A NOTE ON THE SPLITTING PRINCIPLE

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ABSTRACT. We offer a new perspective on the splitting principle. We give an easy proof that applies to all classical types of vector bundles and in fact to G-bundles for any compact connected Lie group G. The perspective gives precise calculational information and directly ties the splitting principle to the specification of characteristic classes in terms of classifying spaces.

In the algebraic topology proseminar at Chicago, a student, Nils Barth, asked for the precise relationship between the splitting principle and the specification of Chern classes in terms of maximal tori. This note gives the quick and more general answer that popped to mind. It should be utterly standard, but it was new to me and to other faculty in the audience. Certainly I have not seen it in print.

Let $T = T^n$ be a maximal torus in a compact connected Lie group G of rank n and let R be a commutative ring in which p is invertible for all primes p such that $H_*(G;\mathbb{Z})$ has p-torsion. Classical results of Borel [3] determine these primes explicitly when G is simply connected and describe how to determine them in terms of the elementary abelian p-subgroups of G in general. Other classical results of Borel [1] imply that $H^*(BG;R)$ is a polynomial ring over R on n even degree generators. The information relevant here is just that $H^*(BG;R)$ is concentrated in even degrees.

By the Bott-Samelson theorem [4], $H^*(G/T;\mathbb{Z})$ has no torsion and is also concentrated in even degrees, hence $H^*(G/T;R)$ is a free R-module. Let EG be a universal principal G-bundle and take $BT = EG/T$ and $BG = EG/G$. Inclusion of orbits gives a G-bundle $p\colon BT \longrightarrow BG$ with fiber G/T. Taking cohomology with coefficients in R henceforward, we see immediately that the Serre spectral sequence of this bundle collapses to give

$$H^*(BT) \cong H^*(BG) \otimes H^*(G/T) \tag{1}$$

as an $H^*(BG)$-module via p^*. In particular, $H^*(BT)$ is a free $H^*(BG)$-module.

Now let ξ be a G-bundle over a space X. For convenience, we assume that X is path connected. Let ξ have classifying map $f\colon X \longrightarrow BG$. Of course, we can think of ξ as a principal G-bundle or as a G-bundle with fiber F for any G-space F. Construct the pullback diagram

$$\begin{array}{ccc} Y & \xrightarrow{\ g\ } & BT \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[\ f\ ]{} & BG. \end{array} \tag{2}$$

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Then $q\colon Y \longrightarrow X$ is a G-bundle with fiber G/T. The action of $\pi_1(X)$ on $H^*(G/T)$ is trivial, since it is the pullback of the action of $\pi_1(BG) = 0$, and the elements of $H^*(G/T)$ are permanent cycles in the Serre spectral sequence of q because they are permanent cycles in the Serre spectral sequence of p. Therefore the Serre spectral sequence of q collapses to give an isomorphism

$$H^*(Y) \cong H^*(X) \otimes H^*(G/T). \tag{3}$$

The edge homomorphism shows that

$$q^*\colon H^*(X) \longrightarrow H^*(Y) \cong H^*(X) \otimes H^*(G/T)$$

is the canonical inclusion, $x \to x \otimes 1$. Some readers might prefer to use the Eilenberg-Moore spectral sequence to obtain these conclusions.

The map p is the universal example for the reduction of the structural group of a G-bundle from G to T. The G-bundle $q^*\xi$ over Y is classified by $f \circ q = p \circ g$ and therefore has a canonical reduction. *We view this reduction of the structural group of $q^*\xi$ as a generalized splitting principle.*

Theorem 1 (Generalized splitting principle)**.** *For a G-bundle ξ over X, there is a G-bundle $q\colon Y \longrightarrow X$ with fiber G/T and a reduction of the structural group of $q^*\xi$ to T such that $H^*(Y) \cong H^*(X) \otimes H^*(G/T)$ and q^* is the canonical inclusion.*

Reinterpreting the diagram (2), the map g classifies a T-bundle ζ that is the fiberwise product of n-circle bundles ζ_i with classifying maps the coordinates g_i of $g\colon Y \longrightarrow BT \cong (BT^1)^n$. The equality $f \circ q = p \circ g$ says that $q^*\xi$ is the G-bundle obtained by extending the structure group of ζ from T to G. If we know p^* on characteristic classes, then we can read off the characteristic classes of $q^*\xi$ from those of the circle bundles ζ_i. That is, for an element α in $H^*(BG)$, thought of as a characteristic class,

$$\alpha(q^*\xi) = q^*\alpha(\xi) = q^*f^*(\alpha) = (g_1, \cdots, g_n)^*p^*(\alpha). \tag{4}$$

Provided that $\pi_1(G)$ is a free Abelian group, similar arguments work in K-theory, using its Serre or Eilenberg-Moore spectral sequence (see e.g. [5, 6]). In particular, with this restriction on $\pi_1(G)$, q^* is a monomorphism in K-theory.

We describe the classical examples, referring the reader to [2] for background. The first three apply to any commutative ring R and also work with evident modifications in K-theory. Write $H^*(BT)$ as the polynomial algebra in the n canonical generators x_i. For example, viewing $BS^1 = \mathbb{CP}^\infty$ as $K(\mathbb{Z}, 2)$, the x_i are the fundamental classes. Let σ_i denote the ith elementary symmetric function in n variables.

Example 2. Take G to be $U(n)$, $T = T^n$ to be the subgroup of diagonal matrices, and the fiber F to be $\mathbb{C}^n$. The universal Chern classes are characterized as the unique elements c_i of $H^*(BU(n))$ such that

$$p^*(c_i) = \sigma_i(x_1, \cdots, x_n).$$

We have a splitting of $q^*\xi$ into the sum of n complex line bundles ζ_i, and

$$c_i(q^*\xi) = \sigma_i(c_1(\zeta_1), \ldots, c_1(\zeta_n)).$$

Example 3. Take G to be $SU(n) \subset U(n)$, $T = T^{n-1} \subset T^n$ to be the subgroup of diagonal matrices of determinant 1, and F to be $\mathbb{C}^n$. With the left square an instance of (2), we have the evident commutative diagram

$$(5) \qquad \begin{array}{ccccc} Y & \xrightarrow{g} & BT^{n-1} & \xrightarrow{i} & BT^n \\ \Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & BSU(n) & \xrightarrow[j]{} & BU(n). \end{array}$$

Here $j^*(c_1) = 0$ and $j^*(c_i) = c_i$ in $H^*(BSU(n))$ for $i > 1$. Via i^*, we can identify $H^*(BT^{n-1})$ as $P[x_1, \cdots, x_n]/(\sigma_1(x_1, \ldots, x_n))$. The interpretation is that an n-plane $U(n)$-bundle ξ with a reduction of its structural group to $SU(n)$ splits along q as the sum of n line bundles whose tensor product is the trivial line bundle.

Example 4. Take G to be $Sp(n)$, T to be the subgroup of diagonal matrices with complex entries, and F to be $\mathbb{H}^n$. Observe that $T \subset U(n) \subset Sp(n)$, where the second inclusion is given by extension of scalars from $\mathbb{C}$ to $\mathbb{H}$. One way to define the symplectic characteristic classes k_i is by

$$j^*(k_i) = \sum_{a+b=2i} (-1)^{a+i} c_a c_b,$$

where $j\colon BU(n) \longrightarrow BSp(n)$ is the induced map of classifying spaces, and this is equivalent to $p^*(k_i) = \sigma_i^2$ in $H^*(BT)$. Here, in the diagram (2), p factors through j. The interpretation of the diagram is that if ξ is a quaternionic n-plane bundle, then $q^*\xi$ splits as the sum of n quaternionic line bundles that are obtained by extension of scalars from n complex line bundles ζ_i. Moreover,

$$k_i(q^*\xi) = \sigma_i^2(c_1(\zeta_1), \cdots, c_1(\zeta_n)).$$

In the following example, we require 2 to be invertible in R.

Example 5. Take G to be $SO(2n + \varepsilon)$ where $n \geq 1$ and $\varepsilon = 0$ or $\varepsilon = 1$. Take $T = T^n \cong SO(2)^n$ embedded in G as n (2×2)–matrices along the diagonal, with a last diagonal entry 1 if $\varepsilon = 1$. Then $H^*(BSO(2n))$ is the polynomial algebra on the Pontryagin classes P_i, $1 \leq i < n$ and the Euler class χ, where $\chi^2 = P_n$, and $H^*(BSO(2n + 1))$ is the polynomial algebra on the P_i, $1 \leq i \leq n$. Observe that $T \subset U(n) \subset SO(n)$, where the second inclusion is given by identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, as usual. One way to define P_i and χ is by

$$j^*(P_i) = \sum_{a+b=2i} (-1)^{a+i} c_a c_b \quad \text{and} \quad j^*(\chi) = c_n,$$

where $j\colon BU(n) \longrightarrow BSO(2n)$ is the induced map of classifying spaces, and this is equivalent to $p^*(P_i) = \sigma_i^2$ and $p^*(\chi) = \sigma_n$ in $H^*(BT)$. Here, in the diagram (2), p factors through j. The interpretation of the diagram is that if ξ is an oriented real $(2n + \varepsilon)$-plane bundle, then $q^*\xi$ splits as the sum of the realifications of n complex line bundles ζ_i and, if $\varepsilon = 1$, a trivial real line bundle. Moreover,

$$P_i(q^*\xi) = \sigma_i^2(c_1(\zeta_1), \cdots, c_1(\zeta_n))$$

and, if $\varepsilon = 0$,

$$\chi(q^*\xi) = \sigma(c_1(\zeta_1), \cdots, c_1(\zeta_n)).$$

We could easily go on to consider the exceptional Lie groups or to consider generalizations to H-spaces and in particular to p-compact Lie groups, with their maximal tori. We leave such examples and elaborations to the interested reader.

Using maximal 2–tori and mod 2 cohomology, we can often obtain an analogous splitting principle in terms of real line bundles. We illustrate in the case of the orthogonal groups, where we have the following analogues of Examples 2 and 3. We now take cohomology with coefficients in the field $\mathbb{F}_2$.

Example 6. Let $O(1)^n \subset O(n)$ be the subgroup of diagonal matrices. Write $H^*(BO(1))$ as the polynomial algebra in the n canonical basis elements x_i. Let $p\colon BO(1)^n \longrightarrow BO(n)$ be the evident bundle with fiber $O(n)/O(1)^n$. The Stiefel-Whitney classes are the unique elements w_i of $H^*(BO(n))$ such that

$$p^*(w_i) = \sigma_i(x_1, \cdots, x_n).$$

The Serre spectral sequence of p has trivial local coefficients and collapses at E_2. Let ξ be a real n-plane bundle over X with classifying map $f\colon X \longrightarrow BO(n)$ and form the pullback

$$\begin{array}{ccc} Y & \xrightarrow{g} & BO(1)^n \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & BO(n). \end{array}$$

The Serre spectral sequence of q collapses at E_2 to give an isomorphism

$$H^*(Y) \cong H^*(X) \otimes H^*(O(n)/O(1)^n),$$

and $q^*\colon H^*(X) \longrightarrow H^*(Y)$ is the canonical inclusion, $x \to x \otimes 1$. We have a splitting of $q^*\xi$ into the sum of n real line bundles ζ_i, and

$$w_i(q^*\xi) = \sigma_i(w_1(\zeta_1), \ldots, w_1(\zeta_n)).$$

Example 7. Let $O(1)^{n-1} \subset O(1)^n$ be the subgroup of $SO(n) \subset O(n)$ consisting of the diagonal matrices of determinant 1. Let ξ be an oriented real n-plane bundle with classifying map $f\colon X \longrightarrow BSO(n)$ and define Y to be the pullback in the left square of the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{g} & BO(1)^{n-1} & \xrightarrow{i} & BO(1)^n \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & BSO(n) & \xrightarrow[j]{} & BO(n). \end{array}$$

Again, we have $H^*(Y) \cong H^*(X) \otimes H^*(SO(n)/O(1)^{n-1})$ with q^* the canonical inclusion. Here $j^*(w_1) = 0$ and $j^*(w_i) = w_i$ in $H^*(BSO(n))$ for $i > 1$. Via i^*, we can identify $H^*(BO(1)^{n-1})$ as $P[x_1, \cdots, x_n]/(\sigma_1)$. The interpretation is that an n-plane $O(n)$-bundle ξ with a reduction of its structural group to $SO(n)$ splits along q as the sum of n line bundles whose tensor product is the trivial line bundle.

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