THE 3-COMPACT GROUP DI(2)

JESPER M. MØLLER

Abstract. The 3-compact group DI(2) is characterized by cohomology algebra with coefficients in the finite field $\mathbb{F}_3$, $H^*(B\,\mathrm{DI}(2);\mathbb{F}_3) = \mathbb{F}_3[x_{12}, x_{16}]$, being the rank 2 Dickson algebra. We show that DI(2) is a totally N-determined 3-compact group, compute the endomorphism monoid of DI(2), and investigate subgroups of DI(2).

1. Introduction

Group number 12 on the Shephard-Todd list [23] of finite irreducible complex reflection groups is $\mathrm{GL}(2,\mathbb{F}_3)$ or $\mathrm{GL}(V)$ where V is a 2-dimensional $\mathbb{F}_3$-vector space. By Clark-Ewing [7], $\mathrm{GL}(V)$ is even a 3-adic reflection group, containing and generated by 12 reflections of order two, and its mod 3 invariant ring is the Dickson algebra,

$$\mathrm{DI}(2) = \mathbb{F}_3[V]^{\mathrm{GL}(V)} = \mathbb{F}_3[x_{12}, x_{16}]$$

which, with V sitting in grading degree two, is polynomial on a generator of degree 12 and a generator of degree 16.

Since Zabrodsky, Smith-Switzer, and Aguadé [27, 24, 1] it has been known that DI(2) is the cohomology algebra of a space.

The question addressed here is: Does there exist a 3-compact group of rank two with Weyl group $\mathrm{GL}(V)$? The answer is presented below in the form of an existence and uniqueness theorem. Although the existence part, as just noted, is not new, the approach, based on the p-compact group technology as developed by Dwyer and Wilkerson [10, 11, 12], *is* new (but admittedly very close to the one used by Aguadé).

Here is an outline of the argument. Let $\check{T} \cong (\mathbb{Z}/3^\infty)^2$ denote a discrete 3-torus of rank two and t the maximal elementary abelian subgroup of $\check{T}$. Assuming the existence of a 3-compact group DI(2) with (discrete) maximal torus $\check{T}$ and Weyl group $\mathrm{GL}(t) = \mathrm{GL}(2,\mathbb{F}_3)$, let's now analyze its properties.

DI(2) must be connected for no nontrivial 3-group can be the homomorphic image of $\mathrm{GL}(2,\mathbb{F}_3)$, and, since $\mathrm{GL}(2,\mathbb{F}_3)$ contains no elementary abelian 3-groups of order 9, any elementary abelian 3-subgroup of DI(2) is, up to conjugacy, contained in the maximal torus. Consequently [21, 2.9], the Quillen category of DI(2) has the form

$$\mathbb{Z}/2 \circlearrowright \lambda \xrightarrow{\mathrm{GL}(2,\mathbb{F}_3)/\Sigma_3} t \circlearrowleft \mathrm{GL}(2,\mathbb{F}_3)$$

where $\lambda < t$ is any nontrivial, proper subgroup, and the morphisms are as indicated. According to Jackowski and McClure [14] $B\,\mathrm{DI}(2)$ is the homotopy colimit of a diagram of topological spaces indexed by (the opposite of) the above Quillen category. In order to figure out what the two spaces in the centralizer diagram for $B\,\mathrm{DI}(2)$ are, we first get a hold on the maximal torus normalizer, $\check{N}$, for DI(2).

The action of $\mathrm{GL}(t)$ on t extends in an essentially unique way to an action on the ambient group $\check{T}$ and the semi-direct product $\check{N} = \check{T} \rtimes \mathrm{GL}(t)$ is the only extension (2.4) realizing this action. Computing centralizers in $\check{N}$, we find that

$$C_{\check{N}}(\lambda) = \check{T} \rtimes S\Sigma_3, \quad C_{\check{N}}(t) = \check{T},$$

where $S\Sigma_3$ is the Weyl group of SU(3), and since these groups are discrete maximal torus normalizers for totally N-determined 3-compact groups [16], the corresponding centralizers in DI(2)

Date: September 2, 1997.
1991 *Mathematics Subject Classification.* 55R35, 55P15, 55P10.
Key words and phrases. Reflection group, totally N-determined p-compact group, automorphism group.

are

$$C_{\mathrm{DI}(2)}(\lambda) = \mathrm{SU}(3), \quad C_{\mathrm{DI}(2)}(t) = T$$

with automorphisms given by the induced automorphisms of the centralizers in $\check{N}$. Thus we conclude that $B\,\mathrm{DI}(2)$, if it exists, must be the homotopy colimit of a diagram of the form

$$(\mathbb{Z}/2)^{\mathrm{op}} \circlearrowright B\,\mathrm{SU}(3) \xleftarrow{(\Sigma_3)^{\mathrm{op}}\backslash\,\mathrm{GL}(2,\mathbb{F}_3)^{\mathrm{op}}} BT \circlearrowleft \mathrm{GL}(2,\mathbb{F}_3)^{\mathrm{op}}$$

with $(\mathbb{Z}/2)^{\mathrm{op}} = \mathbb{Z}/2$ acting on $B\,\mathrm{SU}(3)$ as $\{\psi^{\pm 1}\}$. This is to be understood as a diagram in the homotopy category of $\mathbb{F}_3$-complete topological spaces. After verifying the vanishing of the Dwyer-Kan obstructions to lifting to the category of spaces, we define $B\,\mathrm{DI}(2)$ as the $\mathbb{F}_3$-completion of the homotopy colimit of any realization of the above diagram. This leads to the main result of this note.

Theorem 1.1. *(3.9, 3.10, 3.12) There exists a connected, rank two, center-free, simple, totally N-determined 3-compact group* $\mathrm{DI}(2)$ *with Weyl group* $\mathrm{GL}(2,\mathbb{F}_3)$. $\mathrm{DI}(2)$ *is unique in the sense that any other rank two 3-compact group with Weyl group* $\mathrm{GL}(2,\mathbb{F}_3)$ *in* $\mathrm{GL}(2,\mathbb{Z}_3)$ *is isomorphic as a 3-compact group to* $\mathrm{DI}(2)$.

Since $\mathrm{DI}(2)$ is totally N-determined in the sense of [16, 7.1], it is easy to compute its endomorphism monoid.

Corollary 1.2. $[B\,\mathrm{DI}(2), B\,\mathrm{DI}(2)] = \mathbb{Z}_3^*/\{\pm 1\} \cup \{0\}$.

This follows from (3.11) since by Ishiguro's theorem [20, 5.6] [13, 1.3], $B\,\mathrm{DI}(2)$ is atomic in the very strong sense that any self-map is either nullhomotopic or an equivalence.

2. Group number 12

The abstract group $\mathrm{GL}(2,\mathbb{F}_3)$ of order $48 = 2^4 \cdot 3$ is number 12 on the Shephard-Todd list [23] of finite irreducible complex reflection groups. We shall here investigate this group in its Clark-Ewing [7] incarnation as a rank two 3-adic reflection group.

Let $\sigma, \tau \in \mathrm{GL}(2,\mathbb{Z}_3)$, where $\mathbb{Z}_3$ denotes the ring of 3-adic integers, be the matrices

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

satisfying the relations $\tau^2 = E = \sigma^3, \tau\sigma\tau^{-1} = \sigma^2$. The two subgroups $\langle\sigma,\tau\rangle$ and $\langle\sigma,-\tau\rangle$ of $\mathrm{GL}(2,\mathbb{Z}_3)$ are both abstractly isomorphic to the permutation group Σ_3 but they are not conjugate as subgroups of $\mathrm{GL}(2,\mathbb{Z}_3)$. To see this, let L denote the canonical $\mathrm{GL}(2,\mathbb{Z}_3)$-module, i.e. a free $\mathbb{Z}_3$-module on two generators, and $\check{T}$ the $\mathrm{GL}(2,\mathbb{Z}_3)$-module $(L\otimes\mathbb{Q})/L$. Then the fixed point group $\check{T}^{\langle\sigma,\tau\rangle}$ is cyclic of order 3 while $\check{T}^{\langle\sigma,-\tau\rangle}$ is the trivial group.

Let $r_3\colon \mathrm{GL}(2,\mathbb{Z}_3) \to \mathrm{GL}(2,\mathbb{F}_3)$ denote reduction modulo 3. The images $S\Sigma_3 = r_3(\langle\sigma,\tau\rangle)$ and $P\Sigma_3 = r_3(\langle\sigma,-\tau\rangle)$ are both [9] [25, 10.7.1] abstractly isomorphic to Σ_3 but again they can be seen to be non-conjugate subgroups of $\mathrm{GL}(2,\mathbb{F}_3)$. We let also σ and τ denote the elements $r_3(\sigma)$ and $r_3(\tau)$ of $\mathrm{GL}(2,\mathbb{F}_3)$.

The subgroup $D = \langle S\Sigma_3, P\Sigma_3\rangle = \langle S\Sigma_3, -E\rangle$ of $\mathrm{GL}(2,\mathbb{F}_3)$ generated by $S\Sigma_3$ and $P\Sigma_3$, or, alternatively, by $S\Sigma_3$ and $-E$, is dihedral of order 12.

Define $\{G > H\}$ to be the set of conjugacy classes of subgroups of the group G abstractly isomorphic to the group H.

Lemma 2.1. *Conjugacy classes of subgroups of* $\mathrm{GL}(2,\mathbb{Z}_3)$ *isomorphic to* Σ_3, D, *or* $\mathrm{GL}(2,\mathbb{F}_3)$ *are given by:*

1. *The map* $r_3\colon \{\mathrm{GL}(2,\mathbb{Z}_3) > H\} \to \{\mathrm{GL}(2,\mathbb{F}_3) > H\}$ *is a bijection for* $H = \Sigma_3, D$, *and* $\mathrm{GL}(2,\mathbb{F}_3)$.
2. $\{\mathrm{GL}(2,\mathbb{F}_3) > \Sigma_3\} = \{S\Sigma_3, P\Sigma_3\}$ *and* $\{\mathrm{GL}(2,\mathbb{F}_3) > D\} = \{D\}$.

Proof. The last part of the lemma is not hard to verify. To transfer these observations to $\mathrm{GL}(2,\mathbb{Z}_3)$, we compute some obstruction groups. Let $V = \mathbb{F}_3^2$ denote the canonical $\mathrm{GL}(2,\mathbb{F}_3)$-module. Then $\overline{H}^*(\mathrm{GL}(2,\mathbb{F}_3); V\otimes V^*) = 0 = \overline{H}^*(D; V\otimes V^*)$ for both groups contain $\mathbb{Z}/2$ as a central subgroup. Direct computations show that $H^i(S\Sigma_3; V\otimes V^*) = 0 = H^i(P\Sigma_3; V\otimes V^*)$ for $i = 1,2$. Therefore the obstructions to lifting to $\mathrm{GL}(2,\mathbb{Z}_3)$ and to uniqueness of such lifts vanish [3, 7.1.2].(See [2] for more information.) □

Let now W denote any subgroup of $\mathrm{GL}(2,\mathbb{Z}_3)$ isomorphic to $\mathrm{GL}(2,\mathbb{F}_3)$, for instance the explicit subgroup of $\mathrm{GL}(2,\mathbb{Z}_3)$ from [25, Example 1 p. 201]. The center of W is the subgroup $Z(W) = \{\pm E\}$. The restriction of r_3 to W is [25, 10.7.1] an isomorphism $r_3|W\colon W \to \mathrm{GL}(2,\mathbb{F}_3)$ and if we (in spite of ambiguous notation) put $S\Sigma_3 = (r_3|W)^{-1}(S\Sigma_3)$, $P\Sigma_3 = (r_3|W)^{-1}(P\Sigma_3)$, and also $D = (r_3|W)^{-1}(D)$, then $r_3|W$ is an isomorphism

$$(W; D, S\Sigma_3, P\Sigma_3, Z(W)) \to (\mathrm{GL}(2,\mathbb{F}_3); D, S\Sigma_3, P\Sigma_3, \{\pm E\}) \tag{2.2}$$

of groups with subgroups.

The group W acts on L, $\check{T}$, and on the maximal elementary abelian 3-subgroup, t, of $\check{T}$.

Lemma 2.3. *The cohomology groups $H^*(G;L)$, $H^*(G;L\otimes\mathbb{Q})$, $H^*(G;\check{T})$, and $H^*(W;t)$ are trivial for $G = W$ or $G = D$.*

Proof. Since the functor $H^*(Z(W);-)$ vanishes on all three modules, so does $H^*(G;-)$ by the Lyndon-Hochschild-Serre spectral sequence. □

Corollary 2.4. *Let $\check{N}$ be any extension of G by $\check{T}$ realizing $\check{T}$ as a G-module. Then $\check{N}$ is isomorphic to the semi-direct product $\check{T}\rtimes G$, $G = W, D$.*

Proof. $H^2(G;\check{T}) = 0$ by (2.3). □

Since $\check{T}$ is a characteristic subgroup of $\check{N} = \check{T}\rtimes W$, there is an exact sequence

$$1 \to \mathrm{Aut}_W(\check{T}) \to \mathrm{Aut}(\check{N}) \to \mathrm{Aut}(W),$$

where $\mathrm{Aut}_W(\check{T})$ denotes the group of W-equivariant automorphisms of $\check{T}$, and an exact sequence

$$1 \to \frac{\mathrm{Aut}_W(\check{T})}{Z(W)} \to \frac{\mathrm{Aut}(\check{N})}{W} \to \mathrm{Out}(W)$$

obtained by quotienting out the normal subgroup $W \lhd \mathrm{Aut}(\check{N})$.

Lemma 2.5. [2] *In the above exact sequence* $\mathrm{Out}(W)$ *is cyclic of order two, generated by χ, and* $\mathrm{Aut}(\check{N})/W \cong \mathrm{Aut}_W(\check{T})/Z(W)$ *is isomorphic to $\mathbb{Z}_3^*/\{\pm 1\}$.*

Above, $\chi \in \mathrm{Aut}(\mathrm{GL}(2,\mathbb{F}_3))$ denotes the automorphism given by $\chi(C) = (\det C)C$ for all $C \in \mathrm{GL}(2,\mathbb{F}_3)$. Note that χ has order two and that χ, swapping the two subgroups $S\Sigma_3$ and $P\Sigma_3$ of $\mathrm{GL}(2,\mathbb{F}_3)$, is not inner.

The Weyl group $W_T(X)$ of a connected p-compact group X with maximal torus $T \to X$ may naturally be considered as a subgroup (conjugacy class) of $\mathrm{Aut}(\pi_1(T))$, $\mathrm{Aut}(\check{T})$, or, if p is odd, $\mathrm{Aut}(t)$, where $\check{T}$ is a discrete maximal torus and t the maximal elementary abelian subgroup of $\check{T}$.

It is often convenient to speak of the Weyl group as a subgroup (conjugacy class) of $\mathrm{Aut}(L) = \mathrm{GL}(r,\mathbb{Z}_p)$, $\mathrm{Aut}((\mathbb{Z}/p^\infty)^r)$, or (if p is odd) $\mathrm{Aut}(V) = \mathrm{GL}(r,\mathbb{F}_p)$, where L is a free $\mathbb{Z}_p$-module of rank r and V elementary abelian of order p^r. The following definition is intended to make this usage, which already appeared in (1.1), precise and to stress the point that Weyl groups should be viewed as subgroups of general linear groups.

Definition 2.6. *The connected p-compact group X has Weyl group W in* $\mathrm{GL}(r,\mathbb{Z}_p)$ *if $\theta W\theta^{-1} = W_T(X)$ for some maximal torus $T \to X$ and some isomorphism $\theta\colon L \to \pi_1(T)$.*

This definition apply equally well to subgroups of $\mathrm{Aut}((\mathbb{Z}/p^\infty)^r)$ or (if p is odd) $\mathrm{Aut}(V) = \mathrm{GL}(r,\mathbb{F}_p)$. For instance, the subgroup $S\Sigma_3$ ($P\Sigma_3$) of $\mathrm{GL}(2,\mathbb{F}_3)$ is the Weyl group of the 3-compact group SU(3) (PU(3)). Thus SU(3) and PU(3) do not have the same Weyl group in $\mathrm{GL}(2,\mathbb{F}_3)$. Conversely, any connected 3-compact group with Weyl group $S\Sigma_3$ in $\mathrm{GL}(2,\mathbb{F}_3)$ is isomorphic to SU(3), cf. [6] [21, 4.4, 4.7]. If p is odd and W a reflection group in $\mathrm{GL}(r,\mathbb{Z}_p)$ of order prime to p then there is [9] a unique connected p-compact group with W as its Weyl group in $\mathrm{GL}(r,\mathbb{Z}_p)$.

3. Construction of DI(2)

We shall construct the 3-compact group DI(2) as the homotopy colimit over an index category with just two objects.

For any group G with subgroup H let $\mathbb{I}(G,H)$ denote the category with two objects, 0 and 1, and morphism sets

- $\mathbb{I}(G,H)(0,0) = N_G(H)/H$
- $\mathbb{I}(G,H)(1,1) = G$
- $\mathbb{I}(G,H)(0,1) = G/H$
- $\mathbb{I}(G,H)(1,0) = \emptyset$

and composition of morphisms given by group multiplication in $N_G(H)$ and G and by the right-left actions

$$G \times G/H \times N_G(H)/H \to G/H$$

also determined by the group multiplication. Here is a pictorial presentation

$$N_G(H)/H \circlearrowright 0 \xrightarrow{G/H} 1 \circlearrowleft G$$

of the index category $\mathbb{I}(G,H)$.

A functor $C\colon \mathbb{I}(G,H) \to \mathcal{C}$ ($C\colon \mathbb{I}(G,H)^{\mathrm{op}} \to \mathcal{C}$) into some category $\mathcal{C}$ consists of

- an object $C(0)$ equipped with an action of $N_G(H)/H$ ($(N_G(H)/H)^{\mathrm{op}}$)
- an object $C(1)$ equipped with an action of G (G^{op})
- an $N_G(H)$-equivariant ($N_G(H)^{\mathrm{op}}$-equivariant) morphism $C(0) \to C(1)$ ($C(0) \leftarrow C(1)$).

There is for instance a functor $\mathbb{I}(G,H)^{\mathrm{op}} \to \mathcal{O}(G)$ into the orbit category that takes the object 0 to G/H and the object 1 to $G/\{e\}$. This functor identifies $\mathbb{I}(G,H)^{\mathrm{op}}$ with a full subcategory of $\mathcal{O}(G)$.

Example 3.1. Let M be an $R[\mathbb{I}(G,H)]$-module (a functor from $\mathbb{I}(G,H)$ into the category of R-modules) where R is a field or the ring $\mathbb{Z}_p$ of p-adic integers. Assuming that

- $p \nmid |G\colon H|$
- $S_p = \mathrm{Syl}_p(H)$ is cyclic of order p
- $N_{N_G(H)}(S_p) = N_G(S_p)$

we have [21, 3.8] an exact sequence

$$0 \to \lim^0_{\mathbb{I}(G,H)} M \to M(0)^{N_G(H)/H} \to M(1)^{N_G(H)}/M(1)^G \to \lim^1_{\mathbb{I}(G,H)} M \to 0$$

while $\lim^j_{\mathbb{I}(G,H)} = 0$ for $j > 1$.

Also fixed point functors are naturally functors defined on $\mathbb{I}(G,H)$.

Example 3.2. Let M be an $R[G]$-module. Then M^H is an $N_G(H)/H$-module and the inclusion $M^H \to M$ is $N_G(H)$-equivariant. Let $F(M)$ be the $\mathbb{I}(G,H)$-module defined by these data. With the same assumptions on G, H, and R as in (3.1) we have

$$\lim^j_{\mathbb{I}(G,H)} F(M) = \begin{cases} M^G & j = 0 \\ 0 & j > 0 \end{cases}$$

for the higher limits over $\mathbb{I}(G,H)$ of the functor $F(M)$.

We specialize now to the pair $(G,H) = (W, S\Sigma_3)$ from (2.2), which by [1] satisfies the conditions of (3.1), and consider the $\mathbb{I}(W,S\Sigma_3)^{\mathrm{op}}$-diagram of groups and conjugacy classes of group homomorphisms schematically given by

$$(N_W(S\Sigma_3)/S\Sigma_3)^{\mathrm{op}} \circlearrowright \check{T} \rtimes S\Sigma_3 \xleftarrow{(S\Sigma_3)^{\mathrm{op}} \backslash W^{\mathrm{op}}} \check{T} \circlearrowleft W^{\mathrm{op}} \tag{3.3}$$

where the morphism from left to right is the inclusion, W^{op} acts on $\check{T}$ through the inversion map $w \to w^{-1}$, and $w \in N_W(S\Sigma_3)^{\mathrm{op}}$ takes (t,g) in the semi-direct product to $(w^{-1}t, w^{-1}gw)$. This diagram is centric in the sense that the inclusion induces group homomorphisms of centralizers

$$\check{T}^{S\Sigma_3} = C_{\check{T} \rtimes S\Sigma_3}(\check{T} \rtimes S\Sigma_3) \to C_{\check{T} \rtimes S\Sigma_3}(\check{T}) \xleftarrow{\cong} C_{\check{T}}(\check{T}) = \check{T}$$

with the right morphism an isomorphism. Hence there is an induced $\mathbb{I}(W, S\Sigma_3)$-module

$$N_W(S\Sigma_3)/S\Sigma_3 \circlearrowright \check{T}^{S\Sigma_3} \xrightarrow{W/S\Sigma_3} \check{T} \circlearrowleft W \tag{3.4}$$

of abelian centers. Here, the quotient group $N_W(S\Sigma_3)/S\Sigma_3 = (S\Sigma_3 \times Z(W))/S\Sigma_3 \cong Z(W)$ is cyclic of order two and it acts non-trivially on $\check{T}^{S\Sigma_3} \cong \mathbb{Z}/3$.

Now regard the $\mathbb{I}(W, S\Sigma_3)^{\mathrm{op}}$-diagram (3.3) as a diagram of maximal torus normalizers and consider the corresponding $\mathbb{I}(W, S\Sigma_3)^{\mathrm{op}}$-diagram, BC,

$$(N_W(S\Sigma_3)/S\Sigma_3)^{\mathrm{op}} \circlearrowright B\,\mathrm{SU}(3) \xleftarrow{(S\Sigma_3)^{\mathrm{op}}\backslash W^{\mathrm{op}}} BT \circlearrowleft W^{\mathrm{op}} \tag{3.5}$$

of 3-compact groups and conjugacy classes of morphisms. (Here, $B\,\mathrm{SU}(3)$ is short for $(B\,\mathrm{SU}(3))^\wedge_3$.) This is a central diagram in the sense of Dwyer–Kan [8] and so there is an induced diagram, BZC, of centers

$$(N_W(S\Sigma_3)/S\Sigma_3) \circlearrowright B(\check{T}^{S\Sigma_3}) \xrightarrow{W/S\Sigma_3} BT \circlearrowleft W \tag{3.6}$$

which may also be viewed as the result of applying the functor $B(-)^\wedge_3$ to the $\mathbb{I}(W, S\Sigma_3)$-module ZC of diagram (3.4).

The higher limits of the two $\mathbb{I}(W, S\Sigma_3)$-modules $\pi_1(BZC)$ and $\pi_2(BZC)$ were computed in (3.1).

Lemma 3.7. $\lim^*_{\mathbb{I}(W,S\Sigma_3)} \pi_1(BZC) = 0 = \lim^*_{\mathbb{I}(W,S\Sigma_3)} \pi_2(BZC)$.

The Dwyer–Kan obstruction groups being trivial we know that there exists an $\mathbb{I}(W, S\Sigma_3)^{\mathrm{op}}$-diagram in the category of topological spaces inducing diagram (3.5) in the homotopy category. Define $B\,\mathrm{DI}(2)$ to be the $\mathbb{F}_3$-completion of the homotopy colimit of any such diagram. By construction, $B\,\mathrm{DI}(2)$ comes equipped with a map $Bi\colon BT \to B\,\mathrm{DI}(2)$ which is equivariant up to homotopy with respect to the given action of W on BT and the trivial action on $B\,\mathrm{DI}(2)$. On cohomology, $(Bi)^*$ takes $H^*(B\,\mathrm{DI}(2))$ into the invariant ring $H^*(BT)^W$.

Proposition 3.8. *The homomorphism*

$$(Bi)^*\colon H^*(B\,\mathrm{DI}(2); R) \to H^*(BT; R)^W$$

is an isomorphism for $R = \mathbb{F}_3, \mathbb{Z}_3, \mathbb{Q}_3$.

Proof. This follows from the Bousfield-Kan spectral sequence [5, XII.4.5] and (3.2). □

The convention in force above is that $H^*(-;\mathbb{Q}_3)$, where $\mathbb{Q}_3$ is the field of 3-adic numbers, stands for $H^*(-;\mathbb{Z}_3) \otimes \mathbb{Q}$.

The algebra $H^*(BT;\mathbb{F}_3)^W = H^*(BT;\mathbb{F}_3)^{\mathrm{GL}(2,3)}$ is the Dickson algebra $DI(2)$: A polynomial algebra on a generator x_{12} in degree 12 and a generator $x_{16} = P^1 x_{12}$ in degree 16.

Corollary 3.9. $B\,\mathrm{DI}(2)$ *is a connected, center-free, simple, 3-compact group with maximal torus* $i\colon T \to \mathrm{DI}(2)$*, Weyl group* W *in* $\mathrm{GL}(2, \mathbb{Z}_3)$*, and discrete maximal torus normalizer* $\check{N} = \check{T} \rtimes W$.

Proof. By definition and by (3.8) $B\,\mathrm{DI}(2)$ is an 11-connected $\mathbb{F}_3$-complete space with loop space cohomology $H^*(\Omega B\,\mathrm{DI}(2);\mathbb{F}_3)$ exterior on two generators, in particular, finite. This shows [10] that $\mathrm{DI}(2)$ is a 3-compact group.

The 3-compact group morphism $i\colon T \to \mathrm{DI}(2)$ is a monomorphism for $H^*(BT;\mathbb{F}_3)$ is a finitely generated [3, 1.3.1] [25, 2.3.1] $H^*(B\,\mathrm{DI}(2);\mathbb{F}_3)$-module via $H^*(Bi;\mathbb{F}_3)$. By construction, W is contained in the Weyl group of $\mathrm{DI}(2)$, and since we know from the formula [10, 9.7]

$$\mathbb{Q}_3[x_{12}, x_{16}] = H^*(B\,\mathrm{DI}(2);\mathbb{Q}_3) = H^*(BT;\mathbb{Q}_3)^{W_T(\mathrm{DI}(2))}$$

that [25, 5.5.4] the order of the Weyl group of $\mathrm{DI}(2)$ is $\frac{12}{2} \cdot \frac{16}{2} = 48$, which also is the order of W, the Weyl group of $\mathrm{DI}(2)$ is precisely W.

The (discrete approximation to) the normalizer of the maximal torus of $\mathrm{DI}(2)$ is by (2.4) the semi-direct product $\check{N} = \check{T} \rtimes W$.

The center of $\mathrm{DI}(2)$ is isomorphic [11, 7.1] [16, 3.4] to the center of $\check{N}$ which is trivial. Since also $\pi_1(T) \otimes \mathbb{Q}$ is a simple W-module, $\mathrm{DI}(2)$ is a simple 3-compact group in the sense of [11, 1.2]. □

We now address uniqueness questions, in particular whether $\mathrm{DI}(2)$ is determined by its maximal torus normalizer in the sense of [16, 7.1].

Theorem 3.10. *The 3-compact group* DI(2) *is totally N-determined.*

Proof. Let $\mathbb{A}(\mathrm{DI}(2))$ stand for the Quillen category of conjugacy classes of monomorphisms of elementary abelian 3-groups to DI(2). Note that there is an obvious functor $\mathbb{I}(W, S\Sigma_3) \to \mathbb{A}(\mathrm{DI}(2))$ that takes 0 to the rank one monomorphism $\check{T}^{S\Sigma_3} \to \check{T} \to \mathrm{DI}(2)$ and 1 to the rank two monomorphism $t \to \check{T} \to \mathrm{DI}(2)$. Since the cohomology of $B\,\mathrm{DI}(2)$ embeds into the cohomology of BT, this is an equivalence of categories [21, 2.9]. The centralizer $C_{\mathrm{DI}(2)}(\check{T}^{S\Sigma_3})$ is isomorphic to the N-determined 3-compact group SU(3) for its discrete maximal torus normalizer is $C_{\check{N}}(\check{T}^{S\Sigma_3}) = \check{T} \rtimes S\Sigma_3$. Similarly, the centralizer $C_{\mathrm{DI}(2)}(t)$ is isomorphic to the N-determined 3-compact group T for its discrete maximal torus normalizer is $C_{\check{N}}(t) = \check{T}$. Thus the $\mathbb{A}(\mathrm{DI}(2))$-module $\pi_i(BZC_{\mathrm{DI}(2)})$ is equivalent to the $\mathbb{I}(W, S\Sigma_3)$-module $\pi_i(BZC)$, $i = 1, 2$. Now the vanishing lemma (3.7) together with general results about N-determinism [16, 4.9, 7.15, 7.17] imply that DI(2) is totally N-determined. □

Corollary 3.11. *The group* Aut(DI(2)) *of homotopy classes of self-homotopy equivalences of* $B\,\mathrm{DI}(2)$ *is isomorphic to* $\mathbb{Z}_3^*/\{\pm 1\}$.

Proof. Since DI(2) is totally N-determined, Aut(DI(2)) is isomorphic to $\mathrm{Aut}(\check{N})/W$ which was computed in (2.5). □

DI(2) is the only 3-compact group with $\mathrm{GL}(2, \mathbb{F}_3)$ as Weyl group (in the sense of (2.6)).

Corollary 3.12. *Let X be a connected 3-compact group of rank two and suppose that the Weyl group of X is either W in* $\mathrm{GL}(2, \mathbb{Z}_3)$ *or* $\mathrm{GL}(2, \mathbb{F}_3)$ *in* $\mathrm{GL}(2, \mathbb{F}_3)$*. Then X and* DI(2) *are isomorphic as 3-compact groups.*

Proof. In either case, since there is essentially just one copy of GL(2, 3) in $\mathrm{GL}(2, \mathbb{Z}_3)$, the subgroup $W < \mathrm{GL}(2, \mathbb{Z}_3)$ is the Weyl group of X. Since $H^2(W; \check{T}) = 0$, the maximal torus normalizer of X can only be the semi-direct product $\check{N}$. But this is also the maximal torus normalizer of the N-determined 3-compact group DI(2), so $X \cong \mathrm{DI}(2)$. □

Similarly, $B\,\mathrm{DI}(2)$ is the only $\mathbb{F}_3$-complete space realizing the Dickson algebra as its cohomology algebra.

Corollary 3.13. *Let X be a 3-compact group such that $H^*(BX; \mathbb{F}_3)$ and $H^*(B\,\mathrm{DI}(2); \mathbb{F}_3)$ are isomorphic unstable algebras. Then X and* DI(2) *are isomorphic as 3-compact groups.*

Proof. Note that X is connected and that X and DI(2) have isomorphic Quillen categories by Lannes theory [15, 0.4]. In particular, X is connected of rank two with Weyl group GL(2, 3) in GL(2, 3). Hence X and DI(2) are isomorphic by (3.12). □

4. G_2 as a 3-compact group

We present a construction of BG_2 at the prime $p = 3$ very similar to the construction of DI(2) of the previous section.

Recall that D, which is dihedral of order 12, is the subgroup $D = \langle S\Sigma_3, Z(W)\rangle \cong S\Sigma_3 \times Z(W)$ of W generated by $S\Sigma_3$ and the center. $S\Sigma_3$ is normal in D with quotient $D/S\Sigma_3$ isomorphic to $Z(W)$ which also is the center of D. There is a contravariant functor from the index category $\mathbb{I}(D, S\Sigma_3)$

$$Z(W) \circlearrowright 0 \xrightarrow{D/S\Sigma_3} 1 \circlearrowleft D$$

to the homotopy category of topological spaces given by the diagram, BC,

$$Z(W)^{\mathrm{op}} \circlearrowright B\,\mathrm{SU}(3) \xleftarrow{(S\Sigma_3)^{\mathrm{op}}\backslash D^{\mathrm{op}}} BT \circlearrowleft D^{\mathrm{op}} \tag{4.1}$$

where the group $Z^{\mathrm{op}} = \mathbb{Z}$ acts on $B\,\mathrm{SU}(3)$ as $\{\psi^{\pm 1}\}$ and the two morphisms from right to left are the elements of the orbit $Bi \cdot D^{\mathrm{op}} \subset [BT, B\,\mathrm{SU}(3)]$ for the action of D^{op} on the maximal torus $Bi\colon BT \to B\,\mathrm{SU}(3)$. Since (4.1) is a centric diagram [8] there are associated $\mathbb{I}(D, S\Sigma_3)$-modules

$$Z(W) \circlearrowright Z(\mathrm{SU}(3)) \xrightarrow{D/S\Sigma_3} 0 \circlearrowleft \qquad \circlearrowright 0 \xrightarrow{D/S\Sigma_3} \pi_2(BT) \circlearrowleft D$$

denoted $\pi_1(BZC)$ and $\pi_2(BZC)$, respectively. Since

$$\lim^*_{\mathbb{I}(D,S\Sigma_3)} \pi_1(BZC) = 0 \tag{4.2}$$

because $Z \cong \mathbb{Z}/2$ acts non-trivially on the center $Z(\mathrm{SU}(3)) \cong \mathbb{Z}/3$, and

$$\lim^*_{\mathbb{I}(D,S\Sigma_3)} \pi_2(BZC) = 0 \tag{4.3}$$

by (3.1), we know from [8, 1.1] that there exists an essentially unique realization of (4.1).

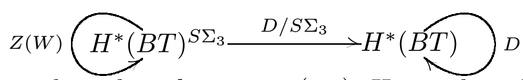
Define BG_2 to be the homotopy colimit of any realization of (4.1). By construction, there is map $Bi\colon BT \to BG_2$ which is homotopy invariant under the action of D^{op} on BT.

Proposition 4.4. *The homomorphism The homomorphism*

$$(Bi)^*\colon H^*(BG_2;R) \to H^*(BT;R)^D$$

is an isomorphism for $R = \mathbb{F}_3, \mathbb{Z}_3, \mathbb{Q}_3$.

Proof. The $\mathbb{I}(D,S\Sigma_3)$-module



is the result of applying the cohomology functor to (4.1). Hence the cohomology of the homotopy colimit is

$$H^*(BG_2) = \lim^0_{\mathbb{I}(D,S\Sigma_3)} F(H^*(BT)) = H^*(BT)^D$$

by the Bousfield-Kan spectral sequence [5, XII.4.5] and (3.2). □

The invariant ring

$$H^*(BT;\mathbb{F}_3)^D = \mathbb{F}_3[x_4, x_{12}]$$

is [25, 7.4 Example 3] polynomial on a generator x_4 in degree 4 and a generator x_{12} in degree 12.

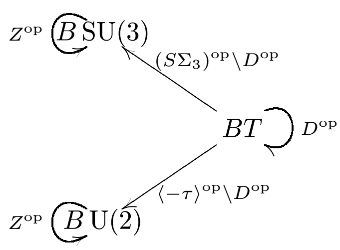
Corollary 4.5. *BG_2 is a connected, center-free, simple, 3-compact group with maximal torus $i\colon T \to G_2$, Weyl group D in $\mathrm{GL}(2,\mathbb{Z}_3)$, and discrete maximal torus normalizer $\check{N} = \check{T} \rtimes D$.*

Proof. The proof is similar to that of (3.9) since also $H^2(D;\check{T}) = 0$ and $Z(\check{T} \rtimes D) = \check{T}^D = 0$. □

Theorem 4.6. *The 3-compact group G_2 is totally N-determined.*

Proof. Noting that the p-compact groups in the centralizer diagram (4.1) for BG_2 are totally N-determined [16, 21], the theorem follows from the vanishing results (4.2, 4.3) and the general N-determinism theory [16, 4.9, 7.15, 7.17]. (Note that in [16, 4.9, 7.17] it is harmless to replace $\mathbb{A}(X)$ by any subcategory $\mathbb{A}$ for which $\operatorname{hocolim}_{\mathbb{A}^{\mathrm{op}}} BC_X(\nu)$ is $H^*\mathbb{F}_p$-equivalent to BX.) □

Remark 4.7. In diagram (4.1) we are using a full subcategory of the Quillen category as index category. The centralizer decomposition based on the complete Quillen category [21, 2.5] $\mathbb{A}(G_2) = \mathbb{I}(D,S\Sigma_3) \cup \mathbb{I}(D,\langle -\tau\rangle)$ is a diagram of the form



where $Z^{\mathrm{op}} = Z \cong \mathbb{Z}/2$ acts on $B\,\mathrm{SU}(3)$ and $B\,\mathrm{U}(2)$ as $\{\psi^{\pm 1}\}$.

Corollary 4.8. *The group $\mathrm{Out}(G_2)$ of homotopy classes of self-homotopy equivalences of BG_2 is isomorphic to $\mathbb{Z}_3^*/\{\pm 1\}$.*

Proof. Since G_2 is totally N-determined, $\mathrm{Out}(G_2)$ is [16, 7.2] isomorphic to the group $\mathrm{Aut}(\check{T} \rtimes D)/D$. □

Corollary 4.9. *Let Y be a connected 3-compact group with Weyl group D in $\mathrm{GL}(2,\mathbb{Z}_3)$ or $r_3(D)$ in $\mathrm{GL}(2,\mathbb{F}_3)$. Then Y and G_2 are isomorphic as 3-compact groups.*

Proof. Since $H^2(D;\check{T}) = 0$ the maximal torus normalizer of Y must be isomorphic to the semi-direct product $\check{T} \rtimes D$ and then Y must be isomorphic to G_2 because G_2 is totally N-determined. □

Corollary 4.10. [22] *Let Y be a 3-compact group such that $H^*(BX;\mathbb{F}_3)$ and $H^*(BG_2;\mathbb{F}_3)$ are isomorphic as unstable algebras. Then Y and G_2 are isomorphic as 3-compact groups.*

Proof. Since $H^1(BX;\mathbb{F}_3) = 0$, X is connected, and since X and G_2 have isomorphic cohomology algebras, X and G_2 have isomorphic Quillen categories [15]. In particular [21, 2.9], X has Weyl group $r_3(D)$ in $\mathrm{GL}(2,\mathbb{F}_3)$. □

5. Subgroups of DI(2)

This section contains some general theory for monomorphisms of between p-compact groups and it is shown that DI(2) contains essentially unique copies of each of the 3-compact groups $\mathrm{SU}(2) \times \mathrm{SU}(2)$, $\mathrm{U}(2)$, $\mathrm{Spin}(5)$, $\mathrm{SU}(3)$, $\mathrm{PU}(3)$, and G_2.

Let X_1 and X_2 be two connected p-compact groups of the same rank. Let $j_1 \colon N_1 \to X_1$ and $j_2 \colon N_2 \to X_2$ be normalizers of maximal tori $i_1 \colon T_1 \to X_1$ and $i_2 \colon T_2 \to X_2$.

Consider the homomorphism [17, 3.11]

$$N \colon \mathrm{Mono}(X_1, X_2) \to \mathrm{Mono}(N_1, N_2) \tag{5.1}$$

that to any conjugacy class of a monomorphism $f \colon X_1 \to X_2$ associates the unique conjugacy class $N(f) \colon N_1 \to N_2$ such that

$$\begin{array}{ccc} N_1 & \xrightarrow{N(f)} & N_2 \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle j_2} \\ X_1 & \xrightarrow[f]{} & X_2 \end{array}$$

commutes up to conjugacy. Here, $\mathrm{Mono}(X_1, X_2) \subset [BX_1, BX_2]$ denotes the set of conjugacy classes of monomorphisms of X_1 into X_2 and $\mathrm{Mono}(N_1, N_2)$ denotes the set of conjugacy classes of maps $BN_1 \to BN_2$ inducing monomorphisms on π_1 and isomorphisms on π_2. Note that if $\check{N}_1 \to N_1$ and $\check{N}_2 \to N_2$ are discrete approximations then

$$[B\check{N}_1, B\check{N}_2] = [BN_1, BN_2]$$

so that

$$\mathrm{Mono}(N_1, N_2) = \mathrm{Mono}(\check{N}_1, \check{N}_2)/\check{N}_2$$

consists of conjugacy classes of monomorphisms of $\check{N}_1$ into $\check{N}_2$. For any monomorphism $f \in \mathrm{Mono}(X_1, X_2)$, we let $\check{N}(f) \in \mathrm{Mono}(\check{N}_1, \check{N}_2)$, determined up to conjugacy, denote any discrete approximation to $N(f)$.

Definition 5.2. *The monomorphism $f \in \mathrm{Mono}(X_1, X_2)$ is N-determined if $N^{-1}(N(f)) \subset \mathrm{Mono}(X_1, X_2)$ consists of f alone.*

Let $W_1 = \pi_0(N_1)$ and $W_2 = \pi_0(N_2)$ denote the Weyl groups.

Example 5.3. If $p \nmid |W_1|$, then all monomorphisms are N-determined. Indeed, it is not difficult to see that (5.1) is bijective in this case.

In case $X_1 = X_2$, the map (5.1) is the homomorphism $N \colon \mathrm{Out}(X_1) \to \mathrm{Out}(N_1)$ previously encountered. We say that X_1 has N-determined monomorphisms if this map is injective; if X_1 is totally N-determined N is a bijection. Note that (5.1) is equivariant in the sense that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Mono}(X_1, X_2) \times \mathrm{Out}(X_1) & \longrightarrow & \mathrm{Mono}(X_1, X_2) \\ {\scriptstyle N\times N}\downarrow & & \downarrow{\scriptstyle N} \\ \mathrm{Mono}(N_1, N_2) \times \mathrm{Out}(N_1) & \longrightarrow & \mathrm{Mono}(N_1, N_2) \end{array}$$

relating group actions on sets of monomorphisms.

Proposition 5.4. *Let $i \colon X_1 \to X_2$ be a monomorphism between the two p-compact groups X_1 and X_2 of the same rank. Then the Euler characteristic $\chi(X_2/iX_1) = |W_2 \colon W_1|$ and if*

- *i is N-determined*
- *X_1 is totally N-determined*
- *$\{\check{N}_2 > \check{N}_1\}$ is a one-point set*

then the action $\mathrm{Mono}(X_1, X_2) \times \mathrm{Out}(X_1) \to \mathrm{Mono}(X_1, X_2)$ *is transitive and all monomorphisms of X_1 into X_2 are N-determined.*

Proof. The first part is [17, 3.11]. For the second part, note first that for any $\alpha \in \mathrm{Out}(X_1)$, $i\alpha$ is an N-determined monomorphism. Suppose namely that $N(f) = N(i\alpha) = N(i)N(\alpha)$ for some monomorphism $f\colon X_1 \to X_2$. Then $N(f\alpha^{-1}) = N(f)N(\alpha)^{-1} = N(i)$, so $f\alpha^{-1} = i$ or $f = i\alpha$.

Let now $f\colon X_1 \to X_2$ be any monomorphism and $\check{N}(f)\colon \check{N}_1 \to \check{N}_2$ a representative for the conjugacy class $N(f)$. Since $\check{N}_2$ contains but a single copy of $\check{N}_1$ up to conjugacy and X_1 is totally N-determined, $\check{N}(f) = \check{N}(i)\check{N}(\alpha)$ for some automorphism α of X_1. Then $\check{N}(f) = \check{N}(i\alpha)$ and $f = i\alpha$. □

The third condition is satisfied in case $\check{N}_1 = \check{T}_1 \rtimes W_1$, $\check{N}_2 = \check{T}_2 \rtimes W_2$ are semi-direct products and the set $\{W_1 > W_2\}$ is a one-point set.

Definition 5.5. *For a monomorphism $f\colon Y \to X$ of p-compact groups, let $W_X(f)$ or $W_X(Y)$, the Weyl group of f, denote the component group of the Weyl space $\mathcal{W}_X(Y)$* [12, 4.1, 4.3].

Proposition 5.6. *Let $f\colon Y \to X$ be a monomorphism of p-compact groups.*

1. *If the homomorphism $\pi_0(Z(Y)) \to \pi_0(C_X(Y))$ induced by f is surjective, then the Weyl group $W_X(Y)$ is the isotropy subgroup $\mathrm{Out}(Y)_f$ for the action of $\mathrm{Out}(Y)$ on $f \in \mathrm{Mono}(Y, X)$.*
2. *If f is centric* [8]*, then there is a short exact sequence of loop spaces* [10, 3.2] *$Y \to N_X(Y) \to W_X(Y)$ where $N_X(Y)$ is the normalizer of f* [12, 4.4].

Proof. The monomorphism f determines a fibration

$$\mathcal{W}_X(Y) \to \coprod_{f\circ\alpha\simeq f} \mathrm{map}(BY, BY)_{B\alpha} \xrightarrow{Bf} \mathrm{map}(BY, BX)_{Bf}$$

where the components of the total space are indexed by the isotropy subgroup $\mathrm{Out}(Y)_f$ and the fibre is the Weyl space. The assumptions of the proposition assure that the inclusion of the fibre into the total space is a bijection on π_0. If we make the additional assumption that f be centric, the Weyl space becomes homotopically discrete and the exact sequence of the proposition is the one from [12, 4.6] □

Lemma 5.7. *Let $i\colon X_1 \to X_2$ be a monomorphism and $N(i) \in \mathrm{Mono}(N_1, N_2)$ the induced monomorphism of normalizers. Then the stabilizer subgroup $\mathrm{Out}(N_1)_{N(i)}$ of $N(i)$ is isomorphic to $N_{W_2}(W_1)/W_1$.*

Proof. Note that there is an epimorphism

$$N_{\check{N}_2}(\check{N}_1)/\check{N}_1 \twoheadrightarrow \left(\mathrm{Aut}(\check{N}_1)/\check{N}_1\right)_{N(i)} = \mathrm{Out}(N_1)_{N(i)}$$

given by conjugation by elements of $\check{N}_2$ normalizing $\check{N}_1$. This bijection is actually also injective, hence bijective, for if conjugation by, say, $n_2 \in N_{\check{N}_2}(\check{N}_1)$ agrees with conjugation by some element $n_1 \in \check{N}_1$, then n_1 and n_2 have the same image in W_2, so that n_2 belongs to $\check{N}_1$. This follows because the Weyl groups of the connected p-compact groups X_1 and X_2 are faithfully represented in their maximal tori. Consequently

$$\mathrm{Out}(N_1)_{N(i)} \cong N_{\check{N}_2}(\check{N}_1)/\check{N}_1$$

and this last group is isomorphic to the quotient group $N_{W_2}(W_1)/W_1$ by the projection $\check{N}_2 \twoheadrightarrow W_2$. □

Proposition 5.8. *Let $i\colon X_1 \to X_2$ be an N-determined monomorphism between the two p-compact groups X_1 and X_2 inducing an epimorphism $\pi_0(Z(X_1)) \to \pi_0(C_{X_2}(X_1))$. Then*

$$W_{X_2}(X_1) = N_{W_2}(W_1)/W_1$$

provided X_1 is totally N-determined.

Proof. The assumptions imply that the Weyl group $W_{X_2}(X_1)$ is isomorphic to the stabilizer subgroup $\mathrm{Out}(X_1)_i$ which again is isomorphic to the stabilizer subgroup $\mathrm{Out}(N_1)_{N(i)}$ for the action

$$\mathrm{Mono}(\check{N}_1, \check{N}_2)/\check{N}_2 \times \mathrm{Aut}(\check{N}_1)/\check{N}_1 \to \mathrm{Mono}(\check{N}_1, \check{N}_2)/\check{N}_2$$

of $\mathrm{Out}(N_1)$ on $N(i) \in \mathrm{Mono}(N_1, N_2)$. Now apply (5.7). □

Example 5.9. By (5.3), the inclusion $\check{T}\rtimes\langle -\tau\rangle \rightarrowtail \check{T}\rtimes W$ is realizable by an N-determined monomorphism $i\colon \mathrm{U}(2) \to \mathrm{DI}(2)$. The monomorphism i is centric (because $B\,\mathrm{U}(2) = BT_{h\mathbb{Z}/2}$ and $C_{\mathrm{U}(2)}(T) \cong T \cong C_{\mathrm{DI}(2)}(T)$) so (5.4, 5.6, 5.8)

$$\chi(\mathrm{DI}(2)/\mathrm{U}(2)) = 24 \quad \text{and} \quad W_{\mathrm{DI}(2)}(\mathrm{U}(2)) \cong Z(W)$$

and $\mathrm{Out}(\mathrm{U}(2))$ acts transitively on $\mathrm{Mono}(\mathrm{U}(2), \mathrm{DI}(2))$ since $\{\check{T}\rtimes W > \check{T}\rtimes\langle -\tau\rangle\}$ is a one-point set.

Example 5.10. Similarly, $\{W > \mathbb{Z}/2 \times \mathbb{Z}/2\}$ is a one-point set, so there is an essentially unique monomorphism $i\colon \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{DI}(2)$ realizing the inclusion of $\check{T}\rtimes(\mathbb{Z}/2 \times \mathbb{Z}/2)$ into $\check{T}\rtimes W$. The monomorphism i is N-determined, centric, and

$$\chi(\mathrm{DI}(2)/\mathrm{SU}(2) \times \mathrm{SU}(2)) = 12 \quad \text{and} \quad W_{\mathrm{DI}(2)}(\mathrm{SU}(2) \times \mathrm{SU}(2)) \cong Z(W)$$

and $\mathrm{Out}(\mathrm{SU}(2) \times \mathrm{SU}(2))$ acts transitively on $\mathrm{Mono}(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{DI}(2))$.

Example 5.11. Also $\{W > D_8\} = \{D_8\}$, where D_8 is the dihedral group of order 8. It follows that there exists a unique monomorphism $i\colon \mathrm{Spin}(5) \to \mathrm{DI}(2)$ realizing the inclusion $\check{T}\rtimes D_8 \rightarrowtail \check{T}\rtimes W$. This monomorphism is centric (because $B\,\mathrm{Spin}(5)$ and $B(\check{T}\rtimes D_8)$ are $H^*\mathbb{F}_3$-equivalent), so

$$\chi(\mathrm{DI}(2)/\mathrm{Spin}(5)) = 6 \quad \text{and} \quad W_{\mathrm{DI}(2)}(\mathrm{Spin}(5)) \cong Z(W)$$

and $\mathrm{Out}(\mathrm{Spin}(5))$ acts transitively on $\mathrm{Mono}(\mathrm{Spin}(5), \mathrm{DI}(2))$.

In a situation where a pair of monomorphisms $G \to X_1$ and $G \to X_2$ are given, let us write

$$\mathrm{map}^{BG}(BX_1, BX_2)$$

for the space of maps $BX_1 \to BX_2$ under BG up to homotopy.

Lemma 5.12. *Let $z\colon Z \to X_1$ be a central monomorphism and $i\colon X_1 \to X_2$ any monomorphism inducing an isomorphism $X_1 \cong C_{X_1}(z) \to C_{X_2}(f \circ z)$. Then f induces a homotopy equivalence*

$$\mathrm{map}^{BZ}(BX_1, BX_1) \to \mathrm{map}^{BZ}(BX_1, BX_2)$$

of mapping spaces.

Proof. The spaces $BC_{X_1}(z) = \mathrm{map}(BZ, BX_1)_{Bz}$ and $BC_{X_2}(f \circ z) = \mathrm{map}(BZ, BX_2)_{B(f\circ z)}$ are X_1/Z-spaces and $BC_f(Z)\colon BC_{X_1}(z) \to BC_{X_2}(f \circ z)$ is an X_1/Z-map inducing a map

$$\mathrm{map}^{BZ}(BX_1, BX_1) = BC_{X_1}(z)^{h(X_1/Z)} \to BC_{X_2}(f \circ z)^{h(X_1/Z)} = \mathrm{map}^{BZ}(BX_1, BX_2)$$

of homotopy fixed point spaces. If $C_f(z)$ is an isomorphism, then this map is a homotopy equivalence. □

This happens for instance for $V \to C_X(V) \to X$ so that

$$\mathrm{map}^{BV}(BC_X(V), BC_X(V)) \simeq \mathrm{map}^{BV}(BC_X(V), BX)$$

for any connected p-compact group X, any elementary abelian p-group V, and any monomorphism $V \to X$.

Example 5.13. Let $i\colon \mathrm{SU}(3) \to \mathrm{DI}(2)$ denote the monomorphism arising in the construction (3.5) of $B\,\mathrm{DI}(2)$ as a homotopy colimit. By (5.12), Bi induces a homotopy equivalence

$$\mathrm{map}^{B\mathbb{Z}/3}(B\,\mathrm{SU}(3), B\,\mathrm{SU}(3)) \to \mathrm{map}^{B\mathbb{Z}/3}(B\,\mathrm{SU}(3), B\,\mathrm{DI}(2))$$

where $\mathbb{Z}/3 \to \mathrm{SU}(3)$ is the center, and thus a bijection

$$\mathrm{Out}^+(\mathrm{SU}(3)) \to \mathrm{Mono}(\mathrm{SU}(3), \mathrm{DI}(2)),$$

where $\mathrm{Out}^+(\mathrm{SU}(3))$ consists of the unstable Adams operations ψ^u indexed by units $u \in \mathbb{Z}_3^*$ with $u \equiv 1 \mod 3$. We obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Out}^+(\mathrm{SU}(3)) & \xrightarrow{\cong} & \mathrm{Mono}(\mathrm{SU}(3), \mathrm{DI}(2)) \\ {\scriptstyle N}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle N} \\ \mathrm{Out}(\check{T} \rtimes S\Sigma_3)/S\Sigma_3 & \longrightarrow & \mathrm{Mono}(\check{T} \rtimes S\Sigma_3, \check{T} \rtimes W)/W \end{array}$$

and using (5.7) we see that the kernel of the composition going down and then right is trivial. Thus i is N-determined and [8, 4.2] centric. Consequently,

$$\chi(\mathrm{DI}(2)/\,\mathrm{SU}(3)) = 8 \quad \text{and} \quad W_{\mathrm{DI}(2)}(\mathrm{SU}(3)) \cong Z(W),$$

$\mathrm{Out}(\mathrm{SU}(3))$ acts transitively on $\mathrm{Mono}(\mathrm{SU}(3), \mathrm{DI}(2))$, and all monomorphisms of $\mathrm{SU}(3)$ into $\mathrm{DI}(2)$ are N-determined.

Example 5.14. Similarly, the monomorphism $i\colon \mathrm{SU}(3) \to G_2$ arising in the construction (4.1) of BG_2 as a homotopy colimit is N-determined and centric. Also, $\mathrm{Out}(\mathrm{SU}(3))$ acts transitively on $\mathrm{Mono}(\mathrm{SU}(3), G_2)$ with stabilizer subgroup $W_{G_2}(\mathrm{SU}(3)) = \{\pm E\}$, and $\chi(G_2/\,\mathrm{SU}(3)) = 2$.

Example 5.15. The inclusions of the maximal torus and of $\mathrm{SU}(3)$ into $\mathrm{DI}(2)$ constitute a homotopy coherent set of maps out of the centralizer diagram (4.1) for BG_2 into $B\,\mathrm{DI}(2)$. Observing that both maps are centric one sees first that the Wojtkowiak obstruction groups vanish according to (4.2, 4.3) and next that the resulting map $BG_2 \to B\,\mathrm{DI}(2)$ is a centric monomorphism realizing the inclusion $\check{T} \rtimes D \to \check{T} \rtimes D$ of maximal torus normalizers. As also $\{W > D\} = \{D\}$, we conclude that

$$\chi(\mathrm{DI}(2)/G_2) = 4 \quad \text{and} \quad W_{\mathrm{DI}(2)}(G_2) \cong \{1\}$$

and that $\mathrm{Out}(G_2)$ acts transitively on $\mathrm{Mono}(G_2, \mathrm{DI}(2))$.

Example 5.16. According to [14], $B\,\mathrm{PU}(3)$ is the homotopy colimit of a diagram of the form

$$(N_{P\Sigma_3}(\langle\sigma\rangle)/\langle\sigma\rangle)^{\mathrm{op}} \circlearrowright B(\check{T} \rtimes \langle\tau\rangle)^\wedge_3 \xleftarrow{S_3^{\mathrm{op}} \backslash \mathrm{SL}(V)^{\mathrm{op}}} BV \circlearrowleft W^{\mathrm{op}}$$

where V is elementary abelian of order 3^2, $S_3 = \mathrm{Syl}_3(\mathrm{SL}(V))$, and $(N_{P\Sigma_3}(\langle\sigma\rangle)/\langle\sigma\rangle)^{\mathrm{op}} \cong \langle\tau\rangle^{\mathrm{op}}$ acts on $\check{T} \rtimes \langle\tau\rangle$ by conjugation [21, 2.9]. The obvious map

$$B(\check{T} \rtimes \langle\sigma\rangle) \to B(\check{T} \rtimes W) \to B\,\mathrm{DI}(2) \tag{5.17}$$

is invariant up to homotopy under this action. The Shapiro lemma [18, 4.5] and the identities $C_{\check{T} \rtimes \langle\sigma\rangle}(\check{T}) = \check{T} = C_{\check{T} \rtimes W}(\check{T})$ show that (5.17) in centric. Since $\mathrm{Mono}(V, \mathrm{DI}(2)) = \{*\}$, there is up to conjugation a unique monomorphism $BV \to B\,\mathrm{DI}(2)$, which must respect any group action on BV, and the centralizer of this monomorphism is the maximal torus T of $\mathrm{DI}(2)$. These two maps define a homotopy coherent set of maps out of the centralizer diagram (5.2) for $B\,\mathrm{PU}(3)$. The Wojtkowiak obstruction groups [26] for piecing them together to an actual map out of $B\,\mathrm{PU}(3)$ are $\lim^1$ and $\lim^2$ of the diagram

$$\mathbb{Z}/2 \circlearrowright \check{T}^{\langle\sigma\rangle} \xrightarrow{\mathrm{SL}(V)/S_3} 0 \circlearrowleft \mathrm{SL}(V)$$

where $\mathbb{Z}/2$ acts non-trivially on $\check{T}^{\langle\sigma\rangle} \cong \mathbb{Z}/3$, and $\lim^2$ and $\lim^3$ of the diagram

$$\mathbb{Z}/2 \circlearrowright 0 \xrightarrow{\mathrm{SL}(V)/S_3} L \circlearrowleft \mathrm{SL}(V)$$

where $L = \pi_2(BT)$. Since these obstructions groups vanish, indeed the entire $\lim^* = 0$ in both cases [21, 3.4, 3,9], there exists a unique homotopy class $Bi\colon B\,\mathrm{PU}(3) \to B\,\mathrm{DI}(2)$ extending the two given maps. Also, the restriction of i to the 3-normalizer of the maximal torus is a monomorphism, so i itself is a monomorphism [21, 5.2], and i is centric because the Bousfield-Kan spectral sequence [5, XI.7.1] for $\mathrm{map}(B\,\mathrm{PU}(3), B\,\mathrm{DI}(2))_{Bi}$ shows that this mapping space is weakly contractible. As also $\{\check{T} \rtimes W > \check{T} \rtimes P\Sigma_3\}$ is a one-point set and $\mathrm{PU}(3)$ is totally N-determined [21], (5.4, 5.6, 5.8) show that

$$\chi\,(\mathrm{DI}(2)/\,\mathrm{PU}(3)) = 8 \quad \text{and} \quad W_{\mathrm{DI}(2)}(\mathrm{PU}(3)) = Z(W)$$

and that the group Out(PU(3)) acts transitively on the set Mono(PU(3), DI(2)) of conjugacy classes of monomorphisms.

There is no monomorphism of PU(3) into G_2 for $\mathbb{A}(\mathrm{PU}(3))(V) = \mathrm{SL}(V)$ is too big to be a subgroup of $\mathbb{A}(G_2)(V) = D$. Indeed, no nontrivial compact, connected Lie group admits a proper, center-free subgroup of maximal rank [4].

The next example describes the normalizers of the elementary abelian subgroups of DI(2). Strictly speaking, these normalizers are not 3-compact groups, but rather extended 3-compact groups, in that their component groups are not 3-groups.

We start with a general observation.

Proposition 5.18. *Let $\nu\colon V \to X$ be a monomorphism of an elementary abelian p-group V into a p-compact group X.*

1. *There is a short exact sequence of groups*
$$1 \to \pi_0(C_X(\nu)/V) \to W_X(\nu) \to \mathbb{A}(X)(\nu) \to 1$$
where $C_X(\nu)/V$ is the standard quotient [10, 8.3].
2. *There is a short exact sequence of loop spaces*
$$C_X(\nu) \to N_X(\nu) \to \mathbb{A}(X)(\nu)$$
where $N_X(\nu)$ is the normalizer of ν [12, 4.4].

Proof. Assuming $B\nu\colon BV \to BX$ to be a fibration, consider the induced fibration
$$\mathcal{W}_X(\nu) \to \coprod_{f\in\mathbb{A}(X)(\nu)} \mathrm{map}(BV, BV)_{Bf} \xrightarrow{B\nu} \mathrm{map}(BV, BX)_{B\nu}$$
where the fibre is the Weyl space [12, 4.1] of ν and the components, each one homotopy equivalent to BV, of the total space are indexed by the automorphism group of ν in the Quillen category. The homotopy exact sequences of this fibration and of its sub-fibration
$$C_X(\nu)/V \to BV \to BC_X(\nu)$$
give the exact sequence of groups and show that $B(C_X(\nu)/V)$ is the regular covering space of $B\mathcal{W}_X(\nu)$ corresponding to the normal subgroup $\pi_0(C_X(\nu)/V) \lhd W_X(\nu)$. Thus there is a pull-back diagram
$$\begin{array}{ccc} BC_X(\nu) & \longrightarrow & BN_X(\nu) \\ \downarrow & & \downarrow \\ B(C_X(\nu)/V) & \longrightarrow & B\mathcal{W}_X(\nu), \end{array}$$
where the horizontal maps are regular covering spaces. □

Example 5.19. For any monomorphism $\lambda\colon \mathbb{Z}/3 \to \mathrm{DI}(2)$ there is (5.18) a short exact sequence of loop spaces
$$\mathrm{SU}(3) \to N_{\mathrm{DI}(2)}(\lambda) \to Z(W)$$
where $Z(W) \cong \mathbb{Z}/2$ acts on SU(3) as $\{\psi^{\pm 1}\}$. Thus
$$N_{\mathrm{DI}(2)}(\lambda) = \mathrm{SU}(3) \rtimes Z(W)$$
where $B(\mathrm{SU}(3)\rtimes Z(W))$ denotes the total space of the unique [19, 3.3, 3,7] $B\,\mathrm{SU}(3)$-fibration over $BZ(W)$ realizing the given monodromy action. (It is not essential in [19, §3] that the component group $\pi_0(X)$ be a p-group.) Since the homotopy fixed point space $BZ(\mathrm{SU}(3))^{hZ(W)}$ is contractible, the inclusion $\check{T}\rtimes S\Sigma_3 \rightarrowtail \mathrm{SU}(3)$ extends uniquely to a short exact sequence morphism
$$\begin{array}{ccccc} \check{T}\rtimes S\Sigma_3 & \longrightarrow & N_{\check{T}\rtimes W}(\lambda) & \longrightarrow & Z(W) \\ \downarrow & & \downarrow & & \| \\ \mathrm{SU}(3) & \longrightarrow & N_{\mathrm{DI}(2)}(\lambda) & \longrightarrow & Z(W) \end{array}$$
where $N_{\check{T}\rtimes W}(\lambda) = \check{T}\rtimes\overline{W}(\lambda) = \check{T}\rtimes(S\Sigma_3 \times Z(W))$.

For any monomorphism $\nu\colon (\mathbb{Z}/3)^2 \to \mathrm{DI}(2)$ there is a short exact sequence of loop spaces

$$T \to N_{\mathrm{DI}(2)}(\nu) \to W$$

so $N_{\mathrm{DI}(2)}(\nu)$ is an extended p-compact torus with $\check{T}\rtimes W$ as discrete approximation [11, 3.12].

Example 5.20. The normalizers of the 3-compact subgroups of DI(2) are (5.6.2)

$$N_{\mathrm{DI}(2)}(G_2) = G_2 \quad \text{and} \quad N_{\mathrm{DI}(2)}(X) = X \rtimes Z(W)$$

for $X = \mathrm{U}(2), \mathrm{SU}(2)\times\mathrm{SU}(2), \mathrm{Spin}(5), \mathrm{SU}(3), \mathrm{PU}(3)$ where $Z(W)$ acts on X as $\{\psi^{\pm1}\}$. In each case there is a unique short exact sequence morphism connecting the normalizer in $\check{T}\rtimes W$ of $N_X(T)$ and the normalizer in DI(2) of X. For $X = \mathrm{PU}(3)$, for instance, the picture is

$$\begin{array}{ccccc}
\check{T}\rtimes P\Sigma_3 & \longrightarrow & N_{\check{T}\rtimes W}(\check{T}\rtimes P\Sigma_3) & \longrightarrow & Z(W) \\
\downarrow & & \downarrow & & \| \\
\mathrm{PU}(3) & \longrightarrow & N_{\mathrm{DI}(2)}(\mathrm{PU}(3)) & \longrightarrow & Z(W)
\end{array}$$

where $N_{\check{T}\rtimes W}(\check{T}\rtimes P\Sigma_3) = \check{T}\rtimes(P\Sigma_3 \times Z(W))$. It seems likely that this is another instance of N-determinism.

References

[1] J. Aguadé, *Constructing modular classifying spaces*, Israel J. Math. **66** (1989), 23–40.

[2] K.K.S. Andersen, *Cohomology of Weyl groups with applications to topology*, Master's thesis, Københavns Universitet, August 1997.

[3] D.J. Benson, *Polynomial invariants of finite groups*, London Mathematical Society Lecture Note Series, vol. 190, Cambridge University Press, Cambridge, 1994.

[4] A. Borel and J. De Siebenthal, *Les sous-groupes fermés de rang maximum des groupes de Lie clos*, Comment. Math. Helv. **23** (1949), 200–221.

[5] A.K. Bousfield and D.M. Kan, *Homotopy limits, completions and localizations*, 2nd ed., Lecture Notes in Mathematics, vol. 304, Springer-Verlag, Berlin–Heidelberg–New York–London–Paris–Tokyo, 1987.

[6] C. Broto and A. Viruel, *Homotopy uniqueness of $BPU(3)$*, Preprint, 1995.

[7] A. Clark and J.R. Ewing, *The realization of polynomial algebras as cohomology rings*, Pacific J. Math **50** (1974), 425–434.

[8] W.G. Dwyer and D.M. Kan, *Centric maps and realizations of diagrams in the homotopy category*, Proc. Amer. Math. Soc. **114** (1992), 575–584.

[9] W.G. Dwyer, H.R. Miller, and C.W. Wilkerson, *Homotopical uniqueness of classifying spaces*, Topology **31** (1992), 29–45.

[10] W.G. Dwyer and C.W. Wilkerson, *Homotopy fixed point methods for Lie groups and finite loop spaces*, Ann. of Math. (2) **139** (1994), 395–442.

[11] ———, *The center of a p-compact group*, The Čech Centennial. Contemporary Mathematics, vol. 181 (Providence, Rhode Island) (M. Cenkl and H. Miller, eds.), American Mathematical Society, 1995, pp. 119–157.

[12] ———, *Product splittings for p-compact groups*, Fund. Math. **147** (1995), 279–300.

[13] S. Jackowski, J. McClure, and R. Oliver, *Self homotopy equivalences of classifying spaces of compact connected Lie groups*, Fund. Math. **147** (1995), 99–126.

[14] S. Jackowski and J.E. McClure, *Homotopy decompostion of classifying spaces via elementary abelian subgroups*, Topology **31** (1992), 113–132.

[15] Jean Lannes, *Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire*, Inst. Hautes Études Sci. Publ. Math. (1992), no. 75, 135–244, With an appendix by Michel Zisman.

[16] J.M. Møller, *Deterministic p-compact groups*, To appear in the proceedings of the Algebraic Topology Year at the Fields Institute.

[17] ———, *Normalizers of maximal tori*, To appear in Math. Z.

[18] ———, *Homotopy Lie groups*, Bull. Amer. Math. Soc. **32** (1995), 413–428.

[19] ———, *Extensions of p-compact groups*, Algebraic Topology: New Trends in Localization and Periodicity. Progress in Mathematics, vol. 136 (Basel–Boston–Berlin) (C. Broto, C. Casacuberta, and G. Mislin, eds.), Birkhäuser, 1996, pp. 307–327.

[20] ———, *Rational isomorphisms of p-compact groups*, Topology **35** (1996), 201–225.

[21] ———, *$PU(p)$ as a p-compact group*, Preprint, April 1997.

[22] D. Notbohm, *Homotopy uniqueness of classifying spaces of compact connected Lie groups at primes dividing the order of the Weyl group*, Topology **33** (1994), 271–330.

[23] G.C. Shephard and J.A. Todd, *Finite unitary reflection groups*, Can. J. Math **6** (1954), 274–304.

[24] L. Smith and R.M. Switzer, *Realizability and nonrealizability of Dickson algebras*, Proc. Amer. Math. Soc. **89** (1983), 303–313.

[25] Larry Smith, *Polynomial invariants of finite groups*, Research Notes in Mathematics, vol. 6, A K Peters Ltd., Wellesley, MA, 1995.

[26] Z. Wojtkowiak, *On maps from* $\operatorname{holim} F$ *to* Z, Algebraic Topology. Barcelona 1986. Lecture Notes in Mathematics, vol. 1298 (Berlin–Heidelberg–New York–London–Paris–Tokyo) (J. Aguadé and R. Kane, eds.), Springer-Verlag, 1987, pp. 227–236.

[27] A. Zabrodsky, *On the realization of invariant subgroups of* $\pi_*(X)$, Trans. Amer. Math. Soc. **285** (1984), no. 2, 467–496.

Matematisk Institut, Universitetsparken 5, DK–2100 København Ø
E-mail address: moller@math.ku.dk