

Cobordism of involutions revisited, revisited

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ABSTRACT. Boardman’s work [3,4] on Conner and Floyd’s five-halves conjecture looks remarkably contemporary in retrospect. This note reexamines some of that work from a perspective proposed recently by Greenlees and Kriz.

1. Unoriented involutions

A large part of Boardman’s argument can be summarized as a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{N}_{\mathbb{Z}/2\mathbb{Z}*}^{geo} & \longrightarrow & \bigoplus_{i \geq 0} \mathcal{N}_i(BO_{*-i}) & \xrightarrow{\partial} & \tilde{\mathcal{N}}_{*-1}(B\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0 \\
 & & \downarrow & & \downarrow J & & \downarrow \\
 0 & \longrightarrow & \mathcal{N}_*[[w]] & \longrightarrow & \mathcal{N}_*((w)) & \longrightarrow & \mathcal{N}_{*-1}[w^{-1}] \longrightarrow 0 .
 \end{array}$$

of short exact sequences: here $\mathcal{N}_*((w))$ is the ring of homogeneous formal Laurent series over the unoriented cobordism ring, in an indeterminate w of cohomological degree one; such series are permitted only finitely many negative powers of w . The lower left-hand arrow is inclusion of the ring of formal power series, and the lower right-hand arrow is the quotient homomorphism. The maps across the top are geometric: the left-hand arrow sends the class of an unoriented manifold with involution to its class in the relative bordism of unoriented $\mathbb{Z}/2\mathbb{Z}$ -manifolds-with-boundary with action free on the boundary, while the right-hand arrow ∂ sends such a manifold to its boundary, regarded as a manifold with involution classified by a map to $B\mathbb{Z}/2\mathbb{Z}$. It is a theorem of Conner and Floyd [7 §22] that such a relative manifold with involution is cobordant to the normal ball bundle of its fixed-point set, so the group in the top row center is the sum of bordism groups of classifying spaces for orthogonal groups, as indicated.

REMARKS:

- i) Thom showed that unoriented cobordism is a generalized Eilenberg-Mac Lane spectrum, but this is not true of equivariant unoriented bordism, cf. [8];
- ii) the homomorphism ∂ is **not** a derivation;

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iii) the right-hand vertical arrow is an isomorphism, while the left-hand vertical arrow sends a closed manifold with involution V to the bordism class of its Borel construction, regarded as an element of $\mathcal{N}^{-*}(B\mathbb{Z}/2\mathbb{Z})$, cf. [13];

iv) the middle vertical homomorphism J is a ring homomorphism; its construction [4, Theorem 9] is one of Boardman's innovations. He concludes that it is in fact a monomorphism [4, Corollary 17], so $\mathcal{N}_{\mathbb{Z}/2\mathbb{Z}*}^{geo}$ might be described as the subring of classes $[V]$ in the relative bordism group such that $J[V]$ is 'holomorphic' in w . The homomorphism J is constructed using an inverse system of truncated projective spaces; around the same time Mahowald [1,11] called attention to the remarkable properties of a similar construction in **framed** bordism, and went on to consider the properties of this inverse limit ...

v) The class of the interval $[-1, +1]$ (with sign reversal as involution) defines an element of the relative bordism group which maps under J to w^{-1} , cf. [5].

In modern terminology the diagram displays the Tate $\mathbb{Z}/2\mathbb{Z}$ -cohomology [[9]; Swan probably also deserves mention here, cf. [14]] of the forgetful map from geometric $\mathbb{Z}/2\mathbb{Z}$ -equivariant unoriented cobordism to ordinary unoriented cobordism; this is very closely related to the construction used by Kriz [10 Corollary 1.4] to compute the homotopy-theoretic $\mathbb{Z}/p\mathbb{Z}$ -equivariant bordism groups $MU_{\mathbb{Z}/p\mathbb{Z}*}^{hot}$. In fact the direct limit of the system defined by suspending the diagram with respect to a complete family of $\mathbb{Z}/2\mathbb{Z}$ -representations (cf. [15]) has top row

$$0 \rightarrow \mathcal{N}_{\mathbb{Z}/2\mathbb{Z}*}^{hot} \rightarrow \mathcal{N}_*(BO)[u, u^{-1}] \rightarrow \tilde{\mathcal{N}}_*(B\mathbb{Z}/2\mathbb{Z}) \rightarrow 0,$$

which is identical [aside from obvious modifications] with Kriz's. It follows, in particular, that the stabilization map

$$\mathcal{N}_{\mathbb{Z}/2\mathbb{Z}*}^{geo} \rightarrow \mathcal{N}_{\mathbb{Z}/2\mathbb{Z}*}^{hot}$$

is injective.

2. Complex circle actions

Boardman has an explicit formula for his J -homomorphism: if $[V]$ is the cobordism class of a closed manifold with involution, then

$$J[V] = \pi^{-1} \sum_{k \geq 0} w^k p_* \partial(w^{-k-1}V),$$

where $p_* : \mathcal{N}_*(B\mathbb{Z}/2\mathbb{Z}) \rightarrow \mathcal{N}_*$ sends a manifold with free involution to its quotient, and

$$\pi := \sum_{k \geq 0} [\mathbb{R}P_k] w^k,$$

by [4 Theorem 14]. This formula has a natural interpretation in terms of formal group laws, but the idea is easier to explain in the context of the \mathbb{T} -equivariant Tate cohomology of MU , which fits in an exact sequence

$$0 \rightarrow MU^{-*}(B\mathbb{T}) \rightarrow t_{\mathbb{T}}(MU_*) = MU_*((c)) \rightarrow MU_{*-1}(B\mathbb{T}) \rightarrow 0;$$

here the Chern class c plays the role of the Stiefel-Whitney class w in the unoriented case. Since the complex cobordism ring is torsion-free, we can introduce denominators with impunity: the analogue of w^{-1} is the class c^{-1} of the two-disk, viewed

as a complex-oriented manifold with circle action free on the boundary. Boardman observes that

$$p_*\partial w^{-k-1} = [\mathbb{R}P_k] ;$$

similarly, we have

$$p_*\partial c^{-k-1} = [\mathbb{C}P_k]$$

in the complex-oriented situation.

Proposition: The homomorphism $p_*\partial$ is the formal residue at the origin with respect to an additive coordinate for the formal group law of MU .

Proof: The idea is to write c as a formal power series

$$c = z + \text{terms of higher order ,}$$

with coefficients chosen so that

$$\text{res}_{z=0}c^{-k-1} = [\mathbb{C}P_k] .$$

Clearly

$$\sum_{k \geq 0} \text{res}_0 c^{-k-1} u^k = \text{res}_0 \frac{c^{-1}}{1 - c^{-1}u} = \sum_{k \geq 0} [\mathbb{C}P_k] u^k = \log'_{MU}(u) .$$

On the other hand, if $c = f(z)$ then

$$\text{res}_0(f(z) - u)^{-1} = \frac{1}{2\pi i} \int \frac{dz}{f(z) - u}$$

which equals

$$\frac{1}{2\pi i} \int \frac{(f^{-1})'(c)dc}{c - u} = (f^{-1})'(u) ;$$

thus $c = \exp_{MU}(z)$.

REMARKS:

i) In the context of remark iv) above, Boardman's formula for J on 'holomorphic' elements V is thus essentially the same as Cauchy's. His use of the symbol π suggests that this analogy was not far from his mind.

ii) The formal residue was introduced in Quillen's Bulletin announcement [12], but seems to have since disappeared from algebraic topology. Perhaps that paper deserves another look as well.

iii) These constructions suggest that while the Chern class c is a natural uniformizing parameter for algebraic questions about complex cobordism, its inverse may be more natural for geometric questions. We thus have two reasonable coordinates on cobordism, centered at the south and north poles of the Riemann sphere, overlapping in the temperate region defined by Tate cohomology.

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