DERIVED CATEGORIES IN ALGEBRA AND TOPOLOGY

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I will give a philosophical overview of some joint work with Igor Kriz (in algebra), with Tony Elmendorf and Kriz (in topology), and with John Greenlees (in equivariant topology). I will begin with a description of some foundational issues before saying anything about the applications. This is not the best way to motivate people, but I must explain the issues involved in order to describe what we have done. Let me just say that the emphasis I shall give to an analogy between algebra and topology is not just an expository device. The algebraic work that I will describe both illuminates the deeper topological theory and has applications to algebraic geometry.

We begin by displaying an analogy that is familiar to topologists. It is the starting point of our work.

ALGEBRA	TOPOLOGY
a commutative ring k	the sphere spectrum S
differential graded k-modules	spectra
tensor product	smash product
internal hom $\mathrm{Hom}(X,Y)$	function spectrum $F(X,Y)$
dual $DX = \mathrm{Hom}(X,k)$	dual $DX = F(X,S)$
projective k-module	CW spectrum
finitely generated projective	finite CW spectrum
$\mathrm{Hom}(X,Y)\otimes E \cong \mathrm{Hom}(X,Y\otimes E)$ $DX\otimes E \cong \mathrm{Hom}(X,E)$ (X or E fin. gen. projective)	$F(X,Y)\wedge E \simeq F(X,Y\wedge E)$ $DX\wedge E \simeq F(X,E)$ (X or E a finite CW spectrum)
hyperhomology	$E_q(X) \equiv \pi_q(X\wedge E)$
hypercohomology	$E^q(X) \equiv \pi_{-q}F(X,E)$

Note that the right column already encodes the important topological theory of Spanier-Whitehead duality: if X is a finite CW spectrum, then

$$E_*(DX) \cong E^{-*}(X).$$

Provisionally, we regard the columns as providing ground categories in which to study homotopical algebra. The usual starting point in algebra is a commutative differential graded k-algebra, or DGA, A. The usual starting point in topology is a commutative ring spectrum R. This is an algebraic structure defined not in the ground category

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of spectra but rather in its "derived category", which is called the stable homotopy category of spectra and denoted by $\bar{h}\mathscr{S}$. A map of spectra is called a weak equivalence if it induces an isomorphism on homotopy groups, and $\bar{h}\mathscr{S}$ is constructed from the homotopy category of spectra by formally inverting the weak equivalences. Thus a ring spectrum is a spectrum R together with a product $\phi : R \wedge R \longrightarrow R$ and unit $\eta : S \longrightarrow R$ such that the following diagrams commute in $\bar{h}\mathscr{S}$:

$$\begin{tikzcd} S \wedge R \arrow[r, "\eta\wedge 1"] \arrow[dr, "\simeq"'] & R \wedge R \arrow[d, "\phi"] & R \wedge S \arrow[l, "1\wedge\eta"'] \arrow[dl, "\simeq"] \\ & R & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R \wedge R \wedge R \arrow[r, "1\wedge\phi"] \arrow[d, "\phi\wedge 1"'] & R \wedge R \arrow[d, "\phi"] \\ R \wedge R \arrow[r, "\phi"] & R \end{tikzcd}$$

The unlabelled equivalences are canonical isomorphisms in $\bar{h}\mathscr{S}$ that give the unital property, and we have suppressed such an associativity isomorphism in the second diagram. Intuitively, these diagrams commute only up to homotopy. Similarly, there is a transposition isomorphism $\tau : E \wedge F \longrightarrow F \wedge E$ in $\bar{h}\mathscr{S}$, and R is commutative if the following diagram commutes in $\bar{h}\mathscr{S}$:

$$\begin{tikzcd} R \wedge R \arrow[rr, "\tau"] \arrow[dr, "\phi"'] & & R \wedge R \arrow[dl, "\phi"] \\ & R & \end{tikzcd}$$

An R-module is a spectrum M together with a map $\mu : R \wedge M \longrightarrow M$ such that the following diagrams commute in $\bar{h}\mathscr{S}$:

$$\begin{tikzcd} S \wedge M \arrow[r, "\eta\wedge 1"] \arrow[dr, "\simeq"'] & R \wedge M \arrow[d, "\mu"] \\ & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R \wedge R \wedge M \arrow[r, "1\wedge\mu"] \arrow[d, "\phi\wedge 1"'] & R \wedge M \arrow[d, "\mu"] \\ R \wedge M \arrow[r, "\mu"] & M \end{tikzcd}$$

The analog in algebra is a (differential graded) module M over a DGA A. A map of A-modules is a quasi-isomorphism if it induces an isomorphism on homology. The derived category $\mathscr{D}_A = \bar{h}\mathscr{M}_A$ of A-modules is constructed from the homotopy category of A-modules by formally inverting the quasi-isomorphisms.

The categories $\mathscr{M}_A$ of A-modules and $\mathscr{S}$ of spectra admit suspension functors and cofiber, or mapping cone, constructions that lead to long exact sequences of homological or homotopical invariants. For a map $f \colon M \longrightarrow N$, the sequence

$$M \xrightarrow{f} N \longrightarrow Cf \longrightarrow \Sigma M$$

is called an "exact triangle" and leads to a "triangulation" of the derived category. Analogously, in topology, we would like to have a triangulated category of R-modules for a ring spectrum R. However, this fails hopelessly with the definitions just given. The cofiber of a map $f \colon M \longrightarrow N$ of R-modules need not be an R-module. We can find a map $R \wedge Cf \longrightarrow Cf$ such that the following diagram commutes in $\bar{h}\mathscr{S}$, but the map depends on a choice of homotopy making the left square commute, and the associativity diagram that we required of R-modules generally fails to commute for Cf.

$$\begin{tikzcd} R \wedge M \arrow[r] \arrow[d] & R \wedge N \arrow[r] \arrow[d] & R \wedge Cf \arrow[r] \arrow[d] & R \wedge \Sigma M \arrow[d] \\ M \arrow[r] & N \arrow[r] & Cf \arrow[r] & \Sigma M \end{tikzcd}$$

More deeply, when R is commutative, we would like to construct a smash product $M \wedge_R N$ analogous to the tensor product $M \otimes_A N$ of A-modules in algebra. It is far from clear how to begin. The algebraic constructions are easy because of the good properties of the concrete underlying category of k-modules. Specifically, $\mathscr{M}_k$ is symmetric monoidal in the sense that its tensor product is associative, commutative, and unital up to coherent natural isomorphism. The smash product in the category $\mathscr{S}$ of spectra is not associative, commutative, or unital. It only becomes so on passage to the derived category $\bar{h}\mathscr{S}$, which is symmetric monoidal. It is this limitation on the smash product that forced the homotopical definitions of ring and module spectra that we gave above.

There is a significant difference in paradigm: algebraic topologists are entirely comfortable working with fuzzy objects in the stable homotopy category, with no point-set level models in mind. Algebraic geometers work with more concrete objects, and they wouldn't dream of taking a "ring in the derived category" seriously, as topologists routinely make use of ring spectra.

The theory that I will describe gives both a new point-set level topological theory of rings and modules and a new algebraic theory of algebras and modules up to homotopy. These allow a far more precise analogy than the one displayed above. The new topological theory allows the wholesale importation of techniques of commutative algebra into stable homotopy theory. Applications include:

• A homotopical replacement for the Baas-Sullivan theory of manifolds with singularities as a tool for the construction of new spectra from cobordism spectra.

• New generalized universal coefficient and Künneth spectral sequences.

• New constructions of topological Hochschild homology and topological cyclic homology.

• The construction of equivariant versions of such module spectra over the complex cobordism spectrum MU as the Brown-Peterson and Morava K-theory spectra.

• A completion theorem analogous to the Atiyah-Segal completion theorem in K-theory that applies to module spectra over MU.

The new algebraic theory leads to the construction of a sensible site in which to define "integral mixed Tate motives" in algebraic geometry, realizing a program that was proposed by Deligne.

These applications are described in our announcements [1] and [2]. However, those notes say nothing about the actual constructions, and my purpose here is to give an intuitive introduction to the foundations that lead to these applications.

I shall sketch some topological definitions to give substance to the discussion. This is a distillation of an introduction to the stable homotopy category and is intended to give some feeling for the issues involved. Recall that the smash product $X \wedge Y$ of based spaces X and Y is the quotient $X \times Y/X \vee Y$; we write $F(X, Y)$ for the function space of based maps $X \longrightarrow Y$.

A "universe" U is a countably infinite dimensional real inner product space. It suffices to think of $U = \mathbb{R}^\infty$. If V and W are finite dimensional sub inner product spaces of U and $V \subset W$, we let $W - V$ denote the orthogonal complement of V in W. For a based space X, we let $\Sigma^V X = X \wedge S^V$ and $\Omega^V X = F(S^V, X)$, where S^V is the one-point compactification of V.

A "prespectrum" T is a collection of based spaces TV and based maps

$$\sigma\colon \Sigma^{W-V} TV \longrightarrow TW$$

that satisfy an evident transitivity condition. We write

$$\tilde{\sigma}\colon TV \longrightarrow \Omega^{W-V} TW$$

for the adjoint of σ. A prespectrum is a "spectrum" if each map $\tilde{\sigma}$ is a homeomorphism. (We generally write E for a spectrum and T for a prespectrum.) A map of prespectra is a collection of maps $f\colon TV \longrightarrow T'V$ that are strictly compatible with the structure maps σ; a map $E \longrightarrow E'$ of spectra is a weak equivalence if each $f\colon EV \longrightarrow E'V$ is a weak equivalence of spaces.

Let $\mathscr{P}U$ and $\mathscr{S}U$ denote the categories of prespectra and of spectra indexed on U. There is a "spectrification" functor $L\colon \mathscr{P}U \longrightarrow \mathscr{S}U$ that is left adjoint to the forgetful functor $\ell\colon \mathscr{S}U \longrightarrow \mathscr{P}U$. This is analogous to sheafification from presheaves to sheaves. Constructions made on prespectra are transported to spectra via (L, ℓ). For example, the smash product of a prespectrum T and a based space X is given by $(T \wedge X)(V) = (TV) \wedge X$. The smash product of a spectrum E and a based space X is then $E \wedge X = L(\ell E \wedge X)$. Typically, this procedure is necessary for functors that are left adjoints, whereas functors that are right adjoints preserve spectra. For example, $F(X, T)(V) = F(X, TV)$ gives the function prespectrum of a based space and a prespectrum; if T is a spectrum, then so is $F(X, T)$.

Since we have based cylinders $E \wedge I_+$, where the plus denotes adjunction of a disjoint basepoint, we have the notion of a homotopy between maps of spectra. There results a homotopy category $h\mathscr{S}U$ of spectra indexed on U, and we obtain $\bar{h}\mathscr{S}U$ by adjoining inverses to the weak equivalences; we abbreviate $\bar{h}\mathscr{S}U$ to $\bar{h}\mathscr{S}$ when U is understood. The suspension and loop functors, $\Sigma E = E \wedge S^1$ and $\Omega E = F(S^1, E)$, become inverse equivalences of categories on $\bar{h}\mathscr{S}$. It is in that sense that $\bar{h}\mathscr{S}$ is a "stable category".

There is a functor Σ^∞ from based spaces to spectra specified by $\Sigma^\infty X = \{Q\Sigma^V X\}$, where $QY = \cup \Omega^V \Sigma^V Y$. It is left adjoint to the zeroth space functor $\Omega^\infty E = E(0)$. We think of $\Sigma^\infty X$ as the stabilization of the space X. Spaces of the form $E(0)$ are called infinite loop spaces.

There is a theory of CW-spectra that is analogous to the theory of CW complexes. The only twist is that we have negative dimensional spheres and our cells must be allowed to take positive and negative dimensions. The stable category $\bar{h}\mathscr{S}$ is equivalent to the homotopy category of CW spectra and cellular maps.

We now come to the crux of the matter: the construction of smash products of prespectra and spectra. The obvious definitions would seem to be

$$(T \wedge T')(V \oplus V') = TV \wedge T'V' \text{ and } E \wedge E' = L(\ell E \wedge \ell E').$$

This makes sense and works, provided that it is interpreted in the right way. In fact, this constructs $\wedge$ as a functor $\mathscr{S}U \times \mathscr{S}U' \longrightarrow \mathscr{S}(U \oplus U')$ for a pair of universes U and U'. Similarly, we can define explicit function spectra $F(E', E'')$, where F is a functor $\mathscr{S}U' \times \mathscr{S}U'' \longrightarrow \mathscr{S}U$. The changes of universe are essential. We refer to this operation as an "external" smash product; it is associative and commutative.

To internalize, we choose a linear isometry $f : U \oplus U \longrightarrow U$ and construct a functor $f_* : \mathscr{S}(U \oplus U) \longrightarrow \mathscr{S}U$. We then define an internal smash product by $\wedge = f_* \circ \wedge$. Two choices of f give equivalent functors on passage to stable categories, and this independence of the choice leads to the proof that the stable category level smash product is associative and commutative. It is this internal smash product that was relevant to the product $R \wedge R \longrightarrow R$ in our original definition of a ring spectrum.

We can collect all choices of f into a single parameter space for smash products and so eliminate the apparent dependence on f. More generally, we can construct such parameter spaces for j-fold smash products. Thus let $\mathscr{L}(j) = \mathscr{I}(U^j, U)$ be the space of linear isometries $U^j \to U$. This is a contractible space with a free action of the symmetric group Σ_j. We have a system of maps

$$\mathscr{L}(k) \times \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k) \longrightarrow \mathscr{L}(j_1 + \cdots + j_k),$$

$$(g, f_1, \ldots, f_k) \longrightarrow g \circ (f_1 \oplus \cdots \oplus f_k).$$

These maps are suitably associative, unital, and equivariant. These laws are codified in the general notion of an operad, and operads whose jth space is Σ_j-free and contractible for each j are called E_∞ operads.

We can construct "twisted half-smash product" functors $\mathscr{L}(j) \ltimes E^{(j)}$, where $E^{(j)}$ denotes the j-fold external smash power of E. These are functors $\mathscr{S}U \longrightarrow \mathscr{S}U$. They are the spectrum level analogs of the evident functors

$$\mathscr{L}(j) \ltimes X^{(j)} \equiv \mathscr{L}(j)_+ \wedge X^{(j)}$$

on based spaces X, and we have

$$\Sigma^\infty(\mathscr{L}(j) \ltimes X^{(j)}) \cong \mathscr{L}(j) \ltimes (\Sigma^\infty X)^{(j)}.$$

We can now give the fundamental definitions [4, 3]: an E_∞ ring spectrum is a spectrum R together with maps

$$\theta_j : \mathscr{L}(j) \ltimes R^{(j)} \longrightarrow R$$

that are suitably associative, unital, and equivariant *on the point set level*. This is as close to a commutative and associative ring spectrum as one can hope to get. A module (or E_∞ module) over R is a spectrum M together with maps

$$\lambda_j : \mathscr{L}(j) \ltimes (R^{(j-1)} \wedge M) \longrightarrow M$$

that, with the θ_j, are suitably associative, unital, and equivariant.

Most of the important cohomology theories in algebraic topology are represented by E_∞ ring spectra. Examples include the sphere spectrum S; the Eilenberg-MacLane spectra HA for discrete commutative rings A; the Thom spectra MO, MU, and MSp; the connective K-theory spectra kO and kU; and the algebraic K-theory spectra KA of discrete commutative rings. Many other examples are constructed via multiplicative infinite loop space theory [5]; that theory allows one to construct E_∞ ring spectra from E_∞ ring spaces, which in turn arise from suitable categories with $\oplus$ and $\otimes$. Recent work of Hopkins, Miller, McClure, Kriz, Elmendorf, Vogt, Schwänzl, and others has given still more examples. A very rich theory of E_∞ rings, including "cell theory" (Hopkins) and "Postnikov systems" (Kriz), is now emerging. Much of it depends on the theory I am about to describe.

The definitions just given are the right ones, but they are rather hard to work with. Our recent breakthrough recasts these notions in a far more conceptual and workable form. To explain the idea, we return to our analogy and consider its algebraic side.

Operads of (differential graded) k-modules are defined by replacing Cartesian products of spaces with tensor products of k-modules. They can be obtained, for example, by applying the normalized singular k-chain functor to an operad of spaces. An operad $\mathscr{C}$ of k-modules is an E_∞ operad if $\mathscr{C}(j)$ is a free $k[\Sigma_j]$-resolution of k for each j; the chain operad of an E_∞ operad of spaces is an example. To fix ideas, we agree to let $\mathscr{C}$ denote the chain operad so obtained from our topological E_∞ operad $\mathscr{L}$.

We define an E_∞ k-algebra to be a k-module A together with maps

$$\theta_j : \mathscr{C}(j) \otimes A^{(j)} \longrightarrow A,$$

where $A^{(j)}$ denotes the j-fold tensor power of A; these maps must satisfy associativity, unity, and equivariance relations exactly like those in the definition of E_∞ ring spectra. Modules over such algebras are defined similarly in terms of maps

$$\lambda_j : \mathscr{C}(j) \otimes (A^{(j-1)} \otimes M) \longrightarrow M.$$

These definitions are forced by examples from algebraic geometry. Deligne, seeking foundations for an integral theory of mixed Tate motives, asked me if Bloch's Chow complex of an algebraic variety, which is a simplicial abelian group with a partially defined product, might give rise to a quasi-isomorphic E_∞ algebra, and, if so, if there might then be a good derived category of modules over an E_∞ algebra. Kriz and I gave positive answers to these questions. Unless k is a field of characteristic zero, one cannot hope to replace the resulting E_∞ algebras by quasi-isomorphic genuine DGA's. The present level of generality is essential. As an aside, the topological theory applied to the algebraic K-theory spectra of fields gives an alternative site for a possible theory of mixed Tate motives.

As another digression, operads and their actions are now playing a serious role in differential geometry and mathematical string theory. Here E_n operads, related to n-fold loop spaces, play a fundamental role. Different types of discrete operads define different types of algebras, such as Lie algebras, and Lie algebras up to homotopy characterized by actions by appropropriate Lie-like chain operads are also playing an important role. Recall that a module over a Lie algebra is the same thing as a module over its universal enveloping algebra, which is an associative algebra. Precisely mimicking the proof, one can show that, for any operad $\mathscr{C}$ and $\mathscr{C}$-algebra A, there is an associative universal enveloping DGA $U(A)$ such that an A-module is the same thing as a $U(A)$-module.

Now return to our particular operad $\mathscr{C}$. The ground ring k is a $\mathscr{C}$-algebra via augmentations, and its universal enveloping algebra turns out to be $U(k) = \mathscr{C}(1)$. That is, as one can easily check directly from the formal definitions, if we regard k as an E_∞ k-algebra, we find that an E_∞ k-module is the same thing as a $\mathscr{C}(1)$-module. Analogously, in topology, S is an E_∞ ring spectrum, and an E_∞ S-module is the same thing as a spectrum with an "action of the monoid $\mathscr{L}(1)$" defined in terms of an associative and unital action map $\mathscr{L}(1) \ltimes M \longrightarrow M$.

A fundamental idea at this point is to switch ground categories from k-modules and spectra to E_∞ k-modules and S-modules. Here a miracle occurs. We define a new tensor product of E_∞ k-modules in algebra or smash product of E_∞ S-modules in topology. In algebra, the definition is:

$$M \boxtimes N \equiv \mathscr{C}(2) \otimes_{\mathscr{C}(1)\otimes\mathscr{C}(1)} M \otimes N.$$

In more detail, instances of the operad structure maps

$$\mathscr{C}(k) \times \mathscr{C}(j_1) \times \cdots \times \mathscr{C}(j_k) \longrightarrow \mathscr{C}(j_1 + \cdots + j_k)$$

give a left action of $\mathscr{C}(1)$ and a right action of $\mathscr{C}(1)\otimes\mathscr{C}(1)$ on $\mathscr{C}(2)$. The latter action is used to make sense of the displayed tensor product, and the former action gives the new tensor product a structure of $\mathscr{C}(1)$-module. This is already remarkable: the algebra $\mathscr{C}(1)$ is not commutative, so it is rather surprising to have an internal tensor product on its modules. This much would be true for any operad. The real miracle is that, with our particular choice of operad $\mathscr{C}$, this tensor product turns out to be associative and commutative, with a natural unit equivalence $\lambda\colon k \boxtimes M \longrightarrow M$.

We can now define an E_∞ k-algebra A to be a $\mathscr{C}(1)$-module together with a product $\phi : A \boxtimes A \longrightarrow A$ and unit $\eta : k \longrightarrow A$ such that the evident associativity, commutativity, and unit diagrams (like those displayed at the start) are commutative. Similarly, we define an A-module M to be a $\mathscr{C}(1)$-module together with an action $\mu : A \boxtimes M \longrightarrow M$ such that the evident associativity and unit diagrams commute. Moreover, we can define the tensor product of A-modules M and N with actions μ and ν to be the coequalizer (or difference cokernel) displayed in the diagram

$$\begin{tikzcd} M \boxtimes A \boxtimes N \arrow[r, shift left, "\mu\boxtimes\mathrm{Id}"] \arrow[r, shift right, "\mathrm{Id}\boxtimes\nu"'] & M \boxtimes N \arrow[r] & M \boxtimes_A N. \end{tikzcd}$$

There is a concomitant internal hom functor $Hom_A(M,N)$; it is defined as an appropriate equalizer.

Actually, one can be more categorically precise about this. For $\mathscr{C}(1)$-modules M with a given "unit map" $\eta\colon k \longrightarrow M$, we can define a variant, $\boxdot$ say, of the product $\boxtimes$ which is not only associative and commutative, but also unital with unit k. The modified product is defined by the pushout diagram

$$\begin{tikzcd} k \boxtimes N \cup_{k\boxtimes k} M \boxtimes k \arrow[r, "\lambda\cup\lambda"] \arrow[d, "\eta\boxtimes\mathrm{Id} + \mathrm{Id}\boxtimes\eta"'] & M \cup_k N \arrow[d] \\ M \boxtimes N \arrow[r] & M \boxdot N. \end{tikzcd}$$

An E_∞ k-algebra is precisely the same thing as a commutative monoid in the symmetric monoidal category of unital $\mathscr{C}(1)$-modules. There is a similar way to be more precise about the notion of an A-module.

The topological theory works just the same way. For (E_∞) S-modules M and N, we can make sense of the definition

$$M \wedge_S N \equiv \mathscr{L}(2) \ltimes_{\mathscr{L}(1)\times\mathscr{L}(1)} M \wedge N.$$

This is again an S-module, and this smash product over S is an associative and commutative operation with a natural unit equivalence $\lambda\colon S \wedge_S M \longrightarrow M$. We redefine an E_∞ ring spectrum to be an S-module with a product $\phi : R \wedge_S R \longrightarrow R$ and unit $\eta : S \longrightarrow R$ such that, with $\wedge$ replaced by $\wedge_S$, the diagrams that we gave at the start commute. The point is that the commutation now makes sense and is required on the point set level, that is, in the category of S-modules. We define R-modules similarly in terms of action maps $\mu : R \wedge_S M \longrightarrow M$, and we define the smash product over R of R-modules M and N to be the coequalizer

$$\begin{tikzcd} M \wedge_S R \wedge_S N \arrow[r, shift left, "\mu\wedge\mathrm{Id}"] \arrow[r, shift right, "\mathrm{Id}\wedge\nu"'] & M \wedge_S N \arrow[r] & M \wedge_R N. \end{tikzcd}$$

There is a concomitant right adjoint function R-module $F_R(M,N)$.

Again, there is a variant of the smash product over S, $\star_S$ say, that is defined on S-modules M with unit maps $\eta\colon S \longrightarrow M$ and that is associative, commutative, and unital. It is defined by a pushout diagram just like that defining $\boxdot$. An E_∞ ring spectrum is precisely a commutative monoid in the symmetric monoidal category of unital S-module spectra. There is a similar way to be more precise about R-modules. Here another miracle occurs: these simple conceptual definitions turn out to be equivalent to the pre-existing definitions in terms of actions by the linear isometries operad $\mathscr{L}$. This allows use of the older theory to supply examples, which can then be studied algebraically by means of our new theory.

In particular, we can mimic the theory of cell spectra to develop a theory of cell R-modules. A map of R-module spectra is said to be a weak equivalence if it is a weak equivalence as a map of spectra. The derived stable homotopy category of R-modules, $\bar{h}\mathscr{M}_R$, is constructed from the homotopy category of R-modules by formally inverting the weak equivalences, and it is equivalent to the homotopy category of cell R-modules. It is a triangulated category, and it is symmetric monoidal under the derived smash product of R-modules. It provides the starting point for the various applications that we listed at the start.

We think of the sphere spectrum S as a universal ground ring. For any E_∞ ring spectrum R, R-modules are S-modules by neglect of structure. The stable homotopy category $\bar{h}\mathscr{M}_S$ provides an improved substitute for the stable homotopy category $\bar{h}\mathscr{S}$ that we started with.

Theorem 1. The forgetful functor $\mathscr{M}_S \longrightarrow \mathscr{S}$ induces an equivalence of categories $\bar{h}\mathscr{M}_S \longrightarrow \bar{h}\mathscr{S}$. For S-modules M and N, there are natural isomorphisms in $\bar{h}\mathscr{S}$

$$M \wedge N \simeq M \wedge_S N \text{ and } F(M,N) \simeq F_S(M,N).$$

The topology now feeds back into algebra in a most amusing fashion. The standard treatment of tensor products and internal hom functors in the derived category of differential graded modules M over a DGA A entails the use of suitable projective resolutions of such modules. These are awkward to deal with for general, unbounded, modules. These difficulties disappear if one mimics the topologists' treatment of smash products and function spectra in the stable homotopy category. There is a very simple theory of cell A-modules which provides a substitute for projective resolutions. Here free A-modules on one generator substitute for sphere spectra as the domains of attaching maps of cells. Topological results such as Whitehead's theorem and Brown's representability theorem transcribe directly into algebra. Every A-module M is quasi-isomorphic to a cell A-module, and the derived category $\mathscr{D}_A$ is equivalent to the homotopy category of cell A-modules. Moreover, this treatment works equally well in the more general context of modules over E_∞ k-algebras A. Here again, the derived category $\mathscr{D}_A$ of A-modules is triangulated, and it is symmetric monoidal under the derived tensor product. It is just such a generalized derived category that is needed to realize Deligne's program for defining a good category of integral mixed Tate motives.

To summarize, we display the more sophisticated and precise analogy between algebra and topology that emerges from our discussion. For a (differential graded) k-module M, write

$$M_q = H_q(M) = M^{-q}.$$

For a spectrum M, write

$$M_q = \pi_q(M) = M^{-q}.$$

ALGEBRA	TOPOLOGY
E_∞ k modules $= \mathscr{C}(1)$-modules	E_∞ S-modules $= \mathscr{L}(1)$-modules
"tensor product" $\boxtimes$	smash product over S
internal hom $\mathrm{Hom}_\boxtimes(X,Y)$	function spectrum $F_S(X,Y)$
E_∞ k-algebra A = commutative monoid in the category of unital $\mathscr{C}(1)$-modules	E_∞ ring spectrum R = commutative monoid in the category of unital $\mathscr{L}(1)$-modules
A-module	R-module
quasi-isomorphism	weak equivalence
derived category $\mathscr{D}_A$	stable homotopy category $h\mathscr{M}_R$
cell A-module	cell R-module
finite cell A-module	finite cell R-module
tensor product over A	smash product over R
internal hom $\mathrm{Hom}_A(M,N)$	function R-module $F_R(M,N)$
dual $D_A(M) = \mathrm{Hom}_A(M,A)$	dual $D_R(M) = F_R(M,R)$
$\mathrm{Hom}_A(X,Y)\otimes_A E \simeq \mathrm{Hom}_A(X, Y\otimes E)$ $D_A X \otimes_A E \simeq \mathrm{Hom}_A(X,E)$ (X or E a finite cell A-module)	$F_R(X,Y)\wedge_R E \simeq F_R(X, Y\wedge_R E)$ $D_R X \wedge_R E \simeq F_R(X,E)$ (X or E a finite cell R-module)
spectral sequence $\mathrm{Tor}^{A_*}(M_*,N_*) \Longrightarrow (M\otimes_A N)_*$	spectral sequence $\mathrm{Tor}^{R_*}(M_*,N_*) \Longrightarrow (M\wedge_R N)_*$
spectral sequence $\mathrm{Ext}_{A_*}(M_*,N^*) \Longrightarrow Hom_A(M,N)^*$	spectral sequence $\mathrm{Ext}_{R_*}(M_*,N^*) \Longrightarrow F_R(M,N)^*$

In one important case, the analogy reduces to an equivalence of derived categories in algebra and topology.

Theorem 2. Let A be a commutative ring. Then A-modules M can be realized functorially by Eilenberg-Mac Lane spectra HM that are modules over the E_∞ ring spectrum HA, and

$$\mathrm{Tor}^A_*(M,N) \cong (HM\wedge_{HA} HN)_* \text{ and } \mathrm{Ext}^*_A(M,N) \cong F_{HA}(HM,HN)^*$$

as A-modules. Further, the stable homotopy category of HA-modules is equivalent to the derived category of A-modules.

The essential point is that $HM \wedge_{HA} HN$ and $F_{HA}(HM, HN)$ are equivalent to *derived* tensor product and Hom functors in the category of chain complexes of A-modules. The spectral sequences at the end of our displayed analogy are the appropriate generalizations to E_∞ algrabras and E_∞ ring spectra of the isomorphisms of the theorem. In topology, they specialize to give generalized Künneth and universal coefficients spectral sequences.

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