E_∞-ALGEBRAS AND p-ADIC HOMOTOPY THEORY

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Abstract. Let $\mathbf{F}_p$ denote the field with p elements and $\bar{\mathbf{F}}_p$ its algebraic closure. We show that the singular cochain functor with coefficients in $\bar{\mathbf{F}}_p$ induces a contravariant equivalence between the homotopy category of connected p-complete nilpotent spaces of finite p-type and a full subcategory of the homotopy category of E_∞ $\bar{\mathbf{F}}_p$-algebras. Draft: January 26, 1998, 17:28

Introduction

Since the invention of localization and completion of topological spaces, it has proved extremely useful in homotopy theory to view the homotopy category from the perspective of a single prime at a time. The work of Quillen, Sullivan, and others showed that, viewed rationally, homotopy theory becomes completely algebraic. In particular, Sullivan showed that an important subcategory of the homotopy category of rational spaces is contravariantly equivalent to a subcategory of the homotopy category of commutative differential graded $\mathbf{Q}$-algebras, and that the functor underlying this equivalence is closely related to the singular cochain functor. In this paper, we offer a similar theorem for p-adic homotopy theory.

Since the non-commutativity of the multiplication of the $\mathbf{F}_p$ singular cochains is visible already on the homology level in the Steenrod operations, it would be naive to think that any reasonably useful subcategory of the p-adic homotopy category could be equivalent to a category of commutative differential graded algebras. We must instead look to a more sophisticated class of algebras, E_∞ algebras [12]. In fact, it turns out that even the category of E_∞ $\mathbf{F}_p$-algebras is not quite sufficient; rather we consider E_∞ algebras over the algebraic closure $\bar{\mathbf{F}}_p$ of $\mathbf{F}_p$. We prove the following theorem.

Main Theorem. *The singular cochain functor with coefficients in $\bar{\mathbf{F}}_p$ induces a contravariant equivalence from the homotopy category of connected nilpotent p-complete spaces of finite p-type to a full subcategory of the homotopy category of E_∞ $\bar{\mathbf{F}}_p$-algebras.*

The homotopy category of connected p-complete nilpotent spaces of finite p-type is a full subcategory of the p-adic homotopy category, the category obtained from the category of spaces by formally inverting those maps that induce isomorphisms on singular homology with coefficients in $\mathbf{F}_p$. We remind the reader that a connected space is p-complete, nilpotent, and of finite p-type if and only if its Postnikov tower has a principal refinement in which each fiber is of type $K(\mathbf{Z}/p\mathbf{Z}, n)$ or $K(\mathbf{Z}^\wedge_p, n)$, where $\mathbf{Z}^\wedge_p$ denotes the p-adic integers.

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By the homotopy category of E_∞ algebras, we mean the category obtained from the category of algebras over a particular but unspecified E_∞ $\bar{\mathbf{F}}_p$ operad by formally inverting the maps in that category that are quasi-isomorphisms of the underlying differential graded $\bar{\mathbf{F}}_p$-modules, the maps that induce an isomorphism of homology groups. It is well-known that up to equivalence, this category does not depend on the operad chosen. We refer the reader to [12, I] for a good introduction to operads, E_∞ operads, and E_∞ algebras.

Comparison with Other Approaches. The papers [9, 11, 20] and the unpublished ideas of [6] all compare p-adic homotopy theory to various homotopy categories of algebras (or coalgebras). We give a short comparison of these results to the results proved here.

The first announced results along the lines of our Main Theorem appeared in [20]. The arguments there are not well justified, however, and some of the results appear to be wrong.

More recently, [9, 11] have compared the p-adic homotopy category with the homotopy categories of simplicial cocommutative coalgebras and cosimplicial commutative algebras. In particular, [9] proves that the p-adic homotopy category embeds as a full subcategory of the homotopy category of cocommutative simplicial $\bar{\mathbf{F}}_p$-coalgebras. It is straightforward to describe the relationship between this theorem and our Main Theorem. There is a functor from the homotopy category of simplicial cocommutative coalgebras to the homotopy category of E_∞ algebras given by normalization of the dual cosimplicial commutative algebra [10] (see also §1 below). Applied to the singular simplicial chains of a space, we obtain the singular cochain complex of that space. Our Main Theorem implies that on the subcategory of nilpotent spaces of finite p-type, this refined functor remains a full embedding. This gives an affirmative answer to the question asked in [11, 6.3].

The unpublished ideas of [6] for comparing the p-adic homotopy category to the homotopy category of E_∞ ring spectra under the Eilenberg-Mac Lane spectrum $H\bar{\mathbf{F}}_p$, would give a "brave new algebra" version of our Main Theorem. A proof of such a comparison can be given along similar lines to the proof of our Main Theorem. We sketch the argument in Appendix B. A direct comparison between this approach and our Main Theorem would require a comparison of the homotopy category of E_∞ $H\bar{\mathbf{F}}_p$ ring spectra and the category of E_∞ $\bar{\mathbf{F}}_p$-algebras, and also an identification of the composite functor from spaces to E_∞ differential graded $\bar{\mathbf{F}}_p$-algebras as the singular cochain functor. We will provide this comparison and this identification in [14] and [15].

1. Outline of the Paper

Since the main objects we work with in this paper are the cochain complexes, it is convenient to grade differential graded modules "cohomologically" with the differential raising degrees. This makes the cochain complexes concentrated in non-negative degrees, but forces E_∞ operads to be concentrated in non-positive degrees. Along with this convention, we write the homology of a differential graded module M as H^*M. We work almost exclusively with ground ring $\bar{\mathbf{F}}_p$; throughout this paper, C^*X and H^*X always denote the cochain complex and the cohomology of X taken with coefficients in $\bar{\mathbf{F}}_p$. We write $C^*(X;\mathbf{F}_p)$ and $H^*(X;\mathbf{F}_p)$ for the cochain complex and the cohomology of X with coefficients in $\mathbf{F}_p$ or $C^*(X;k)$ and $H^*(X;k)$ for these with coefficients in the field k.

The first prerequisite to the Main Theorem is recognizing that the singular cochain functor can be regarded as a functor into the category of $\mathcal{E}$-algebras for some E_∞ $\bar{\mathbf{F}}_p$-operad $\mathcal{E}$. In fact, for the purpose of this paper, the exact construction of this structure does not matter so long as the (normalized) cochain complex of a simplicial set is naturally an $\mathcal{E}$-algebra. However, we do need to know that such a structure exists. This can be shown as follows.

The work of Hinich and Schechtman in [10] gives the singular cochain complex of a space or the cochain complex of a simplicial set the structure of a "May algebra", an algebra over an acyclic operad $\mathcal{Z}$, the "Eilenberg-Zilber" operad. Unfortunately, $\mathcal{Z}$ is not an E_∞ operad since it is not Σ-free and since it is non-zero in both positive and negative degrees. Nevertheless, when we take coefficients in a field, both these deficiencies are trivial to overcome. Let $\bar{\mathcal{Z}}$ be the "(co)-connective cover" of $\mathcal{Z}$: $\bar{\mathcal{Z}}(n)$ is the differential graded $\bar{\mathbf{F}}_p$-module that is equal to $\mathcal{Z}(n)$ in degrees less than zero, equal to the kernel of the differential in degree zero, and zero in positive degrees. The operadic multiplication of $\mathcal{Z}$ lifts to $\bar{\mathcal{Z}}$, making it an acyclic operad. Tensoring $\bar{\mathcal{Z}}$ with an E_∞ operad $\mathcal{C}$ gives an E_∞ operad $\mathcal{E}$ and a map of operads $\mathcal{E} \to \mathcal{Z}$. The cochain complex of a simplicial set then obtains the natural structure of an algebra over the E_∞ operad $\mathcal{E}$.

We write $\mathscr{E}$ for the category of $\mathcal{E}$-algebras. Since we are assuming that the functor C^* from spaces to $\mathcal{E}$-algebras factors through the category of simplicial sets, we can work simplicially. As is fairly standard, we refer to the category obtained from the category of simplicial sets by formally inverting the weak equivalences as *the homotopy category*; this category is equivalent to the category of Kan complexes and homotopy classes of maps and to the category of CW spaces and homotopy classes of maps. Since the cochain functor converts $\mathbf{F}_p$-homology isomorphisms and in particular weak equivalences of simplicial sets to quasi-isomorphisms of $\mathcal{E}$-algebras, the (total) derived functor exists as a contravariant functor from the homotopy category to the homotopy category of $\mathcal{E}$-algebras. We prove in Section 2 that this functor has a right adjoint U. The functor U provides the inverse equivalence in the Main Theorem.

Precisely, the adjoint U is a contravariant functor from the homotopy category of E_∞ $\bar{\mathbf{F}}_p$-algebras to the homotopy category, and we have a canonical isomorphism

$$\mathscr{H}o(X, UA) \cong \bar{h}\mathscr{E}(A, C^*X)$$

for a simplicial set X and an $\mathcal{E}$-algebra A. Here and elsewhere $\mathscr{H}o$ denotes the homotopy category and $\bar{h}\mathscr{E}$ denotes the homotopy category of $\mathcal{E}$-algebras. We write u_X for the "unit" of the adjunction $X \to UC^*X$. For the purposes of this paper, we say that a simplicial set X is *resolvable by E_∞ $\bar{\mathbf{F}}_p$-algebras* or just *resolvable* if the map u_X is an isomorphism in the homotopy category. In Section 3, we prove the following two theorems.

Theorem 1.1. *Let X be the limit of a tower of Kan fibrations $\cdots \to X_n \to \cdots X_0$. Assume that the canonical map from H^*X to $\operatorname{Colim} H^*X_n$ is an isomorphism. If each X_n is resolvable, then X is resolvable.*

Theorem 1.2. *Let X, Y, and Z be connected simplicial sets of finite p-type, and assume that Z is simply connected. Let $X \to Z$ be a map of simplicial sets, and let $Y \to Z$ be a Kan fibration. If X, Y, and Z are resolvable, then so is the fiber product $X \times_Z Y$.*

These theorems allow us to argue inductively up towers of principal Kan fibrations. The following theorem proved in Section 4 provides a base case.

Theorem 1.3. *$K(\mathbf{Z}/p\mathbf{Z}, n)$ and $K(\mathbf{Z}_p^{\wedge}, n)$ are resolvable for $n \geq 1$.*

We conclude that every connected nilpotent p-complete simplicial set of finite p-type is resolvable. The Main Theorem is now an elementary categorical consequence:

$$\mathscr{H}o(X, Y) \cong \mathscr{H}o(X, UC^*Y) \cong \bar{h}\mathscr{E}(C^*Y, C^*X)$$

for X, Y connected nilpotent p-complete simplicial sets of finite p-type.

We mention here one more result in this paper. This result is needed in the proof of Theorem 1.3 but appears to be of independent interest. The work of [17] provides the homology of E_∞ algebras in characteristic p with operations P^s and βP^s (when $p > 2$) for $s \in \mathbf{Z}$. It follows from a check of the axioms and the identification of βP^0 as the Bockstein that when these operations are applied to the cochain complex of a simplicial set they perform the Steenrod operation of the same names, where we understand P^s to be the zero operation for $s < 0$ and the identity for $s = 0$. The "algebra of all operations" $\mathfrak{B}$ therefore surjects onto the Steenrod algebra $\mathfrak{A}$ with kernel containing the two-sided ideal generated by $1 - P^0$. The following theorem describes the precise relationship between $\mathfrak{B}$ and $\mathfrak{A}$.

Theorem 1.4. *The left ideal of $\mathfrak{B}$ generated by $(1-P^0)$ is a two-sided ideal whose quotient $\mathfrak{B}/(1-P^0)$ is canonically isomorphic to $\mathfrak{A}$.*

The analogue of the Main Theorem for fields other than $\bar{\mathbf{F}}_p$ is discussed in Appendix A. In particular, we show that the analogue of the Main Theorem does not hold when $\bar{\mathbf{F}}_p$ is replaced by any finite field.

2. Construction of the Functor U

In this section, we construct the functor U whose restriction provides the inverse equivalence of the Main Theorem. In fact, U is constructed as the derived functor of a functor from the category of $\mathcal{E}$-algebras to the category of simplicial sets that we also denote as U. We begin by observing that the cochain functor C^* from simplicial sets to $\mathcal{E}$-algebras is an adjoint.

Consider the cosimplicial simplicial set $\Delta = \Delta[\,\cdot\,]$ given by the standard simplexes. Then $C^*\Delta[\,\cdot\,]$ is a simplicial $\mathcal{E}$-algebra. For an arbitrary set S, write $P(S, C^*\Delta[n])$ for the product of copies of $C^*\Delta[n]$ indexed on S. Then for a simplicial set X, $P(X, C^*\Delta[\,\cdot\,])$ is a cosimplicial simplicial $\mathcal{E}$-algebra. Write $M(X, C^*\Delta)$ for the *end*, the equalizer in the category of $\mathcal{E}$-algebras of the diagram

$$\prod_n P(X_n, C^*\Delta[n]) \rightrightarrows \prod_{\substack{f\colon \mathbf{m}\to\mathbf{n} \\ f \text{ in } \mathbf{\Delta}^{\mathrm{op}}}} P(X_n, C^*\Delta[m]).$$

By construction $M(X, C^*\Delta)$ is an $\mathcal{E}$-algebra, contravariantly functorial in the simplicial set X.

Proposition 2.1. *The cochain functor C^* is canonically naturally isomorphic to $M(-, C^*\Delta)$ as a functor from simplicial sets to $\mathcal{E}$-algebras.*

Proof. For each element of X_m, there is a canonical map $\Delta[m] \to X_m$, and the collection of all such maps induces a map of $\mathcal{E}$-algebras

$$C^*X \to \prod_m P(X_m, C^*\Delta[m]).$$

By naturality, this map factors through the equalizer to induce a map of $\mathcal{E}$-algebras $C^*X \to M(X, C^*\Delta)$. The underlying differential graded k-module of an equalizer of $\mathcal{E}$-algebras is the equalizer of the underlying differential graded k-modules. It follows that the induced map $C^*X \to M(X, C^*\Delta)$ is an isomorphism of the underlying differential graded k-modules and therefore an isomorphism of $\mathcal{E}$-algebras. $\square$

The description of C^* given by Proposition 2.1 makes it easy to recognize C^* as an adjoint. For an $\mathcal{E}$-algebra A, let UA be the simplicial set whose set of n-simplices U_nA is the mapping set $\mathscr{E}(A, C^*\Delta[n])$. Clearly UA is a contravariant functor of A. For a simplicial set X, the set of simplicial maps from X to UA, $\mathbf{\Delta}^{\mathrm{op}}\mathbf{Set}(X, UA)$ is by definition the end of the cosimplicial simplicial set $\mathbf{Set}^m_n(X, UA) = \mathbf{Set}(X_m, U_nA)$ that in cosimplicial degree m and simplicial degree n consists of the set of maps of sets from X_m to U_nA. Consider the cosimplicial simplicial bijection

$$\begin{aligned}\mathbf{Set}(X_m, U_nA) &= \mathbf{Set}(X_m, \mathscr{E}(A, C^*\Delta[n])) \\ &\cong \mathscr{E}(A, \prod_{X_m} C^*\Delta[n]) = \mathscr{E}(A, P(X_m, C^*\Delta[n])).\end{aligned}$$

Passing to ends gives a bijection

$$\mathbf{\Delta}^{\mathrm{op}}\mathbf{Set}(X, UA) \cong \mathscr{E}(A, C^*X)$$

natural in A and X. Thus, we have proved the following proposition.

Proposition 2.2. *The functors U and C^* are contravariant right adjoints between the category of simplicial sets and the category of $\mathcal{E}$-algebras.*

In [13, §5], we studied the homotopical properties of adjoint functors between a closed model category and a category of E_∞ algebras. Since the discussion there was in terms of covariant functors, we apply it to U, C^* viewed as an adjoint pair between the category of $\mathcal{E}$-algebras and the opposite to the category of simplicial sets. As such, U is the left adjoint. Taking the closed model category structure on the opposite category of simplicial as the one opposite to the standard one [19] on the category of simplicial sets, the "fibrations" are the maps opposite to monomorphisms and the "weak equivalences" are the maps opposite to weak equivalences. It follows that the functor C^* converts "fibrations" to surjections and "weak equivalences" to quasi-isomorphisms. It then follows from [13, 1.9,1.10] that the left derived functor of $U\colon \mathscr{E} \to (\mathbf{\Delta}^{\mathrm{op}}\mathbf{Set})^{\mathrm{op}}$ exists and is adjoint to the right derived functor of $C^*\colon (\mathbf{\Delta}^{\mathrm{op}}\mathbf{Set})^{\mathrm{op}} \to \mathscr{E}$. When we regard U as a contravariant functor, this derived functor is the right derived functor, and we obtain the following proposition.

Proposition 2.3. *The (right) derived functor of U exists and gives an adjunction $\bar{h}\mathscr{E}(A, C^*X) \cong \mathscr{H}o(X, UA)$.*

Applying [13, 1.10] again, we obtain the following proposition, which is needed in the proofs of Theorems 1.1 and 1.2 in the next section.

Proposition 2.4. *The functor U converts relative cell inclusions of $\mathcal{E}$-algebras to Kan fibrations of simplicial sets.*

According to [13, 1.9], the derived functor of U is constructed by first approximating an arbitrary $\mathcal{E}$-algebra with a cell $\mathcal{E}$-algebra [13, 1.2] and then applying U. This gives us the following proposition.

Proposition 2.5. *Let X be a simplicial set and $A \to C^*X$ a quasi-isomorphism, where A is a cell $\mathcal{E}$-algebra. The unit of the derived adjunction $X \to UC^*X$ is represented by the map $X \to UA$.*

Instead of using the standard model structure on the category of simplicial sets, we can use the "$H_*(-;\mathbf{F}_p)$-local" model structure constructed in [1]. In this structure, the cofibrations remain the monomorphisms but the weak equivalences are the $\mathbf{F}_p$-homology equivalences. Since the functor C^* has the stronger property of converting $\mathbf{F}_p$-homology isomorphisms to quasi-isomorphisms, the derived adjunction factors as an adjunction between the homotopy category of $\mathcal{E}$-algebras and the p-adic homotopy category. Although we do not need it in the remainder of our work, we see that the functor U has the following strong $H_*(-;\mathbf{F}_p)$-local homotopy properties.

Proposition 2.6. *The functor U converts relative cell inclusions to $H_*(-;\mathbf{F}_p)$-local fibrations. For a cell $\mathcal{E}$-algebra A, UA is an $H_*(-;\mathbf{F}_p)$-local simplicial set.*

3. The Fibration Theorems

In this section, we prove Theorems 1.1 and 1.2 that allow us to construct resolvable simplicial sets out of other resolvable simplicial sets. The proofs proceed by choosing cell $\mathcal{E}$-algebra approximations and applying Proposition 2.4 of the previous section.

Proof of Theorem 1.1. By [13, 2.1], a map of $\mathcal{E}$-algebras can be factored as a relative cell inclusion followed by an acyclic surjection. Applying this to the $\mathcal{E}$-algebras C^*X_n, we can construct the following commutative diagram of $\mathcal{E}$-algebras.

$$\begin{tikzcd}
A_0 \arrow[r, tail] \arrow[d, two heads, "\sim"'] & A_1 \arrow[r, tail] \arrow[d, two heads, "\sim"'] & \cdots \arrow[r, tail] & A_n \arrow[r, tail] \arrow[d, two heads, "\sim"'] & \cdots \\
C^*X_0 \arrow[r] & C^*X_1 \arrow[r] & \cdots \arrow[r] & C^*X_n \arrow[r] & \cdots
\end{tikzcd}$$

Here as in [13] the arrows "$\overset{\sim}{\twoheadrightarrow}$" denote acyclic surjections and the arrows "$\rightarrowtail$" denote relative cell inclusions. Let $A = \operatorname{Colim} A_n$. From the universal property, we obtain a map $A \to C^*X$. The assumption that $H^*X = \operatorname{Colim} H^*X_n$ then implies that the map $A \to C^*X$ is a quasi-isomorphism.

Applying the functor U, we see that UA is the limit of UA_n. We have the following commutative diagram.

$$\begin{tikzcd}
\cdots \arrow[r, two heads] & X_n \arrow[r, two heads] \arrow[d, "\sim"'] & \cdots \arrow[r, two heads] & X_1 \arrow[r, two heads] \arrow[d, "\sim"'] & X_0 \arrow[d, "\sim"'] \\
\cdots \arrow[r, two heads] & UA_n \arrow[r, two heads] & \cdots \arrow[r, two heads] & UA_1 \arrow[r, two heads] & UA_0
\end{tikzcd}$$

The bottom row is a tower of Kan fibrations by Proposition 2.4 and the vertical maps are weak equivalences by Proposition 2.5 and the assumption that the X_n are resolvable. It follows that the map of the limits $X \to UA$ is a weak equivalence, and we conclude that A is resolvable. □

The proof of Theorem 1.2 is similar, but needs in addition the identification given in [13, 1.5] of the homology of the pushout of cell $\mathcal{E}$-algebras over relative cell inclusions in terms of the E_∞ torsion product of [12]. To apply this, we need the

following result proved in Section 7 relating the E_∞ torsion product to the usual differential torsion product.

Lemma 3.1. *Let X, Y, and Z be as in Theorem 1.2. The E_∞ torsion product $\operatorname{Tor}_*^{C^*Z}(C^*X, C^*Y)$ is canonically isomorphic to the usual differential torsion product $\operatorname{Tor}_*^{C^*Z}(C^*X, C^*Y)$. Under this isomorphism, the composite*

$$\operatorname{Tor}_*^{C^*Z}(C^*X, C^*Y) \to H^*(C^*X \amalg_{C^*Z} C^*Y) \to H^*(C^*(X \times_Z Y)) = H^*(X \times_Z Y)$$

is the Eilenberg–Moore map.

Proof of Theorem 1.2. Using [13, 1.7], we can choose cell $\mathcal{E}$-algebras A, B, C, quasi-isomorphisms $A \to C^*Z$, $B \to C^*X$, $C \to C^*Y$, and relative cell inclusions $A \to B$, $A \to C$ such that the following diagram commutes.

$$\begin{tikzcd}
B \arrow[d, "\sim"'] & A \arrow[l] \arrow[r] \arrow[d, "\sim"'] & C \arrow[d, "\sim"'] \\
C^*X & C^*Z \arrow[l] \arrow[r] & C^*Y
\end{tikzcd}$$

Let $D = B \amalg_A C$ and consider the map $D \to C^*(X \times_Z Y)$. By Lemma 3.1 and well-known results on the Eilenberg–Moore map (e.g. [21, 3.2]), the map $D \to C^*(X \times_Z Y)$ is a quasi-isomorphism. It follows that the unit of the derived adjunction is represented for $X \times_Z Y$ as the map $X \times_Z Y \to UD$. We have the following commutative diagram.

$$\begin{tikzcd}
X \times_Z Y \arrow[rr] \arrow[dr] \arrow[dd] & & Z \arrow[dr] \arrow[dd] & \\
& UD \arrow[rr] \arrow[dd] & & UC \arrow[dd] \\
Y \arrow[rr] \arrow[dr] & & X \arrow[dr] & \\
& UB \arrow[rr] & & UA
\end{tikzcd}$$

The assumption that X, Y, and Z are resolvable implies that all four maps between the top and bottom squares are weak equivalences, and we conclude that $X \times_Z Y$ is resolvable. □

4. A Model for $C^*K(\mathbf{Z}/p\mathbf{Z}, n)$

In this section, we prove Theorem 1.3 that $K(\mathbf{Z}/p\mathbf{Z}, n)$ and $K(\mathbf{Z}^\wedge_p, n)$ are resolvable for $n \geq 1$. We prove the resolvability of $K(\mathbf{Z}/p\mathbf{Z}, n)$ by constructing an explicit cell $\mathcal{E}$-algebra model of $C^*K(\mathbf{Z}/p\mathbf{Z}, n)$ that lets us analyze the unit of the derived adjunction. The case of $\mathbf{Z}^\wedge_p$ follows easily from the case of $\mathbf{Z}/p\mathbf{Z}$ and the work of the previous section.

The construction of our cell model requires the use of the generalized Steenrod operations for E_∞ algebras [12, §I.7], [17]. The theory of [17] gives $\mathbf{F}_p$-linear (but not $\bar{\mathbf{F}}_p$-linear) operations on the homology of an $\mathcal{E}$-algebra. In this section, we only need the operation P^0. This operation preserves degree and performs the p-th power operation on elements in degree zero. Using this fact, naturality, and the fact

that the operations commute with "suspension" [17, 3.3], the following observation can be proved by the argument of [17, 8.1].

Proposition 4.1. *For any simplicial set X, the operation P^0 on H^*X induced by the $\mathcal{E}$-algebra structure is the identity on elements of H^*X in the image of $H^*(X;\mathbf{F}_p)$.*

In Section 5, we describe all of the $\mathcal{E}$-algebra Steenrod operations on H^*X in terms of the usual Steenrod operations on $H^*(X;\mathbf{F}_p)$.

For $n \geq 1$, let K_n be a model for $K(\mathbf{Z}/p\mathbf{Z}, n)$ such that the set of n-simplices of K_n is $\mathbf{Z}/p\mathbf{Z}$, e.g. the "minimal" model [16, p. 100]. Then we have a fundamental cycle k_n of C^nK_n which represents the cohomology class in H^nK_n that is the image of the fundamental cohomology class of $H^n(K(\mathbf{Z}/p\mathbf{Z}, n);\mathbf{F}_p)$. Write $\bar{\mathbf{F}}_p[n]$ for the differential graded $\bar{\mathbf{F}}_p$-module consisting of $\bar{\mathbf{F}}_p$ in degree n and zero in all other degrees, and let $\bar{\mathbf{F}}_p[n] \to C^*K_n$ be the map of differential graded $\bar{\mathbf{F}}_p$-modules that sends $1 \in \bar{\mathbf{F}}_p$ to k_n. Let $\mathbb{E}$ denote the free functor from differential graded $\bar{\mathbf{F}}_p$-modules to $\mathcal{E}$-algebras. We obtain an induced map of $\mathcal{E}$-algebras $a\colon \mathbb{E}\bar{\mathbf{F}}_p[n] \to C^*K_n$ that sends the fundamental class i_n of $\mathbb{E}\bar{\mathbf{F}}_p[n]$ to the fundamental class k_n of C^*K_n.

The operation P^0 is not the identity on the fundamental homology class of $\mathbb{E}\bar{\mathbf{F}}_p[n]$. We obtain our cell $\mathcal{E}$-algebra model of C^*K_n, by forcing $(1-P^0)[i_n]$ to be zero as follows. Let p_n be an element of $\mathbb{E}\bar{\mathbf{F}}_p[n]$ that represents $(1-P^0)[i_n]$. Since $(1-P^0)[k_n]$ is zero in H^*K_n, $a(p_n)$ is a boundary in C^nK_n. Choose an element q_n of $C^{n-1}K_n$ such that $dq_n = a(p_n)$. Write $C\bar{\mathbf{F}}_p[n]$ for the differential graded $\bar{\mathbf{F}}_p$-module that is $\bar{\mathbf{F}}_p$ in dimensions $n-1$ and n and zero in all other dimensions, with the differential $\bar{\mathbf{F}}_p \to \bar{\mathbf{F}}_p$ the identity. We have a canonical map $q_n\colon C\bar{\mathbf{F}}_p[n] \to C^*K_n$ sending the generators to q_n and $a(p_n)$. We have a canonical map $\bar{\mathbf{F}}_p[n] \to C\bar{\mathbf{F}}_p[n]$, and a map $p_n\colon \bar{\mathbf{F}}_p[n] \to \mathbb{E}\bar{\mathbf{F}}_p[n]$ that sends the generator 1 to the element p_n. The diagram of differential graded k-modules on the left below then commutes.

$$\begin{tikzcd} \bar{\mathbf{F}}_p[n] \arrow[r] \arrow[d, "p_n"'] & C\bar{\mathbf{F}}_p[n] \arrow[d, "q_n"] \\ \mathbb{E}\bar{\mathbf{F}}_p[n] \arrow[r, "a"'] & C^*K_n \end{tikzcd} \qquad \begin{tikzcd} \mathbb{E}\bar{\mathbf{F}}_p[n] \arrow[r] \arrow[d, "p_n"'] & \mathbb{E}C\bar{\mathbf{F}}_p[n] \arrow[d, "q_n"] \\ \mathbb{E}\bar{\mathbf{F}}_p[n] \arrow[r, "a"'] & C^*K_n \end{tikzcd}$$

It follows that the diagram of $\mathcal{E}$-algebras on the right above commutes. Let B_n be the $\mathcal{E}$-algebra obtained from the following pushout diagram in the category of $\mathcal{E}$-algebras.

$$\begin{tikzcd} \mathbb{E}\bar{\mathbf{F}}_p[n] \arrow[r] \arrow[d, "p_n"'] & \mathbb{E}C\bar{\mathbf{F}}_p[n] \arrow[d, "q_n"] \\ \mathbb{E}\bar{\mathbf{F}}_p[n] \arrow[r, "a"'] & B_n \end{tikzcd}$$

We therefore obtain a map $\alpha\colon B_n \to C^*K_n$. We prove the following theorem in Section 6.

Theorem 4.2. *The map $\alpha\colon B_n \to C^*K_n$ is a quasi-isomorphism.*

Corollary 4.3. *K_n is resolvable.*

Proof. Applying U to the pushout diagram that defines B_n, we obtain the following pullback diagram of simplicial sets.

$$\begin{tikzcd} UB_n \arrow[r] \arrow[d, two heads] & U\mathbb{E}C\bar{\mathbf{F}}_p[n] \arrow[d, two heads] \\ U\mathbb{E}\bar{\mathbf{F}}_p[n] \arrow[r, "Up_n"'] & U\mathbb{E}\bar{\mathbf{F}}_p[n] \end{tikzcd}$$

The vertical maps are Kan fibrations since the inclusion $\mathbb{E}\bar{\mathbf{F}}_p[n] \to \mathbb{E}C\bar{\mathbf{F}}_p[n]$ is a relative cell inclusion. The following two propositions then imply that UB_n is a $K(\mathbf{Z}/p\mathbf{Z}, n)$.

By the theorem, the unit of the derived adjunction $K_n \to UC^*K_n$ is represented by the map $K_n \to UB_n$. To see that it is a weak equivalence, it suffices to see that the induced map on π_n is an isomorphism. The p distinct homotopy classes of maps from S^n to K_n induce maps $C^*K_n \to C^*S^n$ that differ on homology. It follows that the composite maps $B_n \to C^*S^n$ differ on homology and are therefore different maps in $\bar{h}\mathscr{E}$. We conclude from the adjunction isomorphism $\bar{h}\mathscr{E}(B_n, C^*S^n) \cong \mathscr{H}o(S^n, UB_n)$ that the map $K_n \to UB_n$ is injective on π_n, and is therefore an isomorphism on π_n. □

Proposition 4.4. *$U\mathbb{E}C\bar{\mathbf{F}}_p[n]$ is contractible.*

Proof. $\mathbb{E}C\bar{\mathbf{F}}_p[n]$ is a cell $\mathcal{E}$-algebra and so $U\mathbb{E}C\bar{\mathbf{F}}_p[n]$ and the map $\bar{\mathbf{F}}_p \to \mathbb{E}C\bar{\mathbf{F}}_p[n]$ is a quasi-isomorphism, so the map $U\mathbb{E}C\bar{\mathbf{F}}_p[n] \to U\bar{\mathbf{F}}_p = *$ is a weak equivalence of Kan complexes. □

Proposition 4.5. *$U\mathbb{E}\bar{\mathbf{F}}_p[n]$ is a $K(\bar{\mathbf{F}}_p, n)$ and the map Up_n induces on π_n the map $1 - \Phi$, where Φ denotes the Frobenius automorphism of $\bar{\mathbf{F}}_p$.*

Proof. We have canonical isomorphisms

$$U\mathbb{E}\bar{\mathbf{F}}_p[n] = \mathscr{E}(\mathbb{E}\bar{\mathbf{F}}_p[n], C^*\Delta) \cong \mathscr{M}(\bar{\mathbf{F}}_p[n], C^*\Delta),$$

where $\mathscr{M}$ denotes the category of differential graded $\bar{\mathbf{F}}_p$-modules. Thus $U\mathbb{E}\bar{\mathbf{F}}_p[n]$ is the simplicial set which in dimension m is the set of cocycles in $C^n\Delta[m]$. This is the minimal $K(\bar{\mathbf{F}}_p, n)$ [16, p. 100–101].

The map of simplicial sets $\Delta[n] \to \Delta[n]/\partial\Delta[n]$ induces a bijection

$$\mathscr{E}(\mathbb{E}\bar{\mathbf{F}}_p[n], C^*\Delta[n]) \cong \mathscr{E}(\mathbb{E}\bar{\mathbf{F}}_p[n], C^*(\Delta[n]/\partial\Delta[n])).$$

On the other hand, since $C^{n-1}(\Delta[n]/\partial\Delta[n]) = 0$, we have a canonical identification

$$\mathscr{E}(\mathbb{E}\bar{\mathbf{F}}_p[n], C^*(\Delta[n]/\partial\Delta[n])) \cong \mathscr{M}(\bar{\mathbf{F}}_p[n], C^*(\Delta[n]/\partial\Delta[n])) \cong H^n(\Delta[n]/\partial\Delta[n]).$$

By naturality, the map $H^n(\Delta[n]/\partial\Delta[n]) \to H^n(\Delta[n]/\partial\Delta[n])$ induced by p_n must be $1 - P^0$. Under the isomorphism

$$H^n(\Delta[n]/\partial\Delta[n]) \cong H^n(\Delta[n]/\partial\Delta[n]; \mathbf{F}_p) \otimes \bar{\mathbf{F}}_p \cong \bar{\mathbf{F}}_p,$$

we can identify the operation $1 - P^0$ as $1 - \Phi$ by Proposition 4.1 and the Cartan formula [17, 2.7ff]. □

We complete the proof of Theorem 1.3 by deducing that $K(\mathbf{Z}_p^\wedge, n)$ is resolvable for $n \geq 1$.

Proof of Theorem 1.3. We see by induction and Theorem 1.2 that $K(\mathbf{Z}/p^m\mathbf{Z}, n)$ is resolvable for $n \geq 1$ by considering the following fiber square

$$\begin{tikzcd} K(\mathbf{Z}/p^m\mathbf{Z}, n) \arrow[r] \arrow[d, two heads] & PK(\mathbf{Z}/p\mathbf{Z}, n+1) \arrow[d, two heads] \\ K(\mathbf{Z}/p^{m-1}\mathbf{Z}, n) \arrow[r] & K(\mathbf{Z}/p\mathbf{Z}, n+1), \end{tikzcd}$$

where $PK(\mathbf{Z}/p\mathbf{Z}, n+1)$ is some contractible simplicial set with a Kan fibration to $K(\mathbf{Z}/p\mathbf{Z}, n+1)$. Since $K(\mathbf{Z}_p^\wedge, n)$ can be constructed as the limit of a tower of Kan fibrations

$$\cdots \to K(\mathbf{Z}/p^m\mathbf{Z}, n) \to \cdots \to K(\mathbf{Z}/p\mathbf{Z}, n),$$

and the natural map $H^*K(\mathbf{Z}_p^\wedge, n) \to \operatorname{Colim} H^*K(\mathbf{Z}/p^m\mathbf{Z}, n)$ is an isomorphism, we conclude from Theorem 1.1 that $K(\mathbf{Z}_p^\wedge, n)$ is resolvable. $\square$

5. The Algebra of Generalized Steenrod Operations

The key to the proof of Theorem 4.2 is a study of the algebra of all generalized Steenrod operations of [17]. Precisely, let $\mathfrak{B}$ be the free associative $\mathbf{F}_p$-algebra generated by the P^s and (if $p > 2$) the βP^s [17, 2.2,§5] for all $s \in \mathbf{Z}$ modulo the two-sided ideal consisting of those operations that are zero on all "Adem objects" [17, 4.1] of "$\mathscr{C}(p,\infty)$" of [17, 2.1]. The Adem objects of $\mathscr{C}(p,\infty)$ include all E_∞ algebras over any E_∞ k-operad for any commutative $\mathbf{F}_p$-algebra k. We refer to $\mathfrak{B}$ as the algebra of all operations. In this section, we prove Theorem 1.4 and provide the main results needed in the next section to prove Theorem 4.2. We use the standard arguments effective in studying the Steenrod and Dyer-Lashoff algebras to analyze the structure of $\mathfrak{B}$.

Definition 5.1. We define length, admissibility, and excess as follows

(i) $p = 2$: Consider sequences $I = (s_1, \ldots, s_k)$. The sequence I determines the operation $P^I = P^{s_1} \cdots P^{s_k}$. We define the *length* of I to be k. Say that I is *admissible* if $s_j \geq 2s_{j+1}$ for $1 \leq j < k$. We define the *excess* of I by

$$e(I) = s_k + \sum_{j=1}^{k-1}(s_j - 2s_{j+1}) = s_1 - \sum_{j=2}^{k} s_j$$

(ii) $p > 2$: Consider sequences $I = (\epsilon_1, s_1, \ldots, \epsilon_k, s_k)$ such that ϵ_i is 0 or 1. The sequence I determines the operation $P^I = \beta^{\epsilon_1}P^{s_1} \cdots \beta^{\epsilon_k}P^{s_k}$, where $\beta^0 P^s$ means P^s and $\beta^1 P^s$ means βP^s. We define the *length* of I to be k. Say that I is *admissible* if $s_j \geq ps_{j+1} + \epsilon_{j+1}$. We define the *excess* of I by

$$e(I) = 2s_k + \epsilon_1 + \sum_{j=1}^{k-1}(2s_j - 2ps_{j+1} - \epsilon_{j+1}) = 2s_1 + \epsilon_1 - \sum_{j=2}^{k}(2s_j(p-1) + \epsilon_j)$$

In either case, by convention, the empty sequence determines the identity operation, has length zero, is admissible, and has excess $-\infty$. If I and J are sequences, we denote by (I, J) their concatenation.

Proposition 5.2. *The set $\{P^I \mid I \text{ is admissible}\}$ is a basis of the underlying $\mathbf{F}_p$-module of $\mathfrak{B}$.*

Proof. It follows from the Adem relations [17, 4.7] that the set generates $\mathfrak{B}$ as a $\mathbf{F}_p$-module. Linear independence follows by examination of the action on $H^*(\mathbb{G}\mathbf{F}_p[n])$ as n gets large, where $\mathbb{G}$ denotes the free $\mathcal{G}$-algebra functor for some E_∞ $\mathbf{F}_p$-operad $\mathcal{G}$. This follows for example from [18, 2.2 or 2.6]. □

Proposition 5.3. *If $s > 0$ then $P^{-s}(P^0)^s = 0$ and (if $p > 2$) $\beta P^{-s}(P^0)^s = 0$ (P^{-s} or βP^{-s} composed with s factors of P^0).*

Proof. The Adem relations [17, 4.7] for $\beta^\epsilon P^{-s}P^0$ when $s > 0$ are given by

$$\beta^\epsilon P^{-s}P^0 = \sum_{i=-\infty}^{\infty} (-1)^{-s-i}(pi - s, s - i - 1)\beta^\epsilon P^{-(s-i)}P^{-i},$$

where we understand $\epsilon = 0$ when $p = 2$, and the notation (j, k) denotes the binomial coefficient $(j + k)!/(j!k!)$ when $j \geq 0$ and $k \geq 0$ and zero if either $j < 0$ or $k < 0$. The coefficient $(pi - s, s - i - 1)$ therefore can only be non-zero when $s/p \leq i \leq s-1$. Then $P^{-1}P^0 = 0$ and $\beta P^{-1}P^0 = 0$ since the binomial coefficients are zero for all values of i. Assume by induction that $P^{-t}(P^0)^t = 0$ for all t such that $1 \leq t < s$; we see that $P^{-s}P^0$ and $\beta P^{-s}P^0$ are both in the left ideal generated by $\{P^{-t} \mid 1 \leq t < s\}$ and hence by the inductive hypothesis are annihilated by $(P^0)^{s-1}$; therefore, $P^{-s}(P^0)^s = 0$ and $\beta P^{-s}(P^0)^s = 0$. □

We can now prove the first half of Theorem 1.4.

Proposition 5.4. *The left ideal of $\mathfrak{B}$ generated by $(1 - P^0)$ is a two-sided ideal.*

Proof. By the previous proposition it suffices to show that for every admissible sequence I, $(1 - P^0)P^I$ is an element of the left ideal generated by $(1 - P^0)$. Let $I = (\epsilon_1, s_1, \ldots, \epsilon_k, s_k)$ be an admissible sequence (where if $p = 2$ each $\epsilon_j = 0$ and we think of this sequence as $(s_1, \ldots, s_k)$). If $s_k < 0$ then by what we have already shown,

$$P^I = P^I(1 - (P^0)^{-s_k}) = P^I(1 + P^0 + \cdots + (P^0)^{-s_k-1})(1 - P^0)$$

is in the ideal and hence $(1-P^0)P^I$ is as well. We can therefore assume that $s_k \geq 0$, and it follows from admissibility that $s_j \geq 0$ for all j. We proceed by induction on k, the length of I.

The statement is trivial for $k = 0$ (the empty sequence); now assume by induction that the statement holds for all sequences J of length less than k. We can write I as the concatenation $((\epsilon, s), J)$ for some sequence J of length $k - 1$. If $s = 0$, the Adem relation for $P^0\beta P^0$ is $P^0\beta P^0 = \beta P^0 P^0$, and we see that

$$(1 - P^0)P^I = (1 - P^0)\beta^\epsilon P^0 P^J = \beta^\epsilon P^0(1 - P^0)P^J$$

is in the ideal by induction. For $s > 0$, the Adem relation for P^0P^s takes the form

$$P^0P^s = \sum (-1)^i(-pi, (p-1)s + i - 1)P^{s-i}P^i.$$

When $i > 0$ the binomial coefficient is zero, when $i = 0$ we get the term P^sP^0, and when $i < 0$ we get terms of the form binomial coefficient times $P^{s-i}P^i$ that we know from the work above are in the ideal; therefore, we can write $P^0P^s = P^sP^0 + \alpha(1 - P^0)$ for some α. An entirely similar argument shows that $P^0\beta P^s$ can also be written $P^0\beta P^s = \beta P^sP^0 + \alpha(1 - P^0)$ for some α. It follows that

$$(1 - P^0)P^I = (1 - P^0)\beta^\epsilon P^sP^J = (\beta^\epsilon P^s + \alpha)(1 - P^0)P^J$$

is in the ideal by induction, and this completes the argument. □

For the other half of Theorem 1.4, we need a canonical map from $\mathfrak{B}$ to the Steenrod algebra $\mathfrak{A}$. It can be shown [17, 10.5] that the Steenrod operations on the cohomology of a simplicial set arise from the action of $\mathfrak{B}$ from a $\mathscr{C}(p,\infty)$ structure on the cochains with coefficients in $\mathbf{F}_p$. However, it is important for our purposes to relate the action of $\mathfrak{B}$ obtained from the $\mathcal{E}$-algebra structure to the Steenrod algebra. The previous proposition implies that if x is an element of a left $\mathfrak{B}$-module that is fixed by P^0, then the submodule $\mathfrak{B}x$ generated by x is fixed by P^0. It follows from this observation and Proposition 4.1 that for any simplicial set X, the $\mathbf{F}_p$-submodule $H^*(X;\mathbf{F}_p)$ of H^*X is a $\mathfrak{B}$-submodule. It then follows from the axioms that uniquely identify the Steenrod operations that the action of P^s on $H^*(X;\mathbf{F}_p)$ coincide with the Steenrod operations of the same name. Furthermore, by looking at C^*K_n, it is possible to identify βP^s as the composite of the operation P^s and the Bockstein. Thus, we understand the canonical map $\mathfrak{B}\to\mathfrak{A}$ as follows.

Proposition 5.5. *Let k be a commutative $\mathbf{F}_p$-algebra and let $\mathcal{G}$ an E_∞ operad of differential graded k-algebras. For any $\mathcal{G}$-algebra structure on $C^*(X;k)$ that is natural in the simplicial set X, the operations P^s and (for $p>2$) βP^s act on an element of $H^*(X;\mathbf{F}_p)\subset H^*(X;k)$ by the Steenrod operations of the same name.*

Remark 5.6. The previous proposition and the Cartan formula [17, 2.2] allow the identification the operations on H^*X in terms of the Steenrod operations. When X is of finite p-type, $H^*X\cong H^*(X;\mathbf{F}_p)\otimes_{\mathbf{F}_p}\bar{\mathbf{F}}_p$, and so every element of H^nX can be written as a linear combination $a_1x_1+\cdots+a_mx_m$ for some elements $x_1,\dots,x_m$ in $H^*(X;\mathbf{F}_p)$ and $a_1,\dots,a_m$ in $\bar{\mathbf{F}}_p$. Then

$$\beta^\epsilon P^s(a_1x_1+\cdots+a_mx_m)=\Phi(a_1)\beta^\epsilon P^s x_1+\cdots+\Phi(a_m)\beta^\epsilon P^s x_m,$$

where Φ denotes the Frobenius automorphism of $\bar{\mathbf{F}}_p$. In general H^*X is the limit of H^*X_α where X_α ranges over the finite subcomplexes of X.

Proof of Theorem 1.4. The map $\mathfrak{B}\to\mathfrak{A}$ is clearly surjective. Since the relation $P^0=1$ holds in $\mathfrak{A}$, the map $\mathfrak{B}\to\mathfrak{A}$ factors through the ring $\mathfrak{B}/(1-P^0)$ and certainly remains surjective. To see that it is injective, observe that by what we have shown, $\mathfrak{B}/(1-P^0)$ is generated as an $\mathbf{F}_p$-module by those P^I for admissible sequences $I=(\epsilon_1,s_1,\dots,\epsilon_k,s_k)$ such that $s_j>0$ for each j; the image of these elements in $\mathfrak{A}$ form an $\mathbf{F}_p$-module basis, and in particular are linearly independent. $\square$

6. Unstable Modules over $\mathfrak{B}$

In this section, we prove Theorem 4.2. The proof is based on a comparison of free unstable modules over $\mathfrak{A}$ with free unstable modules over $\mathfrak{B}$.

Definition 6.1. A module M over $\mathfrak{B}$ is *unstable* if for every $x\in M$ of degree d, and for every admissible sequence I with $e(I)>d$, $P^Ix=0$.

Observe that a module over the Steenrod algebra is unstable if and only if it is unstable as a module over $\mathfrak{B}$. Also observe that if $M=H^*A$ for an object of $\mathscr{C}(p,\infty)$, e.g. an E_∞ k-algebra A for a commutative $\mathbf{F}_p$-algebra k, then M is unstable [17, 5.(3)–(4)]. In the statement of the following proposition, the *enveloping algebra* of an unstable $\mathfrak{B}$-module is the free graded commutative algebra modulo the relation that the p-th power operation (the *restriction*) of any element is its p-th power under the multiplication in the ring.

Proposition 6.2. *If $\mathcal{G}$ is an E_∞ $\mathbf{F}_p$-operad, then $H^*\mathbb{G}\mathbf{F}_p[n]$ is the enveloping algebra of the free unstable $\mathfrak{B}$-module on one generator in degree m. $H^*\mathbb{E}\bar{\mathbf{F}}_p[n]$ is the extended $\bar{\mathbf{F}}_p$-algebra on the enveloping algebra of the free unstable $\mathfrak{B}$-module on one generator in degree n.*

Proof. The argument of [18, 2.6] applies to prove the first statement. The second statement follows from the first. $\square$

We denote by $\mathcal{A}_n^{\mathrm{un}}$ and $\mathcal{B}_n^{\mathrm{un}}$ the free unstable $\mathfrak{A}$ and $\mathfrak{B}$-modules on one generator in degree n. In the following proposition, $(1-P^0)$ denotes the map of $\mathfrak{B}$-modules $\mathcal{B}_n^{\mathrm{un}} \to \mathcal{B}_n^{\mathrm{un}}$ that sends the generator to $1-P^0$ times the generator.

Proposition 6.3. *For $n \geq 1$, the sequence*

$$0 \to \mathcal{B}_n^{\mathrm{un}} \xrightarrow{(1-P^0)} \mathcal{B}_n^{\mathrm{un}} \to \mathcal{A}_n^{\mathrm{un}} \to 0$$

is exact and split in the category of restricted $\mathbf{F}_p$-modules.

Proof. The fact that $\mathcal{B}_n^{\mathrm{un}} \to \mathcal{A}_n^{\mathrm{un}}$ is onto is clear since it is a map of $\mathfrak{B}$-modules and $\mathcal{A}_n^{\mathrm{un}}$ is generated as a $\mathfrak{B}$-module by the image of the generator of $\mathcal{B}_n^{\mathrm{un}}$. Similarly, exactness in the middle is clear from examination of the $\mathbf{F}_p$-module bases of $\mathfrak{B}$ and $\mathfrak{A}$. Thus, it remains to show that the map $\mathcal{B}_n^{\mathrm{un}} \to \mathcal{B}_n^{\mathrm{un}}$ is injective and split in the category of restricted $\mathbf{F}_p$-modules.

We proceed by writing an explicit splitting $f \colon \mathcal{B}_n^{\mathrm{un}} \to \mathcal{B}_n^{\mathrm{un}}$ in the category of restricted $\mathbf{F}_p$-modules as follows. It suffices to specify f on $P^I b_n$ for each admissible $I = (\epsilon_1, s_1, \dots, \epsilon_k, s_k)$ with $e(I) \leq n$. If $s_k < 0$, choose

$$f(P^I b_n) = P^I(1 + P^0 + (P^0)^2 + \cdots)b_n.$$

This is well-defined by Proposition 5.3. If $\epsilon_k + s_k > 0$ or if I is empty, then choose $f(P^I b_n)$ to be zero. Let $n(I)$ denote the largest number n such that the subsequence $(\epsilon_{k-n+1}, s_{k-n+1}, \dots, \epsilon_k, s_k)$ is all zeros; if $\epsilon_k \neq 0$ or $s_k \neq 0$ then $n(I)$=0. We have chosen $f(P^I)$ when $n(I)$ is zero, we now proceed by induction on $n(I)$ to choose $f(P^I)$ for $n(I) > 0$. When $n(I) > 0$, we can write I as the concatenation $(J, (0,0))$ where $n(J) = n(I) - 1$; choose

$$f(P^I b_n) = -P^J b_n + f(P^J b_n).$$

It is immediate from the construction and the fact that p-th power operations do not change the excess that f is a map of restricted $\mathbf{F}_p$-modules. We need to verify that the composite of f and the map $(1-P^0)$ is the identity. Let us denote by M_- the $\mathbf{F}_p$-submodule of $\mathcal{B}_n^{\mathrm{un}}$ generated by $P^I b_n$ for those $I = (\epsilon_1, s_1, \dots, \epsilon_k, s_k)$ with $s_k < 0$; let us denote by M_+ the $\mathbf{F}_p$-submodule generated by $P^I b_n$ for those $I = (\epsilon_1, s_1, \dots, \epsilon_k, s_k)$ with $s_k \geq 0$ or $k = 0$; clearly $\mathcal{B}_n^{\mathrm{un}}$ is the internal direct sum $M_- \oplus M_+$. The map $(1-P^0)$ sends $P^I b_n$ to $P^I(1-P^0)b_n$; it clearly sends M_+ into M_+, and it follows from Propositions 5.2 and 5.3 that it sends M_- to M_-. Since on M_-, f sends αb_n to $\alpha(1 + P^0 + (P^0)^2 + \cdots)b_n$, it follows that the composite on M_- sends αb_n to $\alpha(1-P^0)(1 + P^0 + (P^0)^2 + \cdots)b_n = \alpha b_n$. To see that the composite is the identity on M_+, it suffices to check it on a standard basis element, $P^J b_n$, where $J = (\epsilon_1, s_1, \dots, \epsilon_k, s_k)$ is an admissible sequence with $e(J) \leq n$ and $s_k \geq 0$. Write I for the concatenation $(J, (0,0))$. Observe that I is admissible and $e(I) = e(J) \leq n$, so

$$f(P^J(1-P^0)b_n) = f(P^J b_n) - f(P^I b_n) = f(P^J b_n) - (-P^J b_n + f(P^J b_n)) = P^J b_n.$$

It follows that the composite is the identity. $\square$

Proof of Theorem 4.2. Let $\mathbb{V}$ denote the composite of the enveloping algebra functor and the functor $\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} (-)$. Since this is the free functor from restricted $\mathbf{F}_p$-modules to graded commutative $\bar{\mathbf{F}}_p$-algebras, it preserves colimits. To avoid confusion, let us note the (isomorphic) image of $\mathcal{B}_n^{\mathrm{un}}$ in $\mathcal{B}_n^{\mathrm{un}}$ under the map $(1-P^0)$ discussed above by I_n; by Proposition 6.3, $\mathcal{B}_n^{\mathrm{un}}$ is isomorphic as a restricted $\mathbf{F}_p$-module to the direct sum $I_n \oplus \mathcal{A}_n^{\mathrm{un}}$, and it follows that the ring $\mathbb{V}\mathcal{B}^{\mathrm{un}}$ is isomorphic to the ring $\mathbb{V}I_n \otimes \mathbb{V}\mathcal{A}_n^{\mathrm{un}}$. We therefore obtain isomorphisms

$$\mathrm{Tor}_{*,*}^{\mathbb{V}I_n}(\bar{\mathbf{F}}_p, \mathbb{V}\mathcal{B}_n^{\mathrm{un}}) \cong \bar{\mathbf{F}}_p \otimes_{\mathbb{V}I_n} \mathbb{V}\mathcal{B}_n^{\mathrm{un}} \cong \mathbb{V}\mathcal{A}_n^{\mathrm{un}}$$

where the first map is the projection from the torsion product to the tensor product and the second map is induced by $\mathbb{V}$ applied to the quotient map $\mathcal{B}_n^{\mathrm{un}} \to \mathcal{A}_n^{\mathrm{un}}$.

On the other hand, Proposition 6.2 identifies $H^*\mathbb{E}\bar{\mathbf{F}}_p[n]$ as $\mathbb{V}\mathcal{B}_n^{\mathrm{un}}$. It is well-known that $H^*K_n = H^*K(\mathbf{Z}/p\mathbf{Z}, n)$ can be identified with $\mathbb{V}\mathcal{A}_n^{\mathrm{un}}$. We see from Proposition 5.5 that the map a from $\mathbb{E}\bar{\mathbf{F}}_p[n]$ to C^*K_n in the construction of B_n induces on homology groups the map $\mathbb{V}\mathcal{B}_n^{\mathrm{un}} \to \mathbb{V}\mathcal{A}_n^{\mathrm{un}}$ obtained by applying $\mathbb{V}$ to the quotient map $\mathcal{B}_n^{\mathrm{un}} \to \mathcal{A}_n^{\mathrm{un}}$. Likewise, the map $p_n \colon \mathbb{E}\bar{\mathbf{F}}_p[n] \to \mathbb{E}\bar{\mathbf{F}}_p[n]$ in the construction of B_n induces the map $\mathbb{V}\mathcal{B}_n^{\mathrm{un}} \to \mathbb{V}\mathcal{B}_n^{\mathrm{un}}$ obtained by applying $\mathbb{V}$ to the map $(1-P^0) \colon \mathcal{B}_n^{\mathrm{un}} \to \mathcal{B}_n^{\mathrm{un}}$; in other words, we can identify the map induced by p_n on homology as the inclusion $\mathbb{V}I_n \to \mathbb{V}\mathcal{B}_n^{\mathrm{un}}$. By [13, 1.5], the spectral sequence of [12, V.7.3] calculates the homology groups of the pushout B_n. This spectral sequence has E^2 term $\mathrm{Tor}_{*,*}^{\mathbb{V}I_n}(\bar{\mathbf{F}}_p, \mathbb{V}\mathcal{B}_n^{\mathrm{un}})$. From the discussion of the last paragraph, we see that this spectral sequence degenerates at E^2 with no extension problems and that the map from B_n to C^*K_n is a weak equivalence. □

7. The E_∞ Torsion Product and the Eilenberg–Moore Map

In this section we prove Lemma 3.1. The proof consists of an adaptation of the results of [13] to compare a bar construction in the category of $\mathcal{E}$-algebras to the cochain complex of the cobar construction of spaces.

Recall that for maps of simplicial sets $X \to Z$ and $Y \to Z$, the cobar construction $\mathrm{Cobar}^\bullet(X, Z, Y)$ is the cosimplicial simplicial set that is given in cosimplicial degree n by

$$\mathrm{Cobar}^n(X, Z, Y) = X \times \underbrace{Z \times \cdots \times Z}_{n \text{ factors}} \times Y$$

with face maps induced by diagonal maps and degeneracies by projections. The cochain complex $C^*(\mathrm{Cobar}^\bullet(X, Z, Y))$ is then a simplicial $\mathcal{E}$-algebra. The normalization $N(C^*(\mathrm{Cobar}^\bullet(X, Z, Y)))$ is a differential graded $\bar{\mathbf{F}}_p$-module; there is a canonical map from the (usual) differential torsion product to the homology

$$\mathrm{Tor}^{C^*Z}(C^*X, C^*Y) \to H^*(N(C^*(\mathrm{Cobar}^\bullet(X, Z, Y)))),$$

which is an isomorphism when X, Y, and Z are of finite p-type. On the other hand, considering $X \times_Z Y$ as a cosimplicial simplicial set constant in the cosimplicial direction, the inclusion $X \times_Z Y \to X \times Y$ induces a map of cosimplicial simplicial sets $X \times_Z Y \to \mathrm{Cobar}^\bullet(X, Z, Y)$ and therefore a map of differential graded $\bar{\mathbf{F}}_p$-modules

$$N(C^*(\mathrm{Cobar}^\bullet(X, Z, Y))) \to C^*(X \times_Z Y).$$

The composite map

$$\mathrm{Tor}^{C^*Z}(C^*X, C^*Y) \to H^*(N(C^*(\mathrm{Cobar}^\bullet(X, Z, Y)))) \to H^*(X \times_Z Y)$$

is the Eilenberg–Moore map.

The corresponding construction in the category of $\mathcal{E}$-algebras is the bar construction. For $\mathcal{E}$-algebra maps $A \to B$ and $A \to C$, the bar construction $\beta_\bullet(B, A, C)$ is the simplicial $\mathcal{E}$-algebra that is given in simplicial degree n by

$$\beta_n(B, A, C) = B \amalg \underbrace{A \amalg \cdots \amalg A}_{n \text{ factors}} \amalg C.$$

Regarding $B \amalg_A C$ as a constant simplicial $\mathcal{E}$-algebra, the map $B \amalg C \to B \amalg_A C$ induces a map of simplicial $\mathcal{E}$-algebras $\beta_\bullet(B, A, C) \to B \amalg_A C$ and therefore a map of differential graded $\bar{\mathbf{F}}_p$-modules $N(\beta_\bullet(B, A, C)) \to B \amalg_A C$. According to [13, 1.6], when A is a cell $\mathcal{E}$-algebra, and $A \to B$ and $A \to C$ are relative cell inclusions, the natural map

$$N(\beta_\bullet(B, A, C)) \to B \amalg_A C$$

is a quasi-isomorphism.

The proof of Lemma 4.2 is a straightforward comparison of these two constructions.

Proof of Lemma 4.2. Using [13, 1.7], we can find cell $\mathcal{E}$-algebras A, B, C, relative cell inclusions $A \to B$, $A \to C$, and quasi-isomorphisms $A \to Z$, $B \to X$, $C \to Y$, such that the following diagram commutes.

$$\begin{tikzcd}
B \arrow[d, "\sim"'] & A \arrow[l, tail] \arrow[r, tail] \arrow[d, "\sim"'] & C \arrow[d, "\sim"'] \\
C^*X & C^*Z \arrow[l] \arrow[r] & C^*Y
\end{tikzcd}$$

The various projection maps of $X \times (Z \times \cdots \times Z) \times Y$ induce a map

$$\begin{aligned} B \amalg (A \amalg \cdots \amalg A) \amalg C \to C^*X \amalg (C^*Z \amalg \cdots \amalg C^*Z) \amalg C^*Y & \\ \to C^*(X \times (Z \times \cdots \times Z) \times Y). & \end{aligned}$$

By [13, 1.4] and the Künneth theorem, the composite above is a quasi-isomorphism. We obtain a degreewise quasi-isomorphism of simplicial $\mathcal{E}$-algebras

$$\beta_\bullet(B, A, C)) \to C^*(\mathrm{Cobar}^\bullet(X, Z, Y))$$

and therefore a quasi-isomorphism of differential graded $\bar{\mathbf{F}}_p$-modules

$$N(\beta_\bullet(B, A, C)) \to N(C^*(\mathrm{Cobar}^\bullet(X, Z, Y)))$$

that makes the following diagram commute.

$$\begin{tikzcd}
N(\beta_\bullet(B, A, C)) \arrow[r, "\sim"] \arrow[d, "\sim"'] & N(C^*(\mathrm{Cobar}^\bullet(X, Z, Y))) \arrow[d] \\
B \amalg_A C \arrow[r] & C^*(X \times_Z Y)
\end{tikzcd}$$

By [13, 1.5], $H^*(B \amalg_A C)$ is the E_∞ torsion product $\mathrm{Tor}^{C^*Z}(C^*X, C^*Y)$, and under this identification, the map $B \amalg_A C \to C^*X \amalg_{C^*Z} C^*Y$ is the canonical map $\mathrm{Tor}^{C^*Z}(C^*X, C^*Y) \to C^*X \amalg_{C^*Z} C^*Y$. The lemma now follows. $\square$

Appendix A. Other Fields

We use the techniques developed in the body of the paper to discuss when the analogue of the Main Theorem holds for a field k. We prove the following theorem. In this theorem, Φ denotes the Frobenius endomorphism on a field of positive characteristic.

Theorem A.1. *Let k be a field. The singular cochain functor with coefficients in k induces an equivalence between the homotopy category of $H_*(-;k)$-local [1] nilpotent spaces of finite k-type and a full subcategory of the homotopy category of E_∞ k-algebras if and only if k satisfies one of the following two conditions*
(i) *$k = \mathbf{Q}$, the field of rational numbers.*
(ii) *k has positive characteristic and $1 - \Phi$ is surjective.*

It follows in particular that the analogue of the Main Theorem does not hold when k is a finite field. The smallest field of characteristic p for which $1 - \Phi$ is surjective is the fixed field in $\bar{\mathbf{F}}_p$ of $\mathbf{Z}^\wedge_p \lhd \mathrm{Gal}(\bar{\mathbf{F}}_p/\mathbf{F}_p)$.

For an arbitrary field k, there is no difficulty in providing a natural $\mathcal{E}_k$-algebra structure on the cochains of simplicial sets, for some E_∞ k-operad $\mathcal{E}_k$. For example the work of [10] and the construction described in Section 1 produce such a structure. Write $\mathscr{E}_k$ for the category of $\mathcal{E}_k$-algebras. We can form the adjoint functor $U(-;k)$ from $\mathcal{E}_k$-algebras to simplicial sets by the simplicial mapping set

$$U_\bullet(A;k) = \mathscr{E}(A, C^*(\Delta[\,\cdot\,];k)).$$

Arguing as in Section 2, we obtain the following proposition.

Proposition A.2. *The functors $C^*(-;k)$ and $U(-;k)$ are contravariant right adjoints between the category of $\mathcal{E}_k$-algebras and the category of simplicial sets. Their right derived functors exist and give an adjunction between the homotopy category of $\mathcal{E}_k$-algebras and the homotopy category.*

We say that a simplicial set is k-resolvable if the unit of the derived adjunction $X \to U(C^*(X;k);k)$ is an isomorphism in the homotopy category. As an elementary consequence of the previous proposition, we see that $C^*(-;k)$ gives an equivalence as in the statement of the theorem if and only if every connected $H_*(-;k)$-local nilpotent simplicial set of finite k-type is k-resolvable. The base field $\bar{\mathbf{F}}_p$ is irrelevant in Sections 3 and 7, and the arguments there apply to prove the following propositions that allow us to argue inductively up principally refined Postnikov towers.

Proposition A.3. *Let $X = \mathrm{Lim}\, X_n$ be the limit of a tower of Kan fibrations. Assume that the canonical map from $H^*(X;k)$ to $\mathrm{Colim}\, H^*(X_n;k)$ is an isomorphism. If each X_n is k-resolvable, then X is k-resolvable.*

Proposition A.4. *Let X, Y, and Z be connected simplicial sets of finite k-type, and assume that Z is simply connected. Let $X \to Z$ be a map of simplicial sets, and let $Y \to Z$ be a Kan fibration. If X, Y, and Z are k-resolvable, then so is the fiber product $X \times_Z Y$.*

A connected space is nilpotent $H_*(-;k)$-local and of finite k-type if and only if its Postnikov tower has a principal refinement with fibers:
(i) $K(\mathbf{Q}, n)$ when k is characteristic zero.
(ii) $K(\mathbf{Z}/p\mathbf{Z}, n)$ or $K(\mathbf{Z}^\wedge_p, n)$ when k is characteristic $p > 0$.

By the argument in Section 4, $K(\mathbf{Z}_p^\wedge, n)$ is easily seen to be k-resolvable when $K(\mathbf{Z}/p\mathbf{Z}, n)$ is. The theorem is therefore a consequence of the following two propositions.

Proposition A.5. *Let k be a field of characteristic zero. $K(\mathbf{Q}, n)$ is k-resolvable if and only if $k = \mathbf{Q}$.*

Proof. Write $\mathbb{E}$ for the free $\mathcal{E}_k$-algebra functor. Let $a\colon \mathbb{E}k[n] \to C^*(K(\mathbf{Q}, n); k)$ be any map of $\mathcal{E}_k$-algebras that sends the fundamental class of $k[n]$ to the fundamental class of $H^*(K(\mathbf{Q}, n); \mathbf{Q}) \subset H^*(K(\mathbf{Q}, n); k)$. Since k is characteristic zero, it is easy to see that a is a quasi-isomorphism, so the unit of the derived adjunction is represented by the map $K(\mathbf{Q}, n) \to U\mathbb{E}k[n]$. It is straightforward to check that $U\mathbb{E}k[n]$ is a $K(k, n)$ and the map $K(\mathbf{Q}, n) \to K(k, n)$ induces on π_n the inclusion $\mathbf{Q} \subset k$. $\square$

Proposition A.6. *Let k be a field of characteristic $p > 0$. $K(\mathbf{Z}/p\mathbf{Z}, n)$ is k-resolvable if and only if $1 - \Phi$ is surjective on k.*

Proof. We can construct a model $B_{n;k}$ for $C^*(K_n, k)$ exactly as in Section 4 and prove that the map $\alpha_k\colon B_n \to C^*(K_n; k)$ is a quasi-isomorphism just as in Section 6. We are therefore reduced to checking when the map $K_n \to UB_{n;k}$ is a weak equivalence. Again, we have $UB_{n;k}$ given by a Kan fibration square

$$\begin{tikzcd} UB_{n;k} \arrow[r] \arrow[d, two heads] & U\mathbb{E}Ck[n] \arrow[d, two heads] \\ U\mathbb{E}k[n] \arrow[r, "Up_n"'] & U\mathbb{E}k[n]. \end{tikzcd}$$

The argument of Proposition 4.5 then applies to show that $U\mathbb{E}k[n]$ is a $K(k, n)$ and the map Up_n induces on π_n the map $1 - \Phi$. It follows that $UB_{n;k}$ is a $K(\mathbf{Z}/p\mathbf{Z}, n)$ if and only if $1 - \Phi$ is surjective. When $1 - \Phi$ is surjective, it is straightforward to verify that the map $K_n \to UB_{n;k}$ is a weak equivalence. $\square$

Appendix B. E_∞ Ring Spectra under $H\bar{\mathbf{F}}_p$

We sketch how the arguments in this paper can be modified to prove the following unpublished theorem of W. G. Dwyer and M. J. Hopkins [6] comparing the p-adic homotopy category with the homotopy category of E_∞ $H\bar{\mathbf{F}}_p$ ring spectra.

Theorem B.1. (Dwyer–Hopkins) *The free mapping spectrum functor $F((-)_+, \bar{\mathbf{F}}_p)$ induces an equivalence between the homotopy category of connected nilpotent spaces of finite p-type and a full subcategory of the homotopy category of E_∞ $H\bar{\mathbf{F}}_p$ ring spectra.*

By the homotopy category of E_∞ $H\bar{\mathbf{F}}_p$ ring spectra, we mean the category obtained from the category of E_∞ ring spectra under the (cofibrant) E_∞ ring spectrum $H\bar{\mathbf{F}}_p$ by formally inverting the weak equivalences. The free mapping spectrum $F(X_+, H\bar{\mathbf{F}}_p)$ is naturally an E_∞ ring spectrum with an E_∞ ring map

$$H\bar{\mathbf{F}}_p = F(*_+, H\bar{\mathbf{F}}_p) \to F(X_+, H\bar{\mathbf{F}}_p)$$

induced by the collapse map $X \to *$. The functor $F((-)_+, \bar{\mathbf{F}}_p)$ therefore takes values in the category of E_∞ $H\bar{\mathbf{F}}_p$ ring spectra. This functor is the spectrum analogue of the singular chain complex. Its right derived functor represents unreduced

ordinary cohomology with coefficients in $\bar{\mathbf{F}}_p$ in the sense that there is a canonical map $H^*(X;\bar{\mathbf{F}}_p) \to \pi_{-*}(F(X_+,\bar{\mathbf{F}}_p))$ that is an isomorphism if X is a CW complex.

It is convenient for us to use a modern variant of the category of E_∞ $H\bar{\mathbf{F}}_p$ ring spectra, the category of commutative $H\bar{\mathbf{F}}_p$-algebras, a certain subcategory introduced in [8]. The forgetful functor from commutative $H\bar{\mathbf{F}}_p$-algebras to E_∞ $H\bar{\mathbf{F}}_p$ ring spectra induces an equivalence of homotopy categories. We have a commutative $H\bar{\mathbf{F}}_p$-algebra variation of the free mapping spectrum functor, given by

$$FX = S \wedge_{\mathcal{L}} F(X_+, H\bar{\mathbf{F}}_p).$$

There is a natural map $FX \to F(X_+, H\bar{\mathbf{F}}_p)$ that is always a weak equivalence, and so it suffices to prove that the functor F induces an equivalence between the homotopy category of connected nilpotent spaces of finite p-type and a full subcategory of the homotopy category of commutative $H\bar{\mathbf{F}}_p$-algebras. We denote the category of commutative $H\bar{\mathbf{F}}_p$-algebras as $\mathscr{C}$. By [8, VII.4.10], $\mathscr{C}$ is a closed model category with weak equivalences the weak equivalences of the underlying spectra; we denote its homotopy as $\bar{h}\mathscr{C}$.

The commutative $H\bar{\mathbf{F}}_p$-algebra FX is the "cotensor" of $H\bar{\mathbf{F}}_p$ with X [8, VII.2.9]. In general, the cotensor A^X of a commutative $H\bar{\mathbf{F}}_p$-algebra A with the space X is the commutative $H\bar{\mathbf{F}}_p$-algebra that solves the universal mapping problem $\mathscr{C}(-,A^X) \cong \mathscr{U}(X,\mathscr{C}(-,A))$, where $\mathscr{U}$ denotes the category of (compactly generated and weakly Hausdorff) spaces. Similarly, the tensor $A \otimes X$ of A with the space X is the commutative $H\bar{\mathbf{F}}_p$-algebra that solves the universal mapping problem $\mathscr{C}(A\otimes X,-) \cong \mathscr{U}(X,\mathscr{C}(A,-))$. Clearly, when they exist A^X and $A \otimes X$ are unique up to canonical isomorphism, and [8, VII.2.9] guarantees that they exist for any A and any X. The significance of the identification of FX as the tensor is in the following proposition.

Proposition B.2. *The functor $T\colon \mathscr{C} \to \mathscr{U}$ defined by $TA = \mathscr{C}(A, H\bar{\mathbf{F}}_p)$ is a continuous contravariant right adjoint to F. In other words, there is a homeomorphism $\mathscr{U}(X,TA) \cong \mathscr{C}(A,FX)$, natural in the space X and the commutative $H\bar{\mathbf{F}}_p$-algebra A.*

We have introduced the notion of tensor here to take advantage of [8, VII.4.16] that identifies the tensor $A \otimes I$ as a Quillen cylinder object when A is cofibrant. This allows us to relate the homotopies in the sense of Quillen with topological homotopies defined in terms of $(-)\otimes I$ or in terms of paths in mapping spaces. In particular, since all objects in $\mathscr{C}$ are fibrant, it follows that the natural transformation $\pi_0(\mathscr{C}(A,-)) \to \bar{h}\mathscr{C}(A,-)$ is an isomorphism when A is cofibrant. Since the adjunction isomorphism $\mathscr{U}(X,TA) \cong \mathscr{C}(A,FX)$ is a homeomorphism, letting X vary over the spheres, we obtain the following proposition.

Proposition B.3. *The functor T preserves weak equivalences between cofibrant objects.*

As a slight generalization of the proof of [8, VII.4.16], it is elementary to check that when A is a cofibrant object of $\mathscr{C}$ and $A \to B$ is a cofibration, the map $(A\otimes I) \amalg_A B \to B \otimes I$ is an acyclic cofibration and therefore (since every object is fibrant) the inclusion of a retract. Applying T and using the tensor adjunction, we obtain the following proposition.

Proposition B.4. *The functor T converts cofibrations to fibrations.*

The functors F and T are therefore a model category adjunction. In particular, we obtain the following proposition.

Proposition B.5. *The (right) derived functors F and T exist and give a contravariant right adjunction $\mathscr{H}o(X, TA) \cong \bar{h}\mathscr{C}(A, FX)$.*

For the purposes of this section, let us say that a space X is $H\bar{\mathbf{F}}_p$-resolvable if the unit of the derived adjunction $X \to TFX$ is a weak equivalence. Thus, we need to show that if X is a connected nilpotent p-complete space of finite p-type, then X is $H\bar{\mathbf{F}}_p$-resolvable. Again, we work by induction up principally refined Postnikov towers. The following analogue of Theorem 1.1 can be proved from Proposition B.4 by essentially the same argument used to prove Theorem 1.1 from Proposition 2.4.

Proposition B.6. *Let $X = \operatorname{Lim} X_n$ be the limit of a tower of Serre fibrations. Assume that the canonical map from H^*X to $\operatorname{Colim} H^*X_n$ is an isomorphism. If each X_n is $H\bar{\mathbf{F}}_p$-resolvable, then X is resolvable.*

We have in addition the following analogue of Theorem 1.2.

Theorem B.7. *Let X, Y, and Z be connected spaces of finite p-type, and assume that Z is simply connected. Let $X \to Z$ be a map, and let $Y \to Z$ be a Serre fibration. If X, Y, and Z are $H\bar{\mathbf{F}}_p$-resolvable, then so is the fiber product $X \times_Z Y$.*

The proof of the this theorem is essentially the same in outline as the proof of Theorem 1.2. The analogue of Lemma 3.1 can be proved by observing that the bar construction of the cofibrant approximations in $\mathscr{C}$ is equivalent to the (thickened) realization of F applied to the cobar construction of the singular simplicial sets on the spaces X, Y, and Z. Some fiddling with the filtration induced by the cosimplicial direction of the cobar construction and the filtration induced by the skeletal filtration of the singular simplicial sets allows the identification of $\operatorname{Tor}^{FZ}_{-*}(FX, FY)$ as $\operatorname{Tor}^{C^*Z}_*(C^*X, C^*Y)$ and the composite map

$$\operatorname{Tor}^{C^*Z}_*(C^*X, C^*Y) \cong \operatorname{Tor}^{FZ}_{-*}(FX, FY) \to \pi_{-*}F(X \times_Z Y) \cong H^*(X \times_Z Y)$$

as the Eilenberg–Moore map.

To complete the proof of Theorem B.1, we need to see that $K(\mathbf{Z}/p\mathbf{Z}, n)$ is $H\bar{\mathbf{F}}_p$-resolvable. It then follows as in Section 1.3 that $K(\mathbf{Z}^\wedge_p, n)$ is $H\bar{\mathbf{F}}_p$-resolvable and by induction up principal Postnikov towers that every connected nilpotent p-complete space of finite p-type is $H\bar{\mathbf{F}}_p$-resolvable. The remainder of the appendix is devoted to sketching a proof of the following theorem.

Theorem B.8. *For $n \geq 1$, $K(\mathbf{Z}/p\mathbf{Z}, n)$ is $H\bar{\mathbf{F}}_p$-resolvable.*

The homotopy groups of a commutative $H\bar{\mathbf{F}}_p$-algebra have an action by the algebra $\mathfrak{B}$, and it is elementary to show that the "free" commutative $H\bar{\mathbf{F}}_p$-algebra on the spectrum S^{-n}, denoted $\mathbb{P}S^{-n}_{H\bar{\mathbf{F}}_p}$ in [8] is the extended $\bar{\mathbf{F}}_p$-algebra on the enveloping algebra of the free unstable $\mathfrak{B}$-module on one generator in degree n. We construct a commutative $H\bar{\mathbf{F}}_p$-algebra B_n as the commutative $H\bar{\mathbf{F}}_p$-algebra that makes the following diagram a pushout in $\mathscr{C}$.

$$\begin{tikzcd}
\mathbb{P}S^{-n}_{\bar{\mathbf{F}}_p} \arrow[r] \arrow[d, "p_n"'] & \mathbb{P}CS^{-n}_{\bar{\mathbf{F}}_p} \arrow[d] \\
\mathbb{P}S^{-n}_{\bar{\mathbf{F}}_p} \arrow[r] & B_n
\end{tikzcd}$$

Here p_n is any map in the unique homotopy class that on homotopy groups sends the fundamental class of $\pi_{-n}S^{-n}_{H\bar{\mathbf{F}}_p}$ to $1 - P^0$ applied to the fundamental class. Choosing a map $a\colon \mathbb{P}S^{-n}_{\bar{\mathbf{F}}_p} \to FK(\mathbf{Z}/p\mathbf{Z}, n)$ that represents the fundamental class of $H^n(K(\mathbf{Z}/p\mathbf{Z}, n))$, and a null homotopy $\mathbb{P}CS^{-n}_{\bar{\mathbf{F}}_p} \to FK(\mathbf{Z}/p\mathbf{Z}, n)$ for the map $p_n \circ a\colon \mathbb{P}S^{-n}_{\bar{\mathbf{F}}_p} \to FK(\mathbf{Z}/p\mathbf{Z}, n)$, we obtain an induced map $B_n \to FK(\mathbf{Z}/p\mathbf{Z}, n)$.

Proposition B.9. *For $n \geq 1$, the map $B_n \to FK(\mathbf{Z}/p\mathbf{Z}, n)$ is a weak equivalence.*

The proof uses the Eilenberg–Moore spectral sequence of [8, IV.4.1] in place of the Eilenberg–Moore spectral sequence of [12, V.7.3], but otherwise is the same as the proof of Theorem 4.2.

Since B_n is a cofibrant commutative $H\bar{\mathbf{F}}_p$-algebra, the unit of the derived adjunction is represented by the map $K(\mathbf{Z}/p\mathbf{Z}, n) \to TB_n$ adjoint to the map constructed above. Since B_n is defined as a pushout of a cofibration, Proposition B.4 allows us to identify TB_n as the pullback of fibration. Looking at the mapping spaces and using the freeness adjunction, we see that TB_n is the homotopy fiber of an endomorphism on $K(\bar{\mathbf{F}}_p, n)$. Write α_n for the induced endomorphism on $\bar{\mathbf{F}}_p$. To see that TB_n is a $K(\mathbf{Z}/p\mathbf{Z}, n)$, it suffices to show that α_n is $1 - \Phi$. Once we know that TB_n is a $K(\mathbf{Z}/p\mathbf{Z}, n)$, the argument of Corollary 4.3 shows that the map $K(\mathbf{Z}/p\mathbf{Z}, n) \to TB_n$ is a weak equivalence, completing the proof of Theorem B.8.

Unfortunately, the simple algebraic argument given in Proposition 4.5 to identify α_n as $1 - \Phi$ in the algebraic case does not have a topological analogue. Here we must use the topology to make this identification. The key observation is that the commutative $H\bar{\mathbf{F}}_p$-algebras B_n are related by “suspension”. We make this precise in the following proposition. For this proposition, note that the definition of B_n makes sense for $n = 0$, although the map $B_0 \to FK(\mathbf{Z}/p\mathbf{Z}, 0)$ may not be a weak equivalence.

Proposition B.10. *For $n > 0$, B_{n-1} is homotopy equivalent as a commutative $H\bar{\mathbf{F}}_p$-algebra to the pushout of the following diagram*

$$\begin{tikzcd} B_n \arrow[r] \arrow[d] & B_n \otimes S^1 \\ H\bar{\mathbf{F}}_p \end{tikzcd}$$

where the map $B_n \to H\bar{\mathbf{F}}_p$ is the augmentation $B_n \to FK(\mathbf{Z}/p\mathbf{Z}, n) \to F* = H\bar{\mathbf{F}}_p$ induced by the inclusion of the basepoint of $K(\mathbf{Z}/p\mathbf{Z}, n)$ and the map $B_n \to B \otimes S^1$ is induced by the inclusion $* \to S^1$.

For an augmented commutative $H\bar{\mathbf{F}}_p$-algebra A, denote the analogous pushout for A as $\Sigma_{\mathscr{C}}A$. If we give $\mathbb{P}S^{-n}_{H\bar{\mathbf{F}}_p}$ the augmentation induced by applying $\mathbb{P}$ to the map $S^n_{H\bar{\mathbf{F}}_p} \to *$, then $\Sigma_{\mathscr{C}}\mathbb{P}S^{-n}_{H\bar{\mathbf{F}}_p}$ is canonically isomorphic to $\mathbb{P}S^{-n+1}_{H\bar{\mathbf{F}}_p}$. This gives us a canonical suspension homomorphism $\sigma\colon \tilde{\pi}_{-n}A \to \tilde{\pi}_{-n+1}\Sigma_{\mathscr{C}}A$, where $\tilde{\pi}_*$ is the kernel of the augmentation map $\pi_*A \to \pi_*H\bar{\mathbf{F}}_p$. The following proposition is closely related to and can be deduced from [17, 3.3].

Proposition B.11. *The suspension homomorphism σ commutes with the operation P^s for all s.*

We can choose the map p_n in the construction of B_n to be augmented for the augmentation described on $\mathbb{P}S^n_{H\bar{\mathbf{F}}_p}$ above. Then it follows from the previous proposition that $\Sigma_{\mathscr{C}} p_n$ is homotopic to p_{n-1}. This observation can be used to prove Proposition B.10.

It follows from Proposition B.10 that TB_{n-1} is the loop space of TB_n. In fact, we see from the discussion above that the fiber sequence for TB_{n-1}

$$TB_{n-1} \to K(\bar{\mathbf{F}}_p, n-1) \to K(\bar{\mathbf{F}}_p, n-1)$$

is the loop of the corresponding fiber sequence for TB_n. In particular, α_n and α_{n-1} are the same endomorphisms of $\bar{\mathbf{F}}_p$. Since P^0 performs the p-th power map on classes in degree zero, α_0 is $1 - \Phi$. We conclude that α_n is $1 - \Phi$.

References

[1] A. K. Bousfield, "The Localization of Spaces with Respect to Homology," *Topology* 14 (1975), pp. 133–150.

[2] A. K. Bousfield, "On the Homology Spectral Sequence of a Cosimplicial Space," *Amer. J. Math.* 109 (1987), pp. 361–394.

[3] A. K. Bousfield, V. K. A. M. Gugenheim, *On PL De Rham Theory and Rational Homotopy Type, Memoirs Amer. Math. Soc.* 179, 1976.

[4] W. G. Dwyer, "Strong Convergence of the Eilenberg-Moore Spectral Sequence," *Topology* 13 (1974), pp. 255–265.

[5] W. G. Dwyer, "Exotic Convergence of the Eilenberg-Moore Spectral Sequence," *Illinois J. Math.* 19 (1975), pp. 607–617.

[6] W. G. Dwyer, M. J. Hopkins, unpublished work, 1992.

[7] W. G. Dwyer, J. Spalinski, "Homotopy Theories and Model Categories," in I. M. James, ed, *Handbook of Algebraic Topology*, Elsevier Science B.V., 1995.

[8] A. D. Elmendorf, I. Kriz, M. A. Mandell, J. P. May, *Rings, Modules, and Algebras in Stable Homotopy Theory, Amer. Math. Soc. Surveys & Monographs,* vol. 47, 1996. Errata `http://www.math.uchicago.edu/~mandell/ekmmerr.dvi`.

[9] P. G. Goerss, "Simplicial Chains over a Field and p-Local Homotopy Theory," *Math. Z.* 220 #4 (1995), pp. 523–544.

[10] V. A. Hinich, V. V. Schechtman, "On Homotopy Limit of Homotopy Algebras," *K-Theory, Arithmetic, and Geometry, Springer Lecture Notes 1289* (1987), pp. 240–264.

[11] I. Kriz, "p-Adic Homotopy Theory," *Top. & Appl.* 52 (1993), pp. 279–308.

[12] I. Kriz, J. P. May, *Operads, Algebras, Modules, and Motives, Asterisque* 233, 1995.

[13] M. A. Mandell, "The Homotopy Theory of E_∞ Algebras," submitted in conjunction with this manuscript.

[14] M. A. Mandell, "Algebraization of E_∞ Ring Spectra," in preparation.

[15] M. A. Mandell, "Natural E_∞ Multiplications on Cochain Complexes," in prepartion.

[16] J. P. May, *Simplicial Objects in Algebraic Topology*, Van Nostrand, 1967.

[17] J. P. May, "A General Algebraic Approach to Steenrod Operations," *Steenrod Algebra and its Applications: A Conference to Celebrate N. E. Steenrod's Sixtieth Birthday, Springer Lecture Notes* 168 (1970), pp. 153–231.

[18] J. E. McClure, "The mod p K-Theory of QX," *H_∞ Ring Spectra and their Applications, Springer Lecture Notes* 1176 (1986), pp. 291–377.

[19] D. G. Quillen, *Homotopical Algebra, Springer Lecture Notes* 43, 1967.

[20] V. A. Smirnov, "Homotopy Theory of Coalgebras," *Izv. Akad. Nauk SSSR Ser. Mat.* 49 (1985), pp. 1302–1321, trans. in *Math. USSR–Izv.* 27 (1986), pp. 575–592.

[21] L. Smith, "Homological Algebra and the Eilenberg-Moore Spectral Sequence," *Trans. Amer. Math. Soc.* 129 (1967), pp. 58–93.

[22] D. Sullivan, "Infinitesimal Computations in Topology," *Publ. Math. I. H. E. S.* 47 (1978), pp. 269–331.

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