E_∞-ring structures for Tate spectra

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1 Introduction.

Let G be a compact Lie group and k_G a G spectrum (as defined in [3, Section I.2]). Greenlees and May ([2]) have defined an associated G-spectrum $t(k_G)$ called the *Tate spectrum*. They observe that if k_G is a ring G-spectrum then there is an induced ring G-spectrum structure on $t(k_G)$, and that if k_G is homotopy-commutative then $t(k_G)$ will also be homotopy-commutative (see [2, Proposition 3.5]). It is therefore natural to ask whether an equivariant E_∞-ring structure on k_G induces an equivariant E_∞-ring structure on $t(k_G)$ (we will recall the definition in a moment). We offer both positive and negative answers to this question.

On the positive side, we show that $t(k_G)$ inherits a structure which is somewhat weaker than an equivariant E_∞-ring structure, but which should be adequate for most practical purposes. To explain this, let us recall from [3, Example VII.1.4] that to each G-universe U is associated an equivariant operad $\mathcal{L}(U)$. Let us fix a complete G universe U and let V denote the trivial G-universe U^G. An equivariant E_∞-ring structure is defined to be an action of an equivariant operad equivalent to $\mathcal{L}(U)$ (see [3, Definitions VII.2.1 and VII.1.2 and Remark VII.1.3]). Let us define an E'_∞-ring structure to be an action of an equivariant operad equivalent to $\mathcal{L}(V)$; since G acts trivially on $\mathcal{L}(V)$ we can rephrase this by saying that an E'_∞-ring structure is an action of a nonequivariant E_∞ operad through G-maps. Since there is a map of operads $\mathcal{L}(V) \to \mathcal{L}(U)$, an equivariant E_∞ structure specializes to an E'_∞ structure. On the other hand, Remark VII.2.5 of [3] shows that if k_G is an E'_∞-ring spectrum then the fixed point spectra $(k_G)^H$ have (nonequivariant) E_∞-ring structures which are consistent as H varies; this is likely to be the point most relevant for applications.

Our positive result is:

Theorem 1 *If k_G is an E'_∞-ring spectrum then so is $t(k_G)$; in particular all fixed-point spectra $(t(k_G))^H$ are nonequivariant E_∞-ring spectra.*

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The proof of Theorem 1 will show that the diagram in Proposition 3.5 of [2] is a diagram of E'_∞-ring spectra.

To state our negative result we need to recall the definition of $t(k_G)$. Let EG be a contractible free G-CW complex and let $\tilde{E}G$ denote the G-space defined by the cofiber sequence

$$EG_+ \to S^0 \to \tilde{E}G$$

(here $+$ denotes a disjoint basepoint). Let $F(EG_+, k_G)$ be the function spectrum of maps from EG_+ to k_G ([3, Definition I.3.2]). Then $t(k_G)$ is defined to be the G-spectrum

$$F(EG_+, k_G) \wedge \tilde{E}G.$$

Let us write ι for the natural map $S^0 \to \tilde{E}G$.

Theorem 2 *Let G be a finite cyclic group and let k_G be any G-spectrum. Suppose that $t(k_G)$ has an equivariant E_∞-ring structure whose unit factors (up to equivariant homotopy) through $\Sigma_G^\infty \iota$. Then $t(k_G)$ must be equivariantly contractible.*

This implies that if k_G is a ring G-spectrum for which $t(k_G)$ is not equivariantly contractible, then $t(k_G)$ cannot have an equivariant E_∞-ring structure whose underlying ring G-spectrum structure is compatible with that of k_G under the natural map $k_G \to t(k_G)$. In particular, the underlying ring G-spectrum structure of $t(k_G)$ cannot be that defined in [3, Proposition 3.5]. Thus it seems that there is no natural way to give $t(k_G)$ an equivariant E_∞-ring structure.

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2 Proof of Theorem 1.

Theorem 1 is an immediate consequence of the following two lemmmas, of which the second is well-known. Let us recall from [3, Definition VII.2.7] that, given an equivariant operad $\mathcal{C}$, a $\mathcal{C}_0$ space is an action of $\mathcal{C}$ in the category of based G-spaces; that is, it is a based G-space X with based G-maps

$$(\mathcal{C}_j)_+ \wedge_{\Sigma_j} X^{(j)} \to X$$

(here $^{(j)}$ denotes j-fold smash product) satisfying the same compatability conditions that are used to define an equivariant $\mathcal{C}$-space. In particular, this definition makes sense if $\mathcal{C}$ is a nonequivariant operad provided with the trivial G-action; it then says that $\mathcal{C}$ acts on X through G-maps.

Lemma 3 *There is a nonequivariant E_∞ operad $\mathcal{C}$ for which $\tilde{E}G$ is an equivariant $\mathcal{C}_0$ space.*

Lemma 4 *Let $\mathcal{C}$ be any equivariant operad.*

(a) *If k_G is a $\mathcal{C}$-ring spectrum (that is, if it has an equivariant action of $\mathcal{C}$) then so is $F(Y_+, k_G)$ for any G-space Y.*

(b) *If h_G is a $\mathcal{C}$-ring spectrum and X is a $\mathcal{C}_0$-space then $h_G \wedge X$ is a $\mathcal{C}$-ring spectrum.*

Theorem 1 follows from Lemma 4(b) if we let h_G be $F(EG_+, k_G)$ and X be $\tilde{E}G$.

Proof of Lemma 4. In each case, we specify the structural maps which constitute the $\mathcal{C}$-action; the fact that they satisfy the necessary compatibility relations is a straightforward application of the methods of [3, Sections VI.1–VI.3].

For part (a) the structural map

$$\xi_j : \mathcal{C}_j \ltimes F(Y_+, k_G)^{(j)} \to F(Y_+, k_G)$$

is the adjoint of the composite

$$\begin{aligned} &Y_+ \wedge \mathcal{C}_j \ltimes F(Y_+, k_G)^{(j)} \xrightarrow{\Delta\wedge 1} (Y_+)^{(j)} \wedge \mathcal{C}_j \ltimes F(Y_+, k_G)^{(j)} \\ &\xrightarrow{\cong} \mathcal{C}_j \ltimes ((Y_+)^{(j)} \wedge F(Y_+, k_G)^{(j)}) \xrightarrow{1\ltimes e} \mathcal{C}_j \ltimes k_G^{(j)} \xrightarrow{\xi'_j} k_G; \end{aligned}$$

here Δ is the diagonal map of Y, the isomorphism is that of [3, Proposition VI.1.5], e is the evaluation map, and ξ'_j is the structural map of k_G.

For part (b) the structural map

$$\xi_j : \mathcal{C}_j \ltimes (h_G \wedge X)^{(j)} \to h_G \wedge X$$

is the composite

$$\mathcal{C}_j \ltimes (h_G \wedge X)^{(j)} = \mathcal{C}_j \ltimes (h_G^{(j)} \wedge X^{(j)}) \xrightarrow{\delta} (\mathcal{C}_j \ltimes h_G^{(j)}) \wedge (\mathcal{C}_{j+} \wedge X^{(j)}) \xrightarrow{\xi'_j\wedge\xi''_j} h_G \wedge X,$$

where δ is the map given in Definition VI.3.5 of [3] and ξ'_j, ξ''_j are the structural maps for h_G and X. QED

Proof of Lemma 3. First let us observe that $\tilde{E}G$ is nonequivariantly contractible and that for any nontrivial subgroup H of G the H-fixed set $(\tilde{E}G)^H$ is exactly S^0; the same is true for $(\tilde{E}G)^{(j)}$ since the smash product of spaces commutes with H-fixed sets.

Let Map_*^G denote based G-maps. Restriction to the G-fixed set gives a map

$$\phi : \mathrm{Map}_*^G(\tilde{E}G^{(j)}, \tilde{E}G) \to \mathrm{Map}_*(S^0, S^0)$$

which we claim is a weak equivalence. Assuming this for the moment, let $\mathcal{C}'_j$ be the space $\phi^{-1}(\mathrm{id})$. Then the spaces $\mathcal{C}'_j$ with the evident composition operations γ form an operad $\mathcal{C}'$ and $\tilde{E}G$ is a $\mathcal{C}'_0$-space. The only thing preventing $\mathcal{C}'$ from being a nonequivariant E_∞ operad is that the action of Σ_j on $\mathcal{C}'_j$ may not be free. To remedy this let $\mathcal{C}''$ be any nonequivariant E_∞ operad and define $\mathcal{C}$ to be $\mathcal{C}' \times \mathcal{C}''$, acting on $\tilde{E}G$ via the projection $\mathcal{C}' \times \mathcal{C}'' \to C'$.

It only remains to prove the claim that ϕ is a weak equivalence. First we observe that the reduced diagonal map

$$\Delta : \tilde{E}G \to \tilde{E}G^{(j)}$$

is a weak equivalence on each fixed-point set, and is therefore a G-homotopy equivalence by the equivariant Whitehead theorem. It follows that

$$\Delta^* : \mathrm{Map}_*^G(\tilde{E}G, \tilde{E}G) \to \mathrm{Map}_*^G(\tilde{E}G^{(j)}, \tilde{E}G)$$

is a homotopy equivalence, so it suffices to verify the claim when $j = 1$.

To handle this case, we map the cofiber sequence

$$EG_+ \to S^0 \to \tilde{E}G$$

into $\tilde{E}G$ to get a fiber sequence

$$\mathrm{Map}_*^G(\tilde{E}G, \tilde{E}G) \to \mathrm{Map}_*^G(S^0, \tilde{E}G) \to \mathrm{Map}_*^G(EG_+, \tilde{E}G).$$

The middle term is equal to S^0, so it suffices to show that the third term is weakly contractible. For this we recall that the functor $\mathrm{Map}_*^G(EG_+, -)$ takes G-maps which are nonequivariant weak equivalences to weak equivalences (for example, this follows from [1, XI.5.6] since $\mathrm{Map}_*^G(EG_+, -)$ is a special case of the holim construction). Since $\tilde{E}G$ is nonequivariantly contractible we see that $\mathrm{Map}_*^G(EG_+, \tilde{E}G)$ is weakly contractible and we are done. QED

3 Proof of Theorem 2.

As motivation for the proof of Theorem 2, we first explain why the operad $\mathcal{C}'$ constructed in the proof of Lemma 3 is not equivalent to the linear isometries operad $\mathcal{L}U$. Let $G = Z/2$

for simplicity and consider the $G \times \Sigma_2$-spaces $\mathcal{L}U_2$ and $\mathcal{C}'_2$. Let H be the diagonal copy of $Z/2$ in $G \times \Sigma_2 = Z/2 \times Z/2$. We claim that $\mathcal{L}U_2$ has H fixed points but $\mathcal{C}'_2$ has none; this certainly implies that $\mathcal{L}U_2$ and $\mathcal{C}'_2$ are not $G \times \Sigma_2$-equivalent. To see that $\mathcal{L}U_2$ has H-fixed points we need only show that there is an H-equivariant linear isometry from $U \oplus U$ to U; but this is obvious since as an H-representation $U \oplus U$ is a complete H-universe, and is therefore H-isomorphic to U. (We note for later use that $(\mathcal{L}U_2)^H$ is in fact contractible by [3, Lemma II.1.5]). On the other hand, if $\mathcal{C}'_2$ had an H-fixed point then there would be a $G \times \Sigma_2$-equivariant map

$$\tilde{E}G^{(2)} \to \tilde{E}G$$

(with Σ_2 acting trivially on the target) which extends the identity map of S^0, and passing to H-fixed points would give a (nonequivariant) map $(\tilde{E}G^{(2)})^H \to S^0$ which extends the identity map of S^0. But this is impossible since $(\tilde{E}G^{(2)})^H$ is contractible: there is a (nonequivariant) homeomorphism

$$\tilde{E}G \to (\tilde{E}G^{(2)})^H$$

which takes x to $x \wedge gx$, where g is the generator of G.

The proof of Theorem 2 is a variant of the same idea. For simplicity, we begin with the case $G = Z/2$. Suppose that $t(k_G)$ has an equivariant E_∞-ring structure whose unit η factors through $\Sigma^\infty_G \iota$. Then there is a G-homotopy commutative diagram of G-spectra

$$\begin{array}{ccc} \mathcal{L}U_2 \ltimes_{\Sigma_2} (S^0_G)^{(2)} & \xrightarrow{1 \ltimes \Sigma^\infty_G \iota^{(2)}} \mathcal{L}U_2 \ltimes_{\Sigma_2} (\Sigma^\infty_G \tilde{E}G)^{(2)} \to & \mathcal{L}U_2 \ltimes_{\Sigma_2} t(k_G)^{(2)} \\ \downarrow{\scriptstyle \xi_2} & & \downarrow{\scriptstyle \xi'_2} \\ S^0_G & \xrightarrow{\eta} & t(k_G), \end{array}$$

where ξ_2 and ξ'_2 are the structural maps for S^0_G and $t(k_G)$. Next we recall that the upper-left corner of this diagram is an equivariant suspension spectrum, so that we may pass to the adjoint to get a G-homotopy commutative diagram of spaces. More precisely, [3, Proposition VI.5.3] gives an isomorphism

$$\mathcal{L}U_2 \ltimes_{\Sigma_2} (S^0)^{(2)} \cong \Sigma^\infty_G (\mathcal{L}U_{2+} \wedge_{\Sigma_2} (S^0_G)^{(2)})$$

which carries ξ_2 to the composite

$$\Sigma^\infty_G (\mathcal{L}U_{2+} \wedge (S^0)^{(2)}) = \Sigma^\infty_G (\mathcal{L}U_2/\Sigma_2)_+ \xrightarrow{\Sigma^\infty_G \pi} \Sigma^\infty_G S^0;$$

here π is the evident projection $(\mathcal{L}U_2/\Sigma_2)_+ \to S^0$. Thus the adjoint of the diagram above has the form

$$\begin{array}{ccc}
\mathcal{L}U_{2+}\wedge_{\Sigma_2}(S^0)^{(2)} & \xrightarrow{1\wedge_{\Sigma_2}\iota^{(2)}} \mathcal{L}U_{2+}\wedge \tilde{E}G^{(2)} \to & \Omega^\infty_G(\mathcal{L}U_2 \ltimes {}_{\Sigma_2} t(k_G)^{(2)}) \\
\downarrow = & & \\
(\mathcal{L}U_2/\Sigma_2)_+ & & \downarrow \Omega^\infty_G \xi'_2 \\
\downarrow \pi & & \\
S^0 & \xrightarrow{\tilde{\eta}} & \Omega^\infty_G t(k_G)
\end{array}$$

For our purposes, the important thing about this diagram is that $\tilde{\eta}\circ\pi$ factors, up to G-homotopy, through $\mathcal{L}U_{2+}\wedge\tilde{E}G^{(2)}$. Precomposing with the projection

$$\mathcal{L}U_{2+}\wedge(S^0)^{(2)} \to \mathcal{L}U_{2+}\wedge_{\Sigma_2}(S^0)^{(2)}$$

we see that the composite

$$\mathcal{L}U_{2+}\wedge(S^0)^{(2)} = (\mathcal{L}U_2)_+ \xrightarrow{\pi} S^0 \xrightarrow{\tilde{\eta}} \Omega^\infty_G t(k_G) \tag{1}$$

(where we have again written π for the evident projection) factors up to $G\times\Sigma_2$-homotopy through $\mathcal{L}U_{2+}\wedge\tilde{E}G^{(2)}$. Now let H be the diagonal copy of $Z/2$ in $G\times\Sigma_2$. Passing to the H-fixed points of (1) (and noting that the H-fixed points of $\Omega^\infty_G t(k_G)$ are the same as the G-fixed points since Σ_2 acts trivially) we see that the composite

$$(\mathcal{L}U_2^H)_+ \xrightarrow{\pi^H} S^0 \xrightarrow{\tilde{\eta}^G} (\Omega^\infty_G t(k_G))^G \tag{2}$$

factors up to (nonequivariant) homotopy through

$$\mathcal{L}U^H_{2+}\wedge(\tilde{E}G^{(2)})^H.$$

But we have shown in the first paragraph of this section that $(\tilde{E}G^{(2)})^H$ is contractible, so the composite (2) is (nonequivariantly) homotopy trivial. We also showed in the first paragraph that $\mathcal{L}U_2^H$ is contractible, so π^H is an equivalence, and we conclude that

$$\tilde{\eta}^G : S^0 \to (\Omega^\infty_G t(k_G))^G$$

is homotopy trivial. This means that $\tilde{\eta}$ is G-homotopy trivial, and passing to the adjoint we see that η itself is G-homotopy trivial. But η is the unit of the equivariant E_∞ ring $t(k_G)$, so $t(k_G)$ must be equivariantly contractible, as was to be shown.

So far we have assumed that G is $Z/2$. When G is cyclic of order n, with generator g, one need only repeat the same argument with $\mathcal{L}U_2$ replaced by $\mathcal{L}U_n$, Σ_2 replaced by Σ_n, and H replaced by the subgroup of $G\times\Sigma_n$ generated by (g,σ), where σ is an n-cycle. QED

References

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