DIAGRAM SPACES, DIAGRAM SPECTRA, AND FSP'S

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Contents

This is one of several papers in which we explore the interrelationships among certain categories that can be taken as models for spectra and (highly structured) ring spectra. In this paper, we study certain categories of "diagram ring spectra" that are isomorphic to corresponding categories of "functors with smash product (FSP's)". The notion of an FSP was introduced by Bökstedt [2], and his use of FSP's to define topological Hochschild homology established their convenience and importance in stable homotopy theory. Versions of FSP's had been defined earlier: in different language, what we call FSP's in the category of symmetric spectra were defined by Gunnarson [7], and what we call FSP's in the category of orthogonal spectra were defined by the second author and others [18, 17] in the early 1970's.

As we shall see, FSP's are defined in terms of "external smash products". It is a crucial insight of Jeff Smith that external smash products can be internalized. We show that each category of generalized FSP's is isomorphic to the category of monoids in an associated symmetric monoidal category of diagram spectra. Such monoids are what we mean by diagram ring spectra. Smith introduced the category of symmetric spectra, showed that it is symmetric monoidal, and observed that its externally defined FSP's and internally defined monoids give isomorphic categories. We study a general form of the construction. In fact, the relevant categorical framework was already in place by 1970, in work of Day [5].

Hovey, Shipley, and Smith [8] have studied the category of symmetric spectra and its homotopy theory, and the papers [21] and [23] go further with the homotopical study of its ring spectra. There is a coordinate-free analogue of the category of symmetric spectra, which we call the category of orthogonal spectra; it was introduced by May [17, §5] (who called its objects $\mathscr{I}_*$-prespectra). The names come from the fact that actions by symmetric groups and by orthogonal groups are built into the

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spaces that comprise symmetric spectra and orthogonal spectra. Still another category of the type that we consider is the category of Γ-spaces. That category was introduced by Segal [22], and its homotopy theory was studied by Anderson [1] and Bousfield and Friedlander [3]. Its symmetric monoidal structure and concomitant theory of ring spectra have been studied by Lydakis [13] and Schwede [20].

In this paper, we explore the formal structure that is common to such categories. We recall some standard facts and terminology about symmetric monoidal categories in Section 1. We introduce functor categories $\mathscr{D}\mathscr{T}$ of $\mathscr{D}$-spaces in Section 2. Here $\mathscr{D}$ is a suitable domain category for the space-valued functors we shall study, and $\mathscr{T}$ is the category of based spaces. We discuss symmetric monoidal functor categories $\mathscr{D}\mathscr{T}$ of $\mathscr{D}$-spaces in Section 3. Our focus is on comparisons between such categories as $\mathscr{D}$ varies. In Section 4, we define forgetful and prolongation functors $\mathbb{U} : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{C}\mathscr{T}$ and $\mathbb{P} : \mathscr{C}\mathscr{T} \longrightarrow \mathscr{D}\mathscr{T}$ associated to suitable maps $\mathscr{C} \longrightarrow \mathscr{D}$ of domain categories; $\mathbb{U}$ and $\mathbb{P}$ are right and left adjoint.

We show how functors with smash product, or $\mathscr{D}$-FSP's, fit into this framework in Section 5. Briefly, there is an external smash product that takes a pair of $\mathscr{D}$-spaces to a $(\mathscr{D} \times \mathscr{D})$-space, and there is a related internal smash product that takes a pair of $\mathscr{D}$-spaces to a $\mathscr{D}$-space. A $\mathscr{D}$-FSP is defined in terms of the external smash product, and it determines and is determined by a $\mathscr{D}$-monoid (or $\mathscr{D}$-ring) defined with respect to the internal smash product.

A $\mathscr{D}$-monoid R has an associated category of R-modules. We refer to the corresponding structure defined in terms of the external smash product as a $\mathscr{D}$-spectrum over R. In Section 6, we interpret the category of R-modules, alias the category of $\mathscr{D}$-spectra over R, as the category of $\mathscr{D}_R$-spaces for a domain category $\mathscr{D}_R$ constructed from $\mathscr{D}$ and R. We discuss adjoint forgetful and prolongation functors $\mathbb{U}$ and $\mathbb{P}$ between categories of $\mathscr{C}$-spectra and $\mathscr{D}$-spectra in Section 7.

Finally, in Section 8, we specialize to the examples that we are most interested in. For particular domain categories $\mathscr{D}$, we fix a canonical $\mathscr{D}$-monoid S that is related to spheres and obtain the category $\mathscr{D}\mathscr{S}$ of $\mathscr{D}$-spectra over S. It is symmetric monoidal when S is commutative. In a continuation of this paper [16], we shall study model structures on such categories and compare them homotopically.

We have chosen to work with functors that take values in based spaces because some of our motivating examples make little sense simplicially. However, everything in this paper can be adapted without difficulty to functors that take values in the category of based simplicial sets. The simplicially minded reader will understand "spaces" to mean "simplicial sets" and "continuous" to mean "simplicial" throughout. In fact, all of our constructions apply verbatim to functors that take values in any symmetric monoidal category that is tensored and cotensored over either topological spaces or simplicial sets. Examples of such symmetric monoidal functor categories arise in other fields, such as algebraic geometry.

1. Symmetric monoidal categories

We first fix some language to avoid later confusion. A monoidal category is a category $\mathscr{D}$ together with a product $\square = \square_{\mathscr{D}} : \mathscr{D} \times \mathscr{D} \longrightarrow \mathscr{D}$ and a unit object $u = u_{\mathscr{D}}$ such that $\square$ is associative and unital up to coherent natural isomorphism; $\mathscr{D}$ is symmetric monoidal if $\square$ is also commutative up to coherent natural isomorphism. See [9, 10, 15] for discussions of the precise meaning of coherence. A symmetric monoidal category $\mathscr{D}$ is *closed* if it has internal hom objects $F(d, e)$ with adjunction isomorphisms

$$\mathscr{D}(d\Box e, f) \cong \mathscr{D}(d, F(e, f)).$$

There are evident notions of monoids in monoidal categories and commutative monoids in symmetric monoidal categories. The highly structured ring spectra in any of the modern approaches to stable homotopy theory are exactly the monoids and commutative monoids in the relevant symmetric monoidal ground category. To compare such objects in different ground categories, we need language to describe when functors and natural transformations preserve monoids and commutative monoids.

Definition 1.1. A functor $F : \mathscr{A} \longrightarrow \mathscr{B}$ between monoidal categories is *lax monoidal* if there is a map $\lambda : u_{\mathscr{B}} \longrightarrow F(u_{\mathscr{A}})$ and there are maps

$$\phi : F(A)\Box_{\mathscr{B}} F(A') \longrightarrow F(A\Box_{\mathscr{A}} A')$$

that specify a natural transformation $\phi : \Box_{\mathscr{B}} \circ (F \times F) \longrightarrow F \circ \Box_{\mathscr{A}}$; it is required that all coherence diagrams relating the associativity and unit isomorphisms of $\mathscr{A}$ and $\mathscr{B}$ to the maps λ and ϕ commute. If $\mathscr{A}$ and $\mathscr{B}$ are symmetric monoidal, then F is *lax symmetric monoidal* if all coherence diagrams relating the associativity, unit, and commutativity isomorphisms of $\mathscr{A}$ and $\mathscr{B}$ commute. The functor F is *strong monoidal* or *strong symmetric monoidal* if λ and ϕ are isomorphisms.

The definition is incomplete in that we have not specified the relevant "coherence diagrams", but the intuition should be clear enough. See [9, 10] for details. The direction of the arrows λ and ϕ in the definition leads to the following conclusion.

Lemma 1.2. *If $F : \mathscr{A} \longrightarrow \mathscr{B}$ is lax monoidal and M is a monoid in $\mathscr{A}$ with unit $\eta : u_{\mathscr{A}} \longrightarrow M$ and product $\mu : M\Box_{\mathscr{A}} M \longrightarrow M$, then $F(M)$ is a monoid in $\mathscr{B}$ with unit $F(\eta) \circ \lambda : u_{\mathscr{B}} \longrightarrow F(u_{\mathscr{A}}) \longrightarrow F(M)$ and product*

$$F(\mu) \circ \phi : F(M)\Box_{\mathscr{B}} F(M) \longrightarrow F(M\Box_{\mathscr{A}} M) \longrightarrow F(M).$$

If $F : \mathscr{A} \longrightarrow \mathscr{B}$ is lax symmetric monoidal and M is a commutative monoid in $\mathscr{A}$, then $F(M)$ is a commutative monoid in $\mathscr{B}$.

We also need the concomitant notion of a monoidal natural transformation. Here we needn't use an adjective "lax" or "strong" since the definition is the same for either lax or strong monoidal functors.

Definition 1.3. Let F and G be lax monoidal or lax symmetric monoidal functors $\mathscr{A} \longrightarrow \mathscr{B}$. A natural transformation $\alpha : F \longrightarrow G$ is *monoidal* if the following diagrams commute:

$$\begin{array}{ccc} & u_{\mathscr{B}} & \\ {\scriptstyle\lambda_F}\swarrow & & \searrow{\scriptstyle\lambda_G} \\ F(u_{\mathscr{A}}) & \xrightarrow{\ \alpha\ } & G(u_{\mathscr{A}}) \end{array} \quad \text{and} \quad \begin{array}{ccc} F(A)\Box_{\mathscr{B}} F(A') & \xrightarrow{\alpha\Box\alpha} & G(A)\Box_{\mathscr{B}} G(A') \\ {\scriptstyle\phi_F}\downarrow & & \downarrow{\scriptstyle\phi_G} \\ F(A\Box_{\mathscr{A}} A') & \xrightarrow{\ \alpha\ } & G(A\Box_{\mathscr{A}} A'). \end{array}$$

The following assertion is obvious from the definition and the previous lemma.

Lemma 1.4. *If α is monoidal and A is a monoid in $\mathscr{A}$, then $\alpha : F(A) \longrightarrow G(A)$ is a map of monoids in $\mathscr{B}$. If α is symmetric monoidal and A is a commutative monoid in $\mathscr{A}$, then $\alpha : F(A) \longrightarrow G(A)$ is a map of commutative monoids in $\mathscr{B}$.*

2. Categories of $\mathscr{D}$-spaces

Spaces will mean compactly generated weak Hausdorff spaces throughout. One reference is [19]; a thorough treatment is given in [11, App]. We let $\mathscr{T}$ denote the category of based spaces. The category $\mathscr{T}$ is a closed symmetric monoidal topological category under the smash product and function space functors, written $X \wedge Y$ and $F(X,Y)$; its unit object is S^0. Conceptually, it is a distinctive feature of the topological, as opposed to simplicial, setting that the internal hom spaces $F(X,Y)$ and the categorical hom spaces $\mathscr{T}(X,Y)$ coincide.

Let $\mathscr{D}$ be a topological category. We assume that $\mathscr{D}$ is based, in the sense that it has a given initial and terminal object $*$. Thus the space $\mathscr{D}(d,e)$ of maps $d \to e$ is based with basepoint $d \to * \to e$. When $\mathscr{D}$ is given as an unbased category, we implicitly adjoin a base object $*$; in other words, we then understand $\mathscr{D}(d,e)$ to mean the union of the unbased space of maps $d \to e$ in $\mathscr{D}$ and a disjoint basepoint. The base object of $\mathscr{T}$ is a one-point space. By a functor between based categories, we always understand a functor that carries base objects to base objects; that is, we take this as part of our definition of "functor". A functor $F : \mathscr{D} \longrightarrow \mathscr{D}'$ between topological categories is continuous if $F : \mathscr{D}(d,e) \longrightarrow \mathscr{D}'(Fd, Fe)$ is a continuous map for all d and e; a natural transformation is continuous if its evaluation on each object is a continuous map.

Definition 2.1. A *$\mathscr{D}$-space* is a continuous functor $T : \mathscr{D} \longrightarrow \mathscr{T}$. Let $\mathscr{D}\mathscr{T}$ denote the category of $\mathscr{D}$-spaces and continuous natural maps between them.

We think of a $\mathscr{D}$-space as a diagram of spaces whose shape is specified by $\mathscr{D}$. The category of $\mathscr{D}$-spaces is complete and cocomplete, with limits and colimits constructed levelwise. It is also tensored and cotensored. For a $\mathscr{D}$-space T and based space X, the tensor $T \wedge X$ is given by the levelwise smash product and the cotensor $F(X,T)$ is given by the levelwise function space. Thus

$$\mathscr{D}\mathscr{T}(T \wedge X, T') \cong \mathscr{T}(X, \mathscr{D}\mathscr{T}(T,T')) \cong \mathscr{D}\mathscr{T}(T, F(X,T')). \tag{2.2}$$

We define homotopies between maps of $\mathscr{D}$-spaces by use of the cylinders $T \wedge I_+$.

Spaces and $\mathscr{D}$-spaces are related by a system of adjoint pairs of functors.

Definition 2.3. For an object d of $\mathscr{D}$, define the *evaluation functor* $Ev_d : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{T}$ by $Ev_d T = T(d)$ and define the *shift suspension functor* $F_d : \mathscr{T} \longrightarrow \mathscr{D}\mathscr{T}$ by $(F_d X)(e) = \mathscr{D}(d,e) \wedge X$. The functors F_d and Ev_d are left and right adjoint,

$$\mathscr{D}\mathscr{T}(F_d X, T) \cong \mathscr{T}(X, Ev_d T). \tag{2.4}$$

Moreover, Ev_d is covariantly functorial in d and F_d is contravariantly functorial in d. We write $Ev_d^{\mathscr{D}}$ and $F_d^{\mathscr{D}}$ when necessary to avoid confusion.

Notation 2.5. We use the alternative notation $d^* = F_d S^0$. Thus $d^*(e) = \mathscr{D}(d,e)$ and $F_d X = d^* \wedge X$; d^* is the *$\mathscr{D}$-space represented by the object d.*

Recall that a skeleton $sk\mathscr{D}$ of a category $\mathscr{D}$ is a full subcategory with one object in each isomorphism class. The inclusion $sk\mathscr{D} \longrightarrow \mathscr{D}$ is an equivalence of categories. When $\mathscr{D}$ is topological and has a small skeleton $sk\mathscr{D}$, $\mathscr{D}\mathscr{T}$ is a topological category. The set $\mathscr{D}\mathscr{T}(T,T')$ of maps $T \longrightarrow T'$ is topologized as the equalizer displayed in the diagram

$$\mathscr{D}(T,T') \longrightarrow \prod_d F(T(d), T'(d)) \underset{\tilde{\nu}}{\overset{\tilde{\mu}}{\rightrightarrows}} \prod_{\alpha: d \to e} F(T(d), T'(e)) \ ,$$

where the products run over the objects and morphisms of $sk\mathscr{D}$. For $f = (f_d)$, the αth component of $\tilde{\mu}(f)$ is $T'(\alpha) \circ f_d$ and the αth component of $\tilde{\nu}(f)$ is $f_e \circ T(\alpha)$. By an immediate comparison of represented functors, this implies that any $\mathscr{D}$-space T can be written as the coend of the contravariant functor d^* of d and the given covariant functor T.

Lemma 2.6. *Let $\mathscr{D}$ have a small skeleton $sk\mathscr{D}$ and let T be a $\mathscr{D}$-space. Then the evaluation maps $\varepsilon : d^* \wedge T(d) \longrightarrow T$ induce a natural isomorphism*

$$\int^{d \in sk\mathscr{D}} d^* \wedge T(d) \longrightarrow T.$$

Explicitly, T is isomorphic to the coequalizer of the parallel arrows in the diagram

$$\bigvee_{d,e} e^* \wedge \mathscr{D}(d,e) \wedge T(d) \underset{\mathrm{id} \wedge \varepsilon}{\overset{\varepsilon \wedge \mathrm{id}}{\rightrightarrows}} \bigvee_d d^* \wedge T(d) \xrightarrow{\ \varepsilon\ } T,$$

where the wedges run over pairs of objects and objects of $sk\mathscr{D}$ and the parallel arrows are wedges of smash products of identity and evaluation maps.

3. Symmetric monoidal categories of $\mathscr{D}$-spaces

Let $\mathscr{D}$ be a symmetric monoidal (based) topological category with unit object u and continuous product $\square$. The reader may want to glance ahead to Section 8, where the examples we have in mind are displayed.

Definition 3.1. For $\mathscr{D}$-spaces T and T', define the *"external" smash product* $T \bar{\wedge} T'$ by

$$T \bar{\wedge} T' = \wedge \circ (T \times T') : \mathscr{D} \times \mathscr{D} \longrightarrow \mathscr{T};$$

thus, for objects d and e of $\mathscr{D}$, $(T \bar{\wedge} T')(d,e) = T(d) \wedge T'(e)$. For a $\mathscr{D}$-space T' and a $(\mathscr{D} \times \mathscr{D})$-space T'', define the *external function $\mathscr{D}$-space* $\bar{F}(T', T'')$ by

$$\bar{F}(T', T'')(d) = \mathscr{D}(T', T''\langle d \rangle),$$

where $T''\langle d \rangle(e) = T''(d,e)$. Then, for $\mathscr{D}$-spaces T and T' and a $(\mathscr{D} \times \mathscr{D})$-space T'',

$$(\mathscr{D} \times \mathscr{D})\mathscr{T}(T \bar{\wedge} T', T'') \cong \mathscr{D}\mathscr{T}(T, \bar{F}(T', T'')). \tag{3.2}$$

Lemma 3.3. *There is a natural isomorphism*

$$F_d X \bar{\wedge} F_e Y \longrightarrow F_{(d,e)}(X \wedge Y).$$

Proof. Using (3.2), (2.4), and the definitions, we see that

$$(\mathscr{D} \times \mathscr{D})\mathscr{T}(F_d X \bar{\wedge} F_e Y, T) \cong \mathscr{T}(X \wedge Y, T(d,e)) \cong (\mathscr{D} \times \mathscr{D})\mathscr{T}(F_{(d,e)}(X \wedge Y), T)$$

for a $(\mathscr{D} \times \mathscr{D})$-space T. □

Now assume further that our given symmetric monoidal category $\mathscr{D}$ has a small skeleton $sk\mathscr{D}$; $sk\mathscr{D}$ inherits a symmetric monoidal structure such that the inclusion $sk\mathscr{D} \subset \mathscr{D}$ is strong symmetric monoidal.

We internalize the external smash product $T \bar{\wedge} T'$ by taking its topological left Kan extension along $\square$ [15, Ch.X]. This gives the category of $\mathscr{D}$-spaces a smash product $\wedge$ under which it is a closed symmetric monoidal topological category. For an object d of $\mathscr{D}$, let $\square/d$ denote the category of objects $\square$-over d; its objects are the maps $\alpha : e\square f \to d$ and its morphisms are the pairs of maps $(\phi, \psi) : (e,f) \longrightarrow (e', f')$ such that $\alpha'(\phi \square \psi) = \alpha$. This category inherits a topology from $\mathscr{D}$, and a map $d \to d'$ induces a continuous functor $\square/d \longrightarrow \square/d'$.

Definition 3.4. Let T and T' be $\mathscr{D}$-spaces. Define the *internal smash product* $T \wedge T'$ to be the topological left Kan extension of $T \bar{\wedge} T'$ along $\Box$. It is characterized by the universal property

$$\mathscr{D}\mathscr{T}(T \wedge T', T'') \cong (\mathscr{D} \times \mathscr{D})\mathscr{T}(T \bar{\wedge} T', T'' \circ \Box). \tag{3.5}$$

On an object d, it is specified explicitly as the colimit

$$(T \wedge T')(d) = \operatorname{colim}_{e \Box f \to d} T(e) \wedge T'(f)$$

indexed on $\Box/d$; this makes sense since $\Box/d$ has a small cofinal subcategory. When $\mathscr{D}$ itself is small, $(T \wedge T')(d)$ can also be described as the coend

$$(T \wedge T')(d) = \int^{(e,f) \in \mathscr{D} \times \mathscr{D}} \mathscr{D}(e \Box f, d) \wedge (T(e) \wedge T'(f))$$

with its topology as a quotient of $\vee_{(e,f)} \mathscr{D}(e \Box f, d) \wedge (T(e) \wedge T'(f))$. By the functoriality of colimits, maps $d \to d'$ in $\mathscr{D}$ induce maps $(T \wedge T')(d) \longrightarrow (T \wedge T')(d')$ that make $T \wedge T'$ into a $\mathscr{D}$-space.

Definition 3.6. Let T, T', and T'' be $\mathscr{D}$-spaces. Define the *internal function $\mathscr{D}$-space* $F(T', T'')$ by

$$F(T', T'') = \bar{F}(T', T'' \circ \Box).$$

Then (3.1) and (3.5) immediately imply the adjunction

$$\mathscr{D}(T \wedge T', T'') \cong \mathscr{D}(T, F(T', T'')). \tag{3.7}$$

Lemma 3.8. *There is a natural isomorphism*

$$F_d X \wedge F_e Y \longrightarrow F_{d \Box e}(X \wedge Y).$$

Proof. Using (3.5), (3.7), (2.4), and the definitions, we see that

$$\mathscr{D}\mathscr{T}(F_d X \wedge F_e Y, T) \cong \mathscr{T}(X \wedge Y, T(d \Box e)) \cong \mathscr{D}\mathscr{T}(F_{d \Box e}(X \wedge Y), T)$$

for a $\mathscr{D}$-space T. $\Box$

Theorem 3.9. *The category $\mathscr{D}\mathscr{T}$ is closed symmetric monoidal under $\wedge$ and the internal function $\mathscr{D}$-space functor F; its unit object is u^*.*

The proof is formal and will be omitted; see Day [5]. In Section 8, the smash product of $\mathscr{D}$-spaces is displayed explicitly in some of the most interesting examples.

4. Forgetful and prolongation functors on diagram spaces

We wish to compare the categories $\mathscr{D}\mathscr{T}$ as $\mathscr{D}$ varies.

Definition 4.1. Let $\mathscr{C}$ and $\mathscr{D}$ be (based) topological categories. We say that $\mathscr{D}$ is a *category under $\mathscr{C}$* if we are given a continuous functor $\iota : \mathscr{C} \longrightarrow \mathscr{D}$. In practice, ι is faithful; we often regard it as an embedding of categories and omit it from the notations. We say that $\mathscr{D}$ is a *symmetric monoidal category under $\mathscr{C}$* if ι is a strong symmetric monoidal functor.

We assume that $\mathscr{D}$ is a category under $\mathscr{C}$ and that $\mathscr{C}$ is skeletally small in the rest of this section.

Definition 4.2. Define the *forgetful functor* $\mathbb{U} : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{C}\mathscr{T}$ on $\mathscr{D}$-spaces Y by letting $(\mathbb{U}Y)(c) = Y(\iota c)$. Define the *prolongation functor* $\mathbb{P} : \mathscr{C}\mathscr{T} \longrightarrow \mathscr{D}\mathscr{T}$ on $\mathscr{C}$-spaces T by letting $\mathbb{P}T$ be the topological left Kan extension of T along ι. It is characterized by the universal property

$$\mathscr{D}\mathscr{T}(\mathbb{P}T, Y) \cong \mathscr{C}\mathscr{T}(T, \mathbb{U}Y). \tag{4.3}$$

Let ι/d be the topological category of objects ι-over d; its objects are the maps $\alpha : \iota c \longrightarrow d$ in $\mathscr{D}$ and its morphisms are the maps $\psi : c \longrightarrow c'$ in $\mathscr{C}$ such that $\alpha'(\iota\psi) = \alpha$. On an object d, $\mathbb{P}T$ is specified explicitly as the colimit

$$\mathbb{P}T(d) = \operatorname{colim}_{\iota c \to d} T(c)$$

indexed on ι/d. If $\mathscr{C}$ is small, $\mathbb{P}T(d)$ can also be described as the coend

$$\mathbb{P}T(d) = \int^{c \in \mathscr{C}} \mathscr{D}(\iota c, d) \wedge T(c). \tag{4.4}$$

The name "prolongation" is motivated by the following special case.

Lemma 4.5. *If $\iota : \mathscr{C} \longrightarrow \mathscr{D}$ is fully faithful, then the unit $\eta : \mathrm{Id} \longrightarrow \mathbb{U}\mathbb{P}$ of the adjunction (4.3) is a natural isomorphism. Thus $\mathbb{P}$ prolongs T to a functor defined on $\mathscr{D}$ that restricts to T on $\mathscr{C}$.*

Proof. For $c \in \mathscr{C}$, the identity map of ιc is a terminal object in $\iota/\iota c$. □

The evident relation $Ev_c\mathbb{U}Y = Y(\iota c) = Ev_{\iota c}Y$ implies the following observation.

Lemma 4.6. *For an object c of $\mathscr{C}$, $\mathbb{P}F_cX$ is naturally isomorphic to $F_{\iota c}X$.*

By an h-cofibration $i : A \longrightarrow X$ of $\mathscr{D}$-spaces, we understand a map that satisfies the Homotopy Extension Property (HEP). That is, for every map $f : X \longrightarrow Y$ and homotopy $h : A \wedge I_+ \longrightarrow Y$ such that $h_0 = f \circ i$, there is a homotopy $\tilde{h} : X \wedge I_+ \longrightarrow Y$ such that $\tilde{h}_0 = f$ and $\tilde{h} \circ (i \wedge \mathrm{id}) = h$. The universal test case is the mapping cylinder $Y = Mi = X \cup_i (A \wedge I_+)$, with the evident f and h, in which case $\tilde{h}$ is a retraction $X \wedge I_+ \longrightarrow MI$. The following easy lemma is crucial to our work.

Lemma 4.7. *The functors $\mathbb{U}$ and $\mathbb{P}$ preserve colimits, smash products with spaces, and h-cofibrations.*

Proof. Since colimits of $\mathscr{C}$-spaces and $\mathscr{D}$-spaces are defined levelwise, as are smash products with spaces, the first two preservation properties are clear, and the third follows by the retract of mapping cylinders criterion. □

We are especially interested in the multiplicative properties of $\mathbb{U}$ and $\mathbb{P}$. We now assume that $\mathscr{D}$ is a symmetric monoidal category under $\mathscr{C}$ and that both $\mathscr{C}$ and $\mathscr{D}$ are skeletally small. Note that we require ι to be strong rather than just lax symmetric monoidal.

Proposition 4.8. *The functor $\mathbb{U} : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{C}\mathscr{T}$ is lax symmetric monoidal and the functor $\mathbb{P} : \mathscr{C}\mathscr{T} \longrightarrow \mathscr{D}\mathscr{T}$ is strong symmetric monoidal. Moreover, the following diagrams commute, where T and T' are $\mathscr{C}$-spaces and Y and Y' are $\mathscr{D}$-spaces:*

$$\begin{array}{ccc} T \wedge T' & & \\ {\scriptstyle \eta\wedge\eta}\big\downarrow & \overset{\eta}{\searrow} & \\ \mathbb{U}\mathbb{P}T \wedge \mathbb{U}\mathbb{P}T' & \longrightarrow & \mathbb{U}\mathbb{P}(T \wedge T') \end{array} \qquad \begin{array}{ccc} \mathbb{P}\mathbb{U}Y \wedge \mathbb{P}\mathbb{U}Y' & \longrightarrow & \mathbb{P}\mathbb{U}(Y \wedge Y'). \\ {\scriptstyle \varepsilon\wedge\varepsilon}\big\downarrow & \overset{\varepsilon}{\swarrow} & \\ T \wedge T' & & \end{array}$$

Proof. Observe that left Kan extension also gives a functor

$$\mathbb{P} : (\mathscr{C} \times \mathscr{C})\mathscr{T} \longrightarrow (\mathscr{D} \times \mathscr{D})\mathscr{T}.$$

A direct comparison of colimits shows that

$$\mathbb{P}(T \,\bar{\wedge}\, T') \cong \mathbb{P}T \,\bar{\wedge}\, \mathbb{P}T', \tag{4.9}$$

and it is trivial to check the analogous isomorphism

$$\mathbb{U}(Y \,\bar{\wedge}\, Y') \cong \mathbb{U}Y \,\bar{\wedge}\, \mathbb{U}Y'. \tag{4.10}$$

We have unit and product isomorphisms $\lambda : u_{\mathscr{D}} \longrightarrow \iota u_{\mathscr{C}}$ and $\phi : \square_{\mathscr{D}} \circ (\iota \times \iota) \longrightarrow \iota \circ \square_{\mathscr{C}}$. For $(\mathscr{D} \times \mathscr{D})$-spaces Z, ϕ induces a natural isomorphism

$$\mathbb{U}(Z \circ \square_{\mathscr{D}}) \cong (\mathbb{U}Z) \circ \square_{\mathscr{C}}. \tag{4.11}$$

The unit isomorphism $\mathbb{P}u^*_{\mathscr{C}} \cong u^*_{\mathscr{D}}$ is given by Lemma 4.6, and the unit map $u^*_{\mathscr{C}} \longrightarrow \mathbb{U}u^*_{\mathscr{D}}$ is its adjoint. The defining universal properties of $\wedge$ and $\mathbb{P}$, together with (4.9) and (4.11), give a natural isomorphism

$$\mathscr{D}\mathscr{T}(\mathbb{P}T \wedge \mathbb{P}T', Y) \xrightarrow{\cong} \mathscr{D}\mathscr{T}(\mathbb{P}(T \wedge T'), Y),$$

and this implies the product isomorphism $\mathbb{P}T \wedge \mathbb{P}T' \cong \mathbb{P}(T \wedge T')$. Note the direction of the displayed arrow: $\mathbb{P}$ would not even be lax monoidal if ι were only lax, rather than strict, monoidal. Similarly, the defining universal properties of $\wedge$ and $\mathbb{P}$, together with (4.10) and (4.11), give a composite natural map

$$\begin{aligned}
\mathscr{D}\mathscr{T}(Y \wedge Y', Y \wedge Y') \quad &\cong \quad (\mathscr{D} \times \mathscr{D})\mathscr{T}(Y \,\bar{\wedge}\, Y', (Y \wedge Y') \circ \square_{\mathscr{D}}) \\
&\xrightarrow{\varepsilon^*} \quad (\mathscr{D} \times \mathscr{D})\mathscr{T}(\mathbb{P}\mathbb{U}(Y \,\bar{\wedge}\, Y'), (Y \wedge Y') \circ \square_{\mathscr{D}}) \\
&\cong \quad (\mathscr{C} \times \mathscr{C})\mathscr{T}(\mathbb{U}(Y \,\bar{\wedge}\, Y'), \mathbb{U}((Y \wedge Y') \circ \square_{\mathscr{D}})) \\
&\cong \quad (\mathscr{C} \times \mathscr{C})\mathscr{T}(\mathbb{U}Y \,\bar{\wedge}\, \mathbb{U}Y', \mathbb{U}(Y \wedge Y') \circ \square_{\mathscr{C}}) \\
&\cong \quad \mathscr{C}\mathscr{T}(\mathbb{U}Y \wedge \mathbb{U}Y', \mathbb{U}(Y \wedge Y')).
\end{aligned}$$

The product map $\mathbb{U}Y \wedge \mathbb{U}Y' \longrightarrow \mathbb{U}(Y \wedge Y')$ is the image of the identity map of $Y \wedge Y'$ along this composite. Note that one cannot expect this map to be an isomorphism. The commutativity of the diagrams displayed in the statement is formal. □

5. Diagram spectra and functors with smash product

Fix a skeletally small symmetric monoidal category $\mathscr{D}$. We have the symmetric monoidal category $\mathscr{D}\mathscr{T}$ of $\mathscr{D}$-spaces, and we consider its monoids and commutative monoids and their modules and algebras. These are defined in terms of the internal smash product in $\mathscr{D}\mathscr{T}$, and we shall explain their reinterpretations in terms of the more elementary external smash product $\bar{\wedge}$. The proofs of the comparisons are easy direct applications of the defining universal properties of $\wedge$ (3.5) and F_d (2.3).

Recall the definitions in Section 1. We have the category of lax monoidal functors $\mathscr{D} \longrightarrow \mathscr{T}$ and monoidal transformations and its full subcategory of lax symmetric monoidal functors. These are the structures defined in terms of the external smash product that correspond to monoids and commutative monoids in $\mathscr{D}\mathscr{T}$.

Proposition 5.1. *The category of monoids in $\mathscr{D}\mathscr{T}$ is isomorphic to the category of lax monoidal functors $\mathscr{D} \longrightarrow \mathscr{T}$. The category of commutative monoids in $\mathscr{D}\mathscr{T}$ is isomorphic to the category of lax symmetric monoidal functors $\mathscr{D} \longrightarrow \mathscr{T}$.*

Proof. Let $R : \mathscr{D} \longrightarrow \mathscr{T}$ be lax monoidal. We have a unit map $\lambda : S^0 \longrightarrow R(u)$ and product maps $\phi : R(d) \wedge R(e) \longrightarrow R(d\Box e)$ that make all coherence diagrams commute. We may view ϕ as a natural transformation $R \,\bar{\wedge}\, R \longrightarrow R \circ \Box$. By the defining properties of F_u and $\wedge$, λ and ϕ determine and are determined by maps $\tilde{\lambda} : u^* \longrightarrow R$ and $\tilde{\phi} : R \wedge R \longrightarrow R$ that give R a structure of monoid in $\mathscr{D}\mathscr{T}$. □

Now assume given a lax monoidal functor $R : \mathscr{D} \longrightarrow \mathscr{T}$. Regarding R as a monoid in $\mathscr{D}\mathscr{T}$, we have an evident notion of a (right) R-module M defined in terms of a map $M \wedge R \longrightarrow M$. The external version of an R-module is called a $\mathscr{D}$-spectrum over R.

Definition 5.2. A *$\mathscr{D}$-spectrum over R* is a $\mathscr{D}$-space $T : \mathscr{D} \longrightarrow \mathscr{T}$ together with a continuous natural transformation $\sigma : T \,\bar{\wedge}\, R \longrightarrow T \circ \Box$ such that the composite

$$T(d) \cong T(d) \wedge S^0 \xrightarrow{\text{id} \wedge \lambda} T(d) \wedge R(u) \xrightarrow{\sigma} T(d\Box u) \cong T(d)$$

is the identity and the following diagram commutes:

$$\begin{array}{ccc} T(d) \wedge R(e) \wedge R(f) & \xrightarrow{\sigma \wedge \text{id}} & T(d\Box e) \wedge R(f) \\ \Big\downarrow {\scriptstyle \text{id} \wedge \phi} & & \Big\downarrow {\scriptstyle \sigma} \\ T(d) \wedge R(e\Box f) & \xrightarrow[\sigma]{} & T(d\Box e\Box f). \end{array}$$

Proposition 5.3. *The category of (right) R-modules is isomorphic to the category of $\mathscr{D}$-spectra over R.*

We let $\mathscr{D}\mathscr{S}_R$ denote the category of $\mathscr{D}$-spectra over R, regarding it interchangably as the category of (right) R-modules.

We can mimic the definitions of tensor product and Hom functors in algebra.

Definition 5.4. For a right R-module T and a left R-module T', define $T \wedge_R T'$ to be the coequalizer in the category of $\mathscr{D}$-spaces (constructed levelwise) displayed in the diagram

$$T \wedge R \wedge T' \underset{\text{id} \wedge \mu'}{\overset{\mu \wedge \text{id}}{\rightrightarrows}} T \wedge T' \longrightarrow T \wedge_R T',$$

where μ and μ' are the given actions of R on T and T'.

Definition 5.5. For right R-modules T' and T'', define $F_R(T', T'')$ to be the equalizer displayed in the following diagram of $\mathscr{D}$-spaces:

$$F_R(T', T'') \longrightarrow F(T', T'') \underset{\omega}{\overset{\mu^*}{\rightrightarrows}} F(T' \wedge R, T'').$$

Here $\mu^* = F(\mu, \text{id})$ and ω is the adjoint of the composite

$$F(T', T'') \wedge T' \wedge R \xrightarrow{\varepsilon \wedge \text{id}} T'' \wedge R \xrightarrow{\nu} T'',$$

where μ and ν are the actions of R on T' and T''.

In the rest of this section, we assume that our given R is a commutative monoid in $\mathscr{D}\mathscr{T}$; that is, R is a lax symmetric monoidal functor $\mathscr{D} \longrightarrow \mathscr{T}$. In this case, the categories of left and right R-modules are isomorphic. Moreover, $T \wedge_R T'$ and

$F_R(T,T')$ inherit R-module structures from T or, equivalently, T'. For R-modules T, T', and T'',

$$\mathscr{D}\mathscr{S}_R(T \wedge_R T', T'') \cong \mathscr{D}\mathscr{S}_R(T, F_R(T', T'')). \tag{5.6}$$

Theorem 5.7. *When R is commutative, the category $\mathscr{D}\mathscr{S}_R$ of R-modules is closed symmetric monoidal with unit R under the smash product $\wedge_R$ and the function R-module functor F_R.*

Definition 5.8. A (commutative) R-algebra is a (commutative) monoid in $\mathscr{D}\mathscr{S}_R$.

The external version of an R-algebra is called a $\mathscr{D}$-FSP (functor with smash product) over R. We write τ consistently for symmetry isomorphisms.

Definition 5.9. A $\mathscr{D}$-FSP over R is a $\mathscr{D}$-space T together with a unit map $\eta : R \longrightarrow T$ of $\mathscr{D}$-spaces and a continuous natural product map $\mu : T \,\bar{\wedge}\, T \longrightarrow T \circ \Box$ of functors $\mathscr{D} \times \mathscr{D} \longrightarrow \mathscr{T}$ such that the composite

$$T(d) \cong T(d) \wedge S^0 \xrightarrow{\mathrm{id} \wedge \lambda} T(d) \wedge R(u) \xrightarrow{\mathrm{id} \wedge \eta} T(d) \wedge T(u) \xrightarrow{\mu} T(d \Box u) \cong T(d)$$

is the identity and the following unity, associativity, and centrality of unit diagrams commute:

$$\begin{array}{ccc} R(d) \wedge R(e) & \xrightarrow{\eta \wedge \eta} & T(d) \wedge T(e) \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \mu} \\ R(d \Box e) & \xrightarrow[\eta]{} & T(d \Box e), \end{array}$$

$$\begin{array}{ccc} T(d) \wedge T(e) \wedge T(f) & \xrightarrow{\mu \wedge \mathrm{id}} & T(d \Box e) \wedge T(f) \\ {\scriptstyle \mathrm{id} \wedge \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ T(d) \wedge T(e \Box f) & \xrightarrow[\mu]{} & T(d \Box e \Box f), \end{array}$$

and

$$\begin{array}{ccccc} R(d) \wedge T(e) & \xrightarrow{\eta \wedge \mathrm{id}} & T(d) \wedge T(e) & \xrightarrow{\mu} & T(d \Box e) \\ {\scriptstyle \tau}\downarrow & & & & \downarrow{\scriptstyle T(\tau)} \\ T(e) \wedge R(d) & \xrightarrow[\mathrm{id} \wedge \eta]{} & T(e) \wedge T(d) & \xrightarrow[\mu]{} & T(e \Box d) \end{array}$$

A $\mathscr{D}$-FSP is commutative if the following diagram commutes, in which case the centrality of unit diagram just given commutes automatically:

$$\begin{array}{ccc} T(d) \wedge T(e) & \xrightarrow{\mu} & T(d \Box e) \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle T(\tau)} \\ T(e) \wedge T(d) & \xrightarrow[\mu]{} & T(e \Box d). \end{array}$$

A $\mathscr{D}$-FSP over R is a $\mathscr{D}$-spectrum over R with additional structure.

Lemma 5.10. *A $\mathscr{D}$-FSP over R has an underlying $\mathscr{D}$-spectrum over R with structure map*

$$\sigma = \mu \circ (\mathrm{id} \,\bar{\wedge}\, \eta) : T \,\bar{\wedge}\, R \longrightarrow T \circ \Box.$$

Proposition 5.11. *The category of R-algebras is isomorphic to the category of $\mathscr{D}$-FSP's over R. The category of commutative R-algebras is isomorphic to the category of commutative $\mathscr{D}$-FSP's over R.*

6. An interpretation of diagram spectra as diagram spaces

Let $\mathscr{D}$ be symmetric monoidal and fix a lax monoidal functor $R : \mathscr{D} \longrightarrow \mathscr{T}$. We do not require R to be lax symmetric monoidal in general, although that is the case of greatest interest. We reinterpret the category $\mathscr{D}\mathscr{S}_R$ of $\mathscr{D}$-spectra over R, alias the category of right R-modules, as the category $\mathscr{D}_R\mathscr{T}$ of $\mathscr{D}_R$-spaces, where $\mathscr{D}_R$ is a lax monoidal category constructed from $\mathscr{D}$ and R. If R is a lax symmetric monoidal functor, then $\mathscr{D}_R$ is a symmetric monoidal category. In this case, we can reinterpret the smash product $\wedge_R$ of R-modules as the smash product in the category of $\mathscr{D}_R$-spaces. Conceptually, this simplifies the theory by reducing the study of $\mathscr{D}$-spectra over R to a special case of the general study of functor categories.

Just as in algebra, for a $\mathscr{D}$-space T, $T \wedge R$ is the free R-module generated by T. Recall the represented functors d^* from Notations 2.5 and remember that they behave contravariantly with respect to d.

Construction 6.1. We construct a monoidal category $\mathscr{D}_R$ and a strong monoidal functor $\delta : \mathscr{D} \longrightarrow \mathscr{D}_R$. The objects of $\mathscr{D}_R$ are the objects of $\mathscr{D}$. For objects d and e of $\mathscr{D}$, the space of morphisms $d \longrightarrow e$ in $\mathscr{D}_R$ is

$$\mathscr{D}_R(d,e) = \mathscr{D}\mathscr{S}_R(e^* \wedge R, d^* \wedge R),$$

and composition is inherited from composition in $\mathscr{D}\mathscr{S}_R$. Thus $\mathscr{D}_R$ may be identified with the full subcategory of $\mathscr{D}\mathscr{S}_R^{op}$ whose objects are the free R-modules $d^* \wedge R$. We specify δ on morphisms by smashing maps given by the contravariant functoriality of d^* with the identity map of R. Observe that we have adjunction isomorphisms

$$\mathscr{D}_R(d,e) \cong (d^* \wedge R)(e).$$

On objects, the product $\Box_R$ in $\mathscr{D}_R$ is the product $\Box$ of $\mathscr{D}$. The unit object is the unit object u of $\mathscr{D}$. The product $f \Box_R f'$ of morphisms $f : e^* \wedge R \longrightarrow d^* \wedge R$ and $f' : e'^* \wedge R \longrightarrow d'^* \wedge R$ is

$$f \wedge_R f' : (e \Box e')^* \wedge R \cong (e \wedge R) \wedge_R (e'^* \wedge R) \longrightarrow (d \wedge R) \wedge_R (d'^* \wedge R) \cong (d \Box d')^* \wedge R.$$

Observe that

$$(d^* \wedge R)(e) \cong \mathrm{colim}_{\alpha : f \Box g \longrightarrow e}\, \mathscr{D}(d,f) \wedge R(g). \tag{6.2}$$

Taking α to be the canonical isomorphism $e \Box u \cong e$ and using the unit $\lambda : S^0 \longrightarrow R(u)$, we can identify δ on morphisms as the evident map

$$\mathscr{D}(d,e) = \mathscr{D}(d,e) \wedge S^0 \longrightarrow \mathscr{D}(d,e) \wedge R(u) \longrightarrow (d^* \wedge R)(e).$$

By inspection of definitions, δ is a strong monoidal functor.

Proposition 6.3. *The categories $\mathscr{D}\mathscr{S}_R$ of $\mathscr{D}$-spectra over R and $\mathscr{D}_R\mathscr{T}$ of $\mathscr{D}_R$-spaces are isomorphic.*

Proof. Taking α in (6.2) to be the identity map of $d \Box e$ and using the identity map $d \longrightarrow d$, we obtain an inclusion $\nu : R(e) \longrightarrow (d^* \wedge R)(d \Box e)$. Let T be a $\mathscr{D}_R$-space. Pullback along δ gives T a structure of $\mathscr{D}$-space. Pullback along ν of the evaluation map $\mathscr{D}_R(d, d \Box e) \wedge T(d) \longrightarrow T(d \Box e)$ gives the components $T(d) \wedge R(e) \longrightarrow T(d \Box e)$ of a map $T \bar{\wedge} R \longrightarrow T \circ \Box$. Via (3.5), this gives an action of R on T. These two actions determine the original action of $\mathscr{D}_R$. Indeed, working conversely, if T is an R-module and $\alpha : f\Box g \longrightarrow e$ is a morphism of $\mathscr{D}$, then the composites displayed in the following diagram pass to colimits to define the evaluation maps $(d^* \wedge R)(e) \wedge T(d) \longrightarrow T(e)$ of a functor $T : \mathscr{D}_R \longrightarrow \mathscr{T}$.

$$\begin{array}{ccc}
\mathscr{D}(d,f) \wedge R(g) \wedge T(d) & & \\
\Big\downarrow{\scriptstyle \mathrm{id}\wedge\tau} & & \\
\mathscr{D}(d,f) \wedge T(d) \wedge R(g) & \xrightarrow{\varepsilon\wedge\mathrm{id}} & T(f) \wedge R(g) \\
\Big\downarrow{\scriptstyle ((-)\Box\, \mathrm{id})\wedge\mu} & & \Big\downarrow{\scriptstyle \mu} \\
\mathscr{D}(d\Box g, f\Box g) \wedge T(d\Box g) & \xrightarrow[\varepsilon]{} & T(f\Box g) \\
 & & \Big\downarrow{\scriptstyle T(\alpha)} \\
 & & T(e).
\end{array}$$

Here ε is the evaluation map of T and μ is the action of R on T. This gives the desired isomorphism of categories. □

In the commutative case, we have the following important addenda.

Proposition 6.4. *If $R : \mathscr{D} \longrightarrow \mathscr{T}$ is a lax symmetric monoidal functor, then $\mathscr{D}_R$ is a symmetric monoidal category under $\Box_R$, $\delta : \mathscr{D} \longrightarrow \mathscr{D}_R$ is a strong symmetric monoidal functor, and the isomorphism of categories $\mathscr{D}\mathscr{S}_R \cong \mathscr{D}_R\mathscr{T}$ is an isomorphism of symmetric monoidal categories.*

Proof. Inspections of definitions give the statements about $\mathscr{D}_R$ and δ. To show that the smash products agree under the isomorphism between $\mathscr{D}\mathscr{S}_R$ and $\mathscr{D}_R\mathscr{T}$, we can either compare the definitions of the respective smash products directly, or we can compare the defining universal properties. Note that the unit $(u_{\mathscr{D}_R})^*$ of the smash product of $\mathscr{D}_R$-spaces is isomorphic to R since

$$(u_{\mathscr{D}_R})^*(d) = \mathscr{D}_R(u_{\mathscr{D}_R}, d) = ((u_{\mathscr{D}})^* \wedge R)(e) \cong R(e). \quad \Box$$

Remark 6.5. If $R = (u_{\mathscr{D}})^*$, then $\delta : \mathscr{D} \longrightarrow \mathscr{D}_R$ is an identification. That is, as in any symmetric monoidal category, $\mathscr{D}$-spaces admit a unique structure of module over the unit for the smash product.

7. Forgetful and prolongation functors on diagram spectra

We use the categories $\mathscr{D}_R$ to reduce comparisons of categories of diagram spectra to comparisons of categories of diagram spaces. Return to the context of Section 4 and assume given a symmetric monoidal category $\mathscr{D}$ under $\mathscr{C}$, with strong symmetric monoidal functor $\iota : \mathscr{C} \longrightarrow \mathscr{D}$, where both $\mathscr{C}$ and $\mathscr{D}$ are skeletally small.

Proposition 7.1. *Let $R : \mathscr{D} \longrightarrow \mathscr{T}$ be a lax monoidal functor and let $Q = R \circ \iota : \mathscr{C} \longrightarrow \mathscr{T}$. Then $\iota : \mathscr{C} \longrightarrow \mathscr{D}$ extends to a strong monoidal functor $\kappa : \mathscr{C}_Q \longrightarrow \mathscr{D}_R$. If R is lax symmetric monoidal, then κ is strong symmetric monoidal.*

Proof. Using (6.2) to write the morphism sets of $\mathscr{C}_Q$ and $\mathscr{D}_R$ as colimits, we see immediately that smash products of maps $\iota : \mathscr{C}(a,b) \longrightarrow \mathscr{D}(\iota a, \iota b)$ and identity maps of the spaces $Q(c) = R(\iota c)$ pass to colimits to give the required extension. □

Applied to $\kappa : \mathscr{C}_Q \longrightarrow \mathscr{D}_R$, the results of Section 4 give a prolongation functor $\mathbb{P}$ from $\mathscr{C}_Q$-spaces (= $\mathscr{C}$-spectra over Q) to $\mathscr{D}_R$-spaces (= $\mathscr{D}$-spectra over R) that is left adjoint to the evident forgetful functor $\mathbb{U}$. Just as Proposition 6.4 gives two equivalent descriptions of the smash product of R-modules (when R is symmetric), so we have two equivalent descriptions of the prolongation of Q-modules, the second of which does not make use of the categories $\mathscr{C}_Q$ and $\mathscr{D}_R$.

Lemma 7.2. *Consider $\mathbb{P} : \mathscr{C}\mathscr{T} \longrightarrow \mathscr{D}\mathscr{T}$. If $Q : \mathscr{C} \longrightarrow \mathscr{T}$ is a lax (symmetric) monoidal functor, then $\mathbb{P}Q : \mathscr{D} \longrightarrow \mathscr{T}$ is a lax (symmetric) monoidal functor and $\mathbb{P}$ restricts to a functor from the category of $\mathscr{C}$-spectra over Q to the category of $\mathscr{D}$-spectra over $\mathbb{P}Q$. Moreover, the adjunction (4.3) restricts to an adjunction*

$$\mathscr{D}\mathscr{S}_{\mathbb{P}Q}(\mathbb{P}T, Y) \cong \mathscr{C}\mathscr{S}_Q(T, \mathbb{U}Y). \tag{7.3}$$

Proof. The first statement is immediate from Proposition 4.8. For the second statement, we must show that if T is a Q-module and Y is a $\mathbb{P}Q$-module, then a map $f : \mathbb{P}T \longrightarrow Y$ of $\mathscr{D}$-spaces is a map of $\mathbb{P}Q$-modules if and only if its adjoint $\tilde{f} : T \longrightarrow \mathbb{U}Y$ is a map of Q-modules. The proof is a pair of diagram chases that ultimately boil down to use of the pair of triangles displayed in Proposition 4.2. □

Lemma 7.4. *Let $Q = R \circ \iota = \mathbb{U}(R)$, where R is a lax (symmetric) monoidal functor $\mathscr{D} \longrightarrow \mathscr{T}$. The adjoint $\varepsilon : \mathbb{P}Q = \mathbb{P}\mathbb{U}R \longrightarrow R$ of the identity map of $\mathbb{U}R$ is a map of (commutative) monoids in $\mathscr{D}\mathscr{T}$, hence an R-module Y may be regarded as a $\mathbb{P}Q$-module by pullback of the action along ε. Passage to coequalizers specifies an extension of scalars functor*

$$(-) \wedge_{\mathbb{P}Q} R : \mathscr{D}\mathscr{S}_{\mathbb{P}Q} \longrightarrow \mathscr{D}\mathscr{S}_R$$

such that, for R-modules T and $\mathbb{P}Q$-modules T',

$$\mathscr{D}\mathscr{S}_R(T \wedge_{\mathbb{P}Q} R, T') \cong \mathscr{D}\mathscr{S}_{\mathbb{P}Q}(T, T').$$

When R is symmetric, the pullback of action functor is lax symmetric monoidal and the extension of scalars functor is strong symmetric monoidal.

Proof. The proof is formally the same as for extension of scalars in algebra. □

Now the uniqueness of adjoints gives the following result.

Proposition 7.5. *Let $Q = R \circ \iota$. Then $\mathbb{P} : \mathscr{C}_Q\mathscr{T} \longrightarrow \mathscr{D}_R\mathscr{T}$ agrees under the isomorphisms of its source and target with the composite of $\mathbb{P} : \mathscr{C}\mathscr{S}_Q \longrightarrow \mathscr{D}\mathscr{S}_{\mathbb{P}Q}$ and extension of scalars $\mathscr{D}\mathscr{S}_{\mathbb{P}Q} \longrightarrow \mathscr{D}\mathscr{S}_R$.*

8. Examples of diagram spectra

It is high time that we specialized the general abstract theory to the examples of interest in stable homotopy theory. Here we change our point of view. So far, we have considered general lax monoidal functors $R : \mathscr{D} \longrightarrow \mathscr{T}$, usually symmetric. Now we focus on a particular, canonical, choice, which we denote by S, or $S_{\mathscr{D}}$ when necessary for clarity, to suggest spheres. It is a faithful functor in all of our examples.

We take S^n to be the one-point compactification of $\mathbb{R}^n$; the one-point compactification of $\{0\}$ is S^0, and it is convenient to let $S^n = *$ if $n < 0$. Similarly, for a finite dimensional real inner product space V, we take S^V to be the one-point compactification of V. Our first example is elementary, but crucial to the theory.

Example 8.1 (Prespectra)**.** Let $\mathscr{N}$ be the (unbased) category of non-negative integers, with only identity morphisms between them. The symmetric monoidal structure is given by addition, with 0 as unit. An $\mathscr{N}$-space is a sequence of based spaces. The canonical functor $S = S_{\mathscr{N}}$ sends n to S^n. It is strong monoidal, but it is *not* symmetric since permutations of spheres are not identity maps. This is the source of difficulty in defining the smash product in the stable homotopy category. A *prespectrum* is an $\mathscr{N}$-spectrum over S. Let $\mathscr{P}$ denote the category of prespectra. Since S^n is canonically isomorphic to the n-fold smash power of S^1, the category of prespectra defined in this way is isomorphic to the usual category of prespectra, whose objects are sequences of based spaces T_n and based maps $\Sigma T_n \longrightarrow T_{n+1}$.

The shift suspension functors to $\mathscr{N}$-spectra are given by $(F_m X)_n = X \wedge S^{n-m}$. The smash product of $\mathscr{N}$-spaces is given by

$$(T \wedge T')_n = \bigvee_{p=0}^{n} T_p \wedge T'_{n-p}.$$

The category $\mathscr{N}_S$ such that an $\mathscr{N}$-spectrum is an $\mathscr{N}_S$-space has morphism spaces

$$\mathscr{N}_S(m,n) = S^{n-m}.$$

Because $S_{\mathscr{N}}$ is not symmetric, the category of $\mathscr{N}$-spectra does not have a smash product that makes it a symmetric monoidal category. For all other $\mathscr{D}$ that we consider, the functor $S_{\mathscr{D}}$ is a strong symmetric monoidal embedding $\mathscr{D} \longrightarrow \mathscr{T}$. Therefore the category of $\mathscr{D}$-spectra over S is symmetric monoidal.

Example 8.2 (Symmetric spectra)**.** Let Σ be the (unbased) category of finite sets $\mathbf{n} = \{1, \dots, n\}$, $n \geq 0$, and their permutations; thus there are no maps $\mathbf{m} \longrightarrow \mathbf{n}$ for $m \neq n$, and the set of maps $\mathbf{n} \to \mathbf{n}$ is the symmetric group Σ_n. The symmetric monoidal structure is given by concatenation of sets and block sum of permutations, with $\mathbf{0}$ as unit. The canonical functor $S = S_\Sigma$ sends $\mathbf{n}$ to S^n. A *symmetric spectrum* is a Σ-spectrum over S. Let $\Sigma\mathscr{S}$ denote the category of symmetric spectra. Define a strong symmetric monoidal faithful functor $\iota : \mathscr{N} \longrightarrow \Sigma$ by sending n to $\mathbf{n}$ and observe that $S_{\mathscr{N}} = S_\Sigma \circ \iota$. In effect, we have made S_Σ symmetric by adding permutations to the morphisms of $\mathscr{N}$. The idea of doing this is due to Jeff Smith.

The shift suspension functors to symmetric spectra are given by

$$(F_m X)(\mathbf{n}) = \Sigma_{n+} \wedge_{\Sigma_{n-m}} (X \wedge S^{n-m}).$$

The smash product of Σ-spaces is given by

$$(T \wedge T')(\mathbf{n}) \cong \bigvee_{p=0}^{n} \Sigma_{n+} \wedge_{\Sigma_p \times \Sigma_{n-p}} T(\mathbf{p}) \wedge T(\mathbf{n} - \mathbf{p})$$

as a Σ_n-space. Implicitly, we are considering the set of partitions of the set $\mathbf{n}$. If we were considering the category of all finite sets k, we could rewrite this as

$$(T \wedge T')(k) = \bigvee_{j \subset k} T(j) \wedge T'(k - j),$$

and this reinterpretation explains the associativity and commutativity of $\wedge$. The category Σ_S such that a Σ-spectrum is a Σ_S-space has morphism spaces

$$\Sigma_S(\mathbf{m}, \mathbf{n}) = \Sigma_{n+} \wedge_{\Sigma_{n-m}} S^{n-m}.$$

Example 8.3. The functor S_Σ is the case $X = S^1$ of the strong symmetric monoidal functor $S_X : \Sigma \longrightarrow \mathscr{T}$ that sends $\mathbf{n}$ to the n-fold smash power $X^{(n)}$ for a based space X. Moreover, the S_X give all strong symmetric monoidal functors $\Sigma \longrightarrow \mathscr{T}$. Applied to S_X, our theory constructs a symmetric monoidal category of "S_X-modules". The homotopy theory of these categories is relevant to localization theory.

Example 8.4 (Orthogonal spectra)**.** Let $\mathscr{I}$ be the (unbased) category of finite dimensional real inner product spaces and linear isometric isomorphisms; there are no maps $V \longrightarrow W$ unless $\dim V = \dim W = n$ for some $n \geq 0$, when the space of morphisms $V \longrightarrow W$ is homeomorphic to the orthogonal group $O(n)$. The symmetric monoidal structure is given by direct sums, with $\{0\}$ as unit. The canonical functor $S = S_{\mathscr{I}}$ sends V to S^V. An *orthogonal spectrum* is an $\mathscr{I}$-spectrum over S. Let $\mathscr{I}\mathscr{S}$ denote the category of orthogonal spectra. Define a strong symmetric monoidal faithful functor $\iota : \Sigma \longrightarrow \mathscr{I}$ by sending $\mathbf{n}$ to $\mathbb{R}^n$ and using the standard inclusions $\Sigma_n \longrightarrow O(n)$. Observe that $S_\Sigma = S_{\mathscr{I}} \circ \iota$.

The shift suspension functors to orthogonal spectra are given on $W \supset V$ by

$$(F_V X)(W) = O(W)_+ \wedge_{O(W-V)} (X \wedge S^{W-V}),$$

where $W - V$ is the orthogonal complement of V in W; an analogous description applies whenever $\dim W \geq \dim V$, and $(F_V X)(W) = *$ if $\dim W < \dim V$. Note that we can restrict attention to the skeleton $\{\mathbb{R}^n\}$ of $\mathscr{I}$. For an inner product space V of dimension n, choose a subspace V_p of dimension p for each $p \leq n$. The smash product of $\mathscr{I}$-spaces is given by

$$(T \wedge T')(V) \cong \bigvee_{p=0}^{n} O(V)_+ \wedge_{O(V_p)\times O(V-V_p)} T(V_p) \wedge T'(V - V_p)$$

as an $O(V)$-space. This describes the topology correctly, but to see the associativity and commutativity of $\wedge$, we can rewrite this set-theoretically as

$$(T \wedge T')(V) = \bigvee_{W \subset V} T(W) \wedge T'(V - W).$$

The category $\mathscr{I}_S$ such that an $\mathscr{I}$-spectrum is an $\mathscr{I}_S$-space has morphism spaces

$$\mathscr{I}_S(V, W) = O(W)_+ \wedge_{O(W-V)} S^{W-V}$$

for $V \subset W$.

This example admits several variants. For instance, we can use real vector spaces and their isomorphisms, without insisting on inner product structures and isometries, or we can use complex vector spaces.

Example 8.5. Let V have dimension n and let $TO(V)$ be the Thom space of the tautological n-plane bundle over the Grassmannian of n-planes in $V \oplus V$. Then $TO(-)$ is the object function of a lax symmetric monoidal functor $TO : \mathscr{I} \longrightarrow \mathscr{T}$.

The equivariant generalization of this example was exploited in [6]. Our formal theory applies to functors like TO, but we focus on the canonical functors $S_{\mathscr{D}}$.

Example 8.6 ($\mathscr{W}$-spaces)**.** It is tempting to take $\mathscr{D} = \mathscr{T}$, but that does not have a small skeleton. We can take $\mathscr{D}$ to be any skeletally small based subcategory of $\mathscr{T}$ that contains S^0 and is closed under smash products, taking $S_{\mathscr{D}}$ to be the inclusion $\mathscr{D} \longrightarrow \mathscr{T}$. In particular, we can take $\mathscr{D}$ to be the category $\mathscr{W}$ of based spaces homeomorphic to finite CW complexes. The theory works equally well if we redefine $\mathscr{W}$ in terms of countable rather than finite CW complexes. We have evident strong symmetric monoidal faithful functors $\Sigma \longrightarrow \mathscr{W}$ and $\mathscr{I} \longrightarrow \mathscr{W}$ under which $S_{\mathscr{W}}$ restricts to S_Σ and $S_{\mathscr{I}}$.

The shift suspension functors to $\mathscr{W}$-spaces are given by

$$(F_Y X)(Z) = F(Y, Z) \wedge X.$$

The simplicial counterpart of this example is treated in detail by Lydakis [14]. This example suggests an alternative way of viewing Σ and $\mathscr{I}$.

Remark 8.7. It is sometimes convenient, and sometimes inconvenient, to change point of view and think of the objects of Σ and $\mathscr{I}$ as the spheres S^n and S^V, thus thinking of Σ and $\mathscr{I}$ as subcategories of $\mathscr{W}$. With this point of view, $\Box$ is a subfunctor of $\wedge$ and S is the inclusion of a symmetric monoidal subcategory.

All of our examples so far are categories under $\mathscr{N}$. However, our last example is not of this type.

Example 8.8 ($\mathscr{F}$-spaces = Γ-spaces)**.** Let $\mathscr{F}$ be the category of finite based sets $\mathbf{n}^+ = \{0, 1, \dots, n\}$ and all based maps, where 0 is the basepoint. This is the opposite of Segal's category Γ [22]. This category is based with base object the one point set $\mathbf{0}^+$. Take $\Box$ to be the smash product of finite based sets; to be precise, we order the non-zero elements of $\mathbf{m}^+ \wedge \mathbf{n}^+$ lexicographically. The unit object is $\mathbf{1}^+$. The canonical functor $S_{\mathscr{F}}$ sends $\mathbf{n}^+$ to $\mathbf{n}^+$ regarded as a discrete based space; it is the restriction to $\mathscr{F}$ of the functor $S_{\mathscr{W}}$.

In contrast to the cases of symmetric spectra and orthogonal spectra, the action of $S_{\mathscr{D}}$ required of $\mathscr{D}$-spectra gives no additional data when $\mathscr{D} = \mathscr{F}$ or $\mathscr{D} = \mathscr{W}$.

Lemma 8.9. *Let $S_{\mathscr{D}} : \mathscr{D} \longrightarrow \mathscr{T}$ be an embedding of $\mathscr{D}$ as a full symmetric monoidal subcategory of $\mathscr{T}$. Then a $\mathscr{D}$-space T admits a unique structure of $\mathscr{D}$-spectrum, and the categories of $\mathscr{D}$-spaces and $\mathscr{D}$-spectra are isomorphic. In particular, this applies to $\mathscr{D} = \mathscr{F}$ and $\mathscr{D} = \mathscr{W}$.*

Proof. This is an instance of Remark 6.5, but it is worthwhile to explain it explicitly. Let X and Y be spaces in $\mathscr{D}$, omit the embedding $S_{\mathscr{D}}$ from the notation, and write $\wedge$ for $\Box$. The action $\sigma : T(X) \wedge Y \longrightarrow T(X \wedge Y)$ is the adjoint of the composite

$$Y \xrightarrow{\alpha} \mathscr{T}(X, X \wedge Y) = \mathscr{D}(X, X \wedge Y) \xrightarrow{T} \mathscr{T}(T(X), T(X \wedge Y)),$$

where $\alpha(y)(x) = x \wedge y$. The equality holds because $\mathscr{D}$ is a full subcategory of $\mathscr{T}$, and the map T is continuous by our requirement that $\mathscr{D}$-spaces be continuous functors. For the uniqueness, let $S^0 = \{*, 1\}$ and observe that an element $y \in Y$ determines the based map $\tilde{y} : S^0 \longrightarrow Y$ with $\tilde{y}(1) = y$. The naturality and unit conditions in the definition of an action σ force the relation

$$\sigma(t \wedge y) = \sigma \circ (\mathrm{id} \wedge \tilde{y})(t \wedge 1) = T(\mathrm{id} \wedge \tilde{y})(t \wedge 1)$$

for $t \in T(X)$. This agrees with our definition of σ and proves its uniqueness. $\Box$

Special cases of $\mathscr{D}$-FSP's (over $S_{\mathscr{D}}$) were first defined by different authors.

Remark 8.10. Up to nomenclature, $\mathscr{D}$-FSP's were first introduced as follows.

1. A $\mathscr{T}$-FSP is an FSP as introduced by Bökstedt in [2], although his definition was simplicial and he imposed convergence and connectivity conditions.

2. A commutative $\mathscr{I}$-FSP is an $\mathscr{I}_*$-prefunctor as defined by May, Quinn, and Ray [18]; this was the earliest definition of this general type.
3. A Σ-FSP is a symmetric ring spectrum as defined by Smith [8]. This notion first appeared under the name "strictly associative ring spectrum" in Gunnarson [7]. The name "FSP defined on spheres" has also been used.
4. An $\mathscr{F}$-FSP is a Gamma-ring as defined by Lydakis and Schwede [13, 20].

Different kinds of FSP's arise naturally in different applications. In Bökstedt's construction of topological Hochschild homology, it is natural to use Σ-FSP's. In the early applications of [4, 18, 17] and in recent equivariant applications [6], it is essential to work with $\mathscr{I}$-FSP's. In other applications, $\mathscr{F}$-FSP's are essential [20]. We shall compare these categories homotopically in the sequel [16].

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