Braids, trees, and operads

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ABSTRACT. The space of unordered configurations of distinct points in the plane is aspherical, with Artin's braid group as its fundamental group. Remarkably enough, the space of ordered configurations of distinct points on the real projective line, modulo projective equivalence, has a natural compactification (as a space of equivalence classes of trees) which is also (by a theorem of Davis, Januszkiewicz, and Scott) aspherical. The classical braid groups are ubiquitous in modern mathematics, with applications from the theory of operads to the study of the Galois group of the rationals. The fundamental groups of these new configuration spaces are not braid groups, but they have many similar formal properties. This talk [at the Gdansk conference on algebraic topology 05-06-01] is an introduction to their study.

1. The bubbletree operad and quantum cohomology

1.0 Work on **conformal field theories** leads physicists to an interest in configuration spaces

$$\mathrm{Config}^{n+1}\mathbb{C}P_1 \sim \mathrm{Config}^{n}\mathbb{C}\ .$$

of points on the complex projective line. They are most interested in the **quotients** of these spaces by the action of $\mathrm{PGl}_2(\mathbb{C})$. The points are noncoincident, so both the spaces and the group are noncompact, and taking the quotient is tricky: it leads naturally to a compactification

$$\overline{\mathcal{M}}_{0,n}(\mathbb{C}) \sim \mathrm{Config}^{n}(\mathbb{C}P_1)/\mathrm{PGl}_2(\mathbb{C})\ .$$

The physicists discovered a **repulsive potential** among these points: pushing two together creates a bubble onto which they escape [13].

1.1 Thus $\overline{\mathcal{M}}_{0,n}(\mathbb{C})$ is the moduli space of marked genus zero **stable** algebraic curves (which have (at worst) double points, and at least three marked points on each irreducible component).

Example: $\overline{\mathcal{M}}_{0,4}(\mathbb{C}) \cong \mathbb{C}P_1$ via the classical cross-ratio. Note, $\overline{\mathcal{M}}_{0,3}(\mathbb{C})$ is a point: a configuration of three points on $\mathbb{C}P_1$ is **rigid**.

These spaces are very nice in some ways: they are compact manifolds, with cohomology concentrated in even dimension, and no torsion [8].

1991 *Mathematics Subject Classification.* Primary 55R80, Secondary 14N35, 20F36.
The author was supported in part by the NSF.

Operads, by example:

1.2 An operad $\mathcal{O}_* = \{\mathcal{O}_k, k \geq 1\}$ is a collection of spaces together with some **composition** maps

$$\mathcal{O}_n \times \mathcal{O}_{i_1} \times \cdots \times \mathcal{O}_{i_n} \to \mathcal{O}_i$$

(where $i = \sum i_k$) satisfying some axioms ...

ex. i) $\overline{\mathcal{M}}_{0,*+1}(\mathbb{C})$

ex. ii) $\mathrm{End}_n(X) = \mathrm{Maps}(X^n, X)$

is the **endomorphism operad** of an object X in a monoidal category. Composition sends a map from X^n to X and a tuple of maps from X^{i_*} to X to a composition

$$X^i = X^{i_1} \times \ldots X^{i_n} \to X^n \to X \ .$$

A morphism $\mathcal{O}_* \to \mathrm{End}_*(X)$ of operads makes X into an $\mathcal{O}_*$-**algebra**.

ex. iii) Br_n = Artin's braid group on n strings

defines the **braid operad**, with **cabling**

$$\mathrm{Br}_n \times \mathrm{Br}_{i_1} \times \ldots \mathrm{Br}_{i_n} \to \mathrm{Br}_i$$

as composition.

Monoidal functors preserve operads; hence the homology of an operad (in spaces) is an operad in graded modules [9].

1.3 Theorem (WDVV, Kontsevich [5]): The (rational) homology of a smooth projective algebraic variety V is an $H_*(\overline{\mathcal{M}}_{0,*+1}(\mathbb{C}))$-operad algebra.

[This led to the solution of the 19th-century enumerative geometry problem of classification of lines with specified incidence in $\mathbb{C}P_2$.]

The **construction** of this algebra structure uses the Gromov-Witten invariants: There are (compact) moduli spaces $GW_k(V)$ of holomorphic maps from genus zero stable curves with k marked points, to V. These spaces have many components, indexed by degree $[h : C \to V] \in H_2(V, \mathbb{Z})$. There is also an **evaluation** map

$$GW_k(V) \to \overline{\mathcal{M}}_{0,k}(\mathbb{C}) \times V^k$$

which defines a cycle

$$GW_k \in H_*(\overline{\mathcal{M}}_{0,k}(\mathbb{C})) \otimes H_*(V)^{\otimes k} \ .$$

[Actually the coefficients lie in the Novikov ring $\Lambda = \mathbb{Q}[H_2(V, \mathbb{Z})]$, but this will be supressed.] Using Poincaré duality, we can rewrite GW_{k+1} as an element of

$$\mathrm{Hom}(H_*(\overline{\mathcal{M}}_{0,k+1}(\mathbb{C})), \mathrm{Hom}(H_*(V)^{\otimes k}, H_*(V)))$$

which then defines a morphism

$$H_*(\overline{\mathcal{M}}_{0,k+1}(\mathbb{C})) \to \mathrm{End}_k(H_*(V))$$

of operads, QED.

In particular, the point $\overline{\mathcal{M}}_{0,3}(\mathbb{C})$ defines a **quantum multiplication**

$$H_*(V, \Lambda) \otimes_\Lambda H_*(V, \Lambda) \to H_*(V, \Lambda)$$

which is usually not standard ...

2. Devadoss's mosaic operad and Fukaya's Lagrangian cohomology

2.0 The moduli space [2]

$$\mathrm{Config}^n(\mathbb{R}P_1)/\mathrm{PGl}_2(R) \sim \overline{\mathcal{M}}_{0,n}(\mathbb{R})$$

of configurations of points on the circle can be pictured as a space of **trees** or **mosaics** of hyperbolic polygons. The points have an intrinsic cyclic order, and $\{\overline{\mathcal{M}}_{0,*}(\mathbb{R})\}$ is naturally a **cyclic** operad [6]. I am indebted to Sasha Voronov, for pointing out that the analogous complex operad is also cyclic!

2.1 Fukaya considers a compact symplectic manifold (M, ω) together with an oriented Lagrangian submanifold L (i.e. of half the dimension of M, such that $\omega|_L = 0$; some subtle issues involving the Stiefel-Whitney class $w_2(L)$ will be ignored.) I **conjecture** that the following is a theorem; something slightly weaker (cf. below) has been proved by Fukaya and his school [4]:

For a generic almost-complex structure compatible with ω, there are compact oriented moduli spaces FO_k of pseudo-holomorphic hyperbolic polygons

$$(P, \partial P) \to (M, L)$$

together with evaluation maps

$$FO_k \to \overline{\mathcal{M}}_{0,k}(\mathbb{R}) \times L^k$$

which define an action of $H_*(\overline{\mathcal{M}}_{0,*+1}(\mathbb{R}))$ on $H_*(L, \Lambda)$ (where now $\Lambda = \mathbb{Q}[H_2(M, \mathbb{Z})]$).

2.2 The (co)homology of these spaces is **not** known (but cf. [14]). However, we can draw some beautiful pictures [2,3].

Grothendieck, in his *Esquisse*, says $\overline{\mathcal{M}}_{0,5}(\mathbb{C})$ is 'un petit joyaux'. Its real points $\overline{\mathcal{M}}_{0,5}(\mathbb{R})$ map to $\overline{\mathcal{M}}_{0,4}(\mathbb{R}) \times \overline{\mathcal{M}}_{0,4}(\mathbb{R}) = T^2$ by selecting two distinct subsets of four points. To get the full space, we need to blow up (ie, add crosscaps) at the three configurations defined by triple coincidences, resulting in $T^2 \# 3\mathbb{R}P^2$.

There is a more symmetric picture, defined by blowing up four points on $\mathbb{R}P^2$. Both pictures are **tesselated** by pentagons, though this is easier to see in the second picture. This is a regular polytope with twelve pentagonal faces: it is the dodecahedron's 'evil twin'.

2.3 In general, there is a surjective map

$$\Sigma_k \times_{D_k} K_{k-3} \to \overline{\mathcal{M}}_{0,k}(\mathbb{R})$$

(where D_k is the dihedral group of order $2k$) which is 2^n to 1 on codimension n faces: in general, these moduli spaces are tesselated by Stasheff **associahedra**.

There is a commutative diagram

$$\begin{array}{ccc} \mathrm{Config}^*(\mathbb{R}) & \longrightarrow & \mathrm{Config}^*(\mathbb{C}) \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{0,*+1}(\mathbb{R}) & \longrightarrow & \overline{\mathcal{M}}_{0,*+1}(\mathbb{C}) \end{array}$$

The space in the upper right corner is homotopy-equivalent to the little disks operad, and the space in the upper left corner is the classical A_∞ operad $\{\Sigma_* \times K_{*-1}\}$ (made **permutative**, ie endowed with an action of the symmetric group. The left vertical map defines the tesselation; thus the mosaic operad is a kind of (quasi-commutative) quotient of the A_∞ operad. Fukaya shows that the A_∞ operad acts on Floer cohomology, but I believe that action passes through this quotient. The diagram above is a fiber product of spaces, but it is not quite a fiber product of operads.

2.4 Theorem of Davis, Januszkiewicz, and Scott [1]: this tesselation defines a piecewise negatively curved metric on $\overline{\mathcal{M}}_{0,*}(\mathbb{R})$; these spaces are therefore $K(\pi, 1)$'s!

Remark: Devadoss [3] has recently shown (using similar methods) that the Fulton-MacPherson compactification of $\mathrm{Config}^*(\mathbb{R}P_1)$ is also aspherical!

3. Operads in groups (and groupoids)

3.0 Observation: The fundamental group of an operad is an operad in groups ... provided you're careful about basepoints.

Example $\{1, \ldots, n \in \mathbb{C}\}$ (with that order) defines a basepoint $* \in \mathrm{Config}^n(\mathbb{C})$; but the natural action of Σ_n moves it around (by changing the order).

Recall that a space has a fundamental **groupoid**, with respect to a system of basepoints: it is a category, with the points as objects, and homotopy classes of paths between them as morphisms.

Note also that a surjective homomorphism $\phi : G \to H$ of groups defines a groupoid $[H/G]$ with H as set of objects, and

$$\mathrm{mor}(h_0, h_1) = \{g \in G \mid \phi(g)h_0 = h_1\}$$

as morphisms. With this notation,

$$\pi(\mathrm{Config}^n(\mathbb{C}) \text{ rel } \Sigma_n(*)) \cong [\Sigma_n/\mathrm{Br}_n]$$

where $\mathrm{Br}_n \to \Sigma_n$ is the standard homomorphism. Thus the fundamental groupoid of the little disks operad is the braid operad. I owe thanks to Dan Christensen and J.P. Meyer, for explaining this to me.

3.1 Definition: The braid **category B** has integers n as objects, with Br_n as its endomorphisms. [There are no morphisms between distinct integers.] This is a (universal) braided monoidal category, with tensor product $\mathbf{B} \times \mathbf{B} \to \mathbf{B}$ defined by juxtaposition $(n, m) \mapsto n + m$.

A standard construction [7] defines a functor

$$[\Sigma_n/\mathrm{Br}_n] \to \mathrm{Func}(\mathbf{B}^n, \mathbf{B})$$

which makes the category $\mathbf{B}$ an **algebra** over the braid operad. More generally, any braided monoidal category is an algebra over the braid operad.

The operad $\overline{\mathcal{M}}_{0,*+1}(\mathbb{R})$ defines a similar category: there is an exact sequence

$$\pi_1(\overline{\mathcal{M}}_{0,*+1}(\mathbb{R})) \rightarrowtail \pi_1(\overline{\mathcal{M}}_{0,*+1}(\mathbb{R})_{h\Sigma_*}) \twoheadrightarrow \Sigma_*$$

in which the fundamental group of the homotopy quotient plays the role of the braid group. There is a similar tensor category **D**, which is a kind of universal example of an algebra over the associated operad in groupoids.

3.2 Here are some questions and speculations:

i) do these fundamental groups act in some natural way on Fukaya's cohomology?

ii) does **D** have an interpretation in terms of cyclic operads with a trace in the sense of Markl [11]?

iii) are these fundamental groups in some sense Galois groups for solutions of Calogero-Moser systems [of points moving on the line [12]] analogous to the role played by the braid groups in the Knizhnik-Zamolodchikov equations?

iv) The Grothendieck-Teichmüller group [10] is in some sense the automorphisms of the braid operad. Do automorphisms of the operad in groupoids defined by the moduli spaces $\overline{\mathcal{M}}_{0,*}(\mathbb{R})$ have any similar properties?

iv) Does the rank of $H_1(\overline{\mathcal{M}}_{0,k+1}(\mathbb{R}))$ equal $\binom{n}{3}$?

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