# THE HOMOTOPY THEORY OF $E_\infty$ ALGEBRAS

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ABSTRACT. Let $k$ be a commutative ring and let $\mathcal{C}$ be the operad of differential graded $k$-modules obtained as the singular $k$-chains of the linear isometries operad [4, §V.9]. We show that the category of $\mathcal{C}$-algebras is a proper closed model category. We use the amenable description of the coproduct in this category [4, V.3.4] to analyze the coproduct of and develop a homotopy theory for algebras over an arbitrary $E_\infty$ operad. Draft: January 26, 1998, 17:26.

## INTRODUCTION

Recent years have brought increasing interest in operads and in algebras over operads. The monograph [4] provides an exposition of the basic theory of operads and makes a particular study of the algebras over $E_\infty$ operads and the modules over these algebras. For a particular $E_\infty$ operad $\mathcal{C}$, [4] obtains a good homotopy theory on the categories of modules. In this paper, we study the "homotopy theory" of $E_\infty$ algebras themselves.

Ideally, we would like to show that for an $E_\infty$ operad $\mathcal{E}$, the category of $\mathcal{E}$-algebras forms a closed model category. Although we have not succeeded in this for a general $E_\infty$ operad, we do provide a closed model structure on the category of $\mathcal{C}$-algebras, for the $E_\infty$ operad $\mathcal{C}$ of [4, §V.9]. In general, we prove that the category of $\mathcal{E}$-algebras has many of the useful properties expected from a closed model category. We construct and make sense of "homotopies" of $E_\infty$ algebras; these allow us to analyze the homotopical properties of adjoint functors as done for model categories in [1, 7]. In addition, we analyze the coproducts and pushouts of $E_\infty$ algebras and find sufficient conditions for them to behave as expected: we prove that in favorable cases the pushout has homology given by the "$E_\infty$ torsion product" of [4].

We present an application of this theory in [5], to provide a characteristic $p$ version of Sullivan's theorem comparing the unstable rational homotopy category to the category of commutative differential graded algebras over the rational numbers. Over the rational numbers, $E_\infty$ algebras are essentially the same as commutative differential graded algebras; for a precise statement see [4, II.1.5]. Over a field of characteristic $p$, the category of commutative differential graded algebras cannot have a good homotopy theory; for example, the free commutative differential graded algebra on a contractible differential graded module is never quasi-isomorphic to the ground field. On the other hand, the homotopy theory we develop in this paper for categories of $E_\infty$ algebras allows us to deduce in [5] that the homotopy category of connected $p$-complete nilpotent spaces of finite $p$-type are contravariantly equivalent to a full subcategory of the homotopy category of $E_\infty$ $\bar{\mathbb{F}}_p$-algebras, where $\bar{\mathbb{F}}_p$ denotes the algebraic closure of the field with $p$ elements.

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## 1. Summary of Results

We take this section to summarize for easy reference the main results proved in this paper. Throughout, $k$ denotes a fixed but arbitrary commutative ring that we take as ground ring. All constructions are formed in the category of $k$-modules, differential graded $k$-modules, or their more structured subcategories. In particular "$\otimes$" always denotes the tensor product over $k$.

We begin with the operad $\mathcal{C}$ of [4, §V.9]. For $\mathcal{C}$, the fundamental result is the following theorem, proved in Section 3.

**Theorem 1.1.** *Let $k$ be a commutative ring and let $\mathcal{C}$ be the $E_\infty$ operad of differential graded $k$-modules of* [4, §V.9]. *The category of $\mathcal{C}$-algebras is a proper closed model category, with weak equivalences the quasi-isomorphisms and fibrations the surjections.*

The cofibrations in this category also admit a straightforward description as the retracts of "relative cell inclusions". The relative cell inclusions are the maps obtained by a cell attachment process:

**Definition 1.2.** Let $\mathcal{G}$ be an operad and let $\mathbb{G}$ denote the free $\mathcal{G}$-algebra functor. A map of $\mathcal{G}$-algebras $f\colon A \to B$ is *relative cell $\mathcal{G}$-algebra inclusion* if there exists a sequence of $\mathcal{G}$-algebra maps $A = A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} \cdots$ such that

(i) $B \cong \operatorname{Colim} i_n$ under $A$.
(ii) Each map $i_n$ is formed as a pushout of $\mathcal{G}$-algebras

$$\begin{tikzcd} \mathbb{G}X_n \arrow[r] \arrow[d] & \mathbb{G}CX_n \arrow[d] \\ A_n \arrow[r, "i_n"'] & A_{n+1} \end{tikzcd}$$

where $X_n$ is a free differential graded $k$-module with zero differential, $CX_n$ is the cone on $X_n$ [4, p. 58], and $X_n \to CX_n$ is the canonical inclusion.

We say that a $\mathcal{G}$-algebra $A$ is a cell $\mathcal{G}$-algebra if the initial map $\mathcal{G}(0) \to A$ is a relative cell inclusion.

As mentioned in the introduction, we are presently unable to obtain closed model structures on the categories of algebras over $E_\infty$ operads other than $\mathcal{C}$. The problem is the difficulty in analyzing the coproduct. To explain this, consider the simplest contractible differential graded $k$-module $Ck[n]$ that consists of $k$ in degrees $n$ and $n+1$ and zero in all other degrees. The obstruction for the existence of a closed model structure on the category of $\mathcal{G}$-algebras is the behavior of the coproduct of $\mathbb{G}Ck[n]$ with arbitrary $\mathcal{G}$-algebras. In Section 2, we prove the following proposition.

**Proposition 1.3.** *Let $\mathcal{G}$ be an operad. The following are equivalent.*

(i) *For every $\mathcal{G}$-algebra $A$ and every integer $n$, the inclusion $A \to A \amalg \mathbb{G}Ck[n]$ is a quasi-isomorphism.*
(ii) *The category of $\mathcal{G}$-algebras is a closed model category with weak equivalences the quasi-isomorphisms and fibrations the surjections.*

We are able to verify condition 1.3.(i) for the $E_\infty$ operad $\mathcal{C}$, but not for any other $E_\infty$ operad. On the other hand, if we restrict to the case of cell algebras, we can analyze coproducts in the category of $\mathcal{E}$-algebras for an arbitrary $E_\infty$ operad $\mathcal{E}$. The following theorem is proved in Section 6.

**Theorem 1.4.** *Let $\mathcal{E}$ be an $E_\infty$ operad. If $A$ and $B$ are cell $\mathcal{E}$-algebras then there is a canonical isomorphism $H_*(A \amalg B) \cong \mathrm{Tor}_*(A, B)$.*

We can also prove a relative version of the previous result. Part V of [4] describes a generalization of the differential torsion product over a differential graded $k$-algebra to an "$E_\infty$ torsion product" over an $E_\infty$ $k$-algebra. For a $\mathcal{C}$-algebra, this is constructed directly; for an $E_\infty$ algebra over a different $E_\infty$ operad, this is constructed by first replacing by an equivalent $\mathcal{C}$-algebra. We prove the following theorem in Section 6.

**Theorem 1.5.** *Let $\mathcal{E}$ be an $E_\infty$ operad. If $A$ is a cell $\mathcal{E}$-algebra and $A \to B$ and $A \to C$ are relative cell inclusions, then there is a canonical isomorphism $H_*(B \amalg_A C) \cong \mathrm{Tor}^A_*(B, C)$.*

An Eilenberg–Moore spectral sequence for the calculation of this $E_\infty$ torsion product is given in [4, V.7.3]. In certain cases, this type of spectral sequence can be constructed directly from the $\mathcal{E}$-algebras, using the *bar construction*. For $\mathcal{E}$-algebras $A$, $B$, $C$ and $\mathcal{E}$-algebra maps $A \to B$, $A \to C$, we can form a simplicial $\mathcal{E}$-algebra $\beta^{\mathcal{E}}_\bullet(B, A, C)$ that in degree $n$ is given by

$$\beta^{\mathcal{E}}_n(B, A, C) = B \amalg \underbrace{A \amalg \cdots \amalg A}_{n \text{ factors}} \amalg C.$$

The face maps are induced by the codiagonal maps $A \amalg A \to A$, $B \amalg A \to B \amalg B \to B$, and $A \amalg C \to C \amalg C \to C$. The degeneracies are induced by the inclusions of the form $X \amalg Y \to X \amalg A \amalg Y$ for each factor of $A$ in the coproduct. From a simplicial differential graded $k$-module, we obtain a differential graded $k$-module by *normalization* (called "totalization" in [4, §II.V]), the obvious generalization of forming the normalized chain complex of a simplicial abelian group. In the case of a simplicial $\mathcal{E}$-algebra, the normalization obtains a $\mathcal{E}$-algebra structure via the shuffle map [4, p. 51]. Write $\beta^{\mathcal{E}}(B, A, C)$ for the $\mathcal{E}$-algebra obtained as the normalization of $\beta^{\mathcal{E}}_\bullet(B, A, C)$. Regarding $B \amalg_A C$ as a constant simplicial $\mathcal{E}$-algebra, the natural map $B \amalg C \to B \amalg_A C$ induces a natural map of simplicial $\mathcal{E}$-algebras $\beta^{\mathcal{E}}_\bullet(B, A, C) \to B \amalg_A C$ and therefore a map of $\mathcal{E}$-algebras $\beta^{\mathcal{E}}(B, A, C) \to B \amalg_A C$. In Section 7 we prove the following theorem giving a sufficient condition for this map to be a quasi-isomorphism.

**Theorem 1.6.** *Let $\mathcal{E}$ be an $E_\infty$ operad. If $A$, $B$, and $C$ are cell $\mathcal{E}$-algebras and $A \to B$ and $A \to C$ are relative cell inclusions, then the natural map $\beta^{\mathcal{E}}(B, A, C) \to B \amalg_A C$ is a quasi-isomorphism.*

Since $\beta^{\mathcal{E}}(B, A, C)$ is the normalization of a simplicial differential graded $k$-module, it has a canonical filtration by differential graded $k$-modules. We therefore obtain from Theorem 1.6 a strongly convergent spectral sequence for the calculation of $H_*(B \amalg_A C)$. When $k$ is a field or when $H_*A$ and $H_*B$ are flat $k$-modules, Theorem 1.4 allows the easy identification of the $E^2$-term as $\mathrm{Tor}^{H_*A}_{*,*}(H_*B, H_*C)$.

In order to apply the previous theorems in general, we need to be able to replace arbitrary $\mathcal{E}$-algebras with quasi-isomorphic cell $\mathcal{E}$-algebras and to replace arbitrary maps with quasi-isomorphic relative cell inclusions. For this, we provide the following observation, proved in Section 2 by the small object argument.

**Proposition 1.7.** *If $A \to B$ and $A \to C$ are maps of $\mathcal{E}$-algebras, then we can form a commutative diagram*

$$\begin{tikzcd} B' \arrow[d, two heads, "\sim"] & A' \arrow[l, tail] \arrow[r, tail] \arrow[d, two heads, "\sim"] & C' \arrow[d, two heads, "\sim"] \\ B & A \arrow[l] \arrow[r] & C \end{tikzcd}$$

*where $A'$, $B'$, and $C'$ are cell $\mathcal{E}$-algebras, the maps labeled "$\overset{\sim}{\twoheadrightarrow}$" are surjective quasi-isomorphisms, and the maps labeled "$\rightarrowtail$" are relative cell inclusions.*

The use of the symbol "$\sim$" and the arrows "$\rightarrowtail$", "$\twoheadrightarrow$" in the previous proposition provide a convenient shorthand notation in diagrams. We formalize this here for use in the rest of this paper.

**Notation 1.8.** The arrow "$\rightarrowtail$" indicates a map that is known to be or is assumed to be a relative cell inclusion. The arrow "$\twoheadrightarrow$" indicates a map that is known to be or is assumed to be a fibration. The symbol "$\sim$" decorating an arrow indicates a map that is known to be or is assumed to be a quasi-isomorphism.

Although not enough to establish a model category structure, these results on coproducts are sufficient to develop a notion of (Quillen left) homotopy in the category for which cell $\mathcal{E}$-algebras satisfy the analogue of the Whitehead theorem. Aside from this, the most useful feature of a closed model structure would provide is the criterion [7, Theorem 4-3] for an adjoint pair of functors to induce an adjunction on homotopy categories; we show in Section 5 that the categories of $E_\infty$ algebras satisfy the following version of this theorem.

**Theorem 1.9.** *Let $L\colon \mathscr{E} \to \mathscr{M}$ and $R\colon \mathscr{M} \to \mathscr{E}$ be left and right adjoints between the category $\mathscr{E}$ of algebras over an $E_\infty$ operad $\mathcal{E}$ and a closed model category $\mathscr{M}$.*

(i) *If $L$ converts relative cell inclusions between cell $\mathcal{E}$-algebras to cofibrations and $R$ converts fibrations to surjections, then the left derived functor of $L$ and the right derived functor of $R$ exist and are adjoint. Moreover, $L$ converts quasi-isomorphisms between cell $\mathcal{E}$-algebras to weak equivalences, and the restriction of the left derived functor of $L$ to the cell $\mathcal{E}$-algebras is naturally isomorphic to the derived functor of the restriction of $L$.*

(ii) *Suppose that (i) holds and in addition for any cell $\mathcal{E}$-algebra $A$ and any fibrant object $Y$ in $\mathscr{M}$, a map $A \to RY$ is a quasi-isomorphism if and only if the adjoint $LA \to Y$ is a weak equivalence. Then the left derived functor of $L$ and the right derived functor of $R$ are inverse equivalences.*

We have in addition the following useful variants. For simplicity of statement, we say that a map is an *acyclic relative cell inclusion* if it is both a relative cell inclusion and a quasi-isomorphism. Likewise, we say that a map is an *acyclic surjection* if it is both a surjection and a quasi-isomorphism.

**Theorem 1.10.** *The hypothesis of 1.9.(i) is equivalent to each of the following.*

(i) *$L$ converts relative cell inclusions between cell $\mathcal{E}$-algebras to cofibrations and acyclic relative cell inclusions between cell $\mathcal{E}$-algebras to acyclic cofibrations.*

(ii) *$R$ converts fibrations to surjections and acyclic fibrations to acyclic surjections.*

A dual version of Theorem 1.9 also holds with slightly stronger hypotheses. The exact statement is given in Section 5 as Theorem 5.3.

Although we are primarily interested in $E_\infty$ algebras, Proposition 1.3 can be used to obtain closed model category structures on categories of algebras over other kinds of operads. The following theorem proved in Section 13 implies in particular that when $\mathcal{G}$ is an $A_\infty$ operad, the category of $\mathcal{G}$-algebras is a closed model category. The proof is independent of the work on $E_\infty$ algebras of Sections 3–12.

**Theorem 1.11.** *If $\mathcal{G}$ is the operad associated to a non-Sigma operad* [4, I.1.2.(i)], *then the category of $\mathcal{G}$-algebras is a closed model category with weak equivalences the quasi-isomorphisms and fibrations the surjections.*

## 2. Relative Cell Inclusions and the Small Object Argument

In this section, $\mathcal{G}$ denotes an arbitrary operad of differential graded $k$-modules. We prove Propositions 1.3 and 1.7 and develop some of the properties of relative cell inclusions needed for the proof of the remaining results. The main step is given by Quillen's small object argument [7, p. II.3.3–4]. We use it in the following form.

**Proposition 2.1.** *A map of $\mathcal{G}$-algebras $f\colon A \to B$ can be factored $f = p \circ i$ where $i$ is a relative cell inclusion and $p$ is an acyclic surjection.*

*Proof.* Let $A_0 = A$ and let $p_0\colon A_0 \to B$ be the map $f$. We construct $A_{n+1}$, $p_{n+1}$ inductively as follows. We form $A_{n+1}$ as a pushout $A_n \amalg_{\mathbb{G}X_n} \mathbb{G}CX_n$, where $X_n$ is a free differential graded $k$-module with zero differential. We choose $X_n$ to have a generator in degree $m$ for each pair $(x, y)$ where $x$ is a degree $m$ element of $A_n$ and $y$ is an element of $B$ whose differential is $p_n x$. The map $\mathbb{G}X_n \to A_n$ is induced by the tautological map $X_n \to A_n$, and the map $p_{n+1}\colon A_{n+1} \to B$ is induced by the tautological map $CX_n \to B$: the maps that send a basis element to the element $x$ or $y$ that indexes it. Write $A'$ for $\operatorname{Colim} A_n$. Let $i\colon A \to A'$ be the inclusion, and let $p\colon A' \to B$ be the colimit of the maps $p_n$. By definition $i$ is a relative cell inclusion. The map $p_1$ contains in its image all the cycles of $B$, and so $p_2$ is a surjection. The map $p$ is therefore both surjective and surjective on homology. It is injective on homology since $H_*A' = \operatorname{Colim} H_*A_n$ and the kernel of the map $H_*A_n \to H_*B$ maps to zero in $H_*A_{n+1}$. □

In commuting homology with the sequential colimit of $\mathcal{G}$-algebras in the previous proof, we have implicitly used the following fact.

**Proposition 2.2.** *Filtered colimits of $\mathcal{G}$-algebras are formed in the category of differential graded $k$-modules.*

*Proof.* The monad $\mathbb{G}$ that defines $\mathcal{G}$-algebras commutes with filtered colimits. □

Proposition 1.7 is an immediate consequence of Proposition 2.1: Form $A'$ by factoring the initial map $\mathcal{G}(0) \to A$. We obtain $B'$, $C'$ and the relative cell inclusions $A' \to B'$ and $A' \to C'$ by factoring the composite maps $A' \to A \to B$ and $A' \to A \to C$. The following proposition implies that $B'$ and $C'$ are cell $\mathcal{G}$-algebras.

**Proposition 2.3.** *The composite of relative cell inclusions is a relative cell inclusion.*

*Proof.* The proposition is proved by rearranging the order in which the "cells" are attached as in the proof of [4, III.1.2]. Consider relative cell inclusions $A \to B$ and $B \to C$. Choose $A_n$ and $X_n$ as in Definition 1.2, and choose $B_n$ and $Y_n$ such that $B_0 = B$, $B_{n+1} = B_n \amalg_{\mathbb{G}Y_n} \mathbb{G}CY_n$ and $C \cong \operatorname{Colim} B_n$. Let $D_{m,0} = A_m$

and let $Z_{m,0} = X_m$. Then $\operatorname{Colim} D_{m,0} \cong B = B_0$; we form $D_{m,n+1}$ and $Z_{m,n+1}$ with $B_{n+1} \cong \operatorname{Colim}_m D_{m,n+1}$ inductively as follows. Let $D_{0,n+1} = A$. Choose a basis for $Y_{n+1}$. For each basis element $y$ choose $m_y$ such that the image of $y$ under the map $Y_n \to B_n$ is in the image of $D_{m_y,n}$. Let $Z_{m,n+1}$ be the direct sum of $Z_{m,n}$ and the submodule of $Y_n$ generated by those basis elements $y$ with $m_y = m$, and let $D_{m+1,n+1} = D_{m,n+1} \amalg_{\mathbb{G}Z_{m,n+1}} \mathbb{G}CZ_{m,n+1}$. A check of universal properties identifies $\operatorname{Colim}_m D_{m,n+1}$ as $B_{n+1}$. Let $Z_m = \operatorname{Colim}_n Z_{m,n}$, $D_0 = A$, and $D_{m+1} = D_m \amalg_{\mathbb{G}Z_m} \mathbb{G}CZ_m$. A check of universal properties identifies $D_m$ as $\operatorname{Colim}_n D_{m,n}$ and therefore identifies $\operatorname{Colim}_m D_m$ as $\operatorname{Colim}_n B_n \cong C$. This shows that the map $A \to C$ is a relative cell inclusion. □

We begin the proof of Proposition 1.3 with the following fact. Recall [1, 3.1.2] that given a solid diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

the map $i$ is said to have the *left lifting property* with respect to the map $p$ if there exists a map $B \to X$ represented by the dotted arrow above that makes the whole diagram commute.

**Proposition 2.4.** *A map has the left lifting property with respect to the collection of acyclic surjections if and only if it is the retract of a relative cell inclusion.*

*Proof.* The retracts of relative cell inclusions have the left lifting property with respect to the acyclic surjections of $\mathcal{G}$-algebras since the maps $X_n \to CX_n$ have the left lifting property with respect to the acyclic surjections of differential graded $k$-modules.

On the other hand let $f$ be a map that has the left lifting property with respect to the acyclic surjections of $\mathcal{G}$-algebras. Factor $f$ as $p \circ i$ as in Proposition 2.1. Then we can find a lift in the following diagram

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"'] & A' \arrow[d, two heads, "p", "\sim"'] \\ B \arrow[r, "\mathrm{id}"'] \arrow[ur, dotted] & B. \end{tikzcd}$$

This expresses $f$ as a retract of $i$, a relative cell inclusion. □

Proposition 2.1 gives one factorization required by the axioms of a closed model category. The following proposition provides the other factorization in the case when $\mathcal{G}$ satisfies condition 1.3.(i).

**Proposition 2.5.** *A map of $\mathcal{G}$-algebras $f\colon A \to B$ can be factored $f = p \circ i$ where $p$ is a surjection and $i$ is a relative cell inclusion that has the left lifting property with respect to the collection of surjections. If $\mathcal{G}$ satisfies 1.3.(i) then $i$ can be chosen to be a quasi-isomorphism.*

*Proof.* Let $X$ be the free differential graded $k$-module with zero differential that has one generator $x_b$ in dimension $n$ for each element $b$ of $B$ in dimension $n+1$. We use $Cx_b$ to denote the unique element of the differential graded $k$-module $CX$ whose differential is the image of $x_b$ under the canonical inclusion $X \to CX$. We obtain a surjection of differential graded $k$-modules $CX \to B$, induced by sending $Cx_b$ to $b$. The map $A \amalg \mathbb{G}CX \to B$ is then a surjection. The map $A \to A \amalg \mathbb{G}CX$ clearly has the necessary lifting property. Writing $A \to A \amalg \mathbb{G}CX$ as the composite of $A \to A \amalg \mathbb{G}X$ and $A \amalg \mathbb{G}X \to A \amalg \mathbb{G}CX$ shows that it is a relative cell inclusion.

We show that the map $A \to A \amalg \mathbb{G}CX$ is a quasi-isomorphism when $\mathcal{G}$ satisfies 1.3.(i). Consider the filtered system $\langle B_\alpha \rangle$ of finite subsets of homogeneous elements of $B$ ordered by inclusion. Letting $X_\alpha$ be the free differential graded $k$-module with zero differential that has one generator $x_b$ in dimension $n$ for each element $b$ of $B_\alpha$ that is in dimension $n+1$ in $B$, we obtain a filtered system $A_\alpha = A \amalg \mathbb{G}CX_\alpha$ whose colimit is $A \amalg \mathbb{G}CX$. Any map in this system $A_\alpha \to A_\beta$ can be factored into a finite sequence of maps,

$$A_\alpha = A_{\alpha_1} \to \cdots \to A_{\alpha_m} = A_\beta$$

in which each $B_{\alpha_{j+1}}$ has a single element not in $B_{\alpha_j}$. Then $A_{\alpha_{j+1}} \cong A_{\alpha_j} \amalg \mathbb{G}Ck[n]$ for some $n$ depending on $j$. Since we are assuming that condition 1.3.(i) holds, all maps in the filtered system $\langle A_\alpha \rangle$ are quasi-isomorphisms, and it follows that the map from $A$ to the colimit $A \amalg \mathbb{G}CX$ is a quasi-isomorphism. $\square$

The previous proposition allows the proof of the following lifting property by the standard retraction argument, as in the proof of Proposition 2.4.

**Proposition 2.6.** *Suppose $\mathcal{G}$ satisfies 1.3.(i). If $f\colon A \to B$ is an acyclic relative cell inclusion, then $f$ satisfies the left lifting property with respect to the collection of surjections.*

Proposition 1.3 is now an easy consequence.

*Proof of Proposition 1.3.* Assume that (i) holds. We take the cofibrations to be the retracts of the relative cell inclusions; we need to verify MC1–5 of [1, p. 12]. Conditions MC1–3 are well-known. Condition MC4 is given by Propositions 2.4 and 2.6. Condition MC5 is given by Propositions 2.1 and 2.5.

Assume that (ii) holds. The map $A \to A \amalg \mathbb{G}Ck[n]$ has the left lifting property with respect to the collection of surjections, and must therefore be a quasi-isomorphism. $\square$

## 3. The Proof of Theorem 1.1

The proof of Theorem 1.1 relies heavily on the theory of modules over a $\mathcal{C}$-algebra developed in [4]. The category underlying these is the category of $\mathcal{C}(1)$-modules; these are the modules for $k$ regarded as a $\mathcal{C}$-algebra. The special properties of the operad $\mathcal{C}$ allow the construction of a symmetric weakly monoidal product $\boxtimes$. For $\mathcal{C}(1)$-modules $M$ and $N$, there is a canonical map $\mathrm{Tor}_*(M,N) \to H_*(M \boxtimes N)$ that is an isomorphism in favorable cases [4, V.1.9].

The product $\boxtimes$ satisfies all the axioms of a symmetric monoidal product with the minor modification that the given natural transformation $k \boxtimes (-) \to \mathrm{Id}$ is not an isomorphism. To correct the minor difficulties this causes, [4] introduces the category of "unital" $\mathcal{C}(1)$-modules. A unital $\mathcal{C}(1)$-module is a $\mathcal{C}(1)$-module $M$ with a chosen $\mathcal{C}(1)$-module map $k \to M$ thought of as a unit. This allows the construction of "mixed products" $\lhd$, $\rhd$ and a "unital product" $\boxdot$ with stronger unit properties; see [4, §V.2] for details. In particular, the product $\boxdot$ is a symmetric monoidal product on the category of unital $\mathcal{C}(1)$-modules [4, V.2.11]. The commutative monoids for $\boxdot$ are exactly the $\mathcal{C}$-algebras [4, V.3.9], and therefore $\boxdot$ provides the coproduct of $\mathcal{C}$-algebras.

For a $\mathcal{C}$-algebra $A$, an $A$-module is a $\mathcal{C}(1)$-module $M$ with a suitably associative and unital map $A \lhd M \to M$ [4, V.3.3]. The category of $A$-modules has a symmetric weak monoidal product $\boxtimes_A$, and the $E_\infty$ torsion product $\mathrm{Tor}^A_*(M,N)$ of $A$-modules $M$ and $N$ is defined as the homology of the left derived functor of $\boxtimes_A$. Note that for $A = k$, "$\mathrm{Tor}^k_*(M,N)$" in this sense is canonically isomorphic to the usual differential torsion product of the underlying differential graded $k$-modules [4, V.1.9].

A unital $A$-module is an $A$-module $M$ with a chosen $A$-module map $A \to M$. A unital version $\boxdot_A$ of the $\boxtimes_A$-product makes the category of unital $A$-modules symmetric monoidal. For unital $A$-modules $M$ and $N$, there is a canonical map $M \boxtimes_A N \to M \boxdot_A N$ and therefore a canonical map $\mathrm{Tor}^A_*(M,N) \to H_*(M \boxdot_A N)$. A commutative monoid $B$ in the category of unital $A$-modules is the same thing as a $\mathcal{C}$-algebra $B$ and a map of $\mathcal{C}$-algebras $A \to B$. It follows that for maps of $\mathcal{C}$-algebras $A \to B$ and $A \to C$, the pushout $B \amalg_A C$ in the category of $\mathcal{C}$-algebras is given by $B \boxdot_A C$. Thus, we have a canonical map $\mathrm{Tor}^A_*(B,C) \to H_*(B \amalg_A C)$.

**Theorem 3.1.** *Let $A \to B$ and $A \to C$ be maps of $\mathcal{C}$-algebras and assume that $A \to B$ is a relative cell inclusion. Then the natural map $\mathrm{Tor}^A_*(B,C) \to H_*(B \amalg_A C)$ is an isomorphism.*

Theorem 1.1 is an immediate consequence of the previous theorem. The map $k \to \mathbb{C}Ck[n]$ is a quasi-isomorphism and a relative cell inclusion, so the map

$$H_*A \cong \mathrm{Tor}_*(A,k) \to \mathrm{Tor}_*(A, \mathbb{C}Ck[n]) \cong H_*(A \amalg \mathbb{C}Ck[n])$$

is an isomorphism. Proposition 1.3 gives the closed model structure. Left properness follows from the fact that for a relative cell inclusion $A \to B$, a quasi-isomorphism $A \to C$ induces an isomorphism

$$H_*B \cong \mathrm{Tor}^A_*(B,A) \to \mathrm{Tor}^A_*(B,C) \cong H_*(B \amalg_A C).$$

Right properness follows from the fact that limits of $\mathcal{C}$-algebras are created in the category of differential graded $k$-modules and the fact that the fibrations are surjections. The proof of Theorem 3.1 occupies the remainder of this section.

We prove Theorem 3.1 by identifying a large class of unital $A$-modules for which the $\boxdot_A$ product is particularly well-behaved.

**Definition 3.2.** We say that a unital $A$-module $M$ is $\boxdot_A$-flat if the canonical map $\mathrm{Tor}^A_*(M,N) \to H_*(M \boxdot_A N)$ is an isomorphism for every unital $A$-module $N$.

The following proposition lists some elementary properties of $\boxdot_A$-flat modules that can be proved by the standard arguments.

**Proposition 3.3.** *Let $A$ be a $\mathcal{C}$-algebra and let $M$ be a unital $A$-module.*

(i) *$A$ is $\boxdot_A$-flat.*
(ii) *If $M$ is the retract of a $\boxdot_A$-flat unital $A$-module then $M$ is $\boxdot_A$-flat.*
(iii) *If $M$ is $A$-module chain homotopy equivalent rel $A$ to a $\boxdot_A$-flat unital $A$-module then $M$ is $\boxdot_A$-flat.*
(iv) *If $M$ is a filtered colimit of $\boxdot_A$-flat unital $A$-modules, then $M$ is $\boxdot_A$-flat.*
(v) *If $M$ is the normalization of a simplicial unital $A$-module that is $\boxdot_A$-flat in each degree then $M$ is $\boxdot_A$-flat.*
(vi) *If $M \cong A \boxdot N$ for a $\boxdot$-flat unital $\mathcal{C}(1)$-module $N$ then $M$ is $\boxdot_A$-flat.*

(vii) *If $M \cong N \boxdot_B P$ for some $\boxdot_A$-flat unital $A \boxdot B$-module $N$ and some $\boxdot_B$-flat unital $B$-module $P$, for some $\mathcal{C}$-algebra $B$, then $M$ is $\boxdot_A$-flat.*

The following lemma is proved in Section 8.

**Lemma 3.4.** *If $Z$ is a bounded below free differential graded $k$-module (or more generally, a cell $k$-module* [4, III.1.1]*), then the unital $\mathcal{C}(1)$-module $\mathbb{C}Z$ is $\boxdot$-flat.*

We actually use the following variation.

**Proposition 3.5.** *If $Z$ is a free differential graded $k$-module with zero differential, then $\mathbb{C}CZ$ is $\boxdot_{\mathbb{C}Z}$-flat.*

*Proof.* Consider the simplicial $\mathcal{C}$-algebra $\beta_\bullet = \beta^{\mathcal{C}}_\bullet(\mathbb{C}Z, \mathbb{C}Z, \mathbb{C}CZ)$, and write $\beta$ for the normalization, the $\mathcal{C}$-algebra $\beta^{\mathcal{C}}(\mathbb{C}Z, \mathbb{C}Z, \mathbb{C}CZ)$. The inclusion of $\mathbb{C}Z$ into the coproduct $\mathbb{C}Z \amalg \mathbb{C}CZ = \beta_0$ makes $\beta$ a $\mathcal{C}$-algebra under $\mathbb{C}Z$ and therefore in particular a unital $\mathbb{C}Z$-module. In simplicial degree $n$, $\beta_\bullet$ is given by

$$\beta_n = \mathbb{C}Z \boxdot \underbrace{\mathbb{C}Z \boxdot \cdots \boxdot \mathbb{C}Z}_{n \text{ factors}} \boxdot \mathbb{C}CZ.$$

It therefore follows from Lemma 3.4 and 3.3.(iv)–(vi) that $\beta$ is $\boxdot_{\mathbb{C}Z}$-flat. There is a canonical map of $\mathcal{C}$-algebras $\beta \to \mathbb{C}CZ$. To prove that $\mathbb{C}CZ$ is $\boxdot_{\mathbb{C}Z}$-flat, we construct a section as a map of $\mathcal{C}$-algebras under $\mathbb{C}Z$. Let $\alpha_\bullet$ be the simplicial differential graded $k$-module

$$\alpha_n = Z \oplus \underbrace{Z \oplus \cdots \oplus Z}_{n \text{ factors}} \oplus CZ,$$

and let $\alpha$ be its normalization. Then there is a canonical identification of $\beta_\bullet$ as $\mathbb{C}\alpha_\bullet$ and therefore a canonical map of differential graded $k$-modules $\alpha \to \beta$. On the other hand,

$$\alpha \cong (Z \otimes I) \cup_Z CZ$$

where $I$ is the "unit interval differential graded $k$-module" [4, p. 58]. The inclusion of $CZ$ in $\alpha$ above is not a map under $Z$, but it is straightforward to construct a map $CZ \to \alpha$ that is a map under $Z$ and with the further property that the composite

$$CZ \to \alpha \to \beta \to \mathbb{C}CZ$$

is the usual inclusion. This induces a map $\mathbb{C}CZ \to \mathbb{C}\alpha = \beta$ of $\mathcal{C}$-algebras under $\mathbb{C}Z$ whose composite with the map $\beta \to \mathbb{C}CZ$ is the identity. □

Theorem 3.1 is an immediate consequence of the following proposition.

**Proposition 3.6.** *Let $A$ and $B$ be $\mathcal{C}$-algebras and let $A \to B$ be a relative cell inclusion. Then $B$ is $\boxdot_A$-flat.*

*Proof.* Let $A_n$ and $X_n$ be as in Definition 1.2. By 3.3.(iv), it suffices to show that each $A_n$ is $\boxdot_A$-flat. This follows by induction from Proposition 3.5 and 3.3.(vii). □

## 4. The Homotopy Theory of $E_\infty$ Algebras

In this section, $\mathcal{E}$ denotes an arbitrary $E_\infty$ operad. We show that for cell $\mathcal{E}$-algebras, the factorization and lifting properties in the axioms for a closed model category hold. We use these properties to construct a homotopy theory of $\mathcal{E}$-algebras for which the analogue of the Whitehead theorem holds. This allows us to describe the derived category $\mathscr{E}[\mathscr{Q}^{-1}]$ obtained by formally inverting the quasi-isomorphisms in terms of the category cell $\mathcal{E}$-algebras and homotopy classes of maps. We use this description to prove the adjunction theorems in the next section.

The following lemma is the main building block we need for the results of this section. It is proved in Section 6 by comparing the $\mathcal{E}$-algebra coproduct with the $\mathcal{C}$-algebra coproduct via the functor $V$ of [4, V.1.7].

**Lemma 4.1.** *If $B$ is a cell $\mathcal{E}$-algebra then the inclusion $B \to B \amalg \mathbb{E}Ck[n]$ is a quasi-isomorphism for every $n$.*

Lemma 4.1 leads to the following proposition.

**Proposition 4.2.** *Let $A$ be a cell $\mathcal{E}$-algebra. A map $f\colon A \to B$ has the left lifting property with respect to the collection of surjections if and only if $f$ is the retract of an acyclic relative cell inclusion.*

*Proof.* Factor $f$ as $p \circ i$ using Proposition 2.5. Using Lemma 4.1, the second part of the proof of Proposition 2.5 shows that $i$ is a quasi-isomorphism. Now if we assume that $f$ is an acyclic relative cell inclusion, then applying the argument of Proposition 2.6 with this $f$, $p$, $i$ shows that $f$ is a retract of $i$, and so has the required lifting property. Conversely, if $f$ has the lifting property then we can find a lift in the following diagram

$$\begin{tikzcd} A \arrow[r, tail, "i", "\sim"'] \arrow[d, "f"'] & A' \arrow[d, two heads, "p"] \\ B \arrow[r, "\mathrm{id}"'] \arrow[ur, dotted] & B. \end{tikzcd}$$

Such a lift expresses $f$ as a retract of $i$, an acyclic relative cell inclusion. $\square$

As a consequence of Proposition 4.2, we obtain the following result.

**Proposition 4.3.** *Let $A$ be a cell $\mathcal{E}$-algebra and let $A \to B$ and $A \to C$ be relative cell inclusions. If $A \to B$ is a quasi-isomorphism, then so is the map $C \to B \amalg_A C$.*

*Proof.* Applying Proposition 4.2, we see that the map $C \to B \amalg_A C$ has the left lifting property with respect to the collection of surjections. Applying Proposition 4.2 again, we conclude that $C \to B \amalg_A C$ is a quasi-isomorphism. $\square$

**Definition 4.4.** Let $A$ be a $\mathcal{E}$-algebra. A *cylinder object* for $A$ consists of a $\mathcal{E}$-algebra $IA$ together with a relative cell inclusion $\partial_0 \amalg \partial_1 \colon A \amalg A \to IA$ and a weak equivalence $\sigma\colon IA \to A$ such that the composites $\sigma \circ \partial_0$ and $\sigma \circ \partial_1$ are each the identity on $A$. We say that a cylinder object is *nice* if the map $\sigma$ is a surjection. If $f_0$, $f_1$ are maps $A \to B$, a *(nice) homotopy* from $f_0$ to $f_1$ is a map $f\colon IA \to B$ for some (nice) cylinder object $IA$ such that $f_i = f \circ \partial_i$, $i = 0, 1$. We say that maps $f_0$ and $f_1$ are *(nicely) homotopic* if there exists a (nice) homotopy from $f_0$ to $f_1$. Write $\pi(A, B)$ for the set of $\mathcal{E}$-algebra maps from $A \to B$ modulo the equivalence relation generated by "nicely homotopic".

It follows from Proposition 2.1 that nice cylinder objects always exist. We have used the equivalence relation generated by nice homotopies in the definition of $\pi(A, B)$, since, as we shall see, nice homotopies have better composition properties than the more basic notion of homotopies. On the other hand, we have made the definition of homotopies above in analogy with the definition of (left) homotopies in the motivating example of closed model categories. This weaker notion is essential, just as it is for closed model categories, because functors that preserve "cofibrations" tend not to preserve "fibrations" and then fail to preserve a relation like "nice homotopy". In the special case when $A$ is a cell $\mathcal{E}$-algebra, it turns out that homotopies and nice homotopies are essentially equivalent and two maps $A \to B$ represent the same element of $\pi(A, B)$ if and only if they are nicely homotopic. We state this as the following proposition.

**Proposition 4.5.** *Let $A$ be a cell $\mathcal{E}$-algebra. Then maps $f_0, f_1 \colon A \to B$ are homotopic if and only if they are nicely homotopic. Moreover, "nicely homotopic" is an equivalence relation on the set of maps $A \to B$.*

*Proof.* Clearly nicely homotopic maps are homotopic. Let $IA \to B$ be a homotopy from $f_0$ to $f_1$. Factor the map $IA \to A$ using Proposition 2.5 as $IA \to I'A \to A$. Then $I'A$ is a nice cylinder object and the map $IA \to I'A$ has the left lifting property with respect to the collection of surjections by Proposition 4.2. It follows that we can lift the map $IA \to B$ to a map $I'A \to B$ that provides a nice homotopy. For the second statement, clearly "homotopic" is reflexive (factor through $A$) and symmetric (reverse the indexes of $\partial_0$ and $\partial_1$); Proposition 4.3 with the argument of [7, Lemma 1-3] allows us to glue cylinder objects and shows that the relation is symmetric. □

**Proposition 4.6.** *If $g_0, g_1 \colon B \to C$ are (nicely) homotopic maps then for any map $h \colon C \to D$, $h \circ g_0$ and $h \circ g_1$ are (nicely) homotopic. If $g_0, g_1$ are nicely homotopic maps, then for any map $f \colon A \to B$, $g_0 \circ f$ and $g_1 \circ f$ are nicely homotopic.*

*Proof.* Let $g \colon IB \to C$ be a homotopy from $g_0$ to $g_1$; then $h \circ g$ is a homotopy from $h \circ g_0$ to $h \circ g_1$. Now assume that $IB$ is a nice cylinder object. Choose a nice cylinder object for $A$. Proposition 2.4 allows us to construct a lift in the following diagram.

$$\begin{tikzcd}
A \amalg A \arrow[r, "f \amalg f"] \arrow[d, tail] & B \amalg B \arrow[r] & IB \arrow[d, two heads, "\sim"] \\
IA \arrow[r] & A \arrow[r, "f"'] & B
\end{tikzcd}$$

Such a lift is a nice homotopy from $g_0 \circ f$ to $g_1 \circ f$. □

It follows that there is a well-defined composition law

$$\pi(A, B) \times \pi(B, C) \to \pi(A, C),$$

and we can therefore form a category $\pi\mathscr{E}$ whose objects are the $\mathcal{E}$-algebras and whose set of morphisms from $A$ to $B$ is given by $\pi(A, B)$. Here we have done slightly better than the general theory of closed model categories since the map from any $\mathcal{E}$-algebra to the final object 0 is a surjection. This puts us in the analogue of the situation when all objects are fibrant.

The following proposition is the analogue of [7, Lemma 1-7].

**Proposition 4.7.** *Let $A$ be a cell $\mathcal{E}$-algebra. If $g \colon B \to C$ is an acyclic surjection, then $\pi(A, g) \colon \pi(A, B) \to \pi(A, C)$ is a bijection.*

*Proof.* By Proposition 2.4, the map is a surjection. By Proposition 4.5, it suffices to show that maps $f_0, f_1 \colon A \to B$ are homotopic whenever $g \circ f_0$ and $g \circ f_1$ are homotopic. Let $h\colon IA \to C$ be a homotopy from $g \circ f_0$ to $g \circ f_1$. A lift in the following diagram

$$\begin{tikzcd}
A \amalg A \arrow[r, "f_0 \amalg f_1"] \arrow[d, rightarrowtail] & B \arrow[d, "\sim"] \\
IA \arrow[r, "h"'] \arrow[ur, dotted] & C
\end{tikzcd}$$

gives a homotopy from $f_0$ to $f_1$. ☐

**Proposition 4.8.** *If $A$ and $B$ are cell $\mathcal{E}$-algebras and $f\colon A \to B$ is a quasi-isomorphism, then the image of $f$ in $\pi\mathscr{E}$ is an isomorphism.*

*Proof.* It follows from Proposition 4.7 that the proposition holds in the special case when $f$ is an acyclic surjection. By factoring $f$ as in Proposition 2.1, it therefore suffices to consider the case when $f$ is an acyclic relative cell inclusion. By Proposition 4.2, we can find a lift $g$ in the following diagram.

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}"] \arrow[d, rightarrowtail, "f", "\sim"'] & A \arrow[d] \\
B \arrow[r] \arrow[ur, dotted, "g"'] & 0
\end{tikzcd}$$

Then $g$ is a quasi-isomorphism, and we can factor it as an acyclic relative cell inclusion $g'\colon B \to A'$ followed by an acyclic surjection $h\colon A' \to A$. Arguing as above, the map $g'$ has a retraction $f'\colon A' \to B'$, and we obtain the following commutative diagram of quasi-isomorphisms.

$$\begin{tikzcd}
A \arrow[rr, "\mathrm{id}"] \arrow[dd, rightarrowtail, "f"'] & & A \\
& A' \arrow[ur, "h"] \arrow[dr, "f'"] & \\
B \arrow[ur, rightarrowtail, "g'"] \arrow[rr, "\mathrm{id}"'] & & B
\end{tikzcd}$$

Since $h$ is an acyclic surjection between cell $\mathcal{E}$-algebras, it is an isomorphism in $\pi\mathscr{E}$, and it follows that $f$ and $g$ are inverse isomorphisms in $\pi\mathscr{E}$. ☐

The $\mathcal{E}$-algebra analogue of the Whitehead theorem is an easy consequence.

**Theorem 4.9.** *(The Whitehead Theorem) Let $A$ be a cell $\mathcal{E}$-algebra. A quasi-isomorphism $f\colon B \to C$ induces an isomorphism $\pi(A,B) \to \pi(A,C)$.*

*Proof.* By Proposition 1.7, we can form a commutative diagram

$$\begin{tikzcd}
B' \arrow[r, rightarrowtail, "f'", "\sim"'] \arrow[d, "b"', "\sim"] & C' \arrow[d, "c", "\sim"'] \\
B \arrow[r, "\sim", "f"'] & C,
\end{tikzcd}$$

where $B'$ and $C'$ are cell $\mathcal{E}$-algebras. The theorem now follows from Proposition 4.7 and Proposition 4.8. ☐

Let $\pi\mathscr{E}_c$ denote the full subcategory of $\pi\mathscr{E}$ consisting of the cell $\mathcal{E}$-algebras. Let $\mathscr{E}[\mathscr{Q}^{-1}]$ denote the category obtained from the category of $\mathcal{E}$-algebras by formally inverting the quasi-isomorphisms. The canonical functor $\gamma\colon \mathscr{E} \to \mathscr{E}[\mathscr{Q}^{-1}]$ clearly factors through $\pi\mathscr{E}$, and so we have a functor $\bar{\gamma}\colon \pi\mathscr{E}_c \to \mathscr{E}[\mathscr{Q}^{-1}]$. As an immediate consequence of the Whitehead theorem, we have the following corollary, which is the analogue of [7, Theorem 1-1'].

**Corollary 4.10.** *The functor* $\bar{\gamma}\colon \pi\mathscr{E}_c \to \mathscr{E}[\mathscr{Q}^{-1}]$ *is an equivalence.*

## 5. Categories of $E_\infty$ Algebras and Adjoint Functors

We prove Theorem 1.9 by following the proof of the analogous statement for closed model categories given in [1, 9.7]. Following this proof, we need three additional observations beyond the work of the previous section. The first is Theorem 1.10.

*Proof of Theorem 1.10.* This is proved just as [1, 9.8]. The only thing we still need to observe is that the acyclic surjections and the surjections of $\mathcal{E}$-algebras can be characterized as the maps that have the right lifting property with respect to the collection of relative cell inclusions between cell $\mathcal{E}$-algebras and the collection of acyclic relative cell inclusions between cell $\mathcal{E}$-algebras, respectively. These characterizations follow from the fact that the acyclic surjections and the surjections of differential graded $k$-modules can be characterized as the maps that have the right lifting property with respect to the inclusions $k[n] \to Ck[n]$ and $k[n] \to I[n]$, respectively, for all $n$. Here $[n]$ is the shift functor, and $I$ is the "unit interval differential graded $k$-module" [4, p. 58]. □

The second additional observation is the lemma of K. Brown [1, 9.9] and its dual.

**Proposition 5.1.** *Let $L$ and $R$ be as in 1.9.(i). Then $L$ converts all quasi-isomorphisms between cell $\mathcal{E}$-algebras to weak equivalences and $R$ converts all weak equivalences between fibrant objects to quasi-isomorphisms.*

*Proof.* The argument for [1, 9.9] and its dual applies. □

Finally, we need to be able to convert "right homotopies" into homotopies. The proof is the same as the analogous statement for model categories, except that we do not know that "right cylinder objects" always exist.

**Proposition 5.2.** *Let $A$ be a cell $\mathcal{E}$-algebra. Let $B$, $B^I$ be $\mathcal{E}$-algebras, $s\colon B \to B^I$ a quasi-isomorphism and $d_0 \times d_1\colon B^I \to B \times B$ a surjection such that $d_i \circ s$ is the identity for $i = 0, 1$. Let $f_0$, $f_1$ be maps $A \to B$. There exists a map $h\colon A \to B^I$ that makes the following diagram commute if and only if $f_0$ is homotopic to $f_1$.*

$$\begin{tikzcd} & B^I \arrow[d, "d_0 \times d_1"] \\ A \arrow[ur, "h"] \arrow[r, "f_0 \times f_1"'] & B \times B \end{tikzcd}$$

*Proof.* The "if" part is proved as follows. Let $f\colon IA \to B$ be a homotopy from $f_0$ to $f_1$, and consider the following diagram.

$$\begin{tikzcd} A \arrow[r, "s\circ f_0"] \arrow[d, tail, "\partial_0"', "\sim"] & B^I \arrow[d, two heads] \\ IA \arrow[r, "(f_0\circ\sigma)\times f"'] & B\times B \end{tikzcd}$$

Choose a lift $g$ and let $h = g\circ\partial_1$. Then $h$ makes the required diagram commute.

For the "only if" part, note that the maps $d_0$ and $d_1$ induce isomorphisms $\pi(A, B^I) \cong \pi(A, B)$ by Proposition 4.7. Since each $d_i$ is a retract of the map $s$, $s$ induces an isomorphism $\pi(A,B) \to \pi(A, B^I)$ and we see that $d_0$ and $d_1$ induce the same isomorphism. Thus, $f_0 = d_0\circ h$ and $f_1 = d_1 \circ h$ are the same element of $\pi(A,B)$ and it follows from Proposition 4.5 that they are homotopic. $\square$

*Proof of Theorem 1.9.* The argument for the analogous theorem for closed model categories in [1, 9.7] now applies. $\square$

The following theorem is the dual version of Theorem 1.9. The stronger hypotheses are needed because we lack the lift property of Proposition 4.2 for arbitrary $\mathcal{E}$-algebras.

**Theorem 5.3.** *Let $L\colon \mathscr{M} \to \mathscr{E}$ and $R\colon \mathscr{E}\to\mathscr{M}$ be left and right adjoints between a closed model category $\mathscr{M}$ and the category $\mathscr{E}$ of algebras over an $E_\infty$ operad $\mathcal{E}$.*

(i) *If $R$ converts surjections to fibrations and quasi-isomorphisms to weak equivalences, then the left derived functor of $L$ and the right derived functor of $R$ exist and are adjoint.*

(ii) *Suppose that (i) holds and in addition for any cofibrant object $X$ of $\mathscr{M}$ and any $\mathcal{E}$-algebra $B$, a map $X \to RB$ is a weak equivalence if and only if the adjoint $LX \to B$ is a quasi-isomorphism. Then the left derived functor of $L$ and the right derived functor of $R$ are inverse equivalences.*

*Proof.* By Proposition 2.4, $L$ converts cofibrations into retracts of cell inclusions. By Proposition 4.2, $L$ converts acyclic cofibrations between cofibrant objects to retracts of acyclic relative cell inclusions between retracts of cell objects. The proof of [1, 9.8] applies to show that $L$ converts any weak equivalence between cofibrant objects to a quasi-isomorphism between retracts of cell $\mathcal{E}$-algebras. It follows that the left derived functor $\mathbf{L}$ of $L$ exists and can be constructed by cofibrant approximation followed by application of $L$. The right derived functor $\mathbf{R}$ of $R$ clearly exists by the universal property of the map $\mathscr{E}\to\mathscr{E}[\mathscr{Q}^{-1}]$ and is just a factorization of the functor $R$ through this map.

Let $X$ be a cofibrant object of $\mathscr{M}$ and let $B$ be a cell object of $\mathcal{E}$. A left homotopy in $\mathscr{M}$ between maps $X \to RB$ gives a homotopy in $\mathscr{E}$ between the adjoint maps $LX \to B$. On the other hand, factoring the diagonal map $B \to B\times B$ by Proposition 2.5, we obtain a cell $\mathcal{E}$-algebra $B^I$ and maps $s\colon B\to B^I$ and $d_0\times d_1\colon B^I \to B\times B$, which by Proposition 4.2 satisfy the hypotheses of Proposition 5.2. Proposition 5.2 then implies that for homotopic maps $LX \to B$ in $\mathscr{E}$, the adjoint maps $X\to RB$ are right homotopic in $\mathscr{M}$. It follows that we have an isomorphism $[X, RB] \cong \pi(LX, B)$, natural in cofibrant objects $X$ and cell objects $B$. Since $\mathbf{R}$ is naturally isomorphic to $\mathbf{R}\circ\bar\gamma\circ\tilde{Q}$, where $\bar\gamma$ and $\tilde{Q}$ are the equivalences of Corollary 4.10, it follows that $\mathbf{L}$ and $\mathbf{R}$ are adjoint.

Finally, assume the hypothesis of (ii). When $X$ is cofibrant in $\mathscr{M}$, the derived unit map $X \to \mathbf{RL}X$ is represented by the unit map $X \to RLX$ whose adjoint is the identity map on $RX$ and therefore a quasi-isomorphism. For a $\mathcal{E}$-algebra $B$, choose a cofibrant approximation $X \to RB$. Then the counit map $\mathbf{LR}B \to B$ is represented by the map $LX \to B$, whose adjoint, the map $X \to RB$, is by assumption a weak equivalence. It follows that the unit and counit of the derived adjunction are isomorphisms. Thus, the functors $\mathbf{L}$ and $\mathbf{R}$ are inverse equivalences. □

## 6. The Proofs of Lemma 4.1 and Theorems 1.4 and 1.5

The proofs of Lemma 4.1 and Theorems 1.4 and 1.5 are of a similar nature and are based on applying the lifting properties of Section 2 to the various natural transformations between the operad "push forward" and related functors of [4, p. 136]. We begin by developing notation for these functors and natural transformations.

Let $\mathcal{O}$ and $\mathcal{P}$ be $E_\infty$ operads and let $\mathcal{S}$ be the $E_\infty$ operad $\mathcal{O}\otimes\mathcal{P}$. Let $\mathbb{O}$, $\mathbb{P}$, and $\mathbb{S}$ be the free $\mathcal{O}$, $\mathcal{P}$, and $\mathcal{S}$-algebra functors. The augmentations of $\mathcal{O}$ and $\mathcal{P}$ induce maps of operads $\mathcal{S} \to \mathcal{O}$ and $\mathcal{S} \to \mathcal{P}$ and therefore maps of the associated monads. In this situation, embedingwe can form a "two-sided monadic bar construction" [4, §II.4]: For a map of monads $\mathbb{A} \to \mathbb{B}$ and an $\mathbb{A}$-algebra $X$, the two-sided bar construction $B_\bullet(\mathbb{B},\mathbb{A},X)$ is the simplicial $\mathbb{B}$-algebra that is given in simplicial degree $n$ by

$$B_n(\mathbb{B},\mathbb{A},X) = \mathbb{B}\underbrace{\mathbb{A}\cdots\mathbb{A}}_{n\text{ factors}}X,$$

natural in $\mathbb{A} \to \mathbb{B}$ and $X$. Consider the functors

$$O_\bullet = B_\bullet(\mathbb{O},\mathbb{S},-), \qquad P_\bullet = B_\bullet(\mathbb{P},\mathbb{S},-), \qquad \text{and} \qquad S_\bullet = B_\bullet(\mathbb{S},\mathbb{S},-)$$

from $\mathcal{S}$-algebras to $\mathcal{O}$, $\mathcal{P}$, and $\mathcal{S}$-algebras respectively. Normalization makes these into functors $O$, $P$, and $S$ from $\mathcal{S}$-algebras to $\mathcal{O}$, $\mathcal{P}$, and $\mathcal{S}$-algebras.

The maps of monads $\mathbb{S} \to \mathbb{O}$ and $\mathbb{S} \to \mathbb{P}$ induce natural transformations $o\colon S \to O$ and $p\colon S \to P$. Since for any differential graded $k$-module $X$, $\mathbb{S}X \to \mathbb{O}X$ and $\mathbb{S}X \to \mathbb{P}X$ are quasi-isomorphisms, the maps $S_\bullet A \to O_\bullet A$ and $S_\bullet A \to P_\bullet A$ are degreewise quasi-isomorphisms for every $\mathcal{S}$-algebra $A$, and so the natural transformations $o$ and $p$ are quasi-isomorphisms for all $\mathcal{S}$-algebras. As in [4, II.4.2], we have a natural transformation $\sigma\colon S \to \mathrm{Id}$ which is a quasi-isomorphism for all $\mathcal{S}$-algebras. The maps of operads $\mathcal{S} \to \mathcal{O}$ and $\mathcal{S} \to \mathcal{P}$ allow us to regard $\mathcal{O}$-algebras and $\mathcal{P}$-algebras as $\mathcal{S}$-algebras; we typically omit notation for these functors, but when important, we denote them as $U_{\mathcal{O}}$ and $U_{\mathcal{P}}$. The natural transformations

$$B_\bullet(\mathbb{O},\mathbb{S},U_{\mathcal{O}}-) \to B_\bullet(\mathbb{O},\mathbb{O},-) \to \mathrm{Id}_{\mathscr{O}} \text{ and } B_\bullet(\mathbb{P},\mathbb{S},U_{\mathcal{P}}-) \to B_\bullet(\mathbb{P},\mathbb{P},-) \to \mathrm{Id}_{\mathscr{P}}$$

induce natural transformations $\omega\colon OU_{\mathcal{O}} \to \mathrm{Id}$ and $\pi\colon PU_{\mathcal{P}} \to \mathrm{Id}$ such that the following diagrams commute.

$$\begin{tikzcd} S \arrow[rr, "o"] \arrow[dr, "\sigma"'] & & O \arrow[dl, "\omega"] \\ & \mathrm{Id} & \end{tikzcd} \qquad \begin{tikzcd} S \arrow[rr, "p"] \arrow[dr, "\sigma"'] & & P \arrow[dl, "\pi"] \\ & \mathrm{Id} & \end{tikzcd}$$

It follows that the maps $\omega$ and $\pi$ are quasi-isomorphisms for all $\mathcal{O}$-algebras and all $\mathcal{P}$-algebras respectively.

The proofs of the results in this section are based on the following lifting lemma.

**Lemma 6.1.** *Let $A$, $B$, $C$ be cell $\mathcal{P}$-algebras and let $A \to B$, $A \to C$ be relative cell inclusions. Suppose given a commutative diagram of $\mathcal{O}$-algebras*

$$\begin{tikzcd}
B' \arrow[d, two heads, "\sim"] & A' \arrow[l, tail] \arrow[r, tail] \arrow[d, two heads, "\sim"] & C' \arrow[d, two heads, "\sim"] \\
OB & OA \arrow[l, tail] \arrow[r, tail] & OC,
\end{tikzcd}$$

*with $A'$, $B'$, $C'$ cell $\mathcal{O}$-algebras. Then there are quasi-isomorphisms of $\mathcal{P}$-algebras $A \to PA'$, $B \to PB'$, $C \to PC'$ such that the diagram*

$$\begin{tikzcd}
B \arrow[d, "\sim"] & A \arrow[l, tail] \arrow[r, tail] \arrow[d, "\sim"] & C \arrow[d, "\sim"] \\
PB' & PA' \arrow[l, tail] \arrow[r, tail] & PC'
\end{tikzcd}$$

*commutes and the composite $B \amalg_A C \to P(B' \amalg_{A'} C') \to PO(B \amalg_A C)$ is a quasi-isomorphism.*

Lemma 4.1 is now an easy consequence.

*Proof of Lemma 4.1.* Let $\mathcal{O} = \mathcal{C}$ and $\mathcal{P} = \mathcal{E}$, $A = \mathcal{E}(0)$, and $C = \mathbb{E}Ck[n]$. Construct $A'$, $B'$, $C'$ by Proposition 1.7. Applying Lemma 6.1, we obtain quasi-isomorphisms $B \to PB'$ and $\mathbb{E}Ck[n] \to C'$ that make the following diagram commute.

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & PB' \arrow[r] \arrow[d] & POB \arrow[d] \\
B \amalg \mathbb{E}Ck[n] \arrow[r] & P(B' \amalg_{A'} C') \arrow[r] & PO(B \amalg \mathbb{E}Ck[n])
\end{tikzcd}$$

The top row consists of quasi-isomorphisms and the composite map in the bottom row is a quasi-isomorphism by the lemma. The middle vertical map is a quasi-isomorphism by the work of Section 3. It follows that the map $B \to B \amalg \mathbb{E}Ck[n]$ is a quasi-isomorphism. ☐

For the proof of Theorems 1.4 and 1.5, recall that for maps of $\mathcal{E}$-algebras $A \to B$ and $A \to C$, $\mathrm{Tor}^A_*(B,C)$ is defined as $\mathrm{Tor}^{VA}_*(VB,VC)$, where $V$ is the functor of [4, V.1.7]. Choose cell $\mathcal{C}$-algebras $A'$, $B'$, and $C'$ by applying Proposition 1.7 to the diagram $VB \leftarrow VA \to VC$, i.e. $A' \to B'$, $A' \to C'$ are relative cell inclusions, and $A' \to VA$, $B' \to VB$, $C' \to VC$ are acyclic surjections. The work of Section 3 then implies that we can define $\mathrm{Tor}^A_*(B,C)$ as $H_*(B' \amalg_{A'} C')$. Theorems 1.4 and 1.5 are then consequences of the following more specific theorem.

**Theorem 6.2.** *Let $A$, $B$, $C$, $A'$, $B'$, $C'$ be as above. If $A$ is a cell $\mathcal{E}$-algebra and the maps $A \to B$, $A \to C$ are relative cell inclusions, then the canonical map $B' \amalg_{A'} C' \to V(B \amalg_A C)$ is a quasi-isomorphism.*

*Proof.* Let $\mathcal{G}$ be $\mathcal{C} \otimes \mathcal{E}$. The functor $V$ is the functor $N_*(B_\bullet(\mathbb{C}, \mathbb{G}, U_{\mathcal{E}}-))$ from $\mathcal{E}$-algebras to $\mathcal{C}$-algebras described in the previous section; denote by $W$ the corresponding functor $N_*(B_\bullet(\mathbb{E}, \mathbb{G}, U_{\mathcal{C}}-))$ from $\mathcal{C}$-algebras to $\mathcal{E}$-algebras. Applying Lemma 6.1 with $\mathcal{O} = \mathcal{C}$ and $\mathcal{P} = \mathcal{E}$, we obtain quasi-isomorphisms $A \to WA'$, $B \to WB'$, $C \to WC'$, and a map $B \amalg_A C \to W(B' \amalg_{A'} C')$ such that the composite map $B \amalg_A C \to WV(B \amalg_A C)$ is a quasi-isomorphism. In particular, the map $B \amalg_A C \to W(B' \amalg_{A'} C')$ induces an injection on homology.

Factor the map $A \to WA'$ as $A \to \bar{A} \to WA'$ as in Proposition 2.4, and factor the maps $B \amalg_A \bar{A} \to WB'$ and $C \amalg_A \bar{A} \to WC'$ through $\bar{B}$ and $\bar{C}$ by Proposition 2.4. By Proposition 4.3, the maps $B \to \bar{B}$ and $C \to \bar{C}$ are quasi-isomorphisms. We have the following commutative diagram.

$$\begin{tikzcd}
B \arrow[d, tail, "\sim"] & A \arrow[l, tail] \arrow[r, tail] \arrow[d, tail, "\sim"] & C \arrow[d, tail, "\sim"] \\
\bar{B} \arrow[d, "\sim"] & \bar{A} \arrow[l, tail] \arrow[r, tail] \arrow[d, "\sim"] & \bar{C} \arrow[d, "\sim"] \\
WB' & WA' \arrow[l] \arrow[r] & WC'
\end{tikzcd}$$

Applying Lemma 6.1 with $\mathcal{O} = \mathcal{E}$ and $\mathcal{P} = \mathcal{C}$, we obtain quasi-isomorphisms $A' \to V\bar{A}$, $B' \to V\bar{B}$, $C' \to V\bar{C}$, and a map $B' \amalg_{A'} C' \to V(\bar{C} \amalg_{\bar{A}} \bar{B})$ such that the composite $B' \amalg_{A'} C' \to VW(B' \amalg_{A'} C')$ is a quasi-isomorphism. Since $V$ preserves homology, it follows that the map $\bar{B} \amalg_{\bar{A}} \bar{C} \to W(B' \amalg_{A'} C')$ induces a surjection on homology.

The map $B \amalg_A C \to W(B' \amalg_{A'} C')$ factors as the composite of the map $B \amalg_A C \to \bar{B} \amalg_{\bar{A}} \bar{C}$ and the map $\bar{B} \amalg_{\bar{A}} \bar{C} \to W(B' \amalg_{A'} C')$. By Proposition 4.3, the map $B \amalg_A C \to \bar{B} \amalg_{\bar{A}} \bar{C}$ is a quasi-isomorphism. The map $B \amalg_A C \to W(B' \amalg_{A'} C')$ is then both an injection and a surjection on homology and is therefore a quasi-isomorphism. The map $W(B' \amalg_{A'} C') \to WV(B \amalg_A C)$ is therefore a quasi-isomorphism. Since $W$ preserves homology, it follows that the map $B' \amalg_{A'} C' \to V(B \amalg_A C)$ is a quasi-isomorphism. $\square$

Before beginning the proof of Lemma 6.1, we need the following observation.

**Proposition 6.3.** *The functors $O$, $P$, and $S$ preserve surjections. The natural transformations $o$, $\omega$, $p$, $\pi$, and $\sigma$ are surjections.*

*Proof.* The first statement follows from the fact that the functors $\mathbb{O}$, $\mathbb{P}$, $\mathbb{S}$, and normalization preserve surjections. The map $\sigma$ is a split surjection of the underlying differential graded $k$-module. The second statement for the other maps follows from the fact that for any differential graded $k$-module $X$, the maps $\mathbb{S}X \to \mathbb{O}X$ and $\mathbb{S}X \to \mathbb{P}X$ are surjections. This can be seen from the fact that for each $n$ the augmentations $\mathcal{O}(n) \to k$ and $\mathcal{P}(n) \to k$ are surjections. $\square$

*Proof of Lemma 6.1.* By Proposition 6.3, the natural quasi-isomorphism $PSA \to A$ is a surjection. Thus, we can choose a lift $A \to PSA$ in the following diagram

$$\begin{tikzcd}
\mathcal{P}(0) \arrow[r] \arrow[d, tail] & PSA \arrow[d, "\sim"] \\
A \arrow[r, "\mathrm{id}"'] \arrow[ur, dotted] & A
\end{tikzcd}$$

and lifts $B \to PSB$ and $C \to PSC$ in the following diagrams.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail] & PSA \arrow[r] & PSB \arrow[d, "\sim"] \\
B \arrow[rr, "\mathrm{id}"'] \arrow[urr, dotted] & & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d, tail] & PSA \arrow[r] & PSC \arrow[d, "\sim"] \\
C \arrow[rr, "\mathrm{id}"'] \arrow[urr, dotted] & & C
\end{tikzcd}$$

The universal property of the pushout gives a map $B \amalg_A C \to PS(B \amalg_A C)$. Composing with the natural transformation $PS \to PO$, we obtain the following commutative diagram.

$$\begin{tikzcd} & B \amalg_A C \arrow[dl, "g"'] \arrow[d] \arrow[dr, "\mathrm{id}"] & \\ PO(B \amalg_A C) & PS(B \amalg_A C) \arrow[l, "\sim"] \arrow[r, "\sim"'] & B \amalg_A C \end{tikzcd}$$

It follows that the map $g$ is a quasi-isomorphism. Since the functor $P$ preserves quasi-isomorphisms and surjections, the maps $PA' \to POA$, $PB' \to POB$, and $PC' \to POC$ are acyclic surjections. Thus we can choose diagonal lifts in the following diagrams.

$$\begin{tikzcd} \mathcal{P}(0) \arrow[r] \arrow[d, tail] & PA' \arrow[d, two heads, "\sim"] \\ A \arrow[r, "\sim"'] \arrow[ur, dotted] & POA \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, tail] & PB' \arrow[d, two heads, "\sim"] \\ B \arrow[r, "\sim"'] \arrow[ur, dotted] & POB \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, tail] & PC' \arrow[d, two heads, "\sim"] \\ C \arrow[r, "\sim"'] \arrow[ur, dotted] & POC \end{tikzcd}$$

The bottom horizontal quasi-isomorphisms are the composites of the maps $A \to PSA$, $B \to PSB$, $C \to PSC$ chosen above and the natural transformations $PS \to PO$. Note that the lifts $A \to POA$, $B \to POB$, $C \to POC$ must be quasi-isomorphisms. The universal property of the pushout gives a map $h\colon B \amalg_A C \to P(B' \amalg_{A'} C')$ such that the following diagram commutes

$$\begin{tikzcd} B \amalg_A C \arrow[d, "g"', "\sim"] \arrow[dr, "h"] & \\ PO(B \amalg_A C) & P(B' \amalg_{A'} C'), \arrow[l, "Pf"] \end{tikzcd}$$

where $f$ is the canonical map $B' \amalg_{A'} C' \to O(B \amalg_A C)$. $\square$

## 7. The Proof of Theorem 1.6

Theorem 1.6 proceeds by a comparison of the bar construction $\beta^{\mathcal{E}}$ in the category of $\mathcal{E}$-algebras with the bar construction $\beta^{\mathcal{C}}$ in the category of $\mathcal{C}$-algebras. We begin with the proof of Theorem 1.6 in the case of the operad $\mathcal{C}$.

**Proposition 7.1.** *Let $A'$ be a cell $\mathcal{C}$-algebra, and let $A' \to B'$, $A' \to C'$ be relative cell inclusions. Then the natural map $\beta^{\mathcal{C}}(B', A', C') \to B' \amalg_{A'} C'$ is a quasi-isomorphism.*

*Proof.* It follows from the definition of $\boxdot_{A'}$ that the functor $B' \boxdot_{A'} (-)$ commutes with normalization, and so $\beta^{\mathcal{C}}(B', A', C') \cong B' \boxdot_{A'} \beta^{\mathcal{C}}(A', A', C')$. The map $\beta^{\mathcal{C}}(A', A', C') \to C'$ is a quasi-isomorphism since it is a chain homotopy equivalence of the underlying differential graded $k$-modules. Since $A' \to B'$ is a relative cell inclusion, $B'$ is $\boxdot_{A'}$-flat, and the composite

$$\beta^{\mathcal{C}}(B', A', C') \cong B' \boxdot_{A'} \beta^{\mathcal{C}}(A', A', C') \to B' \boxdot_{A'} C' = B' \amalg_{A'} C'$$

is therefore a quasi-isomorphism. $\square$

Let $V$ denote the functor from $\mathcal{E}$-algebras to $\mathcal{C}$-algebras of [4, V.1.7]. Given $\mathcal{E}$-algebras $A$, $B$, and $C$ as in Theorem 1.6, find cell $\mathcal{C}$-algebras $A'$, $B'$, $C'$ to make the following commutative diagram.

$$\begin{tikzcd}
B' \arrow[d, "\sim"] & A' \arrow[l, tail] \arrow[r, tail] \arrow[d, "\sim"] & C' \arrow[d, "\sim"] \\
VB & VA \arrow[l] \arrow[r] & VC
\end{tikzcd}$$

The maps $B' \amalg (A' \amalg \cdots \amalg A') \amalg C \to V(B \amalg (A \amalg \cdots \amalg A) \amalg C)$ induce a map of simplicial $\mathcal{C}$-algebras

$$\beta^{\mathcal{C}}_\bullet(B', A', C') \to V\beta^{\mathcal{E}}_\bullet(B, A, C)$$

By Theorem 6.2, this map is a degreewise quasi-isomorphism. The map of simplicial $\mathcal{E}$-algebras from $\beta^{\mathcal{E}}_\bullet(B, A, C)$ to the constant simplicial $\mathcal{E}$-algebra $B \amalg_A C$ induces a map of simplicial $\mathcal{C}$-algebras

$$V(\beta^{\mathcal{E}}_\bullet(B, A, C)) \to V(B \amalg_A C)$$

that makes the following diagram commute.

$$\begin{tikzcd}
\beta^{\mathcal{C}}(B', A', C') \arrow[r] \arrow[d] & B' \amalg_{A'} C' \arrow[d] \\
N_*(V(\beta^{\mathcal{E}}_\bullet(B, A, C))) \arrow[r] & V(B \amalg_A C)
\end{tikzcd}$$

Here $N_*(\,\cdot\,)$ denotes normalization. Since the normalization of a degreewise quasi-isomorphism is a quasi-isomorphism, the left vertical arrow is a quasi-isomorphism. The top horizontal arrow is a quasi-isomorphism by Proposition 7.1, and the right vertical arrow is a quasi-isomorphism by Theorem 6.2. It follows that the bottom horizontal map is a quasi-isomorphism.

Now let $S$ denote the functor $S(-) = N_*(B_\bullet(\mathbb{S}, \mathbb{S}, -))$ as in the previous section, where $\mathcal{S}$ is the operad $\mathcal{C} \otimes \mathcal{E}$. As above, we have natural quasi-isomorphisms of $\mathcal{S}$-algebras $S(-) \to V(-)$ and $S(-) \to \mathrm{Id}$. From the following commutative diagram

$$\begin{tikzcd}
N_*(V(\beta^{\mathcal{E}}_\bullet(B, A, C))) \arrow[d, "\sim"'] & N_*(S(\beta^{\mathcal{E}}_\bullet(B, A, C))) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d] & \beta^{\mathcal{E}}(B, A, C) \arrow[d] \\
V(B \amalg_A C) & S(B \amalg_A C) \arrow[l, "\sim"] \arrow[r, "\sim"'] & B \amalg_A C,
\end{tikzcd}$$

we see that the map $\beta^{\mathcal{E}}(B, A, C) \to B \amalg_A C$ is a quasi-isomorphism. This completes the proof of Theorem 1.6.

## 8. The Proof of Lemma 3.4

This section reduces Lemma 3.4 to a statement purely in terms of the operad $\mathcal{C}$, Lemma 8.2 below. The proof of Lemma 8.2 occupies the next two sections.

Let $A$ be a unital $\mathcal{C}(1)$-module and let $Z$ be a differential graded $k$-module. The unital $\mathcal{C}(1)$-module $\mathbb{C}Z \boxdot A$ has a direct sum decomposition

$$\mathbb{C}Z \boxdot A = A \oplus \Big(\bigoplus_{m>0} (\mathcal{C}(m) \otimes_{k[\Sigma_m]} Z^{(m)}) \rhd A\Big).$$

Here $Z^{(m)}$ denotes the $m$-th tensor power $Z \otimes \cdots \otimes Z$. By [4, p. 106], there is a natural map, canonical up to chain homotopy,

$$(\mathcal{C}(m) \otimes_{k[\Sigma_m]} Z^{(m)}) \otimes A \to (\mathcal{C}(m) \otimes_{k[\Sigma_m]} Z^{(m)}) \rhd A.$$

Lemma 3.4 reduces to showing that this map is a quasi-isomorphism: the assumption on $Z$ in Lemma 3.4 forces the tensor product on the left to have homology the differential torsion product [4, §III.4]. Thus, Lemma 3.4 is a consequence of the following proposition.

**Proposition 8.1.** *If $A$ is a unital $\mathcal{C}(1)$-module and $Z$ is a differential graded $k$-module, then the map*

$$(\mathcal{C}(m) \otimes_{k[\Sigma_m]} Z^{(m)}) \otimes A \to \mathcal{C}(m) \otimes_{k[\Sigma_m]} Z^{(m)} \rhd A$$

*is a chain homotopy equivalence of differential graded $k$-modules.*

By [4, V.9.2] and the definition of $\rhd$ [4, V.2.1], the $\rhd$-product on the right above is given by the following pushout diagram.

$$\begin{tikzcd}
\mathcal{C}(m+1) \otimes_{k[\Sigma_m]\otimes\mathcal{C}(1)} (Z^{(m)} \otimes k) \arrow[r] \arrow[d, "\lambda"'] & \mathcal{C}(m+1) \otimes_{k[\Sigma_m]\otimes\mathcal{C}(1)} (Z^{(m)} \otimes A) \arrow[d] \\
\mathcal{C}(m) \otimes_{k[\Sigma_m]} Z^{(m)} \arrow[r] & (\mathcal{C}(m) \otimes_{k[\Sigma_m]} Z^{(m)}) \rhd A
\end{tikzcd}$$

On the other hand, by inspection, the tensor product on the left above is given by the following pushout diagram.

$$\begin{tikzcd}
(\mathcal{C}(m) \otimes \mathcal{C}(1)) \otimes_{k[\Sigma_m]\otimes\mathcal{C}(1)} (Z^{(m)} \otimes k) \arrow[r] \arrow[d, "\tilde\lambda"'] & (\mathcal{C}(m) \otimes \mathcal{C}(1)) \otimes_{k[\Sigma_m]\otimes\mathcal{C}(1)} (Z^{(m)} \otimes A) \arrow[d] \\
\mathcal{C}(m) \otimes_{k[\Sigma_m]} Z^{(m)} \arrow[r] & (\mathcal{C}(m) \otimes_{k[\Sigma_m]} Z^{(m)}) \otimes A
\end{tikzcd}$$

In the top diagram, the map labeled $\lambda$ is induced by the last operad degeneracy map $\mathcal{C}(m+1) \to \mathcal{C}(m)$; in the bottom diagram, the map labeled $\tilde\lambda$ is induced by the degeneracy map $\mathcal{C}(1) \to k$. The top and bottom diagram differ only in the replacement of the right $(k[\Sigma_m] \otimes \mathcal{C}(1))$-module $\mathcal{C}(m+1)$ in the top diagram with the right $(k[\Sigma_m] \otimes \mathcal{C}(1))$-module $\mathcal{C}(m) \otimes \mathcal{C}(1)$ in the bottom diagram.

The natural map in Proposition 8.1 is induced by the map of right $(k[\Sigma_m]\otimes\mathcal{C}(1))$-modules

$$\mu\colon \mathcal{C}(m) \otimes \mathcal{C}(1) \cong \langle t \rangle \otimes (\mathcal{C}(m) \otimes \mathcal{C}(1)) \to \mathcal{C}(2) \otimes (\mathcal{C}(m) \otimes \mathcal{C}(1)) \to \mathcal{C}(m+1),$$

where the first map is induced by the inclusion of a certain zero cycle $t$ in $\mathcal{C}(2)$ and the second map is the operad multiplication [4, p. 106]. Since the vertical maps in the bottom pushout diagram above are isomorphisms, clearly we can describe the map in Proposition 8.1 in terms of a map of the pushout diagrams above. The map of the lower left entries is induced by the map

$$\bar\mu\colon \mathcal{C}(m) \cong \langle t \rangle \otimes (\mathcal{C}(m) \otimes \mathcal{C}(0)) \to \mathcal{C}(2) \otimes (\mathcal{C}(m) \otimes \mathcal{C}(0)) \to \mathcal{C}(m),$$

since this is the unique map of right $k[\Sigma_m]$-modules that makes the diagram

$$\begin{tikzcd} \mathcal{C}(m) \otimes \mathcal{C}(1) \arrow[r, "\mu"] \arrow[d] & \mathcal{C}(m+1) \arrow[d] \\ \mathcal{C}(m) \arrow[r, "\bar{\mu}"'] & \mathcal{C}(m) \end{tikzcd}$$

commute. We prove the following lemma in the next two sections.

**Lemma 8.2.** *The map $\mu$ is a chain homotopy equivalence of right $(k[\Sigma_m] \otimes \mathcal{C}(1))$-modules. The homotopy inverse $\nu\colon \mathcal{C}(m+1) \to \mathcal{C}(m) \otimes \mathcal{C}(1)$ can be chosen so that the composite*

$$\mathcal{C}(m+1) \to \mathcal{C}(m) \otimes \mathcal{C}(1) \to \mathcal{C}(m)$$

*is the last degeneracy map. The chain homotopies $G\colon \mathcal{C}(m) \otimes \mathcal{C}(1) \to \mathcal{C}(m) \otimes \mathcal{C}(1)$ and $H\colon \mathcal{C}(m+1) \to \mathcal{C}(m+1)$ can be chosen such that there are right $\Sigma_m$-module chain homotopies $J, K\colon \mathcal{C}(m) \to \mathcal{C}(m)$ from the identity to $\bar{\mu}$ that make the following diagrams commute.*

$$\begin{tikzcd} \mathcal{C}(m) \otimes \mathcal{C}(1) \arrow[r, "G"] \arrow[d] & \mathcal{C}(m) \otimes \mathcal{C}(1) \arrow[d] \\ \mathcal{C}(m) \arrow[r, "J"'] & \mathcal{C}(m) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}(m+1) \arrow[r, "H"] \arrow[d] & \mathcal{C}(m+1) \arrow[d] \\ \mathcal{C}(m) \arrow[r, "K"'] & \mathcal{C}(m) \end{tikzcd}$$

*Here the vertical maps are induced by the operad degeneracy maps $\mathcal{C}(1) \to k$ and $\mathcal{C}(m+1) \to \mathcal{C}(m)$.*

The map $\nu$ from Lemma 8.2 induces a map of the pushout diagrams above from the top diagram to the bottom diagram and therefore a map of differential graded $k$-modules

$$(\mathcal{C}(m) \otimes_{k[\Sigma_m]} Z^{(m)}) \rhd A \to (\mathcal{C}(m) \otimes_{k[\Sigma_m]} Z^{(m)}) \otimes A.$$

The homotopies $G$, $J$ and $H$, $K$ induce chain homotopies on the diagrams above that give chain homotopies on

$$(\mathcal{C}(m) \otimes_{k[\Sigma_m]} Z^{(m)}) \otimes A \qquad \text{and} \qquad (\mathcal{C}(m) \otimes_{k[\Sigma_m]} Z^{(m)}) \rhd A.$$

from the composite maps to the identity. This proves Proposition 8.1.

## 9. The Linear Isometries Operad

The proof of Lemma 8.2 requires Lemmas 9.1 and 9.2 below regarding the linear isometries operad. Lemma 9.1 is the equivariant version of [4, V.9.3]. Lemma 9.2 provides the additional compatibility with the degeneracy maps that we need for our application to Lemma 8.2. These two lemmas together provide a topological version of Lemma 8.2. In the next section, we show how to convert the topology into algebra.

Let $U = \mathbf{R}^\infty = \operatorname{Colim} \mathbf{R}^n$ regarded as an inner product space. Recall that the linear isometries operad, $\mathcal{L}$, is the operad whose $n$-th space is the space of linear isometries from $U^n = U \oplus \cdots \oplus U$ to $U$ with product $\gamma$ given by composition. See [3, §I.3,§XI] for details. The space $\mathcal{L}(m+1)$ has a right action of the monoid $\Sigma_m \times \mathcal{L}(1)$ by letting $\mathcal{L}(1)$ act on the last copy of $U$ in $U^{m+1}$ and letting $\Sigma_m$ act of the first $m$ copies. If $\alpha$ is an element of $\mathcal{L}(m+1)$, we can ignore the first $m$ copies of $U$ in $U^{m+1}$ and associate to $\alpha$ an element of $\mathcal{L}(1)$. Similarly, we can ignore the last copy of $U$ and associate to $\alpha$ an element of $\mathcal{L}(m)$. This association defines a $\Sigma_m \times \mathcal{L}(1)$-equivariant map of spaces

$$h\colon \mathcal{L}(m+1) \to \mathcal{L}(m) \times \mathcal{L}(1).$$

If we choose an isometric isomorphism $t\colon U^2 \to U$, then for every element $(\delta, \epsilon)$ in $\mathcal{L}(m) \times \mathcal{L}(1)$, the multiplication $\gamma(t;\delta,\epsilon) = t \circ (\delta \oplus \epsilon)$ gives an element of $\mathcal{L}(m+1)$. Again, it is easy to see that this defines a $\Sigma_m \times \mathcal{L}(1)$-equivariant map of spaces

$$g\colon \mathcal{L}(m) \times \mathcal{L}(1) \to \mathcal{L}(m+1).$$

**Lemma 9.1.** *The maps $g$ and $h$ are inverse $\Sigma_m \times \mathcal{L}(1)$-equivariant homotopy equivalences.*

*Proof.* We use 1 to denote the identity map on $U$ regarded as an element of $\mathcal{L}(1)$. Write $i_1$ for the isometric embedding of $U$ in $U^{m+1}$ as the last copy and $i_m$ for the isometric of $U^m$ in $U^{m+1}$ as the first $m$-copies. By ignoring the second copy of $U$ in $U^2$, we can associate to $t$ an element $\hat{t}_1$ in $\mathcal{L}(1)$, and by ignoring the first copy, we obtain an element $\hat{t}_2$ in $\mathcal{L}(1)$. In this notation, $h$ is the map

$$h\colon \alpha \mapsto (\alpha \circ i_m, \alpha \circ i_1),$$

and $h \circ g$ is the map that takes an element $(\delta, \epsilon)$ in $\mathcal{L}(m) \times \mathcal{L}(1)$ to $(\hat{t}_1 \circ \delta, \hat{t}_2 \circ \epsilon)$. Since $\mathcal{L}(1)$ is contractible, we can choose paths $\phi_j$ in $\mathcal{L}(1)$ from 1 to $\hat{t}_j$ for $j = 1, 2$. Composition on the left on $\mathcal{L}(m) \times \mathcal{L}(1)$ with the path $(\phi_1, \phi_2)$ in $\mathcal{L}(1) \times \mathcal{L}(1)$ gives a $\Sigma_m \times \mathcal{L}(1)$-equivariant homotopy from the identity on $\mathcal{L}(m) \times \mathcal{L}(1)$ to $h \circ g$.

Consider the space $\mathcal{A}$ of isometric automorphisms of $U^m \oplus U^m$ and denote by $\mathcal{A}^{\Sigma_m}$ the subspace of the $\Sigma_m$-equivariant maps, where we give $U^m \oplus U^m$ the diagonal $\Sigma_m$-action. It is straightforward to write explicitly a path in $\mathcal{A}^{\Sigma_m}$ from the identity to the map that switches the copies of $U^m$. Choose $\psi$ to be such a path. We use the path $\psi$ to construct the homotopy for $\mathcal{L}(m+1)$.

The map $g \circ h$ acts on an element $\alpha$ of $\mathcal{L}(m+1)$ as follows:

$$g \circ h\colon \alpha \mapsto t \circ (\alpha \circ i_m \oplus \alpha \circ i_1).$$

Consider $t \circ (\alpha \circ i_m \oplus \alpha)$. This is a linear isometry from $U^m \oplus U^m \oplus U$ to $U$. We can compose on the right with $\psi \oplus 1$ to obtain a path of linear isometries $U^m \oplus U^m \oplus U$ to $U$. Writing $i$ for the inclusion of $U^m \oplus U$ in $U^m \oplus U^m \oplus U$ as the last $m+1$ copies of $U$,

$$t \circ (\alpha \circ i_m \oplus \alpha) \circ (\psi \oplus 1) \circ i$$

gives a path in $\mathcal{L}(m+1)$ from $\hat{t}_2 \circ \alpha$ to $g(h(\alpha))$. Composition on the left with the path $\phi_2$ provides a path from $\alpha$ to $\hat{t}_2 \circ \alpha$. These paths assemble to a homotopy from the identity to $g \circ h$. This is a $\Sigma_m \times \mathcal{L}(1)$-equivariant homotopy since $(\psi \oplus 1) \circ i$ is a path through $\Sigma_m \times \mathcal{L}(1)$-equivariant maps. $\square$

Write $\hat{t}_1$ as in the proof above for the element of $\mathcal{L}(1)$ obtained from $t$ by ignoring the second copy of $U$ in $U^2$. Let $\bar{g}\colon \mathcal{L}(m) \to \mathcal{L}(m)$ be the map that takes the element $\delta$ of $\mathcal{L}(m)$ to the element $\hat{t}_1 \circ \delta$. Then the following diagrams commute.

$$\begin{tikzcd} \mathcal{L}(m) \times \mathcal{L}(1) \arrow[r, "g"] \arrow[d] & \mathcal{L}(m+1) \arrow[d] \\ \mathcal{L}(m) \arrow[r, "\bar{g}"'] & \mathcal{L}(m) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L}(m+1) \arrow[r, "h"] \arrow[d] & \mathcal{L}(m) \times \mathcal{L}(1) \arrow[d] \\ \mathcal{L}(m) \arrow[r, "\mathrm{id}"'] & \mathcal{L}(m) \end{tikzcd}$$

Here the vertical maps are the projection $\mathcal{L}(m)\times\mathcal{L}(1)\to\mathcal{L}(m)$ and the last operad degeneracy map $\mathcal{L}(m+1)\to\mathcal{L}(m)$. The following lemma explains the relationship of these diagrams to the homotopies above.

**Lemma 9.2.** *There are $\Sigma_m$-equivariant homotopies $J$ and $K$ from the identity of $\mathcal{L}(m)$ to $\bar{g}$ that make the following diagrams commute*

$$\begin{tikzcd} \mathcal{L}(m)\times\mathcal{L}(1)\times I \arrow[r, "G"] \arrow[d] & \mathcal{L}(m)\times\mathcal{L}(1) \arrow[d] \\ \mathcal{L}(m)\times I \arrow[r, "J"'] & \mathcal{L}(m) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L}(m+1)\times I \arrow[r, "H"] \arrow[d] & \mathcal{L}(m+1) \arrow[d] \\ \mathcal{L}(m)\times I \arrow[r, "K"'] & \mathcal{L}(m), \end{tikzcd}$$

*where $G$ and $H$ are the homotopies constructed in the proof of Lemma 9.1.*

*Proof.* Let $\phi_1$, $\phi_2$, and $\psi$ be the paths in the proof of Lemma 9.1. The homotopy $J$ is given by composition on the left with the path $\phi_1$. Write $\iota$ for the inclusion of $U^m$ in $U^m\oplus U^m$ as the last $m$ compies of $U$. The homotopy $K$ takes the element $\beta$ in $\mathcal{L}(m)$ along the path $\phi_2\circ\beta$ followed by the path $t\circ(\beta\oplus\beta)\circ\psi\circ\iota$. □

## 10. The Proof of Lemma 8.2

Recall from [4, §V.9] that the operad $\mathcal{C}$ is built out of the singular chain complex of the linear isometries operad. We have that $\mathcal{C}(n)=C_*\mathcal{L}(n)$, and the multiplication

$$\mathcal{C}(n)\otimes(\mathcal{C}(j_1)\otimes\cdots\otimes\mathcal{C}(j_n))\to\mathcal{C}(j)$$

is the composite of the shuffle map [2, I.5.3]

$$\mathcal{C}(n)\otimes(\mathcal{C}(j_1)\otimes\cdots\otimes\mathcal{C}(j_n))\to C_*(\mathcal{L}(n)\times(\mathcal{L}(j_1)\times\cdots\times\mathcal{L}(j_n)))\to\mathcal{C}(j_1+\cdots+j_n)$$

and the map induced by the multiplication of $\mathcal{L}$. We remind the reader that the shuffle map is an associative and commutative natural transformation [2, I.5.10].

By associativity, it follows that the map $\mu$ described in Section 8 is the composite of the shuffle map $\mathcal{C}(m)\otimes\mathcal{C}(1)\to C_*(\mathcal{L}(m)\times\mathcal{L}(1))$ and the map induced by the map $g$ described in Section 9. We define the map $\nu$ for Lemma 8.2 as the composite of the map induced by the map $h$ of Section 9 with the Alexander–Whitney map

$$\mathcal{C}(m+1)\xrightarrow{C_*h}C_*(\mathcal{L}(m)\times\mathcal{L}(1))\to\mathcal{C}(m)\otimes\mathcal{C}(1).$$

We obtain the homotopies $G$, $H$, $J$, and $K$ for Lemma 8.2 from the corresponding homotopies in Lemmas 9.1 and 9.2 as follows.

Since the composite of the shuffle map followed by the Alexander–Whitney map is the identity, the composite $\nu\circ\mu$ is $C_*(h\circ g)$ and we take the chain homotopy $G$ for Lemma 8.2 from the identity to $\nu\circ\mu$ to be the chain homotopy induced by the homotopy $G$ of Lemma 9.1. Likewise, we can take the chain homotopies $J$ and $K$ for Lemma 8.2 to be the chain homotopies obtained from the homotopies $J$ and $K$ of Lemma 9.2.

The composite $\mu\circ\nu$ is not $C_*(g\circ h)$ but rather the composite

$$C_*g\circ\sigma\circ\rho\circ C_*h,$$

where $\rho$ denotes the Alexander–Whitney map and $\sigma$ denotes the shuffle map. We obtain the homotopy $H$ in Lemma 8.2 using the chain homotopy $\Phi$ of [2, II.2] from the identity to $\sigma\circ\rho$ and the chain homotopy induced by the homotopy $H$ of Lemma 9.1.

Both the Alexander–Whitney map and the shuffle map are isomorphisms when one of the tensor factors comes from a constant simplicial $k$-module. In addition, the homotopy $\Phi$ is zero when one of the factors is constant. It follows from the topological diagrams in Lemma 9.2 that the algebraic diagrams

$$\begin{tikzcd}
\mathcal{C}(m)\otimes\mathcal{C}(1) \arrow[r, "G"] \arrow[d] & \mathcal{C}(m)\otimes\mathcal{C}(1) \arrow[d] \\
\mathcal{C}(m) \arrow[r, "J"'] & \mathcal{C}(m)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}(m+1) \arrow[r, "H"] \arrow[d] & \mathcal{C}(m+1) \arrow[d] \\
\mathcal{C}(m) \arrow[r, "K"'] & \mathcal{C}(m)
\end{tikzcd}$$

commute. Likewise, it follows from naturality that the map $\nu$ and the chain homotopies we have constructed are $\Sigma_m$-equivariant. Lemma 8.2 is then completed by showing that the map $\nu$ is a map of right $\mathcal{C}(1)$-modules and that the chain homotopies above are chain homotopies of right $\mathcal{C}(1)$-modules.

For the chain homotopies $G$, $J$, and $K$, this is relatively straightforward. Denote by $I$ the "unit interval differential graded $k$-module" [4, p. 58], the free differential graded $k$-module with a generator $[I]$ in degree 1, and generators $[0]$ and $[1]$ in degree zero with $d[I]=[1]-[0]$. A chain homotopy $X\to Y$ is the same thing as a map of differential graded $k$-modules $X\otimes I\to Y$. Let $I_\bullet$ denote the simplicial $k$-module free on the simplicial set $\Delta[1]$; so $I$ is the normalization of $I_\bullet$. We denote normalization of a simplicial $k$-module $X_\bullet$ as $N_*(X_\bullet)$. Then the chain homotopy $N_*(X_\bullet)\otimes I\to N_*(Y_\bullet)$ induced by a simplicial homotopy $X_\bullet\times I_\bullet\to Y_\bullet$ is just the composite with the shuffle map and normalization:

$$N_*(X_\bullet)\otimes I \xrightarrow{\sigma} N_*(X_\bullet\times I_\bullet)\to N_*(Y_\bullet).$$

It therefore follows from the associativity of the shuffle map that the chain homotopies $G$, $J$, and $K$ are chain homotopies of right $\mathcal{C}(1)$-modules.

To see that the map $\nu$ is a map of right $\mathcal{C}(1)$-modules, we need an associativity relation between the shuffle map and the Alexander–Whitney map. The following lemma is proved in the next section.

**Lemma 10.1.** *Let $\sigma$ denote the shuffle map and let $\rho$ denote the Alexander-Whitney map. For any simplicial $k$-modules $A_\bullet$, $B_\bullet$, $C_\bullet$, the following diagram commutes.*

$$\begin{tikzcd}
N_*(A_\bullet\otimes B_\bullet)\otimes N_*(C_\bullet) \arrow[r, "\sigma"] \arrow[d, "\rho\otimes\mathrm{Id}"'] & N_*(A_\bullet\otimes B_\bullet\otimes C_\bullet) \arrow[d, "\rho"] \\
N_*(A_\bullet)\otimes N_*(B_\bullet)\otimes N_*(C_\bullet) \arrow[r, "\mathrm{Id}\otimes\sigma"'] & N_*(A_\bullet)\otimes N_*(B_\bullet\otimes C_\bullet)
\end{tikzcd}$$

Let us denote by $S_\bullet X$ the simplicial $k$-module free on the singular simplicial set of a space $X$, so that $C_*X=N_*(S_\bullet X)$. Then plugging in $A_\bullet=S_\bullet\mathcal{L}(m)$, $B_\bullet=S_\bullet\mathcal{L}(1)$, $C_\bullet=S_\bullet\mathcal{L}(1)$, and composing with $S_\bullet$ applied to the multiplication $\mathcal{L}(1)\times\mathcal{L}(1)\to\mathcal{L}(1)$, we see that the Alexander-Whitney map

$$\rho\colon C_*(\mathcal{L}(m)\otimes\mathcal{L}(1))\to\mathcal{C}(m)\otimes\mathcal{C}(1)$$

is a map of right $\mathcal{C}(1)$-modules, and it follows that $\nu$ is as well.

To see that the chain homotopy $H$ respects the right $\mathcal{C}(1)$-module structure, it suffices to see that the homotopy $\Phi$ from the identity to $\sigma\circ\rho$ does. This is a consequence of the following lemma proved in Section 12, again plugging in $A_\bullet=S_\bullet\mathcal{L}(m)$, $B_\bullet=S_\bullet\mathcal{L}(1)$, $C_\bullet=S_\bullet\mathcal{L}(1)$.

**Lemma 10.2.** *Let $\Phi$ be the natural chain homotopy of* [2, II.2] *from the identity to $\sigma \circ \rho$. For any simplicial $k$-modules $A_\bullet$, $B_\bullet$, $C_\bullet$, the following diagram commutes.*

$$\begin{tikzcd}
N_*(A_\bullet \otimes B_\bullet) \otimes N_*(C_\bullet) \arrow[r, "\sigma"] \arrow[d, "\Phi \otimes \mathrm{Id}"'] & N_*(A_\bullet \otimes (B_\bullet \otimes C_\bullet)) \arrow[d, "\Phi"] \\
N_*(A_\bullet \otimes B_\bullet) \otimes N_*(C_\bullet) \arrow[r, "\sigma"'] & N_*(A_\bullet \otimes (B_\bullet \otimes C_\bullet))
\end{tikzcd}$$

This completes the proof of Lemma 8.2.

## 11. The Proof of Lemma 10.1

Write $f = \rho \circ \sigma$, $g = (\mathrm{Id} \otimes \sigma) \circ (\rho \otimes \mathrm{Id})$. We show that $f = g$ by direct calculation. For a homogeneous element of $N_*(A_\bullet \otimes B_\bullet) \otimes N_*(C_\bullet)$ of the form $a_m \otimes b_m \otimes c_n$, (where the subscript denotes the degree), the formulas for $\sigma$ and $\rho$ yield the following formulas for $f$ and $g$.

$$f(a_m \otimes b_m \otimes c_n) = \sum_{i=0}^{m+n} \sum_{\substack{(\mu,\nu)\text{ an} \\ (m,n)\text{-shuffle}}} (-1)^{\tau(\mu)} \tilde{\partial}^{m+n-i} s_\nu a \otimes \partial_0^i s_\nu b \otimes \partial_0^i s_\mu c$$

$$g(a_m \otimes b_m \otimes c_n) = \sum_{i=0}^{m} \sum_{\substack{(\alpha,\beta)\text{ an} \\ (m-i,n)\text{-shuffle}}} (-1)^{\tau(\alpha)} \tilde{\partial}^{m-i} a \otimes s_\beta \partial_0^i b \otimes s_\alpha c$$

In the above formulas, $\tilde{\partial}$ denotes the last face map, $\partial_j$ for an element in degree $j$, so for example

$$\tilde{\partial}^{m-i} x = \partial_{i+1} \circ \cdots \circ \partial_m x$$

when $x$ has degree $m$. The notation $s_\mu$ denotes the composite of degeneracies $s_{\mu_1}$, $s_{\mu_2}$, etc, i.e. for an $(m,n)$-shuffle $(\mu,\nu)$,

$$s_\mu = s_{\mu_m} \circ \cdots \circ s_{\mu_1} \qquad s_\mu = s_{\nu_n} \circ \cdots \circ s_{\nu_1}.$$

When $i > m$ in the formula for $f$, $\tilde{\partial}^{m+n-i}$ consists of fewer than $n$ face maps whereas $s_\nu$ consists of $n$ degeneracy maps; it follows that in this case $\tilde{\partial}^{m+n-i} s_\nu a$ is zero in the normalized complex $N_*(\,\cdot\,)$. Since

$$\tilde{\partial}^{m+n-i} s_j = \begin{cases} \tilde{\partial}^{m+n-i-1} & \text{when } j \geq i \\ s_j \tilde{\partial}^{m+n-i} & \text{when } j < i \end{cases}$$

it follows that $\tilde{\partial}^{m+n-i} s_\nu a$ is zero unless $\mu_j = j-1$ for $1 \leq j \leq i$. In the remaining cases when $\tilde{\partial}^{m+n-i} s_\nu a$ is non-zero, it is equal to $\tilde{\partial}^{m-i} a$, and

$$\partial_0^i s_{\nu_n} \cdots s_{\nu_0} b = s_{\nu_n - i} \cdots s_{\nu_1 - i} \partial_0^i b, \qquad \partial_0^i s_{\mu_m} \cdots s_{\mu_1} c = s_{\mu_m - i} \cdots s_{\mu_{i+1} - i} c.$$

Matching up the terms, we conclude that $f = g$.

## 12. The Proof of Lemma 10.2

Mixing the terminology from Eilenberg and Mac Lane [2] with the modern terminology and notation, we define a "monotonic operator" to be a map in $\Delta^{\mathrm{op}}$ regarded as an operation on simplicial $k$-modules. More generally, an "operator" is any natural homomorphism

$$(A_1)_{p_1} \otimes \cdots \otimes (A_n)_{p_n} \to (A_1)_{q_1} \otimes \cdots \otimes (A_n)_{q_n}$$

where $A_1, \ldots, A_n$ denote simplicial $k$-modules and the subscripts $p_m$ and $q_m$ denote simplicial degrees. Any operator can be written uniquely as a linear combination of tensor products of monotonic operators. If $\alpha$ is an operator from degree $p_1, \ldots, p_n$ to degree $q_1, \ldots, q_n$, we define the "derived operator", $\alpha'$ from degree $p_1+1, \ldots, p_n+1$ to degree $q_1+1, \ldots, q_n+1$, by the following prescription. The derived operator of $\partial_i$ is $\partial_{i+1}$; the derived operator of $s_i$ is $s_{i+1}$; for operators $\alpha$ and $\beta$,

$$(\alpha \otimes \beta)' = \alpha' \otimes \beta'$$

and whenever $\alpha \circ \beta$ or $\alpha + \beta$ is defined,

$$(\alpha \circ \beta)' = \alpha' \circ \beta' \qquad (\alpha + \beta)' = \alpha' + \beta'.$$

It is straightforward to verify that this is well-defined. Finally, we say that a monotonic operator is "frontal" if it can be written in the form

$$s_{i_m} \cdots s_{i_1} \partial_{j_n} \cdots \partial_{j_1}$$

where $i_m > \cdots > i_1 \geq 0$ and $j_1 > \cdots j_n > 0$, i.e. if it does not require the zeroth face operation when written in standard form. We call an operator frontal if it can be written as the linear combination of tensor products of frontal monotonic operators. It is easy to check that the composition of frontal operators is frontal and that all derived operators are frontal. Furthermore, if $\alpha$ is frontal then $s_0\alpha = \alpha' s_0$. Departing from the terminology of [2], we say an operator $\alpha$ from degree $p_1, \ldots, p_n$ to degree $q_1, \ldots, q_n$ is "degenerate" if $q_1 = \cdots = q_n$ and $\alpha$ can be written as a linear combination of operators of the form

$$(s_i \otimes \cdots \otimes s_i) \circ \beta$$

where $\beta$ is some operator from degree $p_1, \ldots, p_n$ to degree $q_1 - 1, \ldots, q_n - 1$. Note that since derivation commutes with composition, the derived operator of a degenerate operator is degenerate.

The advantage of this point of view is that the unnormalized shuffle map $\sigma$ restricted to a given degree is a frontal operator. The concept of derived operator allows an inductive definition of the shuffle map [2, I.5.7]. We are primarily interested in the shuffle map

$$N_*(A_\bullet \otimes B_\bullet) \otimes N_*(C_\bullet) \to N_*(A_\bullet \otimes B_\bullet \otimes C_\bullet)$$

so it is convenient to regard this as an inhomogeneous sum of operators $\sigma_{p,p,q}$ from degree $p, p, q$ to degree $p+q, p+q, p+q$. In terms of these operators, the inductive definition is as follows.

$$\sigma_{p,p,q} = \begin{cases} \sigma'_{p-1,p-1,q} \circ (1 \otimes 1 \otimes s_0) + (-1)^p \sigma'_{p,p,q-1}(s_0 \otimes s_0 \otimes 1) & p > 0, q > 0 \\ \sigma'_{p-1,p-1,0} \circ (1 \otimes 1 \otimes s_0) & p > 0, q = 0 \\ \sigma'_{0,0,q-1}(s_0 \otimes s_0 \otimes 1) & p = 0, q > 0 \\ \mathrm{Id} & p = 0 = q \end{cases}$$

If we use the convention that $\sigma'_{-1,-1,q} = 0 = \sigma'_{p,p,-1}$, this reduces to the simpler formula

$$\sigma_{p,p,q} = \sigma'_{p-1,p-1,q} \circ (1 \otimes 1 \otimes s_0) + (-1)^p \sigma'_{p,p,q-1}(s_0 \otimes s_0 \otimes 1) \qquad p+q > 0.$$

Let $h = \sigma \circ \rho$; restricted to any given degree, $h$ determines an operator $h_{p,p}$ from degree $p, p$ to itself. Regarding $h$ as a map

$$N_*(A_\bullet \otimes (B_\bullet \otimes C_\bullet)) \to N_*(A_\bullet \otimes (B_\bullet \otimes C_\bullet))$$

we also obtain an operator $h_{p,p,p}$ from degree $p,p,p$ to itself. By Lemma 10.1 and the associativity of the shuffle map, the diagram

$$\begin{tikzcd} N_*(A_\bullet \otimes B_\bullet) \otimes N_*(C_\bullet) \arrow[r, "\sigma"] \arrow[d, "h\otimes \mathrm{Id}"'] & N_*(A_\bullet \otimes (B_\bullet \otimes C_\bullet)) \arrow[d, "h"] \\ N_*(A_\bullet \otimes B_\bullet) \otimes N_*(C_\bullet) \arrow[r, "\sigma"'] & N_*(A_\bullet \otimes (B_\bullet \otimes C_\bullet)) \end{tikzcd}$$

commutes. This is equivalent to the observation that

$$\sigma_{p,p,q} \circ (h_{p,p} \otimes 1) - h_{p+q,p+q,p+q} \circ \sigma_{p,p,q}$$

is degenerate. It follows that

$$\sigma'_{p,p,q} \circ (h'_{p,p} \otimes 1) - h'_{p+q,p+q,p+q} \circ \sigma'_{p,p,q}. \tag{12.1}$$

is degenerate.

The chain homotopy $\Phi$ is defined [2, II.2.13] by $\Phi_{0,0} = 0$ and

$$\Phi_{p,p} = -\Phi'_{p-1,p-1} + h'_{p,p} \circ (s_0 \otimes s_0) \qquad p > 0.$$

When we apply this operator to $A_p \otimes (B_\bullet \otimes C_\bullet)_p$, we can regard it as an operator $A_p \otimes B_p \otimes C_p \to A_{p+1} \otimes B_{p+1} \otimes C_{p+1}$ that we denote as $\Phi_{p,p,p}$. Note that $\Phi_{p,p}$ and $\Phi_{p,p,p}$ are frontal operators. In this notation, the assertion of the lemma for this chain homotopy $\Phi$ is equivalent to the assertion that

$$\sigma_{p+1,p+1,q} \circ (\Phi_{p,p} \otimes 1) - \Phi_{p+q,p+q,p+q} \circ \sigma_{p,p,q}$$

is degenerate. We prove this by induction on $p$ and induction on $q$ for fixed $p$.

When $p = 0$, $\Phi_{p,p} = 0$, so it suffices to show that $\Phi_{q,q,q} \circ \sigma_{0,0,q}$ is degenerate. This latter assertion follows from [2, II.2.4] since the image of $\sigma_{0,0,q}$ applied to $A_\bullet \otimes B_\bullet \otimes C_\bullet$ lies in the image of the shuffle map $\sigma$ applied to $A_0 \otimes \mathrm{diag}(B_\bullet \otimes C_\bullet)_q$.

Now consider the case when $p > 0$. By induction, we can assume the assertion for degree $p-1, q$ and derive it to conclude that

$$\sigma'_{p,p,q} \circ (\Phi'_{p-1,p-1} \otimes 1) - \Phi'_{p+q-1,p+q-1,p+q-1} \circ \sigma'_{p-1,p-1,q}$$

is degenerate. If $q > 0$, the assertion holds for degree $p, q-1$ by induction on $q$, and we conclude that

$$\sigma'_{p+1,p+1,q-1} \circ (\Phi'_{p,p} \otimes 1) - \Phi'_{p+q-1,p+q-1,p+q-1} \circ \sigma'_{p,p,q-1}$$

is degenerate. In the case when $q = 0$, this last expression is zero by our convention that $\sigma'_{n,n,-1} = 0$. Therefore, for any $q$, composing these formulas on the right with $(1 \otimes 1 \otimes s_0)$ and $(s_0 \otimes s_0 \otimes 1)$ respectively, adding with a sign, and making the substitution

$$\sigma_{p,p,q} = \sigma'_{p-1,p-1,q} \circ (1 \otimes 1 \otimes s_0) + (-1)^p \sigma'_{p,p,q-1} \circ (s_0 \otimes s_0 \otimes 1)$$

we see that

$$\sigma'_{p,p,q}(\Phi'_{p-1,p-1} \otimes s_0) - (-1)^{p+1} \sigma'_{p+1,p+1,q-1}(\Phi'_{p,p} \otimes 1)(s_0 \otimes s_0 \otimes 1) \\ - \Phi'_{p+q-1,p+q-1,p+q-1}\sigma_{p,p,q}$$

is degenerate. Using the fact that $\Phi$ is a frontal operator, we see that

$$\sigma'_{p,p,q}(\Phi'_{p-1,p-1} \otimes s_0) - (-1)^{p+1} \sigma'_{p+1,p+1,q-1}(s_0 \otimes s_0 \otimes 1)(\Phi_{p,p} \otimes 1) \\ - \Phi'_{p+q-1,p+q-1,p+q-1}\sigma_{p,p,q}$$

is degenerate. Using the definition of $\Phi$ and the fact that $\sigma$ is frontal, we can make the substitution

$$\begin{aligned}-\Phi'_{p+q-1,p+q-1,p+q-1}\sigma_{p,p,q} &= \Phi_{p+q,p+q,p+q}\sigma_{p,p,q} - h'_{p+q,p+q,p+q}(s_0 \otimes s_0 \otimes s_0)\sigma_{p,p,q} \\ &= \Phi_{p+q,p+q,p+q}\sigma_{p,p,q} - h'_{p+q,p+q,p+q}\sigma'_{p,p,q}(s_0 \otimes s_0 \otimes s_0).\end{aligned}$$

Since expression (12.1) is degenerate, we have that

$$h'_{p+q,p+q,p+q}\sigma'_{p,p,q}(s_0 \otimes s_0 \otimes s_0) \qquad \text{and} \qquad \sigma'_{p,p,q}(h'_{p,p} \otimes 1)(s_0 \otimes s_0 \otimes s_0)$$

differ by a degenerate operator, and we conclude that

$$\begin{aligned}\sigma'_{p,p,q}(\Phi'_{p-1,p-1} \otimes s_0) - (-1)^{p+1}\sigma'_{p+1,p+1,q-1}(s_0 \otimes s_0 \otimes 1)(\Phi_{p,p} \otimes 1) \\ + \Phi_{p+q,p+q,p+q}\sigma_{p,p,q} - \sigma'_{p,p,q}(h'_{p,p} \otimes 1)(s_0 \otimes s_0 \otimes s_0)\end{aligned}$$

is degenerate. Combining the first and last terms and using the definition of $\Phi$, we see that

$$\begin{aligned}-\sigma'_{p,p,q}(\Phi_{p,p} \otimes s_0) - (-1)^{p+1}\sigma'_{p+1,p+1,q-1}(s_0 \otimes s_0 \otimes 1)(\Phi_{p,p} \otimes 1) \\ + \Phi_{p+q,p+q,p+q}\sigma_{p,p,q}\end{aligned}$$

is degenerate. Finally, using the inductive description of $\sigma$, we see that

$$-\sigma_{p+1,p+1,q}(\Phi_{p,p} \otimes 1) + \Phi_{p+q,p+q,p+q}\sigma_{p,p,q}$$

is degenerate. This completes the proof.

## 13. Closed Model Categories of Non-Sigma Operads

Recall from [4, I.1.2.(i)] the definition of a non-Sigma operad. A non-Sigma operad is defined in terms of the same diagrams as an operad but without any actions of the symmetric groups. For a non-Sigma operad $\bar{\mathcal{G}}$, we obtain an associated operad $\mathcal{G}$, by defining $\mathcal{G}(n) = \bar{\mathcal{G}}(n) \otimes k[\Sigma_n]$, using block sum of permutations for the operad multiplication of the symmetric groups part. The monad $\mathbb{G}$ associated to the non-Sigma operad $\bar{\mathcal{G}}$ is

$$\mathbb{G}X = \bigoplus_{n \geq 0} \bar{\mathcal{G}}(n) \otimes X^{(n)},$$

where $X^{(n)}$ denotes the $n$-th tensor power $X \otimes \cdots \otimes X$, and we understand $X^{(0)} = k$.

In order to prove Theorem 1.11, it suffices to show that if $A$ is a $\bar{\mathcal{G}}$-algebra and $Z$ is a contractible differential graded $k$-module, then the map $A \to A \amalg \mathbb{G}Z$ is a homotopy equivalence of the underlying differential graded $k$-modules. A check of universal properties shows that the following diagram is a coequalizer in the category of differential graded $k$-modules

$$\mathbb{G}(\mathbb{G}A \oplus Z) \rightrightarrows \mathbb{G}(A \oplus Z) \longrightarrow A \amalg \mathbb{G}Z.$$

Here one map on the left is $\mathbb{G}\theta$, where $\theta$ is the action map $\mathbb{G}A \to A$. The other is the map of $\bar{\mathcal{G}}$-algebras induced by the maps of differential graded $k$-modules $\mathbb{G}A \to \mathbb{G}(A \oplus Z)$ and $Z \to \mathbb{G}(A \oplus Z)$. Expanding out the definition of $\mathbb{G}$ in terms of $\bar{\mathcal{G}}$, we see that

$$\begin{aligned}\mathbb{G}(\mathbb{G}A \oplus Z) &= \bigoplus_{n\geq m\geq 0} \bigoplus_{\binom{n}{m}\text{ copies}} \bar{\mathcal{G}}(n) \otimes \big((\mathbb{G}A)^{(n-m)} \otimes Z^{(m)}\big) \\ \mathbb{G}(A \oplus Z) &= \bigoplus_{n\geq m\geq 0} \bigoplus_{\binom{n}{m}\text{ copies}} \bar{\mathcal{G}}(n) \otimes \big(A^{(n-m)} \otimes Z^{(m)}\big).\end{aligned}$$

We have rearranged the tensor factors in order to write a compact formula above, but the original arrangement must be retained for interpreting the maps in the coequalizer in terms of the non-Sigma operad multiplication.

Both maps in the coequalizer preserve the "power" of $Z$: they both send direct summands of $\mathbb{G}(\mathbb{G}A \oplus Z)$ containing $Z^{(m)}$ to direct summands of $\mathbb{G}(A \oplus Z)$ containing the same power $Z^{(m)}$. It follows that the differential graded $k$-module of $A \amalg \mathbb{G}Z$ breaks up as a direct sum $\oplus_{m\geq 0} Y_m$, where $Y_m$ corresponds to the image of the summands where $Z^{(m)}$ appears. Note that the piece $Y_0$ is the coequalizer of the diagram

$$\mathbb{G}\mathbb{G}A \rightrightarrows \mathbb{G}A$$

and so is isomorphic to $A$. In fact, the splitting of $A \amalg \mathbb{G}Z$ as $Y_0 \oplus (\oplus_{m>0} Y_m)$ is just the splitting obtained from the retraction

$$A \to A \amalg \mathbb{G}Z \to A$$

where the section is the inclusion of $A$ into the coproduct and the retraction is induced by the map $Z \to 0$. Thus, it suffices to show that $Y_m$ is contractible for $m \geq 1$. However, $Z^{(m)}$ is a contractible differential graded $k$-module. Choosing a contraction of $Z^{(m)}$ specifies contractions of

$$\bigoplus_{n\geq m} \bar{\mathcal{G}}(n) \otimes \big((\mathbb{G}A)^{(n-m)} \otimes Z^{(m)}\big) \quad \text{and} \quad \bigoplus_{n\geq m} \bar{\mathcal{G}}(n) \otimes \big(A^{(n-m)} \otimes Z^{(m)}\big)$$

that commute with both maps in the coequalizer and therefore specify a contraction of $Y_m$.

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