A Proof Of The Hilbert-Smith Conjecture

by

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Dedicated to the memory of Deane Montgomery

Abstract

The Hilbert-Smith Conjecture states that if G is a locally compact group which acts effectively on a connected manifold as a topological transformation group, then G is a Lie group. A rather straightforward proof of this conjecture is given. The motivation is work of Chernavsky ("Finite-to-one mappings of manifolds", *Trans. of Math.* Sk. 65 (107), 1964.). His work is generalized to the orbit map of an effective action of a p-adic group on compact connected n-manifolds with the aid of some new ideas. There is no attempt to use Smith Theory even though there may be similarities. Smith's exact sequences are not used.

1. *Introduction.*

In 1900, Hilbert proposed twenty-three problems [8]. For an excellent discussion concerning these problems, see the *Proceedings of Symposia In Pure Mathematics* concerning "Mathematical Developments Arising From Hilbert Problems" [3]. The abstract by C.T. Yang [22] gives a review of Hilbert's Fifth Problem "*How is Lie's concept of continuous groups of transformations of manifolds approachable in our investigation without the assumption of differentiability*?" Work of von Neumann [40] in 1933 showed that differentiability is not completely dispensable. This with results of Pontryagin [35] in 1939 suggested the specialized version of Hilbert's problem: *If G is a topological group and a topological manifold, then is G topologically isomorphic to a Lie group*? This is generally regarded as Hilbert's Fifth Problem. The first partial result was given by Brouwer [26] in 1909-1910 for locally euclidean groups of dimension ≤ 2. The best known partial results were given for compact locally euclidean groups and for commutative locally euclidean groups by von Neumann [40] and Pontryagin [35], respectively.

In 1952, the work of Gleason [5] and Montgomery-Zippin [13] proved: *Every locally euclidean group is a Lie group.* This solved Hilbert's Fifth Problem.

A more general version of Hilbert's Fifth Problem is the following:

If G is a locally compact group which acts effectively on a connected manifold as a topological transformation group, then is G a Lie group? The Hilbert-Smith Conjecture states that the answer is yes.

Papers of Montgomery [34] in 1945 and Bochner-Montgomery [1] in 1946 established the partial result: *Let G be a locally compact group which acts effectively on a differentiable manifold M such that for any $g \in G$, $x \mapsto gx$ is a differentiable transformation of M. Then G is a Lie group and (G, M) is a differentiable transformation group.* Another partial result was given by a theorem of Yamabe [43] and a theorem of Newman [15] as follows: *If G is a compact group which acts effectively on a manifold and every element of G is of finite order, then G is a finite group.*

It has been shown [14] that an affirmative answer to the generalized version of Hilbert's Fifth Problem is equivalent to a negative answer to the following: *Does there exist an effective action of a p-adic group on a manifold*?

It is proved here that the answer to this question is No. Thus, the *Hilbert-Smith Conjecture* is true, i.e., *A locally compact group acting effectively on a connected n-manifold must be a Lie group.*

A brief review of some of the consequences of efforts to solve this problem is given below. There are examples in the literature of *effective* actions of an infinite compact 0-dimensional topological group G (*each $g \in G$-$\{$identity$\}$ moves some point*) on locally connected continua. The classic example of Kolmogoroff [29] in 1937, is one where G operates effectively but not *strongly effectively* [24] on a 1-dimensional locally connected continuum (Peano continuum) such that the orbit space is 2-dimensional. In 1957, R.D. Anderson [24] proved that *any* compact 0-dimensional topological group G can act strongly effectively as a transformation group on the (Menger) universal 1-dimensional curve M such that either (1) the orbit space is homeomorphic to M or (2) the orbit space is homeomorphic to a regular curve. A related result is the example of Keldys [55] of a one-dimensional continuum with a zero-dimension and open mapping onto the square.

The motivation for the proof of The Hilbert-Smith Conjecture is the remarkable work of A.V. Chernavsky [27]. He proved that if f is a finite-to-one open and closed mapping on a connected metric manifold M^n onto a Hausdorff space Y, then

(1) there is a natural number k so that for each $x \in M^n$, the cardinality of $f^{-1}f(x) \leq k$ (bounded multiplicity).

(2) the elements of maximal multiplicity form a dense open set in M^n, and

(3) for each open set U of M^n, there is $\epsilon > 0$ such that if f is any finite-to-one open and closed mapping of M^n onto some Hausdorff space Y and f is not a homeomorphism, then for some $x \in U$, diam $f^{-1}f(x) \geq \epsilon$ (Newman's Property).

In 1960, C.T. Yang [44] proved that if a p-adic group, A_p, acts effectively as a transformation group on X (a locally compact Hausdorff space of homology dimension not greater than n), then the homology dimension of the orbit space X/A_p is not greater than $n+3$. If X is an n-manifold, then the homology dimension of X/A_p is $n+2$. If A_p acts strongly effectively (freely) on an n-manifold X, then the dimension of X/A_p is either $n+2$ or infinity. At about the same time (1961), Bredon, Raymond, and Williams [25] proved the same results using different methods. There are, of course, actions by p-adic groups on p-adic solenoids and actions by p-adic solenoids on certain spaces. See [25] for some of these results.

In 1961, Frank Raymond published the results of his study of the orbit space M/A_p *assuming* an effective action by A_p (as a transformation group) on an n-manifold M. Later (1967), Raymond [38] published work on two problems in the theory of generalized manifolds which are related to the (generalized) Hilbert Fifth Problem.

In 1963, Raymond and Williams [39] gave examples of compact metric spaces X^n of dimension n and an action by a p-adic group, A_p, on X^n such that dim $X^n/A_p = n+2$. Work related to and used in [39] is the paper [41] by Williams. In [41], Williams answers a question of Anderson [24; p. 799] by giving a free action by a compact 0-dimensional group G on a 1-dimensional Peano continuum P with dim $P/G = 2$.

In 1976, I described [32; 33] what I called p-adic polyhedra which admit periodic homeomorphisms of period p. Proper inverse systems $\{P_i, \phi_i\}$ of p-adic n_i-polyhedra have the property that the inverse limit $X = \varprojlim P_i$ admits a free action by a p-adic group.

In 1980, one of my students, Alan J. Coppola [28] generalized results of C.T. Yang [44] which involve homologically analyzing p-adic actions. Coppola formalized these so that homological calculations could be done in a more algorithmic manner. He defined a p-adic transfer homomorphism and used it to produce all of the relevant Smith-Yang exact sequences which are used to homologically analyze Z_{p^r}-actions on compact metric spaces. Coppola studied p-adic actions on homologically uncomplicated spaces. In particular, he proved that if X is a compact metric A_p-space of homological dimension no greater than n and X is homologically locally connected, then the $(n+3)$-homology of any closed subset $A \subset X/A_p$ vanishes.

In 1983, Robinson and I proved Newman's Theorem for finite-to-one open and closed mappings on manifolds [10]. We formalized Newman's Property (and variations) and studied this property for discrete open and closed mappings on generalized continua in 1984 [11].

In 1985, H-T Ku, M-C Ku, and Larry Mann investigated in [30] the connections between

Newman's Theorem involving the size of orbits of group actions on manifolds and the Hilbert-Smith Conjecture. They establish Newman's Theorem (Newman's Property [11]) for actions of compact *connected* non-Lie groups such as the p-adic solenoid.

In 1997, D. Repovš and E.V. Ščepin [51] gave a proof of the Hilbert-Smith Conjecture for actions by Lipschitz maps. See also related work by Shchepin [52]. In the same year, Iozke Maleshick [53] proved the Hilbert-Smith conjecture for Hölder actions.

In 1999, Gaven J. Martin [54] announced a proof of The Hilbert-Smith Conjecture for quasiconformal actions on Riemannian manifolds and related spaces.

The crucial idea that works here is M.H.A. Newman's idea used in his proof that for a given compact connected n-manifold M, there is an $\epsilon > 0$ such that if h is any periodic homeomorphism of period p, a prime > 1, of M onto itself, then there is some $x \in M$ such that the orbit of x, $\{x, h(x), \ldots, h^{p-1}(x)\}$, has diameter $\geq \epsilon$. It is well known that the collection of orbits under the action of a transformation group G on a compact Hausdorff space X is a *continuous decomposition* of X.

The works [20; 21] of David Wilson and John Walsh [18] show that there exist continuous decompositions of n-manifolds M^n, $n \geq 3$, into Cantor sets. This paper shows that such decompositions can not be equivalent to those induced by any action of a p-adic transformation group A_p acting on M^n.

I owe a special debt of gratitude to Patricia Tulley McAuley who has been extremely helpful in reading drafts of numerous attempts to solve this problem and who has provided helpful insights with regard to Čech homology. I am not only most deeply indebted to the work of A.V. Chernavsky [27] but also extremely grateful to him for the numerous important and helpful suggestions which he made after a careful reading of a manuscript and in stimulating conversations with him on a visit to Istanbul Bilgi University in January, 2002. I also wish to thank Eric Robinson, David C. Wilson, and John J. Walsh for having read earlier attempts some of which worked for $n = 3$.

OUTLINE OF THE PROOF

It is well known that if a locally compact group G acts effectively on a connected n-manifold M and G is not a Lie group, then there is a subgroup H of G isomorphic to a p-adic group A_p which acts effectively on M. Thus, the Hilbert-Smith Conjecture can be established by proving that there is no effective action by a p-adic group A_p on a connected n-manifold M. The conjecture is proved by the following theorem. As seen later, there is no loss of generality, in assuming that M is compact, orientable, and without boundary. The proof given below can be adopted to the situation where M is a *locally compact*, orientable, and without boundary by replacing finite open coverings of M and M/A_p with locally finite open coverings with the same properties since the orbit map $\phi : M \to M/A_p$ is open, closed, and proper.

Definition. An n-manifold (M, d) is said to have Newman's Property w.r.t. the class $L(M, p)$ (as stated below) iff there is $\epsilon > 0$ such that for any $\phi \in L(M; p)$, there is some $x \in M$ such that diam $\phi^{-1}\phi(x) \geq \epsilon$ using the metric d on M.

Theorem. *If $L(M, p)$ is the class of all orbit mappings $\phi : M \to M/A_p$ where A_p acts effectively on a compact, connected, and orientable n-manifold M and each $h \in A_p$ is homotopic to the identity (hence, h has degree one and preserves the orientation of M), then M has Newman's Property w.r.t. $L(M, p)$.*

It is well known that M does not have Newman's Property w.r.t. $L(M, p)$. Hence, $L(M, p) = \emptyset$ and the Hilbert-Smith Conjecture is true.

Lemma 2. *(A consequence of a Theorem of Floyd [4].) Suppose that M is a compact connected orientable n-manifold. There is a finite open covering W_1 of M such that (1) order $W_1 = n+1$ and (2) there is a finite open refinement W_2 of W_1 which covers M such that if W is any finite open covering of M refining W_2, then $\pi_{W_1} : \check{H}_n(M) \to H_n(W_1)$ maps $\check{H}_n(M)$ isomorphically onto the image of the projection $\pi_{WW_1} : H_n(W) \to H_n(W_1)$.*

[Here, if U is either a finite or locally finite open covering of M, then $H_n(U)$ is the n^{th} simplicial homology group of the nerve $N(U)$ of U. The coefficient group is always Z_p and $\check{H}_n(M)$ denotes the n^{th} Čech homology of M.]

Now, choose $U = W_1$, and a finite open covering W of M which star refines W_2 where W_1 and W_2 satisfy Lemma 2.

Let ϵ be the Lebesque number of W_2. Choose $\phi \in L(M,p)$ such that diam $\phi^{-1}\phi(x) < \epsilon$ for each $x \in M$. Construct the sequences of coverings $\{V^m\}$ and $\{U^m\}$ as in Lemma 5 below where V^1 star refines W_2 and order $U^m = n+1$ along with projections α_m, β_m, and π_m yielding the following commutative diagram:

$$\begin{array}{ccccccccccc}
\leftarrow & H_n(V^m) & \xleftarrow{\beta_m^*} & H_n(U^m) & \xleftarrow{\alpha_m^*} & H_n(V^{m+1}) & \xleftarrow{\beta_{m+1}^*} & H_n(U^{m+1}) & & \leftarrow & \\
& \nu_m \uparrow & \beta_m^* \swarrow & & \nwarrow \alpha_m^* & \nu_{m+1} \uparrow & \beta_{m+1}^* \swarrow & & & \nwarrow \alpha_{m+1}^* & \\
\leftarrow & H_n(V_n^m) & & \xleftarrow{\pi_m^*} & & H_n(V_n^{m+1}) & & \xleftarrow{\pi_{m+1}^*} & & H_n(V_n^{m+2}) & \leftarrow
\end{array}$$

Here ν_m is the natural map of an n-cycle in $H_n(V_n^m)$, the n^{th} simplicial homology group of the n-skeleton of the nerve $N(V^m)$ of V^m, into its homology class in $H_n(V^m)$. The other maps are those induced by the projections α_m, β_m and π_m. The upper sequence, of course, yields the Čech homology group $\check{H}_n(M)$ as its inverse limit. Furthermore, it can be easily shown, using the diagram, that $\check{H}_n(M)$ is isomorphic to the inverse limit $G = \varprojlim H_n(V_n^m)$, of the lower sequence. Specifically, $\gamma : \check{H}_n(M) \to G$ defined by $\gamma(\Delta) = \{\beta_m^*(\pi_{U_m}(\Delta))\}$ is an isomorphism of $\check{H}_n(M)$ onto G. We shall use the isomorphism in what follows and for convenience we shall let $\gamma(\Delta) = \{z_m^n(\Delta)\}$, i.e. $z_m^n(\Delta) = \beta_m^*(\pi_{U^m}(\Delta)) \in H_n(V_n^m)$. The group $\varprojlim H_n(V_n^m)$ is used because the operator σ (introduced later) is applied to actual n-cycles rather than to elements of a homology class. This is the reason for the sequence $\{U^m\}$.

An operator σ_m is defined on the n chains of $N(V^{m+1})$ for each m. The operator σ_m maps n-cycles to n-cycles and commutes with the projections $\pi_m^* : H_n(V_n^{m+1}) \to H_n(V_n^m)$

and, hence, induces an automorphism on $\pi_{V^{m+1}}(\check{H}_n(M)) \subset H_n(V_n^{m+1})$. See Lemmas 6 and 7.

Distinguished families of n-simplices in $N(V^m)$ are defined. Now, let $z_m = z_m^n(\Delta)$ where Δ is the generator of $G \cong Z_p$. For each n-simplex δ^n in $z_m(\Delta)$, there is a unique distinguished family S_j^m of n-simplices in $N(V^m)$ which contains δ^n. If C_j is the collection of all n-simplices in $z_m(\Delta)$ which are in S_j^m, then the sum of the coefficients of those members of C_j (as they appear in $z_m(\Delta)$) is 0 mod p. Take the projection $\pi_{V^m U}$ from V^m to $U = W_1$. Hence, $\pi_{V^m U}$ has the property that all members of a distinguished family S_j^m of n-simplices in $N(V^m)$ project to the same simplex in $N(U_1)$. Thus, the projection of those members of $z_m(\Delta)$ which are in S_j^m project to the same simplex δ_j in $N(U_1)$ and the coefficient of δ_j is 0 mod p. Thus, $\pi_{V^m U} : H_n(V_n^m) \to H_n(U)$ takes the nontrivial n-cycle $z_m(\Delta)$ to the $0-n$ cycle mod p. This violates the conclusion of Lemma 2. Thus, M has Newman's Property w.r.t. the class $L(M,p)$. Hence, ϵ is a Newman's number and the Theorem is proved.

It is well known that if A_p acts effectively on a compact connected n-manifold M, then given any $\epsilon > 0$, there is an effective action of A_p on M such that diam $\phi^{-1}\phi(x) < \epsilon$ for each $x \in M$. That is, M fails to have Newman's property w.r.t. the class $L(M,p)$. It follows that A_p can not act effectively on a compact connected n-manifold M. *Consequently, the Hilbert-Smith Conjecture is true.*

Details of the proof follow.

2. *Some properties of the orbit mapping of an effective action by A_p on a compact connected orientable n-manifold M.*

Suppose that ϕ is the orbit mapping of a p-adic group A_p acting effectively as a transformation group on an orientable n-manifold $M^n = M$ where p is a prime larger than 1. By [12; 21], there is a sequence $A_p = H_0 \supset H_1 \supset H_2 \supset \ldots$ of open (and closed) subgroups of A_p which closes down on the identity e of A_p such that when $j > i$, H_i/H_j is a cyclic group of order p^{j-i}. Let $h_{ij} : A_p/H_j \to A_p/H_i$ and $h_i : A_p \to A_p/H_i$ be homomorphisms induced by the inclusion homomorphisms (quotient homomorphisms) on A_p and A_p/H_j. Then $\{A_p/H_i; h_{ij}\}$ is an inverse system and $\{h_i\}$ gives an isomorphism of A_p onto $\varprojlim A_p/H_i$. Now, let $a \in A_p - H_i$. For each natural number i, let a_i be the coset aH_i in A_p/H_i. Then a_i is a periodic homeomorphism of M/H_i onto M/H_i with a_i^q being the identity mapping where $q = p^i$ is the period of a_i. Consequently, H_i acts as a transformation group on M and A_p/H_i acts as a cyclic transformation group on M/H_i.

As above, let $\{H_i\}$ be a sequence of open (and closed) subgroups of A_p such that (a) $H_i \supset H_{i+1}$ for each i, (b) if $j \geq i$, then H_i/H_j is a cyclic group of order p^{j-i}, and (c) A_p/H_i is a cyclic group of order p^i. Since A_p acts effectively on M (a compact connected n-manifold), the cyclic group A_p/H_i acts effectively on M/H_i with orbit space M/A_p.

It follows that if $\epsilon > 0$, then there is a natural number j such that $H_j \cong A_p$ acts effectively on M such that if $\phi_j : M \to M/H_j$ is the orbit map of the action, then diam $\phi^{-1}\phi(x) < \epsilon$ for each $x \in M$. Observe that if H is a non empty open and closed subgroup of A_p, then for some i, $H = H_i$.

Observe that if A_p acts effectively as a transformation group on M where M is an orientable connected metric n-manifold, then some orbit is infinite and not discrete. This follows from a Theorem of Chernavsky [27] that if $\phi : M \to Y$ (Hausdorff) is a discrete open and closed (continuous) mapping, then there is a natural number k such that cardinality of $\phi^{-1}\phi(x) \leq k$ for each $x \in M$ (bounded multiplicity). Furthermore, the union of the orbits of maximal cardinality is a dense open set W in M. The stability group of all the points in W is a certain H_v and, thus, H_v acts as the identity on M. Consequently, the action of A_p is not effective contrary to the hypothesis. Thus, the orbit mapping $\phi : M \to M/A_p$ where A_p acts effectively on M is not a discrete open and closed mapping. Hence, some orbit is infinite and not discrete.

The following lemma is crucial to defining certain coverings of M with distinguished families of open sets.

Lemma 1. *Suppose that ϕ is the orbit mapping $\phi : M \to M/A_p$ where A_p acts effectively on a compact orientable connected n-manifold M and each $h \in A_p$ is homotopic to the identity (hence, h has degree one and preserves the orientation of M). For each $z \in M/A_p - \phi(F_\phi)$ and $\epsilon > 0$ where $F_\phi = \{x \mid \phi^{-1}\phi(x) = x\}$, there is a connected open set U such that*

(1) diam $U < \epsilon$,

(2) $z \in U$, and

(3) $\phi^{-1}(U) = \bigcup_{i=1}^{p^s} U_i$ where s is a natural number such that

(a) U_i is a component of $\phi^{-1}(U)$ for each i, $1 \leq i \leq p^s$,

(b) $\bar{U}_i \cap \bar{U}_j = \emptyset$ for $i \neq j$,

(c) $\phi(U_i) = U$ for each i, and

(d) U_1 is homeomorphic to U_j for each j, $1 < j \leq p^s$ (by maps compatible with the projections $\phi \mid U_j$). [The homeomorphism taking U_1 to U_j is a power of a fixed element $g \in A_p - H_1$ where $\{H_i\}$ closes down on the identity in A_p and g preserves the orientation of M. This g is used in Lemma 6.]

Proof. Since $z \in M/A_p - \phi(F_\phi)$, $\phi^{-1}(z)$ is non degenerate. It follows from Whyburn's theory of open mappings and light open mappings [19] that for $\epsilon > 0$, there is a connected open set U such that (1) diam $U < \epsilon$, (2) $z \in U$, and (3) $\phi^{-1}(U)$ consists of a finite number (larger than one) of components $U_1, U_2, \cdots, U_m$ such that $\bar{U}_i \cap \bar{U}_j = \emptyset$ for $i \neq j$, and $\phi(U_i) = U$ for each i.

For each U_j, a component of $\phi^{-1}(U)$, there is an open and closed subgroup G_j of A_p which is the largest subgroup of A_p which leaves U_j invariant and the map induced by ϕ maps U_j/G_j onto U. Since G_j is a normal subgroup of A_p, $G_i = G_j$ for each i and j.

Furthermore, A_p/G_j is a cyclic group of order p^s where s is a natural number. There are p^s pairwise distinct components of $\phi^{-1}(U)$. (See [36: Lemma 2]). It follows that $G_1 = H_i$ for some i where $\{H_i\}$ is the sequence of open and closed subgroups of A_p which closes down on the identity $e \in A_p$ (mentioned above) and $s = i$. Let $a \in A_p - H_i$ such that aH_i generates the cyclic group A_p/H_i. For each natural number i, let a_i be the coset aH_i in A_p/H_i. Thus, a_i is a periodic homeomorphism of M/H_i onto M/H_i with $a_i^q = e$ where $q = p^i$ is the period of a_i.

Let $f_i : M \to M/H_i$ be the orbit map of the action of H_i on M and $g_i : M/H_i \to M/A_p$ be the orbit map of the action of the cyclic group A_p/H_i on M/H_i. That is, $\phi = g_i f_i$. There are p^i cosets $\{v_m H_i\}_{m=1}^{p^i}$ where $v_1 = e$ (the identity) such that for each $x \in U_1$,

(1) $\phi^{-1}\phi(x) = \bigcup_{m=1}^{p^i} v_m H_i(x)$ where $v_m H_i(x) = \{h(x) \mid h \in v_m H_i\}$,

(2) $v_m H_i(x) \in U_m/H_i$,

(3) v_m is an orientation preserving homeomorphism of M onto M, and

(4) if $A_p/H_i = \{a_i, a_i^1, a_i^2, \cdots, a_i^{p^i - 1}\}$ (a cyclic group), then there are elements $k_1, k_2, \cdots, k_{p^i}$ where $k_1 = e$ such that

   (a) $k_m(\pi_i(x)) = \pi_i(v_m H_i(x))$ where π_i maps $\phi^{-1}(U)$ onto $\phi^{-1}(U)/H_i$ and

   (b) k_m maps U_1/H_i homeomorphically onto U_m/H_i with $k_m = v_m H_i \in A_p/H_i$ which is a homeomorphism of M/H_i onto M/H_i.

Thus, $v_m(x) \in U_m$. Let $z \in U_m$. Hence, $\pi_i(z) \in U_m/H_i$ and $(v_m H_i)^{-1}(\pi_i(z)) = v_m^{-1} H_i(\pi_i(z)) \in U_1/H_i$. Consequently, $v_m^{-1} H_i(\pi_i(z)) = \pi_i(v_m^{-1}(z)) \in U_1/H_i$ which implies that $v_m^{-1}(z) \in U_1$. Finally, $v_m(v_m^{-1}(z)) = z$ and v_m maps U_1 homeomorphically onto U_m.

Lemma 1 is proved.

3. *Special coverings, distinguished families, and distinguished subfamilies.*

Let $L(M,p) = \{\phi \mid \phi$ is the orbit mapping of an effective action of a p-adic group A_p (p a prime with $p > 1$) on a compact connected metric orientable n-manifold without boundary, $\phi : M \to M/A_p\}$. For each $\phi \in L(M,p)$, let $F_\phi = \{x \mid x \in M$ and $\phi^{-1}\phi(x) = x\}$, the fixed set of the action A_p. It would simplify the proof of lemmas which follow to know that M is triangulable. Without this knowledge, a theorem of E.E. Floyd is used.

Notation. Throughout this paper, $\check{H}_n(X)$ will denote the n^{th} Čech homology group of X with coefficients in Z_p, the integers mod p, p a fixed prime larger than 1. Also, $H_n(K)$ will denote the n^{th} simplicial homology of a finite simplicial complex K, with coefficients in Z_p. If U is a finite open covering of a space X, then $N(U)$ denotes the nerve of U, $H_n(U)$ is the n^{th} simplicial homology group of $N(U)$, and π_U the usual projection homomorphism $\pi_U : \check{H}_n(X) \to H_n(U)$.

Definition. If f is a mapping of M onto Y, then an open covering U of M is said to be a *saturated open covering* (more precisely, saturated w.r.t. f) iff for each $u \in U$, $f^{-1}f(u) = u$. That is, u is an open inverse set.

The next lemma follows.

Lemma 2. *Suppose that M is a compact connected metric n-manifold. There is a finite open covering W_1 of M such that*

(1) order $W_1 = n + 1$ and

(2) there is a finite open refinement W_2 of W_1 which covers M such that if W is any finite open covering of M refining W_2, then $\pi_{W_1} : \check{H}_n(M) \to H_n(W_1)$ maps $\check{H}_n(M)$ isomorphically onto the image of the projection $\pi_{WW_1} : H_n(W) \to H_n(W_1)$.

Proof. Adapt Theorem (3.3) of [4] to the situation here and use (2.5) of [4].

If M is triangulable, then there is a sufficiently fine triangulation T such that if U consists of the open stars of the vertices of T, then $\pi_U : \check{H}_n(M) \to H_n(U)$ is an isomorphism onto (where, of course, $\check{H}_n(M) \cong Z_p$).

Standing Hypothesis: In the following, M is a compact, connected, and orientable metric n-manifold. Also, $L(M,p)$ is the class of all orbit mappings $\phi : M \to M/A_p$ where A_p acts effectively on M and each $h \in A_p$ is homotopic to the identity (hence, h has degree one and preserves the orientation of M). The finite open coverings W_1 and W_2 which satisfy Lemma 2 will be used in certain lemmas and constructions which follow. Suppose also that $Y = M/A_p$ has a countable basis $Q = \{B_i\}_{i=1}^{\infty}$ such that (a) for each i, B_i is connected and uniformly locally connected and (b) if H is any subcollection of Q and $\bigcap_{h \in H} h \neq \emptyset$, then $\bigcap_{h \in H} h$ is connected and uniformly locally connected (a consequence of a theorem due to Bing and Floyd [50]). All of the open sets used in Y below to construct coverings R^n of Y not related to $\phi(F)$ are in Q.

Lemma 3. *Suppose that $\phi \in L(M,p)$ and $F = F_\phi$. Then there is a finite open irreducible covering W_F which covers F such that*

(1) W_F star refines W_2 (as in Lemma 2) and if $w \in W_F$, then $\phi(w) \in Q$,

(2) order $W_F \leq n+1$,

(3) W_F is a saturated open covering of F,

(4) if $BdF = \partial F = F -$ interior F and $W_{\partial F} = \{w \mid w \in W_F$ and $w \cap B \neq \emptyset\}$, then order $W_{\partial F} \leq n$,

(5) for $w \in W_F$ such that $w \cap$ int $F \neq \emptyset$, then either $w \in W_{\partial F}$ or int $F \supset \bar{w}$, and

(6) if δ^n is an n-simplex in the nerve, $N(W_F)$, the nerve of W_F, then the nucleous, $N[\delta^n]$, of δ^n lies in int F (the interior of F), indeed, int $F \supset \overline{N[\delta^n]}$.

NOTE: An n-simplex is $\{u_0, u_1, \cdots, u_n\} = \delta^n$ where $u_i \in W_F$ and $N[\delta^n] = \bigcap_{j=0}^{n} u_j$.

Proof. If int $F = \emptyset$, then dim $F \leq n-1$ and there is a finite irreducible open covering W'_F of F such that order $W'_F \leq n$ and has properties (1), (4), (5), and (6). For each $w' \in W'_F$, let $w =$ union of all $\phi^{-1}\phi(x)$ such that $x \in w'$ and $w' \supset \phi^{-1}\phi(x)$. It follows that w is open [19] since ϕ is open and closed. Hence, $W_F = \{w \mid w' \in W'_F\}$ satisfies (1) - (6).

Assume that int $F \neq \emptyset$. Cover $\partial F = BdF$ with a finite irreducible collection C of open saturated sets such that order $C \leq n$ and C star refines W_2. Cover $F - C^*$ (C^* denotes the union of the elements of C) with a finite irreducible collection D of open saturated sets such that order $D \leq n+1$, D star refines W_2, and if $d \in D$, then int $F \supset \bar{d}$. Let $C = \{c_1, c_2, \cdots, c_s\}$ and $D = \{d_1, d_2, \cdots, d_t\}$. By a Theorem [47, p. 158, use the corresponding theorem for compact metric spaces], there is an open covering $V = \{v_i \mid 1 \leq i \leq s+t\}$ such that order $V \leq n+1$, $c_i \supset v_i$ for $1 \leq i \leq s$, and $d_i \supset v_i$ for $1 \leq i \leq t$.

Let $W_F = \{w_i \mid w_i$ is the union of all $\phi^{-1}\phi(x)$ such that $x \in v_i$, $1 \leq i \leq s+t$, and $v_i \supset \phi^{-1}\phi(x)\}$. Clearly, W_F satisfies (1)-(6). The lemma is proved.

A proof of the necessity of the Theorem in 47 for compact metric spaces. A compact metric space (X, d) has dimension $\leq n$ if and only if for each finite irreducible open covering $U_1, U_2, \cdots, U_k$, there is a finite open covering $V_1, V_2, \cdots, V_k$ such that (1) $U_i \supset V_i$ for each i, $1 \leq i \leq k$, and (2) order $\{V_1, V_2, \cdots, V_k\} \leq n+1$.

Proof of the necessity. Suppose that dim $X \leq n$. Given $U = \{U_1, U_2, \cdots, U_k\}$, a finite irreducible covering of X. By a Theorem in 47, there is a finite irreducible open refinement $G = \{G_1, G_2, \cdots, G_m\}$ such that (1) for each i, there is some j such that $U_i \supset G_j$ and $U_t \not\supset G_j$ for any $t \neq i$ and (2) order $G \leq n+1$. Let V_1 be the union of all G_i such that $U_1 \supset G_i$ and let V_i be the union of all G_j such that $U_i \supset G_j$ and $U_t \not\supset G_j$ for $1 \leq t < i$, that is, $V_t \not\supset G_j$. It follows that order $V \leq n+1$ where $V = \{V_1, V_2, \cdots, V_k\}$. Clearly, $U_i \supset V_i$ for $1 \leq i \leq k$.

Next, we extend W_F to a *special covering* V of M (defined below) by covering $M - \bigcup_{w \in W_F} w$ in a special way. Recall The Standing Hypothesis. We use the following lemma.

Lemma 4. *Suppose that* $\phi \in L(M,p)$. *Then there exists a finite open covering* R *of* $Y = M/A_p$ *such that*

(1) *if* $y \in \phi(F)$ *where* $F = F_\phi$, *then there is* $r \in R$ *such that* $y \in r = \phi(w)$ *for some* $w \in W_F$, *(as described in Lemma 3),*

(2) *if* $\phi(W_F) = \bigcup_{w \in W_F} \phi(w)$ *and* $y \in Y - \phi(W_F)$, *then there is* $r \in R$ *such that* $y \in r$, $r \in Q$ *where* Q *is the basis in The Standing Hypothesis,* $\bar{r} \cap \phi(F) = \emptyset$, $\phi^{-1}(r) = r_1 \cup r_2 \cup \cdots \cup r_q$, $q = p^t$ *for some natural number* t, *such that for each* $i = 1, 2, \ldots, q$, r_i *is a component of* $\phi^{-1}(r)$, r_i *maps onto* r *under* ϕ, $\overline{r_i} \cap \overline{r_j} = \emptyset$ *for* $i \neq j$, *and* r_i *is homeomorphic to* r_j *for each* i *and* j *with a homeomorphism compatible with the projection* ϕ *(indeed, there is an orientation preserving homeomorphism, an element of* A_p, *which takes* r_i *onto* r_j*),*

(3) R *is irreducible, and*

(4) *if* $r_x \in R$ *and* $r_x \neq \phi(w)$ *for any* $w \in W_F$, $r_y \in R$, $r_x \cap r_y \neq \emptyset$, $\phi^{-1}(r_x)$ *consists of exactly* p^{m_x} *components,* $\phi^{-1}(r_y)$ *consists of exactly* p^{m_y} *components, and* $m_x \geq m_y$, *then each component of* $\phi^{-1}(r_y)$ *meets exactly* $p^{m_x - m_y}$ *components of* $\phi^{-1}(r_x)$.

Proof. Obtain W_F using Lemma 3. Since $Y - \phi(W_F)$ is compact, use Lemma 1 to obtain a finite irreducible covering R' of $Y - \phi(W_F)$ of sets r satisfying the conditions of the lemma such that R' star refines $\{\phi(u) \mid u \in W_2\}$. Property (4) of the conclusion of Lemma 4 is satisfied by using the compactness of Y and choosing R' such that each $r \in R'$ has sufficiently small diameter and $r \in Q$ (the basis in The Standing Hypothesis). Let $R = R' \cup \{\phi(w) \mid w \in W_F\}$. The lemma is established.

Let $V' = \{\bar{c} \mid c$ is a component of $\phi^{-1}(r)$ for some $r \in R$ such that $r \neq \phi(w)$ for any $w \in W_F\} \cup \{\bar{w} \mid w \in W_F\}$ star refines W_2, and $V = \{v \mid \bar{v} \in V'\}$. The irreducible finite open covering $V = V^1$ of M which contains $W_F = W_F^1$ generated by the irreducible open covering R of Y in Lemma 4 is just the first step in establishing Lemma 5 below.

Definition. For each $r_i \in R^1$, $\phi^{-1}(r_i) = \bigcup_{j=1}^{t_i^1} f_{ij}^1$ and $\{f_{ij}^1\}_{j=1}^{t_i^1} = F_i^1$ is called a *distinguished family of open sets* in V^1 where f_{ij}^1 is a component of $\phi^{-1}(r_i)$. If $r \neq \phi(w)$ for any $w \in W_F^1$, then $t_i^1 = p^{s_i}$ where s_i is a natural number. If $r = \phi(w)$ for some $w \in W_F^1$, then $t_i^1 = 1$.

Lemma 5. *There are sequences $\{V^m\}$ and $\{U^m\}$ of finite open coverings of M cofinal in the collection of all open coverings of M such that*

(1) V^{m+1} star refines U^m,

(2) V^1 star refines W_2 of Lemma 2,

(3) U^m star refines V^m,

(4) order $U^m = n+1$,

(5) $\{mesh\ V^m\} \to 0$

(6) V^m is generated by a finite open covering R^m of $Y = M/A_p$, and if W_F^m is the subcollection of V^m which covers $F = F_\phi$, then V^m, W_F^m, and R^m have the properties stated in Lemma 4 where R^m replaces R, V^m replaces V, and W_F^m replaces W_F,

(7) there are projections $\pi_m : V^{m+1} \to V^m$ such that

(a) $\pi_m = \beta_m \alpha_m$ where $\alpha_m : V^{m+1} \to U^m$ and $\beta_m : U^m \to V^m$,

(b) π_m takes each distinguished family $\{f_{ij}^{m+1}\}_{j=1}^{t_i^{m+1}}$ in V^{m+1} (defined in a manner like those defined for V^1 and V^2 below) onto a distinguished family $\{f_{sj}^m\}_{j=1}^{t_s^m}$ in V^m,

(c) π_m extends to a simplicial mapping (also, π_m) of $N(V^{m+1})$ into $N(V^m)$ such that if δ^n is an n-simplex in $N(V^{m+1})$ and if $\pi_m(\delta^n) = \sigma^n$, an n-simplex in $N(V^m)$, then $N[\delta^n] \cap F \neq \emptyset$ if and only if $N[\sigma^n] \cap F \neq \emptyset$. (Also, α_m and β_m denote the extensions of α_m and β_m to simplicial mappings $\alpha_m : N(V^{m+1}) \to N(U^m)$ and $\beta_m : N(U^m) \to N(V^m)$ where

$\pi_m = \beta_m \alpha_m$.)

(d) $\pi_m : V^{m+1} \to V^m$ is equivariant relative to the natural actions of certain cyclic groups whose orders are powers of p and these groups are generated by some $g \in A_p$, g not the identity. Also, π_m induces $\pi_m^* : N(V^{m+1}) \to N(V^m)$ which is equivariant relative to some cyclic group Z_{p^s} which acts on $N(V^{m+1})$ and projects to Z_{p^t} which acts on $N(V^m)$.

The proof of Lemma 5, although straightforward, is long and tedious. The existence of $V = V^1$ in Lemma 4 (which star refines W_2) generated by $R = R^1$ is an initial step of a proof using mathematical induction. Additional first steps are described below. These should help make it clear how the induction is completed to obtain a proof of Lemma 5.

The finite open covering V can be partitioned into either the distinguished families $F_i^1 = \{f_{ij}^1\}_{j=1}^{t_i^1}$ the elements of which are the components of $\phi^{-1}(r_i)$ for some $r_i \in R^1$ with $r_i \neq \phi(w)$ for each $w \in W_F = W_F^1$ and $t_i^1 = p^{s_i}$ or distinguished families of singletons $\{w\}$ where $w \in W_F^1$. As defined above, $V = V^1$ is generated by $R = R^1$.

Clearly, V^1 star refines W_2. Observe that $V_F^1 = \{v \mid v \in V^1 \text{ and } v \cap F \neq \emptyset\} = W_F^1$ and that $V_B^1 = \{v \mid v \in V_F^1 \text{ and } v \cap BdF \neq \emptyset\} = W_B^1$. Also, order $V_B^1 \leq n$ and order $V_F^1 \leq n+1$. Note that order V^1 may be larger than $n+1$ since if ϕ is the orbit mapping of an effective action by a p-adic transformation group, then dim $Y = n+2$ or ∞ [22].

The covering V^1 is defined to be a *special covering of M w.r.t. ϕ* generated by R. Of course, ϕ is fixed throughout this discussion as in the statements of Lemmas 3 and 4. Recall that it follows from Lemma 1, that if $\{f_{kj}^1\}_{j=1}^{q}$ and $\{f_{mj}^1\}_{j=1}^{s}$ are two non singleton families (those containing more than one element) in V^1 such that for some i and t, $f_{ki}^1 \cap f_{mt}^1 \neq \emptyset$, then for each j, the number of elements of $\{f_{mj}^1\}_{j=1}^{s}$ which have a non empty intersection with f_{kj}^1 is a constant c_k and for each j, the number of elements of $\{f_{kj}^1\}_{j=1}^{q}$ which have a non empty intersection with f_{mj}^1 is a constant c_m where $c_k = p^b$,

$b \geq 0$, and $c_m = p^d$, $d \geq 0$. A singleton family (which contains exactly one element) either meets each member of a non singleton family or meets no member of a non singleton family.

Construction Of U^1 Of Order $n+1$ Which Refines V^1

The reason that the sequence $\{U^m\}$ is constructed is to prove (using the definitions of α_m, β_m, and $\pi_m = \beta_m \alpha_m$) that the inverse limit of the n^{th} simplicial homology of the n-skeleta of the nerve of V^m is Z_p which permits the application of σ (defined below) to actual n-cycles. The operator σ can not be applied (as defined) to elements of a homology group.

The next step is to describe a special refinement U^1 of V^1 which has order $n+1$ and other crucial properties. First, construct an auxiliary covering $\hat{U}^1$.

List the *non degenerate* distinguished families of V^1 as $F_1^1, F_2^1, \cdots, F_{n_1}^1$ where $F_i^1 = \{f_{ij}^1\}_{j=1}^{t_i^1}$ where $t_i^1 = p^{b_i}$ for some b_i. Recall that f_{ij}^1 is homeomorphic to f_{it}^1 for each i, j, and t that makes sense. Since R^1 (which generates V^1) is irreducible, it follows that if $f_{ij}^1 \in F_i^1$ and $f_{st}^1 \in F_s^1$ where F_i^1 and F_s^1 are distinguished families in V^1 with $i \neq s$, then f_{ij}^1 and f_{st}^1 are *independent*, that is, $f_{ij}^1 \not\supset f_{st}^1$ and $f_{st}^1 \not\supset f_{ij}^1$.

For each i, $1 \leq i \leq n_1$, choose a closed and connected subset K_i^Y in $\phi(f_{i1}^1) = r_i^1 \in R^1$ where $F_i^1 = \{f_{ij}^1\}_{j=1}^{t_i^1}$ such that

(1) $K_{ij}^M = \phi^{-1}(K_i^Y) \cap f_{ij}^1$, K_{ij}^M is homeomorphic to K_{is}^M for any s and j that makes sense,

(2) $K^M = \{\text{int } K_{ij}^M \mid 1 \leq i \leq n_1 \text{ and } 1 \leq j \leq t_i^1\}$ covers $M - \cup W_F^1$ and $\check{K}^Y = \{\text{int } K_i^Y \mid 1 \leq i \leq n_1\}$ covers $Y - \bigcup_{w \in W_F^1} \phi(w)$, and

(3) K_{ij}^M is connected, $1 \leq j \leq t_i^1$.

To see that this is possible, choose a closed subset A_i of r_i^1, $1 \leq i \leq n_1$, such that $A = \{A_i \mid 1 \leq i \leq n_1\}$ covers $Y - \bigcup_{w \in W_F^1} \phi(w) = Y'$ [47; 49]. Note that there exists a natural number k such that $\{A_i = r_i^1 - N_{\frac{1}{k}}(\partial r_i^1) \mid 1 \leq i \leq n_1\}$ covers Y'. To see this, suppose that for each k, there is $x_k \notin \bigcup_{i=1}^{n_1}(r_i^1 - N_{\frac{1}{k}}(\partial r_i^1))$. Since Y' is compact, there is a subsequence $\{x_{n(k)}\}$ of $\{x_k\}$ which converges to $x \in r_q^1$ for some q. There is some m such that $x \in r_q^1 - N_{\frac{1}{m}}(\partial r_q^1)$ and $x \in$ interior$(r_q^1 - N_{\frac{1}{m+1}}(\partial r_q^1))$ which leads to a contradiction. Choose $p_i \in r_i^1$ and let $C_m(p_i)$ be the component of $r_i^1 - N_{\frac{1}{m}}(\partial r_i^1)$ which contains p_i. It will be shown that $\bigcup_{m=1}^{\infty} C_m(p_i) = r_i^1$. Suppose that there is $q \in r_i^1$ such that $q \notin \bigcup_{m=1}^{\infty} C_m(p_i)$. Since r_i^1 is uniformly locally connected and locally compact ($\bar{r}_i^1$ is a Peano continuum), there is a simple arc $p_i q$ from p_i to q in r_i^1. Consequently, for some m, $r_i^1 - N_{\frac{1}{m}}(\partial r_i^1) \supset p_i q$ which is in $C_m(p_i)$. This is contrary to the assumption above. Hence, $\bigcup_{m=1}^{\infty} C_m(p_i) = r_i^1$. For $p_i \in r_i^1$ fixed as above, choose $p_{ij} \in \phi^{-1}(p_i) \cap f_{ij}^1$ for $1 \le j \le t_i^1$. Let $C_m^i(p_{ij})$ be the component of $\phi^{-1}(C_m(p_i))$ which is in f_{ij}^1 and contains p_j. It will be shown that $\bigcup_{m=1}^{\infty} C_m^i(p_{ij}) = f_{ij}^1$. If this is false, then there is a $q_j \in f_{ij}^1 - \bigcup_{m=1}^{\infty} C_m^i(p_{ij})$. Since f_{ij}^1 is ulc and locally compact, there is a simple arc $p_{ij}q_j$ from p_{ij} to q_j in f_{ij}^1. Now, $r_i^1 \supset \phi(p_{ij}q_j)$. For m large enough, $C_m(p_i) \supset \phi(p_{ij}q_j)$ and some component of $\phi^{-1}\phi(p_{ij}q_j)$ contains $p_{ij}q_j$ and lies in $C_m^i(p_{ij})$. This is contrary to the assumption above. Hence, $\bigcup_{m=1}^{\infty} C_m^i(p_{ij}) = f_{ij}^1$.

Let $K_i^Y = \text{int } C_m(p_i)$ where m is sufficiently large that (1) and (3) are satisfied and also

(4) int $C_m(p_i) \supset A_i$. Then (2) is also satisfied, that is, the components of $\phi^{-1}(C_m(p_i))$ may be taken as K_{ij}^M, $1 \le i \le n_1$.

Next, shrink the elements of $\phi(W_F^1) = \{\phi(w) \mid w \in W_F^1\}$ as follows: Order W_F^1 as $w_1^1, w_2^1, \ldots, w_{n_2}^1$ and choose a natural number s such that $\{K_{n_1+i} = \phi(w_i^1) - N_{\frac{1}{s}}(\partial\phi(w_i^1)) \mid 1 \le i \le n_2\}$ has the property that $\{\text{int } K_{n_1+i} \mid 1 \le i \le n_2\}$ covers $\bigcup_{w \in W_F^1} \phi(w) - \bigcup_{i=1}^{n_1} \text{int } K_i^Y$.

It is well known that a metric space X has dim $X \le n$ if and only if X has a sequence $\{G_i\}$ of open coverings of X such that

(1) G_{i+1} refines G_i for each i,

(2) order $G_i \leq n+1$ for each i, and

(3) mesh $G_i < \frac{1}{i}$.

If X is a manifold, then the elements of G_i can be chosen to be connected and uniformly locally connected. If X is a triangulable n-manifold, then it is easy to see this by using barycentric subdivisions of a triangulation of X. Choose such a sequence $\{G_i\}$ of open coverings of M such that $\{\phi(g) \mid g \in G_1\}$ star refines $K^Y = \{\text{int } K_i^Y \mid 1 \leq i \leq n_1 + n_2\}$.

The next step is to show how to choose some G_i from which $\hat{U}^1$ will be chosen and later modified to give U^1 with the desired properties.

Let ϵ be the Lebesque number of the covering K^Y. Choose G_t, mesh $G_t < \frac{1}{t}$, such that diam $\phi(g) < \frac{\epsilon}{8}$ for $g \in G_t$.

Define

$$G(y) = \{g \mid g \in G_t \text{ and } y \in \phi(g)\}.$$

Statement 1. If $y \in Y$, then there is s, $1 \leq s \leq n_1 + n_2$, such that int $K_s^Y \supset \bigcup_{g \in G(y)} \overline{\phi(g)}$.

Proof. It follows from the choice of ϵ, t.

Choose $\hat{U}^1 \subset G_t$ such that $\hat{U}^1$ is an irreducible finite covering of M. Define

$$\hat{U}(y) = \{g \mid g \in \hat{U}^1 \text{ and } y \in \phi(g)\}.$$

Since G_t is an open covering of M with connected open sets, mesh $G_t < \frac{1}{t}$, and order $G_t \leq n+1$, we obtain:

Statement 2. $\hat{U}^1$ has the following properties:

(1) order $\hat{U}^1 \leq n+1$ and $\hat{U}^1$ star refines V^1,

(2) if $u \in \hat{U}^1$, then u is connected, and

(3) if $y \in Y$ then there is s, $1 \leq s \leq n_1 + n_2$, such that int $K_s^Y \supset \bigcup_{g \in \hat{U}(y)} \overline{\phi(g)}$.

Construction Of U^1 Which Refines $\hat{U}^1$ In A Special Way

For any collection S of sets, let $\cup S$ be the union of sets in S and $\cap S$ be their intersection.

For each $y \in Y$, let $Q(y) = \phi(\hat{U}(y)) = \{\phi(u) \mid u \in \hat{U}^1 \text{ and } y \in \phi(u)\}$. There are at most a finite number of such sets distinct from each other. Order these sets as $Q_1, Q_2, \cdots, Q_{m_1}$ such that for $i < j$, $Q_i \neq Q_j$ and card $Q_i \geq$ card Q_j. Let $O_i = \cap Q_i - \bigcup_{j<i}(\overline{\cap Q_j})$ where $\cap Q_i = \{x \mid x \in \phi(u) \text{ for each } \phi(u) \in Q_i\}$, $1 \leq i \leq m_1$. For each i, $1 \leq i \leq m_1$, let $B_{i\phi(u)} = (\partial O_i) \cap \partial\phi(u)$ where $\phi(u) \in Q_i$ and $(\partial O_i) \cap \partial\phi(u) \neq \emptyset$. There are at most a finite number of such non empty closed sets distinct from each other. Let $B_1, B_2, \cdots, B_{m_2}$ denote all those sets distinct from each other. For each i, let $1 \leq i \leq m_2$, there is $u \in \hat{U}^1$ and a closed subset C_{iu} of ∂u such that $\phi(C_{iu}) = B_i$. Let $B = \bigcup_{i=1}^{m_2} B_i$. For each $y \in B$, let $D(y) = \{B_t \mid y \in B_t\}$. There are at most a finite number of such non empty sets distinct from each other. Order these as $D_1, D_2, \cdots, D_{m_3}$ such that if $i < j$, then $D_i \neq D_j$ and card $D_i \geq$ card D_j.

It follows from the definition that $\phi^{-1}(B)$ is closed and contains no open set. Hence, dimension $\phi^{-1}(B) \leq n - 1$.

For each $y \in Y$, choose $0 < \epsilon_y < (\frac{1}{8})\min\{\{\rho(y, B_i) \mid y \notin B_i,\ 1 \leq i \leq m_2\} \cup \{\rho(\cap D_i, \cap D_j) \mid (\cap D_i) \cap (D_j) = \emptyset\}\}$ and cover B with a finite irreducible collection E of ϵ_y-neighborhoods for $y \in B$. Choose t_0 sufficiently large such that if $g \in G_{t_0}$ and $\phi(g) \cap B \neq \emptyset$, then there is $e \in E$ such that $e \supset \overline{\phi(g)}$.

Construct a finite irreducible collection H' of open sets such that

(1) H' covers $\phi^{-1}(B) - \cup \overset{x}{W}{}^{2}_{F}$ where $\overset{x}{W}{}^{2}_{F}$ is defined below,

(2) H' refines G_{t_0},

(3) H' star refines $\hat{U}^1$ and $\{\phi(u) \mid u \in H'\} = \phi(H')$ star refines $\phi(\hat{U}^1) = \{\phi(u) \mid u \in \hat{U}^1\}$,

(4) order $H' \leq n$,

(5) if $h \in H'$ and $\phi(h) \cap (\cap D_t) \neq \emptyset$ for the smallest t and $B_s \notin D_t$, then $\overline{\phi(h)} \cap B_s = \emptyset$,

(6) if H'_1 denotes the subcollection of H' which covers $\phi^{-1}(\cap D_1) - \cup \overset{x}{W}{}^2_F$, H'_i denotes the subcollection of H' which covers $\phi^{-1}(\cap D_i) - \left(\left(\bigcup_{j=1}^{i-1} (\cup H'_j) \right) \cup (\cup \overset{x}{W}{}^2_F) \right)$ and $h \in H'_i$, $1 < i \leq m_0$, then $\overline{\phi(h)} \cap (\cap D_t) = \emptyset$ for $1 \leq t < i$,

(7) if $N[\delta^k]$ is the nucleous of a k-simplex δ^k_1 in the nerve, $N(H')$, of H' and $N[\delta^k_2]$ is the nucleous of a k-simplex δ^k_2 in $N(H')$, then closure ($N[\delta^k_1]$ – the union of all $\overline{N[\delta^i]}$ where δ^i is an i-simplex in $N(H')$ with $k < i \leq n-1$) does not meet closure ($N[\delta^k_2]$ – the union of all $\overline{N[\delta^i]}$ where $k < i \leq n-1$) whenever $\delta^k_1 \neq \delta^k_2$, and

(8) if $y \in B$, $y \notin B_x$, and $y \in \phi(h)$ where $h \in H'$, then $\overline{\phi(h)} \cap B_x = \emptyset$.

To see that this is possible, consider the following: If $B_s \notin D_1$, then $B_s \cap (\cap D_1) = \emptyset$; otherwise, $B_s \cup D_1$ has cardinality greater than D_1 which contradicts the ordering of the D_i. Cover $\phi^{-1}(\cap D_1)$ with a finite irreducible open cover H'_1 having Properties (2)-(4), (6)-(8) and the property that if $h \in H'_1$ and $B_s \notin D_1$, then $\overline{\phi(h)} \cap B_s = \emptyset$. Thus, H'_1 satisfies Property (5).

Continue using mathematical induction. Suppose that H'_i have been constructed for $1 \leq i < j$ such that $\bigcup_{i=1}^{j-1} H'_i$ satisfy Properties (2)-(5), (7), (8), and Property (6) that if $h \in H'_i$, $1 < i \leq j-1$, then $\overline{\phi(h)} \cap (\cap D_t) = \emptyset$ for $1 \leq t < i$. Let H'_j be a finite irreducible open cover of $\phi^{-1}(\cap D_j) - \bigcup_{i=1}^{j-1} (\cup H'_i)$ having properties (2), (3), (7), (8) and the property that if $B_s \notin D_i$, $1 \leq i \leq j$, then $B_s \cap (\cap D_j) = \emptyset$; otherwise, $B_s \cup D_j$ has cardinality $k >$ card D_j and $B_s \in D_i$ for some i, $1 \leq i \leq j$, and if $h \in H'_j$, then $\overline{\phi(h)} \cap B_s = \emptyset$. This yields Property (5). Furthermore, if $h \in H'_i$, $1 < i \leq j$, then $\phi(h) \cap (\cap D_t) = \emptyset$ for $1 \leq t < i$ (Property (6)). Consequently, $\bigcup_{i=1}^{j} H'_i$ has Properties (2), (3), and (5)-(8) where in (5), $1 \leq t \leq j$.

Use Nagata's Theorem as in the proof of Lemma 3 to shrink each element of $\bigcup_{i=1}^{j} H_i'$ to obtain H_i'', $1 \le i \le j$, such that $\bigcup_{i=1}^{j} H_i''$ covers $\bigcup_{i=1}^{j} \phi^{-1}(\cap D_i)$ and has Property (4) while keeping properties (2), (3), and (5)-(8). For convenience of notation, suppose $\bigcup_{i=1}^{j} H_i'$ has properties (2)-(8) and covers $\bigcup_{i=1}^{j} \phi^{-1}(\cap D_i)$. It follows by mathematical induction that $H_0' = \bigcup_{i=1}^{m_0} H_i'$ covers $\phi^{-1}(B)$ and has properties (2)-(8). There is no loss of generality in assuming that if $h \in H_0'$ and $h \cap \text{int } F \neq \emptyset$, then either int $F \supseteq \bar{h}$ or $h \cap \partial F \neq \emptyset$ and that if $h \in H_0'$ and $h \cap (M - F) \neq \emptyset$, then either $M - F \supseteq \bar{h}$ or $h \cap \partial(F) \neq \emptyset$.

Note: A finite open covering C of a closed subset N of M such that dim $N = n - 1$ where the elements of C are open relative to N and order $C = n$ can be extended to a collection C' of open sets in M such that card $C =$ card C', C' covers M and order $C' = n$. See [48].

Let $H_0^F = \{h \mid h \in H_0' \text{ and } h \cap F \neq \emptyset\}$. Let $H_0^F(\partial F) = \{h \mid H_0^F \text{ and } h \cap \partial F \neq \emptyset\}$. Let W_1 be a finite irreducible open covering of ∂F such that $W_1 \supseteq H_0^F(\partial F)$, star refines both $\hat{U}^1$ and W_F^1, and if $w \in W_1 - H_0^F(\partial F)$, then $\bar{w} \cap \phi^{-1}(B) = \emptyset$. By the use of Nagata's Theorem as in the proof of Lemma 3, there is an open irreducible refinement H_0'' of $H_0' - (\{h \mid h \in H_0', h \cap F \neq \emptyset, \text{ and } h \cap \partial F = \emptyset\} \cup W_1)$ such that (1) order $H_0'' = n$, (2) if $H_0''(\partial F) = \{h \mid h \in H_0'' \text{ and } h \cap \partial F \neq \emptyset\}$, then order of $H_0''(\partial F) = n$, and (3) H_0'' covers $\phi^{-1}(B) - \text{int } F$ and has Properties (2)-(8) above. Now, let W_2 be an irreducible open covering of $F - \bigcup H_0''(\partial F)$ such that if $w \in W_2$ then int $F \supseteq w$. Again, by the use of Nagata's Theorem as in the proof of Lemma 3, there is an open irreducible covering Q of $\phi^{-1}(B) \cup F$ such that (1) $\overset{x}{W}{}_F^2 = \{q \mid q \in Q \text{ and } q \cap F \neq \emptyset\}$ has order $n+1$,
(2) $\overset{x}{W}{}_{\partial F}^2 = \{q(q \in \overset{x}{W}{}_F^2 \text{ and } q \cap \partial F \neq \emptyset\}$ has order n,
(3) $H' = \{q \mid q \in Q \text{ and } q \cap (\phi^{-1}(B) - \bigcup \overset{x}{W}{}_F^2) \neq \emptyset\}$ covers $\phi^{-1}(B) - \bigcup \overset{x}{W}{}_F^2$, (4) order

$H' = n$, and (5) H' has properties (1)-(8) above.

Cover $M - (\cup H' \cup (\cup \overset{x}{W}{}^2_F))$ with a finite irreducible collection C of open sets which star refines $\hat{U}^1$. By the method used in the proof of Lemma 3, the elements of H', C, and $\overset{x}{W}{}^2_F$ can be shrunk to collections H'', C', and $\hat{W}^2_F$, respectively, such that (9) $H'' \cup C' = G$ covers $M - \cup \hat{W}^2_F$ irreducibly, $\hat{W}^2_F$ covers F, order $G \leq n+1$, order $H'' \leq n$, H'' covers $\phi^{-1}(B) - \cup \hat{W}^2_F$, if $g \in G - H''$, then $\bar{g} \cap \phi^{-1}(B) = \emptyset$ and $\bar{g} \cap F = \emptyset$, order $(G \cup \hat{W}^2_{\partial F}) \leq n+1$, order $\hat{W}^2_{\partial F} \leq n$, and order $(G \cup \hat{W}^2_F) \leq n+1$. For the sake of notation, suppose that $H' = H''$ and $\overset{x}{W}{}^2_F = \hat{W}^2_F$ (Here, the elements of $\hat{W}^2_F$ may not be saturated, and by the notation the elements of $\overset{x}{W}{}^2_F = \hat{W}^2_F$ may not be saturated, i.e., if $w' \in \overset{x}{W}{}^2_F$, then w' may not equal $\phi^{-1}\phi(w')$.).

Recall that if δ^n is an n-simplex in $N(\overset{x}{W}{}^2_F)$, the nerve of $\overset{x}{W}{}^2_F$, then $N[\delta^n]$, the nucleous of δ^n has the property that int $F \supset \overline{N[\delta^n]}$. Suppose, without loss of generality, that if δ^n is an n-simplex in $N(G)$, then $\overline{N[\delta^n]} \cap F = \emptyset$. This property may be obtained in the construction above mimicing the proof of Lemma 3.

Now, change H' into a cover H'' of $\phi^{-1}(B) - \cup \overset{x}{W}{}^2_F$ adding to it some new open sets, as follows (but keeping Properties (1)-(9)).

Let $\Delta_{n-1} = \{\overline{N[\delta^{n-1}]} \cap \phi^{-1}(B) \mid \delta^{n-1}$ is an $(n-1)$-simplex in $N(H')\}$. Cover the elements of Δ_{n-1} with a finite irreducible collection H_{n-1} of open sets such that

(1) if $h \in H_{n-1}$, then diam $h < \frac{1}{t_0}$ and $H' \cup H_{n-1}$ has Property (5) of H' above,

(2) if $h \in H_{n-1}$, then $\cup H' \supset h$,

(3) if $h \in H_{n-1}$ and $g \in G - H'$, then $\bar{g} \cap \bar{h} = \emptyset$,

(4) if $d \in \Delta_{n-1}$, then at most one member of H_{n-1} meets d (and, therefore, contains d),

(5) the closures of the members of H_{n-1} are pairwise disjoint,

(6) order $\{h - \cup \Delta_{n-1} \mid h \in H'\} \leq n-1$,

(7) order $(H' \cup H_{n-1}) \leq n+1$, and

(8) if δ^n is an n-simplex in $N(H' \cup H_{n-1})$, then $\overline{N[\delta^n]} \cap F = \emptyset$.

For each i, $1 < i \le n$, let $\Delta_{n-i} = \{(\overline{N[\delta^{n-i}]} - \bigcup_{j<i}(\cup H_{n-j})) \cap \phi^{-1}(B) \mid \delta^{n-i}$ is an $(n-i)$-simplex in $N(H')\}$. It is assumed that H' is constructed such that the elements of Δ_{n-i} are pairwise disjoint. Cover Δ_{n-i} with a finite irreducible collection H_{n-i} of open sets such that

(1) if $h \in H_{n-i}$, then diam $h < \frac{1}{t_0}$ and $H' \cup \left(\bigcup_{j=1}^{i} H_{n-j}\right)$ has Property (5) of H' above,

(2) if $h \in H_{n-i}$, then $\cup H' \supset h$,

(3) if $h \in H_{n-i}$ and $g \in G - H'$, then $\bar{g} \cap \bar{h} = \emptyset$,

(4) if $d \in \Delta_{n-i}$, then at most one member of H_{n-i} meets d (and, therefore, contains d),

(5) the closures of the members of H_{n-i} are pairwise disjoint,

(6) order $\left(\bigcup_{j=1}^{i} H_{n-j}\right) \le i-1$ and order $(H_{n-i} \cup H_{n-j})$ for each j, $1 \le j < i$, is less than or equal to 1,

(7) no member of H_{n-i} meets a member of Δ_{n-j}, $1 \le j < i$,

(8) order $\left\{h - \bigcup_{j=1}^{i}(\cup\Delta_{n-j}) \mid h \in H'\right\} \le n-i$, and

(9) order $\left(H' \cup \left(\bigcup_{j=1}^{i} H_{n-i}\right)\right) \le n+1$.

Let $H'' = \left(\bigcup_{i=1}^{n} H_{n-i}\right) \cup H'$. Note that $\bigcup_{i=1}^{n} H_{n-i} = H$ covers $\phi^{-1}(B) - \cup \overset{x}{W}{}^2_F$.

Let $O = \{g \cap \phi^{-1}(0_i) \mid g \in G$ and $1 \le i \le m_1\}$ (recall that $\phi^{-1}(B) \cap \phi^{-1}(O_i) = \emptyset$). Note that $G \supset H'$ but O contains no member of H'. Thus, $U_0^1 = O \cup H$ has the following properties:

(0) U_0^1 covers $\phi^{-1}(B) - \cup \overset{x}{W}{}^2_F$,

(1) U_0^1 star refines $\hat{U}^1$ and $\phi(U_0^1)$ star refines $\phi(\hat{U}^1)$,

(2) order $U_0^1 = n+1$ and order $(U_0^1 \cup \overset{x}{W}{}^2_{\partial F}) \le n+1$,

(3) if $(\cap D_i) \cap (\cap D_j) = \emptyset$, $u \in U_0^1$, $v \in U^1$, $\phi(u) \cap (\cap D_i) \neq \emptyset$, and $\phi(v) \cap (\cap D_j) \neq \emptyset$, then $\overline{\phi(u)} \cap \overline{\phi(v)} = \emptyset$,

(4) if $B_i \cap B_j = \emptyset$, $u \in U_0^1$, $v \in U_0^1$, $\phi(u) \cap B_i \neq \emptyset$, and $\phi(v) \cap B_j \neq \emptyset$, then $\overline{\phi(u)} \cap \overline{\phi(v)} = \emptyset$,

(5) if $u \in U_0^1$, $\phi(u) \cap (\cap D_i) \neq \emptyset$, and $\phi(u) \cap (\cap D_j) \neq \emptyset$, then $(\cap D_i) \cap (\cap D_j) \neq \emptyset$ and $\phi(u) \cap (\cap(D_i \cup D_j)) \neq \emptyset$,

(6) if $u \in U_0^1$ and $\phi(u) \cap B = \emptyset$ where $B = \bigcup_{j=1}^{m_2} B_j$, then $O_i \supset \phi(u)$ for some i, $1 \leq i \leq n_1$ (it follows that $\phi(h) \supset \overline{\phi(u)}$ for all $h \in \hat{U}^1$ such that $\phi(h) \cap \phi(u) \neq \emptyset$),

(7) if $u \in U_0^1$ and $\phi(u) \cap (\cap D_t) \neq \emptyset$ for the smallest t, then $\phi(h) \supset \phi(u)$ for all $h \in \hat{U}^1$ such that there is no $B_q \in D_t$ with the property that $\partial\phi(h) \supset B_q$ and $\phi(h) \cap \phi(u) \neq \emptyset$,

(8) if $w \in \overset{x}{W}{}_F^2$, then $u \not\supset w$ for any $u \in U_0^1$,

(9) if δ^n is an n-simplex in $N(U_0^1)$, then $\overline{N[\delta^n]} \cap F = \emptyset$, and

(10) if $u \in U_0^1$, $h \in \hat{U}^1$, $\phi(h) \cap \phi(u) \neq \emptyset$, and $\phi(h) \not\supset \phi(u)$, then $\phi(u) \cap \partial\phi(h) \neq \emptyset$ (that is, $\phi(u) \cap B_s \neq \emptyset$ for some s where $B \supset B_s$, see (6)).

Notice that (4) and (5) follow from (3).

The proof of (7) follows:

Lemma. *Suppose that $u \in U_0^1$, $y \in \phi(u)$, t is the smallest subscript such that $y \in \cap D_t$, and $B_s \notin D_t$. Then $\overline{\phi(u)} \cap B_s = \emptyset$.*

Proof. Now, $y \notin B_s$; otherwise, there is some k such that $D_k \supset D_t \cup B_s$, $y \in \cap D_k$, and card $D_k >$ card D_t. Hence, $k < t$. This contradicts the choice of t. By Property (5) of H, $\overline{\phi(u)} \cap B_s = \emptyset$. The Lemma is proved.

If (7) is false, then there exists $u \in U_0^1$, $y \in \cap D_t$ for the smallest t, and there exists $h \in \hat{U}^1$ such that $y \in \phi(h)$ and $\phi(h) \not\supset \phi(u)$. This implies that there exists B_s such that $\partial\phi(h) \supset B_s$ and $\phi(u) \cap B_s \neq \emptyset$ by Property (10). Clearly, $B_s \notin D_t$ since $\phi(h) \cap (\cap D_t) \neq \emptyset$.

This contradicts the Lemma which states that $\phi(u) \cap B_s = \emptyset$. It is not difficult to see that U_0^1 has properties (1)-(10).

Let $U^1 = U_0^1 \cup \overset{x}{W}{}_F^2$. This cover has order $n+1$ and $\overset{x}{W}{}_{\partial F}^2$ is the subcollection of U^1 which covers $BdF = \partial F$. Also, if δ^n is an n-simplex in $N(U^1)$, then either $\overline{N[\delta^n]} \cap F = \emptyset$ or int $F \supset \overline{N[\delta^n]}$.

Observe that if $u \in U_0^1$ and $\phi(u) \cap B = \emptyset$, then for some i, $O_i \supset \phi(u)$ by the definition of O; otherwise, $u \in H$.

Construction of V^2 Which Refines U^1

Next, construct V^2. Recall that for each $y \in Y$ we have defined $\hat{U}(y) = \{u \mid u \in \hat{U}^1$ and $y \in \phi(u)\}$. There is some s, $1 \le s \le n_1 + n_2$, such that $K_s^Y \supset \bigcup_{u \in \hat{U}(y)} \overline{\phi(u)}$. (And every K_s^Y is a slightly shrunken member of the covering of Y, which generates V^1).

Let $W_F^2 = \{w \mid w' \in \overset{x}{W}{}_F^2$ and w is the union of all $\phi^{-1}\phi(x)$ such that $x \in w'$ and $w' \supset \phi^{-1}\phi(x)$, that is, w is saturated and $\phi^{-1}\phi(w) = w\}$. For each $y \in B - \phi(\cup W_F^2)$, choose $r_y^2 \in Q$ (the basis for Y described above) such that

(1) $y \in r_y^2$,

(2) if $U(y) = \{u \mid u \in U^1$ (not $\hat{U}^1$) and $y \in \phi(u)\}$, then $\left(\bigcap_{u \in U(y)} \phi(u)\right) \cap \left(\bigcap_{u \in \hat{U}(y)} \phi(u)\right) \supset \bar{r}_y^2$,

(3) diam $r_y^2 < (\frac{1}{8}) \min\{\rho(y, \partial\phi(v)) \mid v \in \hat{U}^1$ and $y \notin \partial\phi(v)\}$, and

(4) $\phi^{-1}(r_y^2) = r_{y1}^2 \cup r_{y2}^2 \cup \cdots \cup r_{yq}^2$, $q = p^{t_q}$ where $t_q \ge 1$, r_{yi}^2 maps onto r_y^2 under ϕ, $\bar{r}_{yi}^2 \cap \bar{r}_{yj}^2 = \emptyset$ for $i \ne j$, and r_{yi}^2 is homeomorphic to r_{yj}^2 for each i and j with a homeomorphism compatible with the projection ϕ (indeed, there is an element of A_p which takes r_{yi}^2 onto r_{yj}^2).

It follows that for each such y, there is $u \in U_0^1 \subset U^1$ such that $u \supset \bar{r}_y^2$. See the proof of Lemma 4 for the existence of r_y^2.

Let R_1^2 denote a finite irreducible collection of such sets r_y^2 which covers $B - \phi(\cup W_F^2)$.

If $y \in Y$ and $y \notin \cup R_1^2$, and $y \notin \cup\phi(W_F^2)$, then choose r_y^2 satisfying (1)-(4) above such that $\bar{r}_y \cap B = \emptyset$ and let R_2^2 denote a finite irreducible cover of $Y - (\cup R_1^2 \cup (\cup\phi(W_F^2))$, consisting of such r_y^2. Let $R^2 = R_1^2 \cup R_2^2 \cup \phi(W_F^2)$ which is an irreducible cover of Y.

Define $V^2 = \{c \mid c$ is a component of $\phi^{-1}(r_{y_i}^2)$ for some i, where $r_{y_i}^2 \in R_1^2 \cup R_2^2\} \cup \phi(W_F^2)$, which is an irreducible cover of M that star refines V^1 and U^1. The collection of components, $\{f_{ij}^2\}_{j=1}^{t_i^2}$, of $\phi^{-1}(r_{y_i}^2)$, $r_{y_i}^2 \in R_1^2 \cup R_2^2$, is a non degenerate distinguished family in V^2, whereas each $w \in W_F^2$, where $w = \phi^{-1}\phi(w)$, is a singleton distinguished family in V^2.

Definitions Of α_1, β_1, And $\pi_1 = \beta_1\alpha_1$

Case (1): Take any $r_i^2 = r_{y_i}^2 \in R_1^2$ chosen for some $y_i \in B - \cup\phi(W_F^2)$. Let $F_i^2 = \{f_{ij}^2\}_{j=1}^{t_i^2}$ be the (non degenerate) distinguished family in V^2 generated by $r_{y_i}^2$. Now, H_1' is the subcollection of H which covers $\phi^{-1}(\cap D_1) - (\cup \overset{x}{W}{}_F^2)$ and H_i' is the subcollection of H which covers $\phi^{-1}(\cap D_i) - \left(\left(\bigcup_{j=1}^{i-1}(\cup H_j')\right) \cup (\cup \overset{x}{W}{}_F^2)\right)$. Either (a) $y_i \in \cap D_t - \bigcup_{j<t}(\cup H_j')$ for some t or (b) $y_i \notin \cap D_t - \bigcup_{j<t}(\cup H_j')$ for some t.

Case (a): Let $e_{y_i} = \min\{t \mid y_i \in \cap D_t - \bigcup_{j<t}(\cup H_j')\}$. By the construction of H, for each j, $1 \le j \le t_i^2$, there is $U_{ij} \in H'_{e_{y_i}}$ such that $U_{ij} \supset \bar{f}_{ij}^2$, $y_i \in \phi(U_{ij}) \cap (\cap D_{e_{y_i}})$, and $\phi(U_{ij}) \cap (\cap D_t) = \emptyset$ for $t < e_{y_i}$. Choose one such U_{ij} for $\bar{f}_{ij}^2$.

Claim: $e_{y_i} = \min\{t \mid u \in H \subset U^1,\ y_i \in \phi(u), \text{ and } \phi(u) \cap (\cap D_t) \neq \emptyset\}$. Let $m = e_{y_i}$. Recall that H_m' is a finite irreducible open covering of $\phi^{-1}(\cap D_m) - \left(\left(\bigcup_{i=1}^{m-1}(\cup H_i')\right) \cup (\cup \overset{x}{W}{}_F^2)\right)$ having Properties (1)-(10). Consequently, if $y_i \in \phi(u)$, $u \in H \subset U$, and $\phi(u) \cap (\cap D_t) = \emptyset$ for $1 \le t < e_{y_i}$. The claim follows.

Let $s_i = \min\{s \mid \text{int } K_s^Y \supset \overline{\phi(u)}$ for all $u \in \hat{U}(y_i)$ and take $F_{s_i}^1 = \{f_{s_ij}^1\}_{j=1}^{t_{s_i}^1}$ for the given F_i^2. Each $f_{ij}^2 \in F_i^2$ is in one and only one member of $F_{s_i}^1$.

Case (b): Let $e_{y_i} = \min\{t \mid y_i \in \cap D_t\}$. In this case, $y_i \in \phi(u)$ where $u \in H_t'$ for some $t < e_{y_i}$. Let $d_{y_i} = \min\{t \mid y_i \in \phi(u),\ u \in H, \text{ and } \phi(u) \cap (\cap D_t) \neq \emptyset\}$. By construction of H, for each j, $1 \leq j \leq t_i^2$, there is $U_{ij} \in H$ such that $U_{ij} \supset \bar{f}_{ij}^2$ and $\phi(U_{ij}) \cap (\cap D_t) \neq \emptyset$ for the smallest t where $d_{y_i} \leq t$. Choose one such U_{ij} for each f_{ij}^2. Let $V(y_i) = \{U_{ij} \mid 1 \leq j \leq t_i^2\}$. By Property (1) of H, there is some $h \in \hat{U}(y_i)$ such that $\phi(h) \supset \bigcup_{j=1}^{t_i^2} \phi(U_{ij})$. Let $\hat{V}(y_i) = \{h \mid h \in \hat{U}(y_i)$ and $\phi(h) \supset \bigcup_{j=1}^{t_i^2} \phi(U_{ij})\}$.

Let $s_i = \min\{s \mid \text{int } K_s^Y \supset \overline{\phi(h)}$ for all $h \in \hat{V}(y_i)$ NOT $\hat{U}(y_i)\}$ s_i is defined differently in Case (a)) and take $F_{s_i}^1 = \{f_{s_i j}^1\}_{j=1}^{t_{s_i}^1}$ for the given F_i^2. Each $f_{ij}^2 \in F_i^2$ is in one and only one member of $F_{s_i}^1$. For each j, $1 \leq j \leq t_i^2$, there is $U_{ij} \in H \subset U_0^1$ by Property (2) above such that $\phi(U_{ij}) \supset \bar{f}_{ij}^2$. Take any such U_{ij} in this case. There is a unique z_{ij}, $1 \leq z_{ij} \leq t_{s_i}^1$, such that $f_{s_i z_{ij}}^1 \supset U_{ij} \supset f_{ij}^2$. To see that $f_{s_i z_{ij}}^1 \supset U_{ij}$, recall that U_0^1 star refines $\hat{U}^1$, and recall Property (3) of the properties of $\hat{U}^1$ (Statement 2 above). Since $\bigcup_{u \in \hat{U}(y_i)} \overline{\phi(u)} \subset \text{int } K_{s_i}^Y \subset r_{s_i}^1$, $r_{s_i}^1 \in R^1$, $\bigcap_{u \in U(y_i)} \phi(u) \supset \bar{r}_{y_i}^2$, $r_{y_i}^2 \in R^2$, and $U_{ij} \supset \bar{f}_{ij}^2$, that is, $U_{ij} \in U(y_i)$, we have $r_{s_i}^1 \supset \phi(U_{ij})$ and $f_{s_i z_{ij}}^1 \supset U_{ij}$.

Let $\alpha_1(f_{ij}^2) = U_{ij}$, $\beta_1(U_{ij}) = f_{s_i z_{ij}}^1$, and $\pi_1(f_{ij}^2) = \beta_1 \alpha_1(f_{ij}^2) = f_{s_i z_{ij}}^1$.

Case (2): Take $r_{y_i} \in R_2^2$; here $y_i \notin B$, $y_i \notin \phi(W_F^2)^*$, and r_{y_i} generates $F_i^2 = \{f_{ij}^2\}_{j=1}^{t_i^2}$ in V^2. Let $s_i = \min\{s \mid \text{int } K_s^Y \supset \bigcup_{u \in \hat{U}(y_i)} \overline{\phi(u)}\}$. Take $F_{s_i}^1 = \{f_{s_i j}^1\}_{j=1}^{t_{s_i}^1}$ for the given F_i^2, then $\phi(f_{s_i j}^1) \supset \bigcup_{u \in \hat{U}(y_i)} \overline{\phi(u)}\}$. For each j, $1 \leq j \leq t_i^2$, choose $U_{ij} \in O \subset U_0^1 \subset U^1$ such that $U_{ij} \supset f_{ij}^2$ (there is such a U_{ij} by Property (2) above). Then $U_{ij} \in \hat{U}(y_i)$ and there is a unique z_{ij}, $1 \leq z_{ij} \leq t_{s_i}^1$, such that $f_{s_i z_{ij}}^1 \supset U_{ij} \supset f_{ij}^2$. Let $\alpha_1(f_{ij}^2) = U_{ij}$, $\beta_1(U_{ij}) = f_{s_i z_{ij}}^1$, and $\pi_1(f_{ij}^2) = \beta_1 \alpha_1(f_{ij}^2) = f_{s_i z_{ij}}^1$.

Case (3): Take now $y_i \in w \in W_F^2$. Since $w \in W_F^2$, there is a unique $w' \in \overset{x}{W}{}_F^2 \subset U^1$ such that $w' \supset w$. Furthermore, there is $s_w = \min\{s \mid K_s^Y \supset \phi(w), n_1 \leq s \leq n_2\}$. Choose $w_{s_w}^1$ in W_F^1 such that K_s^Y is the shrinking of $\phi(w_{s_w}^1)$. Let $\alpha_1(w) = w$ and $\beta_1(w) = w_{s_w}^1$.

It will be shown now that the mappings α_1 and β_1 are well defined.

We fix the choice of open sets in U^1, which are images of elements of V^2 under the mapping α_1, and the question is: whether the definition of β_1 is correct (well defined)?

Case (A): Suppose that β_1 is not well defined and there exist F_i^2 and F_k^2, two different *non degenerate* distinguished families in V^2 such that (a) $s_i \neq s_k$ (if $s_i = s_k$, then β_1 is well defined), $F_{s_i}^1$ is chosen for F_i^2, $F_{s_k}^1$ is chosen for F_k^2, and (b) $U_{ij} = U_{kt} \supset f_{ij}^2 \cup f_{kt}^2$ where $F_i^2 = \{f_{ij}^2\}_{j=1}^{t_i^2}$, $F_k^2 = \{f_{kj}^2\}_{j=1}^{t_k^2}$, where for some j, $1 \leq j \leq t_i^2$, $U_{ij} \in U^1$ is chosen such that $U_{ij} \supset \overline{f_{ij}^2}$, and for some t, $1 \leq t \leq t_k^2$, $U_{kt} = U_{ij} \in U^1$ is chosen such that $U_{kt} \supset \overline{f_{kt}^2}$ as described above.

Case A(1): $U_{ij} = U_{kt} \in O \subset U_0^1 \subset U^1$. Then $y_i \notin B$ and $y_k \notin B$. Indeed, $y_i \in O_m$ and $y_k \in O_m$ for some m, $1 \leq m \leq n_1$. In this case, it follows from Property (6) of the properties of U^1 that for each $u \in \hat{U}(y_i)$, $y_k \in \phi(u)$, and for each $v \in \hat{U}(y_k)$, $y_i \in \phi(v)$. In fact, $O_m \supset \phi(U_{ij}) = \phi(U_{kt})$, $\cap Q_m \supset O_m$ as defined above, and so, $\phi(u) \supset O_m$ and $\phi(v) \supset O_m$. Thus, $\hat{U}(y_i) = \hat{U}(y_k)$ and $s_i = s_k$ contrary to the assumption above.

Case A(2): $U_{ij} = U_{kt} \in H \subset U^1$, $y_i \in B$, and $y_k \in B$.

Case (a): $e_{y_i} = \min \left\{ t \mid y_i \in \cap D_t - \bigcup_{j<t} (\cup H_j') \right\}$. Now, U_{ij} is chosen such that $U_{ij} \in H_{e_{y_i}}'$, $U_{ij} \supset \bar{f}_{ij}^2$, $y_i \in \phi(U_{ij})$, and $\phi(U_{ij}) \cap (\cap D_t) = \emptyset$ for $t < e_{y_i}$. It follows that $e_{y_i} = e_{y_k}$ since $y_k \in \phi(U_{ij}) = \phi(U_{kt})$, $\phi(U_{ij}) \cap (\cap D_{e_{y_i}}) \neq \emptyset$, $\phi(U_{ij}) \cap (\cap D_{e_{y_k}}) \neq \emptyset$, and $\phi(U_{kt}) \cap (\cap D_{e_{y_i}}) \neq \emptyset$. Recall that $s_i = \min\{s \mid \text{int } K_s^Y \supset \overline{\phi(u)} \text{ for all } u \in \hat{U}(y_i)\}$ and $s_k = \min\{s \mid \text{int } K_s^Y \supset \phi(u) \text{ for all } u \in \hat{U}(y_k)\}$

Claim: $\hat{U}(y_i) = \hat{U}(y_k)$. Suppose that $h \in \hat{U}(y_i)$ and $h \notin \hat{U}(y_k)$. Now, $y_i \in \phi(U_{ij})$ and $y_k \in \phi(U_{ij})$. By Property (7) of U_0^1, if $u \in H \subset U_0^1$ and $\phi(u) \cap (\cap D_t) \neq \emptyset$ for the smallest t, then $\phi(h) \supset \phi(u)$ for all $h \in \hat{U}^1$ such that there is no $B_q \in D_t$ with the property that $\partial\phi(h) \supset B_q$ and $\phi(h) \cap \phi(u) \neq \emptyset$. Since $\phi(h) \cap (\cap D_{e_{y_i}}) \neq \emptyset$, it follows that there is no $B_q \in D_{e_{y_i}}$ such that $\partial\phi(h) \supset B_q$. Consequently, $\phi(h) \supset \phi(U_{ij})$ and $h \in \hat{U}(y_k)$. A similar argument shows that if $h \in \hat{U}(y_k)$, then $\phi(h) \supset \phi(U_{ij})$ and $h \in \hat{U}(y_i)$. The claim is proved and $s_i = s_k$ contrary to the assumption above.

Case (b): $e_{y_i} = \min\{t \mid y_i \in \cap D_t\}$ and $d_{y_i} = \min\{t \mid y_i \in \phi(u),\ u \in H, \text{ and } \phi(u) \cap (\cap D_t) \neq \emptyset\}$. Also, U_{ij} is chosen in H, $1 \leq j \leq t_i^2$, $U_{ij} \supset \bar{f}_{ij}^2$, and $\phi(U_{ij}) \cap (\cap D_t) \neq \emptyset$

for the smallest t where $d_{y_i} \leq t$.

If Case (a) applies to y_k, that is, $e_{y_k} = \min\left\{t \mid y_k \in \cap D_t - \bigcup_{j<t}(\cup H'_j)\right\} = d_{y_k} = \min\{t \mid u \in H \subset U^1,\ y_k \in \phi(u), \text{ and } \phi(u) \cap (\cap D_t) \neq \emptyset\}$. It follows that Case (a) applies to y_i since $y_i \in \phi(U_{ij}) = \phi(U_{kt})$, $U_{kt} \in H'_{e_{y_k}}$, and $e_{y_i} = e_{y_k}$. This yields a contradiction. Hence, Case (b) applies to y_k as well.

Now, $e_{y_k} = \min\{t \mid y_k \in \cap D_t\}$ and $d_{y_k} = \min\{t \mid y_k \in \phi(u),\ u \in H, \text{ and } \phi(u) \cap (\cap D_t) \neq \emptyset\}$. Also, U_{kt} is chosen in H, $1 \leq t \leq t_k^2$, $U_{kt} \supset f_{kt}^2$, and $\phi(U_{kt}) \cap (\cap D_t) \neq \emptyset$ for the smallest t where $d_{y_k} \leq t$.

Now, $\hat{V}(y_i) = \left\{h \mid h \in \hat{U}(y_i) \text{ and } \phi(h) \supset \bigcup_{j=1}^{t_i^2} \phi(U_{ij})\right\}$. Also, $\hat{V}(y_k) = \left\{h \mid h \in \hat{U}(y_k) \text{ and } \phi(h) \supset \bigcup_{j=1}^{t_k^2} \phi(U_{kj})\right\}$. Recall that $s_i = \min\{s \mid \operatorname{int} K_s^Y \supset \overline{\phi(h)} \text{ for all } h \in \hat{V}(y_i)\}$ and $s_k = \min\{s \mid \operatorname{int} K_s^Y \supset \overline{\phi(h)} \text{ for all } h \in \hat{V}(y_k)\}$.

Since $y_i \in \phi(U_{ij}) = \phi(U_{kt})$ and $y_k \in \phi(U_{kt})$, it follows by Property (5) of U_0^1 that $\phi(U_{ij}) \cap (\cap(D_{e_{y_i}} \cup D_{e_{y_k}})) \neq \emptyset$. Consequently, $D_{e_{y_i}} = D_{e_{y_k}}$; otherwise, the definitions of e_{y_i} and e_{y_k} are violated.

Claim: $\hat{V}(y_i) = \hat{V}(y_k)$. Suppose that $h \in \hat{V}(y_i)$ and $h \notin \hat{V}(y_k)$. Thus, there is some q, $1 \leq q \leq t_k^2$, such that $\phi(h) \not\supset \phi(U_{kq})$. By Property (10) of U_0^1, there is $B_s \subset B$ such that $\partial\phi(h) \supset B_s$ and $\phi(U_{kq}) \cap B_s \neq \emptyset$. Now, by Property (5), $\phi(U_{kq}) \cap (\cap(D_{e_{y_k}} \cup \{B_s\})) \neq \emptyset$. By the definition of $D_{e_{y_k}}$, it follows that $B_s \in D_{e_{y_k}} = D_{e_{y_i}}$; otherwise, the definition of e_{y_k} is violated. Also, $\phi(h) \supset \phi(U_{ij})$ and $\phi(U_{ij}) \cap (\cap D_{e_{y_i}}) \neq \emptyset$. Consequently, $B_s \subset \partial\phi(h)$ can not be in $D_{e_{y_i}}$ since $\phi(h) \cap (\cap D_{e_{y_i}}) \neq \emptyset$. Thus, it follows that $\phi(h) \supset \phi(U_{kq})$ for each q, $1 \leq q \leq t_k^2$, and $h \in \hat{V}(y_k)$. Hence, $\hat{V}(y_k) \supset \hat{V}(y_i)$. By a similar argument, $\hat{V}(y_i) \supset \hat{V}(y_k)$ and $\hat{V}(y_i) = \hat{V}(y_k)$. The claim is true. Hence, $s_i = s_k$ contrary to the assumption above.

Note that if $y_i \notin B$, then $U_{ij} \notin H$ and if $y_k \in B$, then $U_{kt} \in H$. Thus, $U_{ij} \neq U_{kt}$.

Case (B): Suppose that $u \in V^2$ and $v \in V^2$ are two different singleton distinguished families in V^2 which are, of course, in W_F^2. If $w_{s_u}^1 \in W_F^1$ and $w_{s_v}^1 \in W_F^1$ are chosen for u and v, respectively, then by definition of α_1 and β_1, $\alpha_1(u) = u' \in U^1$, $u' \subset w_{s_u}^1$, $\beta_1(u') = w_{s_u}^1$, $\alpha_1(v) = v' \in U^1$, $v' \subset w_{s_v}^1$, and $\beta_1(v') = w_{s_v}^1$. Clearly, α_1 and β_1 are well defined.

Case (C): Suppose that β_1 is not well defined and there exists F_i^2, a non degenerate distinguished family in V^2 and $w \in W_F^2$, a singleton distinguished family, such that $s_i \neq s_w$, $F_{s_i}^1$ is chosen for F_i^2, $w_{s_w}^1 \in W_F^1$ is chosen for w, and for some j, $1 \leq j \leq t_i^2$, $U_{ij} = w \supset f_{ij}^2 \cup w$. Here, $\phi(U_{ij}) = \phi(w) \supset r_{y_i}$. This contradicts the choice of r_{y_i} such that $\cup\phi(W_F^2) \not\supset r_{y_i}$.

It should be clear that α_1 and β_1 are well defined.

Clearly, β_1 is defined on U^1 since U^1 is irreducible and V^2 refines U^1. Observe that π_1 maps distinguished families onto distinguished families. Observe also that $\pi_1 : V^2 \to V^1$ is equivariant relative to the natural actions of certain cyclic groups whose orders are powers of p and are generated by some $g \in A_p$, g not the identity. A nondegenerate distinguished family $F_i^2 = \{f_{ij}^2\}_{j=1}^{t_i^2}$ where $t_i^2 = p^{s_i}$ in V^2 can be labelled so that for $g \in A_p - H_1$, $g^s(f_{i1}^2) = f_{i1+s}^2$, $0 \leq s \leq p^{s_i} - 1$, and $g^{p^{s_i}}(f_{i1}^2) = f_{i1}^2$. That is, $Z_{p^{s_i}}$ acts on F_i^2. Also, π_1 sends F_i^2 to a nondegenerate distinguished family $F_{m_i}^1 = \{f_{m_ij}^1\}_{j=1}^{t_{m_i}^1}$ where $t_{m_i}^1 = p^{s_{m_i}}$, and $Z_{p^{s_i}}$ acts on $F_{m_i}^1$, and π_1 commutes with the actions. That is, the image of $\pi_1(f_{ij}^2)$ under $Z_{p^{s_i}}$ is the same as the projection by π_1 of the image of f_{ij}^2 under the action by $Z_{p^{s_i}}$. Also, π_1 induces $\pi_1^* : N(V^2) \to N(V^1)$ and is equivariant relative to the action of $Z_{p^{s_i}}$ on $N(V^2)$ where $Z_{p^{s_i}}$ projects to $Z_{p^{m_i}}$ which acts on $N(V^1)$.

The first steps in the proof of Lemma 5 are complete.

With V^i defined (as indicated for $i = 1$ and 2), define U^i in the manner that U^1 is defined for V^1. Use Lemma 4 to obtain a finite open covering R^{i+1} of Y satisfying the conditions of Lemma 4 where V^i replaces W_2, and W_F^{i+1} replaces W_F, and R^{i+1} replaces

R such that R^{i+1} generates a special covering V^{i+1} of M having the properties similar to those described above for V^2 w.r.t. U^1 and V^1 but w.r.t. U^i and V^i. Similarly, define α_i, β_i, and π_i in the manner that α_1, β_1, and π_1 are described above. Extend α_i, β_i, and π_i in the usual manner to the nerves $N(V^{i+1})$, $N(U^i)$, and $N(V^i)$, respectively.

It should be clear that the proof of Lemma 5 can be completed using mathematical induction and the methods employed above.

Orientation Of The Simplices In $N(V^m)$

Next, orient the simplices in $N(V^m)$ for each special covering V^m. Recall that for each special covering V^m, there is associated a covering R^m of Y which generates V^m. Let $R = R^m$ and $V = V^m$. Suppose that $(v_0, v_1, v_2, \cdots, v_k) = \sigma^k$ is a k-simplex in $N(R)$. For each i, $0 \leq i \leq k$, $\phi^{-1}(v_i) = v_{i1} \cup v_{i2} \cup \cdots \cup v_{it_i}$ where $\{v_{i1}, v_{i2}, \cdots, v_{it_i}\}$ is the distinguished family determined by v_i. Consequently, σ^k determines a distinguished family of k-simplices in $N(\phi^{-1}(R))$ where $\phi^{-1}(R) = \{c \mid c$ is a component of $\phi^{-1}(r)$ for $r \in R\}$. If $q \in N[\sigma^k]$, the nucleus or carrier of σ^k, then $\phi^{-1}(q) \cap v_{ij} \neq \emptyset$ for each $i = 1, 2, \cdots, k$ and each $j = 1, 2, \cdots, t_i$. The orientation of a k-simplex $\delta^k = (v_{0j_0}, v_{1j_1}, \cdots, v_{kj_k})$ is concordant with that of σ^k as indicated by the given order of the vertices $(v_{0j_0}, v_{1j_1}, \cdots, v_{kj_k})$ of δ^k. Since $\phi^{-1}(v_i)$ has t_i components, there will be at least t_i k-simplices in $N(\phi^{-1}(R))$ which are mapped to σ^k by the simplicial mapping $\phi^* : N(\phi^{-1}(R)) \to N(R)$ induced by ϕ. This collection of k-simplices is the *distinguished family determined by* σ^k (more precisely, determined by $N[\sigma^k]$). If $N[\sigma^k] \cap \phi(F) = \emptyset$, then (1) $v_i \cap \phi(F) = \emptyset$ for some i, (2) the distinguished family of k simplices in $N(V^m)$ has cardinality p^c for some natural number c, and (3) $N[\sigma^k]$ is connected and ulc. If $N[\sigma^n] \cap \phi(F) \neq \emptyset$ where σ^n is an n-simplex in $N(R)$, then (1) int $\phi(F) \supset N[\sigma^n]$ and (2) σ^n determines a distinguished family of one n-simplex δ^n in $N(V^m)$ and int $F \supset N[\delta^n]$ by construction.

Distinguished Families Of n-Simplices In $N(V^m)$

The distinguished families $F_i^1 = \{f_{ij}^1\}_{j=1}^{t_i^1}$ of members of the covering V^1 generate distinguished families of n-simplices. That is, for distinguished families $F_{k_i}^1$, $0 \le i \le n$, in V^1 such that $\bigcap_{i=1}^{n}(F_{k_i}^1)^* \neq \emptyset$ where $(F_{k_i}^1)^* = \bigcup_{j=1}^{t_i^1} f_{k_i j}^1$, the $F_{k_i}^1$, $0 \le i \le n$, generate a distinguished family of n-simplices consisting of all n-simplices $\{f_{k_0 j_0}^1, f_{k_1 j_1}^1, \cdots, f_{k_n j_n}^1\}$ such that (1) $f_{k_i j_i}^1 \in F_{k_i}^1$, $0 \le i \le n$, and (2) $\bigcap_{i=0}^{n} f_{k_i j_i}^1 \neq \emptyset$. As pointed out above, a distinguished family of n-simplices is determined by an n-simplex σ^n in $N(R)$ and such a family is a lifting of σ^n to $N(V^m)$.

Consider a distinguished family S_k^1 of n-simplices in $N(V^1)$ as defined above such that a distinguished family S_q^2 of n-simplices defined similarly in $N(V^2)$ using distinguished families in V^2 is mapped onto S_k^1 by $\pi_1^* : N(V^2) \to N(V^1)$. By construction, each n-simplex in S_k^1 is the image of exactly p^c n-simplices for fixed c, a non negative integer, where card $S_q^2 = p^{c_q}$, card $S_k^1 = p^{c_k}$, and $c = c_k - c_q$.

For each natural number m, define distinguished families S_i^m of n-simplices in $N(V^m)$ as described above. Each such family S_i^m is the lifting of an n-simplex γ_i^n in $N(R^m)$, that is, $\phi : M \to M/A_p$ induces a mapping $\phi^* : N(V^m) \to N(R^m)$. If γ_i^n is an n-simplex in $N(R^m)$, then $(\phi^*)^{-1}(\gamma_i^n)$ is the union of a distinguished family of n-simplices S_i^m in $N(V^m)$. The family S_i^m is the lifting of γ_i^n in $N(V^m)$. Each member of S_i^m has an orientation concordant with that of γ_i^n.

Observe that if S_i^m and S_j^m are two non degenerate distinguished families of n-simplices such that $S_i^m = \{\delta_{it}^n\}_{t=1}^{p^{c_i}}$, $S_j^m = \{\delta_{jt}^n\}_{t=1}^{p^{c_j}}$, and δ_{it}^n shares and $(n-1)$-face with δ_{js}^n, then for each t, $1 \le t \le p^{c_i}$, there is some s', $1 \le s' \le p^{c_j}$, such that δ_{it}^n shares an $(n-1)$-face with $\delta_{js'}^n$ and, conversely, for each s, $1 \le s \le p^{c_j}$, there is some t, $1 \le t \le p^{c_i}$, such that δ_{js}^n and δ_{it}^n share an $(n-1)$-face.

If an n-simplex δ^n in $N(V^m)$ is such that $F_\phi \supset N[\delta^n]$, the nucleus of δ^n, then δ^n constitutes a singleton distinguished family in $N(V^m)$ as described above.

The n-Skeleton, $N(V_n^m)$, Of $N(V^m)$, And Inverse Limit $H_n(V_n^m) \cong Z_p$

The proof uses the n-skeleta, $N(V_n^{m+1})$, of $N(V^{m+1})$, the nerve of V^{m+1}, and the fact that the n^{th} simplicial homology of $N(V_n^{m+1})$ determine (using the inverse limit) the n^{th} Čech homology group, Z_p, of M.

Let $N(V_n^m)$ denote the n-skeleton of $N(V^m)$. Recall that V^m denotes the special m^{th} covering. The special projections $\pi_m^* : N(V^{m+1}) \to N(V^m)$, factors by $\alpha_m^* : N(V^{m+1}) \to N(U^m)$ and $\beta_m^* : N(U^m) \to N(V^m)$ where $\pi_m^* = \beta_m^* \alpha_m^*$.

Let $H_n(V_n^m)$ denote the n^{th} simplicial homology of $N(V_n^m)$, the n-skeleton of $N(V^m)$. The coefficient group is always Z_p. The following two lemmas give facts concerning the homology which will be needed to finish the proof of the Theorem.

First, consider the following commutative diagram:

$$
\begin{array}{ccccccccccc}
\leftarrow & H_n(V^m) & \overset{\beta_m^*}{\longleftarrow} & H_n(U^m) & \overset{\alpha_m^*}{\longleftarrow} & H_n(V^{m+1}) & \overset{\beta_{m+1}^*}{\longleftarrow} & H_n(U^{m+1}) & & \leftarrow & \\
 & \nu_m \uparrow & \beta_m^* \swarrow & & \nwarrow \alpha_m^* & \nu_{m+1} \uparrow & \beta_{m+1}^* \swarrow & & & \nwarrow \alpha_{m+1}^* & \\
\leftarrow & H_n(V_n^m) & & \overset{\pi_m^*}{\longleftarrow} & & H_n(V_n^{m+1}) & & \overset{\pi_{m+1}^*}{\longleftarrow} & & H_n(V_n^{m+2}) & \leftarrow
\end{array}
$$

Here ν_m is the natural map of a cycle in $H_n(V_n^m)$ into its homology class in $H_n(V^m)$. The other maps are those induced by the projections α_m, β_m and π_m. The upper sequence, of course, yields the Čech homology group $\check{H}_n(M)$ as its inverse limit. Furthermore, it can be easily shown, using the diagram, that $\check{H}_n(M)$ is isomorphic to the inverse limit $G = \varprojlim H_n(V_n^m)$, of the lower sequence. Specifically, $\gamma : \check{H}_n(M) \to G$ defined by $\gamma(\Delta) = \{\beta_m^*(\pi_{U^m}(\Delta))\}$ (where Δ is a generator of $\check{H}_n(M)$) is an isomorphism of $\check{H}_n(M)$ onto G. We shall use the isomorphism in what follows and for convenience we shall let $\gamma(\Delta) = \{z_m^n(\Delta)\}$, i.e. $z_m^n(\Delta) = \beta_m^*(\pi_{U^m}(\Delta)) \in H_n(V_n^m)$.

Let $g \in A_p - H_1$ with A_p as in the Standing Hypothesis. Consequently, g sends an element of V^{m+1} to an element of V^{m+1} and g induces a simplicial homeomorphism $\bar{g}$ of $N(V^{m+1})$ onto $N(V^{m+1})$ and, thus, induces a simplicial homeomorphism $\bar{g}$ of $N(V_n^{m+1})$

onto $N(V_n^{m+1})$. Clearly, π_m^* commutes with $\bar{g}$. Hence, $\bar{g}$ induces a homomorphism of $H_n(V_n^{m+1})$ onto $H_n(V_n^{m+1})$ which commutes with the induced projections. Finally, g induces a homomorphism of Z_p onto Z_p.

Let Δ be the generator of $\check{H}(M)$ where $\Delta = \{z_m^n(\Delta)\}$, a sequence of n-cycles such that $\pi_m^* : z_{m+1}^n(\Delta) \to z_m^n(\Delta)$ where $z_m^n(\Delta)$ is the m^{th} coordinate of Δ, i.e., $\pi_{V_n^m}(\Delta) = z_m^n(\Delta)$ an n-cycle in $N(V_n^m)$ where $N(V_n^m)$ is the n-skeleton of $N(V^m)$. If $s\Delta$, $s \in Z_p$, is any n-cycle in $\check{H}(M)$, then the coordinate n-cycles $z_m^n(s\Delta)$ and $z_m^n(\Delta)$ contain exactly the same n-simplices in $N(V_n^m)$ (see Lemma 7 below).

It follows from Lemma 2 that there is no loss of generality in assuming that $\pi_{V_n^m} : \check{H}(M) \to H_n(V_n^m)$ has the property that $\pi_{V_n^m}(\check{H}(M)) \cong Z_p$ for each natural number m.

The Operator σ

4. *An Operator σ is defined on n-chains.*

This type of operator was first used by P.A. Smith, later by Chernavsky [27], and now in a modified form.

Take A_p (as in the Standing Hypothesis) such that each member of A_p is homotopic to the identity and, therefore, has degree one and preserves the orientation of M. Take $g \in A_p - H_1$. See Lemma 1. Next, define an operator σ [cf. 27, 10] on n-chains. Recall that if δ^n is an n-simplex in $N(V^m)$, then either $N[\delta^n] \subset$ interior of F_ϕ or $N[\delta^n] \cap F_\phi = \emptyset$.

(1) If δ^n is an n-simplex in $N(V^{m+1})$ and int $F \supset N[\delta^n]$, then $\sum_{s=0}^{p-1} g^s(\delta^n) = p\delta^n = 0$ mod p since $g^s(\delta^n) = \delta^n$.

(2) If δ^n is an n-simplex in $N(V^{m+1})$ and $N[\delta^n] \cap F = \emptyset$, then δ^n is in a unique non degenerate distinguished family S_i^{m+1} of n-simplices which has cardinality p^{c_i}.

To see this, observe that if δ^n is a singleton distinguished family in $N(V^m)$, then the "vertices" of δ^n are members of W_F^{m+1} and, consequently, $N[\delta^n] \cap F \neq \emptyset$ by the construction of W_F^{m+1}.

(3) If $\pi_m^*(\delta^n)$ is an n-simplex, then each n-simplex in S_i^{m+1} projects by π_m^* to an n-simplex in $N(V^m)$. If $\pi_m^*(\delta^n)$ is a k-simplex, $k < n$, then each n-simplex in S_i^{m+1} projects under π_m^* to a k-simplex.

Definition. For each n-simplex δ^n in the nerve, $N(V^{m+1})$, of V^m, let $\sigma_m(\delta^n) = \sum_{s=0}^{p-1} g^s(\delta^n)$ where g^0 is the identity homeomorphism. The collection $\{g^s(\delta^n)\}_{s=0}^{p-1}$ is called a distinguished subfamily of the distinguished family in $N(V^{m+1})$ to which δ^n belongs.

Observe that if δ^n is an n-simplex in $N(V^{m+1})$ such that $N[\delta^n] \cap F = \emptyset$ and $\pi_m^*(\delta^n)$ is an n-simplex in $N(V^m)$, then $N[\pi_m^*(\delta^n)] \cap F = \emptyset$ by the construction of W_F^{m+1} and either (1) $\pi_m^*(\delta^n)$ is a singleton family in $N(V^m)$ or (2) $\pi_m^*(\delta^n)$ is in a non degenerate distinguished family S_j^m of n-simplices in $N(V^m)$.

In Case (2), π_m^* maps $\{g^s(\delta^n)\}_{s=0}^{p-1}$ one-to-one onto $\{\pi_m^*(g^s(\delta^n))\}_{s=0}^{p-1} = \{g^s(\pi_m^*(\delta^n))\}_{s=0}^{p-1}$ and $\pi_m^*\sigma_m\delta^n = \sigma_{m-1}\pi_m^*\delta^n$. It should be clear that if δ^n is an n-simplex in $N(V^{m+1})$, then $\pi_m^*\sigma_m\delta^n = \pi_m^*\sum_{s=0}^{p-1} g^s(\delta^n) = \sum_{s=0}^{p-1}\pi_m^* g^s(\delta^n) = \sum_{s=0}^{p-1} g^s(\pi_m^*(\delta^n)) = \sigma_m\pi_m^*\delta^n$. Note that $\pi_m^* g^s = g^s\pi_m^*$. Clearly, σ_m can be extended to any n-chain in $N(V_n^m)$. If δ^n is an n-simplex in $N(V_n^m)$ and $\pi_m^*\delta^n$ is a k-simplex, $k < n$, then $\pi_m^*\sigma_m(\delta^n)$ is a trivial n-chain.

Recall that the members of a distinguished family of k-simplices in $N(V^m)$ have concordant orientations (being the lifting of a k-simplex in $N(R^m)$ where R^m is a certain cover of Y).

Recall that if δ^n is an n-simplex in $N(V^m)$, then either interior $F \supset N[\delta^n]$ or $N[\delta^n] \cap F = \emptyset$. This follows from the construction of V^m.

Lemma 6. [cf. 27] *The special operator σ_m maps n-cycles to n-cycles and σ_m commutes with the special projections on n-chains. If $z_{m+1}^n(\Delta) = z$ is a coordinate n-cycle in $N(V_n^{m+1})$, then either (a) $\sigma_m z = 0$ or (b) $\sigma_m(z) = z$.*

Proof. The homeomorphism g induces a simplicial homeomorphism $\bar{g}$ of $N(V^{m+1})$ onto itself since g maps "vertices" (elements of V^{m+1}) one-to-one onto "vertices".

Let Δ be a non zero n-cycle in $\check{H}_n(M) = \varprojlim H_n(V_n^{m+1}) \cong Z_p$. Let $z = z_{m+1}^n(\Delta)$ be the coordinate n-cycle of Δ in $H_n(V_n^{m+1})$. Let $\sigma = \sigma_m$. Consider $\sigma(z)$ where $z = \sum_{i=1}^{k} c_i \delta_i^n$. Thus, $\sigma z = \sum_{i=1}^{k} c_i \sigma \delta_i^n$ where $\sigma \delta_i^n = \sum_{s=0}^{p-1} g^s(\delta_i^n)$ and $\{g^s(\delta_i^n)\}_{s=0}^{p-1}$ is the distinguished subfamily of n-simplices in $N(V^{m+1})$ associated with δ_i^n. It follows by the construction that there is $z_{m+2}^n(\Delta)$, a coordinate n-cycle in $N(V_n^{m+2})$ such that $g^s(z_{m+2}^n(\Delta))$ maps by π_{m+1}^* onto $g^s(z)$. Thus, $g^s(z)$ for $0 \le s \le p-1$ contains the same n-simplices as z (see Lemma 7). Hence, $\sigma z = \sum_{t=0}^{p-1} z_t$ where $z_0 = z$ and $z_t = \sum_{i=1}^{k} c_i g^t(\delta_i^n)$ where $g^t(\delta_i^n)$ means δ_i^n whenever int $F \supset N[\delta_i^n]$. Since g is a homeomorphism on M, g induces an automorphism on $\pi_{V^{m+1}}(G)$ in $H_n(V_n^m)$ where $G = \varprojlim H_n(V_n^m) \cong Z_p$, $\pi_{V_n^{m+1}}(G) \cong Z_p$, and z_t is an n-cycle. It follows that $\sigma z = \sum_{s=0}^{p-1} g^s(z)$ which is an n-cycle. If g induces the identity automorphism, then $\sigma z = pz = 0 \bmod p$ (the trivial n-cycle). If the induced automorphism is not the identity, then it will be shown that $\sum_{t=1}^{p-1} z_t = 0 \bmod p$ and that $\sigma z = z$, that is, σ is the identity automorphism. Now, $\pi_{V_n^{m+1}}(G) \cong Z_p = \{0, 1, 2, \cdots, p-1\}$. Since g induces an automorphism on $\pi_{V_n^{m+1}}(G)$, it induces an automorphism g_* on Z_p. Let $g_*(1) = x$. Hence, $g_*^s(1) = x^s$. It is well known in number theory that p divides $x^{p-1}-1$. Also, $x^{p-1}-1 = (x-1)(1+x+x^2+\cdots+x^{p-2})$. Consequently, p divides $x(1+x+x^2+\cdots+x^{p-2}) = x+x^2+\cdots+x^{p-1} = \sum_{s=1}^{p-1} x^s = 0 \bmod p$. It follows that $\sigma(z) = \sum_{s=0}^{p-1} g^s(z) = z + \sum_{s=1}^{p-1} g^s(z) = z \bmod p$. Thus, if g does not induce the identity automorphism, then σ is the identity automorphism.

If under the projection $\pi_m^* : H_n(V_n^{m+1}) \to H_n(V_n^m)$, the image of an n-simplex δ^n such that $N[\delta^n] \subset M - F$ is a k-simplex with $k < n$, then the same is true for all members of the distinguished family to which δ^n belongs. Thus, $\pi_m^* \sigma_m \delta^n = \sigma_m \pi_m^* \delta^n = 0$ where $\sigma_m \pi_m^* \delta^n$ is defined above and is a trivial n-chain. If $\pi_m^*(\delta^n)$ is an n-simplex, then by construction the distinguished subfamily of n-simplices with which δ^n is associated is in one-to-one correspondence with the distinguished subfamily of n-simplices with which $\pi_m^*(\delta^n)$ is associated. Thus, $\pi_m^*\sigma_m\delta^n = \sigma_m\pi_m^*\delta^n$. If $N[\delta^n] \subset F_\phi$, then $\pi_m^*\sigma_m\delta^n = 0 = \sigma_m\pi_m^*\delta^n$. Consequently, σ carries over to the n-cycles of M and to $\check{H}(M)$. Lemma 6 is proved.

Lemma 7. *If Δ_1 and Δ_2 are non zero elements of $\check{H}_n(M)$, then for each m, exactly the same simplices appear in the chains $z_m^n(\Delta_1)$ and $z_m^n(\Delta_2)$ which are in the n-dimensional complex, $N(V_n^m)$, the n-skeleton of $N(V^m)$.*

Proof. Suppose that this is not true. Without loss of generality, assume that the n-simplex δ^n appears in $z_m^n(\Delta_1)$ and not in $z_m^n(\Delta_2)$ where $z_m^n(\Delta_1)$ and $z_m^n(\Delta_2)$ are the m^{th} coordinates of Δ_1 and Δ_2, respectively. These coordinates are n-cycles in $N(V_n^m)$. Since $\check{H}_n(M) = Z_p$, assume that Δ_1 generates $\check{H}_n(M)$ and that $\Delta_2 = s\Delta_1$ for some natural number s, $1 \leq s < p$. Consequently, $z_m^n(\Delta_2) = s(z_m^n(\Delta_1))$. It follows that δ^n appears in $s(z_m^n(\Delta_1))$ and hence in $z_m^n(\Delta_2)$ – a contradiction.

Lemma 8. *Suppose that $\Delta \in \check{H}_n(M)$ with $\Delta \neq 0$ and $z = z_{m+1}^n(\Delta) = \pi_{V_n^{m+1}}(\Delta)$, the coordinate n-cycle of Δ in $H_n(V_n^{m+1})$. Let $z = \sum_{i=1}^{q} c_i\delta_i^n$. Then the collection $C_j = \{\delta_{j_1}^n, \delta_{j_2}^n, \cdots, \delta_{j_{t_j}}^n\}$ of all n-simplices in $\{\delta_1^n, \delta_2^n, \cdots, \delta_q^n\}$ which are in a fixed distinguished family S_j^{m+1} of n-simplices in $N(V_n^{m+1})$ have the properties (1) if $x = \sum_{i=1}^{t} c_{j_i}\delta_{j_i}^n$, then either (a) $\sigma_m x = 0$ when g induces the identity automorphism or (b) $\sigma_m x = x$ when g does not induce the identity automorphism and (2) $\sum_{i=1}^{t} c_{j_i} = 0$ mod p. Recall from Lemma 1 that g preserves the orientation of M.*

Proof. By Lemma 6, either (a) $\sigma_m z = 0$ or (b) $\sigma_m(z) = z$.

Case (a): Since $\sigma_m(z) = 0$, it follows that $\sigma_m x = 0$ since for each i, $1 \leq i \leq t$, $\sigma_m\delta_{j_i}^n = \sum_{s=0}^{p-1} g^s(\delta_{j_i}^n)$ where for each s, $0 \leq s \leq p-1$, $g^s(\delta_{j_i}^n)$ is an n-simplex in $C_q \subset S_j^{m+1}$. Choose notation such that

(1) $S_i^{m+1} = \{\delta_j\}_{j=1}^{p^k}$ ($p^k = p^{c_j}$ in earlier notation),

(2) $g\delta_j = \delta_{j+1}$ for $1 \le j < p^k$ and $g\delta_{p^k} = \delta_1$ (g permutes the δ_j in a cyclic order),

and

(3) $x_g = \sum_{i=1}^{p^k} c_i\delta_i$ where $c_i = 0$ iff $\delta_i \notin C_q$ and $c_i = c_{j_i}$ iff $\delta_i = \delta_{j_i}^n \in C_q$.

Let $\sigma = \sigma_m$. Since $\sigma(z) = 0$ and $\sigma(\delta_i) \in C_q$, $1 \le i \le p^k$, it follows that $\sigma(x_q) = 0 \bmod p$. Note that $0 = \sigma(x_q) = c_1\sum_{i=1}^{p}\delta_i + c_2\sum_{i=1}^{p}\delta_{i+1} + \cdots + c_j\sum_{i=1}^{p}\delta_{i+j-1} + \cdots + c_{p^k-p}\sum_{i=1}^{p}\delta_{i+p^k-p-1} + \cdots + c_{p^k}(\delta_{p^k} + \delta_1 + \delta_2 + \cdots + \delta_{p-1})$. Rearrange as

$$\begin{aligned}\sigma(x_q) = {} & (c_1 + c_{p^k} + c_{p^k-1} + \cdots + c_{p^k-p+2})\delta_1 + \\ & (c_2 + c_1 + c_{p^k} + \cdots + c_{p^k-p+1})\delta_2 + \\ & \vdots \\ & (c_p + c_{p-1} + c_{p-2} + \cdots + c_1)\delta_p + \\ & (c_{p+1} + c_p + c_{p-1} + \cdots + c_2)\delta_{p+1} + \\ & \vdots \\ & (c_{p^k} + c_{p^k-1} + \cdots + c_{p^k-p+1})\delta_{p^k}.\end{aligned}$$

Since $\sigma(x_q) = 0$, it follows that the coefficient of δ_i is 0 mod p for $1 \le i \le p^k$. A careful consideration of pairs of successive coefficients of δ_i and δ_{i+1} will give the following result. If $1 \le i \le p^k$, $1 \le j \le p^k$, and $i \equiv j \bmod p$, then $c_i \equiv c_j \bmod p$. If $k > 1$, then $\sum_{i=1}^{p^k} c_i = \sum_{i=1}^{p} c_i + \sum_{i=p}^{2p} c_i + \sum_{i=2p+1}^{3p} c_i + \cdots + \sum_{i=(p^{k-1}-1)p}^{(p^{k-1})p} c_i$ with p^{k-1} summations each of length p. Now, $c_i \equiv c_j \bmod p$ if $i \equiv j \bmod p$ gives that each of the p^{k-1} summations is congruent to 0 mod p. Thus, $\sum_{i=1}^{p^k} c_i = p^{k-1}\left(\sum_{i=1}^{p} c_i\right) \equiv 0 \bmod p$ if $k > 1$. If $k = 1$, then

$x_q = \sum_{i=1}^{p} c_i \delta_i ,$

$$\begin{aligned}\sigma(x_q) = {} & c_1(\delta_1 + \delta_2 + \cdots + \delta_p) \\ & + c_2(\delta_2 + \delta_3 + \cdots + \delta_{p+1}) \\ & \vdots \\ & + c_p(\delta_p + \delta_1 + \delta_2 + \cdots + \delta_{p-1}), \quad \text{rearrange as} \\ & (c_1 + c_2 + \cdots + c_p)\delta_1 + \\ & (c_1 + c_2 + \cdots + c_p)\delta_2 + \\ & \vdots \\ & (c_1 + c_2 + \cdots + c_p)\delta_p ,\end{aligned}$$

and $\sum_{i=1}^{p} c_i \equiv 0 \mod p$.

It is instructive to consider a simple example. Let $p = 3$ and $x_q = \sum_{i=1}^{9} c_i \delta_i$. Thus, $\sigma(x_q) = c_1(\delta_1 + \delta_2 + \delta_3) + c_2(\delta_2 + \delta_3 + \delta_4) + c_3(\delta_3 + \delta_4 + \delta_5) + c_4(\delta_4 + \delta_5 + \delta_6) + c_5(\delta_5 + \delta_6 + \delta_7) + c_6(\delta_6 + \delta_7 + \delta_8) + c_7(\delta_7 + \delta_9 + \delta_9) + c_8(c_8 + c_9 + c_1) + c_9(c_9 + \delta_1 + \delta_2) =$ (by rearrangement) $= (c_1 + c_8 + c_9)\delta_1 + (c_1 + c_2 + c_9)\delta_2 + (c_1 + c_2 + c_3)\delta_3 + (c_2 + c_3 + c_4)\delta_4 + (c_3 + c_4 + c_5)\delta_5 + (c_4 + c_5 + c_6)\delta_6 + (c_5 + c_6 + c_7)\delta_7 + (c_6 + c_7 + c_8)\delta_8 + (c_7 + c_8 + c_9\delta_9$. For each i, $1 \le i \le 9$, the coefficient of $\delta_i = 0 \mod 3$. Observe that from the coefficients of δ_1 and δ_2, it follows that $c_2 \equiv c_8 \mod 9$. The coefficients of δ_2 and δ_3 yield that $c_3 \equiv c_9 \mod 3$. Continuing, $c_1 \equiv c_4$, $c_2 \equiv c_5$, $c_3 \equiv c_6$, $c_4 \equiv c_7$, $c_5 \equiv c_8$, $c_6 \equiv c_9$, and $c_7 \equiv c_1$ all mod 3. Thus, $(c_1 + c_2 + c_3) + (c_4 + c_5 + c_6) + (c_7 + c_8 + c_9) = 0 \mod 3$ since $(c_4 + c_5 + c_6) \equiv (c_1 + c_2 + c_3) \mod 3$, $(c_7 + c_8 + c_9) \equiv (c_4 + c_5 + c_6) \equiv (c_1 + c_2 + c_3) \mod 3$ and $\sum_{i=1}^{9} c_i \equiv 3(c_1 + c_2 + c_3) \equiv 0 \mod 3$.

Case (b): $\sigma_m(x_q) = x_q$. Choose notation as in Case (a). Write $\sigma(x_q)$ as in Case (a), but in this case, $x_q = \sigma(x_q)$ rather than $0 = \sigma(x)$. Consider first the example $p = 3$ and $x_q =$

$\sum_{i=1}^{9} c_i\delta_i$ where $k = 2$. Now, $\sigma(x_q) = (c_1+c_8+c_9)\delta_1+(c_1+c_2+c_9)\delta_2+(c_1+c_2+c_3)\delta_3+(c_2+c_3+c_4)\delta_4+(c_3+c_4+c_5)\delta_5+(c_4+c_5+c_6)\delta_6+(c_5+c_6+c_7)\delta_7+(c_6+c_7+c_8)\delta_8+(c_7+c_8+c_9)\delta_9 = x_q = c_1\delta_1+c_2\delta_2+c_3\delta_3+c_4+\delta_4+c_5\delta_5+c_6\delta_6+c_7\delta_7+c_8\delta_8+c_9\delta_9$. This is an identity. Thus, the coefficient of δ_i on one side is equal mod p to the coefficient of δ_i on the other side. Hence, $c_1 + c_8 + c_9 \equiv c_1 \mod p$, $c_1 + c_2 + c_9 \equiv c_2 \mod p$, $c_1 + c_2 + c_3 \equiv c_3 \mod p$, and so forth. Thus, $\sum_{i=1}^{9} c_i \equiv (c_1+c_8+c_9)+(c_1+c_2+c_9)+(c_1+c_2+c_3)+(c_2+c_3+c_4)+(c_3+c_4+c_5)+(c_4+c_5+c_6)+(c_5+c_6+c_7)+(c_6+c_7+c_8)+(c_7+c_8+c_9) = 3(c_1+c_2+c_3+c_4+c_5+c_6+c_7+c_9) \equiv 0 \mod p$.

Consider the general case as in Case (a) but with $x_q = \sigma(x_q)$ rather than $0 = \sigma(x_q)$. Hence,

$$\begin{aligned} x_q = \sigma(x_q) = {} & (c_1 + c_{p^k} + c_{p^k-1} + \cdots + c_{p^k-p+2})\delta_1+ \\ & (c_2 + c_1 + c_{p^k} + \cdots + c_{p^k-p+1})\delta_2+ \\ & \vdots \\ & (c_p + c_{p-1} + c_{p-2} + \cdots + c_1)\delta_p+ \\ & (c_{p+1} + c_p + c_{p-1} + \cdots + c_2)\delta_{p+1}+ \\ & \vdots \\ & (c_{p^k} + c_{p^k-1} + \cdots + c_{p^k-p+1})\delta_{p^k} = \sum_{i=1}^{p^k} c_i\delta_i. \end{aligned}$$

It follows from this identity that the coefficient of δ_i on one side is equal mod p to the coefficient of δ_i on the other side. Consequently, $\sum_{i=1}^{p^k} c_i = p\sum_{i=1}^{p^k} c_i \mod p = 0 \mod p$ as claimed where $k > 1$. For $k = 1$, $x_q = \sum_{i=1}^{p} c_i\delta_i = \left(\sum_{i=1}^{p} c_i\right)\delta_1 + \left(\sum_{i=1}^{p} c_i\right)\delta_2 + \cdots + \left(\sum_{i=1}^{p} c_i\right)\delta_p$, $c_t \equiv \sum_{i=1}^{p} c_i \mod p$ for each t, $1 \le t \le p$, and $\sum_{i=1}^{p} c_i \equiv p\left(\sum_{i=1}^{p} c_i\right) \mod p = 0 \mod p$. Lemma 8 is proved.

Lemma 9. *If W is any domain in M (connected non-empty open set), then $\check{H}_n(M -$*

$W) = 0$ and if $\Delta \in \check{H}_n(M)$ with $\Delta \neq 0$, then there is a natural number m_0 such that for each $m > m_0$, the carrier of $z_m(\Delta)$ meets W, that is, if $z_m = \sum_{j=1}^{r} g_j \delta_j^n$ where $1 \leq g_j < p$, then for some j, $N[\delta_j^n] \cap W \neq \emptyset$ where $N[\delta_j^n]$ is the nucleus of δ_j^n.

Proof. Here, the fact that $\check{H}_n(M - W) = 0$ is used. Suppose that the lemma is false. There is a subsequence $\{V^{m_i}\}$ cofinal in the collection of all open coverings of M such that $z_{m_i}(\Delta)$ is an n-cycle in $N(V^{m_i} \mid (M - W))$. Thus, $z_{m_i}(\Delta)$ is not a bounding chain in $N(V^{m_i})$ and, hence not a bounding chain in $N(V^{m_i} \mid (M - W))$. Thus, $z_{m_i}(\Delta)$ determines a non zero element in $H_n(V^{m_i} \mid (M - W))$ and $\{z_{m_i}(\Delta)\}$ determines a non zero element of $\check{H}_n(M - W)$. This yields a contradiction. The lemma is proved.

Lemma 10. [cf. 27] *If $K =$ interior F_ϕ, then $K = \emptyset$ and F_ϕ is nowhere dense.*

Proof. Suppose that $K \neq \emptyset$. Let Δ be a non zero n-cycle in $\check{H}_n(M) \cong Z_p$ and $z_m^n(\Delta) = \beta_m^* \pi_{U^m}(\Delta) \in H_n(V_n^m)$. Let $z_m^n(\Delta) = C_1^m + C_2^m$ where C_1^m is an n-chain such that each n-simplex in C_1^m has a nucleus in $M - \bar{K}$ and C_2^m is an n-chain such that each n-simplex in C_2^m has a nucleus in K. It will be shown that both C_1^m and C_2^m are n-cycles. Let $z = z_m^n(\Delta)$.

Let $z = \sum_{i=1}^{k} c_i \delta_i^n$. If int $F_\phi \supset N[\delta_i^n]$ for some i, then $g^s(\delta_i^n) = \delta_i^n$ for each s. Suppose that δ_j^n shares an $(n-1)$-face δ^{n-1} with δ_i^n where int $F_\phi \not\supset N[\delta_j^n]$. Now, δ_j^n belongs to a nondegenerate distinguished family S_j^m of n-simplices in $N(V^m)$. Furthermore, each member of S_j^m shares the same $(n-1)$-face δ^{n-1}. Let $C_j = \{\delta_{j_1}^n, \delta_{j_2}^n, \ldots, \delta_{j_q}^n\}$ denote the collection of all n-simplices in S_j^m such that $\delta_{j_t}^n$, $1 \leq t \leq q$, is in z (with a non zero coefficient). By Lemma 8, $\sum_{t=1}^{q} c_{j_t} = 0 \bmod p$. Thus, the coefficient of δ^{n-1} in ∂C_1^m is 0 mod p.

To show that ∂C_1^m is 0, it suffices to show that each such $(n-1)$ simplex δ^{n-1} in ∂C_1^m which is a face of some δ_i^n where int $F \supset N[\delta_i^n]$ is 0. As shown above, this is the case and C_1^m is an n-cycle. Thus, $\partial C_1^m = 0$ and C_1^m is an n-cycle. It follows that C_2^m is an n-cycle. From the definition of the special projections, $\pi_m^*(C_i^{m+1}) = C_i^m$ for $i = 1, 2$. Thus, we can write $\Delta = \Delta_1 + \Delta_2$ where $z_m^n(\Delta_i) = C_i^m$ for $i = 1, 2$. Now, the nucleus of each simplex in $z_m^n(\Delta_2)$ misses the nonempty open set $M - \bar{K}$ for each m, so by Lemma 7, $\Delta_2 = 0$. Similarly, the nucleus of each simplex in $z_m^n(\Delta_1)$ misses the non-empty open set K for each m. Hence, by Lemma 7, $\Delta_1 = 0$. Consequently, we have $\Delta = 0$, a contradiction. The lemma is proved.

5. *A proof that a p-adic group A_p can not act effectively on a compact connected n-manifold where $\phi : M \to M/A_p$ is the orbit mapping.*

Remarks. If the compact connected n-manifold M has a non empty boundary, then two copies of M can be sewed together by identifying the boundaries in such a way that the result is a compact connected n-manifold M' without boundary. If A_p acts effectively on M, then A_p acts effectively on M'. If M is not an orientable n-manifold, then we can take the double cover of M on which A_p acts effectively if it acts effectively on M. There is no loss of generality in assuming that M is a compact connected orientable n-manifold without boundary.

Definition. An n-manifold (M, d) is said to have Newman's Property w.r.t. the class $L(M, p)$ (as stated above) iff there is $\epsilon > 0$ such that for any $\phi \in L(M; p)$, there is some $x \in M$ such that diam $\phi^{-1}\phi(x) \geq \epsilon$.

Generalizations can be made to metric spaces (X, d) which are locally compact, connected, and lc^n [4] which have domains D such that $\bar{D}$ is compact, lc^n, and $H_n(X, X - D), Z_p) \cong Z_p$.

Theorem. *If $L(M, p)$ is the class of all orbit mappings $\phi : M \to M/A_p$ where A_p acts effectively on a compact, connected, and orientable n-manifold M, then M has Newman's Property w.r.t. $L(M, p)$.*

Proof. There is no loss of generality in assuming that M is orientable and has empty boundary.

By hypothesis, $\check{H}_n(M) \cong Z_p$. Consider a finite open covering $U = W_1$ where W_1 and W_2 satisfy Lemma 2. If $z(\Delta)$ is the V-coordinate of a non-zero n-cycle $\Delta \in \check{H}_n(M)$ where V refines W_2, then $\pi_{VU} z(\Delta) \neq 0$. Let ϵ be the Lebesque number of W_2. Choose $\phi \in L(M,p)$ such that diam $\phi^{-1}\phi(x) < \epsilon$ for each $x \in M$. Construct the special coverings $\{V^m\}$ and the special refinements $\{U^m\}$ as in Lemma 5 such that the star of each distinguished family in V^1 lies in some element of W_2. Furthermore, the special projections π_m are such that if $\{\delta^n_{sj}\}_{j=1}^{t^m_s}$ is a distinguished family of n-simplices in $N(V^m_n)$, then $\pi_{V^m U}$ takes δ^n_{sj}, $1 \leq j \leq t_s$, to the same simplex δ_s in $N(U)$. Now, let $z_m = z^n_m(\Delta)$.

Let $z_m = \sum_{i=1}^{k} c_i \delta^n_i$. By Lemma 10, $F \cap N[\delta^n_i] = \emptyset$ for each i since int $F = \emptyset$. Hence, for each j, $1 \leq j \leq k$, δ^n_j is in a non degenerate distinguished family S^m_j of n-simplices in $N(V^m)$. Let $C_j = \{\delta^n_{j_1}, \delta^n_{j_2}, \cdots, \delta^n_{j_q}\}$ denote the collection of all n-simplices in S^m_j such that $\delta^n_{j_i}$ appears in z_m for $1 \leq i \leq q$ with non zero coefficients. By Lemma 8, $\sum_{i=1}^{q} c_{j_i} = 0$ mod p. Since the n-simplices in C_j are sent by $\pi_{V^m U}$ to a single simplex δ_j in $N(U)$, it follows that the coefficient of δ_j is 0 mod p and, therefore, z is sent by $\pi_{V^m U}$ to the zero n-cycle in $N(U)$. Thus, the projection of z_m by $\pi_{V^m U} : H_n(V^m_n) \to H_n(U)$ takes the nontrivial n-cycle $z_m(\Delta)$ to the 0 n-cycle mod p. This violates the conclusion of Lemma 2. Thus, M has Newman's Property w.r.t. the class $L(M,p)$. Hence, ϵ is a Newman's number and the Theorem is proved.

It is well known that if A_p acts effectively on a compact connected n-manifold M, then given any $\epsilon > 0$, there is an effective action of A_p on M such that diam $\phi^{-1}\phi(x) < \epsilon$ for each $x \in M$. That is, M fails to have Newman's property w.r.t. $L(M,p)$. It follows that A_p can not act effectively on a compact connected n-manifold M.

6. *How to obtain a proof that a p-adic group can not act effectively on a connected n-manifold.*

As indicated above, there is no loss of generality in assuming that M is a connected orientable n-manifold without boundary. If A_p acts effectively on M (which is locally compact), then the orbit map $\phi : M \to M/A_p$ is open and closed with $\phi^{-1}\phi(x)$ compact for each $x \in M$. Hence, ϕ is a proper map (if $M/A_p \supset A$ and A is compact, then $\phi^{-1}(A)$ is compact).

Construct sequences $\{V^m\}$ and $\{U^m\}$ of locally finite open coverings of M by constructing locally finite open coverings R^m of M/A_p in the same manner as in Lemma 3, 4, and 5 where each $r \in R^m$ has a compact closure. Since ϕ is proper, the distinguished families of open sets in V^m generated by members of R^m have the same properties as in Lemmas 1, 3, 4, and 5. The proof follows as in the compact case.

Consequently, the *Hilbert Smith Conjecture* is true.

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