

The algebraic K-theory spectrum of a 2-adic local field

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1 Introduction

Let F be a local field of characteristic zero. Then F is an extension of the ℓ -adic rationals \mathbb{Q}_ℓ for some prime ℓ , of finite degree d . When ℓ is odd, the homotopy type of the étale K-theory spectrum $K_{\acute{e}t}F$ was determined by Bill Dwyer and the author ([1], Theorem 13.3). Let $P_n X$ denote the $(n - 1)$ -connected cover of a spectrum X .

Theorem 1.1 *If ℓ is odd, then $P_1 K_{\acute{e}t}F \cong P_1(\bigvee^d \Sigma bu^\wedge \vee \Sigma K\mathbb{F}^\wedge \vee K\mathbb{F}^\wedge)$.*

Here bu is connective complex K-theory, \mathbb{F} is a certain finite field \mathbb{F}_p with $p \neq \ell$, and $(-)^^\wedge$ denotes Bousfield ℓ -adic completion. The homotopy type of $K\mathbb{F}^\wedge$ is well-understood and depends only on the number of ℓ -power roots of unity in $F_0 = F(\mu_\ell)$, μ_ℓ denoting the ℓ -th roots of unity. If the strong (Dwyer-Friedlander) form of the Lichtenbaum-Quillen conjectures holds for F , $K_{\acute{e}t}F$ can be replaced by the completed algebraic K-theory spectrum KF^\wedge in Theorem 1.1. In that case there is no need to pass to 0-connected covers. This is necessary in the étale case only because of a spurious \mathbb{Z}_ℓ -summand in $\pi_0 K_{\acute{e}t}F$, arising from the Brauer group.

When $\ell = 2$, the methods of [1] apply only when $\sqrt{-1} \in F$. The purpose of the present paper is to determine the 2-adic homotopy type of $K_{\acute{e}t}F$ without this assumption on F . In fact, thanks to the work of Rognes and Weibel [9] on the 2-adic Lichtenbaum-Quillen conjectures, our main theorem yields the homotopy type of KF^\wedge itself, not just the étale K-theory. Let F_∞ denote the infinite cyclotomic extension obtained by adjoining all the 2-power roots of unity, and let $\Gamma'_F = G(F_\infty/F)$. Thus there is a natural embedding $c : \Gamma'_F \rightarrow \mathbb{Z}_2^\times$. Let $\Gamma_F \subset \Gamma'_F$ denote $G(F_\infty/F_0)$, where now $F_0 = F(\sqrt{-1})$, and let q be a prime power such that $q = -c(\gamma_F)$ for some topological generator γ_F of Γ_F . Call F *exceptional* if Γ'_F has an element σ of order two. Then our main result implies (see Theorem 11.4 for the complete statement):

Theorem 1.2 *Let F be an exceptional 2-adic local field. Then there is a weak equivalence*

$$KF^\wedge \cong \begin{cases} Y \vee (\bigvee^{d-1} \Sigma bu^\wedge) & \text{if } d \text{ odd} \\ Z \vee X \vee (\bigvee^{d-2} \Sigma bu^\wedge) & \text{if } d \text{ even} \end{cases}$$

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where the spectra X, Y, Z depend only on the number of 2-power roots of unity in F , and fit into fibre sequences

$$\begin{aligned} X &\longrightarrow Y \longrightarrow K_{\mathbb{F}_q}^\wedge \\ \Sigma K_{\mathbb{F}_q}^\wedge &\longrightarrow X \longrightarrow \Sigma bu^\wedge \\ \Sigma bu^\wedge &\longrightarrow Z \longrightarrow K_{\mathbb{F}_q}^\wedge. \end{aligned}$$

The connecting maps $K_{\mathbb{F}_q}^\wedge \rightarrow \Sigma X$, $\Sigma bu^\wedge \rightarrow \Sigma^2 K_{\mathbb{F}_q}^\wedge$, and $K_{\mathbb{F}_q}^\wedge \rightarrow \Sigma^2 bu^\wedge$ are all nontrivial, and are essentially determined by the Iwasawa theory of F_∞/F .

In the case $F = \mathbb{Q}_2$ this result was first proved by Rognes [8]. In fact we obtain simple explicit formulae for the connecting maps; see below. Note the theorem says that an exceptional F still satisfies Theorem 1.1, up to non-trivial extensions of spectra. If F is non-exceptional but does not contain $\sqrt{-1}$, Theorem 1.1 goes through unchanged (the only difference in the proof is that the role of Thomason's theorem [12] is taken over by the local version of Rognes-Weibel [9]). In this paper, therefore, we assume throughout that F is exceptional.

Passing to zero-th spaces yields the homotopy-type of $(BGLF^+)^\wedge$ (again, see §11 for a more precise statement):

Theorem 1.3 *Let F be an exceptional 2-adic local field. Then there is a weak equivalence*

$$(BGLF^+)^\wedge \cong \begin{cases} \Omega_0^\infty Y \times (U^\wedge)^{d-1} & \text{if } d \text{ odd} \\ \Omega_0^\infty Z \times \Omega_0^\infty X \times (U^\wedge)^{d-2} & \text{if } d \text{ even} \end{cases}$$

where the spaces $\Omega_0^\infty X$, $\Omega_0^\infty Y$, $\Omega_0^\infty Z$ fit into fibre sequences

$$\begin{aligned} \Omega_0^\infty X &\longrightarrow \Omega_0^\infty Y \longrightarrow (BGL_{\mathbb{F}_q}^+)^\wedge \\ B((BGLF^+)^\wedge) &\longrightarrow \Omega_0^\infty X \longrightarrow U^\wedge \\ U^\wedge &\longrightarrow \Omega_0^\infty Z \longrightarrow (BGL_{\mathbb{F}_q}^+)^\wedge. \end{aligned}$$

We sketch the proof of Theorem 1.2. Let $\eta : KF^\wedge \rightarrow \mathbb{H}_{\acute{e}t}(F; \underline{K})^\wedge$ denote the natural map to the Thomason-Jardine hypercohomology spectrum (this spectrum has the same (-1) -connected cover as $K_{\acute{e}t}F$, after completion). The method of [9] shows that η induces a weak equivalence on 0-connected covers, and so induces an isomorphism on completed topological K-theory $\hat{\mathcal{K}}^*$. Corollary 5.6 of [4] then yields at once:

Theorem 1.4

$$\hat{\mathcal{K}}^p KF \cong \begin{cases} \Lambda' \otimes_{\Lambda'_F} \mathbb{Z}_2 & \text{if } p = 0 \\ \Lambda' \otimes_{\Lambda'_F} M_\infty & \text{if } p = -1 \end{cases}$$

Here Λ' is the ring of operations $[\hat{\mathcal{K}}, \hat{\mathcal{K}}]$, and Λ'_F is the pro-group ring of Γ'_F . The action of Λ'_F on the roots of unity gives an embedding $\Lambda'_F \subset \Lambda'$. The Λ'_F -module M_∞ is the “basic Iwasawa module”. It can be defined as the étale or Galois homology group $H_1(F_\infty; \mathbb{Z}_2)$.

Formally, Theorem 1.4 looks the same in the exceptional and non-exceptional cases. But in the non-exceptional case, the Λ' -modules $\hat{\mathcal{K}}^0, \hat{\mathcal{K}}^{-1}$ have projective dimension one. Writing Λ_F for the pro-group ring of Γ_F , we have also that M_∞ modulo its Λ_F -torsion submodule $\mathbb{Z}_2(1)$ is a free Λ_F -module of rank equal to the degree of F_0 over \mathbb{Q}_2 . Theorem 1.1 is then an easy formal consequence of Theorem 1.4. In the exceptional case, on the other hand, both $\hat{\mathcal{K}}^0 KF$ and $\hat{\mathcal{K}}^{-1} KF$ have infinite projective dimension as Λ' -modules. For example, when $F = \mathbb{Q}_2$, $\hat{\mathcal{K}}^0 KF = \mathbb{Z}_2$, with trivial σ -action. In addition, although it is still true that the Iwasawa module fits into an extension

$$0 \longrightarrow \mathbb{Z}_2(1) \longrightarrow M_\infty \longrightarrow N \longrightarrow 0$$

with N free of rank $2d$ as Λ_F -module, N is not free as Λ'_F -module. The problem, then, is to analyze the representation of σ on N and M_∞ . To achieve this we first classify representations of σ over Λ ; that is, Λ -free Λ' -modules. Using this classification we are able to identify N . Here the answer depends on the parity of d , the reason being that the Hilbert symbol $\{-1, -1\}_F$ is nontrivial if and only if d is odd. We also determine the above extension, which is nontrivial, thereby obtaining an explicit description of $\hat{\mathcal{K}}^* KF$ as Λ' -module.

The next step, as in [1], is to define $K^{red}F$ as the fibre of a certain noncanonical reduction map $r : KF^\wedge \longrightarrow K\mathbb{F}_q^\wedge$. We explicitly compute $\hat{\mathcal{K}}^* K^{red}F$, and find that it has projective dimension one as Λ' -module. In other words, despite the fact that $\hat{\mathcal{K}}^* KF$ has infinite projective dimension, KF has a length one “resolution” by spectra whose $\hat{\mathcal{K}}^*$ has projective dimension one. The homotopy-type of $K^{red}F$ is then determined by $\hat{\mathcal{K}}^* K^{red}F$. It turns out that this homotopy-type is independent of the choice of q and r , and moreover the parity of d does not enter into its description. The attaching map $K\mathbb{F}_q \longrightarrow \Sigma K^{red}F$, on the other hand, does depend on the parity of d . In any case, this reduces the proof of the main theorem to pure algebra.

The paper is organized as follows. Section §2 introduces some basic notation, and reviews some pertinent facts from Iwasawa theory. In §3 we classify representations of a group of order two over the Iwasawa algebra, and certain related extensions. This material is of interest in 2-adic K -local homotopy theory, as well as in Iwasawa theory. In §4 we use the classification theorem to compute the structure of M_∞ as Λ'_F -module, and show how this computation yields the (known) structure of the Galois cohomology groups $H_{\acute{e}t}^*(F; \mathbb{Z}_2(n))$. In §5 we review the Rognes-Weibel theorem and some consequences. In §6 we compute $\hat{\mathcal{K}}^* KF$. In particular, we introduce the Λ' -modules $L[1, k]$ that play a central role in the rest of the paper. In §7, 8, 9 we introduce $K^{red}F$, compute its topological K-theory, and determine its homotopy-type. In §10 we explicitly determine the connecting map for the fibre sequence defining $K^{red}F$, at least up to $K(1)$ -localization. In §11 we assemble the results of the previous sections to prove the main theorems.

2 Notation and preliminaries

2.1 Λ -modules and Λ' -modules

Let $\Gamma' = \text{Aut } \mu_\infty \mathbb{C}$, where $\mu_\infty \mathbb{C}$ is the group of 2-power roots of unity in \mathbb{C} . There is a canonical isomorphism $c : \Gamma' \xrightarrow{\cong} \mathbb{Z}_2^\times$ defined by $\gamma(\xi) = \xi^{c(\gamma)}$ for $\xi \in \mu_\infty \mathbb{C}$. Let

$$\Gamma = \{\gamma \in \Gamma' : c(\gamma) = 1 \text{ mod } 4\}$$

In other words, Γ is the group of automorphisms fixing the fourth roots of unity. Thus Γ is topologically cyclic and of index 2 in Γ' . Let γ_0 be a fixed topological generator of Γ , and set $c_0 = c(\gamma_0)$. If we wish to be specific, we take $c_0 = 5$.

Elements a, b of a topological group are said to be *topologically equivalent* if the subgroups topologically generated by a and by b coincide.

Let $\Gamma_k = \{\gamma \in \Gamma : c(\gamma) = 1 \text{ mod } 2^{k+2}\}$. Then Γ_k is also topologically cyclic, with generator $\gamma_k = \gamma_0^{2^k}$. Set $c_k = c(\gamma_k) = c_0^{2^k}$.

To avoid superfluous notation, we will not distinguish between a group of order two and its unique nontrivial element, usually denoted σ . Thus $\Gamma' = \Gamma \times \sigma$, where σ is the unique element of order two in Γ' .

For background on Iwasawa theory, see [7], §5. The *Iwasawa algebra* Λ is the pro-group ring $\mathbb{Z}_2[[\Gamma]]$. It is isomorphic to a power series ring $\mathbb{Z}_2[[T]]$, where $T = \gamma_0 - 1$. Hence Λ is a regular noetherian local ring of Krull dimension two, with maximal ideal $\mathcal{M} = (2, T)$. There is a beautiful classification theorem for finitely-generated Λ -modules. Call a Λ -module N *elementary* if it is a finite direct sum of cyclic modules Λ/f , where f is either zero or a prime power (the primes of Λ are in turn classified using the Weierstrass preparation theorem). Then for any finitely-generated Λ -module M , there is an elementary module N and an *isogeny* $M \rightarrow N$; that is, a homomorphism with finite kernel and cokernel. The cyclic factors of N , as above, are uniquely determined by M .

Since Λ has global dimension two, a Λ -module M has projective dimension at most two. In particular, the invariants $\text{Ext}_\Lambda^p(M, \Lambda)$ are possibly nonzero only for $p = 0, 1, 2$. These invariants are very useful for analyzing further properties of M . The following two propositions are well-known or easy, and are only sketched; see for example §5 of [7] or [1], §7 as indicated. We assume throughout that M is finitely-generated as Λ -module.

Proposition 2.1 *The following are equivalent:*

- a) $\text{pdim}_\Lambda M \leq 1$
- b) M has no nonzero finite submodules
- c) $\text{Ext}_\Lambda^2(M, \Lambda) = 0$
- d) $\text{Tor}_\Lambda^2(M, \mathbb{Z}/2) = 0$

Proof: For the equivalence of (a), (b) and (c), see for example [1], 7.7. The equivalence of (a) and (d) is an easy general fact about local rings.

Let $M^\#$ denote $\text{Hom}_\Lambda(M, \Lambda)$.

Proposition 2.2 *Let M be a Λ -torsionfree Λ -module. Then*

- a) $M^\#$ is a free Λ -module.
- b) The natural map $M \rightarrow M^{\#\#}$ is an injective isogeny.
- c) If $M \rightarrow Q$ is any isogeny to a free module Q , then Q/M depends only on M , and has the same order as $\text{Ext}_\Lambda^1(M, \Lambda)$. In particular, M is free if and only if $\text{Ext}_\Lambda^1(M, \Lambda) = 0$.
- d) Suppose M is a submodule of a free module Q . Then M is free if and only if Q/M has no finite submodules.

Proof: For (a) see [1], 7.8. For (b) and (c) see [1], 7.11. Part (d) follows easily from (c) and Proposition 2.1.

Remark: We call the order of Q/M in Proposition 2.2c the *defect* of M . Note that if M, M' are torsion-free Λ -modules with defect τ, τ' , and $M \subset M'$ with finite index e , then $\tau = e\tau'$.

Let $\Lambda' = \mathbb{Z}_2[[\Gamma']] = \Lambda[\sigma]$. Then Λ' is a noetherian local ring, with maximal ideal $(1 - \sigma, 2, T)$. But Λ' has infinite global dimension, and its module theory is much more complicated than that of Λ . Later we will classify the Λ -free Λ' -modules; for now we only set up some basic notation.

If M is a Λ' -module, we write M^\pm for the ± 1 -eigenspaces of the σ -action on M . We write M^b for the module whose underlying Λ -module is M , but with the twisted σ -action $\sigma \cdot x = -\sigma x$. If N is given only as a Λ -module, we use the same notation for the Λ' -module N with trivial σ -action. For example, combining these last two conventions, Λ^b means Λ with σ acting as -1 .

The n -th Tate twist $M(n)$ is M with the twisted Γ' -action $\gamma \cdot x = c(\gamma)^n \gamma x$. Note that $\mathbb{Z}_2(n) = \mathbb{Z}_2(1) \otimes \dots \otimes \mathbb{Z}_2(1)$ (n factors), and $M(n) = M \otimes_{\mathbb{Z}_2} \mathbb{Z}_2(n)$.

2.2 Local fields

Let F be a 2-adic local field. In other words, F is a finite extension of \mathbb{Q}_2 , of degree d . Let $F_0 = F(\mu_4)$, $F_\infty = F(\mu_\infty)$. We have $\Gamma_F = G(F_\infty/F_0)$, $\Gamma'_F = G(F_\infty/F)$. We assume throughout this paper that F is *exceptional*, meaning that Γ'_F contains an element σ of order two.

Fixing an embedding of the algebraic closure \overline{F} in \mathbb{C} (or at least an embedding of the 2-power roots of unity) yields embeddings $\Gamma_F \subset \Gamma$, $\Gamma'_F \subset \Gamma'$.

Let 2^{a_F} denote the order of the group of 2-adic roots of unity in F_0 . Then $\Gamma_F = \Gamma_k$, where $k = a_F - 2$. Similarly, as topological generator γ_F for Γ_F we can take γ_k . We have $\Lambda_F = \mathbb{Z}_2[[\Gamma_F]] = \mathbb{Z}_2[[T_F]] \subset \Lambda$, etc., where $T_F = \gamma_F - 1$; further variations on this notational scheme should be self-explanatory.

We write q_F for a choice of odd prime power topologically equivalent to $-c(\gamma_F)$. For example, $c(\gamma_{\mathbb{Q}_2}) = c(\gamma_0) = 5$, and we can take $q_{\mathbb{Q}_2} = 3$.

Let $w_m F = a_F + \nu_2 m$, where ν_2 is the 2-adic valuation. Note that $(\mathbb{Z}_2(m))_{\Gamma_F} \cong \mathbb{Z}/2^{w_m F}$.

2.3 Topological K-theory

Let \mathcal{K} denote the periodic complex K-theory spectrum. Then there is a canonical isomorphism $\Lambda' \xrightarrow{\cong} [\hat{\mathcal{K}}, \hat{\mathcal{K}}]$, continuous with respect to the natural profinite topologies. It is

uniquely determined by the property that, for k an odd integer in $\mathbb{Z}_2^\times = \Gamma'$, k maps to the Adams operation ψ^k .

Now for an odd integer q , let J_q denote the homotopy fibre of $\psi^q - 1 : \mathcal{K} \rightarrow \mathcal{K}$, and let j_q denote its (-1) -connected cover. Then the homotopy type of the completions J_q^\wedge, j_q^\wedge depends only on the topological equivalence class of q . For example, $J_{-5}^\wedge \cong J_3^\wedge$. We have immediately from the definitions:

Proposition 2.3 *If $c(\gamma) = q$, at least up to topological equivalence in \mathbb{Z}_2^\times , then*

$$\hat{\mathcal{K}}^p J_q \cong \begin{cases} \Lambda' / (\gamma - 1) & \text{if } p = 0 \\ 0 & \text{if } p = -1 \end{cases}$$

The case of most interest to us will be when q is topologically equivalent to $-c_k$. In that case we have

$$\hat{\mathcal{K}}^0 J_q \cong \Lambda' / (\sigma\gamma_k - 1).$$

Note also that

$$\hat{\mathcal{K}}^0 \Sigma^2 J_q \cong \Lambda' / (\sigma\gamma_k + c_k).$$

Here we have used the fact that double suspension corresponds to Tate twisting on \mathcal{K} -theory, plus the observation that $\sigma\gamma_k$ is acting trivially on the original module and hence as $c(\sigma\gamma_k) = -c(\gamma_k)$ on the Tate twisted module.

Note that J_q^\wedge is an example of an *excellent spectrum* as defined in [1], 4.10: i.e., a $\hat{\mathcal{K}}$ -local spectrum X such that either $\hat{\mathcal{K}}^{-1} X = 0$ and $\hat{\mathcal{K}}^0 X$ is a finitely-generated Λ' -module of projective dimension at most one, or vice versa. In the first case we say that X has *even type*; in the second X has *odd type*. Maps between excellent spectra are given by a simple algebraic formula; see [1], 4.15. In particular, the homotopy-type of an excellent spectrum X is uniquely determined by its $\hat{\mathcal{K}}^* X$ as Λ' -module.

Recall that if q is a prime power, $j_q^\wedge \cong K\mathbb{F}_q^\wedge$.

$\hat{L}(-)$ denotes Bousfield localization with respect to $\mathcal{K} \wedge M\mathbb{Z}/2$, or equivalently, with respect to the Adams summand $K(1)$ of $\mathcal{K} \wedge M\mathbb{Z}/2$. We have $\hat{L}(-) \cong (L_K(-))^\wedge$. Furthermore, \hat{L} is cohomological localization with respect to $\hat{\mathcal{K}}$, because a map of spectra is a $K \wedge M\mathbb{Z}/2$ -equivalence if and only if it induces an isomorphism on $\hat{\mathcal{K}}^*$. For further discussion see §4 of [1].

3 Representations of σ over Λ

Let M denote a finitely-generated Λ -free Λ' -module. In other words, M is a representation of σ over Λ . The main result of this section is Theorem 3.2, which classifies such representations up to isomorphism. We find that the indecomposable representations have rank at most 2, and that the rank 2 indecomposables fall into two infinite families L_n, L'_n . (Only the first family L_n arises in the Iwasawa theory of local fields; see §4.) As a byproduct we show that representations are classified by their Tate homology groups $\hat{H}_*(\sigma; M)$, where the latter are regarded as modules over the principal ideal domain $\Lambda/2$.

In §3.1 we introduce the modules L_n, L'_n , and compute their Tate homology modules. In §3.2 we prove the classification theorem. In §3.3 we define certain extensions \tilde{L}_n of L_n by $\mathbb{Z}_2(1)$. For $n = 1, 2$, these extensions arise in the Iwasawa theory of 2-adic local fields.

3.1 Some examples

If $M = \Lambda$ as Λ -module, then σ acts on M as multiplication by ± 1 . Following the convention established above, we write $M = \Lambda$ when the sign is positive, and $M = \Lambda^b$ when it is negative.

For $n \geq 0$, let L_n denote the rank two representation on which σ acts by the matrix

$$\begin{pmatrix} 1 & T^n \\ 0 & -1 \end{pmatrix}$$

and let L'_n denote the rank two representation on which σ acts by the matrix

$$\begin{pmatrix} -1 & T^n \\ 0 & 1 \end{pmatrix}$$

Except for the isomorphism $L_0 \cong \Lambda' \cong L'_0$, no two of these modules are isomorphic. To see this, recall that the Tate homology groups of a σ -module M are defined by

$$\hat{H}_q(\sigma; M) = \begin{cases} \text{Ker}(1 + \sigma)/\text{Im}(1 - \sigma) & \text{if } q \text{ even} \\ \text{Ker}(1 - \sigma)/\text{Im}(1 + \sigma) & \text{if } q \text{ odd.} \end{cases}$$

Note that for a Λ' -module M , these groups are $\Lambda/2$ -modules. (The Tate cohomology groups are the same, but with the parity of q reversed.) We then have:

Lemma 3.1 *Let $H_q = \hat{H}_q(\sigma; M)$, and let $Q_n = \Lambda/(2, T^n)$. Then there are isomorphisms of Λ -modules as indicated in the following table:*

	\underline{H}_0	\underline{H}_1
\underline{M}		
Λ	0	$\Lambda/2$
Λ^b	$\Lambda/2$	0
Λ'	0	0
L_n	0	Q_n
L'_n	Q_n	0

Proof: The proof is elementary; we sketch the argument for L_n and leave the rest to the reader. By definition L_n is the representation with basis e_1, e_2 and σ acting by $\sigma e_1 = e_1$, $\sigma e_2 = -e_2 + T^n e_1$. Clearly $L_n^+ = \Lambda e_1$, and $(1 + \sigma)L_n = (2, T^n)e_1$. Hence $\hat{H}_1(\sigma; L_n) \cong$

$\Lambda/(2, T^n)$. On the other hand, $L_n^- = \Lambda(T^n e_1 - 2e_2)$ and $(1 - \sigma)L_n = \Lambda(1 - \sigma)e_2 = \Lambda(-T^n e_1 + 2e_2)$. Hence $\hat{H}_0(\sigma; L_n) = 0$.

Thus the modules L_j, L'_k are distinguished by their Tate homology modules. Note also that $L_n \otimes_{\Lambda} \Lambda^b \cong L'_n$.

Remark: Let f be any element of Λ , and let J_f denote the rank two representation with σ acting by the matrix

$$\begin{pmatrix} 1 & f \\ 0 & -1 \end{pmatrix}$$

Then it is easy to see that up to isomorphism, J_f depends only on $f \bmod 2$, up to units (this also follows from the classification theorem below). Thus if $f = 0 \bmod 2$, then $J_f \cong \Lambda \oplus \Lambda^b$; if $f = uT^n \bmod 2$ for some unit u then $J_f \cong L_n$. Analogous remarks apply with the diagonal entries reversed.

3.2 A classification theorem

Theorem 3.2 *Let M be a finitely-generated Λ -free Λ' -module. Then M is isomorphic to a direct sum of the form*

$$\Lambda^a \oplus (\Lambda^b)^b \oplus (\Lambda')^c \oplus L_{i_1} \oplus \dots \oplus L_{i_s} \oplus L'_{j_1} \oplus \dots \oplus L'_{j_t}$$

with $i_t, j_t \geq 1$. Furthermore, (i) this decomposition is unique up to the order of the summands; and (ii) the isomorphism type of M is uniquely determined by the isomorphism type of the $\Lambda/2$ -modules $\hat{H}_0(\sigma; M), \hat{H}_1(\sigma; M)$.

Proof: Let $r = \text{rank}_{\Lambda} M$, and assume $r > 0$. The proof involves three main steps. First we show that all representations are upper triangular (Corollary 3.5). Then we show that the theorem holds when either M^+ or M^- is a Λ -direct summand (Lemma 3.6). Finally, an induction on the rank r completes the proof.

Let $M^e = M^+ \oplus M^- \subset M$ (the ‘‘eigensubmodule’’).

Lemma 3.3 a) M^e is a free Λ -module of rank r

b) M/M^e is a free $\Lambda/2$ -module of rank less than r .

Proof: a) It is clear that M/M^+ is torsion-free as Λ -module. In particular M/M^+ has no nonzero finite Λ -submodules; hence M^+ is Λ -free by Proposition 2.2. Similarly, M^- is Λ -free. Now let $M[1/2] = \Lambda[1/2] \otimes_{\Lambda} M$. Then $M[1/2] = M^+[1/2] \oplus M^-[1/2]$, so $\text{rank}_{\Lambda} M^e = r$.

b) Since $2x = (1 + \sigma)x + (1 - \sigma)x$, we have $2(M/M^e) = 0$. Hence M/M^e is a finitely-generated module over the local principal ideal domain $\Lambda/2$. But any finitely-generated torsion module over $\Lambda/2$ is finite, and M/M^e has no nonzero finite submodules by Proposition 2.2 and part (a). Therefore M/M^e is a free $\Lambda/2$ -module, of rank at most r .

It remains to show that the rank cannot equal r . If it is equal to r , then $2M = M^e$. Now either M^+ or M^- is nonzero; we may suppose M^+ is nonzero. Then there is an $x \in M$ such that $2x \in M^+$ but $x \notin M^+$, which is evidently absurd. This completes the proof of the lemma.

Lemma 3.4 *Either Λ or Λ^b occurs as a Λ -direct summand in M .*

Proof: Consider the exact sequence

$$M^e \otimes_{\Lambda} \mathbb{F}_2 \xrightarrow{i} M \otimes_{\Lambda} \mathbb{F}_2 \xrightarrow{j} (M/M^e) \otimes_{\Lambda} \mathbb{F}_2 \longrightarrow 0.$$

We claim that i is nonzero. It follows that i is nonzero on either $M^+ \otimes_{\Lambda} \mathbb{F}_2$ or $M^- \otimes_{\Lambda} \mathbb{F}_2$. Nakayama's lemma then implies that either Λ or Λ^b occurs as a Λ -direct summand in M . To prove the claim, suppose to the contrary that i is zero. Then j is an isomorphism; hence M/M^e has rank r as $\Lambda/2$ -module, contradicting Lemma 3.3.

Corollary 3.5 *Every finitely-generated Λ -free Λ' -module M admits a filtration*

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots M_{r-1} \subset M_r = M$$

with quotients M_i/M_{i-1} Λ -free of rank one.

Lemma 3.6 *Suppose that either M^+ or M^- is a Λ -direct summand of M . Then M admits a direct sum decomposition of the required form.*

Proof: Without loss of generality we can assume that M^+ is a Λ -direct summand. Hence there is a Λ -split short exact sequence

$$0 \longrightarrow M^+ \longrightarrow M \longrightarrow M/M^+ \longrightarrow 0.$$

Note that for any $y \in M$ we have $\sigma y = -y + (y + \sigma y)$, so σ acts on M/M^+ as multiplication by -1 . Therefore we can choose a basis $e_1, \dots, e_m, f_1, \dots, f_n$ with respect to which σ has block matrix

$$\begin{pmatrix} I & A \\ 0 & -I \end{pmatrix}$$

for some $m \times n$ -matrix A . Now recall that for any commutative ring R and group G , there is a *local-global spectral sequence*

$$H^p(G; Ext_R^q(M, N)) \Rightarrow Ext_{RG}^{p+q}(M, N).$$

This is just the composite functor spectral sequence that arises from factoring $Hom_{RG}(M, -)$ as $(-)^G \circ Hom_R(M, -)$. (The name comes from the fact that G can be interpreted as a Grothendieck site; then this is the spectral sequence relating local and global Ext for sheaves of R -modules.) Here the spectral sequence degenerates to an isomorphism

$$Ext_{\Lambda'}^1((\Lambda^b)^n, \Lambda^m) \cong H^1(\sigma; Hom_{\Lambda}((\Lambda^b)^n, \Lambda^m)) = Hom_{\Lambda/2}((\Lambda/2)^n, (\Lambda/2)^m).$$

In fact the matrix $A \bmod 2$ itself determines M up to isomorphism of extensions. Up to isomorphism as modules, on the other hand, we are free to replace A by any equivalent

matrix CAD . Here $C \in GL_m\Lambda$, $D \in GL_n\Lambda$, and the required isomorphism arises from conjugation by the block matrix

$$\begin{pmatrix} C & 0 \\ 0 & D^{-1} \end{pmatrix}$$

Now since Λ is a local ring, the natural map $GL_n\Lambda \rightarrow GL_n\Lambda/2$ is surjective for all n . Using the standard classification of matrices over a principal ideal domain, we can therefore assume that A is a block matrix

$$\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$$

where E is an $\ell \times \ell$ diagonal matrix with diagonal entries $T^{i_1}, \dots, T^{i_\ell}$. If a is the number of zero rows in A , and b the number of zero columns, we then have

$$M \cong \Lambda^a \oplus (\Lambda^b)^b \oplus L_{i_1} \oplus \dots \oplus L_{i_\ell}.$$

This completes the proof of Lemma 3.6.

We now prove the existence statement of the theorem by induction on r . We use L to denote a direct sum of modules of type L_i , and L' for a direct sum of modules of type L'_i . At the inductive step, using Corollary 3.5 we can assume there is a short exact sequence of Λ' -modules

$$0 \rightarrow N \rightarrow M \rightarrow \Lambda \rightarrow 0$$

where, by inductive hypothesis, $N \cong \Lambda^a \oplus (\Lambda^b)^b \oplus L \oplus L'$. (If the quotient is Λ^b instead of Λ , we simply apply the functor $(-)\otimes_{\Lambda} \Lambda^b$ to reduce to the form given above.)

Suppose first that L is zero. Then $M^- = N^-$, and M^- is a Λ -direct summand. Lemma 3.6 then implies that M admits a direct sum decomposition of the desired form.

Now suppose that L is not zero. We have

$$\text{Ext}_{\Lambda'}^1(\Lambda, L_n) = H^1(\sigma; L_n) = 0$$

by Lemma 3.1. It follows that L is a direct summand of M as Λ' -module; thus M is decomposable and the inductive hypothesis completes the proof.

Finally, write M in the form

$$M \cong \Lambda^a \oplus (\Lambda^b)^b \oplus \Lambda^c \oplus L_{d_1}^{r_1} \oplus \dots \oplus L_{d_j}^{r_j} \oplus (L'_{e_1})^{s_1} \oplus \dots \oplus (L'_{e_k})^{s_k}.$$

Then, using Lemma 3.1, assertions (i) and (ii) of the theorem follow from the observations:

- a is the $\Lambda/2$ -rank of $\hat{H}_1(\sigma; M)$
- b is the $\Lambda/2$ -rank of $\hat{H}_0(\sigma; M)$
- r_i is the number of $\Lambda/(2, T^{d_i})$ factors in $\hat{H}_1(\sigma; M)$

- s_i is the number of $\Lambda/(2, T^{e_i})$ factors in $\hat{H}_0(\sigma; M)$
- c is determined by the equation $a + b + 2c + 2 \sum r_i + 2 \sum s_j = r$

This completes the proof of Theorem 3.2.

3.3 Extensions of L_n

We will need certain extensions of L_n by $\mathbb{Z}_2(1)$, with $n = 1, 2$. Let $n > 0$ and note that

$$\text{Ext}_{\Lambda'}^1(L_n, \mathbb{Z}_2(1)) = H^1(\sigma; \text{Hom}_{\Lambda}(L_n, \mathbb{Z}_2(1))) = \mathbb{Z}/2.$$

Hence there is a unique non-split extension \tilde{L}_n :

$$0 \longrightarrow \mathbb{Z}_2(1) \longrightarrow \tilde{L}_n \longrightarrow L_n \longrightarrow 0.$$

This extension can be described explicitly as follows: As Λ -module $\tilde{L}_n = \mathbb{Z}_2(1) \oplus \Lambda^2$, where Λ^2 has basis e_1, e_2 and $\mathbb{Z}_2(1)$ has \mathbb{Z}_2 -basis e_0 . The action of σ is given by the matrix

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & (T - \alpha)^n \\ 0 & 0 & -1 \end{pmatrix}$$

where $\alpha = c_0 - 1$. (The remark following Lemma 3.1 is helpful here.)

Taking into account Lemma 3.1, we have:

Proposition 3.7 *For $n > 0$, there is a unique non-split extension \tilde{L}_n of L_n by $\mathbb{Z}_2(1)$. Furthermore,*

$$\hat{H}_p(\sigma; \tilde{L}_n) \cong \begin{cases} 0 & \text{if } p \text{ even} \\ \Lambda/(2, T^{n-1}) & \text{if } p \text{ odd} \end{cases}$$

In particular, \tilde{L}_n is σ -acyclic if and only if $n = 1$.

4 Iwasawa theory for exceptional 2-adic local fields

4.1 The Iwasawa module M_∞ as Λ'_F -module

In this section we describe M_∞ explicitly as a Λ'_F -module. Recall that M_∞ is the Galois group of the maximal abelian 2-extension of F_∞ ; in other words, $M_\infty = H_1(F_\infty; \mathbb{Z}_2)$. Recall also that there is a short exact sequence of Λ'_F -modules

$$0 \longrightarrow \mathbb{Z}_2(1) \longrightarrow M_\infty \longrightarrow N \longrightarrow 0,$$

where N is free of rank $2d$ over Λ_F . (See for example [1], 13.2; note that $\Lambda_F = \Lambda'_{F_0}$.) The problem is to identify the representation of σ on N , and to determine the extension as Λ'_F -modules.

Now Λ'_F is abstractly isomorphic to Λ' , so the results of the preceding sections apply equally to Λ'_F -modules. In particular, there are modules $L_{n,F}$ corresponding to L_n , on which σ acts as the matrix

$$\begin{pmatrix} 1 & T_F^n \\ 0 & -1 \end{pmatrix}$$

(It is important, however, to distinguish between $L_{n,F}$ and L_n , as we will later extend scalars to Λ' .) Similarly, there are modules $\tilde{L}_{n,F}$ fitting into nonsplit extensions

$$0 \longrightarrow \mathbb{Z}_2(1) \longrightarrow \tilde{L}_{n,F} \longrightarrow L_{n,F} \longrightarrow 0.$$

Theorem 4.1 *Let F be an exceptional extension of \mathbb{Q}_2 of degree d . Then there are isomorphisms of Λ'_F -modules*

$$M_\infty \cong \begin{cases} \tilde{L}_{2,F} \oplus (\Lambda'_F)^{d-1} & \text{if } d \text{ odd} \\ \tilde{L}_{1,F} \oplus L_{1,F} \oplus (\Lambda'_F)^{d-2} & \text{if } d \text{ even.} \end{cases}$$

Before starting on the proof, we discuss the source of the parity distinction in Theorem 4.1. Let L be a finite abelian extension of F . By local class field theory there is a canonical homomorphism $\psi_{L/F} : F^\times \longrightarrow G(L/F)$, called the *symbol*, that induces an isomorphism

$$(F^\times / N_{L/F} F^\times) \xrightarrow{\cong} G(L/F).$$

In this paper we are only concerned with 2-extensions. Passing to inverse limits yields an isomorphism

$$\psi_F : (F^\times)^\wedge \xrightarrow{\cong} G(\Omega_F/F),$$

where Ω_F is the maximal abelian 2-extension of F and $(F^\times)^\wedge$, the 2-adic completion of the group of units, is isomorphic to $\mathbb{Z}_2^{d+1} \oplus \mathbb{Z}/2$. Let $\tilde{\sigma}$ denote the unique element of order two in $G(\Omega_F/F)$; thus $\tilde{\sigma} = \psi_F(-1)$. Recall also that there are no elements of finite order in the absolute Galois group G_F , since G_F has finite cohomological dimension. Now consider the natural map $\rho : G(\Omega_F/F) \longrightarrow \Gamma'_F = G(F_\infty/F)$.

Proposition 4.2

$$\rho(\tilde{\sigma}) = \begin{cases} \sigma & \text{if } d \text{ odd} \\ 1 & \text{if } d \text{ even.} \end{cases}$$

Proof: Clearly $\rho(\tilde{\sigma})$ is either σ or 1. To determine which alternative holds, we need only compute the action of $\tilde{\sigma}$ on $F_0 = F\sqrt{-1}$. In other words, we need to compute $(\psi_F(-1))(\sqrt{-1})$, or equivalently $(\psi_{F_0/F}(-1))(\sqrt{-1})$. We have

$$(\psi_{F_0/F}(-1))(\sqrt{-1}) = \{-1, -1\}_F \sqrt{-1},$$

where $\{-1, -1\}_F = \pm 1$ is by definition the *Hilbert symbol* (see [6]). Finally, it is well-known that $\{-1, -1\}_F = (-1)^d$. This proves the proposition.

(For the convenience of the reader, we expand on the last part of the argument: Since -1 is not a sum of two squares in \mathbb{Q}_2 , it is not a norm from $\mathbb{Q}_2\sqrt{-1}$. Hence $\{-1, -1\}_{\mathbb{Q}_2} = -1$. In the general case, consider the commutative square

$$\begin{array}{ccc} F^\times & \xrightarrow{\psi} & G(F_0/F) \\ \downarrow N & & \downarrow \cong \\ \mathbb{Q}_2^\times & \xrightarrow{\psi} & G(\mathbb{Q}_{2,0}/\mathbb{Q}_2) \end{array}$$

where N is the norm. Since $N(-1) = (-1)^d$, this yields the formula $\{-1, -1\}_F = (-1)^d$. Other proofs can be given using alternate interpretations of the Hilbert symbol; cf. [6].)

We now begin the proof of Theorem 4.1. Our strategy will be to compute the Tate homology modules $\hat{H}_*(\sigma; M_\infty)$ and $\hat{H}_*(\sigma; N)$; these will determine the isomorphism type of M_∞ , N by Theorem 3.2 and Proposition 3.7.

Consider the Hochschild-Serre spectral sequence in homology for the extension F_∞/F_∞^σ :

$$E_{p,q}^2 = H_p(\sigma; H_q(F_\infty; \mathbb{Z}_2)) \Rightarrow H_{p+q}(F_\infty^\sigma; \mathbb{Z}_2).$$

Recall that $H_q(F_\infty; \mathbb{Z}_2) = 0 = H_q(F_\infty^\sigma; \mathbb{Z}_2)$ for $q > 1$, and for $p > 0$

$$E_{p,0}^2 = \begin{cases} \mathbb{Z}/2 & \text{if } p \text{ odd} \\ 0 & \text{if } p \text{ even.} \end{cases}$$

Hence we have

$$\hat{H}_p(\sigma; M_\infty) = \begin{cases} \mathbb{Z}/2 & \text{if } p \text{ odd} \\ 0 & \text{if } p \text{ even.} \end{cases}$$

Now consider the long exact sequence in Tate homology of the short exact sequence

$$0 \longrightarrow \mathbb{Z}_2(1) \longrightarrow M_\infty \longrightarrow N \longrightarrow 0$$

It follows easily that $\hat{H}_p(\sigma; N) = 0$ for p even, and that there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \hat{H}_p(\sigma; N) \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

for p odd. We also have an exact sequence

$$0 \longrightarrow H_1(\sigma; M_\infty) \longrightarrow H_1(\sigma; N) \xrightarrow{\partial} H_0(\sigma; \mathbb{Z}_2(1)) \longrightarrow (M_\infty)_\sigma \longrightarrow N_\sigma \longrightarrow 0.$$

Since the natural map $\hat{H}_0(\sigma; \mathbb{Z}_2(1)) \longrightarrow H_0(\sigma; \mathbb{Z}_2(1))$ is an isomorphism $\mathbb{Z}/2 \cong \mathbb{Z}/2$, ∂ is onto. This proves:

Lemma 4.3 a) $\hat{H}_p(\sigma; N)$ is zero for p even, and is isomorphic to either $\Lambda_F/(2, T_F^2)$ or $\Lambda_F/(2, T_F) \oplus \Lambda_F/(2, T_F)$ for p odd.

b) $(M_\infty)_\sigma \xrightarrow{\cong} N_\sigma$

c) The extension $0 \rightarrow \mathbb{Z}_2(1) \rightarrow M_\infty \rightarrow N \rightarrow 0$ is nonsplit as Λ'_F -modules.

From (a), Theorem 3.2, and Lemma 3.1 we conclude that N is isomorphic to either $L_{2,F} \oplus \Lambda'_F{}^{d-1}$ or $L_{1,F} \oplus L_{1,F} \oplus \Lambda'_F{}^{d-2}$. In order to distinguish these two cases using Theorem 3.2 and Lemma 3.1, we would have to know the $\Lambda_F/2$ -module structure on $\hat{H}_1(\sigma; N)$. As we do not see how to do this directly, we will instead make use of the invariant $(-)\Gamma'_F$. Note that $(L_{n,F})\Gamma'_F \cong \mathbb{Z}_2 \oplus \mathbb{Z}/2$ for all $n > 0$.

Lemma 4.4

$$N \cong \begin{cases} L_{2,F} \oplus \Lambda'_F{}^{d-1} & \text{if } d \text{ odd} \\ L_{1,F} \oplus L_{1,F} \oplus \Lambda'_F{}^{d-2} & \text{if } d \text{ even} \end{cases}$$

Proof: By the remarks of the preceding paragraph, the torsion subgroup of $N_{\Gamma'_F}$ has order two or four; it suffices to show that the order is two for d odd and four for d even. We first show that $(M_\infty)\Gamma'_F \rightarrow N_{\Gamma'_F}$ is an isomorphism. Consider the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}_2(1)_\sigma & \longrightarrow & (M_\infty)_\sigma \\ \downarrow & & \downarrow \\ \mathbb{Z}_2(1)\Gamma'_F & \longrightarrow & (M_\infty)\Gamma'_F \end{array}$$

The lefthand map is an isomorphism, while the righthand map is surjective. By Lemma 4.3b, the top map is zero. Hence the bottom map is also zero, and $(M_\infty)\Gamma'_F \cong N_{\Gamma'_F}$ as claimed.

To analyze $(M_\infty)\Gamma'_F$ we use the homology Hochschild-Serre spectral sequence for F_∞/F . Since $H_m(F; \mathbb{Z}_2) = 0$ for $m > 1$, and $E_{p,0}^2 = 0$ for $q > 1$, there is an exact sequence

$$0 \rightarrow H_2(\Gamma'_F; \mathbb{Z}_2) \rightarrow (M_\infty)\Gamma'_F \rightarrow H_1(F; \mathbb{Z}_2) \xrightarrow{\rho} H_1(\Gamma'_F; \mathbb{Z}_2) \rightarrow 0.$$

Now $H_2(\Gamma'_F; \mathbb{Z}_2) = \mathbb{Z}/2$, $H_1(F; \mathbb{Z}_2) \cong \mathbb{Z}_2^{d+1} \oplus \mathbb{Z}/2$, and of course $H_1(\Gamma'_F; \mathbb{Z}_2) = \Gamma'_F \cong \mathbb{Z}_2 \oplus \mathbb{Z}/2$. Moreover $H_1(F; \mathbb{Z}_2) = G(\Omega/F)$, and ρ is the natural map of Proposition 4.2. Hence the torsion subgroup of $(M_\infty)\Gamma'_F$ has order two if d odd and order four if d even. This completes the proof of Lemma 4.4.

For d odd we therefore have an extension of the form

$$0 \rightarrow \mathbb{Z}_2(1) \rightarrow M_\infty \rightarrow L_{2,F} \oplus \Lambda'_F{}^{d-1} \rightarrow 0.$$

Since the extension is non-split by Lemma 4.3c, Proposition 3.7 forces the desired isomorphism $M_\infty \cong \tilde{L}_{2,F} \oplus \Lambda'^{d-1}$.

If d is even we conclude that $M_\infty \cong Q \oplus \Lambda'_F{}^{d-2}$, where Q fits into an extension

$$0 \longrightarrow \mathbb{Z}_2(1) \longrightarrow Q \longrightarrow L_{1,F} \oplus L_{1,F} \longrightarrow 0.$$

Now $\text{Ext}_{\Lambda'_F}^1(L_{1,F} \oplus L_{1,F}, \mathbb{Z}_2(1)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, and the natural action of $\text{Aut}_{\Lambda'_F}(L_{1,F} \oplus L_{1,F})$ on this group preserves the isomorphism type of the middle term Q . Moreover, $\text{Aut}_{\Lambda'_F}(L_{1,F} \oplus L_{1,F})$ contains a canonical subgroup $GL_2\mathbb{Z}_2$ that clearly acts transitively on the nonzero elements of Ext . Hence there is only one isomorphism class Q for which the extension is nonsplit, namely $Q \cong \tilde{L}_{1,F} \oplus L_{1,F}$. This completes the proof of Theorem 4.1.

4.2 Iwasawa theory for F_∞^σ

As a corollary we can determine the Iwasawa module more commonly studied by number theorists: $H_1(F_\infty^\sigma; \mathbb{Z}_2)$. This result will not be used in the sequel.

Let N_k denote the ideal $(2, T_F^k) \subset \Lambda_F$, regarded as a Λ_F -module. Equivalently, N_k has a presentation with generators e_1, e_2 and relation $2e_1 = T_F^k e_2$.

Corollary 4.5 (*Iwasawa [2]*)

$$H_1(F_\infty^\sigma; \mathbb{Z}_2) \cong \Lambda_F^{d-1} \oplus N_1$$

Proof: By class field theory $H_1(F_\infty^\sigma; \mathbb{Z}_2)$ is isomorphic to the norm inverse limit of the 2-completed unit groups $((F_n^\sigma)^\times)^\wedge$. Here $F_n^\sigma \subset F_{n+1}^\sigma \dots$ is the usual tower of quadratic extensions with union F_∞^σ . Since F_n^σ has no 2-power roots of unity other than ± 1 , it follows that $H_1(F_\infty^\sigma; \mathbb{Z}_2)$ is 2-torsion-free. Now consider once again the spectral sequence

$$H_p(\sigma; H_q(F_\infty; \mathbb{Z}_2)) \Rightarrow H_{p+q}(F_\infty^\sigma; \mathbb{Z}_2),$$

which yields a short exact sequence

$$0 \longrightarrow (M_\infty)_\sigma \longrightarrow H_1(F_\infty^\sigma; \mathbb{Z}_2) \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Since $(L_k)_\sigma \cong N_k$, as is easily checked, Lemma 4.3b and Lemma 4.4 imply that

$$(M_\infty)_\sigma \cong \begin{cases} \Lambda_F^{d-1} \oplus N_2 & \text{if } d \text{ odd} \\ \Lambda_F^{d-2} \oplus N_1 \oplus N_1 & \text{if } d \text{ even.} \end{cases}$$

Hence $H_1(F_\infty^\sigma; \mathbb{Z}_2)$ is a torsion-free Λ_F -module of rank d . Moreover, $(M_\infty)_\sigma$ has rank d and defect 4. It follows that $H_1(F_\infty^\sigma; \mathbb{Z}_2)$ has defect 2 (see the remark following Proposition 2.2). Since any two nonzero homomorphisms $\Lambda_F^d \rightarrow \mathbb{Z}/2$ are equivalent (up to automorphism of Λ_F^d), any two submodules of index 2 are isomorphic. This proves the corollary.

4.3 An application to Galois cohomology

Next we compute the 2-adic cohomology $H^{p,m}F \equiv H^p(F; \mathbb{Z}_2(m))$. This calculation is standard, but it is worth recording how it follows from Theorem 4.1.

Theorem 4.6

$$H^{0,m}F \cong \begin{cases} \mathbb{Z}_2 & \text{if } m = 0 \\ 0 & \text{if } m \neq 0. \end{cases}$$

$$H^{1,m}F \cong \begin{cases} \mathbb{Z}_2^{d+1} \oplus \mathbb{Z}/2 & \text{if } m = 1 \\ \mathbb{Z}_2^d \oplus \mathbb{Z}/2 & \text{if } m > 1, m \text{ odd} \\ \mathbb{Z}_2^d \oplus \mathbb{Z}/2^{w_m F} & \text{if } m \text{ even.} \end{cases}$$

$$H^{2,m}F \cong \begin{cases} \mathbb{Z}_2 & \text{if } m = 1 \\ \mathbb{Z}/2^{w_{m-1}F} & \text{if } m > 1, m \text{ odd} \\ \mathbb{Z}/2 & \text{if } m \text{ even.} \end{cases}$$

Proof: For $H^{0,m}$ this is obvious. For $H^{2,m}$, we have by local class field theory that

$$H^2(F; \mathbb{Z}_2(m)) \cong H_0(F; \mathbb{Z}_2(1-m)) \cong H_0(\Gamma'_F; \mathbb{Z}_2(1-m)).$$

If $m = 1$, this last group is evidently \mathbb{Z}_2 . If m is even, it is clearly $\mathbb{Z}/2$. If m is odd, we have

$$(\mathbb{Z}_2(1-m))_{\Gamma'_F} = (\mathbb{Z}_2(1-m))_{\Gamma_F} = (\mathbb{Z}_2(m-1))_{\Gamma_F} = \mathbb{Z}/2^{w_{m-1}F}.$$

For $H^{1,m}$ we use the universal coefficient spectral sequence

$$\text{Ext}_{\Lambda'_F}^p(H_q(F_\infty; \mathbb{Z}_2), \mathbb{Z}_2(m)) \Rightarrow H^{p+q}(F; \mathbb{Z}_2(m)).$$

The spectral sequence yields an exact sequence

$$0 \longrightarrow \text{Ext}_{\Lambda'_F}^1(\mathbb{Z}_2, \mathbb{Z}_2(m)) \longrightarrow H^{1,m}F \longrightarrow \text{Hom}_{\Lambda'_F}(M_\infty, \mathbb{Z}_2(m)) \longrightarrow \text{Ext}_{\Lambda'_F}^2(\mathbb{Z}_2, \mathbb{Z}_2(m)).$$

It follows that the torsion subgroup of $H^{1,m}F$ is precisely $\text{Ext}_{\Lambda'_F}^1(\mathbb{Z}_2, \mathbb{Z}_2(m))$, which in turn is isomorphic to

$$H^1(\Gamma'_F; \mathbb{Z}_2(m)) \cong \begin{cases} \mathbb{Z}/2^{w_m F} & \text{if } m \text{ even} \\ \mathbb{Z}/2 & \text{if } m \text{ odd,} \end{cases}$$

and that

$$\text{rank}_{\mathbb{Z}_2} H^{1,m}F = \text{rank}_{\mathbb{Z}_2}(\text{Hom}_{\Lambda'_F}(M_\infty, \mathbb{Z}_2(m)))$$

Now since M_∞ is a finitely-generated Λ_F -module, we have

$$(Hom_{\Lambda'_F}(M_\infty, \mathbb{Z}_2(m))[\frac{1}{2}]) = Hom_{\Lambda'_F[\frac{1}{2}]}(M_\infty[\frac{1}{2}], \mathbb{Z}_2(m))[\frac{1}{2}],$$

Since $M_\infty[\frac{1}{2}] \cong ((\Lambda'_F)^d \oplus \mathbb{Z}_2(1))[\frac{1}{2}]$, regardless of the parity of d , we conclude that in all cases

$$rank_{\mathbb{Z}_2}(Hom_{\Lambda'_F}(M_\infty, \mathbb{Z}_2(m))) = \begin{cases} d & \text{if } m \neq 1 \\ d + 1 & \text{if } m = 1. \end{cases}$$

This completes the proof of Theorem 4.6.

5 The Rognes-Weibel theorem

Let $\mathbb{H}_{\acute{e}t}(F; \underline{K})$ denote the Thomason-Jardine hypercohomology spectrum associated to the algebraic K -theory presheaf \underline{K} on the étale site of F . In essence, $\mathbb{H}_{\acute{e}t}(F; \underline{K})$ is just the homotopy fixed-point spectrum associated to the action of G_F on $K\overline{F}$, although one must take into account the profinite topology on G_F . For background on hypercohomology spectra, see [3]. The following theorem is due to Rognes and Weibel [9]. It verifies the strong form of the 2-adic Lichtenbaum-Quillen conjectures for F .

Theorem 5.1 *The natural augmentation $\eta : KF \rightarrow \mathbb{H}_{\acute{e}t}(F; \underline{K})$ induces a weak equivalence on 0-connected covers of the completions:*

$$P_1 KF^\wedge \xrightarrow{\cong} P_1 \mathbb{H}_{\acute{e}t}(F; \underline{K})^\wedge.$$

In fact [9] deals only with totally imaginary number rings. However, as pointed out to the author by Rognes, the Bloch-Lichtenbaum and Tate spectral sequence argument of [9] carries over immediately to any 2-adic local field. Indeed, even the Tate spectral sequence can be dispensed with here, since all we need is an isomorphism $(H_{\acute{e}t}^2(F_0; \mathbb{Z}_2(n)))_\sigma \cong H_{\acute{e}t}^2(F; \mathbb{Z}_2(n))$. This follows directly from local class field theory (cf. the first step in the proof of Theorem 4.6).

Remark: Since $\hat{L}(-)$ is invariant under passage to connective covers, we have at once an extension of Thomason's theorem [12] to F : $\hat{L}(KF) \cong \hat{L}\mathbb{H}_{\acute{e}t}(F; \underline{K})$. Since F does not contain $\sqrt{-1}$, this case was not covered by [12].

Theorem 5.2 *For $n > 0$ the homotopy groups of KF^\wedge are as follows:*

$$\pi_n KF^\wedge \cong \begin{cases} \mathbb{Z}_2^{d+1} \oplus \mathbb{Z}/2 & \text{if } n = 1 \\ \mathbb{Z}_2^d \oplus \mathbb{Z}/2 & \text{if } n = 1 \pmod{4}, n > 1 \\ \mathbb{Z}_2^d \oplus \mathbb{Z}/2^{w_m F} & \text{if } n = 3 \pmod{4}, n = 2m - 1 \\ \mathbb{Z}/2^{w_m F} & \text{if } n = 0 \pmod{4}, n = 2m \\ \mathbb{Z}/2 & \text{if } n = 2 \pmod{4}. \end{cases}$$

Proof: Given Theorem 5.1, this is a well-known and easy calculation (although here we are putting the cart before the horse, since in fact the proof of Theorem 5.1 proceeds by first computing the homotopy groups): Since $KF^\wedge \rightarrow \mathbb{H}_{\acute{e}t}(F; \underline{K})^\wedge$ is a weak equivalence on 0-connected covers, the groups in question can be computed from the descent spectral sequence. The descent spectral sequence collapses because $cd_{\acute{e}t}F = 2$ and all the odd rows vanish. It follows that

$$\pi_n KF^\wedge \cong \begin{cases} H^{1,m}F & \text{if } n = 2m - 1 \\ H^{2,m+1}F & \text{if } n = 2m \end{cases}$$

Then the result can be read off from Theorem 4.6.

Remark: There is a natural short exact sequence

$$0 \rightarrow H_{\acute{e}t}^2(F; \mathbb{Z}_2(1)) \pi_0 KF^\wedge \rightarrow \pi_0 \mathbb{H}_{\acute{e}t}(F; \underline{K}) \rightarrow H_{\acute{e}t}^0(F; \mathbb{Z}_2(0)) \rightarrow 0$$

with both end terms isomorphic to \mathbb{Z}_2 . Furthermore, this sequence is canonically split by the natural map $\pi_0 KF^\wedge \rightarrow \pi_0 \mathbb{H}_{\acute{e}t}(F; \underline{K})$. Hence there is a fibre sequence

$$KF^\wedge \rightarrow P_0 \mathbb{H}_{\acute{e}t}(F; \underline{K}) \xrightarrow{\epsilon} H\mathbb{Z}_2$$

where $H\mathbb{Z}_2$ is the Eilenberg-MacLane spectrum.

The natural map $\mathbb{H}_{\acute{e}t}(F; \underline{K}) \rightarrow \hat{L}\mathbb{H}_{\acute{e}t}(F; \underline{K})$ induces a weak equivalence on (-1) -connected covers. Hence:

Corollary 5.3 $KF \rightarrow \hat{L}KF$ induces a weak equivalence $P_1 KF^\wedge \xrightarrow{\cong} P_1 \hat{L}KF$.

In view of the above theorem and its corollary, the homotopy-type of KF^\wedge can be determined by studying $\mathbb{H}_{\acute{e}t}(F; \underline{K})$ and $\hat{L}KF$.

6 Topological K-theory of KF^\wedge

By Corollary 5.6 of [4], we have at once:

Theorem 6.1

$$\hat{\mathcal{K}}^p KF \cong \begin{cases} \Lambda' \otimes_{\Lambda'_F} \mathbb{Z}_2 & \text{if } p = 0 \\ \Lambda' \otimes_{\Lambda'_F} M_\infty & \text{if } p = -1. \end{cases}$$

We wish to identify the Λ' -modules of Theorem 6.1 more explicitly, using Theorem 4.1. Let $k = a_F - 2$ as usual, and set

$$f_k = (1 + T)^{2^k} - c_0^{2^k}, \quad g_k = (1 + T)^{2^k} - 1.$$

Define $A_k = \Lambda/f_k$. Here σ acts trivially, following the convention established earlier. Recall also that A_k^\flat then denotes the same Λ -module with σ acting as -1 .

Proposition 6.2 a) $\Lambda' \otimes_{\Lambda'_F} \mathbb{Z}_2 \cong \Lambda/g_k$

b) $\Lambda' \otimes_{\Lambda'_F} \mathbb{Z}_2(1) \cong A_k^b$

c) $\Lambda' \otimes_{\Lambda'_F} L_{n,F} \cong L_{n2^k}$.

Proof: a) We have $\Lambda' \otimes_{\Lambda'_F} \mathbb{Z}_2 = \Lambda \otimes_{\Lambda'_F} \mathbb{Z}_2 \cong \Lambda/(T_F)$. But $T_F = \gamma_F - 1 = \gamma_0^{2^k} - 1 = (1+T)^{2^k} - 1$.

b) This is similar to (a); alternatively, one can simply Tate twist the result of (a).

c) As Λ -modules we have $\Lambda' \otimes_{\Lambda'_F} L_{n,F} = \Lambda' \oplus \Lambda'$, with σ acting as

$$\begin{pmatrix} 1 & T_F^n \\ 0 & -1 \end{pmatrix}$$

Since $T_F = T^{2^k} \pmod{2}$, the result follows from the remark following Lemma 3.1.

Now consider the extension

$$0 \longrightarrow \mathbb{Z}_2(1) \longrightarrow \tilde{L}_{n,F} \longrightarrow L_{n,F} \longrightarrow 0.$$

Applying the functor $\Lambda' \otimes_{\Lambda'_F} (-)$, we get

$$0 \longrightarrow A_k^b \longrightarrow \Lambda' \otimes_{\Lambda'_F} \tilde{L}_{n,F} \longrightarrow L_{n2^k} \longrightarrow 0.$$

Define a Λ' -module $L[n, k]$ as the Λ -module $\Lambda/f_k \oplus \Lambda^2$, with σ acting by the matrix

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & f_k^n \\ 0 & 0 & -1 \end{pmatrix}$$

Then for $k = a_F - 2$, $L[n, k] = \Lambda' \otimes_{\Lambda'_F} \tilde{L}_{n,F}$. This yields the explicit calculation:

Theorem 6.3

$$\hat{\mathcal{K}}^0 KF \cong \Lambda/g_k$$

and

$$\hat{\mathcal{K}}^{-1} KF \cong \begin{cases} L[2, k] \oplus \Lambda'^{d-1} & \text{if } d \text{ odd} \\ L[1, k] \oplus L_{2^k} \oplus \Lambda'^{d-2} & \text{if } d \text{ even.} \end{cases}$$

To complete the picture, we show that the modules $L[n, k]$ are determined up to isomorphism by their Tate homology groups together with the extension given above. Recall that $Q_m = \Lambda/(2, T^m)$.

Proposition 6.4 *Suppose P is a Λ' -module fitting into a short exact sequence*

$$0 \longrightarrow A_k^b \longrightarrow P \longrightarrow L_{n2^k} \longrightarrow 0.$$

Then P is isomorphic to $L[n, k]$ if and only if

$$\hat{H}_p(\sigma; R) \cong \begin{cases} 0 & \text{if } p \text{ even} \\ Q_{(n-1)2^k} & \text{if } p \text{ odd.} \end{cases}$$

In particular, P is σ -acyclic if and only if $P \cong L[1, k]$.

Proof: First observe that

$$\text{Ext}_{\Lambda'}^1(L_{n2^k}, A_k^b) \cong H^1(\sigma; \text{Hom}_{\Lambda}(L_{n2^k}, A_k^b)) = Q_{2^k}.$$

It is then easy to see that if the extension is nonsplit then P is isomorphic to $A_k^b \oplus \Lambda^2$ with σ acting as

$$\begin{pmatrix} -1 & T^i & 0 \\ 0 & 1 & f_k^n \\ 0 & 0 & -1 \end{pmatrix}$$

for a unique i , $0 \leq i < 2^k$. The boundary map

$$\partial : \hat{H}_1(\sigma; L_{n2^k}) = Q_{n2^k} \longrightarrow \hat{H}_0(\sigma; A_k^b) = Q_{2^k}$$

has image $T^i Q_{2^k}$. Hence

$$\hat{H}_p(\sigma; P) \cong \begin{cases} Q_i & \text{if } p \text{ even} \\ Q_{(n-1)2^k+i} & \text{if } p \text{ odd.} \end{cases}$$

Since $L[n, k]$ corresponds to the case $i = 0$, the result follows.

The module $L[1, k]$ will play a central role in the sequel. As Λ -module, $L[1, k] = A_k^b \oplus \Lambda \oplus \Lambda$; we write e_0, e_1, e_2 respectively for the standard generators of the three summands.

Proposition 6.5 *$L[1, k]$ has projective dimension one as Λ' -module. In fact there is a resolution*

$$0 \longrightarrow \Lambda' \xrightarrow{i} \Lambda' \oplus \Lambda' \xrightarrow{j} L[1, k] \longrightarrow 0,$$

where j maps the two generators to e_1, e_2 and $i(1) = (1 + \sigma, -f_k)$. In particular, $L[1, k]$ has a presentation $\{e_1, e_2; (1 + \sigma)e_2 = f_k e_1\}$.

Proof: Since $L[1, k]$ is σ -acyclic and has projective dimension one as Λ -module, it has projective dimension one as Λ' -module by [5], Lemma 4.6. Alternatively, one can check directly that the indicated sequence is short exact.

7 The reduced K-theory spectrum

Let γ_F denote, as usual, a fixed topological generator of Γ_F . Let $\phi_F = \sigma\gamma_F$. Thus ϕ_F topologically generates a subgroup H of Γ'_F , isomorphic to \mathbb{Z}_2 and of index 2, but acting nontrivially on the fourth roots of unity. Recall that q_F is an odd prime power topologically equivalent to $-c(\gamma_F) = c(\sigma\gamma_F)$. From now on we simply write q in place of q_F . (If $F = \mathbb{Q}_2$, and we think in terms of Adams operations, γ corresponds to ψ^5 , $\sigma\gamma$ corresponds to ψ^{-5} , and we can take $q = 3$.) Let $F' = F_\infty^H$, a quadratic extension of F . Since $H \cong \mathbb{Z}_2$, the epimorphism $G_{F'} \longrightarrow H$ admits a splitting $s : H \longrightarrow G_{F'}$. Let $\tilde{H} = s(H)$, and $\tilde{\phi}_F = s(\phi_F)$. Let E denote the fixed field of \tilde{H} . Then the natural map $G_E = \tilde{H} \longrightarrow H = G(F_\infty/F')$ is an isomorphism. It follows that $E_\infty = \overline{F}$.

Proposition 7.1 *There is a weak equivalence $(KE)^\wedge \cong K\mathbb{F}_q^\wedge$.*

Proof: The augmentation $\eta_E : KE \rightarrow \mathbb{H}_{\acute{e}t}(E; \underline{K})$ induces a weak equivalence

$$P_0 KE^\wedge \rightarrow P_0 \mathbb{H}_{\acute{e}t}(E; \underline{K})^\wedge.$$

Since the absolute Galois group G_E is isomorphic to \mathbb{Z}_2 , with generator $\tilde{\phi}_F$, standard arguments show that $\mathbb{H}_{\acute{e}t}(E; \underline{K})$ is weak equivalent to the homotopy fixed-point spectrum of $K\tilde{\phi}_F : K\bar{F} \rightarrow K\bar{F}$. This spectrum in turn is weak equivalent to the homotopy fibre of $\tilde{\phi}_F - 1$. Finally, there is a homotopy commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & K\bar{F}^\wedge & \xrightarrow{\tilde{\phi}_F - 1} & K\bar{F}^\wedge \\ \bar{\theta} \downarrow & & \downarrow \theta & & \downarrow \theta \\ B & \longrightarrow & bu^\wedge & \xrightarrow{\psi^q - 1} & bu^\wedge \end{array}$$

in which A and B are the homotopy fibres and θ is a weak equivalence given by Suslin's theorems [10], [11]. (Here θ depends on a choice of embedding $\mu F \subset \mu\mathbb{C}$.) From the diagram we obtain a weak equivalence $\bar{\theta} : A \rightarrow B$. In fact $\bar{\theta}$ is determined canonically by θ , and in any case $\bar{\theta}$ is unique up to homotopy since $\hat{K}^{-1}A = 0$. Since the (-1) -connected covers of A and B are respectively KE^\wedge and $K\mathbb{F}_q^\wedge$, this completes the proof.

Corollary 7.2

$$\pi_n KE^\wedge \cong \begin{cases} \mathbb{Z}_2 & \text{if } n = 0 \\ \mathbb{Z}/2^{w_m F} & \text{if } n = 3 \bmod 4, n = 2m - 1 \\ \mathbb{Z}/2 & \text{if } n = 1 \bmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore $\hat{K}^{-1}KE = 0$, and $\hat{K}^0 KE \cong \Lambda' / (\sigma\gamma_k - 1)$.

Proof: This is immediate from the theorem together with the known homotopy of $K\mathbb{F}_q^\wedge$. (Alternatively, since $KE^\wedge \rightarrow \mathbb{H}_{\acute{e}t}(E; \underline{K})^\wedge$ is a weak equivalence on (-1) -connected covers, the homotopy can be read off from the descent spectral sequence; the K-theory can be computed similarly using the spectral sequence of [4].)

Now fix E as above and define the associated *reduction map* $r : KF^\wedge \rightarrow K\mathbb{F}_q^\wedge$ as the composite

$$r : KF^\wedge \rightarrow KE^\wedge \xrightarrow{\bar{\theta}} K\mathbb{F}_q^\wedge.$$

The *reduced K-theory spectrum* is the homotopy fibre of r . *A priori*, $K^{red}F$ depends on the choice of q and r . In Corollary 8.2 we will see that the homotopy type of $K^{red}F$ is independent of both choices.

Proposition 7.3 a) If n is odd, $\pi_n(\iota^\wedge)$ induces an isomorphism from the torsion subgroup of $\pi_n K F^\wedge$ to $\pi_n K E^\wedge$. Hence $\pi_n(\iota^\wedge)$ is split surjective for all n .

b)

$$\pi_n K^{red} F \cong \begin{cases} \mathbb{Z}_2^{d+1} & \text{if } n = 1 \\ \mathbb{Z}_2^d & \text{if } n > 1, n \text{ odd} \\ \mathbb{Z}/2^{w_m F} & \text{if } n = 2m, m \text{ even} \\ \mathbb{Z}/2 & \text{if } n = 2m, m \text{ odd.} \end{cases}$$

Proof: Part (b) follows immediately from part (a) and Theorem 5.2. To prove (a), we may replace ι by the induced map on globally fibrant models

$$\mathbb{H}_{\acute{e}t}(F; \underline{K})^\wedge \xrightarrow{(\mathbb{H}_{\acute{e}t}^\wedge)^\wedge} \mathbb{H}_{\acute{e}t}(E; \underline{K})^\wedge$$

If $n = 2m - 1$, $\pi_n(\mathbb{H}_{\acute{e}t}^\wedge)$ can be identified with the natural map $H^{1,m} F \rightarrow H^{1,m} E$. From the associated universal coefficient spectral sequences, as in the proof of Theorem 4.6, we have a commutative diagram

$$\begin{array}{ccc} Ext_{\Lambda'_F}^1(\mathbb{Z}_2, \mathbb{Z}_2(m)) & \longrightarrow & H^{1,m} F \\ \downarrow & & \downarrow \\ Ext_{\Lambda'_E}^1(\mathbb{Z}_2, \mathbb{Z}_2(m)) & \longrightarrow & H^{1,m} E \end{array}$$

in which the bottom arrow is an isomorphism and the top arrow is an isomorphism onto the torsion subgroup. To complete the proof, we need only show that the left arrow is an isomorphism. This arrow can be identified with the restriction map $H^1(\Gamma'_F; \mathbb{Z}_2(m)) \rightarrow H^1(H; \mathbb{Z}_2(m))$, where H is the closed subgroup of Γ'_F generated by ϕ_F . From the Hochschild-Serre spectral sequence of the extension $H \rightarrow \Gamma'_F \rightarrow \sigma$ we see immediately that this restriction map is an isomorphism onto $(H^1(H; \mathbb{Z}_2(m)))^\sigma$. But σ acts trivially on $\mathbb{Z}_2(m)$ for m even, while for m odd we have $H^1(H; \mathbb{Z}_2(m)) = \mathbb{Z}/2$. Hence in all cases we have $(H^1(H; \mathbb{Z}_2(m)))^\sigma = H^1(H; \mathbb{Z}_2(m))$. This completes the proof of the proposition.

8 Topological K-theory of the reduced K-theory spectrum

In this section we compute $\hat{\mathcal{K}}^* K^{red} F$.

Theorem 8.1 $\hat{\mathcal{K}}^0 K^{red} F = 0$ and $\hat{\mathcal{K}}^{-1} K^{red} F \cong L[1, k] \oplus (\Lambda')^{d-1}$ as Λ' -modules. In particular, $\hat{\mathcal{K}}^{-1} K^{red} F$ is σ -acyclic and has projective dimension one as Λ' -module.

Note that the parity of d does not appear in this result, as it did in Theorem 6.3. We also get:

Corollary 8.2 *The homotopy-type of $K^{red}F$ is independent of the choice of prime power q and reduction map r .*

Proof: By the theorem, $\hat{L}K^{red}F$ is an excellent spectrum; hence its homotopy-type is uniquely determined by the Λ' -module $\hat{\mathcal{K}}^{-1}K^{red}F$. Since $K^{red}F \cong P_1\hat{L}K^{red}F$, this proves the corollary.

The proof of the theorem involves two major steps. The first and hardest step is to show that $\hat{\mathcal{K}}^*K^{red}F$ is σ -acyclic (Lemma 8.3). The difficulty is that we are forced to analyze $\hat{\mathcal{K}}^*K^{red}F$, directly, as the fibre sequence defining $K^{red}F$ only yields $\hat{\mathcal{K}}^*K^{red}F$ up to extension.

The second step is purely algebraic. There is an extension

$$0 \longrightarrow \hat{\mathcal{K}}^{-1}KF \longrightarrow \hat{\mathcal{K}}^{-1}K^{red}F \longrightarrow Ker r^* \longrightarrow 0.$$

in which the end terms have been explicitly determined. In Lemma 8.10 we show that any such extension that is σ -acyclic has the desired form.

8.1 The proof of Theorem 8.1: Step 1

In this section we prove:

Lemma 8.3 *$\hat{\mathcal{K}}^{-1}K^{red}F$ is σ -acyclic.*

The proof involves two steps. Let $\tilde{K}^{red}F$ denote the fibre of $KF \rightarrow KE$, so that $K^{red}F$ is the 2-completion of $\tilde{K}^{red}F$. We first interpret the fibre sequence

$$\tilde{K}^{red}F \longrightarrow KF \longrightarrow KE$$

as a fibre sequence of hypercohomology spectra on $F_{\acute{e}t}$ up to connective covers, (Corollary 8.6). This allows us to replace $\hat{\mathcal{K}}^*K^{red}F$ by $\hat{\mathcal{K}}^*\mathbb{H}(F; \mathcal{E})$ for a certain presheaf of spectra \mathcal{E} . Analysis of the presheaf \mathcal{E} then yields the desired acyclicity (Lemma 8.7).

8.1.1 Reduced K-theory vs. hypercohomology

The method is quite general; for the moment, we let E/F denote an arbitrary separable algebraic extension of fields. Consider the morphism $\pi : Spec E \rightarrow Spec F$ and the associated direct image functor $\pi_{\#}$ on presheaves of spectra. Let \mathcal{E} be a presheaf of spectra defined on a big étale site Φ , so that Φ at least contains all separable algebraic extensions of F and all finite products of such extensions. Let \bar{F} denote a fixed separable closure of F .

Lemma 8.4 *Suppose that \mathcal{E} and $\mathbb{H}_{\acute{e}t}(-; \mathcal{E})$ are continuous presheaves on Φ , and that \mathcal{E} is an additive presheaf. Then the natural map*

$$\mathbb{H}(F; \pi_{\#}\mathcal{E}) \longrightarrow \mathbb{H}(E; \mathcal{E})$$

is a weak equivalence.

Proof: This is an instance of the “generalized descent problem”; cf. [5], 6.3. It is enough to show that $\pi_{\#}\eta : \pi_{\#}\mathcal{E} \rightarrow \pi_{\#}\mathbb{H}(-; \mathcal{E})$ is a stalkwise weak equivalence on $F_{\acute{e}t}$. Here there is only one stalk to check, namely, $p : \text{Spec } \overline{F} \rightarrow \text{Spec } F$. Since \mathcal{E} and $\mathbb{H}_{\acute{e}t}(-; \mathcal{E})$ are continuous, there are weak equivalences

$$p^{\#}\mathcal{E} \cong \text{colim}_{\beta} \mathcal{E}(E \otimes_F F_{\beta}) = \mathcal{E}(E \otimes_F \overline{F}) = \text{colim}_{\alpha} \mathcal{E}(E_{\alpha} \otimes_F \overline{F})$$

and similarly for $\mathbb{H}_{\acute{e}t}(-; \mathcal{E})$; here F_{β} and E_{α} range over finite separable extensions. Now $E_{\alpha} \otimes_F \overline{F}$ is a finite product of copies of \overline{F} , so by the additivity assumption we reduce to showing that $\mathcal{E}(\overline{F}) \rightarrow \mathbb{H}(\overline{F}; \mathcal{E})$ is a weak equivalence. This is trivially true, since $\overline{F}_{\acute{e}t}$ is the trivial site.

Remark: This is, of course, only an abstract manifestation of “Shapiro’s lemma”. If we were working on the site defined by a discrete (as opposed to profinite) group G , with subgroup H , we could express the result as follows: In place of \mathcal{E} we would have a G -spectrum X , with $\pi_{\#}X$ the coinduced spectrum $\text{coind}_H^G X \equiv \text{Map}^H(G, X)$. Then the statement of the lemma becomes

$$\mathbb{H}_{\acute{e}t}(G; \text{coind}_H^G X) \cong \mathbb{H}_{\acute{e}t}(H; X),$$

or even more concretely

$$\text{Map}^G(EG_+, \text{Map}^H(G, X)) \cong \text{Map}^H(EH_+, X).$$

In our situation, with $\mathcal{E} = \underline{K}$, the continuity hypothesis holds for \underline{K} and $\mathbb{H}(-; \underline{K})$ by work of Quillen and Thomason, respectively (cf. [3], Theorem 3.8). Hence we have:

Corollary 8.5 $\mathbb{H}(F; \pi_{\#}\underline{K})^{\wedge} \xrightarrow{\cong} \mathbb{H}(E; \underline{K})^{\wedge}$.

Since hypercohomology commutes with objectwise fibre sequences of presheaves, we also have:

Corollary 8.6 *Let \mathcal{E} denote the objectwise homotopy fibre of $\underline{K} \rightarrow \pi_{\#}\underline{K}$ on $F_{\acute{e}t}$. Then there is a commutative diagram of fibre sequences*

$$\begin{array}{ccccc} \tilde{K}^{\text{red}}F & \longrightarrow & KF & \longrightarrow & KE \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{H}(F; \mathcal{E}) & \longrightarrow & \mathbb{H}(F; K) & \longrightarrow & \mathbb{H}(F; \pi_{\#}\underline{K}) \end{array}$$

in which the vertical arrows all induce weak equivalences on 0-connected covers of completions. In particular, $\hat{\mathcal{K}}^*(\mathbb{H}(F; \mathcal{E})) \cong \hat{\mathcal{K}}^* K^{\text{red}}F$.

Here we are guilty of a small abuse of notation, in that $\pi_{\#}\underline{K}$ is really $\pi_{\#}$ of the restriction of \underline{K} on the big site Φ to $E_{\acute{e}t}$. Since E/F is algebraic, this restriction can also be written as $\pi_{\#}\pi^{\#}\underline{K}$, where now \underline{K} is really the restriction to $F_{\acute{e}t}$. But the meaning should be clear from the context, and to avoid cluttering the notation we omit $\pi^{\#}$. Thus $\underline{K} \rightarrow \pi_{\#}\underline{K}$ is really the adjunction unit $\underline{K} \rightarrow \pi_{\#}\pi^{\#}\underline{K}$.

8.1.2 Acyclicity of $\hat{\mathcal{K}}^{-1}_{\mathbb{H}}(F; \mathcal{E})$

Lemma 8.7 $\hat{\mathcal{K}}^{-1}_{\mathbb{H}}(F; \mathcal{E})$ is σ -acyclic.

Proof: By the universal coefficient theorem for K-theory, it is equivalent to show that

$$(\mathcal{K} \wedge \mathcal{N})_{0\mathbb{H}}(F; \mathcal{E})$$

is σ -acyclic, where $\mathcal{N} = \Sigma^{-1}M\mathbb{Z}/2^\infty$ is the desuspension of the Moore spectrum $M\mathbb{Z}/2^\infty$. (For this and subsequent claims about topological K-theory, see [4].) We will use the spectral sequence

$$E_2^{p,-q} = H^p(F; a(\mathcal{K} \wedge \mathcal{N})_q \mathcal{E}) \Rightarrow (\mathcal{K} \wedge \mathcal{N})_{q-p\mathbb{H}}(F; \mathcal{E}),$$

which is defined since F has finite cohomological dimension (cf. [5], Proposition 3.11). Here the sheafification $a(\mathcal{K} \wedge \mathcal{N})_q \mathcal{E}$ may be regarded simply as the Galois module defined by its stalk. We will abbreviate it as \mathcal{A}_q ; thus

$$\mathcal{A}_q = a(\mathcal{K} \wedge \mathcal{N})_q \mathcal{E} = p^*(\mathcal{K} \wedge \mathcal{N})_q \mathcal{E} = (\mathcal{K} \wedge \mathcal{N})_q p^\# \mathcal{E}.$$

Note that \mathcal{A}_q is periodic in q , with period two.

Lemma 8.8 a) $\mathcal{A}_{-1} = 0$

b) \mathcal{A}_{-2} is G_F -acyclic

c) \mathcal{A}_{-2} is σ -acyclic.

(Here acyclic means that the appropriate cohomology groups vanish in positive degrees.)

Proof: Consider the map

$$(\mathcal{K} \wedge \mathcal{N})_q p^\# \underline{K} \longrightarrow (\mathcal{K} \wedge \mathcal{N})_q p^\# \pi_\# \underline{K}$$

for $q = 0, -1$. For $q = 0$, both groups are zero. This is because $(p^\# \underline{K})^\wedge \cong bu^\wedge$, and hence $(\mathcal{K} \wedge \mathcal{N})_0 p^\# \underline{K} = 0$; similarly, $(\mathcal{K} \wedge \mathcal{N})_0 p^\# \pi_\# \underline{K} = \text{coind}_{\tilde{H}}^{G_F}(\mathcal{K} \wedge \mathcal{N})_0 p^\# \underline{K} = 0$. For $q = -1$, the map can be identified with the canonical adjunction unit $(\Lambda')^* \longrightarrow \text{coind}_{\tilde{H}}^{G_F}(\Lambda')^*$, and in particular is injective. This yields (a) and a short exact sequence

$$0 \longrightarrow (\Lambda')^* \xrightarrow{j} \text{coind}_{\tilde{H}}^{G_F}((\Lambda')^*) \longrightarrow \mathcal{A}_{-2} \longrightarrow 0$$

Here $(-)^*$ denotes the continuous Pontrjagin dual of a profinite module. Since the first two terms of the sequence are σ -acyclic, so is the third; this proves (c).

It remains to prove (b). Note first that $\text{coind}_{\tilde{H}}^{G_F}(\Lambda')^*$ is G_F -acyclic: We have

$$H^p(F; \text{coind}_{\tilde{H}}^{G_F}(\Lambda')^*) = H^p(E; (\Lambda')^*).$$

But $\tilde{H} = G_E$ acts on $(\Lambda')^*$ via the inclusion $\tilde{H} \xrightarrow{\cong} H \subset \Gamma'_F$, so $(\Lambda')^*$ is an acyclic H-module.

Second, note that $H^p(F; (\Lambda')^*) = 0$ for $p > 1$: Since $cd_2 F = 2$, we need only consider $p = 2$. Since $cd_2 F_\infty = 1$, this reduces to showing that

$$H^1(\Gamma'_F; H^1(F_\infty; (\Lambda')^*)) = 0.$$

Now by an easy universal coefficient argument,

$$H^1(F_\infty; (\Lambda')^*) = \text{Hom}(H_1(F_\infty; \mathbb{Z}_2), (\Lambda')^*).$$

It follows easily that $H^1(F_\infty; (\Lambda')^*)$ is a coinduced Γ'_F -module, and so is Γ'_F -acyclic, as desired. Part (b) now follows from the long exact cohomology sequence associated to the short exact sequence above. This completes the proof of Lemma 8.8.

By Lemma 8.8ab and the descent spectral sequence, we have

$$(\mathcal{K} \wedge \mathcal{N})_{-2} \mathbb{H}^{\text{ét}}(F; \mathcal{E}) = \mathcal{A}^{G_F}$$

and

$$(\mathcal{K} \wedge \mathcal{N})_{-1} \mathbb{H}^{\text{ét}}(F; \mathcal{E}) = 0.$$

To complete the proof of Lemma 8.7, we need to show \mathcal{A}^{G_F} is σ -acyclic. This will follow from a very general lemma:

Lemma 8.9 *Let Φ be a Grothendieck site with enough points, and let ℓ be a prime such that objects X of Φ have finite cohomological dimension for ℓ -torsion sheaves. Let G be a finite group, and let \mathcal{A} be a sheaf of ℓ -torsion G -modules. Suppose that (i) \mathcal{A} is acyclic for site cohomology $H_\Phi^*(X; -)$ and (ii) the stalks of \mathcal{A} are acyclic G -modules. Then the module of global sections $\mathcal{A}(X)$ is an acyclic G -module.*

Proof: A proof of a somewhat more general result can be found in [5], proof of Theorem 4.4. Here we will provide an interpretation and proof for the case when Φ is the étale site of a field κ . Then $cd_\ell \kappa < \infty$, \mathcal{A} is a discrete ℓ -torsion $(G_\kappa \times G)$ -module, and the assumptions (i)-(ii) are that \mathcal{A} is acyclic for G_κ and G separately. We wish to show that \mathcal{A}^{G_κ} is G -acyclic. There are composite functor or Kunneth spectral sequences

$$H^p(G_\kappa; H^q(G; \mathcal{A})) \Rightarrow H^{p+q}((G_\kappa \times G; \mathcal{A}))$$

and

$$H^p(G; H^q(G_\kappa; \mathcal{A})) \Rightarrow H^{p+q}((G_\kappa \times G; \mathcal{A})).$$

By assumption both spectral sequence collapse to the bottom edge, and hence

$$H^p(G_\kappa; \mathcal{A}^G) = H^p(G; \mathcal{A}^{G_\kappa}).$$

The lefthand side vanishes for $p > cd_\ell \kappa$, and hence similarly for the righthand side. Since G is finite, this forces \mathcal{A}^{G_κ} G -acyclic as desired.

Combining Lemma 8.8bc and Lemma 8.9 shows that \mathcal{A}^{G_F} is σ -acyclic. This completes the proof of Lemma 8.7, and hence also the proof of Lemma 8.3.

8.2 The proof of Theorem 8.1: Step 2

Note that there is an exact sequence

$$0 \longrightarrow \hat{\mathcal{K}}^{-1} KF \longrightarrow \hat{\mathcal{K}}^{-1} K^{red} F \longrightarrow \hat{\mathcal{K}}^0 K_{\mathbb{F}_q} \xrightarrow{r^*} \hat{\mathcal{K}}^0 KF \longrightarrow 0.$$

From this sequence alone we conclude that (i) $\hat{\mathcal{K}}^0 K^{red} F = 0$; (ii) $pdim_{\Lambda} \hat{\mathcal{K}}^{-1} K^{red} F = 1$; and (iii) there is a short exact sequence

$$0 \longrightarrow \hat{\mathcal{K}}^{-1} KF \longrightarrow \hat{\mathcal{K}}^{-1} K^{red} F \longrightarrow Ker r^* \longrightarrow 0.$$

Combining (ii) with the σ -acyclicity shows that $pdim_{\Lambda} \hat{\mathcal{K}}^{-1} K^{red} F = 1$ (see [5], Lemma 4.6, although this part of Theorem 8.1 also falls out as a byproduct of the computation). To describe r^* , let

$$h_k = (1 + T)^{2^k} + 1, \quad B_k = \Lambda / h_k.$$

Since $\hat{\mathcal{K}}^0 K_{\mathbb{F}_q} = \Lambda' / (\sigma(1 + T)^{2^k} - 1)$ and $\hat{\mathcal{K}}^0 KF = \Lambda / ((1 + T)^{2^k} - 1)$, we have

$$Ker r^* = (\hat{\mathcal{K}}^0 K_{\mathbb{F}_q})^- \cong B_k^b.$$

To complete the proof of Theorem 8.1, therefore, we have only to prove the following algebraic lemma.

Lemma 8.10 *Suppose W is a σ -acyclic Λ' -module fitting into an extension*

$$0 \longrightarrow V \longrightarrow W \longrightarrow B_k^b \longrightarrow 0,$$

where V is isomorphic to either $L[2, k] \oplus \Lambda'^{d-1}$ or $L[1, k] \oplus L_{2^k} \oplus \Lambda'^{d-2}$. Then

$$W \cong L[1, k] \oplus \Lambda'^{d-1}.$$

Proof: Note that V has torsion submodule $tV = A_k^b$. Let $\bar{V} = V/tV$, $\bar{W} = W/tV$. (Note that in both cases we are factoring out tV .) Thus there is a short exact sequence

$$0 \longrightarrow \bar{V} \longrightarrow \bar{W} \longrightarrow B_k^b \longrightarrow 0$$

where \bar{V} is either $L_{2^{k+1}} \oplus \Lambda'^{d-1}$ or $L_{2^k} \oplus L_{2^k} \oplus \Lambda'^{d-2}$. Our strategy is to first show that \bar{W} is Λ -free of rank $2d$. Then we can recover the isomorphism type of W from its Tate homology groups.

Lemma 8.11 *$\bar{W}/2$ is free over $\Lambda/2$ of rank $2d$. In particular,*

$$dim_{\mathbb{F}_2}(\mathbb{F}_2 \otimes_{\Lambda} (\bar{W}/2)) = 2d.$$

Proof: Since B_k^b is 2-torsion-free, we have a short exact sequence

$$0 \longrightarrow \overline{V}/2 \longrightarrow \overline{W}/2 \longrightarrow B_k^b/2 \longrightarrow 0.$$

Note that $\overline{V}/2 \cong (\Lambda/2)^{2d}$ and $B_k^b/2 \cong Q_{2^k}$. Let X denote the $(\Lambda/2)$ -torsion submodule of $\overline{W}/2$; it suffices to show that $X = 0$. Consider the composite map

$$\rho : X \longrightarrow \overline{W}/2 \longrightarrow B_k^b/2.$$

Clearly, ρ is injective; we will show that ρ is also the zero homomorphism. Since X injects into $B_k^b/2$, σ acts trivially on X as well as on $B_k^b/2$. Now for any trivial σ -module Y with exponent 2, $\hat{H}_p(\sigma; Y) = Y$ for all p . Hence it suffices to show that $\hat{H}_p(\sigma; \overline{W}/2) \longrightarrow \hat{H}_p(\sigma; B_k^b/2)$ is zero.

Since W is σ -acyclic we have

$$\hat{H}_p(\sigma; \overline{W}) \begin{cases} 0 & \text{if } p \text{ even} \\ Q_{2^k} & \text{if } p \text{ odd} \end{cases}$$

and hence $\hat{H}_p(\sigma; \overline{W}/2) = Q_{2^k}$ for all p . Now consider the exact sequence

$$\hat{H}_{p+1}(\sigma; B_k^b/2) \longrightarrow \hat{H}_p(\sigma; \overline{V}/2) \longrightarrow \hat{H}_p(\sigma; \overline{W}/2).$$

The end terms are both isomorphic to Q_{2^k} , while the middle term is either $Q_{2^{k+1}}$ or $Q_{2^k} \oplus Q_{2^k}$. Here we have used the hypothesis on V and Lemma 3.1. In either case, dimension counting shows the sequence must be short exact. In particular, the second map is onto. It follows that $\hat{H}_p(\sigma; \overline{W}/2) \longrightarrow \hat{H}_p(\sigma; B_k^b/2)$ is zero, as desired.

Lemma 8.12 \overline{W} is Λ -free of rank $2d$.

Proof: Since Λ is a noetherian local ring, and \overline{W} is a finitely-generated Λ -module, it suffices to show that $Tor_1^\Lambda(\mathbb{Z}/2, \overline{W}) = 0$. Consider the 5-term exact sequence of \mathbb{F}_2 -vector spaces

$$\begin{aligned} 0 \longrightarrow Tor_1^\Lambda(\mathbb{Z}/2, \overline{W}) \longrightarrow Tor_1^\Lambda(\mathbb{Z}/2, B_k^b) \longrightarrow \\ Tor_0^\Lambda(\mathbb{Z}/2, \overline{V}) \longrightarrow Tor_0^\Lambda(\mathbb{Z}/2, \overline{W}) \longrightarrow Tor_0^\Lambda(\mathbb{Z}/2, B_k^b) \longrightarrow 0. \end{aligned}$$

The second and fifth terms have dimension one, while the third and fourth terms have dimension $2d$ (for \overline{V} this is clear; for W use Lemma 8.11). Hence $Tor_1^\Lambda(\mathbb{Z}/2, \overline{W}) = 0$. This completes the proof of Lemma 8.12.

We can now prove Lemma 8.10, thereby completing the proof of Theorem 8.1. Since \overline{W} is Λ -free of rank $2d$ with Tate homology as shown above, Theorem 3.2 shows that $\overline{W} \cong L_{2^k} \oplus \Lambda^{d-1}$. Since W is σ -acyclic, we have $W \cong L[1, k] \oplus \Lambda^{d-1}$ by Lemma 6.4.

9 Homotopy-type of the reduced K-theory spectrum

By Theorem 8.1, $\hat{L}K^{red}F$ is an excellent spectrum of odd type. Hence the homotopy-type of $\hat{L}K^{red}F$ is completely determined by the Λ' -module $\hat{\mathcal{K}}^{-1}K^{red}F$. Let X_k denote the excellent spectrum with $\hat{\mathcal{K}}^{-1}X_F = L[1, k]$. The explicit resolution of $L[1, k]$ given in Proposition 6.5 shows that X_k fits into a fibre sequence

$$X_k \longrightarrow \Sigma^{-1}(\hat{\mathcal{K}} \vee \hat{\mathcal{K}}) \xrightarrow{(1+\sigma, -f_k)} \Sigma^{-1}\hat{\mathcal{K}}.$$

Let $k = a_F - 2$ as usual. Then from Theorem 8.1 we have at once:

Theorem 9.1 *There is a weak equivalence*

$$\hat{L}K^{red}F \cong X_k \vee (\bigvee^{\Sigma^{-1}\hat{\mathcal{K}}}).$$

There is another way to describe the spectrum X_k that fits better with the homotopy group calculation of Proposition 7.3.

Proposition 9.2 *There is a fibre sequence*

$$\Sigma \hat{L}K_{\mathbb{F}_q} \longrightarrow X_k \longrightarrow \Sigma^{-1}\hat{\mathcal{K}}$$

inducing a short exact sequence on π_* and on $\hat{\mathcal{K}}^{-1}$. Furthermore, $[\Sigma^{-1}\hat{\mathcal{K}}, \Sigma^2 \hat{L}K_{\mathbb{F}_q}]$ is a nontrivial cyclic Λ -module, and the connecting map $\epsilon : \Sigma^{-1}\hat{\mathcal{K}} \longrightarrow \Sigma^2 \hat{L}K_{\mathbb{F}_q}$ is a generator.

Proof: Recall that $\hat{\mathcal{K}}^{-1}\Sigma \hat{L}K_{\mathbb{F}_q} \cong \Lambda' / (\sigma\gamma_k + c_k)$. Define a surjective homomorphism $\alpha : L[1, k] \longrightarrow \Lambda' / (\sigma\gamma_k + c_k)$ by $\alpha(e_1) = 1$, $\alpha(e_2) = -c_k$ (see Proposition 6.5). Note this makes sense since

$$(1 + \sigma)(-c_k) = (1 + \sigma)\gamma_k = \gamma_k - c_k = f_k$$

in $\Lambda' / (\sigma\gamma_k + c_k)$. Now observe that α maps the torsion submodule A_k^b of $L[1, k]$ isomorphically onto the (-1) -eigenspace of $\Lambda' / (\sigma\gamma_k + c_k)$.

It follows that there is a short exact sequence

$$0 \longrightarrow Ker \alpha \longrightarrow L_{2k} \longrightarrow \Lambda / (\gamma_k + c_k) \longrightarrow 0.$$

Hence $Ker \alpha$ is Λ -free of rank 2 by Proposition 2.2d. On the other hand, $Ker \alpha$ is also σ -acyclic since $L[1, k]$ and $\Lambda' / (\sigma\gamma_k + c_k)$ are σ -acyclic. It follows that $Ker \alpha$ is a free Λ' -module of rank one (this can also be seen by writing down an explicit generator). The map of excellent spectra $\Sigma \hat{L}K_{\mathbb{F}_q} \longrightarrow X_k$ corresponding to α therefore has cofibre $\Sigma^{-1}\hat{\mathcal{K}}$, as desired.

It follows from Proposition 7.3 that the homotopy sequence is short exact. For the last statement of the proposition, note that $\Sigma^{-1}\hat{\mathcal{K}}$ and $\Sigma^2 \hat{L}K_{\mathbb{F}_q}$ are excellent spectra of opposite type, and hence

$$[\Sigma^{-1}\hat{\mathcal{K}}, \Sigma^2 \hat{L}K_{\mathbb{F}_q}] \cong Ext_{\Lambda'}^1(\Lambda' / (\sigma\gamma_k + c_k)) = \Lambda' / (\sigma\gamma_k + c_k).$$

The connecting map ϵ then corresponds to our extension

$$0 \longrightarrow \Lambda' \longrightarrow L[1, k] \longrightarrow \Lambda' / (\sigma\gamma_k + c_k) \longrightarrow 0.$$

It is not hard to show that this extension represents a generator of the σ -fixed submodule of $Ext_{\Lambda'}^1(\Lambda' / (\sigma\gamma_k + c_k))$, which in turn is isomorphic to $\Lambda / (\gamma_k + c_k)$. The proof is omitted since we do not use this fact.

Since $K^{red}F$ is the 0-connected cover of $\hat{L}K^{red}F$, these results determine the homotopy-type of $K^{red}F$ itself. In particular, we have:

Theorem 9.3

$$K^{red}F \cong P_1X_k \bigvee (\bigvee^{d-1} \Sigma bu^\wedge)$$

where P_1X_k fits into fibre sequences

$$\Sigma^2 bu^\wedge \longrightarrow P_1X_k \longrightarrow \Sigma bu^\wedge \bigvee \Sigma bu^\wedge$$

and

$$\Sigma K_{\mathbb{F}_q}^\wedge \longrightarrow P_1X_k \longrightarrow \Sigma bu^\wedge.$$

10 The connecting map $\hat{L}K_{\mathbb{F}_q} \longrightarrow \hat{L}\Sigma K^{red}F$

Consider the fibre sequence

$$K^{red}F \longrightarrow KF^\wedge \xrightarrow{r} K_{\mathbb{F}_q}^\wedge \xrightarrow{\delta} \Sigma K^{red}F.$$

In this section we determine the localized connecting map $\hat{L}\delta$. Since KF^\wedge is essentially a connected cover of $\hat{L}KF$, this will effectively determine the homotopy-type of KF^\wedge .

Now the prime power q and reduction map r are not canonically defined, so there is no point in calculating δ on the nose. Let us call two morphisms $X \longrightarrow Y$ in a category *equivalent* if they are isomorphic in the category of maps from X to Y . Then in the homotopy category of spectra, the homotopy-type of a homotopy fibre depends only on the equivalence class of the map in question. We will identify the equivalence class of $\hat{L}\delta$; here the answer will, of course, depend on the parity of d .

Since $\hat{L}K_{\mathbb{F}_q}$ and $\hat{L}\Sigma K^{red}F$ are excellent spectra of even type, we have

$$[\hat{L}K_{\mathbb{F}_q}, \hat{L}\Sigma K^{red}F] = Hom_{\Lambda'}(\hat{\mathcal{K}}^{-1}K^{red}F, \hat{\mathcal{K}}^0K_{\mathbb{F}_q}) = Hom_{\Lambda'}(L[1, k] \oplus (\Lambda')^{d-1}, \Lambda' / (\sigma\gamma_k - 1)).$$

Furthermore, equivalence classes of maps of spectra correspond to equivalence classes of Λ' -module homomorphisms. We are concerned only with those homomorphisms whose image is the (-1) -eigenspace $B_k^b \subset \Lambda' / (\sigma\gamma_k - 1)$. Since Λ is a local ring and B_k^b is a cyclic Λ -module, any automorphism of B_k^b comes from multiplication by a unit of Λ . In particular, any automorphism of B_k^b extends to an automorphism of $\Lambda' / (\sigma\gamma_k - 1)$. Hence it is enough to classify surjective homomorphisms $L[1, k] \oplus (\Lambda')^{d-1} \longrightarrow B_k^b$, up to equivalence. Note that

such a homomorphism necessarily annihilates the torsion submodule A_k^b , as well as the $+1$ -eigenspace Λ^d of $L[1, k]/A_k^b \oplus (\Lambda')^{d-1}$.

Now recall that $B_k^b = \Lambda/h_k$, where $h_k = (1 + T)^{2^k} + 1$. Note that h_k is an Eisenstein polynomial and hence is irreducible as polynomial; by the Weierstrass preparation theorem h_k is also irreducible in Λ . Thus B_k^b is a discrete valuation ring, with maximal ideal $T B_k^b$.

Theorem 10.1 *Let $\phi : L[1, k] \oplus (\Lambda')^{d-1} \longrightarrow B_k^b$ be a surjective homomorphism. Then ϕ is equivalent to exactly one of the following:*

$$\psi : L[1, k] \oplus (\Lambda')^{d-1} \xrightarrow{\text{proj}} L[1, k] \xrightarrow{\psi_0} B_k^b$$

where $\psi_0(e_2) = 1$;

$$\xi : L[1, k] \oplus (\Lambda')^{d-1} \xrightarrow{\text{proj}} \Lambda' \xrightarrow{\xi_0} B_k^b$$

where $\xi_0(1) = 1$;

$$\phi_i : L[1, k] \oplus (\Lambda')^{d-1} \xrightarrow{\text{proj}} L[1, k] \oplus \Lambda' \xrightarrow{\phi_{i,0}} B_k^b$$

where $\phi_{i,0}(e_2) = T^i$ and $\phi_{i,0}(1) = 1$ ($1 \in \Lambda'$; $0 < i < 2^k$).

Furthermore, ϕ is determined up to equivalence by the isomorphism class of its kernel.

Proof: Let e_1, e_2 denote the standard generators of $L[1, k]$ (see Proposition 6.5), and let y_1, \dots, y_{d-1} denote the standard basis of the free module Λ'^{d-1} . We consider three cases:

Case 1: $\phi(e_2)$ generates B_k^b . Then up to equivalence, we can assume $\phi(e_2) = 1$. Furthermore, letting $a \in \Lambda$ denote an element that reduces to $\phi(y_k)$ in B_k^b , the substitutions $y'_j = y_j - a e_2$ define an automorphism of $L[1, k] \oplus (\Lambda')^{d-1}$. We conclude that ϕ is equivalent to ψ . Note that $\text{Ker } \psi = L[2, k] \oplus \Lambda'^{d-1}$.

Case 2: $\phi(e_2) = 0$. Then $\phi(y_i)$ is a generator of B_k^b for some i , and we can assume $i = 1$ and $\phi(y_1) = 1$. An obvious change of basis in the Λ'^{d-1} summand then shows that ϕ is equivalent to ξ . Note that $\text{Ker } \xi = L[1, k] \oplus L_{2^k} \oplus \Lambda'^{d-2}$.

Case 3: $\phi(e_2)$ is a nonzero nongenerator of B_k^b . Then we can assume $\phi(e_2) = T^i$, $i > 0$. Furthermore, as in case 2 we can assume $\phi(y_1) = 1$ and $\phi(y_i) = 0$ for $i > 1$. Hence, we may as well assume $d = 2$, to simplify the notation. Consider automorphisms of $L[1, k] \oplus \Lambda'$ of the form

$$\begin{pmatrix} I & 0 \\ H & I \end{pmatrix}$$

where $H \in \text{Hom}_{\Lambda'}(L, \Lambda')$. Here $H e_1, H e_2$ can be any elements satisfying $(1 + \sigma)H e_2 = f_k H e_1$. Now if $i \geq 2^k$, say $i = 2^k + j$, then $T^i = T^{2^k} T^j = u f_k T^j \text{ mod } h_k$ for some unit u . Here we we have used the fact that

$$f_k = (1 + T)^{2^k} - c_0^{2^k} = -(c_0^{2^k} + 1) = 2u' = T^{2^k} u \text{ mod } h_k$$

for units u, u' . Setting $He_2 = -uf_k T^j$ and precomposing with the automorphism indicated above yields an equivalent homomorphism with $\phi(e_2) = 0$; hence we are back in Case 2. Thus we can assume $\phi = \phi_i$ for some i , $0 < i < 2^k$.

It remains to show that $\psi, \xi, \phi_1, \dots, \phi_{2^k-1}$ have pairwise nonisomorphic kernels. Let $\mathcal{H}(\phi) = \hat{H}_1(\sigma; (Ker \phi)/A_k^b)$. Then $\mathcal{H}(\psi) = Q_{2^{k+1}}$ and $\mathcal{H}(\xi) = Q_{2^k} \oplus Q_{2^k}$. To compute $\mathcal{H}(\phi_i)$, let $x_1 = (1 + \sigma)y_1$. Then an easy direct computation shows that $\mathcal{H}(\phi)$ has generators e_1, x_1 and relations $T^{2^k} x_1 = 0, T^{2^k} e_1 - T^i x_1 = 0$. Hence $\mathcal{H}(\phi) \cong Q_i \oplus Q_{2^{k+1}-i}$. Therefore the kernels of the homomorphisms listed above are distinguished by the invariant \mathcal{H} , completing the proof of the theorem.

Corollary 10.2 *The connecting map $\delta : \hat{L}K_{\mathbb{F}_q} \rightarrow \Sigma \hat{L}K^{red}F$ is determined up to equivalence by the formula*

$$\hat{\mathcal{K}}^0 \delta \sim \begin{cases} \psi & \text{if } d \text{ odd} \\ \xi & \text{if } d \text{ even.} \end{cases}$$

11 Homotopy-type of KF^\wedge

11.1 $K(1)$ -local homotopy-type

Recall that X_k is the excellent spectrum with $\hat{\mathcal{K}}^{-1} X_k = L[1, k]$, and let $k = a_F - 2$ as usual.

Theorem 11.1 *There is a fibre sequence*

$$X_k \bigvee^{d-1} (\bigvee \Sigma^{-1} \hat{\mathcal{K}}) \rightarrow \hat{L}KF \rightarrow \hat{L}K_{\mathbb{F}_q},$$

where the connecting map $\hat{L}K_{\mathbb{F}_q} \rightarrow \Sigma X_k \bigvee (\bigvee^{d-1} \hat{\mathcal{K}})$ is given by Corollary 10.2.

Here X_k fits into a fibre sequence $\Sigma \hat{L}K_{\mathbb{F}_q} \rightarrow X_k \rightarrow \Sigma^{-1} \hat{\mathcal{K}}$; see Proposition 9.2.

There is another description of $\hat{L}KF$ that shows to what extent the wedge summands of $\hat{L}K^{red}F$ split off from $\hat{L}KF$ itself. Recall from Theorem 10.1 the homomorphisms

$$L[1, k] \rightarrow B_k^b \subset \Lambda' / (\sigma\gamma_k - 1) = \hat{\mathcal{K}}^0 \hat{L}K_{\mathbb{F}_q}$$

and

$$\Lambda' \rightarrow B_k^b \subset \Lambda' / (\sigma\gamma_k - 1) = \hat{\mathcal{K}}^0 \hat{L}K_{\mathbb{F}_q}.$$

Let $\hat{L}K_{\mathbb{F}_q} \rightarrow \Sigma X_k$ and $\hat{L}K_{\mathbb{F}_q} \rightarrow \hat{\mathcal{K}}$ denote the respective induced maps. Let Y_k, Z_k denote the respective fibres.

Theorem 11.2 *If d is odd,*

$$\hat{L}KF \cong Y_k \bigvee^{d-1} (\bigvee \Sigma^{-1} \hat{\mathcal{K}}).$$

If d is even,

$$\hat{L}KF \cong Z_k \bigvee X_k \bigvee^{d-2} (\bigvee \Sigma^{-1} \hat{\mathcal{K}}).$$

Proof: This is clear from Theorem 11.1 and Corollary 10.2.

11.2 Actual homotopy-type

Since KF^\wedge is a semi-connected cover of $\hat{L}KF$, its homotopy-type is effectively determined by Theorem 11.1. In particular we have:

Theorem 11.3 *There is a fibre sequence*

$$P_1X_k \bigvee^{d-1} (\bigvee \Sigma bu^\wedge) \longrightarrow KF^\wedge \longrightarrow K\mathbb{F}_q^\wedge$$

where P_1X_k fits into fibre sequences

$$\Sigma^2 bu^\wedge \longrightarrow P_1X_k \longrightarrow \Sigma bu^\wedge \bigvee \Sigma bu^\wedge$$

and

$$\Sigma K\mathbb{F}_q^\wedge \longrightarrow P_1X_k \longrightarrow \Sigma bu^\wedge.$$

Theorem 11.2 yields an alternate description of the homotopy-type. The cleanest statement is obtained by passing to 0-connected covers.

Theorem 11.4 *If d is odd,*

$$P_1KF^\wedge \cong P_1Y_k \bigvee^{d-1} (\bigvee \Sigma bu^\wedge)$$

where P_1Y_k fits into a fibre sequence

$$P_1X_k \longrightarrow P_1Y_k \longrightarrow P_1K\mathbb{F}_q^\wedge.$$

If d is even,

$$P_1KF^\wedge \cong P_1Z_k \bigvee P_1X_k \bigvee^{d-2} (\bigvee \Sigma bu^\wedge)$$

where P_1Z_k fits into a fibre sequence

$$\Sigma bu^\wedge \longrightarrow P_1Z_k \longrightarrow P_1K\mathbb{F}_q^\wedge.$$

11.3 Homotopy-type of $(BGLF^+)^\wedge$

Passing to zero-th spaces in Theorem 11.1 yields:

Theorem 11.5 *There is a fibre sequence*

$$\Omega_0^\infty X_k \times (U^\wedge)^{d-1} \longrightarrow (BGLF^+)^\wedge \longrightarrow (BGL\mathbb{F}_q^+)^\wedge$$

where $\Omega_0^\infty X_k$ fits into fibre sequences

$$BU^\wedge \longrightarrow \Omega_0^\infty X_k \longrightarrow U^\wedge \times U^\wedge$$

and

$$B((BGLF^+)^\wedge) \longrightarrow \Omega_0^\infty X_k \longrightarrow U^\wedge.$$

Similarly, Theorem 11.2 yields:

Theorem 11.6 *If d is odd,*

$$(BGLF^+)^\wedge \cong \Omega_0^\infty Y_k \times (U^\wedge)^{d-1},$$

where $\Omega_0^\infty Y_k$ fits into a fibre sequence

$$\Omega_0^\infty X_k \longrightarrow \Omega_0^\infty Y_k \longrightarrow (BGL\mathbb{F}_q^+)^\wedge.$$

If d is even,

$$(BGLF^+)^\wedge \cong \Omega_0^\infty Z_k \times \Omega_0^\infty X_k \times (U^\wedge)^{d-2},$$

where $\Omega_0^\infty Z_k$ fits into a fibre sequence

$$U^\wedge \longrightarrow \Omega_0^\infty Z_k \longrightarrow (BGL\mathbb{F}_q^+)^\wedge.$$

11.4 The example $F = \mathbb{Q}_2$

We conclude with a discussion of the example $F = \mathbb{Q}_2$. In this case the homotopy type of KF^\wedge was first determined by Rognes [8]. Theorem 11.1 shows that there are fibre sequences

$$\hat{L}K^{red}\mathbb{Q}_2 \longrightarrow \hat{L}K\mathbb{Q}_2 \longrightarrow K\mathbb{F}_3^\wedge$$

$$\Sigma K\mathbb{F}_3^\wedge \longrightarrow \hat{L}K^{red}\mathbb{Q}_2 \longrightarrow \Sigma^{-1}\hat{\mathcal{K}}$$

In this case, there is another way to describe the attaching maps. Let $G \subset [K\mathbb{F}_3^\wedge, \Sigma\hat{L}K^{red}\mathbb{Q}_2]$ denote the group of maps inducing zero on π_0 , and let $H \subset [\Sigma^{-1}\hat{\mathcal{K}}, \Sigma^2 K\mathbb{F}_3^\wedge]$ denote the group of maps inducing zero on π_n for $n \geq 1$. Then one can show that $G \cong \mathbb{Z}_2 \cong H$, and that the attaching maps of the two fibre sequences above generate G and H , respectively.

Theorem 11.3 and Theorem 11.4 recover one of the main theorems of [8], yielding fibre sequences

$$K^{red}\mathbb{Q}_2^\wedge \longrightarrow K\mathbb{Q}_2^\wedge \longrightarrow K\mathbb{F}_3^\wedge$$

$$\Sigma K\mathbb{F}_3^\wedge \longrightarrow K^{red}\mathbb{Q}_2^\wedge \longrightarrow \Sigma bu^\wedge$$

and

$$\Omega^\infty K^{red}\mathbb{Q}_2^\wedge \longrightarrow (BGL\mathbb{Q}_2^+)^\wedge \longrightarrow (BGL\mathbb{F}_3^+)^\wedge$$

$$B((BGL\mathbb{Q}_2^+)^\wedge) \longrightarrow \Omega^\infty K^{red}\mathbb{Q}_2^\wedge \longrightarrow U^\wedge.$$

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