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$ of the Quillen minimal model $\mathcal{L}_f: \mathcal{L}_X \rightarrow \mathcal{L}_Y$ of f . In Theorem 4.1, which is the basic result of this section, we establish that these two homomorphisms are equivalent. This result and one immediate consequence will occupy the remainder of this section.

The main step is to establish vector space isomorphisms

$$\Phi: \pi_n(\text{map}_*(X, Y; f)) \otimes \mathbb{Q} \rightarrow H_n(\text{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f))$$

and

$$\Psi: \pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q} \rightarrow H_n(\text{Rel}(\text{ad}_{\mathcal{L}_f})),$$

that give the equivalence. In the following, we assume a fixed choice of Quillen minimal model $\mathcal{L}_f: \mathcal{L}_X \rightarrow \mathcal{L}_Y$ for $f: X \rightarrow Y$. Write $\mathcal{L}_X = \mathbb{L}(W; d_X)$ and $\mathcal{L}_Y = \mathbb{L}(V; d_Y)$. To this end, we define group homomorphisms

$$\Phi': \pi_n(\text{map}_*(X, Y; f)) \rightarrow H_n(\text{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f))$$

and

$$\Psi': \pi_n(\text{map}(X, Y; f)) \rightarrow H_n(\text{Rel}(\text{ad}_{\mathcal{L}_f}))$$

from the ordinary homotopy groups to the appropriate vector spaces, for $n \geq 2$. Then the isomorphisms Φ and Ψ are obtained as the rationalizations of these homomorphisms.

Define Φ' as follows. Let $\alpha \in \pi_n(\text{map}_*(X, Y; f))$ be represented by a map $a: S^n \rightarrow \text{map}_*(X, Y; f)$. Then the adjoint $A: S^n \times X \rightarrow Y$ of a has Quillen minimal model $\mathcal{L}_A: \mathcal{L}_{S^n \times X} \rightarrow \mathcal{L}_Y$. Recall from Corollary 3.5 that $\mathcal{L}_{S^n \times X} = \mathbb{L}(W, v, W'; \partial)$. Now, since a is a (based) map into the function space of *based* maps, we have $A \circ i_1 = *: S^n \rightarrow Y$ and $A \circ i_2 = f: X \rightarrow Y$. It follows that we may take the Quillen minimal model of A to be a DG Lie algebra map

$$\mathcal{L}_A: \mathbb{L}(W, v, W'; \partial) \rightarrow \mathcal{L}_Y$$

that satisfies $\mathcal{L}_A(v) = 0$ and $\mathcal{L}_A(w) = \mathcal{L}_f(w)$ for each $w \in W$. See the appendix for justification of this last assertion. Now define a linear map $\theta_A: \mathbb{L}(W) \rightarrow \mathcal{L}_Y$ that increases degree by n as the composition

$$\mathbb{L}(W) \xrightarrow{S} \mathbb{L}(W, v, W') \xrightarrow{\mathcal{L}_A} \mathcal{L}_Y,$$

where $S: \mathbb{L}(W) \rightarrow \mathbb{L}(W, v, W')$ is the derivation from Corollary 3.5. A straightforward check shows that θ_A is an \mathcal{L}_f -derivation in $\text{Der}_n(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$. Furthermore, we have $d_Y \theta_A = (-1)^n \theta_A d_X$ and so θ_A is a cycle in the derivation space. Finally, we set $\Phi'(\alpha) = \langle \theta_A \rangle \in H_n(\text{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f))$. We will establish that Φ' is a well-defined homomorphism, and that its rationalization Φ is an isomorphism, in Theorem 4.1 below.

We define Ψ' , and thus its rationalization Ψ , in a similar manner. Let $\alpha \in \pi_n(\text{map}(X, Y; f))$ be represented by a map $a: S^n \rightarrow \text{map}(X, Y; f)$. Then the adjoint $A: S^n \times X \rightarrow Y$ of a still satisfies $A \circ i_2 = f: X \rightarrow Y$, but the composition $A \circ i_1: S^n \rightarrow Y$ may give a non-trivial element of $\pi_n(Y)$. Correspondingly, the Quillen minimal model of A satisfies $\mathcal{L}_A(w) = \mathcal{L}_f(w)$ for each $w \in W$, but $\mathcal{L}_A(v) \in \mathcal{L}_Y$ is now some non-trivial d_Y -cycle. As before, setting $\theta_A = \mathcal{L}_A \circ S: \mathbb{L}(W) \rightarrow \mathcal{L}_Y$ defines an \mathcal{L}_f -derivation in $\text{Der}_n(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$. Recalling the definition of $\text{Rel}_*(\text{ad}_{\mathcal{L}_f})$ from Section 2, we obtain an element

$$(\theta_A, \mathcal{L}_A(v)) \in \text{Rel}_n(\text{ad}_{\mathcal{L}_f}).$$

Let δ and D denote the differentials in $\text{Rel}_*(\text{ad}_{\mathcal{L}_f})$ and $\text{Der}_n(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$ respectively. Then we have $\delta(\theta_A, \mathcal{L}_A(v)) = (\text{ad}_{\mathcal{L}_f}(\mathcal{L}_A(v)) - D(\theta_A), d_Y \mathcal{L}_A(v))$, and $d_Y \mathcal{L}_A(v) = 0$. We check that $\text{ad}_{\mathcal{L}_f}(\mathcal{L}_A(v)) - D(\theta_A) = 0 \in \text{Der}_{n-1}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$. For $\chi \in \mathcal{L}_X$, we have

$$\begin{aligned} & \text{ad}_{\mathcal{L}_f}(\mathcal{L}_A(v))(\chi) - D(\theta_A)(\chi) \\ &= [\mathcal{L}_A(v), \mathcal{L}_f(\chi)] - d_Y(\mathcal{L}_A \circ S(\chi)) + (-1)^n \mathcal{L}_A \circ S d_X(\chi) \\ &= \mathcal{L}_A([v, \chi] + (-1)^n S d_X(\chi)) - d_Y \mathcal{L}_A(S(\chi)) \\ &= \mathcal{L}_A \partial(S(\chi)) - d_Y \mathcal{L}_A(S(\chi)) \\ &= 0. \end{aligned}$$

Thus $(\theta_A, \mathcal{L}_A(v))$ is a cycle in the relative chain complex. Finally, we set $\Psi'(\alpha) = \langle \theta_A, \mathcal{L}_A(v) \rangle \in H_n(\text{Rel}(\text{ad}_{\mathcal{L}_f}))$.

In the following result, we establish the basic properties of Φ and Ψ .

Theorem 4.1. *Suppose $n \geq 2$. Then we have:*

- (A) Φ' and Ψ' are well-defined homomorphisms;
- (B) Their rationalizations Φ and Ψ are isomorphisms;

(C) *The following square is commutative:*

$$\begin{array}{ccc}
\pi_n(\mathrm{map}_*(X, Y; f)) \otimes \mathbb{Q} & \xrightarrow[\cong]{\Phi} & H_n(\mathrm{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)) \\
j\# \otimes \mathbb{Q} \downarrow & & H(J) \downarrow \\
\pi_n(\mathrm{map}(X, Y; f)) \otimes \mathbb{Q} & \xrightarrow[\Psi]{\cong} & H_n(\mathrm{Rel}(\mathrm{ad}_{\mathcal{L}_f}))
\end{array}$$

Proof. Throughout the proof we will give full details for arguments concerning Φ . The arguments for Ψ are similiar, and we will simply indicate them without details. We will need to use some facts about homotopy in the DG Lie algebra setting. The most complete reference for this material is [Tan83, Ch.II.5]. The appendix to this paper contains a quick overview, and also provides careful justifications of some technical details used in the following proof. We will make free use of the notation concerning DG Lie algebra homotopy reviewed in the appendix.

(A) Φ' *is well-defined:* Suppose that $a \sim b: S^n \rightarrow \mathrm{map}_*(X, Y; f)$ are two representatives of the homotopy class α . The adjoint of the (based) homotopy in $\mathrm{map}_*(X, Y; f)$ from a to b gives a homotopy of their adjoints, $A, B: S^n \times X \rightarrow Y$, that is stationary on the subset $S^n \vee X \subseteq S^n \times X$. Indeed, the homotopy of the adjoints is stationary at the constant map on S^n and is stationary at f on X . Consequently, the corresponding Quillen minimal models $\mathcal{L}_A, \mathcal{L}_B: \mathcal{L}_{S^n \times X} \rightarrow \mathcal{L}_Y$ are homotopic via a DG Lie algebra homotopy

$$\mathcal{H}: \mathbb{L}(W, v, W')_I \rightarrow \mathcal{L}_Y$$

that satisfies $\mathcal{H}(\mathbb{L}(v)_I) = 0$, $\mathcal{H}(\mathbb{L}(sW, \widehat{W})) = 0$, and $\mathcal{H}(w) = \mathcal{L}_f(w)$ for each $w \in W$ (see Lemma A.2 for details). Let $\sigma: \mathbb{L}(W, v, W')_I \rightarrow \mathbb{L}(W, v, W')_I$ be the derivation of degree 1 defined in the appendix. Define a linear map $\Theta: \mathbb{L}(W) \rightarrow \mathcal{L}_Y$ of degree $n + 1$ as the composition $\Theta = \mathcal{H} \circ \sigma \circ S$. A straightforward check, using the fact that $\mathcal{H}(sW, \widehat{W}) = 0$, shows that Θ is an \mathcal{L}_f -derivation. We will show that $D\Theta = \theta_B - \theta_A \in \mathrm{Der}_n(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$. We have $\theta_B = \mathcal{H} \circ \lambda_1 \circ S$ and $\theta_A = \mathcal{H} \circ \lambda_0 \circ S = \mathcal{H} \circ S$. Let J denote the ideal of $\mathbb{L}(W, v, W')_I$ generated by $sW \oplus \widehat{W}$. Since σ vanishes on the generators of J , and D_I preserves the set of generators, J is stable under the composition σD_I . Furthermore, \mathcal{H} is zero on J , since it is zero on the generators. We claim that $(\sigma D_I)^r: W' \rightarrow \mathbb{L}(W, v, W')_I$ has image in J , for each $r \geq 2$. First observe that

$$\begin{aligned}
\sigma D_I(w') &= \sigma \partial(w') = \sigma([v, w] + (-1)^n S d_X(w)) \\
&= [sv, w] + (-1)^n [v, sw] + (-1)^n \sigma S d_X(w).
\end{aligned}$$

Furthermore, the only terms not in J that may appear in $\sigma S d_X(w)$ are terms in the sub-Lie algebra $\mathbb{L}(W, sW')$ that have exactly one occurrence of an element from sW' . On applying D_I to such terms, the only terms still not in J that may appear in $D_I \sigma S d_X(w)$ are terms in the sub-Lie algebra $\mathbb{L}(W, sW', \widehat{W}')$ that have exactly one occurrence of an element either from sW' , or from \widehat{W}' . Since σ is zero on $sW' \oplus \widehat{W}'$, $\sigma D_I \sigma S d_X(w) \in J$. Direct computation shows that both $\sigma D_I([sv, w])$ and $\sigma D_I([v, sw])$ are in J . That is, $(\sigma D_I)^r(w') \in J$, for each $r \geq 2$. Consequently,

\mathcal{H} vanishes on these terms. Finally, using this, we compute that

$$\begin{aligned}
 \theta_B(w) &= \mathcal{L}_B \circ S(w) = \mathcal{H} \circ \lambda_1(w') \\
 &= \mathcal{H}(w' + D_I \sigma(w') + \sigma D_I(w')) + \sum_{r \geq 2} \frac{1}{r!} (\sigma D_I)^r(w') \\
 &= \mathcal{H}(w') + \mathcal{H} D_I \sigma(w') + \mathcal{H}([sv, w] + (-1)^n [v, sw] + (-1)^n \sigma S d_X(w)) \\
 &= \mathcal{L}_A(w') + d_Y \mathcal{H} \sigma(w') + (-1)^n \mathcal{H} \sigma S d_X(w) \\
 &= \theta_A(w) + d_Y \Theta(w) - (-1)^{n+1} \Theta d_X(w).
 \end{aligned}$$

It follows that the difference of derivations $\theta_B - \theta_A = D\Theta$ in $\text{Der}_n(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$. Hence Φ' is well-defined.

Φ' is a homomorphism: Let $\nu: S^n \rightarrow S^n \vee S^n$ denote the usual pinching co-multiplication. Given $\alpha, \beta \in \pi_n(\text{map}_*(X, Y; f))$, the sum $\alpha + \beta$ is the composition $(\alpha | \beta) \circ \nu$. Suppose α, β have adjoints $A, B: S^n \times X \rightarrow Y$, respectively. Let $i_1, i_2: S^n \rightarrow S^n \vee S^n$ denote the inclusions, and let $(A | B)_f: (S^n \vee S^n) \times X \rightarrow Y$ be the map defined by $(A | B)_f \circ (i_1 \times 1) = A$ and $(A | B)_f \circ (i_2 \times 1) = B$. Then the adjoint of $\alpha + \beta$ is $C := (A | B)_f \circ (\nu \times 1): S^n \times X \rightarrow Y$. We focus on identifying the Quillen minimal model of $(A | B)_f$, and it will follow that Φ'_f is a homomorphism.

The map $(A | B)_f: (S^n \vee S^n) \times X \rightarrow Y$ is determined, up to homotopy, as the unique map F that makes the following diagram homotopy commutative:

$$\begin{array}{ccc}
 S^n \times X & & \\
 \downarrow i_1 \times 1 & \searrow A & \\
 (S^n \vee S^n) \times X & \xrightarrow{\quad F \quad} & Y \\
 \uparrow i_2 \times 1 & \nearrow B & \\
 S^n \times X & &
 \end{array}$$

Consequently, any DG Lie algebra map Γ that makes the diagram

$$\begin{array}{ccc}
 \mathbb{L}(W, v, W'; \partial) & & \\
 \downarrow j_1 & \searrow \mathcal{L}_A & \\
 \mathbb{L}(W, v_1, v_2, W_1, W_2; \partial) & \xrightarrow{\quad \Gamma \quad} & \mathcal{L}_Y \\
 \uparrow j_2 & \nearrow \mathcal{L}_B & \\
 \mathbb{L}(W, v, W'; \partial) & &
 \end{array}$$

commute up to DG homotopy is a Quillen model for $(A | B)_f$. In this diagram, $\mathbb{L}(W, v_1, v_2, W_1, W_2; \partial)$ is the Quillen model of $(S^n \vee S^n) \times X$ as described in Theorem 3.3, and j_1, j_2 the obvious inclusions that identify v with v_1 and W' with W_1 , and v with v_2 and W' with W_2 respectively. This characterization of (the Quillen model of) $(A | B)_f$ is explained in detail in [LS03]. Now there is an obvious choice for $\Gamma: \mathbb{L}(W, v_1, v_2, W_1, W_2; \partial) \rightarrow \mathcal{L}_Y$, namely the map that makes the diagram commute. Hence, this map is a Quillen model for $(A | B)_f$. Finally, since $\nu: S^n \rightarrow S^n \vee S^n$ has Quillen model $\mathcal{L}_\nu: \mathbb{L}(v) \rightarrow \mathbb{L}(v_1, v_2)$ given by $\mathcal{L}_\nu(v) = v_1 + v_2$, it follows that $\nu \times 1: S^n \times X \rightarrow (S^n \vee S^n) \times X$ has Quillen model

$\mathcal{L}_{\nu \times 1}: \mathbb{L}(W, v, W'; \partial) \rightarrow \mathbb{L}(W, v_1, v_2, W_1, W_2; \partial)$ given by $\mathcal{L}_{\nu \times 1}(w') = w_1 + w_2$. Thus

$$\mathcal{L}_C(w') = \Gamma \circ \mathcal{L}_{\nu \times 1}(w') = \Gamma(w_1 + w_2) = \theta_A(w) + \theta_B(w).$$

Hence we have $\theta_C = \mathcal{L}_C \circ S = \theta_A + \theta_B$ and it follows that Φ' is a homomorphism.

Ψ' is a well-defined homomorphism: This is established by making small adjustments to the preceding arguments for Φ' . In this case, the homotopy of the adjoints A and B is stationary at f on X , but is not stationary on S^n . The corresponding homotopy \mathcal{H} of Quillen minimal models still satisfies $\mathcal{H}(\mathbb{L}(sW, \widehat{W})) = 0$, and $\mathcal{H}(w) = \mathcal{L}_f(w)$ for each $w \in W$ (see Lemma A.2 for details), but we must allow for non-zero terms in $\mathcal{H}(\mathbb{L}(v)_I)$. Since v is a ∂ -cycle, however, we have $\lambda_1(v) = v + \widehat{v} \in \mathbb{L}(W, v, W')_I$. Therefore, $\mathcal{L}_B(v) = \mathcal{H} \circ \lambda_1(v) = \mathcal{H}(v) + \mathcal{H}D\sigma(v) = \mathcal{L}_A(v) + d_Y(\mathcal{H}(sv))$. Since \mathcal{H} still vanishes on $sW \oplus \widehat{W}$, the composition $\Theta = \mathcal{H} \circ \sigma \circ S$ still defines an \mathcal{L}_f -derivation. Computing exactly as before, we find that $\theta_B(w) = \theta_A(w) + D\Theta(w) + [\mathcal{H}(sv), \mathcal{L}_f(w)]$. It follows that, in the relative complex, we have $\delta(-\Theta, \mathcal{H}(sv)) = (\theta_B - \theta_A, \mathcal{L}_B(v) - \mathcal{L}_A(v))$. So Ψ' is well-defined. To show that Ψ' is a homomorphism, we can use precisely the same argument as for Φ' , and use the additional identity

$$\mathcal{L}_C(v) = \Gamma \circ \mathcal{L}_{\nu \times 1}(v) = \Gamma(v_1 + v_2) = \mathcal{L}_A(v) + \mathcal{L}_B(v)$$

at the final step.

(B) *Φ is a surjection:* Suppose given $\theta \in \text{Der}_n(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$, a cycle derivation of degree n . Define a Lie algebra map $\mathcal{L}_A: \mathbb{L}(W, v, W'; \partial) \rightarrow \mathcal{L}_Y$ by setting

$$\mathcal{L}_A(w) = \mathcal{L}_f(w), \quad \mathcal{L}_A(v) = 0 \quad \text{and} \quad \mathcal{L}_A(w') = \theta(w)$$

for $w \in W$. Just as in the definition of Φ' , $\mathcal{L}_A \circ S$ is an \mathcal{L}_f -derivation and by construction we have $\mathcal{L}_A \circ S = \theta$. We check that \mathcal{L}_A commutes with differentials as follows:

$$\begin{aligned} \mathcal{L}_A(\partial(w')) &= \mathcal{L}_A([v, w]) + (-1)^n S(d_X(w)) \\ &= 0 + (-1)^n \mathcal{L}_A(S(d_X(w))) \\ &= (-1)^n \theta(d_X(w)) \\ &= d_Y(\theta(w)) \\ &= d_Y(\mathcal{L}_A(w')) \end{aligned}$$

Let $A: S^n \times X \rightarrow Y_{\mathbb{Q}}$ be the geometric realization of \mathcal{L}_A , from the correspondence between (homotopy classes of) maps between rational spaces and DG Lie algebra maps between Quillen models. Let $i_1: S^n \rightarrow S^n \times X$ and $i_2: X \rightarrow S^n \times X$ denote the inclusions. Since $\mathcal{L}_A \circ \mathcal{L}_{i_1} = 0$ and $\mathcal{L}_A \circ \mathcal{L}_{i_2} = \mathcal{L}_f$, we have $A \circ i_1 \sim *$ and $A \circ i_2 \sim f_{\mathbb{Q}}$. Altering the geometric realization A up to homotopy, we may assume $A \circ i_1 = *$ and $A \circ i_2 = f_{\mathbb{Q}}$. Thus, the adjoint $a: S^n \rightarrow \text{map}_*(X, Y_{\mathbb{Q}}; f_{\mathbb{Q}})$ of A represents an element $\alpha \in \pi_n(\text{map}_*(X, Y_{\mathbb{Q}}; f_{\mathbb{Q}}))$. Clearly, we have $\Phi(\alpha) = \langle \theta \rangle$, and so Φ is surjective.

Φ is an injection: Since Φ is a homomorphism, it is sufficient to check that $\Phi(\alpha) = 0$ implies $\alpha = 0 \in \pi_n(\text{map}_*(X, Y_{\mathbb{Q}}; f_{\mathbb{Q}}))$. As before, write $\Phi(\alpha) = \langle \theta_A \rangle$ and suppose $\theta_A \in \text{Der}_n(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$ is a boundary so that $\theta_A = D(\Theta)$ for $\Theta \in \text{Der}_{n+1}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$. Define a homotopy

$$\mathcal{G}: \mathbb{L}(W, v, W')_I \rightarrow \mathcal{L}_Y$$

by setting $\mathcal{G} = \mathcal{L}_A$ on $\mathbb{L}(W, v, W')$ (so \mathcal{G} starts at \mathcal{L}_A), $\mathcal{G} = 0$ on $\mathbb{L}(sW, \widehat{W}, sv, \widehat{v})$, and $\mathcal{G}(sw') = \Theta(w)$ on generators $w' \in W'$. We then set $\mathcal{G}(\widehat{w}') = \mathcal{G}(D_I sw') =$

$d_Y \mathcal{G}(sw') = d_Y \Theta(w)$, for $\widehat{w'} \in \widehat{W'}$, and then extend \mathcal{G} as a Lie algebra map, so that \mathcal{G} is a DG Lie algebra map. It is straightforward to check that \mathcal{G} ends at $\mathcal{G} \circ \lambda_1(w) = \mathcal{L}_f(w)$ and $\mathcal{G} \circ \lambda_1(v) = 0$ on $\mathbb{L}(W, v)$. Just as in the proof of (A) above, we observe that because \mathcal{G} is zero on $\mathbb{L}(sW, \widehat{W})$, the composition $\mathcal{G} \circ \sigma \circ S$ acts as a derivation in $\text{Der}_{n+1}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$. Therefore, we have $\mathcal{G} \circ \sigma \circ S(d_X w) = \Theta(d_X w)$ for each $w \in W$. Furthermore, again just as in part (A), we find that \mathcal{G} is zero on terms $(\sigma D_I)^r(w')$, for each $r \geq 2$. Therefore, for $w' \in W'$, this homotopy ends at

$$\begin{aligned}
 \mathcal{G} \circ \lambda_1(w') &= \mathcal{G}(w' + D_I \sigma(w') + \sigma D_I(w')) + \sum_{r \geq 2} \frac{1}{r!} (\sigma D_I)^r(w') \\
 &= \mathcal{G}(w') + \mathcal{G} D_I \sigma(w') + \mathcal{G} \sigma((-1)^n S d_X(w)) \\
 &= \mathcal{G}(w') + d_Y \mathcal{G} \sigma(w') + (-1)^n \mathcal{G} \sigma S d_X(w) \\
 &= \theta_A(w) + d_Y \Theta(w) - (-1)^{n+1} \Theta d_X(w),
 \end{aligned}$$

which is zero. Hence \mathcal{G} ends at the Quillen minimal model of the composition $f \circ p_2: S^n \times X \rightarrow Y$. It follows that $A \sim f \circ p_2: S^n \times X \rightarrow Y_{\mathbb{Q}}$. Taking adjoints, we obtain that $a \sim *: S^n \rightarrow \text{map}_*(X, Y; f)$. Actually, we only obtain this last homotopy as a free homotopy by taking adjoints, since the homotopy between A and $f \circ p_2$ is based, but not necessarily relative to X . However, a based map from a sphere is based-homotopic to the constant map if it is freely homotopic to the constant map [Spa89, p.27]. Thus Φ is injective.

Ψ is an isomorphism: Once again, we need only make slight adjustments to the preceding arguments for Φ . Suppose given (θ, y) a δ -cycle in $\text{Rel}_n(\text{ad}_{\mathcal{L}_f})$. Then $d_Y(y) = 0$ and $D(\theta) = \text{ad}_{\mathcal{L}_f}(y)$, that is, we have

$$d_Y \theta(\chi) - (-1)^n \theta d_X(\chi) = [y, \mathcal{L}_f(\chi)]$$

for $\chi \in \mathcal{L}_X$. In this case, define $\mathcal{L}_A: \mathbb{L}(W, v, W'; \partial) \rightarrow \mathcal{L}_Y$ by $\mathcal{L}_A(w) = \mathcal{L}_f(w)$, $\mathcal{L}_A(v) = y$, and $\mathcal{L}_A(w') = \theta(w)$. Now argue that Ψ is surjective following the same steps as were taken for Φ .

To show injectivity of Ψ , suppose that $(\theta_A, \mathcal{L}_A(v)) \in \text{Rel}_n(\text{ad}_{\mathcal{L}_f})$ is a boundary in the relative complex, so that $(\theta_A, \mathcal{L}_A(v)) = \delta(\Theta, y)$. That is, $\theta_A = \text{ad}_{\mathcal{L}_f}(y) - d_Y \Theta + (-1)^{n+1} \Theta d_X$ and $d_Y(y) = \mathcal{L}_A(v)$. In this case, define the DG homotopy \mathcal{G} by setting $\mathcal{G} = \mathcal{L}_A$ on $\mathbb{L}(W, v, W')$, $\mathcal{G} = 0$ on $\mathbb{L}(sW, \widehat{W})$, $\mathcal{G}(sv) = -y$, $\mathcal{G}(\widehat{v}) = -\mathcal{L}_A(v)$, $\mathcal{G}(sw') = \Theta(w)$, and $\mathcal{G}(\widehat{w'}) = d_Y \Theta(w)$. With this definition, the same steps as were used for Φ may now be followed to show that Ψ is injective.

(C) The commutativity of the diagram follows directly from the definitions. Notice that $j: \text{map}_*(X, Y; f) \rightarrow \text{map}_*(X, Y; f)$ is the fibre inclusion of the evaluation fibration. Therefore, if a is a representative of $\alpha \in \pi_n(\text{map}_*(X, Y; f))$, we may also take a to be a representative of $j_{\#}(\alpha) \in \pi_n(\text{map}(X, Y; f))$. Consequently, the Quillen minimal model of the adjoint of a representative of $j_{\#}(\alpha)$ may be taken as \mathcal{L}_A with $\mathcal{L}_A(v) = 0$. Hence, before rationalizing, we have $\Psi'(j_{\#}(\alpha)) = \langle \theta_A, 0 \rangle = \langle J(\theta_A) \rangle = H(J)(\langle \theta_A \rangle) = H(J) \circ \Phi'(\alpha)$. \square

Remark 4.2. The homomorphisms Φ' and Ψ' , and hence their rationalizations, have certain naturality properties. Post-composition by a map $g: Y \rightarrow Z$ gives a map $g_*: \text{map}_*(X, Y; f) \rightarrow \text{map}_*(X, Z; g \circ f)$. On the other hand, the Quillen minimal model of g induces a chain map $(\mathcal{L}_g)_*: \text{Der}_*(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f) \rightarrow \text{Der}_*(\mathcal{L}_X, \mathcal{L}_Z; \mathcal{L}_{g \circ f})$. In this latter chain complex, we choose $\mathcal{L}_g \circ \mathcal{L}_f$ as the Quillen minimal model for

$g \circ f$. If we denote the homomorphism Φ of Theorem 4.1 by Φ_*^f , so as to include the original map in the notation, then we have the following commutative diagram:

$$\begin{array}{ccc} \pi_n(\mathrm{map}_*(X, Y; f)) \otimes \mathbb{Q} & \xrightarrow{\Phi_*^f} & H_n(\mathrm{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)) \\ (g_*)_{\#} \otimes 1 \downarrow & & \downarrow H((\mathcal{L}_g)_*) \\ \pi_n(\mathrm{map}_*(X, Z; g \circ f)) \otimes \mathbb{Q} & \xrightarrow{\Phi_*^{g \circ f}} & H_n(\mathrm{Der}(\mathcal{L}_X, \mathcal{L}_Z; \mathcal{L}_{g \circ f})) \end{array}$$

Pre-composition by a map $h: W \rightarrow X$ gives a similar naturality property of Φ . Furthermore, Ψ is natural in the same sense.

Schlessinger and Stasheff, and Tanré have constructed a (non-minimal) Quillen model for the universal fibration

$$X \rightarrow B \mathrm{map}_*(X, X; 1) \rightarrow B \mathrm{map}(X, X; 1)$$

in terms of Lie derivations and adjoint maps ([SS, Tan83]). Their result specializes to identify the long exact sequence induced in rational homotopy groups by the universal fibration, in the framework of Quillen models and derivations. The following consequence of Theorem 4.1 extends this specialization of their result, in that it identifies the long exact rational homotopy sequence of a general evaluation fibration.

We first make the following observation.

Lemma 4.3. *Suppose given a diagram of vector spaces*

$$\begin{array}{ccccccccc} B_{n+1} & \xrightarrow{j_{n+1}} & C_{n+1} & \xrightarrow{k_{n+1}} & A_n & \xrightarrow{i_n} & B_n & \xrightarrow{j_n} & C_n \\ \cong \downarrow \beta_{n+1} & & \cong \downarrow \gamma_{n+1} & & \downarrow \alpha_n & & \cong \downarrow \beta_n & & \cong \downarrow \gamma_n \\ Y_{n+1} & \xrightarrow{q_{n+1}} & Z_{n+1} & \xrightarrow{r_{n+1}} & X_n & \xrightarrow{p_n} & Y_n & \xrightarrow{q_n} & Z_n \end{array}$$

for each $n \geq 2$ (that is, a “ladder with every third rung missing”). Suppose the rows are exact, each β_n and γ_n is an isomorphism, and $\gamma_n \circ j_n = q_n \circ \beta_n$ for each n . Then there exist isomorphisms $\alpha_n: A_n \rightarrow X_n$, for $n \geq 2$, which makes the entire ladder commutative.

Proof. This is straightforward. See [LS03, Lem.3.1] for details. \square

Now consider the evaluation fibration $\mathrm{map}_*(X, Y; f) \xrightarrow{j} \mathrm{map}(X, Y; f) \xrightarrow{\omega} Y$ for a map $f: X \rightarrow Y$. By going one step back in the Barratt-Puppe sequence, we obtain a fibration

$$\Omega Y \xrightarrow{\partial} \mathrm{map}_*(X, Y; f) \xrightarrow{j} \mathrm{map}(X, Y; f)$$

and a long exact sequence in homotopy

$$(1) \quad \cdots \xrightarrow{\partial_{\#}} \pi_n(\mathrm{map}_*(X, Y; f)) \xrightarrow{j_{\#}} \pi_n(\mathrm{map}(X, Y; f)) \xrightarrow{\omega_{\#}} \pi_{n-1}(\Omega Y) \xrightarrow{\partial_{\#}} \cdots$$

Here, we are identifying $\pi_{n-1}(\Omega Y)$ with $\pi_n(Y)$ in the usual way. Notice that when $X = Y$ and $f = 1$ the long exact sequence (1) is equivalent to that of the universal fibration [DZ79].

Theorem 4.4. *Let $f: X \rightarrow Y$ be a map between simply connected CW complexes of finite type with X finite. Then the rationalization of the long exact homotopy sequence (1), as far as the term $\pi_1(\Omega Y) \otimes \mathbb{Q}$, is equivalent to the long exact derivation homology sequence of the Quillen minimal model $\mathcal{L}_f: \mathcal{L}_X \rightarrow \mathcal{L}_Y$ of f , that is,*

$$\cdots \xrightarrow{H(\text{ad}_{\mathcal{L}_f})} H_n(\text{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)) \xrightarrow{H(J)} H_n(\text{Rel}(\text{ad}_{\mathcal{L}_f})) \xrightarrow{H(P)} H_{n-1}(\mathcal{L}_Y) \xrightarrow{H(\text{ad}_{\mathcal{L}_f})} \cdots$$

as far as the term $H_1(\mathcal{L}_Y)$.

Proof. Replace the diagrams

$$\begin{array}{ccc} B_n & \xrightarrow{j_n} & C_n \\ \cong \downarrow \beta_n & & \cong \downarrow \gamma_n \\ Y_n & \xrightarrow{q_n} & Z_n \end{array}$$

of Lemma 4.3 with the diagrams

$$\begin{array}{ccc} \pi_n(\text{map}_*(X, Y; f)) \otimes \mathbb{Q} & \xrightarrow{j_\# \otimes \mathbb{Q}} & \pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q} \\ \cong \downarrow \Phi & & \cong \downarrow \Psi \\ H_n(\text{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)) & \xrightarrow{H(J)} & H_n(\text{Rel}(\text{ad}_{\mathcal{L}_f})) \end{array}$$

for $n \geq 2$. Then with the top row replaced by the rationalization of the long exact homotopy sequence (1), and the bottom row by the long exact derivation homology sequence of $\mathcal{L}_f: \mathcal{L}_X \rightarrow \mathcal{L}_Y$, Lemma 4.3 obtains isomorphisms

$$\alpha_n: \pi_n(\Omega Y) \otimes \mathbb{Q} \rightarrow H_n(\mathcal{L}_Y)$$

with appropriate commutativity properties for $n \geq 2$. At the bottom end, we have a diagram

$$\begin{array}{ccccc} \pi_2(\text{map}_*(X, Y; f)) \otimes \mathbb{Q} & \xrightarrow{j_\# \otimes \mathbb{Q}} & \pi_2(\text{map}(X, Y; f)) \otimes \mathbb{Q} & \xrightarrow{\omega_\# \otimes \mathbb{Q}} & \pi_1(\Omega Y) \otimes \mathbb{Q} \\ \cong \downarrow \Phi & & \cong \downarrow \Psi & & \downarrow \alpha_1 \\ H_2(\text{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)) & \xrightarrow{H(J)} & H_2(\text{Rel}(\text{ad}_{\mathcal{L}_f})) & \xrightarrow{H(P)} & H_1(\mathcal{L}_Y) \end{array}$$

Here, we may use the standard identification of $\pi_1(\Omega Y) \otimes \mathbb{Q}$ with $H_1(\mathcal{L}_Y)$ for our last isomorphism α_1 . Then we need only check that the final square commutes. For this, we regard the standard identification $\alpha_1: \pi_1(\Omega Y) \otimes \mathbb{Q} \rightarrow H_1(\mathcal{L}_Y)$ as follows. Suppose $a: S^2 \rightarrow Y$ is a representative of $\eta \in \pi_1(\Omega Y) \otimes \mathbb{Q} = \pi_2(Y) \otimes \mathbb{Q}$. Then we have $\mathcal{L}_a: \mathcal{L}_{S^2} \rightarrow \mathcal{L}_Y$ and we set $\alpha_1(\eta) = \mathcal{L}_a(u)$, where $\mathcal{L}_{S^2} = \mathbb{L}(u)$ with $|u| = 1$. Now suppose given $\zeta \in \pi_2(\text{map}(X, Y; f)) \otimes \mathbb{Q}$ with adjoint $G: S^2 \times X \rightarrow Y_{\mathbb{Q}}$. Then $\omega_\#(\zeta) = [G \circ i_1] \in \pi_2(Y) \otimes \mathbb{Q}$. Hence, we have $H(P) \circ \Psi(\zeta) = H(P)\langle \theta_G, \mathcal{L}_G(v) \rangle = \langle \mathcal{L}_G(v) \rangle = \langle \mathcal{L}_G \circ \mathcal{L}_{i_1}(u) \rangle = \alpha \circ \omega_\#(\zeta)$. Thus we have the equivalence asserted. \square

Remark 4.5. The last part of the proof above extends to give $\alpha_n \circ (\omega_\# \otimes \mathbb{Q}) = H(P) \circ \Psi$ for all $n \geq 2$, if α_n is taken to be the standard identification $\alpha_n: \pi_n(\Omega Y) \otimes \mathbb{Q} \rightarrow H_n(\mathcal{L}_Y)$. However, we have chosen to argue as above to avoid the details involved in showing the remaining squares of the diagram commute with the standard choice

of α_n . Unfortunately, this means that we have not established a *natural* equivalence of sequences in Theorem 4.4.

5. EVALUATION SUBGROUPS AND GOTTLIEB'S QUESTION

As an immediate consequence of Theorem 4.4, we identify the rationalized evaluation subgroup of a map $f: X \rightarrow Y$ in terms of Quillen models. Let $\psi: K \rightarrow L$ be a DG Lie algebra map.

Definition 5.1. The *n*th evaluation subgroup of the DG Lie algebra map $\psi: L \rightarrow K$ is the subgroup

$$G_n(K, L; \psi) = \ker\{H(\text{ad}_\psi): H_{n-1}(K) \rightarrow H_{n-1}(\text{Der}(L, K; \psi))\}$$

of $H_{n-1}(K)$. Notice the shift in degrees. We specialize this to define the Gottlieb group of a DG Lie algebra following Tanré. The *n*th Gottlieb group of the DG Lie algebra (L, d) is the subgroup

$$G_n(L) = \ker\{H(\text{ad}): H_{n-1}(L) \rightarrow H_{n-1}(\text{Der}(L))\}$$

of $H_{n-1}(L)$.

Theorem 5.2. *Let $f: X \rightarrow Y$ be a map between simply connected CW complexes of finite type with X finite. Then the rationalization of the evaluation subgroup of f is isomorphic to the evaluation subgroup of the Quillen model $\mathcal{L}_f: \mathcal{L}_X \rightarrow \mathcal{L}_Y$ of f as in Definition 5.1. That is, for $n \geq 2$ we have*

$$G_n(Y, X; f) \otimes \mathbb{Q} \cong \ker\{H(\text{ad}_{\mathcal{L}_f}): H_{n-1}(\mathcal{L}_Y) \rightarrow H_{n-1}(\text{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f))\}.$$

Proof. From the long exact homotopy sequence of the evaluation fibration, the rationalized evaluation subgroup $G_n(Y, X; f) \otimes \mathbb{Q}$ corresponds to the kernel of $\partial: \pi_{n-1}(\Omega Y) \otimes \mathbb{Q} \rightarrow \pi_{n-1}(\text{map}_*(X, Y; f)) \otimes \mathbb{Q}$. The result follows from Theorem 4.4. \square

Specializing to the identity map we recover

Corollary 5.3 ([Tan83, Cor.VII.4(10)]). *Let X be a simply connected, finite complex. Then the rationalization of the Gottlieb group of X is the Gottlieb group of the Quillen model (\mathcal{L}_X, d_X) of X as in Definition 5.1. That is, for $n \geq 2$ we have*

$$G_n(X) \otimes \mathbb{Q} \cong \ker\{H(\text{ad}_{\mathcal{L}_X}): H_{n-1}(\mathcal{L}_X) \rightarrow H_{n-1}(\text{Der}(\mathcal{L}_X))\}.$$

We apply these identifications to address Gottlieb's question on the difference between the Whitehead centralizer $P_*(X)$ and the Gottlieb group $G_*(X)$. In ordinary homotopy theory, constructing spaces with $G_*(X) \neq P_*(X)$ represents a challenging problem. Ganea gave the first example of inequality in [Gan68]. See [Opr95] for a recent reference and some interesting examples of $G_1(X) \neq P_1(X)$ with X a finite complex.

Recall the definition of the generalized Whitehead center of a map $P_n(Y, X; f) \subseteq \pi_n(Y)$ from the introduction. From the identification of the Samelson product in $\pi_*(\Omega Y) \otimes \mathbb{Q}$ with the product in $H(\mathcal{L}_Y)$, we have

$$P_n(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}}) \cong \ker\{\text{ad}_{H(\mathcal{L}_f)}: H_{n-1}(\mathcal{L}_Y) \rightarrow \text{Der}_{n-1}(H(\mathcal{L}_X), H(\mathcal{L}_Y); H(\mathcal{L}_f))\}.$$

Notice that although the relative evaluation subgroup behaves well with respect to rationalization, in the sense that $G_n(Y, X; f) \otimes \mathbb{Q} = G_*(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}})$ (at least for X finite), the inclusion $P_*(X) \otimes \mathbb{Q} \subseteq P_*(X_{\mathbb{Q}})$ is usually strict.

Rationally, the difference between the n th evaluation subgroup of a map and the generalized Whitehead center can be described precisely. The difference is governed by the “induced derivation” map

$$I: H_n(\mathrm{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)) \rightarrow \mathrm{Der}_n(H(\mathcal{L}_X), H(\mathcal{L}_Y); H(\mathcal{L}_f))$$

which we now introduce.

A D -cycle $\theta \in \mathrm{Der}_n(L, K; \psi)$ commutes (in the graded sense) with the differentials of L and K , and so induces a map $H(\theta) \in \mathrm{Der}_n(H(L), H(K); H(\psi))$ defined by $H(\theta)(\langle \xi \rangle) = \langle \theta(\xi) \rangle$ for ξ a cycle of L . If θ is a D -boundary then it carries cycles of L to boundaries of K . Thus we obtain a linear map

$$I: H_n(\mathrm{Der}(L, K; \psi)) \rightarrow \mathrm{Der}_n(H(L), H(K); H(\psi))$$

given by $I(\langle \theta \rangle) = H(\theta)$ for θ a cycle of $\mathrm{Der}_n(L, K; \psi)$.

Now consider the commutative diagram

$$\begin{array}{ccc} H_n(\mathcal{L}_Y) & \xrightarrow{H(\mathrm{ad}_{\mathcal{L}_f})} & H_n(\mathrm{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)) \\ & \searrow \mathrm{ad}_{H(\mathcal{L}_f)} & \downarrow I \\ & & \mathrm{Der}_n(H(\mathcal{L}_X), H(\mathcal{L}_Y); H(\mathcal{L}_f)). \end{array}$$

Theorem 5.4. *Let $f: X \rightarrow Y$ be a map between simply connected CW complexes of finite type with X a finite complex. For $n \geq 1$, we have*

$$\frac{P_{n+1}(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}})}{G_{n+1}(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}})} \cong \ker(I) \cap \mathrm{im}(H(\mathrm{ad}_{\mathcal{L}_f})).$$

Proof. The map $H(\mathrm{ad}_{\mathcal{L}_f})$ induces a map

$$\overline{H(\mathrm{ad}_{\mathcal{L}_f})}: \frac{\ker(\mathrm{ad}_{H(\mathcal{L}_f)})}{\ker(H(\mathrm{ad}_{\mathcal{L}_f}))} \rightarrow \ker(I) \cap \mathrm{im}(H(\mathrm{ad}_{\mathcal{L}_f}))$$

that is easily checked to be an isomorphism. The result follows from Theorem 5.2 and the above discussion, in which we identify $\ker(H(\mathrm{ad}_{\mathcal{L}_f}))$ with $G_{n+1}(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}})$, and $\ker(\mathrm{ad}_{H(\mathcal{L}_f)})$ with $P_{n+1}(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}})$. \square

Using these notions, it is straightforward to give examples of maps $f: X \rightarrow Y$ which give an inequality $G_*(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}}) \neq P_*(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}})$.

Example 5.5. Suppose that $f: X \rightarrow Y$ is a rationally trivial map. Then $\mathcal{L}_f = 0: \mathcal{L}_X \rightarrow \mathcal{L}_Y$. It follows that $\mathrm{ad}_{\mathcal{L}_f} = 0: \mathcal{L}_Y \rightarrow \mathrm{Der}_*(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$. In this case, we have $G_*(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}}) = P_*(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}}) = \pi_*(Y) \otimes \mathbb{Q}$. However, suppose that $f_{\#} \otimes \mathbb{Q} = 0: \pi_*(X) \otimes \mathbb{Q} \rightarrow \pi_*(Y) \otimes \mathbb{Q}$, or even just that $\mathrm{im}(f_{\#} \otimes \mathbb{Q}) \subseteq P_*(Y_{\mathbb{Q}})$. Then $\mathrm{ad}_{H(\mathcal{L}_f)} = 0: H_*(\mathcal{L}_Y) \rightarrow \mathrm{Der}_*(H(\mathcal{L}_X), H(\mathcal{L}_Y); H(\mathcal{L}_f))$ and so $P_*(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}}) = \pi_*(Y) \otimes \mathbb{Q}$. However, the equality $G_*(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}}) = \pi_*(Y) \otimes \mathbb{Q}$ would only hold if $H(\mathrm{ad}_{\mathcal{L}_f}) = 0$, and there is no particular reason why this should be so. To illustrate this last case, consider $f: \mathbb{C}P^2 \rightarrow S^4$ obtained by pinching out the 2-cell of $\mathbb{C}P^2$. This map has Quillen minimal model

$$\mathcal{L}_f: \mathbb{L}(x_1, x_3; d_X) \rightarrow \mathbb{L}(u_3; d_Y = 0)$$

with $\mathcal{L}_f(x_1) = 0$ and $\mathcal{L}_f(x_3) = u_3$. Here, the subscript of a generator denotes its degree and $d_X(x_1) = 0$, $d_X(x_3) = [x_1, x_1]$. Now $\mathrm{ad}_{\mathcal{L}_f}(u_3) \in \mathrm{Der}_3(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$ is defined by $\mathrm{ad}_{\mathcal{L}_f}(u_3)(x_1) = 0$ and $\mathrm{ad}_{\mathcal{L}_f}(u_3)(x_3) = [u_3, u_3]$. On the other hand, $D = 0$ in $\mathrm{Der}_3(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$. Therefore, $H(\mathrm{ad}_{\mathcal{L}_f})(\langle u_3 \rangle) \neq 0$ and $G_4(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}}) = 0$.

We next give a class of examples for which the equality $G_*(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}}) = P_*(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}})$ holds. For this, we review the notion of *coformality* and some terminology associated with this concept. Suppose that a minimal DG Lie algebra $\mathbb{L}(V; d)$ has a second (or “upper”) grading on the generating subspace $V = \bigoplus_{i \geq 0} V^i$. This extends to a second grading of $\mathbb{L}(V)$ in the obvious way, and we write $\mathbb{L}(V)^i$ for the sub-vector space of $\mathbb{L}(V)$ consisting of all elements of $\mathbb{L}(V)$ of second grading equal to i . We also write $V^{(i)}$ for the sub-vector space of V consisting of all elements of V of second grading less than or equal to i . Then we say that $\mathbb{L}(V; d)$ is a *bigraded* minimal DG Lie algebra if the differential decreases second degree homogeneously by one, that is, if $d(V^0) = 0$ and $d(V^i) \subseteq \mathbb{L}(V)^{i-1}$ for $i \geq 1$. If $\mathbb{L}(V; d)$ is a bigraded minimal DG Lie algebra, then the second grading passes to homology, making $H(\mathbb{L}(V; d))$ a bigraded Lie algebra. We write $H^i(\mathbb{L}(V; d))$ for the sub-vector space of $H(\mathbb{L}(V; d))$ consisting of homology classes represented by cycles of upper degree equal to i , and we have $H(\mathbb{L}(V; d)) = \bigoplus_{i \geq 0} H^i(\mathbb{L}(V; d))$.

Definition 5.6. Let $\mathbb{L}(V; d)$ be a bigraded minimal DG Lie algebra in the above sense. We say $\mathbb{L}(V; d)$ is *coformal* if $H^i(\mathbb{L}(V; d)) = 0$ for $i > 0$, so that $H(\mathbb{L}(V; d)) = H^0(\mathbb{L}(V; d))$. We say that a space X is a *coformal space* if its Quillen minimal model is coformal.

Equivalently, we may define $\mathbb{L}(V; d)$ to be coformal if there exists a quasi-isomorphism $\rho: \mathbb{L}(V; d) \rightarrow (H(L), d = 0)$. The equivalence of these definitions is established by the notion of a *bigraded model* in the DG Lie algebra setting—see [HS79, Sec.3] for the Sullivan model setting. There are many interesting examples of coformal spaces: Moore spaces and more generally rational co-H-spaces, including suspensions; some homogeneous spaces; products and wedges of coformal spaces.

This notion of coformality extends to a map. Suppose that $\phi: \mathbb{L}(V; d) \rightarrow \mathbb{L}(W; d')$ is a map of bigraded minimal DG Lie algebras as defined above. If $\phi(V^i) \subseteq \mathbb{L}(W)^i$ for each $i \geq 0$, then we say that ϕ is a bigraded map.

Definition 5.7. A map $\phi: \mathbb{L}(V; d) \rightarrow \mathbb{L}(W; d')$ of bigraded minimal DG Lie algebras is a *coformal map* if both $\mathbb{L}(V; d)$ and $\mathbb{L}(W; d')$ are coformal, and ϕ is a bigraded map (with respect to the second gradings that display the coformality of $\mathbb{L}(V)$ and $\mathbb{L}(W)$). A map of coformal spaces $f: X \rightarrow Y$ is a *coformal map* if its Quillen minimal model $\mathcal{L}_f: \mathcal{L}_X \rightarrow \mathcal{L}_Y$ is a coformal map of bigraded minimal DG Lie algebras.

Equivalently, we may define $\phi: \mathbb{L}(V; d) \rightarrow \mathbb{L}(W; d')$ to be coformal if there exist quasi-isomorphisms $\rho: \mathbb{L}(V; d) \rightarrow H(\mathbb{L}(V; d))$ and $\rho': \mathbb{L}(W; d') \rightarrow H(\mathbb{L}(W; d'))$ such that the diagram

$$\begin{array}{ccc} \mathbb{L}(V; d) & \xrightarrow{\phi} & \mathbb{L}(W; d') \\ \rho \downarrow \simeq & & \simeq \downarrow \rho' \\ H(\mathbb{L}(V; d)) & \xrightarrow{H(\phi)} & H(\mathbb{L}(W; d')) \end{array}$$

is DG homotopy commutative.

Remark 5.8. Suppose given a map of DG Lie algebras $\Phi: L \rightarrow L'$. By constructing DG Lie algebra minimal models of L and L' , and then using the standard lifting

lemma, we obtain a DG homotopy commutative diagram

$$\begin{array}{ccc} \mathbb{L}(V; d) & \xrightarrow{\phi} & \mathbb{L}(W; d') \\ \rho \downarrow \simeq & & \simeq \downarrow \rho' \\ L & \xrightarrow{\Phi = H(\phi)} & L' \end{array}$$

In this way, it is always possible to realize a Lie algebra map as the homomorphism induced on homology by a coformal map. Notice, however, that there may be many other DG Lie algebra maps $\mathbb{L}(V; d) \rightarrow \mathbb{L}(W; d')$ that induce the same homomorphism on homology as $H(\phi)$. Coformality, therefore, distinguishes a unique DG homotopy class of maps from amongst the various realizations. From this point of view, a coformal map is the simplest realization of its induced homomorphism on homology.

Theorem 5.9. *Let $f: X \rightarrow Y$ be a coformal map between CW complexes of finite type with X finite. Then $P_*(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}}) = G_*(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}})$.*

Proof. Suppose $\mathcal{L}_X = \mathbb{L}(W; d_X)$ and $\mathcal{L}_Y = \mathbb{L}(V; d_Y)$ are coformal, and that \mathcal{L}_f is bigraded. Take $\alpha \in H_n(\mathcal{L}_Y)$. With reference to Theorem 5.4, we show that if $I \circ H(\text{ad}_{\mathcal{L}_f})(\alpha) = 0$, then $H(\text{ad}_{\mathcal{L}_f})(\alpha) = 0$. Since Y is coformal, we may assume $\alpha = \langle \xi \rangle$ for a d_Y -cycle $\xi \in \mathbb{L}(V)^0$. Observe that $\text{ad}_{\mathcal{L}_f}(\xi) \in \text{Der}_n(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$ is then a D -cycle that preserves upper degree. If $I \circ H(\text{ad}_{\mathcal{L}_f})(\alpha) = 0$, then for each d_X -cycle $\chi \in \mathbb{L}(W)$, we have $\text{ad}_{\mathcal{L}_f}(\xi)(\chi) = d_Y(\eta)$ for some $\eta \in \mathbb{L}(V)$. We now use this to construct $\theta \in \text{Der}_{n+1}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$ such that $D(\theta) = \text{ad}_{\mathcal{L}_f}(\xi)$.

Since X is coformal, we have $W = \bigoplus_{i \geq 0} W^i$, and each $w \in W^0$ is a d_X -cycle. Therefore, we have $\text{ad}_{\mathcal{L}_f}(\xi)(w) = d_Y(\eta)$ for some $\eta \in \mathbb{L}(V)$. Furthermore, since \mathcal{L}_Y is bigraded, we may choose $\eta \in \mathbb{L}(V)^1$. Use this to define a linear map $\theta_0: W^0 \rightarrow \mathbb{L}(V)^1$ and extend to an \mathcal{L}_f -derivation $\theta_0 \in \text{Der}_{n+1}(\mathbb{L}(W^0), \mathcal{L}_Y; \mathcal{L}_f)$. By construction, we have $D(\theta_0)(\chi) = d_Y(\theta_0(\chi)) = \text{ad}_{\mathcal{L}_f}(\xi)(\chi)$ for $\chi \in \mathbb{L}(W^0)$.

Assume inductively that $\theta_m \in \text{Der}_{n+1}(\mathbb{L}(W^{(m)}), \mathcal{L}_Y; \mathcal{L}_f)$ is defined, increasing upper degree homogeneously by 1, and satisfying $D(\theta_m) = \text{ad}_{\mathcal{L}_f}(\xi)$ on $\mathbb{L}(W^{(m)})$. For $w \in W^{m+1}$, consider the element $\text{ad}_{\mathcal{L}_f}(\xi)(w) + (-1)^{n+1}\theta_m(d_X w)$. Since $\text{ad}_{\mathcal{L}_f}(\xi)$ is a D -cycle, and $d_X w \in \mathbb{L}(W)^m \subseteq \mathbb{L}(W^{(m)})$, we compute that

$$\begin{aligned} d_Y(\text{ad}_{\mathcal{L}_f}(\xi)(w) + (-1)^{n+1}\theta_m(d_X w)) &= d_Y \text{ad}_{\mathcal{L}_f}(\xi)(w) + (-1)^{n+1} d_Y \theta_m(d_X w) \\ &= (-1)^n \text{ad}_{\mathcal{L}_f}(\xi)(d_X w) + (-1)^{n+1} \text{ad}_{\mathcal{L}_f}(\xi)(d_X w) \\ &\quad + \text{ad}_{\mathcal{L}_f}(\xi) d_X(d_X w) \\ &= 0. \end{aligned}$$

Since $\text{ad}_{\mathcal{L}_f}(\xi)(w) + (-1)^{n+1}\theta_m(d_X w)$ is a d_Y -cycle in $\mathbb{L}(V)^{m+1}$, we again use the coformality of \mathcal{L}_Y —specifically, that $H^+(\mathcal{L}_Y) = 0$ —to conclude that there exists some $\zeta \in \mathcal{L}_Y$ with $d_Y(\zeta) = \text{ad}_{\mathcal{L}_f}(\xi)(w) + (-1)^{n+1}\theta_m(d_X w)$. Furthermore, we may choose $\zeta \in \mathbb{L}(V)^{m+2}$. Clearly, ζ may be chosen so as to depend linearly on w . So use this to define a linear map $\theta_{m+1}: W^{m+1} \rightarrow \mathbb{L}(V)^{m+2}$ with $\theta_{m+1}(w) = \zeta$, and extend θ_m to an \mathcal{L}_f -derivation $\theta_{m+1} \in \text{Der}_{n+1}(\mathbb{L}(W^{(m+1)}), \mathcal{L}_Y; \mathcal{L}_f)$. By construction, we have

$$\begin{aligned} D\theta_{m+1}(w) &= d_Y \theta_{m+1}(w) - (-1)^{n+1} \theta_{m+1}(d_X w) \\ &= d_Y \zeta - (-1)^{n+1} \theta_m(d_X w) = \text{ad}_{\mathcal{L}_f}(\xi)(w) \end{aligned}$$

for $w \in W^{m+1}$ and since $D(\theta_{m+1})$ is an \mathcal{L}_f -derivation, this gives $D(\theta_{m+1})(\chi) = \text{ad}_{\mathcal{L}_f}(\xi)(\chi)$ for $\chi \in \mathbb{L}(W^{(m+1)})$. This completes the induction and gives an \mathcal{L}_f -derivation $\theta \in \text{Der}_{n+1}(\mathbb{L}(W), \mathcal{L}_Y; \mathcal{L}_f)$ that satisfies $D(\theta) = \text{ad}_{\mathcal{L}_f}(\xi)$. The result follows. \square

The following special case is well-known.

Corollary 5.10. *Let X be a simply connected, finite CW complex. If X is coformal, then $P_*(X_{\mathbb{Q}}) = G_*(X_{\mathbb{Q}})$.*

Proof. Restrict Theorem 5.9 to the case $f = 1: X \rightarrow X$. \square

6. THE RATIONALIZED G -SEQUENCE

In this section, we identify the rationalization of the G -sequence mentioned in the introduction. Suppose given a map $f: X \rightarrow Y$. Then we have the commutative square

$$(2) \quad \begin{array}{ccc} \Omega X & \xrightarrow{\Omega f} & \Omega Y \\ \downarrow \partial & & \downarrow \partial \\ \text{map}_*(X, X; 1) & \xrightarrow{f_*} & \text{map}_*(X, Y; f), \end{array}$$

in which the vertical maps are the connecting maps arising from the evaluation fibrations $\omega: \text{map}(X, X; 1) \rightarrow X$ and $\omega: \text{map}(X, Y; f) \rightarrow Y$ as in Section 4. The maps Ωf and f_* lead to long exact homotopy sequences and the vertical maps give homomorphisms of corresponding terms, yielding a homotopy ladder

$$(3) \quad \begin{array}{ccccccc} \cdots & \xrightarrow{p} & \pi_n(\Omega X) & \xrightarrow{(\Omega f)_\#} & \pi_n(\Omega Y) & \xrightarrow{j} & \pi_n(\Omega f) \xrightarrow{p} \cdots \\ & & \downarrow \partial_\# & & \downarrow \partial_\# & & \downarrow \partial_\# \\ \cdots & \xrightarrow{\hat{p}} & \pi_n(\text{map}_*(X, X; 1)) & \xrightarrow{(f_*)_\#} & \pi_n(\text{map}_*(X, Y; f)) & \xrightarrow{\hat{j}} & \pi_n(f_*) \xrightarrow{\hat{p}} \cdots \end{array}$$

in the usual way. Whenever we have such a ladder, with exact rows, there is an associated ‘‘kernel sequence,’’ that is, a sequence obtained by restricting the maps in the top row to the kernels of the vertical rungs. The G -sequence of the map f may be defined, with a shift in degree, as the kernel sequence of the above homotopy ladder. A portion of this construction is shown here:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{p} & G_{n+1}(X) & \xrightarrow{(\Omega f)_\#} & G_{n+1}(Y, X; f) & \xrightarrow{j} & G_{n+1}^{rel}(Y, X; f) \xrightarrow{p} \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \xrightarrow{p} & \pi_n(\Omega X) & \xrightarrow{(\Omega f)_\#} & \pi_n(\Omega Y) & \xrightarrow{j} & \pi_n(\Omega f) \xrightarrow{p} \cdots \\ & & \downarrow \partial_\# & & \downarrow \partial_\# & & \downarrow \Delta \\ \cdots & \xrightarrow{\hat{p}} & \pi_n(\text{map}_*(X, X; 1)) & \xrightarrow{(f_*)_\#} & \pi_n(\text{map}_*(X, Y; f)) & \xrightarrow{\hat{j}} & \pi_n(f_*) \xrightarrow{\hat{p}} \cdots \end{array}$$

Note that the maps in the G -sequence are just the restrictions of the maps in the long exact homotopy sequence of the map $\Omega f: \Omega X \rightarrow \Omega Y$. Thus compositions of consecutive maps in the G -sequence are trivial. However, the sequence of kernels of a commutative ladder of exact sequences need not be exact, and so the G -sequence

is a chain complex (of \mathbb{Z} -modules). The original description given in [WL88b, LW93] (see also [LS03, Sec.1]) represents the G -sequence as an image sequence, in a way obviously equivalent to the above.

Below, we construct a chain complex associated to any DG Lie algebra map that we will show corresponds to the rationalized G -sequence. Observe that the relative chain complex is a functorial construction. That is, given a commutative square

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ \downarrow \alpha & & \downarrow \beta \\ V' & \xrightarrow{\phi'} & W' \end{array}$$

of DG vector space maps we obtain a DG vector space map $(\beta, \alpha) : \text{Rel}(\phi) \rightarrow \text{Rel}(\phi')$ given by $(\beta, \alpha)(w, v) = (\beta(w), \alpha(v))$. This leads to a commutative diagram of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & W_* & \xrightarrow{J} & \text{Rel}_*(\phi) & \xrightarrow{P} & V_{*-1} \longrightarrow 0 \\ & & \downarrow \beta & & \downarrow (\beta, \alpha) & & \downarrow \alpha \\ 0 & \longrightarrow & W'_* & \xrightarrow{J} & \text{Rel}_*(\phi') & \xrightarrow{P} & V'_{*-1} \longrightarrow 0, \end{array}$$

and hence to a commutative ladder of long exact homology sequences.

Now suppose given a map $\psi : L \rightarrow K$ of DG Lie algebras. Apply the above to the commutative square

$$(4) \quad \begin{array}{ccc} L & \xrightarrow{\psi} & K \\ \downarrow \text{ad} & & \downarrow \text{ad}_\psi \\ \text{Der}(L, L; 1) & \xrightarrow{\psi_*} & \text{Der}(L, K; \psi) \end{array}$$

of DG vector spaces. Note that the relative term $\text{Rel}_*(\psi_*)$ is given by

$$\text{Rel}_n(\psi_*) = \text{Der}_n(L, K; \psi) \oplus \text{Der}_{n-1}(L, L; 1)$$

with differential $\delta(\theta_1, \theta_2) = (\psi \circ \theta_2 - D(\theta_1), D(\theta_2))$. We obtain the following commutative ladder of long exact homology sequences:

$$(5) \quad \begin{array}{ccccccc} \cdots & \xrightarrow{H(P)} & H_n(L) & \xrightarrow{H(\psi)} & H_n(K) & \xrightarrow{H(J)} & H_n(\text{Rel}(\psi)) \xrightarrow{H(P)} \cdots \\ & & \downarrow H(\text{ad}) & & \downarrow H(\text{ad}_\psi) & & \downarrow H(\text{ad}_\psi, \text{ad}) \\ \cdots & \xrightarrow{H(\hat{P})} & H_n(\text{Der}(L, L; 1)) & \xrightarrow{H(\psi_*)} & H_n(\text{Der}(L, K; \psi)) & \xrightarrow{H(\hat{J})} & H_n(\text{Rel}(\psi_*)) \xrightarrow{H(\hat{P})} \cdots \end{array}$$

To obtain unambiguous notation, we have written $\hat{J} : \text{Der}_n(L, K; \mathcal{L}_f) \rightarrow \text{Rel}_n(\psi_*)$ and $\hat{P} : \text{Rel}_n(\psi_*) \rightarrow \text{Der}_{n-1}(L)$ for the usual inclusion and projection maps in the lower sequence. We refer to this ladder as the (horizontal) homology ladder arising from the diagram (4).

Theorem 6.1. *Let $f : X \rightarrow Y$ be a map between simply connected CW complexes of finite type, with X finite. The rationalization of the homotopy ladder (3), down*

to the rung $\partial_{\#} \otimes 1: \pi_2(\Omega Y) \otimes \mathbb{Q} \rightarrow \pi_2(\text{map}_*(X, Y; f)) \otimes \mathbb{Q}$, is equivalent to the homology ladder arising from

$$(6) \quad \begin{array}{ccc} \mathcal{L}_X & \xrightarrow{\mathcal{L}_f} & \mathcal{L}_Y \\ \downarrow \text{ad} & & \downarrow \text{ad}_{\mathcal{L}_f} \\ \text{Der}(\mathcal{L}_X, \mathcal{L}_X; 1) & \xrightarrow{(\mathcal{L}_f)_*} & \text{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f), \end{array}$$

down to the rung $H(\text{ad}_{\mathcal{L}_f}): H_2(\mathcal{L}_Y) \rightarrow H_2(\text{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f))$.

Proof. Let $\Phi^Y: \pi_n(\text{map}_*(X, Y; f)) \otimes \mathbb{Q} \rightarrow H_n(\text{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f))$ be the isomorphism defined in the proof of Theorem 4.1. Let Φ^X be the isomorphism obtained in the same way, by specializing to the case in which $Y = X$ and $f = 1$.

Consider the (vertical) homotopy ladder arising from (2), with exact columns the long exact sequences of the evaluation fibration sequences $\Omega X \rightarrow \text{map}_*(X, X; 1) \rightarrow \text{map}(X, X; 1)$ and $\Omega Y \rightarrow \text{map}_*(X, Y; f) \rightarrow \text{map}(X, Y; f)$. From Theorem 4.4 applied to each column, we obtain an equivalence between this ladder and the *vertical* homology ladder arising from (6). It follows that, with the above notation, we have commutative cubes

$$\begin{array}{ccccc} & & H_n(\mathcal{L}_X) & \xrightarrow{H(\mathcal{L}_f)} & H_n(\mathcal{L}_Y) \\ & \nearrow \alpha_X \cong & \downarrow H(\text{ad}) & & \nearrow \alpha_Y \cong \\ \pi_n(\Omega X) \otimes \mathbb{Q} & \xrightarrow{(\Omega f)_{\#} \otimes 1} & \pi_n(\Omega Y) \otimes \mathbb{Q} & & \\ \downarrow \partial_{\#} & & \downarrow \partial_{\#} & & \downarrow H(\text{ad}_{\mathcal{L}_f}) \\ & \nearrow \Phi^X \cong & H_n(\text{Der}(\mathcal{L}_X, \mathcal{L}_X; 1)) & \xrightarrow{H((\mathcal{L}_f)_*)} & H_n(\text{Der}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)) \\ & & \downarrow \partial_{\#} & & \downarrow \partial_{\#} \\ \pi_n(\text{map}_*(X, X; 1)) \otimes \mathbb{Q} & \xrightarrow{(f_{\#})_{\#} \otimes 1} & \pi_n(\text{map}_*(X, Y; f)) \otimes \mathbb{Q} & & \end{array}$$

for $n \geq 2$. Here, α_X and α_Y denote the isomorphisms that are obtained from Lemma 4.3. Strictly speaking, we cannot conclude an equivalence of ladders from Theorem 4.4, due to the non-natural choices made in Lemma 4.3. However, an easy extension of Lemma 4.3 to the setting of an equivalence of ladders overcomes this difficulty—see [LS03, Lem.3.9] for details.

Switching now to horizontal ladders, we may use top and bottom faces of the displayed cubes, together with the extension of Lemma 4.3 referred to already, to obtain the desired equivalence of ladders. \square

Remark 6.2. Notice that Theorem 6.1 contains a description of the long exact rational homotopy sequences of a general map f , and of the induced map f_* . See [LS03, Th.3.3, Th.3.5] for the corresponding descriptions of these sequences in terms of Sullivan minimal models.

In Definition 5.1, we have defined evaluation subgroups of a map of DG Lie algebras, and of a DG Lie algebra. Our cast of characters that appear in the G -sequence is completed by the following:

Definition 6.3. Let $\psi: L \rightarrow K$ be a DG Lie algebra map. The n th relative evaluation subgroup $G_n^{rel}(K, L; \psi)$ of ψ is the subgroup

$$G_n^{rel}(K, L; \psi) = \ker\{H(\text{ad}_\psi, \text{ad}): H_{n-1}(\text{Rel}(\psi)) \rightarrow H_{n-1}(\text{Rel}(\psi_*))\}$$

of $H_n(\text{Rel}(\psi))$. The G -sequence of ψ is the sequence of kernels from the commutative ladder (5). That is, the sequence

$$\cdots \xrightarrow{H(P)} G_n(L) \xrightarrow{H(\psi)} G_n(K, L; \psi) \xrightarrow{H(J)} G_n^{rel}(K, L; \psi) \xrightarrow{H(P)} \cdots$$

Corollary 6.4. Let $f: X \rightarrow Y$ be a map between simply connected CW complexes of finite type, with X finite. Then the rationalization of the G -sequence of f

$$\cdots \longrightarrow G_n(X) \xrightarrow{\Omega_{f\#} \otimes \mathbb{Q}} G_n(Y, X; f) \otimes \mathbb{Q} \xrightarrow{j} G_n^{rel}(Y, X; f) \otimes \mathbb{Q} \longrightarrow \cdots$$

down to the term $G_3(Y, X; f) \otimes \mathbb{Q}$ is equivalent to the G -sequence of the Quillen model $\mathcal{L}_f: \mathcal{L}_X \rightarrow \mathcal{L}_Y$ of f ,

$$\cdots \longrightarrow G_n(\mathcal{L}_X) \xrightarrow{H(\mathcal{L}_f)} G_n(\mathcal{L}_Y, \mathcal{L}_X; \mathcal{L}_f) \xrightarrow{H(J)} G_n^{rel}(\mathcal{L}_Y, \mathcal{L}_X; \mathcal{L}_f) \longrightarrow \cdots$$

down to the term $G_3(\mathcal{L}_Y, \mathcal{L}_X; \mathcal{L}_f)$.

Proof. This follows immediately from Theorem 6.1, since equivalent ladders have equivalent kernel (and image) sequences. \square

In particular, we can add the following to Theorem 5.2

Corollary 6.5. Let $f: X \rightarrow Y$ be a map between simply connected CW complexes of finite type with X finite. Then the rationalized relative evaluation subgroup $G_n^{rel}(Y, X; f) \otimes \mathbb{Q}$ is isomorphic to the relative Gottlieb group of the Quillen model $\mathcal{L}_f: \mathcal{L}_X \rightarrow \mathcal{L}_Y$ of f as in Definition 6.3. That is, for $n \geq 4$

$$G_n^{rel}(Y, X; f) \otimes \mathbb{Q} \cong \ker\{H(\text{ad}_{\mathcal{L}_f}, \text{ad}): H_{n-1}(\text{Rel}(\mathcal{L}_f)) \rightarrow H_{n-1}(\text{Rel}(\mathcal{L}_f^*))\}.$$

Remark 6.6. With a little more work, the equivalence of Corollary 6.4, and hence Corollary 6.5, may be extended to the $G_3^{rel}(Y, X; f)$ term. Under our hypotheses, $G_2(X) \otimes \mathbb{Q} = 0$, and there seems little to be gained by trying to extend Corollary 6.4 beyond this point (cf. the remarks on the low-end terms in [LS03, Rem.3.11]).

As an application of the above, we consider the question of exactness of the G -sequence for cellular extensions. The first example of inexactness for a cellular inclusion of finite complexes was given in [WL88a]. Observe that rational examples of non-exactness delocalize, since exactness in the ordinary setting implies exactness after rationalization.

We focus on the rationalized G -sequence at the $G_n(X)$ term. Following Lee and Woo [LW93], define the ω -homology of a map $f: X \rightarrow Y$ at this term by setting

$$H_n^{a\omega}(Y, X; f) = \frac{\ker\{f_\#: G_n(X) \rightarrow G_n(Y, X; f)\}}{\text{im}\{p: G_{n+1}^{rel}(Y, X; f) \rightarrow G_n(X)\}}.$$

For a single cell-attachment, the following is a complete result for the lowest degree in which the (rationalized) G -sequence can be non-exact at the $G_n(X)$ -term.

Theorem 6.7. Let X be a finite CW complex of dimension n , and $Y = X \cup_\alpha e^{m+1}$ for some $\alpha \in \pi_m(X)$. Suppose the following three conditions hold:

- (1) $\alpha_{\mathbb{Q}} \neq 0$;

- (2) $\alpha_{\mathbb{Q}} \in G_m(X_{\mathbb{Q}})$;
(3) Y is not rationally equivalent to a point.

Then the G -sequence of the inclusion $i: X \rightarrow Y$ is non-exact at the $G_m(X)$ term, and $H_m^{a\omega}(Y, X; i) \otimes \mathbb{Q} = \mathbb{Q}$. Conversely, if any of (1)–(3) do not hold, then $H_m^{a\omega}(Y, X; i) \otimes \mathbb{Q} = 0$, that is,

$$G_{m+1}^{rel}(Y, X; i) \otimes \mathbb{Q} \xrightarrow{p \otimes \mathbb{Q}} G_m(X) \otimes \mathbb{Q} \xrightarrow{i_{\#} \otimes \mathbb{Q}} G_m(Y, X; i) \otimes \mathbb{Q}$$

is exact.

Proof. Clearly, the kernel of $i_{\#} \otimes \mathbb{Q}: \pi_m(X_{\mathbb{Q}}) \rightarrow \pi_m(Y_{\mathbb{Q}})$ is the subspace $\langle \alpha_{\mathbb{Q}} \rangle$ of $\pi_m(X_{\mathbb{Q}})$. If $\alpha_{\mathbb{Q}} = 0$, then $i_{\#} \otimes \mathbb{Q}: \pi_m(X_{\mathbb{Q}}) \rightarrow \pi_m(Y_{\mathbb{Q}})$ is injective, and if $\alpha_{\mathbb{Q}} \notin G_m(X_{\mathbb{Q}})$, then the restriction of $i_{\#} \otimes \mathbb{Q}$ to $G_m(X_{\mathbb{Q}})$ of $i_{\#}$ is injective. In either case, $H_m^{a\omega}(Y, X; i) \otimes \mathbb{Q} = 0$ by definition.

So suppose that (1) and (2) hold. We show that $G_{m+1}^{rel}(Y, X; i) \otimes \mathbb{Q} = 0$ if (3) also holds. For this, we use Corollary 6.5 and show that $H(\text{ad}_{\mathcal{L}_i}, \text{ad}): H_m(\text{Rel}(\mathcal{L}_i)) \rightarrow H_m(\text{Rel}((\mathcal{L}_i)_*))$ is injective.

We may write the Quillen model of $i: X \rightarrow Y$ as an inclusion $\mathcal{L}_i: \mathcal{L}_X \rightarrow \mathcal{L}_Y$, with $\mathcal{L}_X = \mathbb{L}(W)$, $\mathcal{L}_Y = \mathbb{L}(W) \sqcup \mathbb{L}(y)$, and differentials $d_Y|_{\mathbb{L}(W)} = d_X$ and $d_Y(y) = \chi \in \mathbb{L}(W)$ a cycle of degree $(m-1)$ in \mathcal{L}_X whose homology class represents $\alpha_{\mathbb{Q}}$ via the isomorphism $H_{m-1}(\mathcal{L}_X) \cong \pi_m(X) \otimes \mathbb{Q}$ (cf. [FHT01, Sec.24(d)]).

Any cycle $\zeta \in \text{Rel}_m(\mathcal{L}_i) = (\mathcal{L}_Y)_m \oplus (\mathcal{L}_X)_{m-1}$ may be written $\zeta = (\lambda y + \xi, d_Y(\lambda y + \xi))$ for $\lambda \in \mathbb{Q}$ and $\xi \in (\mathcal{L}_X)_m$. Suppose that $H(\text{ad}_{\mathcal{L}_i}, \text{ad})(\langle \zeta \rangle) = 0$. Then $(\text{ad}_{\mathcal{L}_i}, \text{ad})(\zeta) = \delta(\varphi, \theta)$ for some $(\varphi, \theta) \in \text{Rel}_{m+1}((\mathcal{L}_i)_*)$. In particular, we have $\text{ad}_{\mathcal{L}_i}(\lambda y + \xi) = (\mathcal{L}_i)_*(\theta) - D\varphi \in \text{Der}_m(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_i)$. We now use (3), if necessary, to choose an indecomposable $w \in W$ such that χ and w are linearly independent in \mathcal{L}_X . (A choice is only necessary when $d_Y(y)$ is indecomposable. If χ is decomposable, then any indecomposable of W will do.) On this indecomposable, we evaluate as follows:

$$(7) \quad \text{ad}_{\mathcal{L}_i}(\lambda y + \xi)(w) = \lambda[y, w] + [\xi, w]$$

and

$$(8) \quad ((\mathcal{L}_i)_*(\theta) - D\varphi)(w) = \mathcal{L}_i \circ \theta(w) - d_Y \varphi(w) + (-1)^{m+1} \varphi d_X(w)$$

We claim that all terms of (8) are independent of $[y, w]$. First, $\mathcal{L}_i \circ \theta(w) \in \mathbb{L}(W)$. Next, $\varphi(w) \in (\mathbb{L}(W) \sqcup \mathbb{L}(y))_{m+1+|w|}$ may contain terms in $\mathbb{L}(W)$, quadratic terms $[y, w']$ for $w' \in W$ or $[y, y]$ (if m is odd), or terms involving y of bracket-length at least three. Now $d_Y([y, w']) = [\chi, w'] \pm [y, d_Y(w')]$, and $d_Y([y, y]) = 2[\chi, y]$ (if m is odd). By choice, χ and w are linearly independent, and also d_X is decomposable. Hence, all terms of $d_Y \varphi(w)$ are linearly independent of $[y, w]$. Finally, $d_X(w)$ is decomposable and in $\mathbb{L}(W)_{|w|-1}$. Hence, $\varphi d_X(w)$ is in the ideal of $\mathbb{L}(W) \sqcup \mathbb{L}(y)$ generated by elements of W of degree $\leq |w| - 2$. This proves the claim, and shows that if (7) and (8) agree, which must be the case if $H(\text{ad}_{\mathcal{L}_i}, \text{ad})(\langle \zeta \rangle) = 0$, then we have $\lambda = 0$. We have shown that the kernel of $H(\text{ad}_{\mathcal{L}_i}, \text{ad}): H_m(\text{Rel}(\mathcal{L}_i)) \rightarrow H_m(\text{Rel}((\mathcal{L}_i)_*))$ consists of classes represented by cycles of the form $\zeta = (\xi, d_Y(\xi))$ for $\xi \in (\mathcal{L}_X)_m$. Since $d_Y(\xi) = d_X(\xi)$, we may write $\delta(0, \xi) = (\xi, d_X(\xi)) = \zeta$. That is, $H(\text{ad}_{\mathcal{L}_i}, \text{ad}): H_m(\text{Rel}(\mathcal{L}_i)) \rightarrow H_m(\text{Rel}((\mathcal{L}_i)_*))$ is injective and hence by Corollary 6.5, $G_{m+1}^{rel}(Y, X; i) \otimes \mathbb{Q} = 0$. This establishes the first assertion of the theorem.

The second assertion will be established when we handle the case in which (1) and (2) hold, but (3) does not, that is, in which $Y \simeq_{\mathbb{Q}} *$. In this case, we must have $X \simeq_{\mathbb{Q}} S^n$, $m = n$, and $\alpha \in \pi_n(S^n)$ non-trivial. The Quillen model of Y is then $\mathbb{L}(w, y)$ with $|w| = n - 1$, $|y| = n$, and $d_Y(y) = w$. (Notice that this is precisely the case in which we cannot choose a second indecomposable of W independent of χ .) In addition, since we are assuming that (2) holds, we must have m odd, and hence $|w|$ is even and $\mathbb{L}(w)$ is an abelian Lie algebra. It is easy to see that $(y, w) \in \text{Rel}_m(\mathcal{L}_i)$ is a cycle that cannot be a δ -boundary. Define an \mathcal{L}_i -derivation $\varphi \in \text{Der}_{m+1}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_i)$ by setting $\varphi(w) = -\frac{1}{2}[y, y]$. A straightforward check shows that $(\text{ad}_{\mathcal{L}_i}, \text{ad})(y, w) = (\text{ad}_{\mathcal{L}_i}(y), 0) = \delta(\varphi, 0)$ in $\text{Rel}_m((\mathcal{L}_i)_*)$. That is, $\langle y, w \rangle$ represents an element in $G_m^{\text{rel}}(\mathcal{L}_Y, \mathcal{L}_X; \mathcal{L}_i)$. Furthermore, $H(P)(\langle y, w \rangle) = \langle w \rangle$, and so $H(P): G_m^{\text{rel}}(\mathcal{L}_Y, \mathcal{L}_X; \mathcal{L}_i) \rightarrow G_m(\mathcal{L}_X)$ is onto. The result follows. \square

We conclude with an example of vanishing rational ω -homology. In the following result, we use the ideas discussed before Theorem 5.9, concerning the notion of a coformal map.

Theorem 6.8. *Let $f: X \rightarrow Y$ be a coformal map between CW complexes of finite type, with X finite. Then $H_n^{\omega}(Y, X; f) \otimes \mathbb{Q} = 0$, that is,*

$$G_{n+1}^{\text{rel}}(Y, X; f) \otimes \mathbb{Q} \xrightarrow{p \otimes \mathbb{Q}} G_n(X) \otimes \mathbb{Q} \xrightarrow{f \# \otimes \mathbb{Q}} G_n(Y, X; f) \otimes \mathbb{Q}$$

is exact, for each $n \geq 3$.

Proof. We will use Corollary 6.4 and show that $\ker\{H(\mathcal{L}_f)\} \subseteq \text{im}\{H(P)\}$. From Definition 5.7, we assume that both \mathcal{L}_X and \mathcal{L}_Y admit upper (second) gradings with the properties described in Definition 5.6, and that \mathcal{L}_f preserves upper degrees. Let $\alpha = \langle \xi \rangle \in G_n(\mathcal{L}_X)$ satisfy $H(\mathcal{L}_f)(\alpha) = 0$. We assume ξ is of upper degree zero in the bigraded model for \mathcal{L}_X . Furthermore, $\mathcal{L}_f(\xi) = d_Y(y)$ for some $y \in (\mathcal{L}_Y)_{n+1}$ that we may assume is of upper degree 1. Since α is Gottlieb, $\text{ad}(\xi) = D(\psi)$ for some derivation $\psi \in \text{Der}_{n+1}(\mathcal{L}_X, \mathcal{L}_X; 1)$ and using the coformality of \mathcal{L}_X again, we may assume ψ increases upper degree homogeneously by 1. The pair $(y, \xi) \in \text{Rel}_{n+1}(\mathcal{L}_f)$ is a δ -cycle that satisfies $P(y, \xi) = \xi$. We now show that (y, ξ) represents an element in $G_{n+1}^{\text{rel}}(\mathcal{L}_Y, \mathcal{L}_X; \mathcal{L}_f)$, that is, we show the pair $(\text{ad}_{\mathcal{L}_f}(y), \text{ad}(\xi))$ bounds in $\text{Rel}_{n+1}((\mathcal{L}_f)_*)$. Set $\Theta = (\mathcal{L}_f)_*(\psi) - \text{ad}_{\mathcal{L}_f}(y)$, a derivation in $\text{Der}_{n+1}(\mathcal{L}_Y, \mathcal{L}_X; \mathcal{L}_f)$. It is direct to check that $D(\Theta) = d_Y \circ \Theta - (-1)^{n+1}\Theta \circ d_X = 0$. Moreover, Θ increases upper degree homogeneously by 1. Now adapt the proof of Theorem 5.9 to the current situation, by replacing the derivation $\text{ad}_{\mathcal{L}_f}(\xi)$ in that proof by ψ . The inductive argument used there now results in a derivation $\theta \in \text{Der}_{n+2}(\mathcal{L}_X, \mathcal{L}_Y; \mathcal{L}_f)$, constructed in the same way only increasing upper degree by 2, that satisfies $\Theta = D(\theta)$. Then the pair $(\theta, \psi) \in \text{Rel}_{n+2}((\mathcal{L}_f)_*)$ satisfies $\delta(\theta, \psi) = (\text{ad}_{\mathcal{L}_f}(y), \text{ad}(\xi))$. \square

APPENDIX A. SOME DG LIE ALGEBRA HOMOTOPY THEORY

In this appendix, we present some DG Lie algebra homotopy theory. Our main references for this material are [Tan83, Ch.II.5] and part IV of [FHT01]. For completeness and convenience, we recall the basic notions here. The main point of the appendix is to provide details for some facts used in a crucial way to establish our main results. Since this is a technical appendix, we assume a greater degree of familiarity with techniques from rational homotopy theory than in the main body of the paper.

The algebraic notion of homotopy that we use here is “left homotopy” of DG Lie algebra maps, defined in terms of a suitable cylinder object for a free DG Lie algebra $\mathbb{L}(V; d)$. This is another free DG Lie algebra denoted $\mathbb{L}(V)_I$, together with inclusions $\lambda_0, \lambda_1: \mathbb{L}(V) \rightarrow \mathbb{L}(V)_I$ and a projection $p: \mathbb{L}(V)_I \rightarrow \mathbb{L}(V)$ that together satisfy $p \circ \lambda_i = 1$ for $i = 0, 1$. As a DG Lie algebra, we have $\mathbb{L}(V)_I = \mathbb{L}(V, sV, \widehat{V}; D)$, with sV the suspension of V , and \widehat{V} an isomorphic copy of V . The differential D in $\mathbb{L}(V)_I$ extends d , so that $\mathbb{L}(V)$ is a sub-DG Lie algebra, and is defined on the other generators as $D(sv) = \widehat{v}$ and $D(\widehat{v}) = 0$, for $v \in V$. The inclusion $\lambda_0: \mathbb{L}(V) \rightarrow \mathbb{L}(V)_I$ is the obvious inclusion of the sub-DG Lie algebra $\mathbb{L}(V)$. The inclusion $\lambda_1: \mathbb{L}(V) \rightarrow \mathbb{L}(V)_I$, on the other hand, is more involved. Define a derivation $\sigma \in \text{Der}_1(\mathbb{L}(V)_I, \mathbb{L}(V)_I; 1)$ on generators by $\sigma(v) = sv$, $\sigma(sv) = 0$, and $\sigma(\widehat{v}) = 0$, then define $\theta \in \text{Der}_0(\mathbb{L}(V)_I, \mathbb{L}(V)_I; 1)$ as $\theta = [D, \sigma] = D \circ \sigma + \sigma \circ D$. Observe that θ is a cycle, and is locally nilpotent. Therefore, exponentiating θ gives a (DG) automorphism $\exp(\theta): \mathbb{L}(V)_I \rightarrow \mathbb{L}(V)_I$. Finally, define $\lambda_1 = \exp(\theta) \circ \lambda_0$. The projection p is defined in the obvious way as $p(v) = v$, $p(sv) = 0$, and $p(\widehat{v}) = 0$. Given a pair of DG Lie algebra maps $\phi, \psi: \mathbb{L}(V) \rightarrow L$, we say ϕ is homotopic to ψ if there exists a map $\mathcal{H}: \mathbb{L}(V)_I \rightarrow L$ such that $\mathcal{H} \circ \lambda_0 = \phi$ and $\mathcal{H} \circ \lambda_1 = \psi$. In this case, we say \mathcal{H} is a (DG) homotopy from ϕ to ψ .

In addition to the notation established above, which we take as fixed for this appendix, we will use the following conventions for DG Lie algebra maps: We will generally suppress differentials from our notation. Recall the model for $S^n \times X$ from Section 3. We will use J to denote either inclusion $\mathbb{L}(W) \rightarrow \mathbb{L}(W, v, W')$ or $\mathbb{L}(W, v) \rightarrow \mathbb{L}(W, v, W')$, and J_I to denote the corresponding inclusions of cylinder objects. Namely, in the first case, $J_I(w) = w$, $J_I(sw) = sw$, and $J_I(\widehat{w}) = \widehat{w}$, for $w \in W$. We also fix some notation for maps of spaces that we use in this appendix. Let $i_1: S^n \rightarrow S^n \times X$ and $i_2: X \rightarrow S^n \times X$ denote the inclusions, and π_1 the projection onto the first factor of a product of spaces.

The basic correspondence between the notions of homotopy in the topological and algebraic settings is as follows: Maps $f, g: X \rightarrow Y$ of rational spaces are homotopic if and only if their Quillen models $\mathcal{L}_f, \mathcal{L}_g: \mathcal{L}_X \rightarrow \mathcal{L}_Y$ are homotopic in the sense just defined. For the proof of Theorem 4.1, however, we need finer detail than this basic correspondence. Consider the argument to show Φ' is well-defined, for example. Suppose $a, b: S^n \rightarrow \text{map}_*(X, Y; f)$ are homotopic representatives of the same homotopy element, and that their adjoints are $A, B: S^n \times X \rightarrow Y$. Then A and B are homotopic relative to $S^n \vee X$. Indeed, if H denotes the homotopy from A to B that is adjoint to the homotopy from a to b , then the following diagram commutes:

$$(9) \quad \begin{array}{ccc} (S^n \vee X) \times I & \xrightarrow{\pi_1} & S^n \vee X \\ \downarrow (i_1 | i_2) \times 1 & & \downarrow (*|f) \\ S^n \times X \times I & \xrightarrow{H} & Y \end{array}$$

For our argument that Φ' is well-defined, it is crucial that we may assume the DG Lie algebra homotopy that corresponds to H is restricted in a certain way. Specifically, if \mathcal{H} denotes the DG Lie algebra homotopy that corresponds to H ,

then we require that the following diagram commute:

$$(10) \quad \begin{array}{ccc} \mathbb{L}(W, v)_I & \xrightarrow{p} & \mathbb{L}(W, v) \\ J_I \downarrow & & \downarrow (\mathcal{L}_f|_0) \\ \mathbb{L}(W, v, W')_I & \xrightarrow{\mathcal{H}} & \mathcal{L}_Y \end{array}$$

We also need a similar fact for homotopy elements of the unbased mapping space to establish that Ψ' is well-defined. Whilst this translation is intuitively plausible, there are some technical details to be checked.

Our starting point for this is the corresponding result in the Sullivan model setting, which we have proved in our earlier paper. We assume familiarity with the usual notation in that setting, that is, rational homotopy theory from the DG algebra point of view (see [FHT01]). The algebraic notion of homotopy that we used in [LS03] is “right homotopy” defined in terms of a suitable path object for a free DG algebra. For $\Lambda(V)$, this consists of maps

$$(11) \quad \Lambda(V) \xrightarrow{j} \Lambda(V) \otimes \Lambda(t, dt) \xrightarrow{p_0, p_1} \Lambda(V)$$

that satisfy $p_i \circ j = 1$ for $i = 0, 1$. Here, j is the inclusion $j(\chi) = \chi \otimes 1$. The projections are defined by $p_i(t) = i$, $p_i(dt) = 0$, and $p_i(\chi) = \chi$ for $\chi \in \Lambda(V)$. Given a pair of DG algebra maps $\phi, \psi: \Lambda(W) \rightarrow \Lambda(V)$, we say ϕ is *homotopic to* ψ if there exists a map $\mathcal{G}: \Lambda(W) \rightarrow \Lambda(V) \otimes \Lambda(t, dt)$ such that $p_0 \circ \mathcal{G} = \phi$ and $p_1 \circ \mathcal{G} = \psi$. In this case, we say \mathcal{G} is a (DG) homotopy from ϕ to ψ .

We fix more notation: Let \mathcal{M}_X denote the Sullivan minimal model of a space X . Recall that the minimal model of a product of spaces is the tensor product of their minimal models, so that $\mathcal{M}_{S^n} \otimes \mathcal{M}_X$ is a Sullivan model for $S^n \times X$. Let $\mathcal{M}_f: \mathcal{M}_Y \rightarrow \mathcal{M}_X$ denote a Sullivan minimal model of a map $f: X \rightarrow Y$. Let $q_1: \mathcal{M}_{S^n} \otimes \mathcal{M}_X \rightarrow \mathcal{M}_{S^n}$ and $q_2: \mathcal{M}_{S^n} \otimes \mathcal{M}_X \rightarrow \mathcal{M}_X$ denote the projections.

Lemma A.1. *Let $f: X \rightarrow Y$ be a map with a fixed choice of Sullivan model $\mathcal{M}_f: \mathcal{M}_Y \rightarrow \mathcal{M}_X$. Let $A, B: S^n \times X \rightarrow Y$ be maps that restrict to $A \circ i_2 = B \circ i_2 = f: X \rightarrow Y$.*

(i) *Suppose $A \sim B$ via a homotopy H relative to X , that is, suppose the diagram*

$$(12) \quad \begin{array}{ccc} X \times I & \xrightarrow{\pi_1} & X \\ i_2 \times 1 \downarrow & & \downarrow f \\ S^n \times X \times I & \xrightarrow{H} & Y \end{array}$$

commutes. Then there exists a homotopy $\mathcal{G}: \mathcal{M}_Y \rightarrow \mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt)$ from \mathcal{M}_A to \mathcal{M}_B that is “relative to \mathcal{M}_X ,” in the sense that the diagram

$$(13) \quad \begin{array}{ccc} \mathcal{M}_Y & \xrightarrow{\mathcal{G}} & \mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt) \\ \mathcal{M}_f \downarrow & & \downarrow q_2 \otimes 1 \\ \mathcal{M}_X & \xrightarrow{j} & \mathcal{M}_X \otimes \Lambda(t, dt) \end{array}$$

(strictly) commutes.

- (ii) Suppose further that A and B restrict to $A \circ i_1 = B \circ i_1 = *: S^n \rightarrow Y$, and the homotopy H is also relative to S^n , so that (9) commutes. Then there exists a homotopy $\mathcal{G}: \mathcal{M}_Y \rightarrow \mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt)$ from \mathcal{M}_A to \mathcal{M}_B that is “relative to $\mathcal{M}_{S^n \vee X}$,” in the sense that the diagram

$$\begin{array}{ccc} \mathcal{M}_Y & \xrightarrow{\mathcal{G}} & \mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt) \\ (\varepsilon, \mathcal{M}_f) \downarrow & & \downarrow (q_1 \otimes 1, q_2 \otimes 1) \\ \mathcal{M}_{S^n} \oplus \mathcal{M}_X & \xrightarrow{j \oplus j} & \mathcal{M}_{S^n} \otimes \Lambda(t, dt) \oplus \mathcal{M}_X \otimes \Lambda(t, dt) \end{array}$$

(strictly) commutes ($\varepsilon: \mathcal{M}_Y \rightarrow \mathcal{M}_{S^n}$ denotes the map that is zero in positive degrees).

Proof. The first point is proved in [LS03, Lem.A.2]. The argument given there is readily adapted to prove the second point. \square

We will carefully translate this result into the Quillen model setting via the so-called Quillen functor \mathcal{L} . This is a functor from the category of differential graded algebras to the category of differential graded Lie algebras (see [FHT01, Sec.22(e)] or [Tan83, I.1.(7)] for details). Since we only use general properties of this functor, we do not recall its definition here. We do recall that it preserves quasi-isomorphisms and, as a consequence of its definition, takes an injective DG algebra map to a surjective DG Lie algebra map.

Sullivan and Quillen models of a map are related via the Quillen functor. Suppose $f: X \rightarrow Y$ has Sullivan minimal model $\mathcal{M}_f: \mathcal{M}_Y \rightarrow \mathcal{M}_X$. Applying \mathcal{L} gives $\mathcal{L}(\mathcal{M}_f): \mathcal{L}(\mathcal{M}_X) \rightarrow \mathcal{L}(\mathcal{M}_Y)$. Now suppose $\rho_X: \mathcal{L}_X \rightarrow \mathcal{L}(\mathcal{M}_X)$ and $\rho_Y: \mathcal{L}_Y \rightarrow \mathcal{L}(\mathcal{M}_Y)$ are given minimal models. Then any map $\mathcal{L}_f: \mathcal{L}_X \rightarrow \mathcal{L}_Y$ such that $\rho_Y \circ \mathcal{L}_f$ and $\mathcal{L}(\mathcal{M}_f) \circ \rho_X$ are homotopic is a Quillen minimal model for f . Recall that we obtain such maps as follows: There is a standard way of converting the quasi-isomorphism ρ_Y into a surjective quasi-isomorphism. Namely, let $E(\mathcal{L}(\mathcal{M}_Y))$ denote the acyclic, free DG Lie algebra $\mathbb{L}(V, s^{-1}V)$, with V a vector space isomorphic to $\mathcal{L}(\mathcal{M}_Y)$ and $d(v) = s^{-1}v$ for $v \in V$. Let $\widehat{\rho}_Y: \mathcal{L}_Y \sqcup E(\mathcal{L}(\mathcal{M}_Y)) \rightarrow \mathcal{L}(\mathcal{M}_Y)$ be the map that extends ρ_Y on \mathcal{L}_Y , and maps $\widehat{\rho}_Y(v) = v$, $\widehat{\rho}_Y(s^{-1}v) = dv$ for $v \in V$. Then we may lift $\mathcal{L}(\mathcal{M}_f) \circ \rho_X$ through the surjective quasi-isomorphism $\widehat{\rho}_Y$,

$$\begin{array}{ccc} \mathbb{L}(W) & \xrightarrow{\phi_f} & \mathcal{L}_Y \sqcup E(\mathcal{L}(\mathcal{M}_Y)) \xrightarrow{\pi} \mathcal{L}_Y \\ \rho_X \downarrow \simeq & & \simeq \downarrow \widehat{\rho}_Y \swarrow \rho_Y \\ \mathcal{L}(\mathcal{M}_X) & \xrightarrow{\mathcal{L}(\mathcal{M}_f)} & \mathcal{L}(\mathcal{M}_Y) \end{array}$$

to obtain a map ϕ_f that satisfies $\widehat{\rho}_Y \circ \phi_f = \mathcal{L}(\mathcal{M}_f) \circ \rho_X$ (see [Tan83, II.5.(13)] or [FHT01, Prop.22.11] for the standard lifting lemma). A Quillen minimal model for f is then obtained by composing with the projection π to give $\mathcal{L}_f = \pi \circ \phi_f$.

Homotopies are also related via the Quillen functor. However, since it is contravariant, the Quillen functor translates a right homotopy of DG algebra maps into a left homotopy of DG Lie algebra maps, in the following way. By applying \mathcal{L} to (11), we obtain maps

$$\mathcal{L}(\mathcal{M}_X) \xrightarrow{\mathcal{L}(p_0), \mathcal{L}(p_1)} \mathcal{L}(\mathcal{M}_X \otimes \Lambda(t, dt)) \xrightarrow{\mathcal{L}(j)} \mathcal{L}(\mathcal{M}_X)$$

that satisfy $\mathcal{L}(j) \circ \mathcal{L}(p_i) = \mathcal{L}(1) = 1$ for $i = 0, 1$. Since j is an injective quasi-isomorphism, $\mathcal{L}(j)$ is a surjective quasi-isomorphism. Since $\mathcal{L}(j) \circ \mathcal{L}(p_0) \circ \rho_X = \rho_X = \mathcal{L}(j) \circ \mathcal{L}(p_1) \circ \rho_X: \mathcal{L}_X \rightarrow \mathcal{L}(\mathcal{M}_X)$, the linear difference $\mathcal{L}(p_1) \circ \rho_X - \mathcal{L}(p_0) \circ \rho_X: \mathcal{L}_X \rightarrow \mathcal{L}(\mathcal{M}_X \otimes \Lambda(t, dt))$ has image contained in the DG ideal $\ker(\mathcal{L}(j))$ of $\mathcal{L}(\mathcal{M}_X \otimes \Lambda(t, dt))$. Now suppose $\mathcal{L}_X = \mathbb{L}(W)$. Since $\mathcal{L}(j)$ is a surjective quasi-isomorphism, $\ker(\mathcal{L}(j))$ is an acyclic DG ideal. Hence, by a standard argument ([Tan83, Prop.II.5(4)]), we may construct a left-homotopy $G: \mathbb{L}(W)_I \rightarrow \mathcal{L}(\mathcal{M}_X \otimes \Lambda(t, dt))$ from $\mathcal{L}(p_0) \circ \rho_X$ to $\mathcal{L}(p_1) \circ \rho_X$ that in addition satisfies $G(\mathbb{L}(sW, \widehat{W})) \subseteq \ker(\mathcal{L}(j))$ (this last point is key). That is, we have the following commutative diagram,

$$\begin{array}{ccc}
 \mathbb{L}(W) & & \\
 \downarrow \lambda_0 & \searrow \mathcal{L}(p_0) \circ \rho_X & \\
 \mathbb{L}(W)_I & \xrightarrow{G} & \mathcal{L}(\mathcal{M}_X \otimes \Lambda(t, dt)) \\
 \uparrow \lambda_1 & \nearrow \mathcal{L}(p_1) \circ \rho_X & \\
 \mathbb{L}(W) & &
 \end{array}$$

for which $G(\mathbb{L}(sW, \widehat{W})) \subseteq \ker(\mathcal{L}(j))$. Finally, suppose that $\mathcal{G}: \mathcal{M}_Y \rightarrow \mathcal{M}_X \otimes \Lambda(t, dt)$ is a right homotopy of DG algebra maps from ϕ to ψ . Then $\mathcal{L}(\mathcal{G}) \circ G: \mathbb{L}(W)_I \rightarrow \mathcal{L}(\mathcal{M}_Y)$ is a left homotopy of DG Lie algebra maps from $\mathcal{L}(\phi) \circ \rho_X$ to $\mathcal{L}(\psi) \circ \rho_X$.

We now come to the main point of the appendix.

Lemma A.2. *Let $f: X \rightarrow Y$ be a map with a fixed choice of Quillen model $\mathcal{L}_f: \mathcal{L}_X \rightarrow \mathcal{L}_Y$. Let $A, B: S^n \times X \rightarrow Y$ be maps that restrict to $A \circ i_2 = B \circ i_2 = f: X \rightarrow Y$. Suppose $\mathbb{L}(W)$ is the Quillen minimal model of X , and $\mathbb{L}(W, v, W')$ is the Quillen model of $S^n \times X$ (see Corollary 3.5).*

- (i) *Suppose $A \sim B$ via a homotopy H relative to X , that is, suppose the diagram (12) commutes. Then there exists a DG homotopy $\mathcal{H}: \mathbb{L}(W, v, W')_I \rightarrow \mathcal{L}_Y$ from \mathcal{L}_A to \mathcal{L}_B that is “relative to \mathcal{L}_X ,” in the sense that the diagram*

$$\begin{array}{ccc}
 \mathbb{L}(W)_I & \xrightarrow{p} & \mathbb{L}(W) \\
 \downarrow J_I & & \downarrow \mathcal{L}_f \\
 \mathbb{L}(W, v, W')_I & \xrightarrow{\mathcal{H}} & \mathcal{L}_Y
 \end{array}$$

(strictly) commutes.

- (ii) *Suppose further that A and B restrict to $A \circ i_1 = B \circ i_1 = *: S^n \rightarrow Y$, and the homotopy H is also relative to S^n , so that (9) commutes. Then there exists a DG homotopy $\mathcal{H}: \mathbb{L}(W, v, W')_I \rightarrow \mathcal{L}_Y$ from \mathcal{L}_A to \mathcal{L}_B that is “relative to $\mathcal{L}_{S^n \vee X}$,” in the sense that the diagram (10) (strictly) commutes.*

Proof. (i) In the Sullivan model setting, path objects for \mathcal{M}_X and $\mathcal{M}_{S^n} \otimes \mathcal{M}_X$ are related as in the commutative diagrams

$$\begin{array}{ccccc} \mathcal{M}_{S^n} \otimes \mathcal{M}_X & \xrightarrow{j} & \mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt) & \xrightarrow{p_i} & \mathcal{M}_{S^n} \otimes \mathcal{M}_X \\ q_2 \downarrow & & \downarrow q_2 \otimes 1 & & \downarrow q_2 \\ \mathcal{M}_X & \xrightarrow{j} & \mathcal{M}_X \otimes \Lambda(t, dt) & \xrightarrow{p_i} & \mathcal{M}_X \end{array}$$

for $i = 0, 1$. Applying \mathcal{L} to this diagram, and following the construction of the map G as described above, we obtain homotopies $G_X: \mathbb{L}(W)_I \rightarrow \mathcal{L}(\mathcal{M}_X \otimes \Lambda(t, dt))$ and $G_{S^n \times X}: \mathbb{L}(W, v, W')_I \rightarrow \mathcal{L}(\mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt))$. By constructing $G_{S^n \times X}$ so as to extend G_X on $\mathbb{L}(W)_I$, we may assume these maps are compatible, in the sense that $\mathcal{L}(q_2 \otimes 1) \circ G_X = G_{S^n \times X} \circ J_I: \mathbb{L}(W)_I \rightarrow \mathcal{L}(\mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt))$. Combining this with the diagram obtained from applying \mathcal{L} to (13), we obtain a commutative diagram

$$\begin{array}{ccccc} \mathbb{L}(W)_I & \xrightarrow{G_X} & \mathcal{L}(\mathcal{M}_X \otimes \Lambda(t, dt)) & \xrightarrow{\mathcal{L}(j)} & \mathcal{L}(\mathcal{M}_X) \\ J_I \downarrow & & \downarrow \mathcal{L}(q_2 \otimes 1) & & \downarrow \mathcal{L}(\mathcal{M}_f) \\ \mathbb{L}(W, v, W')_I & \xrightarrow{G_{S^n \times X}} & \mathcal{L}(\mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt)) & \xrightarrow{\mathcal{L}(g)} & \mathcal{L}(\mathcal{M}_Y) \end{array}$$

Furthermore, from the construction of G_X , we have $\mathcal{L}(j) \circ G_X(sW, \widehat{W}) = 0$ so that $\mathcal{L}(j) \circ G_X = \rho_X \circ p$. It remains to check that we can preserve the properties of this diagram in lifting the homotopies to \mathcal{L}_Y .

So suppose that \mathcal{L}_f is a given Quillen model that arises in the way described above from $\phi_f: \mathbb{L}(W) \rightarrow \mathcal{L}_Y \sqcup E(\mathcal{L}(\mathcal{M}_Y))$. With the above ingredients, we obtain a diagram

$$\begin{array}{ccc} \mathbb{L}(W)_I & \xrightarrow{\phi_f \circ p} & \mathcal{L}_Y \sqcup E(\mathcal{L}(\mathcal{M}_Y)) \\ J_I \downarrow & \nearrow \mathcal{G}' & \downarrow \simeq \widehat{\rho}_Y \\ \mathbb{L}(W, v, W')_I & \xrightarrow{\mathcal{L}(g) \circ G_{S^n \times X}} & \mathcal{L}(\mathcal{M}_Y), \end{array}$$

which commutes since $\widehat{\rho}_Y \circ \phi_f \circ p = \mathcal{L}(\mathcal{M}_f) \circ \rho_X \circ p = \mathcal{L}(\mathcal{M}_f) \circ \mathcal{L}(j) \circ G_X$. Therefore, we may lift as indicated through the surjective quasi-isomorphism $\widehat{\rho}_Y$ to obtain the homotopy \mathcal{G}' that satisfies $\widehat{\rho}_Y \circ \mathcal{G}' = \mathcal{L}(g) \circ G_{S^n \times X}$. Finally, define $\mathcal{H} = \pi \circ \mathcal{G}'$. This is a homotopy that starts at a Quillen model for A , and ends at one for B , since we have

$$\begin{aligned} \rho_Y \circ \mathcal{G} \circ \lambda_i &= \rho_Y \circ \pi \circ \mathcal{G}' \circ \lambda_i \\ &\sim \widehat{\rho}_Y \circ \mathcal{G}' \circ \lambda_i = \mathcal{L}(g) \circ G_{S^n \times X} \circ \lambda_i \\ &= \mathcal{L}(g) \circ \mathcal{L}(p_i) \circ \rho_{S^n \times X} = \mathcal{L}(p_i \circ \mathcal{G}) \circ \rho_{S^n \times X} \end{aligned}$$

which equals $\mathcal{L}(A) \circ \rho_{S^n \times X}$ for $i = 0$ and $\mathcal{L}(B) \circ \rho_{S^n \times X}$ for $i = 1$. That is, $\mathcal{G} \circ \lambda_0$ and $\mathcal{G} \circ \lambda_1$ may be taken as Quillen models for A and B , respectively.

(ii) The argument for (i) is easily adapted to establish (ii). In this case, begin by constructing a homotopy

$$G_{S^n \vee X}: \mathbb{L}(W, v)_I \rightarrow \mathcal{L}(\mathcal{M}_X \otimes \Lambda(t, dt)) \sqcup \mathcal{L}(\mathcal{M}_{S^n} \otimes \Lambda(t, dt))$$

so that $(j \sqcup j) \circ G_{S^n \vee X} = (\rho_X \sqcup \rho_{S^n}) \circ p: \mathbb{L}(W, v)_I \rightarrow \mathcal{L}(\mathcal{M}_X) \sqcup \mathcal{L}(\mathcal{M}_{S^n})$. Then extend $(\mathcal{L}(q_2 \otimes 1) \mid \mathcal{L}(q_1 \otimes 1)) \circ G_{S^n \vee X}$ to $G_{S^n \times X}: \mathbb{L}(W, v, W')_I \rightarrow \mathcal{L}(\mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt))$. The argument now proceeds as before. \square

Notice that in either case, Lemma A.2 shows that we may choose a Quillen model \mathcal{L}_A of A that restricts to \mathcal{L}_f on $\mathbb{L}(W)$ (the restriction equals \mathcal{L}_f , and is not just homotopic to it). This justifies a fact that we relied upon for the definition of the maps Φ and Ψ in Theorem 4.1.

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