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$\Omega SU(n)$ does not Split in 2 Suspensions, for $n \geq 3$

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0. Introduction

Solving a conjecture of Hopkins and Mahowald, the second author [**Ri**] showed that Mitchell's [**Mi3**] filtration $\{F_{n,k}\}_{k=1}^{\infty}$ of $\Omega SU(n)$ splits stably, analogous to the Snaith [**Sn2**] splitting of BU. Crabb and Mitchell [**C-M**] then gave similar splittings of $\Omega U(n)/O(n)$ and $\Omega U(2n)/Sp(n)$. The first filtration $F_{n,1}$ is the inclusion $\mathbb{C}P^{n-1} \subset \Omega SU(n)$, which was actually known to split off by the work of James [**Ja**], which was refined by Miller [**Mi2**]. James split $\Sigma\mathbb{C}P^{n-1}$ off $SU(n)$, with a map $\mathcal{J}\colon SU(n) \to \Omega^N\Sigma^N\left(\Sigma\mathbb{C}P^{n-1}\right)$, whose loop $\Omega(\mathcal{J}) : \Omega SU(n) \to \Omega^{N+1}\Sigma^{N+1}\mathbb{C}P^{n-1}$ splits $\mathbb{C}P^{n-1}$ stably off $\Omega SU(n)$. Cohen and Peterson [**C-P**], using Dyer-Lashof operations, showed that $N > 2$ for any such map $\mathcal{J}$. However there is no Dyer-Lashof obstruction to factoring $\Omega(\mathcal{J})$ through $\Omega^2\Sigma^2\mathbb{C}P^{n-1}$. That is, the image of $\Omega(\mathcal{J})$ consists only of monomials in $H_*\left(\mathbb{C}P^{n-1};\mathbb{Z}/2\right)$. The question arose: does there exist a map $\rho\colon \Omega SU(n) \to \Omega^2\Sigma^2\mathbb{C}P^{n-1}$ which splits off $\mathbb{C}P^{n-1}$ stably? The existence of such a map ρ would have implied the stable splitting of $\Omega SU(n)$, provided the Mitchell-Segal [**Mi3, Se1**] group completion model $\coprod_k F_{n,k} \subset \Omega U(n)$ has a $\mathcal{C}_2$-*structure* (cf. May [**Ma3**]). Following Snaith [**Sn2**], the k^{th} splitting map could have been constructed as the composite

$$\Omega SU(n) \xrightarrow{\rho} \Omega^2\Sigma^2\mathbb{C}P^{n-1} \xrightarrow{\lambda_k} Q\left(F(\mathbb{R}^2,k)\times_{\Sigma_k}\left(\mathbb{C}P^{n-1}\right)^{k^+}\right)$$
$$\xrightarrow{Q(c_2)} Q\left({F_{n,k}}^+\right) \to Q\left(F_{n,k}/F_{n,k-1}\right).$$

Here $F(\mathbb{R}^2,k)$ is the ordered configuration space of k points in $\mathbb{R}^2$, λ_k is the k^{th} group-completed Hopf invariant [**Sn1, Co, K-P, Se2**], popularized by Segal, and c_2 is one of the $\mathcal{C}_2$-structure maps. It was furthermore speculated that ρ

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could be constructed by embedding $\Omega SU(n)$ in the configuration space model $C\left(\mathbb{R}^2, \mathbb{C}P^{n-1}\right)$ for $\Omega^2\Sigma^2\mathbb{C}P^{n-1}$. We prove here that no such map ρ exists.

THEOREM 0.1. *For $n \geq 3$, there is no homotopy retraction $\Sigma^2\Omega SU(n) \to \Sigma^2\mathbb{C}P^{n-1}$ of the inclusion $\Sigma^2\mathbb{C}P^{n-1} \to \Sigma^2\Omega SU(n)$.*

We prove slightly more, that $\mathbb{C}P^{n-1}$ does not split off the second filtration $F_{n,2}$ after 2 suspensions. Earlier [**M-R**] we proved Theorem 0.1 for the case $n = 3$, so it is not surprising that Theorem 0.1 is true. We feel, however, that our proof for the general case is sufficiently interesting to justify our efforts. Our previous proof used the Barratt-Ganea-Toda relative Hopf invariant to show that the attaching map of $\Sigma^2F_{3,2}$ ($\nu\eta^2$ on the bottom cell in $\Sigma^2\mathbb{C}P^2$) was essential. We note in particular that Theorem 0.1 does not follow from our earlier proof, as there exists a map $\Sigma^2F_{3,2} \to \Sigma^2\mathbb{C}P^4$ extending the inclusion $\mathbb{C}P^2 \hookrightarrow \mathbb{C}P^4$.

Our proof is based on a relation between unstable cohomology operations due to the first author [**Ma1**]. Let Θ be the unstable secondary operation studied by Mahowald and Peterson [**M-P, Ma2**], which arises from the relation $Sq^2Sq^{4m} = 0$ on $\mathbb{Z}/4$ cohomology classes of dimension $4m+1$. For us n is either $2m+1$ or $2m+2$. Mahowald's work [**Ma1**] implies the unstable relation

$$Sq^2\Theta = Sq^{4m+1}Sq^2 + \text{cup product terms.} \tag{1}$$

We write $\Sigma^2F_{n,2}$ as the cofiber of a composite, and assume that a homotopy retraction $\Sigma^2F_{n,2} \to \Sigma^2\mathbb{C}P^{n-1}$ exists, which implies that our composite is null-homotopic, which allows us to construct a "three-cell complex" N. We obtain a contradiction by applying (1) to a cohomology class in N. In order to construct the cohomology class we need the fact, proved with the Chern character in §4, that any stable map $\mathbb{C}P^{n-1} \wedge \mathbb{C}P^{n-1} \to \mathbb{C}P^{n-1}$ is zero in $\mathbb{Z}/4$ cohomology.

We conjecture that the monoid $\coprod_k F_{n,k}$ has a $\mathcal{C}_2$-structure. We think that with even homotopy commutativity we could extend our splitting of $\Omega SU(n)$ [**Ri**] to $\Omega V_{n,k}(\mathbb{C})$. In §2 we prove homotopy commutativity at the first level $\mu\colon F_{n,1} \times F_{n,1} \to F_{n,2}$, which we require to construct our three cell complex N.

The relation (1), Theorem 1.4 below, contains some undetermined cup products that we have recently computed together with Peterson [**M-P-R**]. Our paper with Peterson gives another proof of Theorem 0.1, using the relation (1) to construct an unstable tertiary operation which is nonzero in $\Sigma^2F_{n,2}$.

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1. Relations between unstable secondary operations

We briefly recall the Mahowald-Peterson operations [**M-P, Ma2, M-R**], and the unstable relation (1) of Mahowald [**Ma1**]. See [**M-P-R**] for a more leisurely account. Recall first the *dual EHP sequence* of Whitehead [**Wh2**] and Barcus and Meyer [**B-M**]. For any nice space X, there is a homotopy fibration

$$\Sigma\Omega X \wedge \Omega X \xrightarrow{H_\mu} \Sigma\Omega X \xrightarrow{\sigma} X, \tag{2}$$

where H_μ is the Hopf construction of the loop multiplication. See [**Wh1**, Ch. VIII: Thm. 1.6; ex.1–2,6] for a proof of the following.

Theorem 1.1 (Whitehead, Barcus and Meyer). *In the metastable range $i \le 3\,\mathrm{conn}(X)$, the Serre exact sequence of fibration* (2) *yields an exact sequence of groups*

$$H^{i+1}(X \wedge X) \xrightarrow{\Delta^*} H^{i+1}(X) \xrightarrow{\sigma^*} H^i(\Omega X) \xrightarrow{H_\mu^*} H^i(\Omega X \wedge \Omega X).$$

Let $XP(2)$ be "ΩX-projective plane", which sits in the cofibration sequence

$$\Sigma\Omega X \wedge \Omega X \xrightarrow{H_\mu} \Sigma\Omega X \xrightarrow{h} XP(2) \xrightarrow{\partial} \Sigma^2\Omega X \wedge \Omega X.$$

Let $\overline{\sigma}\colon XP(2) \to X$ be the canonical map extending the evaluation σ. The following result [**St, Mi1**], which we will need in §3, gives an alternative proof of the exact sequence of Theorem 1.1.

Theorem 1.2 (Stasheff, Milgram). *The following diagram is homotopy commutative.*

$$\begin{array}{ccc}
XP(2) & \xrightarrow{\partial} & \Sigma^2\Omega X \wedge \Omega X \\
 & & \big\downarrow{\scriptstyle \text{shuffle}} \\
\big\downarrow{\scriptstyle \overline{\sigma}} & & \Sigma\Omega X \wedge \Sigma\Omega X \\
 & & \big\downarrow{\scriptstyle \sigma\wedge\sigma} \\
X & \xrightarrow{\Delta} & X \wedge X
\end{array}$$

We review the Mahowald-Peterson unstable secondary operation Θ [**M-P, Ma2**]. Let E be the fiber of the map $\alpha^2\colon K(\mathbb{Z}/4, 4m) \to K(\mathbb{Z}/4, 8m)$, giving us a fibration sequence $K(\mathbb{Z}/4, 8m{-}1) \to E \xrightarrow{\pi} K(\mathbb{Z}/4, 4m) \xrightarrow{\alpha^2} K(\mathbb{Z}/4, 8m)$. A nullhomotopy of the composite $Sq^2 \cdot \alpha^2\colon K(\mathbb{Z}/4, 4m) \to K(\mathbb{Z}/2, 8m{+}2)$ defines a secondary operation $\Theta\colon E \to K(\mathbb{Z}/2, 8m{+}1)$. The nullhomotopic composite

$$\Sigma K(\mathbb{Z}/4, 4m{-}1) \xrightarrow{\sigma} K(\mathbb{Z}/4, 4m) \xrightarrow{\alpha^2} K(\mathbb{Z}/4, 8m)$$

defines a lifting $\tilde{\sigma}\colon \Sigma K(\mathbb{Z}/4, 4m{-}1) \to E$. We deduce from [**M-P, Ma2, M-R**]

Theorem 1.3 (Mahowald-Peterson). *There is a choice of the unstable operation Θ so that the composite*

$$\Sigma K(\mathbb{Z}/4, 4m{-}1) \xrightarrow{\tilde{\sigma}} E \xrightarrow{\Theta} K(\mathbb{Z}/2, 8m{+}1)$$

is adjoint to the cohomology class $\alpha \smile Sq^2\alpha \in H^{8m}(K(\mathbb{Z}/4, 4m{-}1); \mathbb{Z}/2)$.

The cohomology class $\Theta \in H^{8m+1}(E;\mathbb{Z}/2)$ defines an unstable cohomology operation with indeterminacy $\operatorname{Im} Sq^2$. We now consider the unstable secondary relation of the first author [**Ma1**], involving the composition of Sq^2 with Θ. We note that since $Sq^2 Sq^2 = 0$ on $\mathbb{Z}/4$ classes, the composite $Sq^2\Theta$ has no indeterminacy. Since we are working with $\mathbb{Z}/4$ cohomology, we need the Bockstein element $\beta \in H^{4m+1}\left(K\left(\mathbb{Z}/4, 4m\right);\mathbb{Z}/2\right)$.

THEOREM 1.4. *The composite of Sq^2 with $\Theta \in H^{8m+1}\left(E;\mathbb{Z}/2\right)$ factors through the base $K\left(\mathbb{Z}/4, 4m\right)$, so that the following diagram*

$$\begin{array}{ccccc} E & \xrightarrow{\ \Theta\ } & K\left(\mathbb{Z}/2, 8m{+}1\right) & \xrightarrow{Sq^2} & K\left(\mathbb{Z}/2, 8m{+}3\right), \\ \big\downarrow{\scriptstyle\pi} & & & \nearrow{\scriptstyle Sq^{4m+1}Sq^2\alpha+\epsilon_1\cdot Sq^3\alpha\smile\alpha+\epsilon_2\cdot\beta\alpha\smile Sq^2\alpha+\epsilon_3\cdot Sq^2\beta\alpha\smile\alpha} & \\ K\left(\mathbb{Z}/4, 4m\right) & & & & \end{array}$$

commutes up to homotopy, where $\epsilon_1, \epsilon_2, \epsilon_3$ are 0 or 1. Thus for any space N and any class $\alpha \in H^{4m}\left(N;\mathbb{Z}/4\right)$ such that $\alpha^2 = 0$, we have

$$Sq^2\Theta(\alpha) = Sq^{4m+1}Sq^2\alpha + \epsilon_1 Sq^3\alpha \smile \alpha + \epsilon_2 \beta\alpha \smile Sq^2\alpha + \epsilon_3 Sq^2\beta\alpha \smile \alpha. \quad (3)$$

PROOF. By the theory of secondary operations, there exists a homotopy class

$$\gamma\colon K\left(\mathbb{Z}/4, 4m\right) \to K\left(\mathbb{Z}/2, 8m{+}3\right)$$

such that $Sq^2\Theta = \pi^*\gamma$. In fact $\gamma \in \Phi_{1,1}\left(\alpha^2\right)$, where $\Phi_{1,1}$ is the secondary operation defined by the relation $Sq^2 \cdot Sq^2 = 0$ on $\mathbb{Z}/4$ classes.

By Theorem 1.1 and Theorem 1.3, the homology suspension of γ is

$$\sigma^*\gamma = Sq^2\Theta(\tilde\sigma) = Sq^2\alpha \smile Sq^2\alpha = Sq^{4m+1}Sq^2\alpha \in H^{8m+2}\left(K\left(\mathbb{Z}/4, 4m{-}1\right);\mathbb{Z}/2\right).$$

By Theorem 1.1, for $X = K\left(\mathbb{Z}/4, 4m\right)$, γ then equals $Sq^{4m+1}Sq^2\alpha$, plus possibly some cup product terms. ∎

2. The Mitchell-Segal group completion model $\coprod_k F_{n,k} \subset \Omega U(n)$

Mitchell [**Mi3**] defined a non-multiplicative filtration $\{F_{n,k}\}$ of $\Omega SU(n)$, where $F_{n,1}$ is the inclusion $\iota\colon \mathbb{C}P^{n-1} \to \Omega SU(n)$. Letting $X_{n,k} = F_{n,k}/F_{n,k-1}$, the splitting referred to in the introduction is $\Sigma^\infty\Omega SU(n) \simeq \Sigma^\infty \bigvee_{k=1}^{\infty} X_{n,k}$. The disjoint union $\coprod_k F_{n,k}$ has an interpretation as a group completion model of $\Omega U(n)$, due to G. Segal [**Se1**]. That is, there is a strictly associative monoid multiplication $\mu_{k,l}\colon F_{n,k} \times F_{n,l} \to F_{n,k+l}$. We show, following the work of Pressley, Mitchell and Segal, that the group-completion model is homotopy commutative at the first level. See [**Pr, Mi3, Se1, C-M, Ri, M-R**] for background material.

Mitchell [**Mi3**, Cor. 2.12] proves that the multiplication map $\mu\colon \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1} \to F_{n,2}$ is a *desingularization* of projective algebraic varieties, that μ is surjective, and injective on the complement of the subspace $G_{n,2,1} \subset \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1}$ of orthogonal lines (l, m) in $\mathbb{C}^n$. Furthermore the restriction of μ to $G_{n,2,1}$ factors as the fiber bundle projection $\rho\colon G_{n,2,1} \to G_{n,2}$ followed by the natural inclusion $G_{n,2} \subset F_{n,2}$. One easily deduces from Mitchell's description the following.

Corollary 2.1. *$F_{n,2}$ is homeomorphic to the identification space $\mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1} \cup_\rho G_{n,2}$; the multiplication map $\mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1} \xrightarrow{\mu} F_{n,2}$ becomes the quotient space projection.*

Let $T\colon \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1} \to \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1}$ be the twist map sending the pair of lines (k, l) to (l, k). We now prove the following result, whose proof is is suggested by result of Mitchell [**Mi3**, Prop. 2.9] that the top Bruhat stratum of $F_{n,2}$ is the total space of the tangent bundle of $\mathbb{C}P^{n-1}$, and that the composite

$$\mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1} \xrightarrow{\mu} F_{n,2} \xrightarrow{\text{collapse}} F_{n,2}/G_{n,2} \cong T\left(\tau \downarrow \mathbb{C}P^{n-1}\right)$$

is the Thom-Pontryagin collapse of the diagonal $\mathbb{C}P^{n-1} \subset \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1}$.

Theorem 2.2. *The following diagram commutes up to homotopy:*

$$\begin{array}{ccc} \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1} & \xrightarrow{\mu} & F_{n,2} \\ {\scriptstyle T}\uparrow & \nearrow{\scriptstyle \mu} & \\ \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1}. & & \end{array}$$

Proof. For perpendicular unit vectors $u, v \in \mathbb{C}^n$, and $\theta \in [0, \pi/2]$, let

$$[u, \theta, v] = (\mathbb{C}\{u\}, \mathbb{C}\{\cos(\theta)u + \sin(\theta)v\}) \in \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1}.$$

A tubular neighborhood of the diagonal $\mathbb{C}P^{n-1} \subset \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1}$ can be given by $\{(l, k) : l \not\perp k\}$, and the complement of this tubular neighborhood is the flag manifold $G_{n,2,1} \subset \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1}$. Note that $[u, \theta, v]$ belongs to the tubular neighborhood when $\theta \in [0, \pi/2)$ and to the complement when $\theta = \pi/2$. A homotopy $H_t\colon \mu \simeq \mu \cdot T$ of the diagram is given by, for $t \in [0, 1]$,

$$H_t\left([u, \theta, v]\right) = \mu\left(\left[\cos(t\theta)u + \sin(t\theta)v, \theta, e^{\pi i t}\left(-\sin(t\theta)u + \cos(t\theta)v\right)\right]\right).$$

One sees easily that H_t is well defined and continuous. Note that if $l \perp k$, then $H_t(l, k) = l \oplus k \in G_{n,2} \subset F_{n,2}$ for all $t \in [0, 1]$. ∎

Mitchell's inclusion $\iota\colon \mathbb{C}P^{n-1} \to F_{n,2}$ is given by left multiplication with the basepoint $0 \times \mathbb{C} \subset \mathbb{C}^n$. By Theorem 2.2 this is homotopic to right multiplication with the basepoint. By the homotopy extension property we have the homotopy commutative diagram

$$\begin{array}{ccc} \mathbb{C}P^{n-1} & \xrightarrow{\iota=\mu(0\times\mathbb{C},\cdot)} & F_{n,2} \\ {\scriptstyle \mathrm{Fold}}\uparrow & & \uparrow{\scriptstyle \mu} \\ \mathbb{C}P^{n-1}\vee\mathbb{C}P^{n-1} & \xrightarrow{\iota} & \mathbb{C}P^{n-1}\times\mathbb{C}P^{n-1}. \end{array}$$

By taking cofibers we have the following result, which we will need in §3.

Corollary 2.3. *There is a map* $\bar{\mu}\colon \mathbb{C}P^{n-1}\wedge\mathbb{C}P^{n-1} \to X_{n,2}$ *making the diagram homotopy commutative*

$$\begin{array}{ccccc} \mathbb{C}P^{n-1} & \xrightarrow{\iota} & F_{n,2} & \xrightarrow{\pi} & X_{n,2} \\ {\scriptstyle \mathrm{Fold}}\uparrow & & {\scriptstyle \mu}\uparrow & & \uparrow{\scriptstyle \bar{\mu}} \\ \mathbb{C}P^{n-1}\vee\mathbb{C}P^{n-1} & \xrightarrow{\iota} & \mathbb{C}P^{n-1}\times\mathbb{C}P^{n-1} & \xrightarrow{\pi} & \mathbb{C}P^{n-1}\wedge\mathbb{C}P^{n-1}. \end{array}$$

3. The proof of the main theorem

We prove that $\Sigma^2\mathbb{C}P^{n-1}$ is not a retract of $\Sigma^2F_{n,2}$, for n at least 3. We suppose that a retraction exists, and derive a contradiction from our unstable cohomology relation applied to a $4m$-dimensional class α in a "3-cell complex" containing $\Sigma^2\mathbb{C}P^{n-1}$, where $n = 2m+1$ if n is odd, and $n = 2m+2$ if n is even. Theorem 0.1 follows immediately from

Theorem 3.1. *For $n \geq 3$, there is no left homotopy inverse $r\colon \Sigma^2F_{n,2} \to \Sigma^2\mathbb{C}P^{n-1}$ of the inclusion $\Sigma^2\mathbb{C}P^{n-1} \to \Sigma^2F_{n,2}$.*

Proof. Suppose that for $n \geq 3$ we have a retraction map $r\colon \Sigma^2F_{n,2} \to \Sigma^2\mathbb{C}P^{n-1}$. Consider the derived Barratt-Puppe homotopy cofibration sequence

$$\mathbb{C}P^{n-1} \xrightarrow{\iota} F_{n,2} \xrightarrow{\pi} X_{n,2} \xrightarrow{\partial} \Sigma\mathbb{C}P^{n-1} \xrightarrow{-\Sigma\iota} \Sigma F_{n,2} \xrightarrow{-\Sigma\pi} \Sigma X_{n,2} \xrightarrow{-\Sigma\partial} \Sigma^2\mathbb{C}P^{n-1} \to \Sigma^2F_{n,2}$$

The existence of the retraction $r\colon \Sigma^2F_{n,2} \to \Sigma^2\mathbb{C}P^{n-1}$ implies that the boundary map $\Sigma\partial\colon \Sigma X_{n,2} \to \Sigma^2\mathbb{C}P^{n-1}$ is nullhomotopic. We will construct our "3-cell complex" by factorizing the nullhomotopic map $\Sigma\partial$ through a space ΣY.

As is well-known the homology of $\Omega SU(n)$ is a polynomial algebra on the homology classes of $\mathbb{C}P^{n-1}$; this is given by the generating map $\iota\colon \mathbb{C}P^{n-1} \to \Omega SU(n)$. Recall also that the homology of $SU(n)$ is an exterior algebra, with a generating map given by the adjoint of the map ι, which we will call by the same name $\iota\colon \Sigma\mathbb{C}P^{n-1} \to X$. A result of Mitchell [**Mi3**, Thm. 2.3] calculates the homology of $F_{n,2}$.

Theorem 3.2 (Mitchell). *The inclusion $F_{n,2} \to \Omega SU(n)$ induces in homology an isomorphism from $H_*(F_{n,2})$ to the monomials of weight ≤ 2; $H_*(X_{n,2})$ is the monomials of weight $= 2$. In homology the multiplication map $\mu\colon \mathbb{C}P^{n-1}\times\mathbb{C}P^{n-1} \to F_{n,2}$ is the map from the weight 2 tensors to the symmetric square of $H_*(\mathbb{C}P^{n-1})$.*

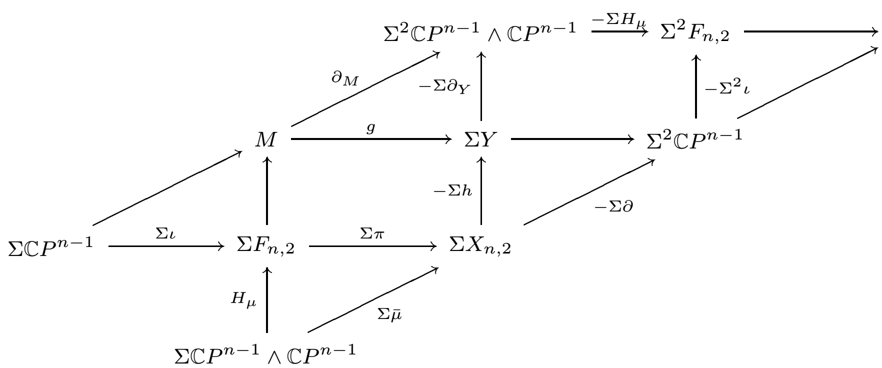
Let Y be the homotopy cofiber of the map $\bar{\mu}$ of §2; we have a cofibration with boundary

$$\mathbb{C}P^{n-1} \wedge \mathbb{C}P^{n-1} \xrightarrow{\bar{\mu}} X_{n,2} \xrightarrow{h} Y \xrightarrow{\partial_Y} \Sigma\mathbb{C}P^{n-1} \wedge \mathbb{C}P^{n-1}.$$

Define M to be the cofiber of the Hopf construction of the multiplication map $\mu\colon \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1} \to F_{n,2}$, giving us a cofibration sequence

$$\Sigma\mathbb{C}P^{n-1} \wedge \mathbb{C}P^{n-1} \xrightarrow{H_\mu} \Sigma F_{n,2} \to M \xrightarrow{\partial_M} \Sigma^2\mathbb{C}P^{n-1} \wedge \mathbb{C}P^{n-1}.$$

Using Corollary 2.3, we have an induced map of cofibers $g\colon M \to \Sigma Y$, sitting in the following homotopy commutative braid

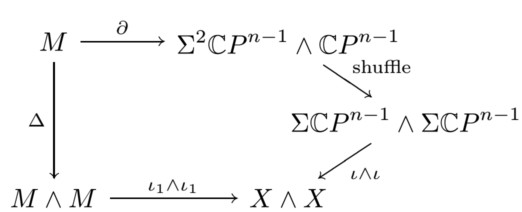


(4)

of cofibration sequences. Our cofiber M is defined similarly to $XP(2)$, for $X = SU(n)$. Therefore we have an induced map of homotopy cofibers $h\colon M \to XP(2)$, giving us the homotopy commutative diagram

$$\begin{array}{ccccccc}
\Sigma\mathbb{C}P^{n-1} \wedge \mathbb{C}P^{n-1} & \xrightarrow{H_\mu} & \Sigma F_{n,2} & \longrightarrow & M & \xrightarrow{\partial_M} & \Sigma^2\mathbb{C}P^{n-1} \wedge \mathbb{C}P^{n-1} \\
\Big\downarrow{\scriptstyle \Sigma\iota\wedge\iota} & & \Big\downarrow{\scriptstyle \iota} & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle \Sigma^2\iota\wedge\iota} \\
\Sigma\Omega X \wedge \Omega X & \xrightarrow{H_\mu} & \Sigma\Omega X & \longrightarrow & XP(2) & \xrightarrow{\partial} & \Sigma^2\Omega X \wedge \Omega X.
\end{array}$$

Now we define $\iota_1\colon M \to X = SU(n)$ to be the composite $M \xrightarrow{h} XP(2) \xrightarrow{\overline{\sigma}} X$. The last square of this diagram combined with Theorem 1.2 and the naturality of the diagonal map allows us to calculate the diagonal map of M. The following diagram



(5)

is homotopy commutative. By assumption (that the retraction r exists), and diagram (4), we have a nullhomotopy of the composite

$$\Sigma\partial\colon \Sigma X_{n,2} \xrightarrow{\Sigma h} \Sigma Y \to \Sigma^2\mathbb{C}P^{n-1}.$$

Let $\gamma\colon \Sigma^2 X_{n,2} \to \Sigma M$ be the resulting coextension of Σh. Let N be the homotopy cofiber of map γ. We then have the the following homotopy commutative diagram.

$$\begin{array}{ccccccc}
\Sigma^2\mathbb{C}P^{n-1}\wedge\mathbb{C}P^{n-1} & \longrightarrow & \Sigma^2\mathbb{C}P^{n-1} & \longrightarrow & N & \longrightarrow & \Sigma^3\mathbb{C}P^{n-1}\wedge\mathbb{C}P^{n-1} \\
\uparrow & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & \nearrow{\scriptstyle -\Sigma\partial_M} & \uparrow{\scriptstyle -\Sigma^2\partial_Y} \\
\Sigma Y & \longrightarrow & \Sigma^2\mathbb{C}P^{n-1} & \longrightarrow & \Sigma M & \xrightarrow{-\Sigma g} & \Sigma^2 Y \\
\uparrow{\scriptstyle \Sigma h} & & \uparrow{\scriptstyle *} & & \uparrow{\scriptstyle \gamma} & & \\
\Sigma X_{n,2} & \longrightarrow & * & \longrightarrow & \Sigma^2 X_{n,2} & &
\end{array} \tag{6}$$

The top row gives us a different cofibration for our "3-cell complex" N.

Now we apply the secondary relation. Let m be $(n-1)/2$ if n is odd, and $n/2-1$ if n is even. Thus n is $2m+1$ or $2m+2$ respectively. By Theorem 4.1 of the next section, there exists a (unique) cohomology class $\alpha \in H^{4m}(N;\mathbb{Z}/4)$ extending $x^{2m-1} \in H^{4m-2}(\mathbb{C}P^{n-1})$. We call $\alpha_0 \in H^{4m-1}(M)$ the restriction of α under the inclusion $\Sigma M \to N$. By Theorem 3.2 and diagram (4) the inclusion $\Sigma\mathbb{C}P^{n-1} \to M$ induces an isomorphism in odd-dimensional homology. Therefore the class α_0 is the pullback of a class $\alpha_1 \in H^{4m-1}(X)$ under the map $\iota_1\colon M \to X = SU(n)$. The Mahowald-Peterson operation Θ is defined on the class α, that is, $\alpha^2 = 0$, since the the even-dimensional cohomology of N maps injectively to the subspace $\Sigma^2\mathbb{C}P^{n-1}$. We apply equation (3) of Theorem 1.4 to the class $\alpha \in H^{4m}(N;\mathbb{Z}/4)$. We first show that all the cup product terms vanish. Recall that for any cofiber $B = X\cup_f CA$ the cup product pairing in $H^*(B)$ factors through a *relative cup product* pairing $\smile\colon H^*(X)\otimes H^*(B) \to H^*(B)$. Using the cofibration of the top row of diagram (6), it is then enough to show that the restrictions to $\Sigma^2\mathbb{C}P^{n-1}$ of the classes $\beta\alpha$, $Sq^2\beta\alpha$ and $Sq^3\alpha$ vanish. But this is clear by dimensional reasons, so the three cup products in equation (3) vanish. By Theorem 1.3, $\Theta\alpha_0$ contains $Sq^2\alpha_0 \smile \alpha_0$. By diagram (5),

$$Sq^2\alpha_0 \smile \alpha_0 = -\partial_M^*\sigma^2\left(x^{2m-1}\otimes x^{2m}\right).$$

Thus $Sq^2\Theta(\alpha)$, which has no indeterminacy, equals the class of the composite

$$N \to \Sigma^3\mathbb{C}P^{n-1}\wedge\mathbb{C}P^{n-1} \xrightarrow{\sigma^3\left(x^{2m}\otimes x^{2m}\right)} K\left(\mathbb{Z}/2, 8m{+}3\right),$$

which is the generator of $H^{8m+3}\left(N;\mathbb{Z}/2\right)$.

But $Sq^{4m+1}Sq^2\alpha = Sq^1Sq^{4m}Sq^2\alpha = 0$, since $H^{8m+2}(N;\mathbb{Z}/2) = 0$. This gives us our contradiction, and completes the proof of Theorem 3.1. ∎

4. K-theory

We prove the following result, which we used in the proof of Theorem 0.1.

THEOREM 4.1. *Any stable map $\mathbb{C}P^n \wedge \mathbb{C}P^n \to \mathbb{C}P^n$ induces the zero map in $\mathbb{Z}/4$ cohomology, for any positive integer n.*

The proof uses complex K-theory and the Chern character. See the books of Adams [**Ad**] or Dyer [**Dy**] for a treatment of the Chern character $\mathrm{ch}\colon \mathrm{K}^0(\cdot) \to H^*(\cdot;\mathbb{Q})$.

The Chern character is natural with respect to stable maps. Given a stable map $f \in \{\mathbb{C}P^n \wedge \mathbb{C}P^n, \mathbb{C}P^n\}$ we have the commutative diagram

$$\begin{array}{ccc} \mathrm{K}^0(\mathbb{C}P^n) & \xrightarrow{f^*} & \mathrm{K}^0(\mathbb{C}P^n \wedge \mathbb{C}P^n) \\ {\scriptstyle \mathrm{ch}}\downarrow & & {\scriptstyle \mathrm{ch}}\downarrow \\ H^*(\mathbb{C}P^n;\mathbb{Q}) & \xrightarrow{f^*} & H^*(\mathbb{C}P^n \wedge \mathbb{C}P^n;\mathbb{Q}) . \end{array} \tag{7}$$

$\mathrm{K}^0(\mathbb{C}P^n) = \mathbb{Z}[x]/\left(x^{n+1}\right)$ and $\mathrm{K}^0(\mathbb{C}P^n \wedge \mathbb{C}P^n) = \mathbb{Z}[x_1, x_2]/\left(x_1^{n+1}, x_2^{n+1}\right)$, where the generator $x = L - 1$ is the virtual bundle of degree zero, L being the Hopf line bundle. Let $t \in H^2(CP^n;\mathbb{Z})$ be the preferred generator. By definition of the Chern character we have

$$\mathrm{ch}(x) = e^t - 1 = \sum_{k=1}^{n} \frac{1}{k!}\, \frac{d^k}{du^k}\left(e^u - 1\right)^i \Big|_{u=0} t^k \in H^*(\mathbb{C}P^n;\mathbb{Q}), \tag{8}$$

using differentiation with respect to the dummy real variable u. Write

$$f^*(x) = \sum_{1\le i,j\le n} a_{i,j}\, x^i \wedge x^j \in \mathrm{K}^0(\mathbb{C}P^n \wedge \mathbb{C}P^n), \qquad \text{for some } a_{i,j} \in \mathbb{Z}.$$

Since the Chern character is a ring homomorphism, diagram (7) gives

$$f^*\left(e^t - 1\right) = \sum_{1\le i,j\le n} a_{i,j}\left(e^t - 1\right)^i \otimes \left(e^t - 1\right)^j \in H^*(\mathbb{C}P^n \wedge \mathbb{C}P^n;\mathbb{Q}). \tag{9}$$

By clearing out the denominators, we obtain the following result.

LEMMA 4.2. *The induced map $f^*\colon H^*(\mathbb{C}P^n;\mathbb{Z}) \to H^*(\mathbb{C}P^n \wedge \mathbb{C}P^n;\mathbb{Z})$ in integral cohomology of our stable map $f \in \{\mathbb{C}P^n \wedge \mathbb{C}P^n, \mathbb{C}P^n\}$ has the expression*

$$f^*(t^\alpha) = \sum_{k+l=\alpha;\ 1\le k,l\le n} C_{k,l}\binom{k+l}{k} t^k \otimes t^l \in H^*(\mathbb{C}P^n \wedge \mathbb{C}P^n;\mathbb{Z}),$$

where

$$C_{k,l} = \sum_{1\le i,j\le n} a_{i,j} \; \frac{d^k}{du^k}\left(e^u-1\right)^i\Big|_{u=0} \; \frac{d^l}{du^l}\left(e^u-1\right)^j\Big|_{u=0}. \qquad (10)$$

We now express f^* in $\mathbb{Z}/4$ cohomology. Let $\delta_{(k>1)} = 1$ when $k > 1$ and 0 when $k = 0, 1$. Similarly define the delta function $\delta_{(k \text{ odd})}$.

LEMMA 4.3. *The coefficient $C_{k,l}$ of Lemma 4.2 satisfies the equation*

$$\begin{aligned} C_{k,l} \equiv\ & a_{1,1} + 2a_{2,1}\,\delta_{(k>1)} + 2a_{1,2}\,\delta_{(l>1)} \\ & + 2a_{3,1}\,\delta_{(k>1)}\,\delta_{(k \text{ odd})} + 2a_{1,3}\,\delta_{(l>1)}\,\delta_{(l \text{ odd})} \pmod 4. \qquad (11)\end{aligned}$$

PROOF. By the chain rule, $\frac{d^k}{du^k}\left(e^u-1\right)^2\Big|_{u=0}$ is even, and $\frac{d^k}{du^k}\left(e^u-1\right)^4\Big|_{u=0} \equiv 0 \pmod 4$. By the product rule, $\frac{d^k}{du^k}\left(e^u-1\right)^i\Big|_{u=0}$ is then even if $i \ge 2$ and divisible by 4 if $i \ge 4$. By the binomial theorem,

$$\begin{aligned} \frac{d^k}{du^k}\left(e^u-1\right)^2\Big|_{u=0} &= 2^k - 2 + \delta_{k,0} \equiv 2\delta_{(k>1)} \pmod 4; \\ \frac{d^k}{du^k}\left(e^u-1\right)^3\Big|_{u=0} &= 3^k - 3\cdot 2^k + 3 - \delta_{k,0} \equiv 2\delta_{(k>1)}\delta_{(k \text{ odd})} \pmod 4. \end{aligned}$$

Substituting these equations into (10) yields Lemma 4.3. ∎

PROOF OF THEOREM 4.1. If $\alpha > n$, we have $f^*(t^\alpha) = 0$, since $t^\alpha = 0 \in H^{2\alpha}(\mathbb{C}P^n;\mathbb{Z})$ for $\alpha > n$. By Lemma 4.2, $C_{k,l} = 0$ if $k+l > n$ and $1 \le k, l \le n$. Thus the right hand side of equation (11) is $\equiv 0 \pmod 4$, for each pair (k,l) with $k + l > n$ and $1 \le k, l \le n$. Theorem 4.1 now follows from "row-reducing" a number of these mod 4 equations. The argument depends on the value of n. For $n = 2$, consider the three equations arising from (k,l) equal to $(2,1)$, $(1,2)$ and $(2,2)$. If $n \ge 4$ is even, let (k,l) equal $(n,1)$, $(1,n)$, $(n,2)$, $(n,3)$ and $(3,n)$. If $n \ge 3$ is odd, let (k,l) equal $(n,1)$, $(1,n)$, $(n,2)$, $(2,n)$ and $(n,3)$. One shows that

$$a_{1,1} \equiv 2a_{1,2} \equiv 2a_{2,1} \equiv 2a_{1,3} \equiv 2a_{3,1} \equiv 0 \pmod 4.$$

Substituting into equation (11) yields $C_{k,l} \equiv 0 \pmod 4$ for all k, l. By Lemma 4.2 and Lemma 4.3, $f^*\colon H^*(\mathbb{C}P^n;\mathbb{Z}/4) \to H^*(\mathbb{C}P^n \wedge \mathbb{C}P^n;\mathbb{Z}/4)$ is the zero map.
∎

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