NILPOTENCE IN THE STEENROD ALGEBRA

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I. Introduction and Notation

While all of the relations in the Steenrod algebra, $\mathcal{A}$, can be deduced in principle from the Adem relations, in practice, it is extremely difficult to determine whether a given polynomial of elements in $\mathcal{A}$ is zero for all but the most elementary cases. In his original paper [Mi] Milnor states "It would be interesting to discover a complete set of relations between the given generators of $\mathcal{A}$". In particular Milnor shows that every positive dimensional element of $\mathcal{A}$ is nilpotent. Thus it would be desirable to find a simple closed form for nilpotence relations in $\mathcal{A}$.

Let $x \in \mathcal{A}$. We say that x has *nilpotence* k, if $x^k = 0$ and $x^{k-1} \neq 0$ (take $x^0 = 1$). In this case we write $\mathrm{Nil}(x) = k$. In this paper we investigate $\mathrm{Nil}(x)$ for several infinite families of Milnor basis elements of $\mathcal{A}$ at the prime 2.

The paper is organized as follows. First, an infinite family of subalgebras and isomorphisms between them are constructed. The isomorphisms are used to produce infinite families of elements having the same nilpotence. Next, we compute strong upper and lower bounds for the nilpotence of Milnor basis elements in these subalgebras. Comparing these bounds and extending to the families produced via the isomorphisms shows that $\mathrm{Sq}(2^m(2^k-1)+1)$ has nilpotence $k+2$. Finally a strong lower bound for the nilpotence of P^s_t is computed for all $s, t \in \mathbb{N}$. The main results are stated and discussed in Sections II and III. Detailed proofs are presented in Section IV.

II. Nilpotence in an Odd Subalgebra of $\mathcal{A}$

There is a doubling isomorphism (see Section IV) which implies that

$$\mathrm{Nil}(\mathrm{Sq}(2r_1, \dots, 2r_m)) \geq \mathrm{Nil}(\mathrm{Sq}(r_1, \dots, r_m))$$

for every Milnor basis element in $\mathcal{A}$. Thus it is natural to begin by asking what the nilpotence of $\mathrm{Sq}(r_1, \dots, r_m)$ is when some or all of the r_i are odd.

We begin by describing a family of isomorphic subalgebras $\mathcal{O}_k \subset \mathcal{A}$ and a family of isomorphisms between them.

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Definition 2.1. Let $k \in \mathbb{N}$. Let $\mathcal{O}_k$ be the $\mathbb{Z}_2$-subspace of $\mathcal{A}$ whose basis is the set of Milnor elements

$$\mathcal{B}_k = \left\{ \mathrm{Sq}(r_1, \dots, r_m) \mid r_i \equiv -1 \pmod{2^{k+1}} \text{ for } i < m, \text{ and } r_m \equiv 1 \pmod{2^{k+1}} \right\}.$$

We will write $\mathcal{O} = \mathcal{O}_0$. Thus we have the vector subspace inclusions

$$\mathcal{A} \supset \mathcal{O} = \mathcal{O}_0 \supset \mathcal{O}_1 \supset \mathcal{O}_2 \supset \dots .$$

Notice $\mathcal{O}$ is just the subspace of $\mathcal{A}$ generated by the Milnor basis elements $\mathrm{Sq}(r_1, \dots, r_m)$ with r_i odd for all $1 \le i \le m$.

Theorem 2.2. *$\mathcal{O}_k$ is a sub-algebra of $\mathcal{A}$ for all $k \in \mathbb{N}$.*

$\mathcal{O}_k$ is not a Hopf subalgebra, but we do not require this for our purposes.

Definition 2.3. Let $\lambda : \mathcal{O} \to \mathcal{O}$ be the $\mathbb{Z}_2$-linear map given by

$$\lambda(\mathrm{Sq}(r_1, \dots, r_m)) = \mathrm{Sq}(2r_1 + 1, 2r_2 + 1, \dots, 2r_{m-1} + 1, 2r_m - 1)$$

on elements of the basis.

For example, $\lambda\left(\mathrm{Sq}(5) + \mathrm{Sq}(3,1,3)\right) = \mathrm{Sq}(9) + \mathrm{Sq}(7,3,5)$.

Theorem 2.4. *λ is an algebra monomorphism.*

If we let $\lambda^{(0)}$ be the identity map on $\mathcal{O}$, and $\lambda^{(k)} = \lambda \circ \lambda^{(k-1)}$ for $k > 1$ then $\lambda^{(k)}$ is also an monomorphism for every k. It is a routine calculation to check that

$$\lambda^{(k)}(\mathrm{Sq}(r_1, \dots, r_m)) = \mathrm{Sq}\left(2^k r_1 + (2^k - 1), \dots, 2^k r_{m-1} + (2^k - 1), 2^k r_m - (2^k - 1)\right) \tag{2.5}$$

Using (2.5) it is elementary to see that $\lambda(\mathcal{O}_k) = \mathcal{O}_{k+1}$ and thus that the restriction of λ to $\mathcal{O}_k$ yields an isomorphism λ_k between $\mathcal{O}_k$ and $\mathcal{O}_{k+1}$. Hence for any $x \in \mathcal{O}$ we have $\mathrm{Nil}(x) = \mathrm{Nil}(\lambda^{(k)}(x))$ for all $k \in \mathbb{N}$. Thus

Corollary 2.6. *Let $\mathrm{Sq}(r_1, \dots, r_m) \in \mathcal{O}$. Then*

$$\mathrm{Nil}\left(\mathrm{Sq}(r_1, \dots, r_m)\right) = \mathrm{Nil}\left(\mathrm{Sq}\left(2^k r_1 + (2^k - 1), \dots, 2^k r_{m-1} + (2^k - 1), 2^k r_m - (2^k - 1)\right)\right)$$

for all $k \in \mathbb{N}$.

In particular, if n is odd then $\mathrm{Nil}\left(\mathrm{Sq}(n)\right) = \mathrm{Nil}\left(\mathrm{Sq}(2^k n - (2^k - 1))\right)$ for all $k \in \mathbb{N}$. For example, since $\mathrm{Nil}\left(\mathrm{Sq}(7)\right) = 4$, every element of the family

$$\mathrm{Sq}(7), \mathrm{Sq}(13), \mathrm{Sq}(25), \mathrm{Sq}(49), \mathrm{Sq}(97), \mathrm{Sq}(193), \dots$$

also has nilpotence 4.

Theorem 2.4 reduces the problem of computing the nilpotence of elements of $\mathcal{O}$ to that of finding the nilpotence of elements in $\mathcal{O} - \mathcal{O}_1$. For the case $m = 1$ this says that the nilpotence of the Milnor elements $\mathrm{Sq}(n)$ with $n \equiv 1 \pmod 4$ is completely determined by the nilpotence of the elements $\mathrm{Sq}(n)$ with $n \equiv -1 \pmod 4$. We begin to attack this question by obtaining a strong upper bound.

Theorem 2.7. *Let* $\mathrm{Sq}(r_1, \dots, r_m) \in \mathcal{O}$. *Then*

$$\mathrm{Nil}(\mathrm{Sq}(r_1, \dots, r_m)) \le \min\left\{ k \mid r_m < 2^{(k-1)m+1} - 1 \right\}.$$

Corollary 2.8. *If n is odd then* $\mathrm{Nil}(\mathrm{Sq}(n)) \le \min\{ k \mid n < 2^k - 1 \}$.

For example $\mathrm{Sq}(15, 31)^4 = 0$ since $31 < 2^{(4-1)2+1} - 1 = 127$. As a possible application, notice that $\mathrm{Sq}(r_1, \dots, r_m)^2 = 0$ whenever $r_m < 2^{m+1} - 1$ and $\mathrm{Sq}(r_1, \dots, r_m) \in \mathcal{O}$. Elements whose square is zero have been useful in the past in developing P_t^s homology theory.

It should be noted that this upper bound appears to be quite good. Computer calculations show that we actually have equality in Corollary 2.8 for every $n \equiv -1 \pmod 4$ less than 143 with the exception of $n = 67$ and $n = 131$ (note also that these exceptions eliminate the possibility that one might actually be able to prove equality in all cases).

We now obtain a lower bound on nilpotence for certain of these elements.

Theorem 2.9. *Let n be odd. Then*

$$\mathrm{Nil}(\mathrm{Sq}(n)) > \max\left\{ k \mid n \equiv -1 \pmod{2^k} \right\}.$$

Combining all of the previous results yields

Theorem 2.10. $\mathrm{Nil}\left(\mathrm{Sq}\left(2^m(2^k - 1) + 1\right)\right) = k + 2$ *for all* $m, k \ge 1$.

Notice for $m = 1$ this implies $\mathrm{Nil}(\mathrm{Sq}(2^k - 1)) = k + 1$.

As an illustration of the theorem consider that $524281 = 2^3(2^{16} - 1) + 1$. Then by Theorem 2.10 we have immediately that $\mathrm{Sq}(524281)^{17} \ne 0$ and $\mathrm{Sq}(524281)^{18} = 0$, which would be a truly monumental computation by usual means.

Table 2.11 gives a comparison between the nilpotence bounds obtained in this section and the actual values of $\mathrm{Nil}(\mathrm{Sq}(n))$ for odd n less than 64. In the table the values labeled Nil are the actual values of $\mathrm{Nil}(\mathrm{Sq}(n))$ obtained from computer calculations. The values labeled High are the upper bounds for $\mathrm{Nil}(\mathrm{Sq}(n))$ obtained from Corollary 2.6 and Corollary 2.8. Similarly, the values labeled Low are the lower bounds obtained from Corollary 2.6 and Theorem 2.9. Finally, the values labeled Gap are just the difference between the upper and lower bounds. Thus the nilpotence is completely determined whenever the gap is zero. This occurs at the values of n given in Theorem 2.10.

Table 2.11: Comparison of Nilpotence Bounds with Computed Values

n	Nil	High	Low	Gap	n	Nil	High	Low	Gap
1	2	2	2	0	33	3	3	3	0
3	3	3	3	0	35	6	6	3	3
5	3	3	3	0	37	5	5	3	2
7	4	4	4	0	39	6	6	4	2
9	3	3	3	0	41	4	4	3	1
11	4	4	3	1	43	6	6	3	3
13	4	4	4	0	45	5	5	4	1
15	5	5	5	0	47	6	6	5	1
17	3	3	3	0	49	4	4	4	0
19	5	5	3	2	51	6	6	3	3
21	4	4	3	1	53	5	5	3	2
23	5	5	4	1	55	6	6	4	2
25	4	4	4	0	57	5	5	5	0
27	5	5	3	2	59	6	6	3	3
29	5	5	5	0	61	6	6	6	0
31	6	6	6	0	63	7	7	7	0

III. Nilpotence of P^s_t

Let $P^s_t = \mathrm{Sq}(r_1, \ldots, r_t)$ where $r_i = 0$ for all $i < t$ and $r_t = 2^s$. There is an old conjecture which has been growing in notoriety ([Da], [Conf]) which says $\mathrm{Nil}\,(\mathrm{Sq}(2^s)) = 2s + 2$ for all k (or equivalently, $\mathrm{Nil}(P^s_1) = 2s + 2$). One naturally might ask what the corresponding conjecture would be for $\mathrm{Nil}(P^s_t)$ for any t. Some sample calculation leads one immediately to the following. Let $\lfloor r \rfloor$ denote the greatest integer less than or equal to the rational number r.

Conjecture 3.1. $\mathrm{Nil}(P^s_t) = 2\lfloor s/t \rfloor + 2$ *for all* $s \geq 0, t \geq 1$.

Our main result regarding this conjecture is

Theorem 3.2. $\mathrm{Nil}(P^s_t) \geq 2\lfloor s/t \rfloor + 2$ *for all* $s \geq 0, t \geq 1$.

This theorem generalizes an original result of Davis [Da], who first proved this theorem for the special case $t = 1$.

It is well known that the conjecture is true if $\lfloor s/t \rfloor = 0$, i.e. if $s < t$. We can also prove the conjecture for $\lfloor s/t \rfloor = 1$.

Theorem 3.3. *If* $\lfloor s/t \rfloor = 1$ *then* $\mathrm{Nil}(P^s_t) = 4$.

The conjecture has been verified by computer calculation for all s, t such that $s+t < 16$ and $s - 2t < 4$ as well as several other cases. For the case $t = 1$ the conjecture was originally verified by Davis for $s \leq 5$ [Da]. A summary of the calculation is given in Table 3.4. It is interesting to note that for many of the 56,627 Milnor basis elements, x, which are a summand of $\mathrm{Sq}(64)^{13}$, the product $\mathrm{Sq}(64) \cdot x$ is nonzero, and yet the sum of all such products is still zero.

The Theorems in this section were first presented in the author's Ph.D. thesis [Mo].

Table 3.4: Powers of $\mathrm{Sq}(2^s)$

	Number of Milnor Basis Elements Which are a Summand of $\mathrm{Sq}(2^s)^k$						
k	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$
1	1	1	1	1	1	1	1
2	0	1	1	2	3	5	8
3		1	3	5	13	32	84
4		0	2	8	27	131	629
5			2	15	72	473	3,632
6			0	7	93	876	11,454
7				8	153	2,222	37,128
8				0	59	2,070	61,447
9					69	3,297	136,738
10					0	1,093	100,618
11						1,313	158,089
12						0	46,325
13							56,627
14							0

IV. Proof of Results

We begin by recalling some results from [Mi] to which we will need to refer in the proofs that follow. The mod 2 Steenrod algebra is a graded $\mathbb{Z}_2$-vector space with basis all formal symbols $\mathrm{Sq}(r_1, r_2, \dots)$ where $r_i \geq 0$ and $r_i > 0$ for finitely many i. As usual, it is convenient to write $\mathrm{Sq}(r_1, \dots, r_m)$ for $\mathrm{Sq}(r_1, \dots, r_m, 0, 0, \dots)$ when $r_m \neq 0$. Let $R = (r_1, \dots, r_m)$. It will also be convenient to write $\mathrm{Sq}(R)$ for the Milnor basis element $\mathrm{Sq}(r_1, \dots, r_m)$.

The product is given by

$$\mathrm{Sq}(r_1, r_2, \dots) \cdot \mathrm{Sq}(s_1, s_2, \dots) = \sum_X \mathrm{Sq}(t_1, t_2, \dots)$$

where the summation is taken over all matrices $X = (x_{ij})$ satisfying:

$$\sum_i x_{ij} = s_j \tag{4.1}$$

$$\sum_j 2^j x_{ij} = r_i \tag{4.2}$$

$$\prod_h (x_{h0}, x_{h-1\,1}, \dots, x_{0h}) \equiv 1 \pmod 2 \tag{4.3}$$

where $(n_1, \dots, n_m)$ is the multinomial coefficient $(n_1 + \cdots + n_m)!/n_1! \cdots n_m!$. We will say such a matrix X is $\mathrm{Sq}(r_1, r_2, \dots)\mathrm{Sq}(s_1, s_2, \dots)$*–allowable*. Each such allowable matrix yields a summand $\mathrm{Sq}(t_1, t_2, \dots)$ given by

$$t_h = \sum_{i+j=h} x_{ij} \tag{4.4}$$

In this case we will say that X is the *matrix associated with* $\mathrm{Sq}(t_1, t_2, \dots)$ (for the product $\mathrm{Sq}(r_1, r_2, \dots) \cdot \mathrm{Sq}(s_1, s_2, \dots)$). The value of x_{00} is never used and may be assumed to be zero.

When evaluating the multinomial coefficients in (4.3) it is well known (e.g. [Ma]) that $(n_1, \dots, n_m)$ is odd if and only if the n_i have disjoint binary expansions. More formally, let $n = \sum_j \alpha_j(n) 2^j$ be the binary expansion of an integer n. Then

Lemma 4.5. *$(n_1, \dots, n_m)$ is odd if and only if for each $k < \infty$, $\alpha_k(n_i) = 1$ for at most one i.*

In particular, if $(n_1, \dots, n_m)$ is odd then at most one of the n_i is odd. We will make frequent use of this fact.

Proof of Theorem 2.2. It suffices to show that $x \cdot y \in \mathcal{O}$ for all $x, y \in \mathcal{B}_0$. We will prove a slightly stronger result which we will need later, namely

Lemma 4.6. *Let $\mathrm{Sq}(r_1, \dots, r_m), \mathrm{Sq}(s_1, \dots, s_n) \in \mathcal{O}$. If $\mathrm{Sq}(t_1, \dots, t_p)$ is a summand of $\mathrm{Sq}(r_1, \dots, r_m)\mathrm{Sq}(s_1, \dots, s_n)$ then $\mathrm{Sq}(t_1, \dots, t_p) \in \mathcal{O}$ and $p = m + n$.*

Proof. Let $X = (x_{ij})$ be the matrix associated with $\mathrm{Sq}(t_1, \dots, t_p)$. $\mathrm{Sq}(r_1, \dots, r_m) \in \mathcal{O}$ implies that r_i is odd for each $1 \le i \le m$. Thus x_{i0} is odd for each $1 \le i \le m$ by (4.2). Combining this with (4.3) shows x_{ij} is even whenever $i + j \le m$, and $j > 0$. Let $d < n$ and assume that x_{mj} is odd for $j \le d$ and x_{ij} is even whenever $i + j \le m + d$, $j > 0$, and $i < m$. Then $\mathrm{Sq}(s_1, \dots, s_n) \in \mathcal{O}$ implies s_{d+1} is odd and thus $x_{m\, d+1}$ is odd by (4.1). Once again invoking (4.3) shows x_{ij} is even whenever $i + j = m + d + 1$, and $j > d + 1$. Thus by finite induction on d we have shown x_{ij} is odd if and only if $j = 0$, $i \le m$ or $i = m$, $j \le n$. Applying (4.4) shows $\mathrm{Sq}(t_1, \dots, t_p) \in \mathcal{O}$. Further $t_p = x_{mn}$ is odd, therefore $p = m + n$. □ □

Proof of Theorem 2.4. It is easy to see from the definition that λ is injective. Let $R = (r_1, \dots, r_m)$, $S = (s_1, \dots, s_n)$, and $T = (t_1, \dots, t_{m+n})$. To show that λ is a homomorphism we will prove that $\mathrm{Sq}(T)$ is a summand of the product $\mathrm{Sq}(R)\mathrm{Sq}(S)$ if and only if $\lambda(\mathrm{Sq}(T))$ is a summand of $\lambda(\mathrm{Sq}(R))\lambda(\mathrm{Sq}(S))$ for every $\mathrm{Sq}(R), \mathrm{Sq}(S) \in \mathcal{O}$. Let $\widehat{X} = (\widehat{x}_{ij})$ be a $\lambda(\mathrm{Sq}(R))\lambda(\mathrm{Sq}(S))$–allowable matrix. As shown in the proof of Lemma 4.6, $\widehat{x}_{ij}$ is odd if and only if $j = 0$, $i \le m$ or $i = m$, $j \le n$. Thus there exist nonnegative integers x_{ij} such that

$$\widehat{x}_{ij} = \begin{cases} 2x_{ij} + 1 & \text{if } j = 0,\ i \le m \text{ or } i = m,\ j < n \\ 2x_{mn} - 1 & \text{if } i = m \text{ and } j = n \\ 2x_{ij} & \text{otherwise.} \end{cases} \tag{4.7}$$

Given such an allowable matrix $\widehat{X}$ we can define the matrix $X = (x_{ij})$. On the other hand, if we are given a $\mathrm{Sq}(R)\mathrm{Sq}(S)$–allowable matrix, $X = (x_{ij})$, we can define a matrix $\widehat{X} = (\widehat{x}_{ij})$ by (4.7). We now wish to show that $\widehat{X}$ is $\lambda(\mathrm{Sq}(R))\lambda(\mathrm{Sq}(S))$–allowable if and only if X is $\mathrm{Sq}(R)\mathrm{Sq}(S)$–allowable. We must verify that each of the conditions (4.1),(4.2), and (4.3) hold for X if and only if they hold for $\widehat{X}$.

Let $1 \le j \le n$ and define $\epsilon_j = \begin{cases} -1 & \text{if } j = n \\ 1 & \text{otherwise} \end{cases}$. Then $\lambda(\mathrm{Sq}(S)) = \mathrm{Sq}(\widehat{s}_1, \dots, \widehat{s}_n)$ where $\widehat{s}_j = 2s_j + \epsilon_j$. Thus checking (4.1) we have

$$\begin{aligned} \sum_{i=0}^{m} \widehat{x}_{ij} = \widehat{s}_j \iff & \left(\sum_{i=0}^{m-1} 2x_{ij} \right) + (2x_{mj} + \epsilon_j) = 2s_j + \epsilon_j \\ \iff & \sum_{i=0}^{m} x_{ij} = s_j \end{aligned}$$

Again letting $\lambda(\mathrm{Sq}(R)) = \mathrm{Sq}(\widehat{r}_1, \dots, \widehat{r}_m)$ we have $\widehat{r}_i = 2r_i + 1$ for $1 \le i < m$ and $\widehat{r}_m = 2r_m - 1$. Verification for (4.2) breaks up into two cases. If $1 \le i < m$ then

$$\begin{aligned} \sum_{j=0}^{n} 2^j \widehat{x}_{ij} = \widehat{r}_i \iff & \left(\sum_{j=1}^{n} 2^j 2x_{ij} \right) + (2x_{i0} + 1) = 2r_i + 1 \\ \iff & \sum_{j=0}^{n} 2^j x_{ij} = r_i \end{aligned}$$

But if $i = m$ we have

$$\begin{aligned} \sum_{j=0}^{n} 2^j \widehat{x}_{mj} = \widehat{r}_m \iff & \left(\sum_{j=0}^{n-1} 2^j (2x_{mj} + 1) \right) + 2^n (2x_{mn} - 1) = 2r_m - 1 \\ \iff & 2 \sum_{j=0}^{n} 2^j x_{mj} + \sum_{j=0}^{n-1} 2^j - 2^n = 2r_m - 1 \\ \iff & 2 \sum_{j=0}^{n} 2^j x_{mj} + (2^n - 1) - 2^n = 2r_m - 1 \\ \iff & \sum_{j=0}^{n} 2^j x_{mj} = r_m \end{aligned}$$

Verification of (4.3) follows easily from the observation that for any multinomial coefficient $(a_1, \dots, a_h)$ we have

$$(a_1, \dots, a_h) \equiv (2a_1 + \gamma_1, \dots, 2a_h + \gamma_h) \pmod 2$$

where $\gamma_i = 1$ for at most one $1 \le i \le h$ and is zero otherwise. This follows immediately from Lemma 4.5. Thus

$$\prod_h (x_{h0}, x_{h-1\,1}, \dots, x_{0h}) \equiv \prod_h (\widehat{x}_{h0}, \widehat{x}_{h-1\,1}, \dots, \widehat{x}_{0h}) \pmod 2$$

Finally let $\mathrm{Sq}(t_1,\ldots,t_{m+n})$ be the summand of $\mathrm{Sq}(R)\mathrm{Sq}(S)$ associated with X and let $\mathrm{Sq}(\widehat{T}) = \mathrm{Sq}(\widehat{t}_1,\ldots,\widehat{t}_{m+n})$ be the summand of $\lambda(\mathrm{Sq}(R))\lambda(\mathrm{Sq}(S))$ associated with $\widehat{X}$. Then by (4.4) for $h < m+n$

$$\begin{aligned}\widehat{t}_h &= \sum_{i+j=h} \widehat{x}_{ij} \\ &= 2\left(\sum_{i+j=h} x_{ij}\right) + 1 \\ &= 2t_h + 1\end{aligned}$$

and $t_{m+n} = \widehat{x}_{mn} = 2x_{mn} - 1 = 2t_{mn} - 1$. Thus $\mathrm{Sq}(\widehat{T}) = \lambda(\mathrm{Sq}(T))$ which completes the proof. □

Proof of Theorem 2.7. Let $R = (r_1,\ldots,r_m)$. It suffices to show that $\mathrm{Sq}(R)^k = 0$ if $r_m < 2^{(k-1)m+1} - 1$. Let $\mathrm{Sq}(T) = \mathrm{Sq}(t_1,\ldots,t_p)$ be any summand of $\mathrm{Sq}(R)^{k-1}$. By Lemma 4.6 $p = (k-1)m$. Let $X = (x_{ij})$ be any $\mathrm{Sq}(R)\mathrm{Sq}(T)$–allowable matrix. As shown in the proof of Lemma 4.6, x_{ij} is odd if $i = m$ and $j \le (k-1)m$. Combining this with (4.2) we have

$$r_m = \sum_{j=0}^{(k-1)m} 2^j x_{mj} \ge \sum_{j=0}^{(k-1)m} 2^j = 2^{(k-1)m+1} - 1.$$

Therefore if $r_m < 2^{(k-1)m+1} - 1$ there are no $\mathrm{Sq}(R)\mathrm{Sq}(T)$–allowable matrices, and hence $\mathrm{Sq}(R)^k = 0$. □

Before continuing we would like to outline an alternate proof of Theorem 3 that lends some insight into what is going on at the cost of being much more tedious.

Let $Q_{t-1} = P_t^0$. It is quite easy to see from the product formula that

$$Q_iQ_j = Q_jQ_i \qquad \text{for all } i,j \in \mathbb{N} \tag{4.8}$$

$$Q_i^2 = 0 \qquad \text{for all } i \in \mathbb{N} \tag{4.9}$$

and that for any $\mathrm{Sq}(r_1,\ldots,r_m)$ with r_i even for all $1 \le i \le m$

$$\mathrm{Sq}(s_1,\ldots,s_m)Q_i = \sum_{j=0}^{m} Q_{j+i}\mathrm{Sq}(s_1,\ldots,s_j - 2^{i+1},\ldots,s_m) \tag{4.10}$$

where we define $\mathrm{Sq}(t_1,\ldots,t_m)$ to be zero if $t_i < 0$ for any i. Notice that (4.10) gives us a way to shift Q_i's from the right side of a Milnor basis element with even entries to the left side. Also notice that the largest Q_j obtainable on the left by shifting a Q_i via (4.10) is Q_{m+i} and that this can only occur if $s_m \ge 2^{i+1}$. For any $\mathrm{Sq}(r_1,\ldots,r_m) \in \mathcal{O}$ we can write

$$\mathrm{Sq}(r_1,\ldots,r_m) = Q_0Q_1\cdots Q_{m-1}\mathrm{Sq}(r_1-1,\ldots,r_m-1)$$

and hence

$$(\mathrm{Sq}(r_1, \dots, r_m))^k = (Q_0 Q_1 \cdots Q_{m-1} \mathrm{Sq}(r_1 - 1, \dots, r_m - 1))^k \tag{4.11}$$

Applying (4.8), (4.9), and (4.10) repeatedly to the right hand side of (4.11) in order to collect all of the Q_i on the left and computing the effect on the m^{th} position in the resulting Milnor elements yields the desired result. We leave this verification to the interested reader.

Proof of Theorem 2.9. Let k be the largest integer such that $n \equiv -1 \pmod{2^k}$. We can write n uniquely in the form $n = 2^k a - 1$ for some odd integer $a \geq 1$. For each $1 \leq h \leq k$ define an h–tuple $R_{n,h} = (r_{n,h,1}, r_{n,h,2}, \dots, r_{n,h,h})$ by

$$r_{n,h,i} = \begin{cases} 2^{k-i}a + 1 & \text{if } 1 \leq i < h \\ 2^{k-i+1}a - 1 & \text{if i=h} \end{cases}$$

For example, for $n = 47$ we have

$$\begin{aligned} R_{47,1} &= (47) \\ R_{47,2} &= (25, 23) \\ R_{47,3} &= (25, 13, 11) \\ R_{47,4} &= (25, 13, 7, 5) \end{aligned}$$

We now wish to show that $\mathrm{Sq}(R_{n,h})$ is a summand of $\mathrm{Sq}(n)^h$ for $1 \leq h \leq k$, and thus that $\mathrm{Sq}(n)^k \neq 0$.

We proceed by finite induction on h. If $h = 1$ then $\mathrm{Sq}(R_{n,1}) = \mathrm{Sq}(n)$, which is clearly a summand of $\mathrm{Sq}(n)^1$. Assume as the induction hypothesis that $\mathrm{Sq}(R_{n,h})$ is a summand of $\mathrm{Sq}(n)^h$ where $h < k$. Suppose $\mathrm{Sq}(R_{n,h+1})$ is a summand of $\mathrm{Sq}(n)\mathrm{Sq}(T)$ for some summand $\mathrm{Sq}(T) = \mathrm{Sq}(t_1, \dots, t_h)$ of $\mathrm{Sq}(n)^h$. Let $X = (x_{ij})$ be the associated matrix. Then by (4.4)

$$x_{1h} = r_{n,h+1,h+1} = 2^{k-h}a - 1$$

and from (4.2)

$$\begin{aligned} n &= \sum_{j=0}^{h} 2^j x_{1j} \\ &= \left(\sum_{j=0}^{h-1} 2^j x_{1j} \right) + 2^h x_{1h} \\ &= \left(\sum_{j=0}^{h-1} 2^j x_{1j} \right) + 2^h \left(2^{k-h}a - 1\right) \\ &= \left(\sum_{j=0}^{h-1} 2^j x_{1j} \right) + n - 2^h + 1. \end{aligned}$$

From which we obtain

$$\sum_{j=0}^{h-1} 2^j x_{1j} = 2^h - 1$$

But once again using the fact from the proof of Lemma 4.6 that x_{1j} is odd for $1 \leq j \leq h$ we conclude that (4.2) is satisfied if and only if $x_{1j} = 1$ for $1 \leq j < h$ (assuming $x_{1h} = 2^{k-h}a - 1$). But from (4.1) and (4.4) with $1 \leq j < h$ we have

$$\begin{aligned} t_j &= x_{0j} + x_{1j} \\ &= (r_{n,h+1,j} - 1) + 1 \\ &= 2^{k-j}a + 1 \\ &= r_{n,h,j} \end{aligned}$$

and

$$\begin{aligned} t_h &= x_{0h} + x_{1h} \\ &= (r_{n,h+1,h} - 1) + r_{n,h+1,h+1} \\ &= ((2^{k-h}a + 1) - 1) + (2^{k-(h+1)+1}a - 1) \\ &= 2^{k-h+1}a - 1 \\ &= r_{n,h,h} \end{aligned}$$

Thus we have shown that if $\mathrm{Sq}(R_{n,h+1})$ is a summand of $\mathrm{Sq}(n)\mathrm{Sq}(T)$ for some summand $\mathrm{Sq}(T) = \mathrm{Sq}(t_1, \ldots, t_h)$ of $\mathrm{Sq}(n)^h$ then $\mathrm{Sq}(T) = \mathrm{Sq}(R_{n,h})$. But by our very construction the matrix X satisfies (4.1) and (4.2) for the product $\mathrm{Sq}(n)\mathrm{Sq}(R_{n,h})$. It also satisfies (4.3) as $r_{n,h,j}$ is always odd and therefore the multinomial coefficient $(1, r_{n,h,j} - 1)$ is odd also. Thus X is $\mathrm{Sq}(n)\mathrm{Sq}(R_{n,h})$–allowable and $\mathrm{Sq}(R_{n,h+1})$ is a summand of $\mathrm{Sq}(n)^{h+1}$, completing the induction and the proof. □

Proof of Theorem 2.10. By Theorem 2.7 we have $\mathrm{Nil}(\mathrm{Sq}(2^k - 1)) \leq k + 1$ and by Theorem 2.9 $\mathrm{Nil}(\mathrm{Sq}(2^k - 1)) > k$. Therefore $\mathrm{Nil}(\mathrm{Sq}(2^k - 1)) = k + 1$. By (2.5)

$$\begin{aligned} \lambda^{(m-1)}\left(\mathrm{Sq}(2^{k+1} - 1)\right) &= \mathrm{Sq}\left(2^{m-1}(2^{k+1} - 1) - (2^{m-1} - 1)\right) \\ &= \mathrm{Sq}\left(2^m(2^k - 1) + 1\right) \end{aligned}$$

for every $m \geq 1, k \geq 0$. Thus

$$\begin{aligned} \mathrm{Nil}\left(\mathrm{Sq}\left(2^m(2^k - 1) + 1\right)\right) &= \mathrm{Nil}\left(\lambda^{(m-1)}\left(\mathrm{Sq}\left(2^{k+1} - 1\right)\right)\right) \\ &= \mathrm{Nil}\left(\mathrm{Sq}\left(2^{k+1} - 1\right)\right) \\ &= k + 2 \end{aligned}$$

□

In order to prove Theorem 3.2 we must first recall the following information from [Mi]. Let $\mathcal{A}_*$ be the Hopf dual of $\mathcal{A}$. $\mathcal{A}_*$ is isomorphic to the polynomial algebra $\mathbb{Z}_2[\xi_1, \xi_2, \dots]$ on generators ξ_i in dimension $2^i - 1$. If $R = (r_1, \dots, r_m)$ we will write ξ^R to mean the monomial $\xi_1^{r_1} \cdots \xi_m^{r_m}$. The basis of monomials ξ^R in $\mathcal{A}_*$ is dual to the Milnor basis for $\mathcal{A}$. As is common we will write $\langle x, y \rangle$ for the evaluation of $y \in \mathcal{A}_*$ on $x \in \mathcal{A}$. Thus

$$\langle \mathrm{Sq}(R), \xi^S \rangle = \begin{cases} 1 & \text{if } R = S \\ 0 & \text{otherwise} \end{cases}$$

The algebra homomorphism $\phi : \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$ by

$$\phi(\xi_k) = \sum_{i+j=k} \xi_i^{2^j} \otimes \xi_j$$

is the dual of the product map in $\mathcal{A}$.

Let E be the exterior subalgebra of $\mathcal{A}$ generated by $\{\, Q_i \mid i \in \mathbb{N} \,\}$. There is a doubling isomorphism $D : \mathcal{A} \to \mathcal{A}//E$ given by

$$D\left(\mathrm{Sq}(s_1, s_2, \dots)\right) = [\mathrm{Sq}(2s_1, 2s_2, \dots)]$$

where $[x]$ denotes the equivalence class in $\mathcal{A}//E$ of $x \in \mathcal{A}$.

Finally, let $\mathcal{A}_n$ be the subalgebra of $\mathcal{A}$ generated by $\left\{\, \mathrm{Sq}\left(2^i\right) \mid i \le n \,\right\}$.

Proof of Theorem 3.2. Let $n, t \in \mathbb{N}$, $t \neq 0$. For each $i \in \mathbb{N}$ let j_i and ϵ_i be the unique integers satisfying $i = 2j_i + \epsilon_i$ where $\epsilon_i \in \{0, 1\}$. Define an integer sequence

$$R_{n,t}(i) = (r_{i,1}, r_{i,2}, r_{i,3}, \dots)$$

recursively on i so that it satisfies the three conditions

$$R_{n,t}(1) = (2^{nt}, 0, 0, \dots) \tag{4.12}$$

$$r_{i,k} = \begin{cases} 2^{-t} r_{i-1,k-1} & \text{if } i \text{ is even} \\ r_{i-1,k} & \text{if } i \text{ is odd} \end{cases} \quad \text{for } k > 1 \text{ and } i > 1 \tag{4.13}$$

$$\sum_{k=1}^{j_i+1} r_{i,k} = 2^{nt+\epsilon_i}. \tag{4.14}$$

Notice that (4.14) is used to compute $r_{i,1}$ after obtaining $r_{i,k}$ for $k > 1$ from (4.13). For example, for $n = 3$ and $t = 2$ (dropping trailing zeros)

$$\begin{aligned} R_{3,2}(1) &= (64) \\ R_{3,2}(2) &= (48, 16) \\ R_{3,2}(3) &= (112, 16) \\ R_{3,2}(4) &= (32, 28, 4) \\ R_{3,2}(5) &= (96, 28, 4) \\ R_{3,2}(6) &= (32, 24, 7, 1) \\ R_{3,2}(7) &= (96, 24, 7, 1) \end{aligned}$$

We will require the following implication of (4.14) for odd i.

$$\begin{aligned}
r_{i,1} &= 2^{nt+1} - \sum_{k=2}^{j_i+1} r_{i,k} \\
&= 2^{nt} + 2^{nt} - \sum_{k=2}^{j_i+1} r_{i-1,k} \\
&= 2^{nt} + \sum_{k=1}^{j_i+1} r_{i-1,k} - \sum_{k=2}^{j_i+1} r_{i-1,k} \\
&= 2^{nt} + r_{i-1,1}
\end{aligned}$$

Define the monomial $\xi^{R_{n,t}(i)} \in \mathcal{A}_*$ by $\xi^{R_{n,t}(i)} = \prod_{k=1}^{j_i+1} \xi_{kt}^{r_{i,k}}$ (notice this is not the same as the definition of ξ^R given before because of the kt subscript). Then for $i > 1$

$$\begin{aligned}
\phi\left(\xi^{R_{n,t}(i)}\right) &= \phi\left(\prod_{k=1}^{j_i+1} \xi_{kt}^{r_{i,k}}\right) \\
&= \prod_{k=1}^{j_i+1} \phi\left(\xi_{kt}\right)^{r_{i,k}} \\
&= \prod_{k=1}^{j_i+1} \left(\xi_{kt} \otimes 1 + \xi_{(k-1)t}^{2^t} \otimes \xi_t + \mathcal{S}_1\right)^{r_{i,k}} \\
&= \prod_{k=1}^{j_i+1} \left(\xi_{kt} \otimes 1 + \xi_{(k-1)t}^{2^t} \otimes \xi_t\right)^{r_{i,k}} + \mathcal{S}_2
\end{aligned}$$

where $\mathcal{S}_1$ is a sum of terms of the form $a \otimes b$ with $b \notin \{1, \xi_t\}$ and $\mathcal{S}_2$ is a sum of terms of the form $a \otimes b$ with $b \neq \xi_t^{2^{nt}}$.

Continuing this derivation with i even yields

$$\begin{aligned}
\phi\left(\xi^{R_{n,t}(i)}\right) &= \left(\prod_{k=1}^{j_i+1} \xi_{(k-1)t}^{2^t r_{i,k}}\right) \otimes \xi_t^{2^{nt}} + \mathcal{S}_3 \\
&= \left(\prod_{k=1}^{j_i+1} \xi_{(k-1)t}^{r_{i-1,k-1}}\right) \otimes \xi_t^{2^{nt}} + \mathcal{S}_3 \\
&= \left(\prod_{k=1}^{j_i} \xi_{kt}^{r_{i-1,k}}\right) \otimes \xi_t^{2^{nt}} + \mathcal{S}_3 \\
&= \xi^{R_{n,t}(i-1)} \otimes \xi_t^{2^{nt}} + \mathcal{S}_3
\end{aligned}$$

where $\mathcal{S}_3$ is a sum of terms of the form $a \otimes b$ with $b \neq \xi_t^{2^{nt}}$ because $\sum_{k=1}^{j_i+1} r_{i,k} = 2^{nt}$.

On the other hand, continuing the derivation with i odd yields

$$\begin{aligned}
\phi\left(\xi^{R_{n,t}(i)}\right) &= (\xi_t \otimes 1 + 1 \otimes \xi_t)^{r_{i,1}} \prod_{k=2}^{j_i+1} \left(\xi_{kt} \otimes 1 + \xi_{(k-1)t}^{2^t} \otimes \xi_t\right)^{r_{i,k}} + \mathcal{S}_2 \\
&= (\xi_t \otimes 1 + 1 \otimes \xi_t)^{2^{nt}} (\xi_t \otimes 1 + 1 \otimes \xi_t)^{r_{i-1,1}} \prod_{k=2}^{j_i+1} \left(\xi_{kt} \otimes 1 + \xi_{(k-1)t}^{2^t} \otimes \xi_t\right)^{r_{i,k}} + \mathcal{S}_2 \\
&= \left(\xi_t^{2^{nt}} \otimes 1 + 1 \otimes \xi_t^{2^{nt}}\right) (\xi_t \otimes 1 + 1 \otimes \xi_t)^{r_{i-1,1}} \prod_{k=2}^{j_i+1} \left(\xi_{kt} \otimes 1 + \xi_{(k-1)t}^{2^t} \otimes \xi_t\right)^{r_{i-1,k}} + \mathcal{S}_2 \\
&= \left(\xi_t^{r_{i-1,1}} \prod_{k=2}^{j_i+1} \xi_{kt}^{r_{i-1,k}}\right) \otimes \xi_t^{2^{nt}} + \left(\xi_t^{2^{nt}} \prod_{k=2}^{j_i+1} \xi_{(k-1)t}^{2^t r_{i-1,k}}\right) \otimes \xi_t^{2^{nt}} + \mathcal{S}_4 \\
&= \left(\prod_{k=1}^{j_i+1} \xi_{kt}^{r_{i-1,k}}\right) \otimes \xi_t^{2^{nt}} + \left(\xi_t^{2^{nt}+2^t r_{i-1,2}} \prod_{k=3}^{j_i+1} \xi_{(k-1)t}^{2^t r_{i-1,k}}\right) \otimes \xi_t^{2^{nt}} + \mathcal{S}_4 \\
&= \xi^{R_{n,t}(i-1)} \otimes \xi_t^{2^{nt}} + \xi_t^{2^{nt+1}} \left(\xi_t^{r_{i-3,1}} \prod_{k=3}^{j_i+1} \xi_{(k-1)t}^{2^t r_{i-1,k}}\right) \otimes \xi_t^{2^{nt}} + \mathcal{S}_4
\end{aligned}$$

where $\mathcal{S}_4$ is a sum of terms of the form $a \otimes b$ with $b \neq \xi_t^{2^{nt}}$ because $\sum_{k=1}^{j_i+1} r_{i-1,k} = 2^{nt}$ and in the last equality we have used the fact that $2^t r_{i-1,2} = r_{i-2,1} = 2^{nt} + r_{i-3,1}$ (taking $r_{0,1} = 0$).

Thus in both cases we have shown that

$$\phi\left(\xi^{R_{n,t}(i)}\right) = \left(\xi^{R_{n,t}(i-1)} + \xi'\right) \otimes \xi_t^{2^{nt}} + \mathcal{S}_5$$

where $\mathcal{S}_5$ is a sum of terms of the form $a \otimes b$ with $b \neq \xi_t^{2^{nt}}$ and ξ' is divisible by $\xi_t^{2^{nt+1}}$ so that its evaluation on all elements of $\mathcal{A}_{t(n+1)-1}$ is zero. This shows that for any $1 \leq i \leq 2n+1$

$$\left\langle \left(P_t^{nt}\right)^i, \xi^{R_{n,t}(i)} \right\rangle = \left\langle \left(P_t^{nt}\right)^{i-1}, \xi^{R_{n,t}(i-1)} \right\rangle \cdot \left\langle P_t^{nt}, \xi_t^{2^{nt}} \right\rangle = \left\langle \left(P_t^{nt}\right)^{i-1}, \xi^{R_{n,t}(i-1)} \right\rangle . \tag{4.15}$$

Noting that $\langle P_t^{nt}, \xi^{R_{n,t}(1)} \rangle = \left\langle P_t^{nt}, \xi_t^{2^{nt}} \right\rangle = 1$ we can use (4.15) and finite induction on i to see that

$$\left\langle \left(P_t^{nt}\right)^i, \xi^{R_{n,t}(i)} \right\rangle \qquad \text{for all } 1 \leq i \leq 2n+1 \;.$$

Thus $\left(P_t^{nt}\right)^{2n+1} \neq 0$ for all $n, t \in \mathbb{N}$, $t \neq 0$.

Invoking the doubling isomorphism we notice that $D\left(P_t^s\right) = \left[P_t^{s+1}\right]$. Since D is an algebra isomorphism we have $D\left(\left(P_t^s\right)^i\right) = \left[\left(P_t^{s+1}\right)^i\right]$. Thus $\left(P_t^s\right)^i \neq 0 \implies D\left(\left(P_t^s\right)^i\right) \neq 0 \implies \left[\left(P_t^{s+1}\right)^i\right] \neq 0 \implies \left(P_t^{s+1}\right)^i \neq 0$. So by induction on w, $\left(P_t^s\right)^i \neq 0 \implies \left(P_t^{s+w}\right)^i \neq 0$ for all $w \in \mathbb{N}$. Since any s can be written uniquely as $s = nt + w$ with $n = \lfloor s/t \rfloor$ we see that $\left(P_t^{nt}\right)^{2n+1} \neq 0 \implies \left(P_t^{nt+w}\right)^{2n+1} \neq 0 \implies \left(P_t^s\right)^{2\lfloor s/t \rfloor + 1} \neq 0$ for all $s, t \in \mathbb{N}$, $t \neq 0$. $\square$

Finally, we can prove Theorem 3.3 by the following Lemma.

Lemma 3. *Let $s, t \in \mathbb{N}$ with $\lfloor s/t \rfloor = 1$ and let $w = s - t$. Then*

(1) $(P^s_t)^2 = \mathrm{Sq}(t_1, t_2, \dots)$ *where*

$$t_i = \begin{cases} 2^w(2^t - 1) & \text{if } i = t \\ 2^w & \text{if } i = 2t \\ 0 & \text{otherwise} \end{cases}$$

(2) $(P^s_t)^3 = \mathrm{Sq}(t_1, t_2, \dots)$ *where*

$$t_i = \begin{cases} 2^w(2^{t+1} - 1) & \text{if } i = t \\ 2^w & \text{if } i = 2t \\ 0 & \text{otherwise} \end{cases}$$

(3) $(P^s_t)^4 = 0$

The proof of this Lemma is an elementary, though tedious, exercise in using the product formula and we shall not present it here. Computer calculations indicate that an analogous method should work for the case $\lfloor s/t \rfloor = 2$ but that this method will not work for the case $\lfloor s/t \rfloor = 3$.

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