Topological gravity in dimensions two and four

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Abstract. Recent work on gravity in two dimensions has a natural generalization to four dimensions.

1. Basic definitions

1.1 The (symmetric monoidal) **two-**category

$$(Gravity)_{d+1}$$

has **objects:** compact oriented d-manifolds, with

- **morphisms** $V_0 \to V_1 : (d+1)$-manifolds W with $\partial W \cong V_0^{op} \sqcup V_1$, and
- diffeomorphisms $\tilde{W} \to W$ as **two-morphisms**.

The category $Mor(V_0, V_1)$ with cobordisms from V_0 to V_1 as objects and diffeomorphisms (equal to the identity on the boundary) as morphisms, is a hom-object in this two-category. Disjoint union defines the **monoidal structure**, and the category has an orientation-reversing **adjoint equivalence** with its opposite.

1.2 The **topological** category

$$(\mathrm{Gravity})_{d+1}$$

has compact **Riemannian** d-manifolds as objects, and the spaces

$$\mathrm{Mor}(V_0, V_1) := \coprod_{V_0^{op} \sqcup V_1 \cong \partial W} (\mathrm{Metrics}/\mathrm{Diff})(W)$$

as its hom-objects. Alternately: a morphism is a $(d+1)$-dimensional cobordism, together with (the equivalence class of) a Riemannian metric on it.

The group of diffeomorphisms which fix a frame at a point acts **freely** on the space of Riemannian metrics on a complete manifold, so the morphism spaces of $(\mathrm{Gravity})_{d+1}$ are roughly just the classifying spaces [7] of the morphism categories of $(Gravity)_{d+1}$.

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1.3 When $d = 1$ the group of diffeomorphisms of a Riemann surface (of genus > 1) has **contractible** components, and its mapping-class group $\Gamma = \pi_0(\mathrm{Diff})$ acts with **finite** isotropy on Teichmüller space, defining a **rational** homology isomorphism

$$B\mathrm{Diff} \sim B\Gamma \sim \mathrm{Teich} \times_\Gamma E\Gamma \to \mathrm{Teich} \times_\Gamma \mathrm{pt} = \mathcal{M} \; .$$

$(\mathrm{Gravity})_{1+1}$ is thus very similar to the category Segal constructed to define conformal field theory.

1.4 A monoidal functor from the topological gravity category to some simpler monoidal category, such as Hilbert spaces and trace-class maps, or modules over a ring spectrum, defines a **theory of topological gravity**. Replacing the Hom-objects in this category with their sets of components leads to TFT's in the sense of Atiyah, and replacing the Hom-objects with their rational homotopy types leads to cohomological field theories.

2. Some examples

2.1 The 'intersection homology' of a connected surface

$$\Sigma \mapsto \ker[H^1(\Sigma, \mathbb{C}) \to H^1(\partial\Sigma, \mathbb{C})]$$

is a simple example: if $\Sigma \circ \Sigma'$ is the composition of two surfaces along a boundary component, then the induced map

$$\tau : B\mathrm{Diff}(\Sigma) \to B\mathrm{U}$$

fits in the commutative diagram

$$\begin{array}{ccc} B\mathrm{Diff}(\Sigma) \times B\mathrm{Diff}(\Sigma') & \longrightarrow & B\mathrm{Diff}(\Sigma \circ \Sigma') \\ \downarrow{\scriptstyle \tau\times\tau} & & \downarrow{\scriptstyle \tau} \\ B\mathrm{U} \times B\mathrm{U} & \xrightarrow{\oplus} & B\mathrm{U} \; . \end{array}$$

This defines a theory of topological gravity with values in a monoidal topological category having **one** object, with the H-space $B\mathrm{U}$ of morphisms. [If we want to fuse along more than one boundary component, though, we need to be more careful.]

This functor has more structure: it takes values in symplectic lattices. The composition

$$B\mathrm{Sp}(\mathbb{Z}) \to B\mathrm{Sp}(\mathbb{R}) \sim B\mathrm{U} \to B(\mathrm{U}/\mathrm{Sp}) \sim \mathrm{SO}/\mathrm{U}$$

is a rational homology isomorphism, so τ lifts to a map

$$\tau_T : B\mathrm{Diff}(\Sigma) \to \mathrm{SO}/\mathrm{U}$$

which sends a surface to its harmonic one-forms with the complex structure defined by the Hodge $*$-operator. This is a form of the Abel-Jacobi-Torelli map

$$\Sigma \mapsto \mathrm{SP}^\infty\Sigma : \mathcal{M} \to \mathcal{A}$$

which sends a surface to its Jacobian; note that the union of Σ with its opposite defines a quaternionic object, which maps to zero.

2.2 Kontsevich-Witten theory is a much deeper example, with the (rationalized) complex cobordism ring-spectrum as target object. A toy version is easy to construct:

Suppose $\partial\Sigma$ has at most one component; then capping it off defines the closed surface $\Sigma_D := \Sigma \circ D$. The cobordism class of the bundle

$$[\Sigma_D] : \Sigma_D \times_{\mathrm{Diff}(\Sigma)} E\mathrm{Diff}(\Sigma) \to B\mathrm{Diff}(\Sigma)$$

is **primitive**, in the sense that it behaves additively under composition: the pull-back

$$\mu^*[\Sigma \circ \Sigma'] \in MU^{-2}(B\mathrm{Diff}(\Sigma) \times B\mathrm{Diff}(\Sigma'))$$

under the composition

$$\mu : \mathrm{Diff}(\Sigma) \times \mathrm{Diff}(\Sigma') \to \mathrm{Diff}(\Sigma \circ \Sigma')$$

is the sum

$$\epsilon^*[\Sigma_D] \otimes 1 + 1 \otimes \epsilon'^*[\Sigma'_D] \ ,$$

where

$$\epsilon : \mathrm{Diff}(\Sigma) \to \mathrm{Diff}(\Sigma_D)$$

is the trivial extension [5].

This says we can pull $\Sigma \circ \Sigma'$ apart as if it were made of taffy: the standard family of quadratic cones in $\mathbb{R}^3$ glued to $(\Sigma^{op} \sqcup \Sigma') \times I$ defines a Diff-equivariant cobordism from $\Sigma \circ \Sigma'$ to $\Sigma_D \sqcup \Sigma'_D$.

If we tensor with $\mathbb{Q}$ [and supress the grading] then the map τ_{kw} representing

$$\exp([\Sigma]) \in MU^*_{\mathbb{Q}}(B\mathrm{Diff}(\Sigma))$$

fits in the homotopy-commutative diagram

$$\begin{array}{ccc} B\mathrm{Diff}(\Sigma)^+ \wedge B\mathrm{Diff}(\Sigma')^+ & \longrightarrow & B\mathrm{Diff}(\Sigma \circ \Sigma')^+ \\ \downarrow{\scriptstyle \tau_{kw} \wedge \tau_{kw}} & & \downarrow{\scriptstyle \tau_{kw}} \\ \mathbf{MU}_{\mathbb{Q}} \wedge_{\mathbb{Q}} \mathbf{MU}_{\mathbb{Q}} & \longrightarrow & \mathbf{MU}_{\mathbb{Q}} \ . \end{array}$$

In both these examples, a surface is sent to some kind of configuration space of points: such constructions take unions to products. Proper Kontsevich-Witten theory involves much more complicated configuration spaces.

These constructions capture much of what's known about the stable cohomology of moduli spaces. Mumford's conjecture, for example, is equivalent to the assertion that τ_{kw} defines an isomorphism on rational cohomology.

2.3 The **Floer homology** $HF^*(Y)$ of a compact 3-manifold Y (e.g. a homology sphere) is defined by the Chern-Simons functional on the space of connections $\mathcal{C}(Y)$ mod gauge equivalence on a (trivial) G-bundle over Y. It is periodically graded, and has a kind of Poincaré pairing. Following Cohen, Jones, and Segal, I assume it is defined by an underlying spectrum $\mathbf{HF}(Y)$. Because the space of connections satisfies

$$\mathcal{C}(Y_0 \sqcup Y_1) = \mathcal{C}(Y_0) \times \mathcal{C}(Y_1) \ ,$$

it follows that

$$\mathbf{HF}(Y_0 \sqcup Y_1) = \mathbf{HF}(Y_0) \wedge \mathbf{HF}(Y_1) \ .$$

Atiyah [1] saw that Floer homology is a topological field theory: when Y bounds Z, the space $\mathcal{A}(Z)$ of Yang-Mills instantons on Z defines (by restriction to ∂Z) a kind of Lagrangian cycle

$$[\mathcal{A}(Z) \to \mathcal{C}(Y)] \in HF^*(Y) \;,$$

and if

$$\partial Z = Y^{op} \sqcup Y', \; \partial Z' = Y'^{op} \sqcup Y''$$

then $[\mathcal{A}(Z)] \wedge [\mathcal{A}(Z')]$ should map to $[\mathcal{A}(Z \cup_Y Z')]$ under the pairing

$$\mathbf{HF}(Y^{op} \sqcup Y') \wedge \mathbf{HF}(Y'^{op} \sqcup Y'') \to \mathbf{HF}(Y^{op} \sqcup Y'') \;.$$

In fact Yang-Mills on Z presupposes a Riemannian metric, and there is a **family**

$$\mathcal{A}(Z) \times_{\mathrm{Diff}(Z)} E\mathrm{Diff}(Z) \to \mathcal{C}(\partial Z) \times B\mathrm{Diff}(Z)$$

of Lagrangian cycles; its hypothetical class

$$\tau_A : B\mathrm{Diff}(Z)^+ \to \mathbf{HF}(\partial Z)$$

should define a theory of topological gravity.

2.4 In honest Kontsevich-Witten theory the analogue $\tau_{\mathbf{kw}}$ of $\exp([\Sigma])$ is the class

$$\sum_{n \geq 0} \langle \overline{\mathcal{M}}_g^n \rangle / n! \in MU^*_{\mathbb{Q}}(\overline{\mathcal{M}}_g) \;,$$

where $\langle \overline{\mathcal{M}}_g^n \rangle$ is the cobordism class of a forgetful map to the Deligne-Mumford space of stable algebraic curves of genus g, from a compactification of the space of smooth curves marked with n distinct points. Its characteristic number polynomial is

$$\Phi_* \mathbf{m}_{tot}(-\nu_{fake}) \in H^*(\overline{\mathcal{M}}_g, MU_{\mathbb{Q}}) \;,$$

where Φ is the forgetful map from the Deligne-Mumford-Knudsen space $\overline{\mathcal{M}}_g^n$, $\mathbf{m}_{tot}$ is the characteristic class defined by the total monomial symmetric function, and $-\nu_{fake}$ is the sum of the tangent line bundles to the modular curve at its marked points. I'm indebted to Gorbunov, Manin, and Zograf for correcting my mistaken assertion [6] that ν_{fake} is the formal normal bundle of Φ : above the divisor $\overline{\mathcal{M}}_g^{n-k+1} \times \overline{\mathcal{M}}_0^{1+k}$ on $\overline{\mathcal{M}}_g^n$ defined by curves with two irreducible components, one of genus zero, these two bundles differ stably by the sum of the pair of tangent lines at the double point.

I claim that $\tau_{\mathbf{kw}}$ respects a monoidal structure defined by Knudsen's gluing map μ: in the simplest case this means that

$$\begin{array}{ccc} \overline{\mathcal{M}}_g^{1+} \wedge \overline{\mathcal{M}}_h^{1+} & \longrightarrow & \overline{\mathcal{M}}_{g+h}^{+} \\ \downarrow & & \downarrow \\ \mathbf{MU}_{\mathbb{Q}} \wedge_{\mathbb{Q}} \mathbf{MU}_{\mathbb{Q}} & \longrightarrow & \mathbf{MU}_{\mathbb{Q}} \end{array}$$

commutes, or equivalently that

$$\mu^* \langle \overline{\mathcal{M}}_{g+h}^n \rangle = \sum_{p+q=n} \langle \overline{\mathcal{M}}_g^{p:1} \rangle \times \langle \overline{\mathcal{M}}_h^{q:1} \rangle \;,$$

where $\langle \overline{\mathcal{M}}_g^{p:r} \rangle \in MU^*_{\mathbb{Q}}(\overline{\mathcal{M}}_g^r)$ is defined by the partially forgetful morphism $\Phi_r : \overline{\mathcal{M}}_g^{p+r} \to \overline{\mathcal{M}}_g^p$. This follows because the diagram

$$\begin{array}{ccc} \coprod_{p+q=n} \overline{\mathcal{M}}_g^{p+1} \times \overline{\mathcal{M}}_h^{1+q} & \longrightarrow & \overline{\mathcal{M}}_{g+h}^n \\ \downarrow {\scriptstyle \tilde{\Phi}_1 \times \tilde{\Phi}_1} & & \downarrow {\scriptstyle \Phi} \\ \overline{\mathcal{M}}_g^1 \times \overline{\mathcal{M}}_h^1 & \xrightarrow{\mu} & \overline{\mathcal{M}}_{g+h} \end{array}$$

is a pullback in the category of smooth stacks; consequently

$$\mu^* \Phi_* = (\tilde{\Phi}_1 \times \tilde{\Phi}_1)_* \tilde{\mu}^* .$$

But $\tilde{\mu}^*$ pulls back tangent lines at marked points to tangent lines at marked points, so

$$\mu^*_{p,q}(\nu^{(n)}_{fake}) = \nu^{(p)}_{fake} \otimes 1 + 1 \otimes \nu^{(q)}_{fake} ,$$

and $\mathbf{m}_{tot}$ is multiplicative, so

$$\mu^* \langle \overline{\mathcal{M}}_{g+h}^n \rangle = \mu^* \Phi_* \mathbf{m}_{tot}(-\nu^{(n)}_{fake})$$

is a sum of terms of the form

$$(\tilde{\Phi}_1 \times \tilde{\Phi}_1)_*(\mathbf{m}_{tot}(-\nu^{(p)}_{fake}) \times \mathbf{m}_{tot}(-\nu^{(q)}_{fake})) , \quad \textbf{QED}.$$

2.5 Topological gravity coupled to the quantum cohomology of a (complex? symplectic?) manifold V is a conjectural theory [8] defined by

$$\sum_{n \geq 0} \langle \overline{\mathcal{M}}_g^{n:k}(V) \rangle / n! \in MU^*_{\mathbb{Q}}(\overline{\mathcal{M}}_g^k \times V^k) .$$

The configuration spaces are now suitable moduli spaces of **stable maps** from curves to V, and the representing morphism

$$\tau_{\mathbf{kw}}(V) : \overline{\mathcal{M}}_g^{k+} \to [V^k, \mathbf{MU}_{\mathbb{Q}}]$$

defines a functor to a monoidal category with objects $\mathbb{N}$, morphisms

$$\mathrm{mor}(j,k) = [V^{j+k}, \mathbf{MU}_{\mathbb{Q}}] ,$$

and compositions defined by a Poincaré trace

$$[V \times V, \mathbf{MU}_{\mathbb{Q}}] \to \mathbf{MU}_{\mathbb{Q}} .$$

There is a natural monoidal functor $n \mapsto [V^n, \mathbf{MU}_{\mathbb{Q}}]$ to the usual category of $\mathbf{MU}_{\mathbb{Q}}$-module spectra; this is the first two-dimensional case in which the target category for a topological gravity theory has not been slightly degenerate.

The simplest case of the monoidal axiom asserts the commutativity of

$$\begin{array}{ccc} \overline{\mathcal{M}}_g^{1+} \wedge \overline{\mathcal{M}}_h^{1+} & \xrightarrow{\mu} & \overline{\mathcal{M}}_{g+h}^{+} \\ \downarrow & & \downarrow \\ [V, \mathbf{MU}_{\mathbb{Q}}] \wedge_{\mathbf{MU}_{\mathbb{Q}}} [V, \mathbf{MU}_{\mathbb{Q}}] & \longrightarrow & \mathbf{MU}_{\mathbb{Q}} ; \end{array}$$

the map from $\overline{\mathcal{M}}_0^3$ to $[V^3, \mathbf{MU}_{\mathbb{Q}}]$ then defines a new (quantum) product. A relative version of this construction uses Gromov-Witten classes

$$\langle \widetilde{\mathcal{M}}_g^{n:k}(V) \rangle \in [\overline{\mathcal{M}}_g^k(V) \times V^n, B\mathbb{T}^n \wedge \mathbf{MU}_{\mathbb{Q}}]_*$$

which record the tangent lines at the marked points; summing over n-fold cap products with a cycle $\mathbf{z} \in H_*(V, H^*(B\mathbb{T}))$ defines a theory based on configuration spaces with marked points restricted to lie on $\mathbf{z}$. This recovers the Gromov-Witten potential and the WDVV family of quantum multiplications.

3. General nonsense

Monoidal functors between monoidal categories form a monoid, much as homomorphisms between abelian groups form an abelian group. Manin and Zograf [4] suggest that we think of these families of theories as parametrized by the Picard group of invertible objects. These objects can be identified with the points of Spec $MU_{\mathbb{Q}}$, which are one-dimensional formal group laws; but there is no natural way to compose them. Kontsevich-Witten theory [2] suggests the natural parametrizing object is the Hopf algebra $\mathcal{Q}$ of Schur Q-functions: these are Hall-Littlewood symmetric functions of the eigenvalues of a positive-definite matrix Λ, evaluated at $t = -1$.

The Kontsevich-Witten genus $MU \to \mathcal{Q}$ defines a formal group law with Q-function coefficients; its exponential (aside from normalization) is the asymptotic expansion as $\Lambda \to +\infty$ of the Mittag-Leffler exponential

$$\sum_{n \geq 0} \frac{\mathrm{Tr}\ \Lambda^n}{\Gamma(1 + \frac{1}{2}n)} \quad .$$

There is a natural **twisted** charge one action of the Virasoro algebra on the Q-functions, and the image under $\tau_{\mathbf{kw}}$ of the fundamental class

$$\exp(\sum [\overline{\mathcal{M}}_g]) \in H_*(\mathrm{SP}^\infty(\coprod \overline{\mathcal{M}}_g), \mathbb{Q})$$

of the space of not-necessarily-connected curves is an sl_2-invariant highest-weight vector, or **vacuum** state.

Recently Eguchi *et al*, Dubrovin, Getzler [3], and others have begun to extend this fundamental result of Kontsevich-Witten theory to topological gravity coupled to quantum cohomology. The relevant Virasoro representation appears to be defined on a group of loops on the torus $H^*(V, \mathbb{R}/\mathbb{Z})$, twisted by the endomorphism $\frac{1}{2}H + tX$, where H, X, Y generate the standard sl_2 action on the Hodge cohomology of V, and $t = c_1(V)/\omega$ depends on the Kähler class.

This twisted torus is mysterious even when V is a point. The adjoint operation on $(\mathrm{Gravity})_{1+1}$ defines an involution on the Picard group of invertible theories, and the analogy with Abel-Jacobi theory suggests that $\mathcal{Q}$ represents its skew-adjoint part.

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