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Schur Q-functions and a Kontsevich-Witten genus

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Abstract. The Virasoro operations in Witten's theory of two-dimensional topological gravity have a homotopy-theoretic interpretation as endomorphisms of an **ordinary** cohomology theory with coefficients in a localization of I. Schur's ring Δ of Q-functions. The resulting theory has many of the features of a vertex operator algebra.

Introduction

Smooth complex curves of genus $g > 1$ form a stack $\mathcal{M}_g$ with a compactification defined by adjoining the divisor of stable singular curves (with double points, but only finitely many automorphisms). Ideas from string theory have led to great progress in understanding the topology of these moduli objects in the large genus limit. This note is concerned with an algebra introduced by Schur in connection with the classification of projective representations of symmetric groups, and its relevance to these spaces. This algebra appears in disguise in classical work on the Riemann moduli space and in more recent work of Witten and Kontsevich on the intersection theory of its Deligne-Mumford compactification $\overline{\mathcal{M}}_g$, but its significance has become clear only in retrospect, as integral cohomology emerges from the fog of $\mathbb{Q}$.

The central construction of topological gravity is a partition function which can be defined geometrically by a family

$$\tau_g : \overline{\mathcal{M}}_g \to \mathbf{MU}\hat{\otimes}\mathbb{Q}[v]$$

of maps to the complex cobordism spectrum tensored with the rationals, and the main result of this paper is the construction of a morphism

$$\mathrm{kw} : \mathbf{MU} \to \mathbf{H}(\Delta[q_1^{-1}])$$

of ring-spectra, which (when composed with τ) sends the fundamental class of the moduli space of stable curves to a highest-weight vector for a naturally defined Virasoro action on $\Delta_{\mathbb{Q}}$. This Kontsevich-Witten genus can be constructed in purely algebraic terms from the theory of formal groups, but its natural context is the topology of moduli spaces. The first section below is a summary of some background from algebraic geometry, emphasizing homotopy theory, while the Hopf algebra of

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Q-functions is the topic of the second. This whole subject is in many ways still quite mysterious, and a final section argues that some features are most natural in an equivariant context. An appendix summarizes the construction of a vertex operator algebra following ideas of A. Baker.

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§1 Background from geometry

1.1. Teichmüller theory describes the Riemann moduli space $\mathcal{M}_g$ as the quotient of a complex $3(g-1)$-dimensional cell by an action of the mapping class group

$$\Gamma_g = \pi_0 \mathrm{Diff}^+(\Sigma_g)$$

of isotopy classes of orientation-preserving diffeomorphisms of a closed surface Σ_g; this action is properly discontinuous and almost free, in the sense that its isotropy groups are finite, so the resulting quotient is an orbifold. The components of the group of diffeomorphisms are contractible, and the map

$$B\mathrm{Diff}^+(\Sigma_g) = B\Gamma_g \to \mathcal{M}_g$$

from the homotopy-theoretic quotient of the Teichmüller action to the geometric quotient is a rational homology isomorphism.

There is no such essentially topological description of the compactification $\overline{\mathcal{M}}_g$, but there are very interesting (proper) forgetful maps

$$\Phi_g^n : \overline{\mathcal{M}}_g^n \to \overline{\mathcal{M}}_g$$

defined on the moduli stack of stable curves marked with n ordered smooth points. [A marked curve is stable, as above, if it has only finitely many automorphisms; the loss of the marking may render an irreducible component of the curve unstable, and the forgetful map is understood to contract such a component to a point.] According to the conventions of Quillen, such a proper complex-oriented map between homology manifolds defines an element

$$[\Phi_g^n] \in MU_{\mathbb{Q}}^{-2n}(\overline{\mathcal{M}}_g)$$

of the ring (tensored with $\mathbb{Q}$) of complex cobordism classes of maps to the moduli space. These cobordism classes can be identified in terms of ordinary cohomology by the ring isomorphism

$$MU_{\mathbb{Q}}^*(X) \to H^*(X, \mathbb{Q}[t_k | k \geq 1])$$

which sends $[\Phi : V \to X]$ to the characteristic number polynomial

$$\sum_I \Phi_* m_I(T_\Phi^*) t^I \ ,$$

where T_Φ^* is the formal cotangent bundle along Φ and Φ_* is the Gysin or transfer map in cohomology; $I = i_1, \dots$ is a multi-index, $t^I = \prod t_k^{i_k}$, and m_I denotes the characteristic class associated to the monomial symmetric function indexed by the partition of $|I| = \sum k i_k$ defined by I. [The t_k correspond to the complete symmetric functions h_k but we keep the standard notation.] This differs from the usual [29 §6.2] definition, which is expressed in terms of the normal bundle rather than the cotangent bundle, but the two constructions are related by a straightforward change of basis for the characteristic classes, and the choice above leads most directly to the formulas of Witten.

1.2. The action of a diffeomorphism of the surface Σ_g on its homology defines an integral symplectic representation of Γ_g, and thus a map

$$B\text{Diff}^+(\Sigma_g) \to BSp(2g, \mathbb{Z}) \; .$$

The group $Sp(2g, \mathbb{Z})$ acts with finite isotropy on the contractible symmetric space $Sp(2g, \mathbb{R})/\mathbb{U}(g)$, with quotient the space $\mathcal{A}_g$ of g-dimensional principally polarized abelian varieties, so the homomorphism induced by this map on rational homology agrees with that defined by the construction which assigns to a Riemann surface its Jacobian. By stability theorems of Harer and Ivanov, these maps are suitably compatible for increasing g, and we can think of them as taking values in the classifying space $BSp(\mathbb{Z})$ of the infinite integral symplectic group. The composition of the obvious maps

$$BSp(\mathbb{Z}) \to BSp(\mathbb{R}) = B\mathbb{U} \to B(\mathbb{U}/\mathbb{O}) = Sp(\mathbb{H})/\mathbb{U}$$

with a final Bott isomorphism is a rational homology isomorphism which (away from two [20 §3.15]) splits a copy of $Sp(\mathbb{H})/\mathbb{U}$ off the group completion $BSp(\mathbb{Z})^+$, cf. [5,30].

1.3.1. One of the main contentions of this paper is that the theory of topological gravity defines maps to the complex cobordism spectrum which are natural analogues, for the compactifications $\overline{\mathcal{M}}_g$, of the classical Abel-Jacobi map described above. The physicists' definitions are motivated by ideas from statistical mechanics: using the language of cobordism, let

$$\tau_g = \sum_{g \geq 0} [\Phi_g^n] \frac{v^n}{n!} \in MU^0_{\mathbb{Q}[v]}(\overline{\mathcal{M}}_g) \; ,$$

with v a bookkeeping variable of (cohomological) degree two. The element τ_g is the class of the space of configurations defined by an indefinite number of distinct but unordered smooth points on a stable curve of genus g; it is tempting to think of this as the space of states of the 'Mumford gas' of free particles on a Riemann surface, with existence as their only attribute. The homotopy class representing this tau-function induces a homomorphism

$$\tau_{g*} : H_*(\overline{\mathcal{M}}_g, \mathbb{Q}) \to H_*(\mathbf{MU}, \mathbb{Q}[v]) = \mathbb{Q}[t_k | k \geq 1][[v]] \; ;$$

the image of the fundamental class $[\overline{\mathcal{M}}_g]$ of the moduli space under this homomorphism is essentially Witten's free energy F_g. [It is convenient to interpret homology with $\mathbb{Q}[v]$ coefficients to be v-adically completed; for a finite complex this implies no change at all.]

1.3.2. To be more precise about this, we will need the slightly more sensitive characteristic number homomorphism which assigns to $[\Phi] \in MU^{-2n}(X)$ the class

$$\text{cl}\; [\Phi] = \sum \Phi_* m_I(T^*_\Phi) t_0^{n-l(I)} t^I \; ,$$

where $l(I) = \sum_{k>0} i_k$ is the length (or number of parts) of the partition I. The ring $\mathbb{Q}[t_k | k \geq 0][t_0^{-1}]$ of cohomology coefficients has now been enlarged to include an invertible polynomial generator of degree zero; this corresponds [cf. §1.5 below] to a natural extension of the Landweber-Novikov algebra of cobordism operations. The monomial symmetric function m_I is a sum

$$\sum x^{d_1}_{\sigma(1)} \cdots x^{d_n}_{\sigma(n)} ,$$

where d_i is a finite sequence of nonnegative numbers with $\sum d_i = |I|$ in which k appears i_k times; it is convenient to think of this sequence as having exactly n terms, with zero appearing $i_0 = n - l(I)$ times. The stable cotangent bundle T^*_Φ along Φ^n_g is the sum of the cotangent line bundles L_i of the modular curve at its marked points, and the characteristic class $m_I(T^*_\Phi)$ is obtained by substituting Euler classes $\mathbf{e}(L_i)$ for the formal variables x_i. Witten's characteristic number [31 §2.4]

$$\langle \tau^I \rangle = (\prod_{1 \leq i \leq n} \mathbf{e}(L_i)^{d_i})[\overline{\mathcal{M}}^n_g]$$

is invariant under permutations of the marked points, so

$$m_I(T^*_\Phi)[\overline{\mathcal{M}}^n_g] = \frac{n!}{I!} \langle \tau^I \rangle ,$$

where $I! = \prod_{k \geq 0} i_k!$. This rational number is the image of

$$\Phi^n_{g*} m_I(T^*_\Phi) \in H^*(\overline{\mathcal{M}}_g, \mathbb{Q})$$

under the Gysin homomorphism defined by the map from $\overline{\mathcal{M}}_g$ to a point; the latter is just evaluation on the fundamental class of $\overline{\mathcal{M}}_g$, and the free energy [31 §2.16] can thus be recognized as the sum over g and n of terms of the form

$$\mathrm{cl}\ ([\Phi^n_g]/n!)[\overline{\mathcal{M}}_g] = \sum \langle \tau^I \rangle \frac{t^I}{I!} .$$

The classical Jacobian is similarly a configuration space of divisors, and the sum

$$j_g = \sum_{n \geq 0} [\mathrm{SP}^n C_g] v^n \in MU^0_{\mathbb{Q}[v]}(\mathcal{M}_g)$$

of symmetric powers of the modular family of curves is the pullback of the product of the universal torus bundle in $MU^{-2g}(BSp(2g, \mathbb{Z}))$ by the coupling constant

$$v^g \sum_{n \geq 0} [\mathbb{C}P(n)] v^n .$$

It is somewhat surprising that no expression seems to be known for the class of this universal bundle.

1.4. The construction τ can be extended to stable curves which are not necessarily connected. From the point of view of homotopy theory the space of unordered configurations of points in X is

$$Q(X) = \coprod_{n \geq 0} E\Sigma_n \times_{\Sigma_n} X^n$$

but over the rationals this has the homotopy type of the infinite symmetric product $\mathrm{SP}^\infty(X)$. We will therefore take

$$Q(\overline{\mathcal{M}}) = Q(\coprod_{g \geq 0} \overline{\mathcal{M}}_g)$$

as a model for the space of stable curves, connected or not; its rational homology is the symmetric tensor algebra on $\oplus_{g \geq 0} H_*(\overline{\mathcal{M}}_g, \mathbb{Q})$, with $\overline{\mathcal{M}}_0$ and $\overline{\mathcal{M}}_1$ interpreted as one-point spaces. It is natural to think of the fundamental homology class of the orbifold $\mathrm{SP}^n\overline{\mathcal{M}}_g$ as the quotient in this tensor algebra of $[\overline{\mathcal{M}}_g]^n$ by $n!$, so

$$[Q(\overline{\mathcal{M}})] = \exp(\sum_{g\geq 0}[\overline{\mathcal{M}}_g]v^{3(g-1)})$$

defines a kind of fundamental class for $Q(\overline{\mathcal{M}})$ with coefficients in $\mathbb{Q}[v]$. Because no marked curve is stable when $3(g-1)+n<0$, surfaces of small genus play a slightly anomalous role in these formulas; for example, it is useful to define $[\overline{\mathcal{M}}_1] = -\frac{1}{12}$.

The hard Lefschetz theorem (or the theory of mixed Hodge structures) defines an action of sl_2 on the rational homology of a projective orbifold, with its fundamental class as a highest weight vector, so the homology of $Q(\overline{\mathcal{M}})$ inherits an action of $sl_2[v,v^{-1}]$ in which the class $[Q(\overline{\mathcal{M}})]$ has conformal weight zero [18 §2.6]. The construction of τ_g extends multiplicatively to define an element

$$\tau \in MU^0_{\mathbb{Q}[v]}(Q(\overline{\mathcal{M}}))$$

which sends this fundamental class to the partition function

$$\tau_W = \tau_*[Q(\overline{\mathcal{M}})] = \exp(\sum_{g\geq 0}F_g) \in \mathbb{Q}[t_k|k\geq 1][[t_0,v]] \ .$$

These results from physics suggest that the (immensely complicated) moduli space of curves has quite interesting homotopy-theoretic approximations, but (unlike the somewhat similar situation in algebraic K-theory) we do not yet understand these stabilizations in terms of universal mapping properties. That τ takes the fundamental class of the moduli space to a highest-weight vector for the natural endomorphisms of $\Delta_{\mathbb{Q}}$ is the central result of Kontsevich-Witten theory, but it is not yet a characterization.

1.5. The parameters t_k have an intrinsic interpretation as polynomial generators for the extended Landweber-Novikov Hopf algebra

$$S = \mathbb{Z}[t_k|\ k\geq 0][t_0^{-1}]$$

of cooperations in complex cobordism. The universal stable cohomology operation is a ring homomorphism

$$MU^*(X) \to MU^* \otimes S$$

and the characteristic number homomorphism of §1.1 can be defined as the composition of this map with the Thom map from cobordism to ordinary cohomology. The universal operation sends the Euler class $\mathbf{e}$ of a complex line bundle to

$$t(\mathbf{e}) = \sum_{k\geq 0} t_k \mathbf{e}^{k+1} \ ;$$

it follows that the algebra S represents the group of formal origin-preserving diffeomorphisms of the line, and that the characteristic number homomorphism induces the classifying map for the universal formal group law

$$X +_S Y = t(t^{-1}(X) + t^{-1}(Y))$$

of additive type. The Lie algebra of formal vector fields on the line is closely related to the Virasoro algebra, but the cobordism ring is more closely related to the untwisted charge one basic representation [21] than to the representation [31 §2.59] defined by topological gravity.

§2 Background from algebra

2.1. The simplest definition of the algebra Δ of Q-functions is as the quotient of the polynomial algebra $\mathbb{Z}[q_k|k \geq 1]$ by the ideal generated by the coefficients of the relation

$$q(T^{\frac{1}{2}})q(-T^{\frac{1}{2}}) = 1 ,$$

where

$$q(T^{\frac{1}{2}}) = \sum_{k \geq 0} q_k T^{\frac{1}{2}k}$$

is a generating function with $q_0 = 1$ [14 §7]. This algebra has a natural grading, which does not fit very comfortably with the conventions of algebraic topology; we will assign the formal variable $T^{\frac{1}{2}}$ cohomological degree one, but we will not assume that elements of odd degree anticommute. The diagonal homomorphism

$$q_i \mapsto \sum_{i=j+k} q_j \otimes q_k$$

makes Δ into a bicommutative Hopf algebra over $\mathbb{Z}$, and the relation

$$(-1)^i q_i^2 = 2q_{2i} + 2 \sum_{i-1 \geq k \geq 1} (-1)^{k-1} q_k q_{2i-k}$$

implies that the square of a generator can be expressed as a sum of monomials in which no q_k appears with exponent greater than one. It follows that Δ is a free module over the integers, with a basis of square-free monomials; similarly, $\Delta[\frac{1}{2}]$ is a polynomial algebra generated by the elements q_{2k+1}, while the reduction of Δ modulo two is an exterior algebra. Being torsion-free, Δ embeds in $\Delta_{\mathbb{Q}} = \Delta \otimes \mathbb{Q}$, and its defining relations imply that the power series

$$\log q(T^{\frac{1}{2}}) = \sum_{i \geq 0} \frac{2x_k}{2k+1} T^{k+\frac{1}{2}}$$

contains only odd powers of $T^{\frac{1}{2}}$. Newton's identity

$$(2k+1)q_{2k+1} = 2(x_0 q_{2k} + x_1 q_{2k-2} + \cdots + x_k)$$

shows that the classes $2x_k$ are integral, e.g. $2x_0 = q_1$ and $2x_1 = 3q_3 - q_1 q_2$.

2.2. The generators q_i of Δ can be interpreted as specializations at $t = -1$ of Hall-Littlewood symmetric functions $q_k(\Lambda; t)$ of the eigenvalues of a positive-definite self-adjoint matrix Λ, defined by

$$q_{\Lambda,t}(T^{\frac{1}{2}}) = \sum q_k(\Lambda; t) T^{\frac{1}{2}k} = \det \frac{1 - \Lambda^{-1} T^{\frac{1}{2}} t}{1 - \Lambda^{-1} T^{\frac{1}{2}}} ;$$

the primitive element x_k thus becomes the power sum $tr\ \Lambda^{-2k-1}$. The canonical symmetric bilinear $\mathbb{Z}[t]$-valued form on the algebra of Hall-Littlewood functions defines a positive-definite inner product on $\Delta_{\mathbb{Q}}$ regarded as a quotient of that algebra [23 III §8.12]. The product and coproduct of Δ are dual with respect to this bilinear form; in fact $\Delta[\frac{1}{2}]$ is a positive self-adjoint Hopf algebra in the sense of Zelevinsky [23 I §5 ex 25]. The classical Q-functions are the orthogonalization of the basis of square-free monomials with respect to this inner product.

2.3. The inner product on $\Delta_{\mathbb{Q}}$ defines a skew bilinear form on the complexification of its space of primitives, which allows us to interpret $\Delta_{\mathbb{Q}}$ as a standard representation of a Heisenberg algebra, or alternately [26, appendix I] of the group of antiperiodic loops on the circle. The latter group has two connected components, and in some ways [13, appendix 8] it is natural to think of Δ as $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-graded algebra, with q_k in homological degree $(k-1, 1)$; but in other contexts the $\mathbb{Z}/2\mathbb{Z}$-grading appears as an action of the Galois group of $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$.

If, using the notation of [32 §1.7], we write

$$2^{\frac{1}{2}} \alpha_{-k-\frac{1}{2}} = -x_k$$

when $k > 0$ then the operators

$$L_0 = \sum_{k>0} \alpha_{-k}\alpha_k + \frac{1}{16} \ , \ L_n = \frac{1}{2} \sum \alpha_k \alpha_{n-k} \ , \ n \neq 0$$

define a twisted [9 §9.4] charge one representation of the Virasoro algebra on $\Delta_{\mathbb{Q}}$. There appear to be deep general connections between self-dual Hopf algebras and vertex operator algebras [1,13].

2.4. It is striking that these Q-functions have been important in algebraic topology for more than a generation: in Cartan's 1960 Seminar [4 §17] the integral homology of $Sp(\mathbb{H})/\mathbb{U} = \Omega^5\mathbb{O}$ appears as the quotient Δ^+ of a polynomial algebra on generators q_k^+ of degree $2k$, modulo the ideal generated by the coefficients of the relation

$$q^+(T)q^+(-T) = 1 \ ,$$

where

$$q^+(T) = \sum_{k \geq 0} q_k T^k \ ;$$

the algebras Δ and Δ^+ thus differ only by a doubling of the grading. The cohomology of the corresponding finite-dimensional homogeneous spaces has been studied more recently along similar lines [16].

In light of the results of Karoubi cited in §1.2, an algebra of Q-functions forms a substantial part of the stable integral homology of the space of Abelian varieties; the primitives x_k^+ are dual to Mumford's classes λ_{2k+1}. The stable rational cohomology of the Riemann moduli space contains a polynomial algebra on the Mumford classes κ_k of degree $2k$, and the dual homology Hopf algebra contains the polynomial algebra [24] on dual primitives $\hat{\kappa}_k$. The map of §1.2 kills the even classes $\hat{\kappa}_{2k}$, while

$$\lambda_{2k-1} \mapsto (-1)^k \frac{B_k}{2k} \kappa_{2k-1} \ ,$$

where B_k is the kth Bernoulli number [29 §6.2]; the even Mumford classes are detected by the constructions of [28]. Stabilization implies that

$$\hat{\kappa}_k \in H_{2k}(\mathcal{M}_g, \mathbb{Q})$$

for sufficiently large g, and (since these classes vanish on decomposables) it follows from the characteristic number formula of §1.1 that

$$\tau_{g*}(\hat{\kappa}_k) = vt_{k+1} \ ;$$

the maps τ_g are however not ring homomorphisms with respect to the usual multiplication [26] on complex cobordism.

2.5. The importance of the theory of symmetric functions in the work of Kontsevich and Witten was discovered by Di Francesco, Itzykson, and Zuber [7 §3.2], but it was Józefiak [17] who saw the connection with Q-functions. In our formalism, their map

$$t_k \mapsto -(2k-1)!!\ tr\ \Lambda^{-2k-1} = -(2k-1)!!x_k\ ,$$

[where the 'odd' factorial $(2k+1)!!$ is the product of the odd integers less than or equal to $2k+1$, with $(-1)!! = 1$ by convention] defines a homomorphism from the extended Landweber-Novikov algebra to $\Delta[\frac{1}{2}]$. The image of the partition function τ_W under this map satisfies a large family of differential equations [22] which can be summarized impressionistically by the assertion that

$$\tilde{\tau}_W(x_i) = \tau_W(x_i + \delta_{i,1})$$

is an sl_2-invariant highest weight vector for the natural Virasoro action on (a completion of) $\Delta_{\mathbb{Q}}$. The formal series τ_W is divergent at $x_i = \delta_{i,1}$, but this claim can be reformulated precisely as

$$\tilde{L}_n \tau_W = 0\ ,\ n \geq -1\ ,$$

where $\tilde{L}_n$ is the linear shift of L_n defined by the map which sends x_1 to $x_1 - 1$, cf. [19]. This defines a charge one vertex operator algebra embedded in a completion of $\Delta_{\mathbb{Q}}$, cf. [8].

There is reason [22 §7, 23 III §7 ex 7] to expect that these results generalize to Hall-Littlewood functions at other roots of unity.

§3 The Kontsevich-Witten genus

3.1. Quillen's theorem establishes a bijection between one-dimensional formal group laws over a commutative ring A, and homomorphisms from the complex cobordism ring to A, so we can define a formal group law

$$X +_Q Y = \mathrm{kw}^{-1}(\mathrm{kw}(X) + \mathrm{kw}(Y))$$

over the localization $\Delta[q_1^{-1}]$, and hence a Q-function valued genus of complex manifolds, by specifying its exponential series to be

$$\mathrm{kw}^{-1}(T) = \sum_{k \geq 0} (2k-1)!! x_0^{-1} x_k T^{k+1}\ .$$

The homomorphism

$$\mathrm{kw} : MU \to S \to \Delta[q_1^{-1}]$$

classifying this group law factors through the map classifying the universal additive law of §1.5 and therefore defines a group law of additive type.

3.2. This Kontsevich-Witten genus is defined by a nonstandard orientation on an ordinary cohomology theory represented by the generalized Eilenberg-Mac Lane space associated to $\Delta[q_1^{-1}]$; its vertex algebra structure is the source of Witten's Virasoro operations.

Some properties of this genus are unfamiliar to the point of pathology: not only is its exponential series integral, for example, but modulo an odd prime it is polynomial as well. The point of this section is that the orientation defining this genus can be interpreted as the formal completion of a coordinate, in the sense of [10 §1.8], on a $\mathbb{T}$-equivariant cohomology theory taking values in sheaves of modules over a certain abelian group object in the category of ringed spaces. This global object is semi-classical, and some of the strangeness of the Kontsevich-Witten genus is a property not of the group object itself, but of a rather inconvenient coordinate on it.

It seems simplest to present this construction in two parts: the first step will define a local version over a basic group object $\mathcal{Q}$. The genus itself will then be defined by a family of group objects parametrized by a Grassmannian of positive-definite self-adjoint matrices.

3.3.1. The basic idea comes from the classical theory of functions of one complex variable: if g is an entire function, of exponential type in every right half-plane, then under certain circumstances there is an asymptotic relation

$$\sum_{n\in\mathbb{Z}} g(n)z^n \sim 0 \; ;$$

such equations may look more familiar written in the form

$$\sum_{n\geq 0} g(n)z^n \sim -\sum_{n\geq 1} g(-n)z^{-n} \; .$$

If $G(z)$ is the left-hand sum, $\check{G}(z^{-1})$ will denote the sum on the right. When g is rational [e.g. $g = 1$] such relations are familiar, but a less trivial example is defined by

$$g(w) = \Gamma(1+\alpha w)^{-1}$$

with $0 < \alpha < 2$. The resulting (Mittag-Leffler) function

$$\exp_\alpha(z) = \sum_{n\geq 0} \frac{z^n}{\Gamma(1+\alpha n)}$$

thus has the asymptotic expansion

$$\exp_\alpha(z) \sim -\sum_{n\geq 1} \frac{z^{-n}}{\Gamma(1-\alpha n)}$$

for z outside a sector of angular width $\alpha\pi$ centered on the positive real axis, cf. [12 §11.3.23]. This function is especially interesting when α is a rational number between zero and one, but we will be concerned only with $\exp_{\frac{1}{2}}(z)$ which, up to normalization, is the Laplace transform of Gaussian measure. Using the duplication formula for the gamma function, its asymptotic expansion takes the form

$$\exp_{\frac{1}{2}}(z) \sim -\pi^{-\frac{1}{2}} z^{-1} \sum_{n\geq 0} (2n-1)!!(-2z^2)^{-n}$$

for z outside a sector of width $\frac{1}{2}\pi$ centered on the positive real axis; in particular the expansion is valid along the entire imaginary axis. The odd function

$$\sin_{\frac{1}{2}}(z) = -\frac{i}{2}[\exp_{\frac{1}{2}}(iz) - \exp_{\frac{1}{2}}(-iz)]$$

therefore satisfies

$$\sin_{\frac{1}{2}}(x) \sim \pi^{-\frac{1}{2}} x^{-1} \sum_{n\geq 0} (2n-1)!!(2x^2)^{-n}$$

as x approaches infinity in either direction along the real axis.

3.3.2. Now consider the behavior on the real line of the entire function

$$\epsilon(z) = (\frac{\pi}{2z})^{\frac{1}{2}} \sin_{\frac{1}{2}}((\frac{z}{2})^{\frac{1}{2}})$$

defined by the power series

$$\frac{\pi^{\frac{1}{2}}}{2} \sum_{n\geq 0} \frac{(-\frac{1}{2}z)^n}{\Gamma(n+\frac{3}{2})} = \sum_{n\geq 0} \frac{(-z)^n}{(2n+1)!!} .$$

The argument above implies that

$$\epsilon(x) \sim \check{\epsilon}(x^{-1}) = \sum_{n\geq 0} (2n-1)!!x^{-n-1}$$

for x large and positive; it follows that ϵ is monotone decreasing for positive x. On the other hand it is clear from its power series expansion that ϵ is monotone decreasing on the negative real axis, so

$$\epsilon : (\mathbb{R}, 0) \to (\mathbb{R}_+, 1)$$

is a bijection. The open interval

$$\mathfrak{Q} = (0, \infty)^+ - \{1\}$$

can now be made an abelian group by interpreting ϵ to be the exponential of a group law defined in a neighborhood of infinity on the projective line: if x and y are large (i. e. nonzero) real numbers, then

$$x +_\infty y = \frac{xy}{x+y}$$

will again be a large real number. It is easy to see that the resulting composition on $P^1(\mathbb{R}) - \{0\}$ is associative, with ∞ as identity element; the inverse of x is just $-x$. We can therefore construct a formal group law $+_{\infty\epsilon}$ over $\mathbb{R}$ by requiring that

$$\epsilon(x) +_{\infty\epsilon} \epsilon(y) \sim \epsilon(x +_\infty y)$$

for x and y large and positive. This group law as the completion at the identity of the analytic composition

$$X +_\epsilon Y = \epsilon(\epsilon^{-1}(X) +_\infty \epsilon^{-1}(Y))$$

with translation-invariant one-form $d\epsilon^{-1}(T)^{-1}$; its exponential is the series $\check{\epsilon}$. The involution

$$[-1]_\epsilon(T) = \epsilon(-\epsilon^{-1}(T))$$

interchanges $(0, 1)$ and $(1, \infty)$, identifying $\mathfrak{Q}$ with the group completion of the semi-group defined on $(0, 1)$ by $+_\epsilon$. This group object is smooth, but not analytic; its exponential map has trivial domain of convergence.

3.3.3. The usual sheaf of smooth real-valued functions defines the structure of a ringed space on $\mathfrak{Q}$, but its law of addition is also compatible with the sheaf of real-analytic functions formally completed at the origin. The composition

$$x \mapsto \Psi(x) = \exp(-\epsilon^{-1}(x)^{-1}) : \mathfrak{Q} \to \mathbb{R}_+^\times$$

is a homomorphism to the real multiplicative group; pulling the sheaf over $\mathbb{G}_m(\mathbb{R})$ defined by $\mathbb{T}$-equivariant K-theory back along this map yields a $\mathbb{T}$-equivariant cohomology theory taking values in the category of sheaves of modules over $\mathfrak{Q}$. This is a kind of ordinary equivariant cohomology theory with coefficients in $\mathcal{Q}$; its Chern-Dold character sends the standard one-dimensional representation of the circle to Ψ, regarded as a section of the structure sheaf. The Hirzebruch genus

$$\frac{x}{\epsilon(x^{-1})} = \pi^{-\frac{1}{2}} \frac{(2x)^{\frac{1}{2}}}{\sin_{\frac{1}{2}}((2x)^{-\frac{1}{2}})}$$

of the associated complex orientation is the reciprocal of a power series in x^{-1} with trivial constant term. It defines a function on $P^1(\mathbb{R})$ analytic aside from a jump discontinuity at 0.

3.4. By replacing the function ϵ with

$$\epsilon_\Lambda(x) = (\frac{\pi}{2x})^{\frac{1}{2}}\ tr\ \sin_{\frac{1}{2}}((\frac{x}{2})^{\frac{1}{2}}\Lambda)$$

we obtain a $\mathbb{T}$-equivariant theory taking values in sheaves over a family of abelian group objects $\mathcal{Q}_\Lambda$ parametrized by equivalence classes of self-adjoint positive-definite matrices Λ. It is remarkable that if Λ is an $np \times np$ matrix and t is a pth root of unity, then it follows from the definition of §2.2 that

$$q_{\Lambda,t}(T^{\frac{1}{2}}) = q_{\Lambda^{-1},t^{-1}}(T^{-\frac{1}{2}})\ ;$$

in particular, if Λ is an endomorphism of an even-dimensional vector space, and $\check{q}(T^{\frac{1}{2}})$ denotes the generating function for the Hall-Littlewood functions at -1 associated to the matrix Λ^{-1}, then

$$\check{q}(T^{-\frac{1}{2}}) = q(T^{\frac{1}{2}})\ .$$

It seems reasonable to expect that this $\mathbb{T}$-equivariant theory is a real version of a theory over the quotient of some completion of $\Delta \otimes \check{\Delta}$ by the coefficients of the relation

$$\check{q}(T^{-\frac{1}{2}})q(-T^{\frac{1}{2}}) = 1\ .$$

Appendix: vertex operator algebras and self-dual Hopf algebras

This appendix has been added in July 1998 to the published version of this paper, which has appeared in 'Homotopy theory via algebraic geometry and representation theory', ed. S. Priddy and M. Mahowald, in Contemporary Mathematics. It is a kind of commentary on Andy Baker's construction of vertex operator algebras associated to even unimodular lattices using ideas from the theory of (bicommutative) Hopf algebras. I suspect that these methods will have further use, and I have tried here to summarize Baker's results in the language of group-valued functors, and to include some references to related work [e. g. on λ-rings [3]] which might otherwise be overlooked.

A.1. The functor which assigns to a commutative ring A, the abelian group (under multiplication) of formal series

$$h(T) = 1 + \sum_{i\geq 1} h_i T^i \in (1 + TA[[T]])^\times := \mathbb{W}_0(A)\ ,$$

is represented by the polynomial Hopf algebra $S := \mathbb{Z}[h_i | i \geq 1]$ with comultiplication

$$\Delta(h_i) = \sum_{i=j+k} h_j \otimes h_k\ .$$

$\mathbb{W}^0(A)$ is essentially the classical Witt ring of A [6 V §2, 2], and its representing algebra can be identified with the usual ring S of symmetric functions. More precisely, there is a natural isomorphism of the set $\mathbb{W}_0(A)$ with the set of ring homomorphisms from S to A, such that the map induced by Δ agrees with multiplication of power series. The functor $\mathbb{W}(A)$ which assigns to A the set of **all** invertible power series over A is represented by the tensor product $S[h_0, h_0^{-1}]$ of the usual ring of symmetric functions with a Hopf algebra representing the multiplicative groupscheme. It will be useful to know that $\mathbb{W}_0(A)$ has a natural commutative ring-structure $*$ characterized by the identity

$$(1 + aT) * (1 + bT) = (1 + abT) .$$

If $\mathbb{L}$ is a free abelian group of finite rank l [e. g. a lattice, with dual $\mathbb{L}^* = \mathrm{Hom}_{\mathrm{ab}}(\mathbb{L}, \mathbb{Z})$], then it is easy to see [for example by choosing a basis] that the functor

$$A \mapsto \mathrm{Hom}_{\mathrm{ab}}(\mathbb{L}, \mathbb{W}_0(A)) = \mathbb{L}^* \otimes_{\mathbb{Z}} \mathbb{W}_0(A)$$

is also represented by a Hopf algebra, which can be taken to be an l-fold tensor product of copies of S. I will write $\otimes^{\mathbb{L}^*} S$ for this representing object, which Baker calls $S(\mathbb{L})$. The point of the definition at the beginning of [1 §3] is to present a coordinate-free definition of this object: a homomorphism from $\mathbb{L}$ to $\mathbb{W}_0(A)$ defines a family $\lambda \mapsto h_i(\lambda)$ of maps from $\mathbb{L}$ to A, such that

$$h_i(\lambda + \lambda') = \sum_{i=j+k} h_j(\lambda) h_k(\lambda') ;$$

the ring $\otimes^{\mathbb{L}^*} S$ thus represents the tensor product functor in a natural way, without specifying a basis. The generating function

$$h^\lambda(T) = \sum_{i \geq 0} h_i(\lambda) T^i$$

is a convenient substitute for such a choice.

It is tempting to think of $\mathbb{L}^*$ as a constant groupscheme, and to interpret $\mathbb{L}^* \otimes_{\mathbb{Z}} \mathbb{W}_0$ as a tensor product in a category of group-valued functors. A natural internal product on a suitable category of commutative and cocommutative Hopf algebras has been studied by Goerss [11] and by Hunton and Turner [14].

In fact $\otimes^{\mathbb{L}^*} S$ is only a part of the vertex operator algebra associated to a lattice; the full construction is usually described as the graded tensor product $\mathbb{Z}[\mathbb{L}] \otimes (\otimes^{\mathbb{L}^*} S)$ of the symmetric functions with the group ring of $\mathbb{L}$. In this context it is natural to think of this group ring as graded, with the element λ in degree $\langle \lambda, \lambda \rangle$, and to give S its usual grading. In view of the discussion above we can define $\mathrm{VOA}(\mathbb{L})$ to be the Hopf algebra representing the functor $\mathbb{L}^* \otimes_{\mathbb{Z}} \mathbb{W}$.

A.2. If H is a commutative and cocommutative Hopf algebra, projective of finite rank over a base ring k, then the module

$$H^* = \mathrm{Hom}_k(H, k)$$

is again a bicommutative Hopf algebra. If

$$A \mapsto \mathrm{Hom}_{k-\mathrm{alg}}(H, A) := \mathbb{H}(A)$$

is the group-valued functor H represents, then its dual Hopf algebra H^* represents the Cartier dual functor

$$A \mapsto \mathrm{Hom}_{\mathrm{gp-valued\ functors}}(\mathbb{H}, \mathbb{G}_m)(A) ,$$

cf. [6 II §1 no. 2.10]. A self-duality on such a (commutative and cocommutative, projective and finite) Hopf algebra can thus be defined either as an isomorphism

$$H \to H^*$$

of Hopf algebras, or as an isomorphism

$$\mathbb{H} \to \mathbb{H}^*$$

of abelian group-valued functors; alternately, such a structure can be defined either as a nondegenerate pairing

$$H \otimes H \to k$$

of algebras [with suitable properties [23 I §5 ex. 25, 13]] or as a nondegenerate pairing

$$\mathbb{H} \times \mathbb{H} \to \mathbb{G}_m$$

of abelian group-valued functors.

Similar dualities exist more generally, in particular in the context of locally finite graded Hopf algebras. I will write S_* for the ring of symmetric functions given its usual grading, and S^* for its graded dual; the Hall inner product discussed by MacDonald then defines an isomorphism

$$S^* \to \check{S}_*$$

of graded algebras, with the convention that $\check{H}_*$ is the graded algebra H_{-*}, i. e. with its grading negated. A self-duality on a locally finite graded Hopf algebra H can thus be defined by a nondegenerate graded pairing

$$H \otimes \check{H} \to k$$

with suitable properties. If we think of the grading on H as an action of the multiplicative group on $\mathbb{H}$, then $\check{\mathbb{H}}$ will have the inverse action.

In terms of group-valued functors, the Hall duality defines a homomorphism

$$\mathbb{W}_0 \times \mathbb{W}_0 \to \mathbb{G}_m$$

which is familiar classically as part of the theory of the Artin-Hasse exponential [6 V §4 no. 4.3]. It seems to be simpler to work not with graded rings but with the formal completion of $\mathbb{W}_0$ at the origin, i. e. to consider the subfunctor $\hat{\mathbb{W}}_0$ defined on complete local rings A by series $h(T)$ with coefficients h_i in the maximal ideal of A such that $h_i \to 0$ as $i \to \infty$. Since $\mathbb{W}_0$ is a commutative ring-valued functor, the duality can be constructed in terms familiar from the theory of Frobenius algebras. It will suffice to define a suitable 'trace' morphism from $\hat{\mathbb{W}}_0$ to $\mathbb{G}_m$; the pairing will then be the result of this trace applied to the Witt ring product. But if $h \in \hat{\mathbb{W}}_0(A)$ then $h(1) \in A^\times$, and

$$h, g \mapsto (h * g)(1)$$

will do the trick ...

A.3. The principal observation in this appendix is that some of Baker's formulae can be simplified considerably by use of the self-dual nature of the underlying Hopf algebra. His formula 2.4 can be summarized as follows: let

$$\mathbb{Y}(h(w)) = h(z + w) \otimes \check{h}(-(z + w)^{-1}) \in (S \otimes \check{S})[[z, z^{-1}]][[w]]$$

be defined by expanding the coefficient in the right-hand term as $\sum_{i \geq 0}(-w)^i z^{-i-1}$; here $\check{h}(T) \in \check{S}[[T]]$ is the analogue of the generating function $h(T)$. This formula extends multiplicatively to define a homomorphism

$$\mathbb{Y} : S \to (S \otimes \check{S})[[z, z^{-1}]]$$

of (graded) ring**oids**, the point being that multiplication is not always defined in the object on the right. The assertion, more precisely, is that for any b and b' in S, the product $\mathbb{Y}(b)\mathbb{Y}(b')$ is in fact defined, and equal to $\mathbb{Y}(bb')$. This is not quite Baker's definition of the vertex operator, but the two agree under the identification of $S \otimes \check{S}$ with $\mathrm{End}(S)$ given by Hall duality: the commutative product in the former algebra is a resource not available in the latter, and it is just what is needed to make sense of the assertion that Baker's formula is multiplicative.

The definition of Cartier duality in terms of an internal Hom functor on the category of commutative groupschemes makes it natural to identify $\mathbb{L} \otimes_{\mathbb{Z}} \mathbb{W}_0$ with $\mathrm{Hom}(\mathbb{L}^* \otimes_{\mathbb{Z}} \mathbb{W}_0, \mathbb{G}_m)$; the isomorphism

$$\mathbb{L}^* \to \mathbb{L}$$

defined by the pairing on a self-dual lattice thus extends to a (grade-negating) isomorphism between $\otimes^{\mathbb{L}^*} S$ and $\otimes^{\mathbb{L}} \check{S}$. The analogue of the generating function $h^\lambda(T)$ is a generating function $\check{h}^\lambda(T) \in (\otimes^{\mathbb{L}} \check{S})[[T]]$, and the formula above for $\mathbb{Y}$ extends immediately to this more general context [1 §3.3]. To define $\mathbb{Y}$ on the whole of $\mathrm{VOA}(\mathbb{L})$, it remains to construct $\mathbb{Y}$ on elements of the group ring of $\mathbb{L}$: Baker's formula is

$$\mathbb{Y}(\lambda) = \lambda h^\lambda(z) \otimes \check{h}^\lambda(-z^{-1}) z^\lambda ,$$

where $z^\lambda \in \mathrm{End}(\mathbb{Z}[\mathbb{L}])[z, z^{-1}]$ is the operation $\mu \mapsto \mu z^{\langle \lambda, \mu \rangle}$ ...

Much of the preceding construction appears to generalize in a relatively straightforward way from the classical algebra of symmetric functions to the Hopf $\mathbb{Z}[t]$-algebra HL of Hall-Littlewood symmetric functions [23 III §5 ex. 8]: this can be defined as the polynomial algebra on generators q_i related to the power sum symmetric functions p_n by the formula

$$q(T) = \sum_{i \geq 0} q_i T^i = \exp(\sum_{n \geq 1} (1 - t^n) p_n \frac{T^n}{n})$$

[23 III §2.10]. As $t \to 0$, $q(T) \to h(T)$, which suggests that Baker's formula for $\mathbb{Y}$ might have an interesting analogue in HL. Taking formal logarithms, we can rewrite that formula in terms of 'reduced' power sums $\tilde{p}_n = p_n/n$ as

$$\mathbb{Y}(\tilde{p}_n) = \sum_{m \in \mathbb{Z} - \{0\}} \frac{1 - t^{|m|}}{1 - t^n} \binom{m}{n} \tilde{p}_m z^{m-n} ,$$

where n is positive and $p_{-m} = (-1)^{m+1} \check{p}_m$ for m negative.

The theory of Q-functions is one of the original motivations for this account: these are elements of the self-dual Hopf algebra which represents the functor

$$A \mapsto \{h \in \mathbb{W}_0(A) \mid h(-T) = h(T)^{-1}\} .$$

An analogous construction exists for any prime p: if ω is a primitive pth root of unity, the subfunctor

$$\{h \in \mathbb{W}_0 \mid \prod_{0 \leq k \leq p-1} h(\omega^k T) = 1\}$$

is also represented by a self-dual Hopf algebra. Expressed in terms of power sums, the relation defining this quotient becomes $p_n = 0$ when p divides n. It is also possible to think of these algebras as quotients of HL defined by specializing t to a root of unity [23 III §7 ex. 7]; perhaps requiring that its order be prime is superfluous. Considerable effort [13,15] has been devoted to constructing aspects of a vertex algebra structure on HL and on the rings of Q-functions, and it may be worth noting that when t is a primitive pth root of unity, the formula above for $\mathbb{Y}$ continues to make sense on the quotient of HL defined by sending p_n to zero when p divides n; but I am not sure if this fits with the thinking in [9 §9.2.51], and at the moment I don't know how to fit a lattice into the picture.

As Baker notes, some constructions of VOA theory are strikingly close to constructions familiar in algebraic topology. As a closing footnote I'd like to suggest the possible relevance of the classifying space $BU_{\mathbb{T}}$ for $\mathbb{T}$-equivariant K-theory in this context: the vertex operator $\mathbb{Y}$ can be interpreted as a homotopy-class of maps

$$BU_{\mathbb{T}} \times BU_{\mathbb{T}} \to BU_{\mathbb{T}}[[z, z^{-1}]] ,$$

or in terms of representation theory as a construction which relates [or 'fuses'] two representations of $\mathbb{T}$ specified at the points $0, 1$ on the projective line, to define a representation at ∞. This seems very close to the point of view of Huang-Lepowsky [and Segal] . . .

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