# Light Open Mappings On Compact $n$-Manifolds Do Not Raise Dimension And A Proof Of The Hilbert-Smith Conjecture

by

Louis F. McAuley

Suppose that $M$ is a compact $n$-manifold and $\phi$ is a light open mapping of $M$ onto a metric space $Y$. It is shown that dim $Y = n$.

The symbol $\rho$ is used for the metric on both $M$ and $Y$. Recall that $\phi$ is light iff for each $x \in M$, $\phi^{-1}\phi(x)$ is totally disconnected.

The following lemma is crucial to defining certain coverings of $M$ with distinguished families of open sets.

**Lemma 1.** *Suppose that $\phi$ is a light open mapping of a compact connected $n$-manifold $M$ onto a metric space $Y$. For each $z \in Y$ and $\epsilon > 0$, there is a connected open set $U$ such that*

*(1) diam $U < \epsilon$,*

*(2) $z \in U$, and*

*(3) $\phi^{-1}(U) = \bigcup_{i=1}^{s} U_i$ where $s$ is a natural number such that*

*(a) $U_i$ is a component of $\phi^{-1}(U)$ for each $i$, $1 \leq i \leq s$,*

*(b) $\bar{U}_i \cap \bar{U}_j = \emptyset$ for $i \neq j$, and*

*(c) $\phi(U_i) = U$ for each $i$.*

*Proof.* It follows from Whyburn's theory of light open mappings [5; p. 148] that for $\epsilon > 0$, there is a connected open set $U$ such that (1) diam $U < \epsilon$, (2) $z \in U$, and (3) $\phi^{-1}(U)$

consists of a finite number of components $U_1, U_2, \cdots, U_s$ such that $\bar{U}_i \cap \bar{U}_j = \emptyset$ for $i \neq j$, and $\phi(U_i) = U$ for each $i$.

**Standing Hypothesis**: In the following, $M$ is a compact metric $n$-manifold. Suppose also that $Y$ has a countable basis $Q = \{B_i\}_{i=1}^{\infty}$ such that (1) for each $i$, $B_i$ is connected and uniformly locally connected and (b) if $H$ is any subcollection of $Q$ and $\bigcap_{h \in H} h \neq \emptyset$, then $\bigcap_{h \in H} h$ is connected and uniformly locally connected (a consequence of a theorem due to Bing and Floyd [1]). All of the open sets used in $Y$ below to construct coverings of $Y$ are in $Q$. The metric $\rho$ is chosen such that for each $\epsilon > 0$ and $x \in M$, $N_\epsilon(x)$, the $\epsilon$-neighborhood of $x$, is connected. Similarly, for $y \in Y$, $N_\epsilon(y)$ is connected.

One can use the alternative below and assume that $n > 1$.

**Alternative Standing Hypothesis**: In the following, $M$ is a compact metric $n$-manifold, $n > 1$. Suppose also that $Y$ has a countable basis $Q = \{B_i\}_{i=1}^{\infty}$ such that for each $i$, $B_i$ is connected and has a connected boundary (a consequence of a Theorem of Jones [3]). All of the open sets used in $Y$ below to construct coverings of $Y$ are in $Q$. The metric $\rho$ is chosen such that for each $\epsilon > 0$ and $x \in M$, $N_\epsilon(x)$, the $\epsilon$-neighborhood of $x$, is connected. Similarly, for $y \in Y$, $N_\epsilon(y)$ is connected [5].

If one uses *The Alternative Standing Hypothesis*, then make note of the following:

It is known that light open mappings on compact metric 1-manifolds are finite-to-one and do not raise dimension [5]. Furthermore, light open mappings on compact metric $n$-manifolds do not lower dimension [4]. Consequently, if $f$ is a light open mapping on a compact metric $n$-manifold $M$ onto a metric space $Y$ with $n > 1$, then $Y$ does not have any local separating points and, hence, by the theorem of Jones [3] has a basis of connected open sets with connected boundaries.

**Lemma 2.** *Suppose that $\phi$ is a light open mapping of $M$ onto $Y$ and $G$ is an open covering of $Y$. Then there exists a finite open covering $R$ of $Y$ which refines $G$ such that*

(1) *if* $y \in Y$, *then there is* $r \in R$ *such that* $y \in r$, $r \in Q$ *where* $Q$ *is the basis in The Standing Hypothesis,* $\phi^{-1}(r) = r_1 \cup r_2 \cup \cdots \cup r_q$, *for some natural number* $q$, *such that for each* $i = 1, 2, \ldots, q$, $r_i$ *is a component of* $\phi^{-1}(r)$, $r_i$ *maps onto* $r$ *under* $\phi$, *and* $\overline{r_i} \cap \overline{r_j} = \emptyset$ *for* $i \neq j$, *and*

(2) $R$ *is irreducible.*

*Proof.* Since $Y$ is compact, use Lemma 1 to obtain a finite irreducible covering $R$ of $Y$ of sets $r$ satisfying the conditions of the lemma.

Let $V^1 = \{c \mid c \text{ is a component of } \phi^{-1}(r) \text{ such that } r \in R = R^1\}$.

**Definition.** For each $r_i^1 \in R^1$, $\phi^{-1}(r_i^1) = \bigcup_{j=1}^{t_i^1} f_{ij}^1$ and $\{f_{ij}^1\}_{j=1}^{t_i^1} = F_i^1$ is called a *distinguished family of open sets* in $V^1$ where $f_{ij}^1$ is a component of $\phi^{-1}(r_i^1)$.

Construction Of $U^1$ Of Order $n+1$ Which Star Refines $V^1$

List the distinguished families of $V^1$ as $F_1^1, F_2^1, \cdots, F_{n_1}^1$ where the degenerate families, if any, are listed last in the ordering.

Recall from the Standing Hypothesis that $Y$ has a countable basis $Q$ with certain properties. Use $Q$ to obtain a finite open star refinement $\hat{R} \subset Q$ of $R^1$ which covers $Y$ such that each $r \in \hat{R}$ is connected and uniformly locally connected (ulc) and inherits certain other properties from $Q$.

Let $\hat{U}^1 = \{c \mid c \text{ is a component of } \phi^{-1}(r) \text{ where } r \in \hat{R}\}$. Clearly, $\hat{U}^1$ star refines $V^1$. Recall also that if $c$ is a component of $\phi^{-1}(r)$, then $\phi(c) = r$.

Construction Of $U^1$ Which Refines $\hat{U}^1$ In A Special Way

For any collection $S$ of sets, let $\cup S$ be the union of sets in $S$ and $\cap S$ be their intersection.

For each $y \in Y$, let $Q(y) = \{r \mid r \in \hat{R} \text{ and } y \in r\}$. There are at most a finite number of such sets distinct from each other. Order these sets as $Q_1, Q_2, \cdots, Q_{m_1}$ such that for $i < j$, $Q_i \neq Q_j$ and card $Q_i \geq$ card $Q_j$. Let $O_i = \cap Q_i - \bigcup_{j<i} (\overline{\cap Q_j})$. For each $i$, $1 \leq i \leq m_1$, let

$B_{ir} = (\partial O_i) \cap \partial r$ where $r \in Q_i$ and $(\partial O_i) \cap \partial r \neq \emptyset$. There are at most a finite number of such non empty closed sets distinct from each other. Let $B_1, B_2, \cdots, B_{m_2}$ denote all those sets distinct from each other. For each $i$, $1 \leq i \leq m_2$, there is $r \in \hat{R}$ such that $\partial r \supset B_i$. Let $B = \bigcup_{i=1}^{m_2} B_i$. Clearly, $B = \bigcup_{r \in \hat{R}} \partial r$. For each $y \in B$, let $D(y) = \{B_t \mid y \in B_t\}$. There are at most a finite number of such non empty sets distinct from each other. Order these as $D_1, D_2, \cdots, D_{m_3}$ such that if $i < j$, then $D_i \neq D_j$ and card $D_i \geq$ card $D_j$.

It follows from the definition that $\phi^{-1}(B)$ is closed and contains no open set. Hence, dimension $\phi^{-1}(B) \leq n - 1$.

Let $e = (\frac{1}{4})\{\min\{\rho(\cap D_i, \cap D_j) \mid (\cap D_i) \cap (\cap D_j) = \emptyset\}$ and $\epsilon$, the Lebesque number of the covering $\hat{R}$ of $Y\}$.

Observe that if $B_s \notin \bigcup_{i=1}^{k} D_i$, then $B_s \cap (\cap D_i) = \emptyset$ for $1 \leq i \leq k$; otherwise, card $(B_s \cup D_i) >$ card $D_i$ and $B_s \in D_j$ for some $j$, $1 \leq j < i$.

Note: A finite open covering $C$ of a closed subset $N$ of $M$ such that dim $N = n - 1$ where the elements of $C$ are open relative to $N$ and order $C = n$ can be extended to a collection $C'$ of open sets in $M$ such that card $C =$ card $C'$, $C'$ covers $M$ and order $C' = n$. See [6]. The open coverings below of subsets $N$ of $M$ such that dim $N \leq n - 1$ consist of open subsets of $M$.

Let $\{D_{1_i}\}_{i=1}^{q_1}$ be the collection of all $D_t$ such that card $D_t = c_1 =$ card $D_1$ (maximum cardinality). Observe that $(\cap D_{1_i}) \cap (\cap D_{1_j}) = \emptyset$ for $i \neq j$.

Cover $\phi^{-1}\left(\bigcup_{i=1}^{q_1} (\cap D_{1_i})\right)$ with a finite irreducible open covering $H_1$ such that

(1) if $h \in H_1$, then diam $\phi(h) < e$,

(2) $H_1$ star refines $\hat{U}^1$ and $\phi(H_1) = \{\phi(h) \mid h \in H_1\}$ star refines $\hat{R}$,

(3) if $H_{1_i}$ is the subcollection of $H_1$ which covers $\phi^{-1}(\cap D_{1_i})$ irreducibly, then $\overline{\cup H_{1_i}} \cap \overline{\cup H_{1_j}} = \emptyset$ for $i \neq j$, and

(4) if for each $i$, $1 \leq i \leq q_1$, $h \in H_{1_i}$, $w \in H_{1_i}$, and $u \in \hat{U}^1$ such that

(a) $\phi(h) \cap \phi(u) \neq \emptyset$,

(b) $\phi(h) \cap \phi(w) \neq \emptyset$, and

(c) $B_s \notin D_{1_i}$ for all $B_s$ such that $\partial\phi(u) \supset B_s$ (alternatively, $\phi(u) \cap (\cap D_{1_i}) \neq \emptyset$), then $\phi(u) \supset \phi(h) \cup \phi(w)$.

Now, (1) and (2) should be clear.

To see (3), observe that if $i \neq j$, then $\phi^{-1}(\cap D_{1_i})$ and $\phi^{-1}(\cap D_{1_j})$ are disjoint compact sets.

To see (4), assume the hypothesis of (4). For each $h \in H_{1_i}$ there exists $y \in \phi(h) \cap (\cap D_{1_i})$. It follows from (1) that $N_e(y) \supset \phi(h)$. Since $B_s \notin D_{1_i}$, $\rho(\cap D_{1_i}), B_s) > 4e$. Thus, $\phi(u) \supset N_{4e}(y)$ since $N_{4e}(y)$ is connected. Also, diam $\phi(w) < e$ and $\phi(w) \cap \phi(h) \neq \emptyset$. Hence, $\phi(u) \supset \phi(h) \cup \phi(w)$ since $\phi(u) \supset N_{4e}(y) \supset \phi(h) \cup \phi(w)$.

Let $c_1, c_2, \cdots, c_{m_4}$ denote the cardinal numbers (distinct from each other) of the cardinality of the collections $D_t$, $1 \leq t \leq m_3$, such that $c_i > c_{i+1}$ for $1 \leq i < m_4$.

Let $\{D_{i_j}\}_{j=1}^{q_i}$ denote the collection of all $D_t$ such that card $D_t = c_i$. If $H_{j-1}$ has been defined, let $\epsilon_j > 0$ be such that $\epsilon_j < (\frac{1}{4}) \min\{e, \epsilon_{j-1}, \delta_t,\ 1 \leq t < j$, where $\delta_t = (\frac{1}{4})\Big\{ \min\Big\{\rho\Big( \cap D_{t_i}, Y - \bigcup_{s=1}^{t-1} \phi(\cup H_s)\Big) \Big| 1 \leq i \leq q_t\Big\}$ and $\min\Big\{\rho\Big( \cap D_{i_a} - \bigcup_{s=1}^{t-1}\phi(\cup H_s), \cap D_{j_b} - \bigcup_{s=1}^{t-1}\phi(\cup H_s)\Big) \Big| \Big( \cap D_{i_a} - \bigcup_{s=1}^{t-1}\phi(\cup H_s)\Big) \cap \Big( \cap D_{j_b} - \bigcup_{s=1}^{t-1}\phi(\cup H_s)\Big) = \emptyset \Big\}\Big\}$. Let $H_j$ be a finite irreducible open covering of $\phi^{-1}\Big(\bigcup_{i=1}^{q_j}(\cap D_{j_i})\Big) - \bigcup_{t=1}^{j-1}(\cup H_t)$ such that

(1) if $h \in H_j$, then diam $\phi(h) < \epsilon_j$ and $\phi(h) \cap \left(\bigcup_{i=1}^{q_{j-1}}(\cap D_{(j-1)_i})\right) = \emptyset$,

(2) $H_j$ star refines $\hat{U}^1$ and $\phi(H_j) = \{\phi(h) \mid h \in H_j\}$ star refines $\phi(\hat{U}^1) = \hat{R}$,

(3) if $H_{j_i}$ is the subcollection of $H_j$ which covers $\phi^{-1}(\cap D_{j_i}) - \bigcup_{t=1}^{j-1}(\cup H_t)$ irreducibly, then $\overline{\cup H_{j_i}} \cap \overline{\cup H_{j_s}} = \emptyset$ for $i \neq s$, and

(4) if for each $i$, $1 \leq i \leq q_i$, $h \in H_{j_i}$, $w_1 \in H_{j_i}$, $w_2 \in H_{j_i}$, and $u \in \hat{U}^1$ such that

(a) $\phi(h) \cap \phi(u) \neq \emptyset$,

(b) $\phi(h) \cap \phi(w_1) \neq \emptyset$,

(c) $\phi(w_1) \cap \phi(w_2) \neq \emptyset$, and

(d) $B_s \notin D_{j_i}$ for all $B_s$ such that $\partial\phi(u) \supset B_s$ (alternatively, $\phi(u) \cap \left(\cap D_{j_i} - \bigcup_{t=1}^{j-1} \phi(\cup H_t)\right) \neq \emptyset$); then $\phi(u) \supset \phi(h) \cup \phi(w_1) \cup \phi(w_2)$.

To see (3), observe that if $i \neq s$, then $\phi^{-1}(\cap D_{j_i}) - \bigcup_{t=1}^{j-1} (\cup H_t)$ and $\phi^{-1}(\cap D_{j_s}) - \bigcup_{t=1}^{j-1} (\cup H_t)$ are disjoint compact sets.

To see (4), assume the hypothesis of (4). Since $h \in H_{j_i}$, it follows that there is $z \in \phi(h) \cap \left(\cap D_{j_i} - \bigcup_{t=1}^{j-1} \phi(\cup H_t)\right)$. Now, $\rho\left(\cap D_{j_i} - \bigcup_{t=1}^{j-1} \phi(\cup H_t), \partial\phi(u)\right) > 4\epsilon_j$. Either (a) $\phi(u) \supset N_{4e}(z)$ or (b) $Y - \overline{\phi(u)} \supset N_{4\epsilon_j}(z)$ since $N_{4\epsilon_j}(z)$ is connected. Since $\phi(u) \cap \phi(h) \neq \emptyset$ and diam $\phi(h) < \epsilon_j$, it follows that $\phi(u) \supset N_{4\epsilon_j}(z)$ and $\phi(u) \supset \phi(h)$. By hypothesis, diam $\phi(w_i) < \epsilon_j$, $i \in \{1, 2\}$. Hence, $\phi(u) \supset \phi(h) \cup \phi(w_1) \cup \phi(w_2)$.

It follows by mathematical induction that $H = \bigcup_{i=1}^{m_4} H_i$ exists which covers $\phi^{-1}(B)$ irreducibly such that

(1) if $h \in H_j$, then diam $\phi(h) < \epsilon_j$, $\phi(h) \cap \left(\bigcup_{i=1}^{q_{j-1}} (D_{(j-1)_i})\right) = \emptyset$,

(2) for each $i$, $1 < i \leq m_4$, $H_i$ is an irreducible open covering of $\phi^{-1}\left(\bigcup_{j=1}^{q_i} (\cap D_{i_j})\right) - \bigcup_{t=1}^{i-1} (\cup H_t)$ and $H_1$ is an irreducible open covering of $\phi^{-1}\left(\bigcup_{i=1}^{q_1} (\cap D_{1_i})\right)$,

(3) $H_j$ star refines $\hat{U}^1$ and $\phi(H_j)$ star refines $\phi(\hat{U}^1) = \hat{R}$,

(4) if $H_{j_i}$ is the subcollection of $H_j$ which covers $\phi^{-1}(\cap D_{j_i}) - \bigcup_{t=1}^{j-1} (\cup H_t)$, then $\overline{\cup H_{j_i}} \cap \overline{\cup H_{j_s}} = \emptyset$ for $i \neq s$, and

(5) if for each $i$, $1 \leq i \leq q_j$, $h \in H_{j_i}$, $w_1 \in H_t$, $j \leq t \leq m_4$, $w_2 \in H_s$, $j \leq s \leq m_4$, and $u \in \hat{U}^1$ such that

(a) $\phi(h) \cap \phi(u) \neq \emptyset$,

(b) $\phi(h) \cap \phi(w_1) \neq \emptyset$,

(c) $\phi(w_1) \cap \phi(w_2) \neq \emptyset$, and

(d) $B_s \notin D_{j_i}$ for all $B_s$ such that $\partial\phi(u) \supset B_s$ (alternatively, $\phi(u) \cap \left( \cap D_{j_i} - \bigcup_{t=1}^{j-1} \phi(\cup H_t) \right) \neq \emptyset$; then $\phi(u) \supset \phi(h) \cup \phi(w_1) \cup \phi(w_2)$.

It should be clear from the arguments above how to obtain properties (1)-(5).

The next step is to shrink $H$ to $H'$ which has order $n$ and retains Properties (1)-(5).

Order the elements of $H$ as $\{h_{11}, h_{12}, \cdots, h_{1x_1}; h_{21}, h_{22}, \cdots, h_{2x_2}; \cdots, h_{m_4 1}, h_{m_4 2}, \cdots, h_{m_4 x_{m_4}}\}$ where the elements of $H_i$ are ordered before the elements of $H_{i+1}$, $1 \leq i < m_4$. A corollary to a Theorem [10; p. 90] states that: A metric space $(X, d)$ has covering dimension $\leq n$ if and only if $X$ has a sequence $\{G_i\}$ such that (1) for each $i$, $G_i$ is an open covering of $X$, (2) for each $i$, $G_{i+1}$ refines $G_i$, and (3) $\{\text{mesh } G_i\} \to 0$. It follows that if $(X, d)$ is a compact metric space with dim $X \leq n$ and $Q$ is any finite irreudcible covering of $X$, then there is a natural number $i$ such that $G_i$ refines $Q$ where $\{G_i\}$ is the squence in the statement above from [10].

Now, dim $\phi^{-1}(B) \leq n-1$. Using [10], it follows that there exists a finite irreducible open covering $G = \{g_1, g_2, \cdots, g_m\}$ of $\phi^{-1}(B)$ (the elements of $G$ are open in $M$) with sufficiently small mesh such that

(1) the subcollection $G_1$ of $G$ which covers $\phi^{-1}(B) \cap \left( \cup H_1 - \bigcup_{j=1}^{m_4} (\cup H_j) \right)$ refines $H_1$,

(2) if $i > 1$, then the subcollection $G_2$ of $G$ which covers $\left( \phi^{-1}\left( \bigcup_{j=1}^{q_i} (\cap D_{i_j}) \right) - \bigcup_{t=1}^{i-1} (\cup G_t) \right) - \bigcup_{j=i+1}^{m_4} (\cup H_j)$ [where $\bigcup_{j=i+1}^{m_4} (\cup H_j) = \emptyset$ when $i = m_4$] refines $H_i$,

(3) for each $i$ and $k$, there is some $j$ such that $h_{ki} \supset g_j$ and $h_{st} \not\supset g_j$ for any ordered pair $(s, t) \neq (k, i)$ (this follows using the irreducibility of $H$), and

(4) order $G \leq n$.

Let $h'_{11}$ be the union of all $g_i$ such that $h_{11} \supset g_i$ and let $h'_{1i}$ be the union of all $g_j$ such that $h_{1i} \supset g_j$ and $h_{1t} \not\supset g_j$ for $1 \leq t < i$, that is, $h'_{1t} \not\supset g_j$. It follows that $H'_1 =$

$\{h'_{11}, h'_{12}, \cdots, h'_{1q_1}\}$ is an irreducible open cover of $\phi^{-1}\left(\bigcup_{i=1}^{q_1} \cap D_{1_i}\right)$. Continue. Let $h'_{21}$ be the union of all $g_i$ such that $h_{21} \supset g_i$ and $h_{1t} \not\supset g_i$, $1 \leq t \leq q_1$. For each $i$, $1 < i \leq q_2$, let $h'_{2i}$ be the union of all $g_j$ such that $h_{2i} \supset g_j$, $h_{1t} \not\supset g_j$, $1 \leq t \leq q_1$, and $h_{2t} \not\supset g_j$ for $1 \leq t < i$. It follows that $H'_2 = \{h'_{21}, h'_{22}, \cdots, h'_{2q_2}\}$ is an irreducible open cover of $\phi^{-1}\left(\bigcup_{i=1}^{q_2} \cap D_{2_i}\right) - \cup H'_1$. To see that $H'_2$ covers $\phi^{-1}\left(\bigcup_{i=1}^{q_2} \cap D_{2_i}\right) - \cup H'_1$ suppose that there is some $x$ in this set such that $x \notin h'_{2t}$ for any $t$, $1 \leq t \leq q_2$. By Property (2) above, $G_2$ refines $H_2$ and covers $\left(\phi^{-1}\left(\bigcup_{j=1}^{q_2}(\cap D_{2j})\right) - \cup G_1\right) - \bigcup_{j=2}^{m_4}(\cup H_j)$. Now, if $x \in g \in G_1$, then $x \in h_{1s}$ for some $s$ since $G_1$ refines $H_1$ and covers $\phi^{-1}(B) \cap \left(\cup H_1 - \bigcup_{j=1}^{m_4}(\cup H_j)\right)$. Hence, $x \notin g$ for any $g \in G_1$. Recall that $\left(\bigcup_{j=1}^{m_4}\overline{\cup H_j}\right) \cap \phi^{-1}\left(\bigcup_{j=1}^{q_2}(\cap D_{2j})\right) = \emptyset$. Thus, by definition of $G_2$, $x \in g$ for some $g \in G_2$ and, consequently, $x \in h_{2t}$ for some $t$ since $G_2$ refines $H_2$. That is, there is a smallest $t$ such that $h_{2t} \supset g$. It can be shown in a similar way that for each $i$, $1 \leq i \leq m_4$, $H'_i$ covers $\phi^{-1}\left(\bigcup_{j=1}^{q_i} D_{i_j}\right) - \bigcup_{t=1}^{i-1}(\cup H'_t)$. Let $H' = \bigcup_{i=1}^{m_4} H'_i$. Clearly, order $H' \leq n$ since order $G \leq n$ and no member of $G$ is in two different elements of $H'_t$, $1 \leq t \leq m_4$. This is similar to a theorem of Nagata in [4].

It is not difficult to see that Properties (1)-(5) are true where $H'_i$ replaces $H_i$, $1 \leq i \leq m_4$. For convenience of notation, suppose that Properties (1)-(5) are true for $H_i$ as stated above.

Shrinking the elements of $H_i$ as above does not violate any of the other properties.

Recall the definition of $Q_i$ and $O_i = \cap Q_i - \bigcup_{j<i}(\overline{\cap Q_j})$. Let $Q_i = \{r_{i_t}\}_{t=1}^{x_i}$. For all possible selections of components $c_{i_t j_t}$ of $\phi^{-1}(r_{i_t})$ such that $\bigcap_{t=1}^{x_i} c_{i_t j_t} \neq \emptyset$, consider $\phi^{-1}(O_i) \cap \left(\bigcap_{t=1}^{x_i} c_{i_t j_t}\right)$ to be an element of $O$. The collection $O$ of all such open sets distinct

from each other is a pairwise disjoint collection which covers $M - \phi^{-1}(B) = \bigcup_{i=1}^{m_1} \phi^{-1}(O_i)$.

Let $U^1 = O \cup H$. Observe that $U^1$ is a finite irreducible open covering of $M$ which star refines $V^1$ and order $U^1 = n+1$. Also, $H$ has Properties (1)-(6) as listed above.

Construction of $V^2$ Which Star Refines $U^1$

Next, construct $V^2$. For each $y \in Y$, define $\hat{U}(y) = \{u \mid u \in \hat{U}^1 \text{ and } y \in \phi(u)\}$. There is some $s$, $1 \leq s \leq z_1$, such that $r_s^1 \supset \bigcup_{u \in \hat{U}(y)} \overline{\phi(u)}$ where $R^1 = \{r_1^1, r_2^1, \cdots, r_{z_1}^1\}$.

For each $y \in B$, choose $r_y^2 \in Q$ (the basis for $Y$ described above) such that

(1) $y \in r_y^2$,

(2) if $U^1(y) = \{u \mid u \in U^1 \text{ and } u \cap \phi^{-1}(y) \neq \emptyset\}$ (that is, $U^1(y)$ covers $\phi^{-1}(y)$), then $\left( \bigcap_{u \in U^1(y)} \phi(u) \right) \cap \left( \bigcap_{u \in \hat{U}(y)} \phi(u) \right) \supset \bar{r}_y^2$,

(3) diam $r_y^2 < (\frac{1}{8}) \min\{\rho(y, \partial\phi(v)) \mid v \in \hat{U}^1 \text{ and } y \notin \partial\phi(v)\}$,

(4) $\phi^{-1}(r_y^2) = r_{y1}^2 \cup r_{y2}^2 \cup \cdots \cup r_{yq}^2$, $r_{yi}^2$ maps onto $r_y^2$ under $\phi$, and $\bar{r}_{yi}^2 \cap \bar{r}_{yj}^2 = \emptyset$ for $i \neq j$, and

(5) if $u \in U^1(y)$, then there exists a component $c$ of $\phi^{-1}(r_y^2)$ such that $u \supset \bar{c}$ and diam $\bar{c} < (\frac{1}{4})\rho(\bar{c}, \partial u)$ and for each component $k$ of $\phi^{-1}(r_y^2)$, there is some $v \in U^1(y)$ such that $v \supset \bar{k}$ and diam $\bar{k} < (\frac{1}{4})\rho(\bar{k}, \partial v)$.

Observe that Property (4) follows from either Lemma 1 or [5; p. 148].

Note that Property (5) may be obtained by using [9; p. 78, Theorem (1.3)]. First, fix $x(u) \in u$ for each $u \in U^1(y)$. Choose $\delta > 0$ such that (a) $\delta < (\frac{1}{4}) \min\{\rho(x(u), \partial u) \mid u \in U^1(y)\}$ and (b) if $x \in \phi^{-1}(y)$, then there is some $v \in U^1(y)$ such that $x \in v$ and $\rho(x, \partial v) < 4\delta$. By [9; p. 78, Theorem (1.3)], it follows that $r_y^2$ can be chosen with sufficiently small diameter such that for each component $c$ of $\phi^{-1}(r_y^2)$, diam $\bar{c} < \delta$. Hence, if $c_{x(u)}$ is the component of $\phi^{-1}(r_y^2)$ which contains $x(u)$, then (a) $u \supset \bar{c}_{x(u)}$, (b) diam $\bar{c}_{x(u)} < (\frac{1}{4})\rho(c_{x(u)}, \partial u)$, and (c) if $k$ is a component of $\phi^{-1}(r_y^2)$, then there is $v \in U^1(y)$ such that $v \supset \bar{k}$ and diam $\bar{k} < (\frac{1}{4})\rho(\bar{k}, \partial v)$.

Let $R_1^2$ denote a finite irreducible collection of such sets $r_y^2$ which covers $B$. If $y \in Y$ and $y \notin \cup R_1^2$, then choose $r_y^2$ satisfying (1)-(5) above such that $\bar{r}_y \cap B = \emptyset$ and let $R_2^2$ denote a finite irreducible cover of $Y - (\cup R_1^2)$, consisting of such $r_y^2$. Let $R^2 = R_1^2 \cup R_2^2$ which is an irreducible cover of $Y$.

Define $V^2 = \{c \mid c$ is a component of $\phi^{-1}(r_{y_i}^2)$ for some $i$, where $r_{y_i}^2 \in R_1^2 \cup R_2^2\}$, which is an irreducible cover of $M$. Observe that Property (5) implies that $V^2$ star refines $U^1$. Now $U^1$ is constructed so that $U^1$ refines $\hat{U}^1$ which star refines $V^1$. Hence, $U^1$ star refines $V^1$. The collection of components, $\{f_{ij}^2\}_{j=1}^{t_i^2}$, of $\phi^{-1}(r_{y_i}^2)$, $r_{y_i}^2 \in R_1^2 \cup R_2^2$, is a distinguished family in $V^2$.

Definitions Of $\alpha_1$, $\beta_1$, And $\pi_1 = \beta_1\alpha_1$

Case (1): $y_i \in B$.

Take any $r_i^2 = r_{y_i}^2 \in R_1^2$ chosen for $y_i \in B$. Let $F_i^2 = \{f_{ij}^2\}_{j=1}^{t_i^2}$ be the distinguished family in $V^2$ generated by $r_{y_i}^2$.

Definition of $e_{y_i}$ for $y_i \in B$

Let $e_{y_i} = \min\{t \mid y_i \in \phi(\cup H_{t_q})$ for some unique $q\}$.

Now, $H_{(e_{y_i})_q} \subset H_{e_{y_i}}$ and $H_{(e_{y_i})_q}$ covers $\phi^{-1}(\cap D_{(e_{y_i})_q}) - \bigcup_{s=1}^{e_{y_i}-1} H_s$ irreducibly. Either (a) $\phi^{-1}(y_i)$ is covered by $H_{(e_{y_i})_q}$ or (b) $\phi^{-1}(y_i)$ is not covered by $H_{(e_{y_i})_q}$.

Case (a): By the choice of $r_{y_i}^2$ which generates $F_i^2$, it follows that for each $j$, $1 \leq j \leq t_i^2$, there is $U_{ij} \in H_{(e_{y_i})_q}$ such that $U_{ij} \supset \bar{f}_{ij}^2$. This follows from Property (5) above. In this case, there is no $u \in O$ such that $u \supset \bar{f}_{ij}^2$.

Case (b): For some $j$, $1 \leq j \leq t_i^2$, choose $U_{ij} \in H_{(e_{y_i})_q}$ if possible such that $U_{ij} \supset \bar{f}_{ij}^2$; if not, then choose $U_{ij} \in H_t$ for the smallest $t$ (where, of course, $t > e_{y_i}$) such that $U_{ij} \supset \bar{f}_{ij}^2$. By definition of $e_{y_i}$, there is some $u \in H_{(e_{y_i})_q}$ such that $u \cap \phi^{-1}(y_i) \neq \emptyset$. By Property (5), there is some $t$, $1 \leq t \leq t_i^2$, such that $u = U_{it} \supset \bar{f}_{it}^2$. If there is no $U_{ij} \in H_{(e_{y_i})_q}$ such that $U_{ij} \supset \bar{f}_{ij}^2$, then clearly there is $u = U_{ij} \in U^1(y_i) \subset H$ such that

$U_{ij} \supset \bar{f}^2_{ij}$ by Property (5) above. Now, $U_{ij} \in H_t \subset H$ for the smallest subscript $t$. By the definition of $e_{y_i}$, it follows that $t > e_{y_i}$.

Definition of $d_{y_i}$ for $y_i \in B$

Let $d_{y_i} = \min\{j \mid D_{j_t} \supset D_{(e_{y_i})_q}$ for the smallest $t\}$.

If $y_i \notin B$, then neither $e_{y_i}$ nor $d_{y_i}$ is defined.

Definition of $\hat{V}(y_i)$

If $y_i \notin B$, then $\hat{V}(y_i) = \hat{U}(y_i)$. For $y_i \in B$, let $\hat{V}(y_i) = \{u \mid u \in \hat{U}^1$ and $\phi(u) \supset \cap D_{(d_{y_i})_t}\}$.

Property P

If $y_i \in B$, then $\hat{V}(y_i) \neq \emptyset$ and if $u \in \hat{V}(y_i)$, then

(1) $u \in \hat{U}(y_i)$, that is, $\hat{U}(y_i) \supset \hat{V}(y_i)$,

(2) $\phi(u) \supset \bigcup_{s=1}^{t_i^2} \phi(U_{is})$,

(3) $\phi(u) \supset \cap D_{(d_{y_i})_t}$,

(4) $\overline{\phi(u)} \supset \cup D_{(d_{y_i})_t}$,

(5) $\overline{\phi(u)} \supset \cup D_{(e_{y_i})_q}$, and

(6) $\phi(u) \supset \cap D_{(e_{y_i})_q} - \bigcup_{t=1}^{e_{y_i}-1} \phi(\cup H_t)$.

*Proof.* Recall that $e_{y_i} = \min\{t \mid y_i \in \phi(\cup H_{t_q})$ for some $q\}$, $d_{y_i} = \min\{j \mid D_{j_t} \supset D_{(e_{y_i})_q}$ for the smallest $t$, and $\hat{V}(y_i) = \{u \mid u \in \hat{U}^1$ and $\phi(u) \supset \cap D_{(d_{y_i})_t}\}$.

Take $u \in \hat{U}^1$ such that $\phi(u) \cap (\cap D_{(d_{y_i})_t}) \neq \emptyset$. It follows that $B_a \notin D_{(d_{y_i})_t}$ for each $B_a$ such that $\partial\phi(u) \supset B_a$. To see this, suppose that $\partial\phi(u) \cap (\cap D_{(d_{y_i})_t}) \neq \emptyset$. Then there exists $B_x$ such that $\partial\phi(u) \supset B_x$ and $B_x \cap (\cap D_{(d_{y_i})_t}) \neq \emptyset$. There exists $c < d_{y_i}$ such that $D_{c_s} \supset D_{(d_{y_i})_t}$ for some unique $s$ where $B_x \in D_{c_s}$. This contradicts the choice of $d_{y_i}$ since $D_{c_s} \supset D_{(d_{y_i})_t} \supset D_{(e_{y_i})_q}$. Hence, $B_a \notin D_{(d_{y_i})_t}$ as claimed.

For any $B_x \in D_{(d_{y_i})_t}$, $B_x = B_{jr} = (\partial O_j) \cap \partial r$ where $r \in Q_j$ for some $j$, $\phi(u) \supset O_j$, and $\phi(u) \in Q_j$. Thus, $\overline{\phi(u)} \supset \cup D_{(d_{y_i})_t}$ and $\phi(u) \supset \cap D_{(d_{y_i})_t}$ since $\partial\phi(u) \cap (\cap D_{(d_{y_i})_t}) = \emptyset$.

This establishes (3), (4), and (5) since $D_{(d_{y_i})_t} \supset D_{(e_{y_i})_q}$.

Take any such $u$ as above. It follows that $\overline{\phi(u)} \supset \cup D_{(e_{y_i})_t}$. Observe that if $B_s \cap (\cap D_{(e_{y_i})_q}) \neq \emptyset$, $B_s \notin D_{(e_{y_i})_q}$, and $\partial\phi(u) \supset B_s$, then $B_s \in D_{a_m}$ for some $a < e_{y_i}$ and the smallest $m$. Thus, $\bigcup_{t=1}^{e_{y_i}-1} \phi(\cup H_t) \supset \cap D_{a_m}$. Note also that $B_s \in D_{(d_{y_i})_t}$.

Now, $B_a \notin D_{(e_{y_i})_q}$ for each $B_a$ such that $\partial\phi(u) \supset B_a$. Consequently, $\phi(u) \supset \cap D_{(e_{y_i})_q} - \bigcup_{t=1}^{e_{y_i}-1} \phi(\cup H_t)$. Hence, (6) is established.

There is $x$, $1 \leq x \leq t_i^2$, such that $U_{ix} \in H_{(e_{y_i})_q}$. Hence, $\phi(u) \cap \phi(U_{ix}) \neq \emptyset$ since $\phi(U_{ix}) \cap \left( \cap D_{(e_{y_i})_q} - \bigcup_{t=1}^{e_{y_i}-1} \phi(\cup H_t) \right) \neq \emptyset$. Take any $U_{iz}$, $1 \leq z \leq t_i^2$. Let $h = U_{ix}$, $w_1 = w_2 = U_{iz} \in H_s$, $e_{y_i} \leq s \leq m_4$. Also, $y_i \in \phi(U_{ix}) \cap \phi(U_{iz})$. By Property (5) of $H$, $\phi(u) \supset \phi(U_{ix}) \cup \phi(U_{iz})$ and $\phi(u) \supset \bigcup_{t=1}^{t_i^2} \phi(U_{it})$. It follows that $y_i \in \phi(u)$ and $u \in \hat{U}(y_i)$. Thus, $\hat{V}(y_i)$ exists and $\hat{U}(y_i) \supset \hat{V}(y_i)$. Hence, (1) and (2) are true and Property P is established.

Definition of $s_i$ for $y_i \in B$

Let $s_i = \min\{s \mid r_s^1 \supset \overline{\phi(u)}$ for all $u \in \hat{V}(y_i)\}$. Take $F_{s_i}^1 = \{f_{s_ij}^1\}_{j=1}^{t_{s_i}^1}$ for the given $F_i^2$. Each $f_{ij}^2 \in F_i^2$ is in one and only one member of $F_{s_i}^1$, say $f_{s_iz_{ij}}^1$. To see that there is a unique $z_{ij}$, $1 \leq z_{ij} \leq t_{s_i}^1$, such that $f_{s_iz_{ij}}^1 \supset U_{ij} \supset \bar{f}_{ij}^2$, recall that $U^1$ star refines $\hat{U}^1$, that $\hat{U}^1$ star refines $V^1$, the definition of $U(y_i)$, and Property (2) in the construction of $V^2$. Since $\bigcup_{u \in \hat{V}(y_i)} \overline{\phi(u)} \subset r_{s_i}^1$, $r_{s_i}^1 \in R^1$, $\bigcap_{u \in U(y_i)} \phi(u) \supset \bar{r}_{y_i}^2$, $\bigcap_{u \in \hat{U}(y_i)} \phi(u) \supset \bar{r}_{y_i}^2$, $r_{y_i}^2 \in R^2$, and $U_{ij} \supset \bar{f}_{ij}^2$, we have $r_{s_i}^1 \supset \phi(U_{ij})$ and $f_{s_iz_{ij}}^1 \supset U_{ij}$ for some unique $z_{ij}$.

Observe that the chosen collection $\{U_{ij}\}_{j=1}^{t_i^2}$ has the property that $U_{ij} \in H_t$ where $t \geq e_{y_i}$.

Let $\alpha_1(f_{ij}^2) = U_{ij}$, $\beta_1(U_{ij}) = f_{s_iz_{ij}}^1$, and $\pi_1(f_{ij}^2) = \beta_1\alpha_1(f_{ij}^2) = f_{s_iz_{ij}}^1$.

Case (2): $y_i \notin B$.

Take $r_{y_i}^2 \in R_2^2$; here $y_i \notin B$ and $r_{y_i}^2$ generates $F_i^2 = \{f_{ij}^2\}_{j=1}^{t_i^2}$ in $V^2$. Let $s_i = \min\{s \mid$

$r^1_s \supset \bigcup_{u \in \hat{U}(y_i)} \overline{\phi(u)}\}$. Take $F^1_{s_i} = \{f^1_{s_i j}\}_{j=1}^{t^1_{s_i}}$ for the given $F^2_i$, then $\phi(f^1_{s_i j}) \supset \bigcup_{u \in \hat{U}(y_i)} \overline{\phi(u)}\}$. For each $j$, $1 \le j \le t^2_i$, choose $U_{ij} \in O \subset U^1$ such that $U_{ij} \supset f^2_{ij}$ (there is such a $U_{ij}$ by Property (2) of $r^2_{y_i}$ in the construction of $V^2$ above). Then $U_{ij} \in U(y_i)$ and there is a unique $z_{ij}$, $1 \le z_{ij} \le t^1_{s_i}$, such that $f^1_{s_i z_{ij}} \supset U_{ij} \supset f^2_{ij}$. Let $\alpha_1(f^2_{ij}) = U_{ij}$, $\beta_1(U_{ij}) = f^1_{s_i z_{ij}}$, and $\pi_1(f^2_{ij}) = \beta_1 \alpha_1(f^2_{ij}) = f^1_{s_i z_{ij}}$.

It will be shown now that the mappings $\alpha_1$ and $\beta_1$ are well defined.

We fix the choice of open sets in $U^1$, which are images of elements of $V^2$ under the mapping $\alpha_1$, and the question is: whether the definition of $\beta_1$ is correct (well defined)?

Suppose that $\beta_1$ is not well defined and there exist $F^2_i$ and $F^2_k$, two different distinguished families in $V^2$ such that (a) $s_i \ne s_k$ (if $s_i = s_k$, then $\beta_1$ is well defined), $F^1_{s_i}$ is chosen for $F^2_i$, $F^1_{s_k}$ is chosen for $F^2_k$, and (b) $U_{ij} = U_{kt} \supset f^2_{ij} \cup f^2_{kt}$ where $F^2_i = \{f^2_{ij}\}_{j=1}^{t^2_i}$, $F^2_k = \{f^2_{kj}\}_{j=1}^{t^2_k}$, where for some $j$, $1 \le j \le t^2_i$, $U_{ij} \in U^1$ is chosen such that $U_{ij} \supset \overline{f^2_{ij}}$, and for some $t$, $1 \le t \le t^2_k$, $U_{kt} = U_{ij} \in U^1$ is chosen such that $U_{kt} \supset \overline{f^2_{kt}}$ as described above.

Case A: $U_{ij} = U_{kt} \in O \subset U^1$. Then $y_i \notin B$ and $y_k \notin B$. Indeed, $y_i \in O_m$ and $y_k \in O_m$ for some $m$, $1 \le m \le n_1$. In this case, it follows from the definition of $O$ that for each $u \in \hat{U}(y_i)$, $\phi(u) \supset O_m$ and for each $v \in \hat{U}(y_k)$, $\phi(v) \supset O_m$. Thus, $\hat{U}(y_i) = \hat{U}(y_k) = \hat{V}(y_i) = \hat{V}(y_k)$, and $s_i = s_k$ contrary to the assumption above.

Case B: $U_{ij} = U_{kt} \in H \subset U^1$, $y_i \in B$, and $y_k \in B$.

There is $x$, $1 \le x \le t^2_i$, such that $U_{ix} \in H_{(e_{y_i})_q}$ and there is $z$, $1 \le z \le t^2_k$, such that $U_{kz} \in H_{(e_{y_k})_s}$. Since $U_{ij} = U_{kt}$, $y_i \in \phi(U_{ij}) \cap \phi(U_{ix})$ and $y_k \in \phi(U_{kt}) \cap \phi(U_{kz})$. Now, $\rho(\cap D_{(e_{y_i})_q}, \cap D_{(e_{y_k})_s}) < e$ since diam $\phi(U_{ix}) < \frac{1}{4e}$, diam $\phi(U_{ij}) < \frac{1}{4e}$, and diam $\phi(U_{kz}) < \frac{1}{4e}$. Consequently, $(\cap D_{(e_{y_i})_q}) \cap (\cap D_{(e_{y_k})_s}) \ne \emptyset$ by the definition of $e$. Observe that $D_{(d_{y_i})_t}$ is the collection of maximal cardinality $d_{y_i}$ that contains $D_{e(y_i)_q}$ with the smallest subscript $t$ as all those collections of cardinality $d_{y_i}$ are ordered) and $D_{(d_{y_k})_r}$ is the collection of maximal cardinality $d_{y_k}$ that contains $D_{(e_{y_k})_s}$ with smallest subscript $r$.

Since $(\cap D_{(e_{y_i})_q}) \cap (D_{(e_{y_k})_s}) \neq \emptyset$, $D_{(d_{y_i})_t} \supset D_{(e_{y_k})_s}$, and $D_{(d_{y_k})_r} \supset D_{(e_{y_i})_q}$, it follows that $d_{y_i} = d_{y_k}$ and $t = r$.

Clearly, $\hat{V}(y_i) = \hat{V}(y_k)$ by the definitions of $\hat{V}(y_i)$ and $\hat{V}(y_k)$. Consequently, $s_i = s_k$ contrary to the assumption above.

Case C: $y_i \notin B$ and $y_k \in B$.

Note that if $y_i \notin B$, then $U_{ij} \notin H$ and if $y_k \in B$, then $U_{kt} \in H$. Thus, $U_{ij} \neq U_{kt}$.

It should be clear that $\alpha_1$ and $\beta_1$ are well defined.

Clearly, $\beta_1$ is defined on $U^1$ since $U^1$ is irreducible and $V^2$ refines $U^1$. Observe that $\pi_1$ maps distinguished families onto distinguished families.

## The Dimension of $Y$ is $n$

For each $i$, let $Z_{s_i}$ denote the union of all $F_i^2$, distinguished families in $V^2$, such that $F_{s_i}^1$ is chosen for $F_i^2$ as described above. That is, $\pi_1(F_i^2) = F_{s_i}^1$. Observe that $\phi^{-1}\phi(Z_{s_i}) = Z_{s_i}$.

Claim: The order $\{Z_{s_i}\}_{i=1}^{\text{card } R^2} \leq n+1$. Let $x \in \bigcap_{j=1}^{m} Z_{s_{i_j}}$ where $s_{i_j} \neq s_{i_k}$, $k \neq j$. Now, for each $i_j$, $x \in f_{i_j t_j}^2 \subset U_{i_j t_j} = \alpha_1(f_{i_j t_j}^2)$. By the well definedness of $\beta_1$, $U_{i_j t_j} \neq U_{i_k t_k}$ for $j \neq k$. Since $U_{i_j t_j} \in U^1$ and order $U^1 \leq n+1$, it follows that $m \leq n+1$ and order $\{Z_{s_i}\}_{i=1}^{\text{card } R^2} \leq n+1$. Thus, $Z = \{\phi(Z_{s_i})\}_{i=1}^{\text{card } R^2}$ is a finite open covering of $Y$ of order $\leq n+1$. Clearly, if $G$ is any finite open covering of $Y$, then there is $R = R^1$ as defined above which covers $Y$ and refines $G$. Hence, $Z$ refines $G$ and order $Z \leq n+1$. Thus, dim $Y \leq n$.

It is not difficult to show that if a $p$-adic group $A_p$ acts effectively on an $n$-manifold $M$, then the orbit mapping $\phi : M \to M/A_p$ is light open and closed. Use the fact that there is a sequence $A_p = H_0 \supset H_1 \supset H_2 \supset \cdots$ of open (and closed) subgroups of $A_p$ which closes down on the identity $e$ of $A_p$ such that when $j > i$, $H_i/H_j$ is a cyclic group of order $p^{j-i}$ and $A_p/H_i$ is a cyclic group of order $p^i$. The cyclic group $A_p/H_i$ acts effectively on $M/H_i$ with orbit space $M/A_p$.

The following theorem is a consequence of the argument above.

**Theorem.** *If a $p$-adic group $A_p$ acts effectively on a compact connected $n$-manifold, then the orbit mapping $\phi : M \to M/A_p = Y$ is a light open mapping and* dim $Y = n$.

## The Hilbert-Smith Conjecture

The Hilbert-Smith Conjecture states that if $G$ is a locally compact group which acts effectively on a connected manifold as a topological transformation group, then $G$ is a Lie group.

It is well known that if a locally compact group $G$ acts effectively on a connected $n$-manifold $M$ and $G$ is not a Lie group [6], then there is a subgroup $H$ of $G$ isomorphic to a $p$-adic group $A_p$ which acts effectively on $M$. Thus, the Hilbert-Smith Conjecture can be established by proving that there is no effective action by a $p$-adic group $A_p$ on a connected $n$-manifold $M$.

C.T. Yang [7] has shown that if a $p$-adic group $A_p$ acts effectively on a compact $n$-manifold $M$, then the dimension of the orbit space $M/A_p = Y$ is $n+2$ or infinity. This contradicts the work in this paper. Hence, there is no such action and the Hilbert-Smith Conjecture is true.

## References


1. Bing, R.H. and Floyd, E.E., *Coverings with connected intersections*, TAMS **69** (1950), 387-391.

2. Browder, F., editor, *Mathematical development arising from Hilbert Problems*, Northern Ill. Univ., 1974, XXVIII, Parts I and II, Proceedings of Symposia in Pure Mathematics.

3. Jones, F.B., *On the Existence of A Small Connected Open Set With A Connected Boundary*, BAMS **68** (1962), 117-119.

4. Nagata, J., *Modern Dimension Theory*, revised and extended edition, Heldermann Verlag, Berlin, 1983.

5. Whyburn, G.T., *Analytic Topology*, AMS Colloq. Publications **28** (1942).

6. Wilder, R.L., *Topology of Manifolds*, AMS Colloquium Publications, Vol. **32** (1949).

7. Yang, C.T., *Hilbert's Fifth Problem and related problems on transformation groups*, Proceedings of Symposia in Pure Mathematics; *Mathematical development arising from Hilbert Problems*, Northern Illinois University, 1974, edited by F. Browder, XXVIII, Part I, 1976, pp. 142-164.

8. ———, *p-adic transformation groups*, Mich. Math. J. **7** (1960), 201-218.

9. Whyburn, G.T., *Topological Analysis*, Princeton University Press, 1958.

10. Nagami, K., *Dimension Theory*, Academic Press, 1970.