$\mathbb{A}^1$-Algebraic topology over a field

Fabien Morel

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Contents

0 Introduction

Let k be a commutative field. In this work we prove in the $\mathbb{A}^1$-homotopy theory of smooth k-schemes [50, 39] the analogues of the following facts:

Theorem 1 *(Brouwer degree) Let $n > 0$ be an integer and let S^n denote the n-sphere. Then for an integer i*

$$\pi_i(S^n) = \begin{cases} 0 \ if \ i < n \\ \mathbb{Z} \ if \ i = n \end{cases}$$

Theorem 2 *(Hurewicz Theorem) For any pointed connected topological space X and any integer $n \geq 1$ the Hurewicz morphism*

$$\pi_n(X) \to H_n(X)$$

is the abelianization if $n = 1$, is an isomorphism if $n \geq 2$ and X is $(n-1)$-connected, and is an epimorphism if $n \geq 3$ and X is $(n-2)$-connected.

Theorem 3 *(Coverings and π_1) Any "reasonable" pointed connected space X admits a universal pointed covering*

$$\tilde{X} \to X$$

It is (up to unique isomorphism) the only pointed simply connected covering of X. Its automorphism group (as unpointed covering) is $\pi_1(X)$ and it is a $\pi_1(X)$-Galois covering.

Theorem 4 *$\pi_1(\mathbb{P}^1(\mathbb{R})) = \mathbb{Z}$ and $\pi_1(\mathbb{P}^n(\mathbb{R})) = \mathbb{Z}/2$ for $n \geq 2$,*
$\pi_1(SL_2(\mathbb{R})) = \mathbb{Z}$ and $\pi_1(SL_{n+1}(\mathbb{R})) = \mathbb{Z}/2$ for $n \geq 2$.

The corresponding complex spaces are simply connected: for $n \geq 1$ one has $\pi_1(\mathbb{P}^n(\mathbb{C})) = \pi_1(SL_n(\mathbb{C})) = *$

Let us denote by Sm_k the category of smooth separated finite type k-schemes. Unless otherwise explicitly stated, we will always consider Sm_k endowed with the Nisnevich topology (see [39, 40]). By a *space over k*, or in short a *space*, we mean a simplicial sheaf of sets on Sm_k.

The main achievement of this work is the understanding of the precise structure of $\mathbb{A}^1$-homotopy sheaves of pointed spaces. Once this is done, the analogues of the previous results will easily follow, as well as a bunch of analogues of classical results and computations.

Given a space $\mathcal{X}$, we denote by $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ the associated sheaf in the Nisnevich topology to the presheaf $U \mapsto Hom_{\mathcal{H}(k)}(U, \mathcal{X})$ where $\mathcal{H}(k)$ denotes the $\mathbb{A}^1$-homotopy category of smooth k-schemes defined in [50, 39].

If $\mathcal{X}$ is pointed, given an integer $n \geq 1$, we denote by $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ the associated sheaf of groups in the Nisnevich topology to the presheaf of groups $U \mapsto Hom_{\mathcal{H}_\bullet(k)}(\Sigma^n(U_+), \mathcal{X})$ where $\mathcal{H}_\bullet(k)$ denotes the pointed $\mathbb{A}^1$-homotopy category on k (see *loc. cit*) and Σ the simplicial suspension. This is a sheaf of abelian groups for $n \geq 2$.

In classical topology, the underlying structure to the corresponding homotopy "sheaves" is quite simple: π_0 is a "discrete" set, π_1 is a"discrete" group and the π_n's, $n \geq 2$, are "discrete" abelian groups.

The following notions are the precise analogues of being "$\mathbb{A}^1$-discrete":

Definition 5 *1) A sheaf of sets $\mathcal{S}$ on Sm_k (in the Nisnevich topology) is said to be* $\mathbb{A}^1$-invariant *if for any $X \in Sm_k$, the map*

$$\mathcal{S}(X) \to \mathcal{S}(\mathbb{A}^1 \times X)$$

induced by the projection $\mathbb{A}^1 \times X \to X$, is a bijection.

2) A sheaf of groups $\mathcal{G}$ on Sm_k (in the Nisnevich topology) is said to be strongly $\mathbb{A}^1$-invariant *if for any $X \in Sm_k$, the map*

$$H^i_{Nis}(X; G) \to H^i_{Nis}(\mathbb{A}^1 \times X; G)$$

induced by the projection $\mathbb{A}^1 \times X \to X$, is a bijection for $i \in \{0, 1\}$.

3) A sheaf M of abelian groups on Sm_k (in the Nisnevich topology) is said to be strictly $\mathbb{A}^1$-invariant *if for any $X \in Sm_k$, the map*

$$H^i_{Nis}(X; M) \to H^i_{Nis}(\mathbb{A}^1 \times X; M)$$

induced by the projection $\mathbb{A}^1 \times X \to X$, is a bijection for any $i \in \mathbb{N}$.

Remark 6 We observe by the very definitions of [39] that a sheaf of set $\mathcal{S}$ is $\mathbb{A}^1$-invariant if and only if it is $\mathbb{A}^1$-local as a space, that a sheaf of groups G is strongly $\mathbb{A}^1$-invariant if and only if the classifying space $B(G) = K(G, 1)$ is an $\mathbb{A}^1$-local space, and that a sheaf of abelian groups M is strictly $\mathbb{A}^1$-invariant if and only if for any $n \in \mathbb{N}$ the Eilenberg-MacLane space $K(M, n)$ is $\mathbb{A}^1$-local.□

The notion of strict $\mathbb{A}^1$-invariance is directly taken from [48, 49]; the $\mathbb{A}^1$-invariant sheaves with transfers of Voevodsky are indeed examples of such sheaves. The Rost's cycle modules [44] give also examples of strictly $\mathbb{A}^1$-invariant sheaves, more precisely of $\mathbb{A}^1$-invariant sheaves with transfers by [11]. Other important examples, which are not of the previous type, are the sheaf $\underline{\mathbf{W}}$ associated to the presheaf of Witt groups $X \mapsto W(X)$ in characteristic $\neq 2$ (this is proven in [42]), or the sheaves $\underline{\mathbf{I}}^n$ of unramified powers of the fundamental ideal used in [33] (still in characteristic $\neq 2$). In fact these sheaves can also be defined in characteristic 2 if one considers the Witt groups of inner product spaces over X studied in [29].

The notion of strong $\mathbb{A}^1$-invariance is new and we will meet important examples of genuine non commutative strongly $\mathbb{A}^1$-invariant sheaves of groups. For instance $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ is non abelian, see 4.3. More generally the $\pi_1^{\mathbb{A}^1}$ of the blow-up of $\mathbb{P}^n$, $n \geq 2$, at several points is highly non-commutative; see [37].

We strongly believe that the notion of $\mathbb{A}^1$-fundamental group sheaf should play a fundamental role in the understanding of $\mathbb{A}^1$-connected projective smooth k-varieties[1] in very much the same way as the usual fundamental group plays a fundamental role in the classification of compact connected differentiable manifolds. The rationally connected smooth projective k-varieties considered in [24] are examples of $\mathbb{A}^1$-connected smooth projective k-varieties, at least with some assumptions on k; see [37].

One of our main global result, which justifies the previous definition, is:

Theorem 7 *Let $\mathcal{X}$ be a pointed space. Then the sheaf of groups $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ is strongly $\mathbb{A}^1$-invariant and for any $n \geq 2$ the sheaf of abelian groups $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is strictly $\mathbb{A}^1$-invariant.*

[1] a space is said to be $\mathbb{A}^1$-connected if its $\pi_0^{\mathbb{A}^1}$ is the point

Remark 8 The proof relies very much on the fact that the base is a field, through Gabber's geometric presentation Lemma [13, 9]. Over a general base the situation is definitely more complicated, at least when the base scheme has dimension at least 2 as pointed out by J. Ayoub [1].□

Remark 9 Recall from [39] that any space $\mathcal{X}$ is the homotopy inverse limit of its Postnikov tower $\{P^n(\mathcal{X})\}_n$ and that if $\mathcal{X}$ is pointed, for each $n \geq 1$ the homotopy fiber of the morphism $P^n(\mathcal{X}) \to P^{n-1}(\mathcal{X})$ is $\mathbb{A}^1$-equivalent to the Eilenberg-MacLane space $K(\pi_n^{\mathbb{A}^1}(\mathcal{X}), n)$. The strongly/strictly $\mathbb{A}^1$-invariant sheaves and their cohomology play exactly the same role as the usual homotopy and singular cohomology groups play in classical algebraic topology.□

We are unfortunately unable to prove the analogue structure result for the $\pi_0^{\mathbb{A}^1}$ which appears to be the most difficult case:

Conjecture 10 *For any space $\mathcal{X}$ the sheaf $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-invariant.*

Remark 11 This conjecture is easy to check for smooth k-schemes of dimension ≤ 1. The case of smooth surfaces is already very interesting and non trivial. To simplify a bit, we assume that k is algebraically closed. Then if X is a projective smooth k-surface which is birationnally ruled over the smooth projective curve C of genus g then:

$$\pi_0^{\mathbb{A}^1}(X) = \begin{cases} * \text{ if } g = 0 \\ C \text{ if } g > 0 \end{cases}$$

For surfaces of general type the conjecture is unknown though it is known in some cases. It is not known for $K3$ surfaces.□

Remark 12 Another interesting general test case is the geometric classifying space $B_{gm}G$ of a smooth algebraic k-group G [39, 47]. From [39], there exists a natural transformation in $X \in Sm_k$: $\theta_G : H^1_{et}(X; G) \to Hom_{\mathcal{H}(k)}(X, B_{gm}G)$. We hope (at least if the group $G^{(0)}$ of irreducible components of G is of order prime to the characteristic of k) that this should induce an isomorphism on associated sheaves $\mathcal{H}^1_{et}(G) \cong \pi_0^{\mathbb{A}^1}(B_{gm}G)$ in the Nisnevich topology (in fact most probably also in the Zariski topology). It doesn't seem to be known in general whether or not this type of sheaves are $\mathbb{A}^1$-invariant.□

Our results rely on the detailed analysis of *unramified sheaves* of groups given in Part 1. Our analysis is done very much in the spirit of a "non-abelian variant" of Rost's cycle modules [44]. These unramified sheaves can be described in terms of their sections on function fields of smooth irreducible k-varieties plus extra structures: "residues" and "specializations".

Our work is entirely elementary except that we use Gabber's geometric presentation Lemma [13, 9] when k is infinite. It is used in Sections 1.2 and 1.3. To prove Theorem 7 in Section 3.1 we also use Gabber's result to prove that the $\mathbb{A}^1$-homotopy sheaves $\pi_n^{\mathbb{A}^1}(\mathcal{X})$, $n \geq 1$, are unramified in the sense of section 1.1. Using the results of section 1.2 we prove that these sheaves are strongly $\mathbb{A}^1$-invariant. The Appendix provides the necessary properties which allow us to reduce the case of a finite field to that of infinite fields. Theorem 14 then implies Theorem 7.

Remark 13 Contrary to Rost's approach [44], the structure involved in our description of unramified sheaves does not use any notion of "transfers". As a consequence over a perfect field one may show that the category of Rost cycle modules (*loc. cit.*) can be described without using transfers in the structure; see section 1.3.

One of the achievements in this work is to define the unramified sheaves of Milnor K-theory on Sm_k, as well as a bunch of related variants like unramified Milnor-Witt K-theory, without using any transfers. We will rather deduce the existence and properties of these transfers in [38]. This yields a completely new answer to this question which was raised in [5] and settled in [22].□

Apart from Gabber's Lemma, one of the main technical tool that we use is the following result (see Section 3.2). It means that the notions of strong and strict $\mathbb{A}^1$-invariance are in fact the "same" for sheaves of abelian groups.

Theorem 14 *Let M be a sheaf of abelian groups on Sm_k. Then:*
M is strongly $\mathbb{A}^1$-invariant $\Leftrightarrow$ M is strictly $\mathbb{A}^1$-invariant.

The Hurewicz theorem (see 3.35 and 3.57), and some of its natural consequences, will easily follow from Theorem 7, at least once the notion of $\mathbb{A}^1$-chain complex and the corresponding notion of $\mathbb{A}^1$-homology sheaves $H_*^{\mathbb{A}^1}(\mathcal{X})$ of a space $\mathcal{X}$ are introduced; see Section 3.3. An important consequence of the Hurewicz theorem is the unstable $\mathbb{A}^1$-connectivity theorem (see Theorem 3.38 in Section 3.3):

Theorem 15 *Let $\mathcal{X}$ be a pointed space and $n \geq 0$ be an integer. If $\mathcal{X}$ is simplicially n-connected then it is $\mathbb{A}^1$-n-connected, meaning that $\pi_i^{\mathbb{A}^1}(\mathcal{X})$ is trivial for $i \leq n$.*

A stable and much weaker version of this result was obtained in [32]. An example of simplicially $(n-1)$-connected pointed space is the n-fold simplicial suspension $\Sigma^n(\mathcal{X})$ of a pointed space $\mathcal{X}$. As $\mathbb{A}^n - \{0\}$ is $\mathbb{A}^1$-equivalent to the simplicial $(n-1)$-suspension $\Sigma^{n-1}(\mathbb{G}m^{\wedge n})$ (see [39]), it is thus $(n-2)$-$\mathbb{A}^1$-connected. In the same way the n-th smash power $(\mathbb{P}^1)^{\wedge n}$, which is $\mathbb{A}^1$-equivalent to the simplicial suspension of the previous one [39], is $(n-1)$-$\mathbb{A}^1$-connected.

Remark 16 In general, the "correct" $\mathbb{A}^1$-connectivity is given by the connectivity of the "corresponding" topological space of real points, through a real embedding of k, rather than the connectivity of its topological space of complex points through a complex embedding. This principle[2] has been a fundamental guide to our work. For instance the pointed algebraic "sphere" $(\mathbb{G}_m)^{\wedge n}$ is not $\mathbb{A}^1$-connected: it must be considered as a "0-dimensional twisted sphere". Observe that its space of real points has the homotopy type of the 0-dimensional sphere $S^0 = \{+1, -1\}$.□

The Hurewicz Theorem implies furthermore that for $n \geq 2$, the first non-trivial $\mathbb{A}^1$-homotopy sheaf of the $(n-2)$-$\mathbb{A}^1$-connected sphere $\mathbb{A}^n - \{0\} \cong_{\mathbb{A}^1} \Sigma^{n-1}(\mathbb{G}_m{}^{\wedge n})$ is its $\pi_{n-1}^{\mathbb{A}^1}$ and is the free strongly (or strictly by Theorem 14) $\mathbb{A}^1$-invariant sheaf of abelian groups generated by the pointed 0-dimensional sphere $(\mathbb{G}_m)^{\wedge n}$. This fact fits closely to classical topology as the first non-trivial homotopy group of the n-dimensional sphere S^n is the free "discrete abelian group" generated by the pointed 0-dimensional sphere S^0. To get the analogue of Theorem 1, it remains thus to describe the free strongly $\mathbb{A}^1$-invariant sheaf $\underline{\mathbf{K}}_n^{MW}$ on $(\mathbb{G}_m)^{\wedge n}$.

For any irreducible smooth k-scheme X with function field F, the abelian group of sections $\underline{\mathbf{K}}_n^{MW}(X)$ injects into $\underline{\mathbf{K}}_n^{MW}(F)$. To define $\underline{\mathbf{K}}_n^{MW}$ we first define its sections $\underline{\mathbf{K}}_n^{MW}(F) =: K_n^{MW}(F)$ on a function field F.

Definition 17 *Let F be a commutative field. The Milnor-Witt K-theory of F is the graded associative ring $K_*^{MW}(F)$ generated by the symbols $[u]$, for*

[2] which owns much to conversations with V. Voevodsky

each unit $u \in F^\times$, of degree $+1$, and one symbol η of degree -1 subject to the following relations:

1 *(Steinberg relation) For each $a \in F^\times - \{1\}$: $[a].[1-a] = 0$*

2 *For each pair $(a,b) \in (F^\times)^2$: $[ab] = [a] + [b] + \eta.[a].[b]$*

3 *For each $u \in F^\times$: $[u].\eta = \eta.[u]$*

4 *Set $h := \eta.[-1] + 2$. Then $\eta \,.\, h = 0$*

This object was introduced in a "complicated way" by the author, until the previous very simple and natural description was found in collaboration with Mike Hopkins: each relation has a natural $\mathbb{A}^1$-homotopic interpretation.

The quotient ring $K_*^{MW}(F)/(\eta)$ is clearly the Milnor K-theory ring $K_*^M(F)$ of F introduced by Milnor in [28]. It is not hard to prove (see section 2.1) that the ring $K_*^{MW}(F)[\eta^{-1}]$ obtained by inverting η is the ring of Laurent polynomials $W(F)[\eta, \eta^{-1}]$ with coefficients in the Witt ring $W(F)$ of non-degenerate symmetric bilinear forms on F (see [29], and [45] in characteristic $\neq 2$). More generally, $K_0^{MW}(F)$ is the Grothendieck-Witt ring $GW(F)$ of non-degenerate symmetric bilinear forms on F. The isomorphism sends the 1-dimensional form $(X,Y) \mapsto uXY$ on F to $< u > := \eta[u] + 1$ (see section 2.1).

Using residue morphisms in Milnor-Witt K-theory and the results of section 1.3, we describe in section 2.2 the unramified sheaf $X \mapsto \underline{\mathbf{K}}_n^{MW}(X)$, where, for X irreducible with function field F, $\underline{\mathbf{K}}_n^{MW}(X) \subset K_n^{MW}(F)$ denotes the kernel of the residues at points in X of codimension 1. In section 2.3 we prove our main computational result:

Theorem 18 *For any field k, for $n \geq 1$, the morphism of sheaves given by mapping an n-tuple of units $(u_1, \ldots, u_n)$ to its associated symbol $[u_1] \ldots [u_n]$*

$$(\mathbb{G}_m)^{\wedge n} \to \underline{\mathbf{K}}_n^{MW}$$

is the universal one to a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups: any morphism of pointed sheaves $(\mathbb{G}_m)^{\wedge n} \to M$ to a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups induces a unique morphism of sheaves of abelian groups

$$\underline{\mathbf{K}}_n^{MW} \to M$$

In other words, for $n \geq 1$, the sheaf $\underline{\mathbf{K}}_n^{MW}$ is the free strongly $\mathbb{A}^1$-invariant sheaf of abelian groups generated by $(\mathbb{G}_m)^{\wedge n}$. Using our Hurewicz Theorem we now obtain the analogue of Theorem 1 we had in mind:

Theorem 19 *For $n \geq 2$ one has a canonical isomorphisms of sheaves*

$$\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}) \cong \pi_n^{\mathbb{A}^1}((\mathbb{P}^1)^{\wedge n}) \cong \underline{\mathbf{K}}_n^{MW}$$

It is not hard to compute for (n,m) a pair of integers the abelian group of morphisms of sheaves of abelian groups from $\underline{\mathbf{K}}_n^{MW}$ to $\underline{\mathbf{K}}_m^{MW}$: it is $\underline{\mathbf{K}}_{m-n}^{MW}(k)$, the isomorphism being induced by the product $\underline{\mathbf{K}}_n^{MW} \times \underline{\mathbf{K}}_{m-n}^{MW} \to \underline{\mathbf{K}}_m^{MW}$. This implies in particular for $n = m$ the:

Corollary 20 *(Brouwer degree) For $n \geq 2$, the canonical morphism*

$$[\mathbb{A}^n - \{0\}, \mathbb{A}^n - \{0\}]_{\mathcal{H}_\bullet(k)} \cong [(\mathbb{P}^1)^{\wedge n}, (\mathbb{P}^1)^{\wedge n}]_{\mathcal{H}_\bullet(k)} \to K_0^{MW}(k) = GW(k)$$

is an isomorphism.

We have denoted here by $[-,-]_{\mathcal{H}_\bullet(k)}$ the set of morphisms in the pointed $\mathbb{A}^1$-homotopy category $\mathcal{H}_\bullet(k)$.

The analogue of the Theory of the Brouwer degree thus assigns to an $\mathbb{A}^1$-homotopy class from an algebraic sphere to itself an element in $GW(F)$; see [35] for a heuristic discussion in case $n = 1$. We observe that in case $n = 1$ there is only an epimorphism $[\mathbb{P}^1, \mathbb{P}^1]_{\mathcal{H}_\bullet(k)} \to GW(k)$ but the group $[\mathbb{P}^1, \mathbb{P}^1]_{\mathcal{H}_\bullet(k)}$ will be entirely understood in Section 4.3.

The class of the Hopf map $\eta \in [\mathbb{A}^2 - \{0\}, \mathbb{P}^1]_{\mathcal{H}_\bullet(k)}$ is indeed closely related to the element η of the Milnor-Witt K-theory. Our computations thus clearly stabilize as follows:

Corollary 21 *[31, 30] Let $\mathcal{SH}(k)$ be the stable $\mathbb{A}^1$-homotopy category of $\mathbb{P}^1$-spectra (or T-spectra) over k [50, 31, 30]. Let $\mathbb{S}^0$ be the sphere spectrum, $(\mathbb{G}_m)$ be the suspension spectrum of the pointed $\mathbb{G}_m$, let $\eta : (\mathbb{G}_m) \to \mathbb{S}^0$ be the (suspension of the) Hopf map and let $M\mathbb{G}L$ be the Thom spectrum [50]. For any integer $n \in \mathbb{Z}$ one has a commutative diagram in which the verticals are canonical isomorphisms:*

$$\begin{array}{ccccc}
[\mathbb{S}^0, (\mathbb{G}_m)^{\wedge n}]_{\mathcal{SH}(k)} & \to & [\mathbb{S}^0, Cone(\eta) \wedge (\mathbb{G}_m)^{\wedge n}]_{\mathcal{SH}(k)} & \cong & [\mathbb{S}^0, M\mathbb{G}L \wedge (\mathbb{G}_m)^{\wedge n}]_{\mathcal{SH}(k)} \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
K_n^{MW}(k) & \twoheadrightarrow & K_n^{MW}(k)/\eta & = & K_n^M(k)
\end{array}$$

Observe that the proof we give here is completely elementary and holds over any field, as opposed to [31, 30], which use the Milnor conjectures and were assuming k is perfect of characteristic $\neq 2$.

Another natural consequence of our work is the theory of $\mathbb{A}^1$-coverings and their relation to $\mathbb{A}^1$-fundamental sheaves of groups. This is discussed in section 4.1. The notion of $\mathbb{A}^1$-covering is quite natural: it is a morphism of spaces having the unique left lifting property with respect to "trivial $\mathbb{A}^1$-cofibrations". The Galois étale coverings of order prime to $char(k)$, or the $\mathbb{G}_m$-torsors are examples of $\mathbb{A}^1$-coverings. We will prove the existence of a universal $\mathbb{A}^1$-covering for any pointed $\mathbb{A}^1$-connected space $\mathcal{X}$, and more precisely the exact analogue of Theorem 3.

This theory is in some sense orthogonal, or "complimentary", to the étale theory of the fundamental group: a 0-$\mathbb{A}^1$-connected space $\mathcal{X}$ has no nontrivial pointed Galois étale coverings [37].

In case $\mathcal{X}$ is not $\mathbb{A}^1$-connected, the étale finite coverings are "captured" by the sheaf $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ which may have non trivial étale covering, like abelian varieties, which are $\mathbb{A}^1$-invariant sheaves.

The universal $\mathbb{A}^1$-covering and the sheaf $\pi_1^{\mathbb{A}^1}$ encode a much more combinatorial and geometrical information than the arithmetical information of the étale one. As we already mentioned we hope this combinatorial object will play a central role in the "$\mathbb{A}^1$-surgery classification" approach to projective smooth $\mathbb{A}^1$-connected k-varieties [37].

We next compute the $\pi_1^{\mathbb{A}^1}$ of $\mathbb{P}^n$, $n \geq 2$ and of SL_n, $n \geq 3$ in sections 4.2:

Theorem 22 *1) For $n \geq 2$, the canonical $\mathbb{G}_m$-torsor $\mathbb{G}_m - (\mathbb{A}^{n+1} - \{0\}) \to \mathbb{P}^n$ is the universal $\mathbb{A}^1$-covering of $\mathbb{P}^n$, and thus yields an isomorphism*

$$\pi_1^{\mathbb{A}^1}(\mathbb{P}^n) \cong \mathbb{G}_m$$

2) One has a canonical isomorphism

$$\pi_1^{\mathbb{A}^1}(SL_2) \cong \underline{\mathbf{K}}_2^{MW}$$

and the inclusions $SL_2 \to SL_n$, $n \geq 3$, induce an isomorphism

$$\underline{\mathbf{K}}_2^{MW}/\eta = \underline{\mathbf{K}}_2^{M} \cong \pi_1^{\mathbb{A}^1}(SL_n)$$

Remark 23 1) In view of [12] it should be interesting to determine the possible $\pi_1^{\mathbb{A}^1}$ of linear algebraic groups.

2) Of course one has for $n \geq 3$, $\pi_1^{\mathbb{A}^1}(SL_n) = \pi_1^{\mathbb{A}^1}(SL_\infty) = \pi_1^{\mathbb{A}^1}(GL_\infty)$. We know from [39, Theorem 3.13 p. 140] that

$$[\Sigma^1(U_+), GL_\infty]_{\mathcal{H}_\bullet(k)} = [\Sigma^2(U_+), \mathbb{Z} \times B(GL_\infty)]_{\mathcal{H}_\bullet(k)} = K_2(U)$$

(where K_2 means the Quillen K_2). Thus our previous computation recover the well-known identification between the associated sheaf (in the Zariski or) Nisnevich topology of K_2 and unramified Milnor K-theory in weight 2.

3) Our computations make clear that the $\mathbb{Z}$ or $\mathbb{Z}/2$ in the statement of Theorem 4 have different "motivic" natures, the fundamental groups of projective spaces being of "weight one" and that of special linear groups of "weight two".□

We finish our paper with a very explicit description of the sheaf $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$, which is the more complicated one! This achieves the proof of the analogue of Theorem 4. The sheaf of groups $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ will be shown to be non-commutative (!) and is thus not equal to $\underline{\mathbf{K}}_1^{MW}$. To describe it, we let $\eta : \mathbb{A}^2 - \{0\} \to \mathbb{P}^1$ be the algebraic Hopf map and we let $\mathbb{P}^\infty$ denote the space obtained by taking the union of all the $\mathbb{P}^n$'s. We have an $\mathbb{A}^1$-fibration sequence (see Section 3.3):

$$\mathbb{A}^2 - \{0\} \to \mathbb{P}^1 \to \mathbb{P}^\infty$$

which gives a non commutative central extension of sheaves of groups

$$0 \to \underline{\mathbf{K}}_2^{MW} = \pi_1^{\mathbb{A}^1}(\mathbb{A}^2 - \{0\}) \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^\infty) = \mathbb{G}_m \to 1$$

which is explicitly described in Section 4.3.

The results of this paper lead to rather natural applications. To cite only a few of them, in [38] we address construction of transfers for "finite $\mathbb{A}^1$-covering". There are two main cases: the case of finite étale coverings which leads to a new construction of transfers in Milnor K-theory (as well as Milnor-Witt K-theory) and the case of $\mathbb{G}_m$-torsor $\mathcal{Y} \to \mathcal{X}$. The latter yields a (stable) transfer morphism of the form: $(\mathbb{G}_m) \wedge (\mathcal{X}_+) \to (\mathcal{Y}_+)$, whose "real-points" gives the usual transfer map for the corresponding $\mathbb{Z}/2$-covering of the real points.

The results of this paper are also used in a fundamental way in our work on the Euler class for algebraic vector bundles [36]. As we already mentioned, these results are also the starting point of our study of $\mathbb{A}^1$-connected smooth projective varieties [37].

Part of our results were discussed and announced in [35].

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Conventions and notations. We fix a base field k. Unless otherwise explicitly stated, no assumption is made on k, neither concerning the characteristic, nor the perfectness, nor the finiteness.

We denote by $\mathcal{F}_k$ the category of finite type separable fields extensions of k. By a discrete valuation v on $F \in \mathcal{F}_k$ we will always mean a *geometric* one, coming from a codimensional 1 point in a smooth model for F. We then let $\mathcal{O}_v \subset F$ denotes its valuation ring, $m_v \subset \mathcal{O}_v$ its maximal ideal and $\kappa(v)$ its residue field.

Recall that Sm_k denotes the category of smooth separated finite type k-schemes, also called smooth k-varieties. We will also use the subcategory $\tilde{Sm}_k$ of Sm_k with the same objects but with only smooth morphisms. It will always be understood that these categories are endowed with the Nisnevich topology [40, 39]. Thus "sheaf" always means, unless otherwise explicitly stated, sheaf in the Nisnevich topology.

We will let $\mathcal{S}et$ denote the category of sets, $\mathcal{A}b$ that of abelian groups. A *space* is a simplicial object in the category of sheaves of sets on Sm_k [39].

We denote by Sm'_k the category of essentially smooth k-schemes. Its objects are k-schemes which are an inverse limit of a left filtering system $(X_\alpha)_\alpha$ with transition morphisms $X_\beta \to X_\alpha$ smooth affine morphisms between smooth k-schemes (see [16]). For instance, for each point $x \in X \in Sm_k$ the local scheme $X_x := Spec(\mathcal{O}_{X,x})$ of X at x, as well as its henselization

$X_x^h := Spec(\mathcal{O}^h_{X,x})$ are essentially smooth k-schemes. In the same way the complement of the closed point in X_x or X_x^h are essentially smooth over k. We will very often make the abuse of saying "smooth k-scheme" instead of essentially smooth k-scheme, if no confusion can arise.

For any scheme X and any integer i we let $X^{(i)}$ denote the set of points in X of codimension i.

Given a presheaf of sets on Sm_k, that is to say a functor $F : (Sm_k)^{op} \to Sets$, and an essentially smooth k-scheme $X = lim_\alpha X_\alpha$ we set $F(X) := colimit_\alpha F(X_\alpha)$. When $X = Spec(A)$ is affine we will also denote this set simply by $F(A)$.

1 Unramified sheaves and strongly $\mathbb{A}^1$-invariant sheaves

1.1 Unramified sheaves of sets

Definition 1.1 *An* unramified presheaf of sets $\mathcal{S}$ *on* Sm_k *(resp. on* $\tilde{Sm_k}$*) is a presheaf of sets* $\mathcal{S}$ *such that the following holds:*

(0) *for any* $X \in Sm_k$ *with irreducible components* X_α*'s,* $\alpha \in X^{(0)}$*, the obvious map* $\mathcal{S}(X) \to \Pi_{\alpha\in X^{(0)}}\mathcal{S}(X_\alpha)$ *is a bijection.*

(1) *for any* $X \in Sm_k$ *and any open subscheme* $U \subset X$ *the restriction map* $\mathcal{S}(X) \to \mathcal{S}(U)$ *is injective if* U *is everywhere dense in* X*;*

(2) *for any* $X \in Sm_k$*, irreducible with function field* F*, the injective map* $\mathcal{S}(X) \hookrightarrow \bigcap_{x\in X^{(1)}} \mathcal{S}(\mathcal{O}_{X,x})$ *is a bijection (the intersection being computed in* $\mathcal{S}(F)$*).*

Remark 1.2 An unramified presheaf $\mathcal{S}$ (either on Sm_k or on $\tilde{Sm_k}$) is automatically a sheaf of sets in the Zariski topology. This follows from **(2)**. We also observe that with our convention, for $\mathcal{S}$ an unramified presheaf, the formula in **(2)** also holds for X essentially smooth over k and irreducible with function field F. We will use these facts freely in the sequel.□

Example 1.3 It was observed in [32] that any strictly $\mathbb{A}^1$-invariant sheaf on Sm_k is unramified in this sense. The $\mathbb{A}^1$-invariant sheaves with transfers of

[48] as well as the cycle modules[3] of Rost [44] give such unramified sheaves. In characteristic $\neq 2$ the sheaf associated to the presheaf of Witt groups $X \mapsto W(X)$ is unramified by [41] (the sheaf associated in the Zariski topology is in fact already a sheaf in the Nisnevich topology).$\square$

Remark 1.4 Let $\mathcal{S}$ be a sheaf of sets in the Zariski topology on Sm_k (resp. on $\tilde{Sm}_k$) satisfying properties **(0)** and **(1)** of the previous definition. Then it is unramified if and only if, for any $X \in Sm_k$ and any open subscheme $U \subset X$ the restriction map $\mathcal{S}(X) \to \mathcal{S}(U)$ is bijective if $X - U$ is everywhere of codimension ≥ 2 in X. We left the details to the reader.$\square$

Remark 1.5 *Base change.* Let $\mathcal{S}$ be a sheaf of sets on $\tilde{Sm}_k$ or Sm_k, let $K \in \mathcal{F}_k$ be fixed and denote by $\pi : Spec(K) \to Spec(k)$ the structural morphism. One may pull-back $\mathcal{S}$ to the sheaf $\mathcal{S}|_K := \pi^*\mathcal{S}$ on $\tilde{Sm}_K$ (or Sm_K accordingly). One easily checks that the sections on a separable (finite type) field extension F of K is nothing but $\mathcal{S}(F)$ when F is viewed in $\mathcal{F}_k$. If $\mathcal{S}$ is unramified so is $\mathcal{S}|_K$: indeed $\pi^*\mathcal{S}$ is a sheaf and satisfies properties **(0)** and **(1)**. We prove **(3)** using the previous Remark.$\square$

Our aim in this subsection is to give an explicit description of unramified sheaves of sets both on $\tilde{Sm}_k$ and on Sm_k in terms of their sections on fields $F \in \mathcal{F}_k$ and some extra structure. We start with the simplest case.

Definition 1.6 *An* unramified $\tilde{\mathcal{F}}_k$-set *consists of:*

(D1) *A functor $\mathcal{S} : \mathcal{F}_k \to \mathcal{S}et$;*

(D2) *For any $F \in \mathcal{F}_k$ and any discrete valuation v on F, a subset*

$$\mathcal{S}(\mathcal{O}_v) \subset \mathcal{S}(F)$$

The previous data are moreover supposed to satisfy the following axioms:

(A1) *If $i : E \subset F$ is a separable extension in $\mathcal{F}_k$, and v is a discrete valuation on F which restricts to a discrete valuation w on E with ramification index 1 then $\mathcal{S}(i)$ maps $\mathcal{S}(\mathcal{O}_w)$ into $\mathcal{S}(\mathcal{O}_v)$ and moreover*

[3]these two notions are indeed quite closed by [11]

if the induced extension $\bar{i} : \kappa(w) \to \kappa(v)$ is an isomorphism, then the following square of sets is cartesian:

$$\begin{array}{ccc} \mathcal{S}(\mathcal{O}_w) & \to & \mathcal{S}(\mathcal{O}_v) \\ \bigcap & & \bigcap \\ \mathcal{S}(E) & \to & \mathcal{S}(F) \end{array}$$

(A2) *Let $X \in Sm_k$ be irreducible with function field F. If $x \in \mathcal{S}(F)$, then x lies in all but a finite number of $\mathcal{S}(\mathcal{O}_x)$'s, where x runs over the set $X^{(1)}$ of points of codimension one.*

We first observe that an unramified sheaf of sets $\mathcal{S}$ on $\tilde{Sm}_k$ defines an unramified $\tilde{\mathcal{F}}_k$-set. First, evaluation on the separable field extensions of k yields a functor:

$$\mathcal{S} : \mathcal{F}_k \to \mathcal{S}et \ , \quad F \mapsto \mathcal{S}(F)$$

For any discrete valuation v on $F \in \mathcal{F}_k$ we observe that $\mathcal{S}(\mathcal{O}_v)$ is a subset of $\mathcal{S}(F)$. We now claim that these data satisfy the axioms **(A1)** and **(A2)** of unramified $\tilde{\mathcal{F}}_k$-set.

Axiom **(A1)** is easily checked by choosing convenient smooth models over k for the essentially smooth k-schemes $Spec(F)$, $Spec(\mathcal{O}_v)$. To prove axiom **(A2)** one observes that any $x \in \mathcal{S}(F)$ comes, by definition, from an element $x \in \mathcal{S}(U)$ for $U \in Sm_k$ an open subscheme of X. Thus any $\alpha \in \mathcal{S}(F)$ lies in all the $\mathcal{S}(\mathcal{O}_x)$ for $x \in X^{(1)}$ lying in U. But clearly there are only finitely many $x \in X^{(1)}$ not lying in U.

This construction defines a "restriction" functor from the category of unramified sheaves of sets on $\tilde{Sm}_k$ to that of unramified $\tilde{\mathcal{F}}_k$-sets.

Proposition 1.7 *The restriction functor from unramified sheaves on $\tilde{Sm}_k$ to unramified $\tilde{\mathcal{F}}_k$-sets is an equivalence of categories.*

Proof. Given an unramified $\tilde{\mathcal{F}}_k$-set $\mathcal{S}$, and $X \in Sm_k$ irreducible with function field F, we define the subset $\mathcal{S}(X) \subset \mathcal{S}(F)$ as the intersection $\bigcap_{x \in X^{(1)}} \mathcal{S}(\mathcal{O}_x) \subset \mathcal{S}(F)$. We extend it in the obvious way for X not irreducible so that property **(0)** is satisfied. Given a smooth morphism $f : Y \to X$ in Sm_k we define a map: $S(f) : \mathcal{S}(X) \to \mathcal{S}(Y)$ as follows. By property **(0)** we may assume X and Y are irreducible with field of fractions E and F respectfully and f is dominant. The map $\mathcal{S}(f)$ is induced by the map $\mathcal{S}(E) \to$

$\mathcal{S}(F)$ corresponding to the fields extension $E \subset F$ and the observation that if $x \in X^{(1)}$ then $f^{-1}(x)$ is a finite set of points of codimension 1 in Y. We check that it is a sheaf in the Nisnevich topology using Axiom **(A1)** and the characterization of Nisnevich sheaves from [39]. It is clearly unramified. Finally to show that we have just constructed the inverse to the restriction functor, we use axiom **(A2)**.□

Remark 1.8 From now on in this paper, we will not distinguish between the notion of unramified $\tilde{\mathcal{F}}_k$-set and that of unramified sheaf of sets on $\tilde{Sm}_k$. If $\mathcal{S}$ is an unramified $\tilde{\mathcal{F}}_k$-set we still denote by $\mathcal{S}$ the associated unramified sheaf of sets on $\tilde{Sm}_k$.□

Definition 1.9 *An* unramified $\mathcal{F}_k$-set $\mathcal{S}$ *is an unramified $\tilde{\mathcal{F}}_k$-set together with the following additional data:*

(D3) *For any $F \in \mathcal{F}_k$ and any discrete valuation v on F such that the residue field $\kappa(v)$ is separable over k, a map $s_v : \mathcal{S}(\mathcal{O}_v) \to \mathcal{S}(\kappa(v))$, called the specialization map associated to v.*

These data should satisfy furthermore the following axioms:

(A3) **(i)** *If $i : E \subset F$ is an extension in $\mathcal{F}_k$, and v is a discrete valuation on F which restricts to a discrete valuation w on E with ramification index 1, then $\mathcal{S}(i)$ maps $\mathcal{S}(\mathcal{O}_w)$ to $\mathcal{S}(\mathcal{O}_v)$ and if the two residue fields are separable over k the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{S}(\mathcal{O}_w) & \to & \mathcal{S}(\mathcal{O}_v) \\ \downarrow & & \downarrow \\ \mathcal{S}(\kappa(w)) & \to & \mathcal{S}(\kappa(v)) \end{array}$$

(ii) *If $i : E \subset F$ is an extension in $\mathcal{F}_k$, and v a discrete valuation on F which restricts to 0 on E then the map $\mathcal{S}(i) : \mathcal{S}(E) \to \mathcal{S}(F)$ has its image contained in $\mathcal{S}(\mathcal{O}_v)$.*

(iii) *if moreover $\kappa(v)$ is separable over k, then if we let $j : E \subset \kappa(v)$ denotes the induced extension the composition $\mathcal{S}(E) \to \mathcal{S}(\mathcal{O}_v) \stackrel{s_v}{\to} \mathcal{S}(\kappa(v))$ is equal to $\mathcal{S}(j)$.*

(A4) **(i)** *For any $X \in Sm_k$, any point $z \in X^{(2)}$ of codimension 2, and for any point $y_0 \in X^{(1)}$ such that $z \in \overline{y}_0$ and such that $\overline{y}_0 \in Sm_k$, the* *map* $s_{y_0} : \mathcal{S}(\mathcal{O}_{y_0}) \to \mathcal{S}(\kappa(y_0))$ *maps* $\bigcap_{y\in(X_z)^{(1)}} \mathcal{S}(\mathcal{O}_y)$ *into* $\mathcal{S}(\mathcal{O}_{\overline{y}_0,z}) \subset \mathcal{S}(\kappa(y_0))$.

(ii) *Moreover if* $\kappa(z)$ *is separable over* k*, the composition*

$$\bigcap_{y\in X^{(1)}} \mathcal{S}(\mathcal{O}_y) \to \mathcal{S}(\mathcal{O}_{z_0}) \to \mathcal{S}(\kappa(z))$$

doesn't depend on the choice of y_0.

Remark 1.10 The Axiom **(A4)** has a special"role". When we will construct unramified Milnor-Witt K-theory in Section 2.2 below, this axiom will appear to be the most difficult to check. In fact the subsection 1.3 is devoted to develop some technic to check this axiom in special case. In Rost's approach [44] this axiom follows from the construction of the Rost's complex for 2-dimensional local smooth k-scheme. However the construction of this complex (even for dimension 2 schemes) requires transfers.□

Now we claim that an unramified sheaf of sets $\mathcal{S}$ on Sm_k defines an unramified $\mathcal{F}_k$-set. From what we have done before, we already have in hands an unramified $\tilde{\mathcal{F}}_k$-set $\mathcal{S}$. Now, for any discrete valuation v on $F \in \mathcal{F}_k$ with residue field $\kappa(v)$ separable over k, there is an obvious map $s_v : \mathcal{S}(\mathcal{O}_v) \to \mathcal{S}(\kappa(v))$, obtained by choosing smooth models over k for the closed immersion $Spec(\kappa(v)) \to Spec(\mathcal{O}_v)$. These together define the data **(D3)**. We now claim that these data satisfy the extra-axioms for unramified $\mathcal{F}_k$-sets. Axiom **(A3)** is easily checked by choosing convenient smooth models for $Spec(F)$, $Spec(\mathcal{O}_v)$ or $Spec(\kappa(v)$.

To check the axiom **(A4)** we use the commutative square:

$$\begin{array}{ccc} \mathcal{S}(X) & \subset & \mathcal{S}(\mathcal{O}_{y_0}) \\ \downarrow & & \downarrow \\ \mathcal{S}(\overline{y_0}) = \mathcal{S}(\mathcal{O}_z) & \subset & \mathcal{S}(\kappa(y_0)) \end{array}$$

and property **(2)**.

Theorem 1.11 *The restriction functor just constructed from unramified sheaves of sets on* Sm_k *to unramified* $\mathcal{F}_k$*-sets is an equivalence of categories.*

The Theorem follows clearly from the following more precise statement:

Lemma 1.12 *Given an unramified $\mathcal{F}_k$-set $\mathcal{S}$, there is a unique way to extend the unramified sheaf of sets $\mathcal{S} : (\tilde{Sm}_k)^{op} \to \mathcal{S}et$ to a sheaf $\mathcal{S} : (Sm_k)^{op} \to \mathcal{S}et$, such that for any discrete valuation v on $F \in \mathcal{F}_k$ with separable residue field, the map $\mathcal{S}(\mathcal{O}_v) \to \mathcal{S}(\kappa(v))$ induced by the sheaf structure is the specialization map $s_v : \mathcal{S}(\mathcal{O}_v) \to \mathcal{S}(\kappa(v))$. This sheaf is automatically unramified.*

Proof. We first define a restriction map $s(i) : \mathcal{S}(X) \to \mathcal{S}(Y)$ for a closed immersion $i : Y \hookrightarrow X$ in Sm_k of codimension 1. If $Y = \amalg_\alpha Y_\alpha$ is the decomposition of Y into irreducible components then $\mathcal{S}(Y) = \Pi_\alpha \mathcal{S}(Y_\alpha)$ and $s(i)$ has to be the product of the $s(i_\alpha) : \mathcal{S}(X) \to \mathcal{S}(Y_\alpha)$. We thus may assume Y (and X) irreducible. We then claim there exits a (unique) map $s(i) : \mathcal{S}(X) \to \mathcal{S}(Y)$ which makes the following diagram commute

$$\begin{array}{ccc} \mathcal{S}(X) & \stackrel{s(i)}{\to} & \mathcal{S}(Y) \\ \cap & & \cap \\ \mathcal{S}(\mathcal{O}_{X,y}) & \stackrel{s_y}{\to} & \mathcal{S}(\kappa(y)) \end{array}$$

where y is the generic point of Y. To check this it is sufficient to prove that for any $z \in Y^{(1)}$, the image of $\mathcal{S}(X)$ through s_y is contained in $\mathcal{S}(\mathcal{O}_{Y,z})$. But z has codimension 2 in X and this follows from the first part of axiom **(A4)**.

Now we have the following:

Lemma 1.13 *Let $i : Z \to X$ be a closed immersion in Sm_k of codimension $d > 0$. Assume there exists a factorization $Z \stackrel{j_1}{\to} Y_1 \stackrel{j_2}{\to} Y_2 \to \cdots \stackrel{j_d}{\to} Y_d = X$ of i into a composition of codimension 1 closed immersions, with the Y_i closed subschemes of X each of which is smooth over k. Then the composition*

$$\mathcal{S}(X) \stackrel{s(j_d)}{\to} \cdots \to \mathcal{S}(Y_2) \stackrel{s(j_2)}{\to} \mathcal{S}(Y_1) \stackrel{s(j_1)}{\to} \mathcal{S}(Z)$$

doesn't depend on the choice of the above factorization of i. We denote this composition by $\mathcal{S}(i)$.

Proof. We proceed by induction on d. For $d = 1$ there is nothing to prove. Assume $d \geq 2$. We may easily reduce to the case Z is irreducible with generic point z. We have to show that the composition

$$\mathcal{S}(X) \stackrel{s(j_d)}{\to} \cdots \to \mathcal{S}(Y_2) \stackrel{s(j_2)}{\to} \mathcal{S}(Y_1) \stackrel{s(j_1)}{\to} \mathcal{S}(Z) \subset \mathcal{S}(\kappa(z))$$

doesn't depend on the choice of the flag $Z \to Y_1 \to \cdots \to \ldots \to X$. We may thus replace X by any open neighborhood Ω of z if we want and even by $Spec(\mathcal{O}_{X,z})$ if necessary.

We first observe that the case $d = 2$ follows directly from the Axiom **(A4)**.

In general as $\mathcal{O}_{X,z}$ is regular of dimension d there exists such an Ω and a sequence of elements $(x_1, \ldots, x_d) \in \mathcal{O}(\Omega)$ which generates the maximal ideal m_v of $A := \mathcal{O}_{X,z}$ and such that the flag

$$Spec(A/(x_1, \ldots, x_d)) \to Spec(A/(x_2, \ldots, x_d) \to \cdots \to Spec(A/(x_d)) \to Spec(A)$$

is the induced flag $Z \cap \Omega \to Y_1 \cap \Omega \to Y_2 \cap \Omega \to \ldots \to \Omega$.

We have thus reduced to proving that given $z \in X^{(d)}$ a point of codimension d, with separable residue field, in a smooth k-scheme X, and with $A = \mathcal{O}_{X,z}$, and given a sequence $(x_1, \ldots, x_d)$ whose associated flag of closed subschemes of $Spec(A)$ consists of smooth k-schemes, the composition

$$\mathcal{S}(A) \to \mathcal{S}(Spec(A/(x_d))) \to \cdots \to \mathcal{S}(Spec(A/(x_2, \ldots, x_d)) \to \mathcal{S}(\kappa)$$

doesn't depend on the choice of $(x_1, \ldots, x_d)$.

As $\kappa(v)$ is separable over k, by [17, Corollaire (**17.12.2**)] the conditions on smoothness on the members of the associated flag to the sequence $(x_1, \ldots, x_d)$ is equivalent to the fact the family $(x_1, \ldots, x_d)$ reduces to a basis of the $\kappa(v)$-vector space $m_v/(m_v)^{\times 2}$.

As a consequence, if $M \in \mathbb{G}L_d(A)$, the sequence $M.(x_i)$ also satisfies this assumption. For instance any permutation on the $(x_1, \ldots, x_d)$ yields an other such sequence. By the case of codimension 2 which was observed above, we see that if we permute x_i and x_{i+1} the compositions $\mathcal{S}(A) \to \mathcal{S}(\kappa(v))$ are the same before or after permutation. We get this way that we may permute as will the x_i's.

Now assume that $(x'_1, \ldots, x'_d)$ is an other sequence in A satisfying the same assumption. Write the x'_i as linear combination in the x_j. We get a matrix $M \in M_d(A)$ with $(x'_i) = M.(x_j)$. This matrix reduces in $M_d(\kappa)$ to an invertible matrix by what we just observed above; thus M itself is invertible. Clearly, one may multiply in a sequence $(x_1, \ldots, x_d)$ by a unit of A an element x_i of the sequence without changing the flag (and thus the composition). Thus we may assume $det(M) = 1$. Now for a local ring A we know that the group $SL_d(A)$ is the group $E_d(A)$ of elementary matrices in A (see [23, Chapter VI Corollary 1.5.3] for instance). Thus M can be written as a product of elementary matrices in $M_d(A)$.

As we already know that our statement doesn't depend on the ordering of a sequence, we have reduced to the following claim: given a sequence $(x_1, \dots, x_d)$ as above and $a \in A$, the $(x_1 + ax_2, x_2, \dots, x_d)$ induces the same composition $\mathcal{S}(A) \to \mathcal{S}(\kappa(v))$ as $(x_1, \dots, x_d)$. But in fact the flags are the same. This proves our claim.□

Now we come back to the proof of the Lemma 1.12. Let $i : Z \to X$ be a closed immersion in Sm_k. By what has been recalled in the previous proof, X can be covered by open subsets U's such that the induced closed immersion $Z \cap U \to U$ admits a factorization as in the statement of the previous Lemma 1.13. Thus for each such U we get a canonical map $s_U : \mathcal{S}(U) \to \mathcal{S}(Z \cap U)$. But applying the same Lemma to the intersections $U \cap U'$, with U' an other such open subset, we see that the s_U are compatible and define a canonical map: $s(i) : \mathcal{S}(X) \to \mathcal{S}(Z)$.

Let $f : Y \to X$ be any morphism between smooth (quasi-projective) k-schemes. Then f is the composition $Y \hookrightarrow Y \times_k X \to X$ of the closed immersion (given by the graph of f) $\Gamma_f : Y \hookrightarrow Y \times_k X$ and the smooth projection $p_X : Y \times_k X \to X$. We set

$$s(f) := \mathcal{S}(X) \overset{s(p_X)}{\to} \mathcal{S}(Y \times_k X) \overset{s(\Gamma_f)}{\to} \mathcal{S}(Y)$$

To check that this defines a functor on $(Sm_k)^{op}$ is not hard. First given a smooth morphism $\pi : X' \to X$ and a closed immersion $i : Z \to X$ in Sm_k, denote by $i" : Z' \to X'$ the inverse image of i through π and by $\pi' : Z' \to Z$ the obvious smooth morphism. Then the following diagram is clearly commutative

$$\begin{array}{ccc} \mathcal{S}(X) & \overset{s(\pi)}{\to} & \mathcal{S}(X') \\ \downarrow s(i) & & \downarrow s(i') \\ \mathcal{S}(Z) & \overset{s(\pi')}{\to} & \mathcal{S}(Z') \end{array}$$

Then, to prove the functoriality, one takes two composable morphism $Z \stackrel{g}{\to} Y \stackrel{f}{\to} X$ and contemplates the diagram

$$\begin{array}{ccccc} Z & \hookrightarrow & Z \times_k Y & \hookrightarrow & Z \times_k Y \times_k X \\ || & & \downarrow & & \downarrow \\ Z & \to & Y & \hookrightarrow & Y \times_k X \\ || & & || & & \downarrow \\ Z & \to & Y & \to & X \end{array}$$

Then one realizes that applying $\mathcal{S}$ and s yields a commutative diagram, proving the claim. Now the presheaf $\mathcal{S}$ on Sm_k is obviously an unramified sheaf on Sm_k as these properties only depend on its restriction to $\tilde{Sm}_k$.□

Remark 1.14 Again, from now on in this paper, we will not distinguish between the notion of unramified $\mathcal{F}_k$-set and that of unramified sheaf of sets on Sm_k. If $\mathcal{S}$ is an unramified $\mathcal{F}_k$-set we still denote by $\mathcal{S}$ the associated unramified sheaf of sets on Sm_k.□

Remark 1.15 The proof of Lemma 1.12 also shows the following. Let $\mathcal{S}$ and $\mathcal{E}$ be sheaves of sets on Sm_k, with $\mathcal{S}$ unramified and $\mathcal{E}$ satisfying conditions **(0)** and **(1)** of unramified presheaves. Then to give a morphism of sheaves $\Phi : \mathcal{E} \to \mathcal{S}$ is equivalent to give a natural transformation $\phi : \mathcal{E}|_{\mathcal{F}_k} \to \mathcal{S}|_{\mathcal{F}_k}$ such that:

1) for any discrete valuation v on $F \in \mathcal{F}_k$, the image of $\mathcal{E}(\mathcal{O}_v) \subset \mathcal{E}(F)$ through ϕ is contained in $\mathcal{S}(\mathcal{O}_v) \subset \mathcal{S}(F)$;

2) if moreover the residue field of v is separable over k, then the induced square commutes:

$$\begin{array}{ccc} \mathcal{E}(\mathcal{O}_v) & \stackrel{s_v}{\to} & \mathcal{E}(\kappa(v)) \\ \downarrow \phi & & \downarrow \phi \\ \mathcal{S}(\mathcal{O}_v) & \to & \mathcal{S}(\kappa(v)) \end{array}$$

We left the details to the reader.□

$\mathbb{A}^1$-invariant unramified sheaves.

Lemma 1.16 *1) Let $\mathcal{S}$ be an unramified sheaf of sets on $\tilde{Sm}_k$. Then $\mathcal{S}$ is $\mathbb{A}^1$-invariant if and only if it satisfies the following:*

For any k-smooth local ring A of dimension ≤ 1 the canonical map $\mathcal{S}(A) \to \mathcal{S}(\mathbb{A}^1_A)$ is bijective.

2) Let $\mathcal{S}$ be an unramified sheaf of sets on Sm_k. Then $\mathcal{S}$ is $\mathbb{A}^1$-invariant if and only if it satisfies the following:

For any $F \in \mathcal{F}_k$ the canonical map $\mathcal{S}(F) \to \mathcal{S}(\mathbb{A}^1_F)$ is bijective.

Proof. 1) One implication is clear. Let's prove the other one. Let $X \in Sm_k$ be irreducible with function field F. In the following commutative square

$$\begin{array}{ccc} \mathcal{S}(X) & \to & \mathcal{S}(\mathbb{A}^1_X) \\ \downarrow & & \downarrow \\ \mathcal{S}(F) & \to & \mathcal{S}(F(T)) \end{array}$$

each map is injective. We observe that $\mathcal{S}(\mathbb{A}^1_X) \to \mathcal{S}(F(T)$ factors as $\mathcal{S}(\mathbb{A}^1_X) \to \mathcal{S}(\mathbb{A}^1_F) \to \mathcal{S}(F(T)$. By our assumption $\mathcal{S}(F) = \mathcal{S}(\mathbb{A}^1_F)$; this proves that $\mathcal{S}(\mathbb{A}^1_X)$ is contained inside $\mathcal{S}(F)$. Now it is sufficient to prove that for any $x \in X^{(1)}$ one has the inclusion $\mathcal{S}(\mathbb{A}^1_X) \subset \mathcal{S}(\mathcal{O}_{X,x}) \subset \mathcal{S}(F)$. But $\mathcal{S}(\mathbb{A}^1_X) \subset \mathcal{S}(\mathbb{A}^1_{\mathcal{O}_{X,x}}) \subset \mathcal{S}(F(T)$, and our assumption gives $\mathcal{S}(\mathcal{O}_{X,x}) = \mathcal{S}(\mathbb{A}^1_{\mathcal{O}_{X,x}})$. This proves the claim.

2) One implication is clear. Let's prove the other one. Let $X \in Sm_k$ be irreducible with function field F. In the following commutative square

$$\begin{array}{ccc} \mathcal{S}(\mathbb{A}^1_X) & \subset & \mathcal{S}(\mathbb{A}^1_F) \\ \downarrow & & || \\ \mathcal{S}(X) & \subset & \mathcal{S}(F) \end{array}$$

each map is injective but maybe the left vertical one. The latter is thus also injective which clearly implies the statement.$\square$

Remark 1.17 Given an unramified sheaf $\mathcal{S}$ of sets on $\tilde{Sm}_k$ with Data **(D3)**, and satisfying the property that for any $F \in \mathcal{F}_k$, the map $\mathcal{S}(F) \to \mathcal{S}(F(T))$ is injective, then $\mathcal{S}$ is an unramified $\mathcal{F}_k$-group if and only if its extension to $k(T)$ is an unramified $\mathcal{F}_{k(T)}$-set.

Indeed, given a smooth irreducible k-scheme X, a point $x \in X$ of codimension d, then $X|_{k(T)}$ is still irreducible $k(T)$-smooth and $\overline{x}|_{k(T)}$ is irreducible and has codimension d in $X|_{k(T)}$. Moreover the maps $M(X) \to M(X|_{k(T)})$, $M(X_x) \to M((X|_{k(T)})_{\overline{x}|_{k(T)}})$, etc.. are injective. So to check the Axioms involving equality between morphisms, etc..., it suffices to check them over $k(T)$ for $M|_{k(T)}$. This allows us to reduce the checking of several Axioms like **(A4)** to the case k is infinite.$\square$

1.2 Unramified sheaves of groups and strong $\mathbb{A}^1$-invariance

Our aim in this section is to study unramified sheaves of groups $\mathcal{G}$ on Sm_k (or on $\tilde{Sm}_k$), as well as their potential strong $\mathbb{A}^1$-invariance property, as well as the comparison between H^1 in Zariski and Nisnevich topology. By an "unramified sheaf of groups" we mean a sheaf of groups on Sm_k (or on $\tilde{Sm}_k$) whose underlying sheaf of sets is unramified in the sense of the previous section.

Let $\mathcal{G}$ be such an unramified sheaf of groups on Sm_k (or $\tilde{Sm}_k$). For any discrete valuation v on $F \in \mathcal{F}_k$ we introduce the pointed set

$$H^1_v(\mathcal{O}_v;\mathcal{G}) := \mathcal{G}(F)/\mathcal{G}(\mathcal{O}_v)$$

and we observe this is a left $\mathcal{G}(F)$-set. More generally for y a point of codimension 1 in $X \in Sm_k$, we set $H^1_y(X;\mathcal{G}) = H^1_y(\mathcal{O}_{X,y};\mathcal{G})$. By axiom **(A2)**, is X is irreducible with function field F the induced left action of $\mathcal{G}(F)$ on $\Pi_{y\in X^{(1)}} H^1_y(X;\mathcal{G})$ preserves the weak-product

$$\Pi'_{y\in X^{(1)}} H^1_y(X;\mathcal{G}) \subset \Pi_{y\in X^{(1)}} H^1_y(X;\mathcal{G})$$

where the weak-product $\Pi'_{y\in X^{(1)}} H^1_y(X;\mathcal{G})$ means the set of families for which all but a finite number of terms are the base point of $H^1_y(X;\mathcal{G})$. By definition, the isotropy subgroup of this action of $\mathcal{G}(F)$ on the base point of $\Pi'_{y\in X^{(1)}} H^1_y(X;\mathcal{G})$ is exactly $\mathcal{G}(X) = \cap_{y\in X^{(1)}}\mathcal{G}(\mathcal{O}_{X,y})$. We will summarize this property by saying that the diagram (of groups, action and pointed set)

$$1 \to \mathcal{G}(X) \to \mathcal{G}(F) \Rightarrow \Pi'_{y\in X^{(1)}} H^1_y(X;\mathcal{G})$$

is "exact" (the double arrow refereing to a left action).

Definition 1.18 *For any point z of codimension 2 in a smooth k-scheme X, we denote by $H^2_z(X;\mathcal{G})$ the orbit set of $\Pi'_{y\in X_z^{(1)}} H^1_y(X;\mathcal{G})$ under the left action of $\mathcal{G}(F)$, where $F \in \mathcal{F}_k$ denotes the field of functions of X_z.*

Now for an irreducible smooth k-scheme X with function field F we may define an obvious "boundary" $\mathcal{G}(F)$-equivariant map

$$\Pi'_{y\in X^{(1)}} H^1_y(X;\mathcal{G}) \to \Pi_{z\in X^{(2)}} H^2_z(X;\mathcal{G}) \tag{1.1}$$

by collecting together the compositions, for each $z \in X^{(2)}$:

$$\Pi'_{y\in X^{(1)}} H^1_y(X;\mathcal{G}) \to \Pi'_{y\in X^{(1)}_z} H^1_y(X;\mathcal{G}) \to H^2_z(X;\mathcal{G})$$

It is not clear in general whether or not the image of the boundary map is always contained in the weak product $\Pi'_{z\in X^{(2)}} H^2_z(X;\mathcal{G})$. We will use the following Axiom depending on $\mathcal{G}$ and an integer d which completes **(A2)**:

(A2') For any smooth k-scheme X of dimension d, the image of the boundary map (1.1) is contained in the weak product $\Pi'_{z\in X^{(2)}} H^2_z(X;\mathcal{G})$.□

Remark 1.19 Given an unramified sheaf $\mathcal{G}$ of groups on $\tilde{Sm}_k$ with Data **(D3)**, and satisfying the property that for any $F \in \mathcal{F}_k$, the map $\mathcal{G}(F) \to \mathcal{G}(F(T))$ is injective, then $\mathcal{G}$ satisfies **(A2')** if and only if its extension to $k(T)$ does. This is done along the same lines as in Remark 1.17.□

We assume from now on that $\mathcal{G}$ satisfies **(A2')**. Altogether we get for X smooth over k, irreducible with function field F, a "complex" $C^*(X;\mathcal{G})$ of groups, action, and pointed sets:

$$1 \to \mathcal{G}(X) \subset \mathcal{G}(F) \rightrightarrows \Pi'_{y\in X^{(1)}} H^1_y(X;\mathcal{G}) \to \Pi_{z\in X^{(2)}} H^2_z(X;\mathcal{G})$$

We set for $X \in Sm_k$: $\mathcal{G}^{(0)}(X) := \Pi'_{x\in X^{(0)}}\mathcal{G}(\kappa(x))$, $\mathcal{G}^{(1)}(X) := \Pi'_{y\in X^{(1)}} H^1_y(X;\mathcal{G})$ and $\mathcal{G}^{(2)}(X) := \Pi'_{z\in X^{(2)}} H^2_z(X;\mathcal{G})$. The correspondence $X \mapsto \mathcal{G}^{(i)}(X)$, $i \leq 2$, can be obviously extended to an unramified presheaf of groups on $\tilde{Sm}_k$, which we still denote by $\mathcal{G}^{(i)}$. Note that $\mathcal{G}^{(0)}$ is a sheaf in the Nisnevich topology. However for $\mathcal{G}^{(i)}$, $i \in \{1,2\}$ it is not the case in general, these are only sheaves in the Zariski topology, as any unramified presheaf.

The complex $C^*(X;\mathcal{G}) : 1 \to \mathcal{G}(X) \to \mathcal{G}^{(0)}(X) \rightrightarrows \mathcal{G}^{(1)}(X) \to \mathcal{G}^{(2)}(X)$ will play in the sequel the role of the (truncated) analogue for $\mathcal{G}$ of the Cousin complex of [9] or of the complex of Rost considered in [44].

Definition 1.20 *Let $1 \to H \subset G \rightrightarrows E \to F$ be a sequence with G a group acting on the set E which is pointed (as a set not as a G-set), with $H \subset G$ a subgroup and $E \to F$ a G-equivariant map of sets, with F endowed with the trivial action. We shall say this sequence is* exact *if the isotropy subgroup of the base point of E is H and if the "kernel" of the pointed map $E \to F$ is equal to the orbit under G of the base point of E.*

We shall say that it is exact in the strong *sense if moreover the map $E \to F$ induces an injection into F of the (left) quotient set ${}_G\backslash E \subset F$.*

By construction $C^*(X;\mathcal{G})$ is exact in the strong sense, for X smooth local of dimension ≤ 2.

Let us denote by $\mathcal{Z}^1(-;\mathcal{G}) \subset \mathcal{G}^{(1)}$ the sheaf theoretic orbit of the base point under the action of $\mathcal{G}^{(0)}$ in the Zariski topology on $\tilde{Sm}_k$. We thus have an exact sequence of sheaves on $\tilde{Sm}_k$ in the Zariski topology

$$1 \to \mathcal{G} \subset \mathcal{G}^{(0)} \rightrightarrows \mathcal{Z}^1(-;\mathcal{G}) \to *$$

As it is clear that $H^1_{Zar}(X;\mathcal{G}^{(0)})$ is trivial (the sheaf $\mathcal{G}^{(0)}$ being flasque), this yields for any $X \in Sm_k$ an exact sequence (of groups and pointed sets)

$$1 \to \mathcal{G}(X) \subset \mathcal{G}^{(0)}(X) \rightrightarrows \mathcal{Z}^1(X;\mathcal{G}) \to H^1_{Zar}(X;\mathcal{G}) \to *$$

in the strong sense.

Remark 1.21 If X is an (essentially) smooth k-scheme of dimension ≤ 1, we thus get a bijection $H^1_{Zar}(X;\mathcal{G}) = {}_{\mathcal{G}^{(0)}(X)}\backslash\mathcal{G}^{(1)}(X)$. For instance, when X is a smooth local k-scheme of dimension 2, and if $V \subset X$ is the complement of the closed point, a smooth k-scheme of dimension 1, we thus get a bijection

$$H^2_z(X;\mathcal{G}) = H^1_{Zar}(V;\mathcal{G})$$

Beware that here the Zariski topology is used. This gives a “concrete” interpretation of the “strange” extra cohomology set $H^2_z(X;\mathcal{G})$.$\square$

For $X \in Sm_k$ let us denote by $\mathcal{K}^1(X;\mathcal{G}) \subset \Pi'_{y\in X^{(1)}} H^1_y(X;\mathcal{G})$ the kernel of the boundary map $\Pi'_{y\in X^{(1)}} H^1_y(X;\mathcal{G}) \to \Pi_{z\in X^{(2)}} H^2_z(X;\mathcal{G})$. The correspondence $X \mapsto \mathcal{K}^1(X;\mathcal{G})$ is a sheaf in the Zariski topology on $\tilde{Sm}_k$. There is an obvious injective morphism of sheaves in the Zariski topology on $\tilde{Sm}_k$: $\mathcal{Z}^1(-;\mathcal{G}) \to \mathcal{K}^1(-;\mathcal{G})$. As $C^*(X;\mathcal{G})$ is exact for any k-smooth local X of dimension ≤ 2, $\mathcal{Z}^1(-;\mathcal{G}) \to \mathcal{K}^1(-;\mathcal{G})$ induces a bijection for any (essentially) smooth k-scheme of dimension ≤ 2.

Remark 1.22 If X is an (essentially) smooth k-scheme of dimension ≤ 2, we thus get that the H^1 of the complex $C^*(X;\mathcal{G})$ is $H^1_{Zar}(X;\mathcal{G})$.$\square$

Now we introduce the following axiom on $\mathcal{G}$:

(A5) **(i)** For any separable finite extension $E \subset F$ in $\mathcal{F}_k$, any discrete valuation v on F which restricts to a discrete valuation w on E with ramification index 1, and such that the induced extension $\bar{i} : \kappa(w) \to \kappa(v)$ is an isomorphism, the commutative square of groups

$$\begin{array}{ccc} \mathcal{G}(\mathcal{O}_w) & \subset & \mathcal{G}(E) \\ \downarrow & & \downarrow \\ \mathcal{G}(\mathcal{O}_v) & \subset & \mathcal{G}(F) \end{array}$$

induces a bijection $H^1_v(\mathcal{O}_v; \mathcal{G}) \to H^1_w(\mathcal{O}_w; \mathcal{G})$.

(ii) For any étale morphism $X' \to X$ between smooth local k-schemes of dimension 2, with closed point respectfully z' and z, inducing an isomorphism on the residue fields $\kappa(z) \cong \kappa(z')$, the pointed map

$$H^2_z(X; \mathcal{G}) \to H^2_{z'}(X'; \mathcal{G})$$

has trivial kernel.□

Remark 1.23 In case $\mathcal{G}$ is an abelian sheaf of groups, the point (ii) of this axioms implies more: using the Mayer-Vietoris exact sequence, we easily see that in fact the map (a group homomorphism indeed) $H^2_z(X; \mathcal{G}) \to H^2_{z'}(X'; \mathcal{G})$ involved in the previous Lemma is also surjective, thus an isomorphism.□

Lemma 1.24 *Let $\mathcal{G}$ be as above. Then the following conditions are equivalent:*

(i) the Zariski sheaf $X \mapsto \mathcal{K}^1(X; \mathcal{G})$ is a sheaf in the Nisnevich topology on $\tilde{Sm}_k$;

(ii) for any smooth k-scheme X of dimension ≤ 2 the comparison map $H^1_{Zar}(X; \mathcal{G}) \to H^1_{Nis}(X; \mathcal{G})$ is a bijection;

(iii) $\mathcal{G}$ satisfies Axiom **(A5)**

Proof. (i) $\Rightarrow$ (ii). Under (i) we know that $X \mapsto \mathcal{Z}^1(X; \mathcal{G})$ is a sheaf in the Nisnevich topology on smooth k-schemes of dimension ≤ 2 (as $\mathcal{Z}^1(X; \mathcal{G}) \to \mathcal{K}^1(X; \mathcal{G})$ is an isomorphism on smooth k-schemes of dimension ≤ 2). The exact sequence in the Zariski topology $1 \to \mathcal{G} \subset \mathcal{G}^{(0)} \rightrightarrows \mathcal{Z}^1(-; \mathcal{G}) \to *$ considered above is then also an exact sequence of sheaves in the Nisnevich topology. The same reasoning as above easily implies (ii), taking into account that $H^1_{Nis}(X; \mathcal{G}^{(0)})$ is also trivial (easy and left to the reader).

(ii) $\Rightarrow$ (iii). Assume (ii). Let's prove **(A5) (i)**. With the assumptions given the square

$$\begin{array}{ccc} Spec(F) & \rightarrow & Spec(\mathcal{O}_v) \\ \downarrow & & \downarrow \\ Spec(E) & \rightarrow & Spec(\mathcal{O}_w) \end{array}$$

is a distinguished square in the sense of [39]. Using the corresponding Mayer-Vietoris type exact sequence and the fact by (ii) that $H^1(X;\mathcal{G}) = *$ for any smooth local scheme X yields immediately that $\mathcal{G}(E)/\mathcal{G}(\mathcal{O}_w) \rightarrow \mathcal{G}(F)/\mathcal{G}(\mathcal{O}_v)$ is a bijection.

Now let's prove **(A5) (ii)**. Set $V = X - \{z\}$ and $V' = X' - \{z'\}$. The square

$$\begin{array}{ccc} V' & \subset & X' \\ \downarrow & & \downarrow \\ V & \subset & X \end{array}$$

is distinguished. From the discussion preceding the Lemma and the interpretation of $H^2_z(X;\mathcal{G})$ as $H^1_{Zar}(V;\mathcal{G})$, the kernel in question is thus the set of (isomorphism classes) of $\mathcal{G}$-torsors over V (indifferently in Zariski and Nisnevich topology as $H^1_{Zar}(V;\mathcal{G}) \cong H^1_{Nis}(V;\mathcal{G})$ by (ii)) which become trivial over V'; but such a torsor can thus be extended to X' and by a descent argument in the Nisnevich topology, we may extend the torsor on V to X. Thus it is trivial because X is local.

(iii) $\Rightarrow$ (i). Now assume Axiom **(A5)**. We claim that Axiom **(A5) (i)** gives exactly that $X \mapsto \mathcal{G}^{(1)}(X)$ is a sheaf in the Nisnevich topology. **(A5) (ii)** is easily seen to be what exactly what is needed to imply that $\mathcal{K}^1(-;\mathcal{G})$ is a sheaf in the Nisnevich topology.□

The monomorphism of Zariski sheaves $\mathcal{Z}^1(-;\mathcal{G}) \rightarrow \mathcal{K}^1(-;\mathcal{G})$ is $\mathcal{G}^{(0)}$-equivariant.

Lemma 1.25 *Assume $\mathcal{G}$ satisfies* **(A5)**. *Fix an integer $d \geq 0$. The following conditions are equivalent:*

(i) For any smooth k-scheme X of dimension $\leq d$ the map $\mathcal{Z}^1(X;\mathcal{G}) \rightarrow \mathcal{K}^1(X;\mathcal{G})$ is bijective;

(ii) For any local smooth k-scheme U of dimension $\leq d$ the map $\mathcal{Z}^1(U;\mathcal{G}) \rightarrow \mathcal{K}^1(U;\mathcal{G})$ is bijective;

(iii) For any local smooth k-scheme U of dimension $\leq d$ with function field F, the complex $C^*(U;\mathcal{G}) : 1 \rightarrow \mathcal{G}(U) \rightarrow \mathcal{G}(F) \Rightarrow \mathcal{G}^{(1)}(U) \rightarrow \mathcal{G}^{(2)}(U)$ is exact.

When this conditions are satisfied, for any smooth k-scheme X of dimension $\leq d$ the comparison map $H^1_{Zar}(X;\mathcal{G}) \to H^1_{Nis}(X;\mathcal{G})$ is a bijection.

Proof. (i) $\Leftrightarrow$ (ii) is clear as both are Zariski sheaves. (ii) $\Rightarrow$ (iii) is proven exactly as in the proof of of (ii) in Lemma 1.24. (iii) $\Rightarrow$ (i) is also clear using the given expressions of the two sides.

If we assume these conditions are satisfied, then

$$_{\mathcal{G}^{(0)}(X)}\backslash \mathcal{Z}^1(X;\mathcal{G}) = H^1_{Zar}(X;\mathcal{G}) \to H^1_{Nis}(X;\mathcal{G}) = \ _{\mathcal{G}^{(0)}(X)}\backslash \mathcal{K}^1(X;\mathcal{G})$$

is a bijection. The last equality follows from the fact that $\mathcal{K}^1(;\mathcal{G})$ is a Nisnevich sheaf and the (easy) fact that $H^1_{Nis}(X;\mathcal{G}^{(0)})$ is also trivial.$\square$

Now we will use one more extra Axiom concerning $\mathcal{G}$ and related to $\mathbb{A}^1$-invariance properties:

(A6) For any localization U of a smooth k-scheme at some point u of codimension ≤ 1, the "complex":

$$1 \to \mathcal{G}(\mathbb{A}^1_U) \subset \mathcal{G}^{(0)}(\mathbb{A}^1_U) \rightrightarrows \mathcal{G}^{(1)}(\mathbb{A}^1_U) \to \mathcal{G}^{(2)}(\mathbb{A}^1_U)$$

is exact. Moreover the morphism $\mathcal{G}(U) \to \mathcal{G}(\mathbb{A}^1_U)$ is an isomorphism.$\square$

Observe that if $\mathcal{G}$ satisfies **(A6)** it is $\mathbb{A}^1$-invariant by Lemma 1.16 (as $\mathcal{G}$ is assumed to be unramified). Observe also that if $\mathcal{G}$ satisfies Axioms **(A2')** and **(A5)**, then we know by Lemma 1.24 that $H^1_{Nis}(\mathbb{A}^1_X;\mathcal{G}) = H^1_{Zar}(\mathbb{A}^1_X;\mathcal{G}) = H^1(\mathbb{A}^1_X;\mathcal{G})$ for X smooth of dimension ≤ 1.

Our main result in this subsection is the following. Observe that in this statement we need to assume that $\mathcal{G}$ is an unramified sheaf of groups on Sm_k (and not only on $\tilde{Sm}_k$). The reason comes from the proof of Lemma 1.30 which uses at some point a restriction to a smooth divisor.

Theorem 1.26 *Assume k is infinite. Let $\mathcal{G}$ be an unramified sheaf of groups on Sm_k that satisfies Axioms* **(A2')**, **(A5)** *and* **(A6)**. *Then it is strongly $\mathbb{A}^1$-invariant. Moreover, for any smooth k-scheme X, the comparison map*

$$H^1_{Zar}(X;\mathcal{G}) \to H^1_{Nis}(X;\mathcal{G})$$

is a bijection.

Remark 1.27 1) When k is a finite field one can show that an unramified sheaf of groups on Sm_k which satisfies Axioms **(A2')**, **(A5)** and **(A6)** is also strongly $\mathbb{A}^1$-invariant. However we can't prove that the comparison map is a bijection, we only know that its kernel is trivial. We won't use these facts.

2) We will show conversely that a strongly $\mathbb{A}^1$-invariant sheaf of groups G on Sm_k, k any field, is always unramified. This is proven when k is perfect in the Appendix, Theorem A.1. The case when k is infinite is done in Corollary 3.8. We prove moreover in Theorem 3.9 that if k is infinite $\mathcal{G}$ satisfies axioms **(A2')**, **(A5)** and **(A6)**.

We thus obtain in the case k is infinite, an equivalence between the category of strongly $\mathbb{A}^1$-invariant sheaves of groups on Sm_k and that of unramified sheaves of groups on Sm_k satisfying axioms **(A2')**, **(A5)** and **(A6)**.

We believe that the result should still hold over a finite field.□

To prove theorem 1.26 we fix an unramified sheaf of groups $\mathcal{G}$ on Sm_k which satisfies the Axioms **(A2')**, **(A5)** and **(A6)**.

We now introduce two properties depending on $\mathcal{G}$ and an integer $d \geq 0$:

(H1) (d) For any local smooth k-scheme of dimension $\leq d$ the complex $1 \to \mathcal{G}(U) \subset \mathcal{G}^{(0)}(U) \rightrightarrows \mathcal{G}^{(1)}(U) \to \mathcal{G}^{(2)}(U)$ is exact.□

(H2) (d) For any localization U of a smooth k-scheme at some point u of codimension $\leq d$, the "complex":

$$1 \to \mathcal{G}(\mathbb{A}^1_U) \subset \mathcal{G}^{(0)}(\mathbb{A}^1_U) \rightrightarrows \mathcal{G}^{(1)}(\mathbb{A}^1_U) \to \mathcal{G}^{(2)}(\mathbb{A}^1_U)$$

is exact.□

(H1) (d) is a reformulation of (ii) of Lemma 1.25. It is a tautology in case $d \leq 2$. **(H2) (d1)** holds by Axiom **(A6)**.

Lemma 1.28 *Let $d \geq 0$ be an integer.*

1) **(H1) (d)** $\Rightarrow$ **(H2) (d)**

2) If k is infinite: **(H2) (d)** $\Rightarrow$ **(H1) (d+1)**

Proof of Theorem 1.26 Assume that k is infinite. Lemma 1.28 implies by an easy induction that properties **(H1)(d)** and **(H2)(d)** hold, for any any d. Lemmas 1.25 and 1.29 below easily imply, form those, the statement of the Theorem.□

Lemma 1.29 *Assume $\mathcal{G}$ is $\mathbb{A}^1$-invariant. Fix an integer $d \geq 0$. The following conditions are equivalent:*

(i) For any smooth k-scheme X of dimension $\leq d$ the map

$$_{\mathcal{G}^{(0)}(X)}\backslash \mathcal{Z}^1(X;\mathcal{G}) = H^1_{Zar}(X;\mathcal{G}) \to H^1_{Zar}(\mathbb{A}^1_X;\mathcal{G}) = \ _{\mathcal{G}^{(0)}(\mathbb{A}^1_X)}\backslash \mathcal{Z}^1(\mathbb{A}^1_X;\mathcal{G})$$

is bijective;

(ii) For any local smooth k-scheme U of dimension $\leq d$

$$_{\mathcal{G}^{(0)}(\mathbb{A}^1_U)}\backslash \mathcal{Z}^1(\mathbb{A}^1_U;\mathcal{G}) = *$$

Proof. The implication (i) $\Rightarrow$ (ii) is clear as for U a smooth local k-scheme $H^1_{Zar}(U;\mathcal{G}) = \ _{\mathcal{G}^{(0)}(U)}\backslash \mathcal{Z}^1(U;\mathcal{G})$ is trivial. The implication (ii) $\Rightarrow$ (i) is proven as follows. (ii) means that $H^1_{Zar}(\mathbb{A}^1_U;\mathcal{G}) = *$ for any local smooth k-scheme U. Fix $X \in Sm_k$ and denote by $\pi : \mathbb{A}^1_X \to X$ the projection. To prove (i) for X it is sufficient to prove that the pointed simplicial sheaf of sets $R\pi_*(B(\mathcal{G}|_{\mathbb{A}^1_X}))$ has trivial π_0. Indeed, its π_1 sheaf is $\pi_*(\mathcal{G}|_{\mathbb{A}^1_X}) = \mathcal{G}|_X$ because $\mathcal{G}$ is $\mathbb{A}^1$-invariant. If the π_0 is trivial, $B(\mathcal{G}|_X) \to R\pi_*(B(\mathcal{G}|_{\mathbb{A}^1_X}))$ is a simplicial weak equivalence which implies the result. Now to prove the $\pi_0 R\pi_*((B(\mathcal{G}|_{\mathbb{A}^1_X})))$ is trivial, we just observe that its stalk a point $x \in X$ is $H^1_{Zar}(\mathbb{A}^1_{X_x};\mathcal{G})$ which is trivial by assumption.□

Proof of Lemma 1.28 Let $d \geq 2$ be an integer (else there is nothing to prove).

Let us prove 1). Assume that **(H1) (d)** holds. Let U be an irreducible smooth k-scheme with function field F. Let us study the following diagram whose middle row is $C^*(\mathbb{A}^1_U;\mathcal{G})$, whose bottom row is $C^*(U;\mathcal{G})$ and whose top row is $C^*(\mathbb{A}^1_F;\mathcal{G})$:

$$\begin{array}{ccccccc}
\mathcal{G}(F) & \subset & \mathcal{G}(F(T)) & \twoheadrightarrow & \Pi'_{y\in(\mathbb{A}^1_F)^{(1)}} H^1_y(\mathbb{A}^1_F;\mathcal{G}) & & \\
\cup & & || & & \uparrow & & \\
\mathcal{G}(\mathbb{A}^1_U) & \subset & \mathcal{G}(F(T)) & \Rightarrow & \Pi'_{y\in(\mathbb{A}^1_U)^{(1)}} H^1_y(\mathbb{A}^1_U;\mathcal{G}) & \to & \Pi'_{z\in(\mathbb{A}^1_U)^{(2)}} H^2_z(\mathbb{A}^1_U;\mathcal{G}) \\
|| & & \cup & & \uparrow & & \uparrow \\
\mathcal{G}(U) & \subset & \mathcal{G}(F) & \Rightarrow & \Pi'_{y\in U^{(1)}} H^1_y(U;\mathcal{G}) & \to & \Pi'_{z\in U^{(2)}} H^2_z(U;\mathcal{G})
\end{array} \tag{1.2}$$

The top horizontal row is exact by Axiom **(A6)**. Assume U is local of dimension $\le d$. The bottom horizontal row is exact by **(H1) (d)**. The middle vertical column can be explicited as follows. The points y of codimension 1 in $\mathbb{A}^1_U$ are of two types: either the image of y is the generic point of U or it is a point of codimension 1 in U; the first set is clearly in bijection with $(\mathbb{A}^1_F)^{(1)}$ and the second one with $U^{(1)}$ through the map $y\in U^{(1)}\mapsto y[T]:=\mathbb{A}^1_y\subset\mathbb{A}^1_U$. For y of the first type, it is clear that the set $H^1_y(\mathbb{A}^1_U;\mathcal{G})$ is the same as $H^1_y(\mathbb{A}^1_F;\mathcal{G})$. As a consequence, $\Pi'_{y\in(\mathbb{A}^1_U)^{(1)}} H^1_y(\mathbb{A}^1_U;\mathcal{G})$ is exactly the product of $\Pi'_{y\in(\mathbb{A}^1_F)^{(1)}} H^1_y(\mathbb{A}^1_F;\mathcal{G})$ and of $\Pi'_{y\in U^{(1)}} H^1_{y[T]}(\mathbb{A}^1_U;\mathcal{G})$.

To prove **(H2)(d)** we have exactly to prove the exactness of the middle horizontal row in (1.2) and more precisely that the action of $\mathcal{G}(F(T))$ on $\mathcal{K}^1(\mathbb{A}^1_U;\mathcal{G})$ is transitive.

Take $\alpha\in\mathcal{K}^1(\mathbb{A}^1_U;\mathcal{G})$. As the top horizontal row is exact, there is a $g\in\mathcal{G}(F(T))$ such that $g.\alpha$ lies in $\Pi'_{y\in U^{(1)}} H^1_{v[T]}(\mathbb{A}^1_U;\mathcal{G})\subset\Pi'_{y\in(\mathbb{A}^1_U)^{(1)}} H^1_y(\mathbb{A}^1_U;\mathcal{G})$, which is the kernel of the vertical $\mathcal{G}(F(T))$-equivariant map $\Pi'_{y\in(\mathbb{A}^1_U)^{(1)}} H^1_y(\mathbb{A}^1_U;\mathcal{G})\to \Pi'_{y\in(\mathbb{A}^1_F)^{(1)}} H^1_y(\mathbb{A}^1_F;\mathcal{G})$

Thus $g.\alpha$ lies in $\mathcal{K}^1(\mathbb{A}^1_U;\mathcal{G})\cap\Pi'_{y\in U^{(1)}} H^1_{y[T]}(\mathbb{A}^1_U;\mathcal{G})\subset\Pi'_{y\in(\mathbb{A}^1_U)^{(1)}} H^1_y(\mathbb{A}^1_U;\mathcal{G})$. Now the obvious inclusion $\mathcal{K}^1(U;\mathcal{G})\subset\mathcal{K}^1(\mathbb{A}^1_U;\mathcal{G})\cap\Pi'_{y\in U^{(1)}} H^1_{y[T]}(\mathbb{A}^1_U;\mathcal{G})$ is a bijection. Indeed, from part 1) of Lemma 1.30 below, $\Pi'_{y\in U^{(1)}} H^1_y(U;\mathcal{G})\subset\Pi'_{y\in U^{(1)}} H^1_{y[T]}(\mathbb{A}^1_U;\mathcal{G})$ is injective and is exactly the kernel of the composition of the boundary map $\Pi'_{y\in U^{(1)}} H^1_{y[T]}(\mathbb{A}^1_U;\mathcal{G})\to\Pi_{z\in(\mathbb{A}^1_U)^{(2)}} H^2_z(\mathbb{A}^1_U;\mathcal{G})$ and the projection

$$\Pi_{z\in(\mathbb{A}^1_U)^{(2)}} H^2_z(\mathbb{A}^1_U;\mathcal{G})\to\Pi_{y\in U^{(1)},z\in(\mathbb{A}^1_y)^{(1)}} H^2_z(\mathbb{A}^1_U;\mathcal{G})$$

This shows that $\mathcal{K}^1(\mathbb{A}^1_U;\mathcal{G})\cap\Pi'_{y\in U^{(1)}} H^1_{y[T]}(\mathbb{A}^1_U;\mathcal{G})$ is contained in $\Pi'_{y\in U^{(1)}} H^1_y(U;\mathcal{G})$.

But now, the right vertical map in (1.2), $\Pi_{z\in U^{(2)}} H^2_z(U;\mathcal{G}) \to \Pi_{z\in(\mathbb{A}^1_U)^{(2)}} H^2_z(\mathbb{A}^1_U;\mathcal{G})$, is induced by the correspondence $z \in U^{(2)} \mapsto \mathbb{A}^1_z \subset \mathbb{A}^1_U$ and the corresponding maps on $H^2_z(-;\mathcal{G})$. By part 2) of Lemma 1.30 below, this map has trivial kernel. This easily implies that $\mathcal{K}^1(\mathbb{A}^1_U;\mathcal{G}) \cap \Pi'_{y\in U^{(1)}} H^1_{y[T]}(\mathbb{A}^1_U;\mathcal{G})$ is contained in $\mathcal{K}^1(U;\mathcal{G})$, proving our claim.

Thus $g.\alpha$ lies in $\mathcal{K}^1(U;\mathcal{G})$. Now by **(H1) (d)** we know there is an $h \in \mathcal{G}(F)$ with $hg.\alpha = *$ as required.

Let us now prove 2). Assume **(H2) (d)** holds. Let's prove **(H1) (d+1)**. Let X be an irreducible smooth k-scheme (of finite type) of dimension $\leq d+1$ with function field F, let $u \in X \in Sm_k$ be a point of codimension $d+1$ and denote by U its associated local scheme, F its function field. We have to check the exactness at the middle of $\mathcal{G}(F) \Rightarrow \Pi'_{y\in U^{(1)}} H^1_y(U;\mathcal{G}) \to \Pi'_{z\in U^{(2)}} H^2_z(U;\mathcal{G})$.

Let $\alpha \in \mathcal{K}^1(U;\mathcal{G}) \subset \Pi'_{y\in U^{(1)}} H^1_y(U;\mathcal{G})$. We want to show that there exists $g \in \mathcal{G}(F)$ such that $\alpha = g.*$. Let us denote by $y_i \in U$ the points of codimension one in U where α is non trivial. Recall that for each $y \in U^{(1)}$, $H^1_y(U;\mathcal{G}) = H^1_y(X;\mathcal{G})$ where we still denote by $y \in X^{(1)}$ the image of y in X. Denote by $\alpha_X \in \Pi'_{y\in X^{(1)}} H^1_y(X;\mathcal{G})$ the canonical element with same support y_i's and same components as α. α_X may not be in $\mathcal{K}^1(X;\mathcal{G})$, but, by Axiom **(A2')**, its boundary its trivial except on finitely many points z_j of codimension 2 in X. Clearly these points are not in $U^{(2)}$, thus we may, up to removing the closure of these z_j's, find an open subscheme Ω' in X which contains u and the y_i's and such that the corresponding element induced by α $\alpha_{\Omega'} \in \Pi'_{y\in\Omega'^{(1)}} H^1_y(X;\mathcal{G})$ is in $\mathcal{K}^1(\Omega';\mathcal{G})$.

As k is infinite, by Gabber's presentation Lemma [13, 9] there exists an open subscheme Ω in Ω', containing u and the y_i's and an étale morphism $\Omega \to \mathbb{A}^1_V$, with V k-smooth of dimension d, such that if $Y \subset \Omega$ denotes the reduced closed subscheme whose generic points are the y_i, the composition $Y \to \Omega \to \mathbb{A}^1_V$ is still a closed immersion (and such that the composition $Y \to \Omega \to \mathbb{A}^1_V \to V$ is a finite morphism).

As U is the localization of Ω at u, the étale morphism $U \to \mathbb{A}^1_V$ induces a morphism of complexes of the form:

$$\begin{array}{ccccc} \mathcal{G}(F) & - & \Pi'_{y\in U^{(1)}} H^1_y(U;\mathcal{G}) & \to & \Pi'_{z\in U^{(2)}} H^2_z(U;\mathcal{G}) \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{G}(E(T)) & - & \Pi'_{y\in (\mathbb{A}^1_V)^{(1)}} H^1_y(\mathbb{A}^1_V;\mathcal{G}) & \to & \Pi'_{z\in (\mathbb{A}^1_V)^{(2)}} H^2_z(\mathbb{A}^1_V;\mathcal{G}) \end{array}$$

where E is the function field of V. Let y'_i be the images of the y_i in $\mathbb{A}^1_V$; these are points of codimension 1 and have the same residue field (because $Y \to \mathbb{A}^1_V$ is a closed immersion). By the axiom **(A5)(i)**, we see that for each i, the map $H^1_{y'_i}(\mathbb{A}^1_V;\mathcal{G}) \to H^1_{y_i}(U;\mathcal{G})$ is a bijection so that there exists in the bottom complex an element $\alpha' \in \Pi'_{y\in(\mathbb{A}^1_V)^{(1)}} H^1_y(\mathcal{G})$ whose image is α. The boundary of this α' is trivial. To show this, observe that if $z \in (\mathbb{A}^1_V)^{(2)}$ is not contained in Y, then the boundary of α' has a trivial component in $H^2_z(\mathbb{A}^1_V;\mathcal{G})$. Moreover, if $z \in (\mathbb{A}^1_V)^{(2)}$ lies in the image of Y in $\mathbb{A}^1_V$, there is, by construction, a unique point z' of codimension 2 in Ω, lying in Y and mapping to z. It has moreover the same residue field as z. The claim now follows from **(A5)(ii)**.

By the inductive assumption **(H2) (d)** we see that α' is of the form $h.*$ in $\Pi'_{y\in(\mathbb{A}^1_V)^{(1)}} H^1_y(\mathbb{A}^1_V;\mathcal{G})$ with $h \in \mathcal{G}(E(T))$. But if g denotes the image of h in $\mathcal{G}(F)$ we have $\alpha = g.*$, proving our claim.□

Lemma 1.30 *Let $\mathcal{G}$ be an unramified sheaf of groups on Sm_k satisfying* **(A2')**, **(A5)** *and* **(A6)**.

1) Let v be a discrete valuation on $F \in \mathcal{F}_k$. Denote by $v[T]$ the discrete valuation in $F(T)$ corresponding to the kernel of $\mathcal{O}_v[T] \to \kappa(v)(T)$. Then the map

$$H^1_v(\mathcal{O}_v;\mathcal{G}) \to H^1_{v[T]}(\mathbb{A}^1_{\mathcal{O}_v};\mathcal{G})$$

is injective and its image is exactly the kernel of

$$H^1_{v[T]}(\mathbb{A}^1_{\mathcal{O}_v};\mathcal{G}) \to \Pi'_{z\in(\mathbb{A}^1_\kappa(v))^{(1)}} H^2_z(\mathbb{A}^1_{\mathcal{O}_v};\mathcal{G})$$

where we see $z \in (\mathbb{A}^1_\kappa(v))^{(1)}$ as a point of codimension 2 in $\mathbb{A}^1_{\mathcal{O}_v}$.

2) For any k-smooth local scheme U of dimension 2 with closed point u, the "kernel" of the map

$$H^2_u(\mathcal{G}) \to H^2_{u[T]}(\mathcal{G})$$

is trivial.

Proof. Part 1) follows immediately from the fact that we know from our axioms the exactness of each row of the Diagram (1.2) is exact for U smooth local of dimension 1.

To prove 2) we shall use the interpretation of $H^2_z(U;\mathcal{G})$, for U smooth local of dimension 2 with closed point z, as $H^1_{Zar}(V;\mathcal{G})$, with V the complement of the closed point u. By Lemma 1.24, we know that $H^1_{Zar}(V;\mathcal{G}) \cong H^1_{Nis}(V;\mathcal{G})$.

Pick up an element α of $H^2_u(U;\mathcal{G}) = H^1_{Nis}(V;\mathcal{G})$ which becomes trivial in $H^2_{u[T]}(\mathbb{A}^1_U;\mathcal{G}) = H^1_{Nis}(V_T;\mathcal{G})$, where $V_T = (\mathbb{A}^1_U)_{u[T]} - u'$, u' denoting the generic point of $A^1_u \subset \mathbb{A}^1_U$. This means that the $\mathcal{G}$-torsor over V become trivial over V_T. As V_T is the inverse limit of the schemes of the form $\Omega - \Omega \cap \overline{u'}$, where Ω runs over the open subschemes of $\mathbb{A}^1_U$ which contains u', we see that there exists such an Ω for which the pull-back of α to $\Omega - \Omega \cap \overline{u'}$ is already trivial. As Ω contains u', $\Omega \cap \overline{u'} \subset \mathbb{A}^1_{\kappa(u)}$ is a non empty dense subset; in case $\kappa(u)$ is infinite, we thus know that there exists a $\kappa(u)$-rational point z in $\Omega \cap \overline{u'}$ lying over u. As $\Omega \to U$ is smooth, it follows from [17, Corollaire 17.16.3 p. 106] that there exists an immersion $U' \to \Omega$ such that $U' \to U$ is étale and whose image contains z. This immersion is then a closed immersion, and up to shrinking a bit U' we may assume that $\Omega \cap \overline{u'} \cap U' = \{z\}$. Thus the cartesian square

$$\begin{array}{ccc} U' - z & \to & U' \\ \downarrow & & \downarrow \\ V & \to & U \end{array}$$

is a distinguished square [39]. And the pull-back of α to $U' - z$ is trivial. Extending it to U' defines a descent data which defines an extension of α to U; thus as any element of $H^1_{Zar}(U;\mathcal{G}) = H^1_{Nis}(U;\mathcal{G})$ α is trivial we get our claim.□

$\mathbb{G}_m$-loop sheaves. Recall the following construction, used for instance by Voevodsky in [48]. Given a presheaf of groups G on Sm_k, we let G_{-1} denote the presheaf of groups given by

$$X \mapsto Ker(G(\mathbb{G}_m \times X) \stackrel{ev_1}{\to} G(X))$$

Observe that if G is a sheaf of groups, so is G_{-1}, and that if G is unramified, so is G_{-1}.

Lemma 1.31 *If G is a strongly $\mathbb{A}^1$-invariant sheaf of groups, so is G_{-1}.*

Proof. One might prove this using our description of those strongly $\mathbb{A}^1$-invariant sheaf of groups given in our first section. We propose another argument. Let $B(G)$ be the simplicial classifying space of G (see [39] for instance). Choose a fibrant resolution $\mathcal{B}(G)$ of $B(G)$. We study the pointed function space

$$R\mathbf{Hom}_{\bullet}(\mathbb{G}_m, B(G)) := \mathbf{Hom}_{\bullet}(\mathbb{G}_m, \mathcal{B}(G))$$

We observe it is fibrant and automatically $\mathbb{A}^1$-local, as $B(G)$ is. Moreover its π_1 sheaf is clearly G_{-1} and its higher homotopy sheaves vanish. Thus the connected component of $R\mathbf{Hom}_{\bullet}(\mathbb{G}_m, B(G))$ is $B(G_{-1})$. We know claim that this space is in fact 0-connected. For this we observe that the simplicial set of sections $\mathbf{Hom}_{\bullet}(\mathbb{G}_m, \mathcal{B}(G))(F)$ over a field $F \in \mathcal{F}_k$ is 0-connected. To prove this it is sufficient to prove that $H^1((\mathbb{G}_m)_F; G) = *$ is the point. This follows from the computation of the set $H^1((\mathbb{G}_m)_F; G)$ using the complex described in the first section, and the fact that $H^1(\mathbb{A}^1_F; G) = *$. Now by [32, Lemma 6.1.3] this fact implies that the space itself is 0-connected.□

1.3 Unramified $\mathbb{Z}$-graded abelian sheaves

In this section we want to give some criteria which imply the Axioms **(A4)** for some type of unramified abelian sheaves. Our method is inspired by Rost [44] but avoids the use of transfers. This section (and part of this paper) grew up in fact from our willingness to construct unramified Milnor-Witt K-theory (as well as Milnor K-theory) without any transfers: using the result of this section, this is achieved in the next section.

Let M_* be a functor $\mathcal{F}_k \to \mathcal{A}b_*$ to the category of $\mathbb{Z}$-graded abelian groups.

Important convention for this section We will make everywhere in this subsection the following additional assumption: M_* is extended from a perfect subfield $k_0 \subset k$ over which k is of finite type. This means that there is a $\mathbb{Z}$-graded functor M'_* on $\mathcal{F}_{k_0}$ such that M is isomorphic to the extension $M'|_k$; see Remark 1.5. This means that the value $M_*(F)$ over F is $M'_*(F|_{k_0})$, where F is considered as a separable finite type extension of k_0. Unless there is a possibility of confusion, we will not mention however the data M'_*, k_0, etc... which will be always understood. One of the advantages of this assumption is that if F is any finite type extension of k, maybe not separable, we may evaluate anyway M_* on F and simply denote by $M_*(F)$ the group $M'_*(F|_{k_0})$. When we will say that M satisfies Axiom "Lambda", we will really mean otherwise explicitly expressed that M'_* satisfies Axiom "Lambda", etc... In the same way, when we assume that M_* is endowed with some Datum, we really mean that M' is endowed with this structure (over k_0).□

We will assume throughout this section that M_* is endowed with the following extra structures.

(D4) (i) For any $F \in \mathcal{F}_k$ a structure of $\mathbb{Z}[F^\times/(F^{\times 2})]$-module on $M_*(F)$, which we denote by $(u, \alpha) \mapsto < u > \alpha \in M_n(F)$ for $u \in F^\times$ and for $\alpha \in M_n(F)$. This structure should be functorial in the obvious sense in $\mathcal{F}_k$.□

(D4) (ii) For any $F \in \mathcal{F}_k$ and any $n \in \mathbb{Z}$, a map $F^\times \times M_{n-1}(F) \to M_n(F)$, $(u, \alpha) \mapsto [u].\alpha$, functorial (in the obvious sense) in $\mathcal{F}_k$.□

(D4) (iii) For any discrete valuation v on $F \in \mathcal{F}_k$ and uniformizing element π a graded epimorphism of degree -1

$$\partial_v^\pi : M_*(F) \to M_{*-1}(\kappa(v))$$

which is functorial, in the obvious sense, with respect to extensions $E \to F$ such that v restricts to a discrete valuation on E, with ramification index 1, if we choose as uniformizing element an element π in E.□

We assume furthermore that the following axioms hold:

(B0) For $(u, v) \in (F^\times)^2$ and $\alpha \in M_n(F)$, one has

$$[uv]\alpha = [u]\alpha + < u > [v]\alpha \qquad \square$$

and moreover $[u][v]\alpha = - < -1 > [v][u]\alpha$.

(B1) For a k-smooth integral domain A with field of fractions F, for any $\alpha \in M_n(F)$, then for all but only finitely many point $x \in Spec(A)^{(1)}$, one has that for any uniformizing element π for x, $\partial_x^\pi(\alpha) \neq 0$.$\square$

(B2) For any discrete valuation v on $F \in \mathcal{F}_k$ with uniformizing element π one has $\partial_v^\pi([u]\alpha) = [\overline{u}]\partial_v^\pi(\alpha) \in M_n(\kappa(v))$ and $\partial_v^\pi(< u > \alpha) =< \overline{u} > \partial_v^\pi(\alpha) \in M_{(n-1)}(\kappa(v))$, for any unit u in $(\mathcal{O}_v)^\times$ and any $\alpha \in M_n(F)$.$\square$

(B3) For any field extension $E \subset F \in \mathcal{F}_k$ and for any discrete valuation v on $F \in \mathcal{F}_k$ which restricts to a discrete valuation w on E, with ramification index e, let $\pi \in \mathcal{O}_v$ be a uniformizing element for v and $\rho \in \mathcal{O}_w$ be a uniformizing element for w. Write $\rho = u\pi^e$, with u a unit in $\mathcal{O}_v$. Then one has for $\alpha \in M_*(E)$, $\partial_v^\pi(\alpha|_F) = e_\epsilon < \overline{u} > (\partial_w^\rho(\alpha))|_{\kappa(v)} \in M_*(\kappa(v))$.$\square$

Here we set for any integer n,

$$n_\epsilon = \sum_{i=1}^{n} < (-1)^{(i-1)} >$$

We observe that as a particular case of **(B3)** we may choose $E = F$ so that $e = 1$ and we get that for any any discrete valuation v on $F \in \mathcal{F}_k$, any uniformizing element π, and any unit $u \in \mathcal{O}_v^\times$, then one has $\partial_v^{u\pi}(\alpha) =< \overline{u} > \partial_v^\pi(\alpha) \in M_{(n-1)}(\kappa(v))$ for any $\alpha \in M_n(F)$.

Thus in case Axiom **(B3)** holds, the kernel of the surjective homomorphism ∂_v^π only depends on the valuation v, not on any choice of π. In that case we then simply denote by

$$M_*(\mathcal{O}_v) \subset M_*(F)$$

this kernel. Axiom **(B1)** is then exactly equivalent to Axiom **(A2)** for unramified $\tilde{\mathcal{F}}_k$-sets. The following is easy:

Lemma 1.32 *Assume M_* satisfies Axioms* **(B1)**, **(B2)** *and* **(B3)**. *Then it satisfies (in each degree) the axioms for a unramified $\tilde{\mathcal{F}}_k$-abelian group. Moreover, it satisfies Axiom* **(A5) (i)**.

We assume from now on (in this section) that M_* satisfies Axioms **(B0)**, **(B1)**, **(B2)** and **(B3)**. Thus we may (and will) consider each M_n as a sheaf of abelian groups on $\tilde{Sm}_k$.

We recall that we denote, for any discrete valuation v on $F \in \mathcal{F}_k$, by $H^1_v(\mathcal{O}_v, M_n)$ the quotient group $M_n(F)/M_n(\mathcal{O}_v)$ and by $\partial_v : M_n(F) \to H^1_v(\mathcal{O}_v, M_n)$ the projection. Of course, if one chooses a uniformizing element π, one gets an isomorphism $\theta_\pi : M_{(n-1)}(\kappa(v)) \cong H^1_v(\mathcal{O}_v, M_n)$ with $\partial_v = \theta_\pi \circ \partial^\pi_v$.

For each discrete valuation v on $F \in \mathcal{F}_k$, and any uniformizing element π set

$$s^\pi_v : M_*(F) \to M_*(\kappa(v)) \ , \ \alpha \mapsto \partial^\pi_v([\pi]\alpha)$$

Lemma 1.33 *Assume M_* satisfies Axioms* **(B0)**, **(B1)**, **(B2)** *and* **(B3)**. *Then for each discrete valuation v the homomorphism $s^\pi_v : M_*(\mathcal{O}_v) \subset M_*(F)$ doesn't depend on the choice of a uniformizing element π.*

Proof. From Axiom **(B0)** we get for any unit $u \in \mathcal{O}^\times$, any uniformizing element π and any $\alpha \in M_n(F)$: $[u\pi]\alpha = [u]\alpha + < u > [\pi]\alpha$. Thus if moreover $\alpha \in M(\mathcal{O}_v)$, one has $s^{u\pi}_v(\alpha) = \partial^{u\pi}_v([u\pi]\alpha) = \partial^{u\pi}_v([u]\alpha) + \partial^{u\pi}_v(< u > [\pi]\alpha) = \partial^{u\pi}_v(< u > [\pi]\alpha)$, as by Axiom **(B2)** $\partial^{u\pi}_v([u]\alpha) = [\overline{u}]\partial^{u\pi}_v(\alpha) = [\overline{u}]0 = 0$. But by the same Axiom **(B2)**, $\partial^{u\pi}_v(< u > [\pi]\alpha) =< \overline{u} > \partial^{u\pi}_v([\pi]\alpha)$, which by Axiom **(B3)** is equal to $< \overline{u} >< \overline{u} > \partial^\pi_v([\pi]\alpha) = \partial^\pi_v([\pi]\alpha)$. This proves the claim.□

We will denote by

$$s_v : M_*(\mathcal{O}_v) \to M_n(\kappa(v))$$

the common value of all the s^π_v's. In this way M_* is endowed with a datum **(D3)**.

We introduce the following Axiom:

(HA) **(i)** For any $F \in \mathcal{F}_k$, the following diagram

$$0 \to M_*(F) \to M_*(F(T)) \xrightarrow{\sum \partial^P_{(P)}} \oplus_{P \in \mathbb{A}^1_F} M_{*-1}(F[T]/P) \to 0$$

is a short exact sequence. Here P runs over the set of irreducible unitary polynomials, and (P) means the associated discrete valuation.□

(HA) (ii) For any $\alpha \in M(F)$, one has $\partial^T_{(T)}([T]\alpha|_{F(T)}) = \alpha$.$\square$

This axiom is obviously related to the Axiom **(A6)**, as it immediately implies that for any $F \in \mathcal{F}_k$, $M(F) \to M(\mathbb{A}^1_F)$ is an isomorphism and $H^1_{Zar}(\mathbb{A}^1_F; M) = 0$.

We next claim:

Lemma 1.34 *Let M_* satisfies all the previous Axioms, including Axioms* **(HA) (i)** *and* **(HA) (ii)** *then Axioms* **(A1) (ii)**, **(A3) (i)** *and* **(A3) (ii)** *hold.*

Proof. The first part of Axiom **(A1) (ii)** follows from Axiom **(B4)**. For the second part we choose a uniformizing element π in $\mathcal{O}_w$, which is still a uniformizing element for $\mathcal{O}_v$ and the square

$$\begin{array}{ccc} M_*(F) & \stackrel{\partial^\pi_v}{\longrightarrow} & M_{(*-1)}(\kappa(v)) \\ \uparrow & & \uparrow \\ M_*(E) & \stackrel{\partial^\pi_w}{\longrightarrow} & M_{(*-1)}(\kappa(w)) \end{array}$$

is commutative by our definition **(D4) (iii)**. Moreover the morphism $M_*(E) \to M_*(F)$ preserve the product by π by **(D4) (i)**.

To prove Axiom **(A3)** we proceed as follows. By assumption we have $E \subset \mathcal{O}_v \subset F$. Choose a uniformizing element π of v. We consider the extension $E(T) \subset F$ induced by $T \mapsto \pi$. The restriction of v is clearly the valuation defined by T on $E[T]$. The ramification index is 1. Using the previous point, we see that we can reduce to the case $E \subset F$ is $E \subset E(T)$ and $v = (T)$. In that case, the claim follows from our Axioms **(HA) (i)** and **(HA) (ii)**.$\square$

From now on, we assume that M_* satisfies all the Axioms previously met in this subsection. We observe that by construction the Axiom **(A5) (i)** is clear.

Fix a discrete valuation v on $F \in \mathcal{F}_k$. We denote by $v[T]$ the discrete valuation on $F(T)$ defined by the divisor $\mathbb{G}_m|_{\kappa(v)} \subset \mathbb{G}_m|_{\mathcal{O}_v}$ whose open complement is $\mathbb{G}_m|_F$. Choose a uniformizing element π for v. Observe that

$\pi \in F(T)$ is still a uniformizing element for $v[T]$.

We want to analyze the following commutative diagram in which the horizontal rows are short exact sequences (given by Axiom **(HA)**):

$$\begin{array}{ccccccccc} 0 & \to & M_*(F) & \to & M_*(F(T)) & \stackrel{\Sigma_P \partial^P_{(P)}}{\longrightarrow} & \oplus_{P\in(\mathbb{A}^1_F)^{(1)}} M_{*-1}(F[T]/P) & \to & 0 \\ & & \downarrow \partial^\pi_v & & \downarrow \partial^\pi_{v[T]} & & \downarrow \Sigma_{P,Q}\partial^{\pi,P}_Q & & \\ 0 & \to & M_{*-1}(\kappa(v)) & \to & M_{*-1}(\kappa(v)(T)) & \stackrel{\Sigma_Q \partial^Q_{(Q)}}{\longrightarrow} & \oplus_{Q\in(\mathbb{A}^1_{\kappa(v)})^{(1)}} M_{*-2}(\kappa(v)[T]/Q) & \to & 0 \end{array} \qquad (1.3)$$

and where the morphisms $\partial^{\pi,P}_Q : M_*(F[T]/P) \to M_{*-1}(\kappa(v)[T]/Q)$ are defined by the diagram.

For this we need the following Axiom:

(B4) Let v be discrete valuation on $F \in \mathcal{F}_k$ and let π be a uniformizing element. Let $P \in (\mathbb{A}^1_F)^{(1)}$ and $Q \in (\mathbb{A}^1_{\kappa(v)})^{(1)}$ be fixed.

(i) If the closed point $Q \in \mathbb{A}^1_{\kappa(v)} \subset \mathbb{A}^1_{\mathcal{O}_v}$ is not in the divisor $D_P \subset \mathbb{A}^1_{\mathcal{O}_v}$ with generic point $P \in \mathbb{A}^1_F \subset \mathbb{A}^1_{\mathcal{O}_v}$ then the morphism $\partial^{\pi,P}_Q$ is zero.

(ii) If Q is in $D_P \subset \mathbb{A}^1_{\mathcal{O}_v}$ and if the local ring $\mathcal{O}_{D_P,Q}$ is a discrete valuation ring with π as uniformizing element then

$$\partial^{\pi,P}_Q = - < -\frac{\overline{P'}}{Q'} > \partial^Q_Q \ : M_*(F[T]/P) \to M_{*-1}(\kappa(v)[T]/Q) \ \ \square$$

We will set $U = Spec(\mathcal{O}_v)$ in the sequel. We first observe that $(\mathbb{A}^1_U)^{(1)} = (\mathbb{A}^1_F)^{(1)} \amalg \{v[T]\}$, where as usual $v[T]$ means the generic point of $\mathbb{A}^1_{\kappa(v)} \subset \mathbb{A}^1_U$. For each $P \in (\mathbb{A}^1_F)^{(1)}$, there is a canonical isomorphism $M_{*-1}(F[T]/P) \cong H^1_P(\mathbb{A}^1_U; M_*)$, as P itself is a uniformizing element for the discrete valuation (P) on $F(T)$. For $v[T]$, there is also a canonical isomorphism $M_{*-1}(\kappa(v)[T]) \cong H^1_{v[T]}(\mathbb{A}^1_U; M_*)$ as π is also a uniformizing element for the discrete valuation $v[T]$ on $F(T)$.

Using the previous isomorphisms, we see that the beginning of the complex $C^*(\mathbb{A}^1_U; M_*)$ (see Section 1.2) is isomorphic to

$$0 \to M_*(\mathbb{A}^1_U) \to M_*(F(T)) \stackrel{\partial^\pi_{v[T]}+\sum_P \partial^P_{(P)}}{\longrightarrow} M_{*-1}(\kappa(v)(T)) \oplus \Big(\oplus_{P\in(\mathbb{A}^1_F)^{(1)}} M_{*-1}(F[T]/P)\Big)$$

The diagram (1.3) can be used to compute the cokernel of the previous morphism $\partial : M_*(F(T)) \to M_{*-1}(\kappa(v)(T)) \oplus \Big(\oplus_{P\in(\mathbb{A}^1_F)^{(1)}} M_{*-1}(F[T]/P)\Big)$. Indeed the epimorphism ∂'

$$M_{*-1}(\kappa(v)(T)) \oplus (\oplus_P M_{*-1}(F[T]/P)) \stackrel{\sum_Q \partial^Q_{(Q)} - \sum_{P,Q} \partial^{\pi,P}_Q}{\longrightarrow} \oplus_{Q\in(\mathbb{A}^1_{\kappa(v)})^{(1)}} M_{*-2}(\kappa(v)[T]/Q)$$

composed with ∂ is trivial, and the diagram

$$M_*(F(T)) \stackrel{\partial}{\to} M_{*-1}(\kappa(v)(T)) \oplus (\oplus_P M_{*-1}(F[T]/P)) \stackrel{\partial'}{\to} \oplus_Q M_{*-2}(\kappa(v)[T]/Q) \to 0 \tag{1.4}$$

is an exact sequence: this is just an obvious reformulation of the properties of (1.3).

Now fix $Q_0 \in (\mathbb{A}^1_{\kappa(v)})^{(1)}$. Let $(\mathbb{A}^1_F)^{(1)}_0$ be the set of P's such that Q_0 lies in the divisor D_P of $\mathbb{A}^1_U$ defined by P.

Lemma 1.35 *Assume M_* satisfies all the previous Axioms. The obvious quotient*

$$M_*(F(T)) \stackrel{\partial}{\to} M_{*-1}(\kappa(v)(T)) \oplus \Big(\oplus_{P\in(\mathbb{A}^1_F)^{(1)}_0} M_{*-1}(F[T]/P)\Big) \stackrel{\partial'_Q}{\to} M_{*-2}(\kappa(v)[T]/Q_0) \to 0$$

of the previous diagram is also an exact sequence.

Proof. Using the snake Lemma, it is sufficient to prove that the image of the composition $\oplus_{P\notin(\mathbb{A}^1_U)^{(1)}_0} M_{*-1}(F[T]/P) \subset \oplus_{P\in(\mathbb{A}^1_U)^{(1)}} M_{*-1}(F[T]/P) \to \oplus_{Q\in(\mathbb{A}^1_{\kappa(v)})^{(1)}} M_{*-2}(\kappa(v)[T]/Q$ is exactly $\oplus_{Q\in(\mathbb{A}^1_{\kappa(v)})^{(1)}-\{Q_0\}} M_{*-2}(\kappa(v)[T]/Q$. Axiom **(B4)(i)** readily implies that this image is contained in
$\oplus_{Q\in(\mathbb{A}^1_{\kappa(v)})^{(1)}-\{Q_0\}} M_{*-2}(\kappa(v)[T]/Q)$.

Now we want to show that the image entirely reaches each $M_{*-2}(\kappa(v)[T]/Q$, $Q \neq Q_0$. For any such Q, there is a P, irreducible, such that Q is $\alpha\overline{P}$, for some unit $\alpha \in \kappa(v)^\times$. Thus Q lies over D_P, but not Q_0. Moreover, (π, P) is a system of generators of the maximal ideal of the local dimension 2 regular ring $(\mathcal{O}_v[T])_{(Q)}$, thus $(\mathcal{O}_v[T]/P)_{(Q)}$ is a discrete valuation ring with uniformizing element the image of π. By Axiom **(B4)(ii)** now, we conclude that $\partial_Q^{\pi,P}$ is onto, proving the claim.□

Now let X be a local smooth k-scheme of dimension 2 with closed point z and function field E. Recall from the beginning of section 1.2 that we denote by $H^2_z(X;M)$ the cokernel of the sum of the residues $M_*(E) \stackrel{\Sigma_{y\in X^{(1)}}\partial_y}{\longrightarrow} \oplus_{y\in X^{(1)}} H^1_y(X;M_*)$. We thus have a canonical exact sequence of the form:

$$0 \to M_*(X) \to M_*(E) \stackrel{\Sigma_{y\in X^{(1)}}\partial_y}{\longrightarrow} \oplus_{y\in X^{(1)}} H^1_y(X;M_*) \stackrel{\Sigma_{y\in X^{(1)}}\partial_z^y}{\longrightarrow} H^2_z(X;M_*) \to 0 \tag{1.5}$$

where the homomorphisms denoted ∂_z^y are defined by the diagram. This diagram is the complex $C^*((\mathbb{A}^1_U)_0;M_*)$.

For X the localization $(\mathbb{A}^1_U)_0$ of $\mathbb{A}^1_U$ at some closed point $Q_0 \in \mathbb{A}^1_{\kappa(v)}$, with $U = Spec(\mathcal{O}_v)$ where v is a discrete valuation on some $F \in \mathcal{F}_k$, we thus get immediately:

Corollary 1.36 *Assume M_* satisfies all the previous Axioms. The complex $C^*((\mathbb{A}^1_U)_0;M_*)$ is canonically isomorphic to exact sequence:*

$$0 \to M_*((\mathbb{A}^1_U)_Q) \to M_*(F(T)) \to M_{*-1}(\kappa(v)(T)) \oplus \Big(\oplus_{P\in(\mathbb{A}^1_F)_0^{(1)}} M_{*-1}(F[T]/P)\Big)$$
$$\to M_{*-2}(\kappa(v)[T]/Q) \to 0$$

This isomorphism provides in particular a canonical isomorphism

$$M_{*-2}(\kappa(v)[T]/Q_0) \cong H^2_{Q_0}(\mathbb{A}^1_U;M_*)$$

Corollary 1.37 *Assume M_* satisfies all the previous Axioms. For each n, the unramified sheaves of abelian groups (on $\tilde{Sm}_k$) M_n satisfies Axiom* **(A2')**.

Proof. From Remark 1.19, it suffices to check this when k is infinite.

Now assume X is a smooth k-scheme. Let $y \in X^{(1)}$ be a point of codimension 1. We wish to prove that given $\alpha \in H^1_y(X;M_*)$, there are only finitely many $z \in X^{(2)}$ such that $\partial_z^y(\alpha)$ is non trivial. But as k is infinite, by Gabber's Lemma, there is an open neighborhood $\Omega \subset X$ of y and an étale morphism $\Omega \to \mathbb{A}^1_V$, for V some open subset of an affine space over k, such that the morphism $\overline{y} \cap \Omega \to \mathbb{A}^1_V$ is a closed immersion.

The complement $\overline{y} - \overline{y} \cap \Omega$ is a closed subset everywhere of > 0-dimension and thus contains only finitely many points of codimension 1 in $\overline{y}$.

For any $z \in (\overline{y} \cap \Omega)^{(1)}$, the étale morphism $\Omega \to \mathbb{A}^1_V$ obviously induces a commutative square

$$\begin{array}{ccc} H^1_y(X; M_*) & \stackrel{\partial^y_z}{\to} & H^2_z(X; M_*) \\ \uparrow \wr & & \uparrow \wr \\ H^1_y(\mathbb{A}^1_V; M_*) & \stackrel{\partial^y_z}{\to} & H^2_z(\mathbb{A}^1_V; M_*) \end{array}$$

(because $\overline{y} \cap \Omega \to \mathbb{A}^1_V$ is a closed immersion), we reduce to proving the claim for the image of y in $\mathbb{A}^1_V$, which clearly follows from our previous results.□

Now that we know that M_* satisfies Axiom **(A2')**, for X a smooth k-scheme with function field E we may define as in section 1.2 a (whole) complex $C^*(X; M_*)$ of the form

$$0 \to M_*(X) \to M_*(E) \stackrel{\Sigma_{y \in X^{(1)}} \partial_y}{\longrightarrow} \oplus_{y \in X^{(1)}} H^1_y(X; M_*) \stackrel{\Sigma_{y,z} \partial^y_z}{\longrightarrow} \oplus_{z \in X^{(2)}} H^2_z(X; M_*) \tag{1.6}$$

We thus get as an immediate consequence:

Corollary 1.38 *Assume M_* satisfies all the previous Axioms. For any discrete valuation v on $F \in \mathcal{F}_k$, setting $U = Spec(\mathcal{O}_v)$, the complex $C^*(\mathbb{A}^1_U; M_*)$ is canonically isomorphic to the exact sequence (1.4):*

$$0 \to M_*(\mathbb{A}^1_U) \to M_*(F(T)) \to M_{*-1}(\kappa(v)(T)) \oplus \Big(\oplus_{P \in (\mathbb{A}^1_F)^{(1)}} M_{*-1}(F[T]/P)\Big)$$

$$\to \oplus_{Q \in (\mathbb{A}^1_U)^{(1)}} M_{*-2}(\kappa(v)[T]/Q) \to 0$$

Consequently, the complex $C^*(\mathbb{A}^1_U; M_*)$ is an exact complex, and in particular, for each n, the unramified sheaves of abelian groups (on $\tilde{Sm}_k$) M_n satisfies Axiom **(A6)**.

Proof. Only the statement concerning Axiom **(A6)** is not completely clear: we need to prove that $M_n(U) \to M_n(\mathbb{A}^1_U)$ is an isomorphism for U a smooth local k-scheme of dimension ≤ 1. The rest of the Axiom is clear. This claim is clear by Axiom **(HA)** for U of dimension 0. We need to prove it for U of the form $Spec(\mathcal{O}_v)$ for some discrete valuation v on some $F \in \mathcal{F}_k$ (observe that for the moment M_* only defines an unramified sheaf on $\tilde{Sm}_k$, and we can only apply point 1) of Lemma 1.16. But this statement follows rather easily by contemplating the diagram (1.3).□

We next prepare the statement of our last Axiom. Let X be a local smooth k-scheme of dimension 2, with field of functions F and closed point z. Consider the complex $C^*(X; M_*)$ associated to X in (1.5). By definition we have a short exact sequence:

$$0 \to M_*(F)/M_*(X) \to \oplus_{y \in X^{(1)}} H^1_y(X; M_*) \to H^2_z(X; M_*) \to 0$$

Let $y_0 \in X^{(1)}$ be such that $\overline{y_0}$ is smooth over k.

The properties of the induced morphism

$$M_*(F)/M_*(X) \to \oplus_{y \in X^{(1)} - \{y_0\}} H^1_y(X; M_*) \tag{1.7}$$

will play a very important role. We first observe:

Lemma 1.39 *Assume M_* satisfies all the previous Axioms (including* **(B4)***). Suppose furthermore that k is infinite. Let X be a local smooth k-scheme of dimension 2, with field of functions F and closed point z, let $y_0 \in X^{(1)}$ be such that $\overline{y_0}$ is smooth over k. Then the homomorphism (1.7) is onto.*

Proof. We first observe (without using that k is infinite) that this property is true for any localization of a scheme of the form $\mathbb{A}^1_U$ at a point z of codimension 2, with $U = Spec(\mathcal{O}_v)$, for some discrete valuation v on F. If $\overline{y_0}$ is $\mathbb{A}^1_{\kappa(v)}$ this is just Axiom **(HA)**. If $\overline{y_0}$ is not $\mathbb{A}^1_{\kappa(v)}$ we observe that the complex $C^*((\mathbb{A}^1_U)_z; M_*)$:

$$M(F(T)) \stackrel{\Sigma_{y \in ((\mathbb{A}^1_U)_z)^1} \partial_y}{\longrightarrow} \oplus_{y \in ((\mathbb{A}^1_U)_z)^{(1)}} H^1_y(X; M) \to H^2_z(\mathbb{A}^1_{\kappa(v)}; M_*) \to 0$$

is isomorphic to the one of Corollary 1.36. By Axiom **(B4)(ii)** we deduce that the map $\partial^y_z : H^1_{y_0}(X; M) \to H^2_z(\mathbb{A}^1_{\kappa(v)}; M_*)$ is surjective. This clearly implies the statement.

To prove the general case we use Gabber's Lemma (and that k is infinite). Let α be an element in $\oplus_{y \in X^{(1)} - \{y_0\}} H^1_y(X; M)$. Let y_1, .., y_r be the points in the support of α. There exists an étale morphism $X \to \mathbb{A}^1_U$, for some local smooth scheme U of dimension 1, and with function field K, such that

$\overline{y_i} \to \mathbb{A}^1_U$ is a closed immersion for each i. But then use the commutative square

$$\begin{array}{ccc} M_*(F) & \overset{\Sigma_{y\in X^1-\{y_0\}}\partial_y}{\longrightarrow} & \oplus_{y\in X^{(1)}-\{y_0\}} H^1_y(X;M_*) \\ \uparrow & & \uparrow \\ M_*(K(T)) & \overset{\Sigma_{y\in((\mathbb{A}^1_U)_z)^1-\{y_0\}}\partial_y}{\longrightarrow} & \oplus_{y\in((\mathbb{A}^1_U)_z)^{(1)}-\{y_0\}} H^1_y(\mathbb{A}^1_U;M_*) \end{array}$$

We now conclude that $\alpha = \Sigma_i \alpha_i$, with $\alpha_i \in H^1_{y_i}(X;M_*) \cong H^1_{y_i}(\mathbb{A}^1_U;M_*)$, $i \in \{1,\dots,r\}$ comes from an element from the bottom right corner. The isomorphism $H^1_{y_i}(X;M_*) \cong H^1_{y_i}(\mathbb{A}^1_U;M_*)$ is a consequence of our definition of $H^1_y(-;M_*)$ and **(D4)(iii)**. The bottom horizontal morphism is onto by the first case we treated. Thus α lies in the image of our morphism.□

Now for our X local smooth k-scheme of dimension 2, with field of functions F and closed point z, with $y_0 \in X^{(1)}$ such that $\overline{y_0}$ is smooth over k, choose a uniformizing element π of y_0 (in $\mathcal{O}_{X,y_0}$). This produces by definition an isomorphism $M_{*-1}(\kappa(y_0)) \cong H^1_{y_0}(X;M_*)$. Now the kernel of the morphism (1.7) is clearly contained in $M_{*-1}(\kappa(y_0)) \cong H^1_{y_0}(X;M_*)$. We may now state our last Axiom:

(B5) Let X be a local smooth k-scheme of dimension 2, with field of functions F and closed point z, let $y_0 \in X^{(1)}$ be such that $\overline{y_0}$ is smooth over k. Choose a uniformizing element π of y_0 (in $\mathcal{O}_{X,y_0}$). Then the kernel of the morphism (1.7) is (identified to a subgroup of $M_{*-1}(\kappa(y_0))$) equal to $M_{*-1}(\mathcal{O}_{y_0,z}) \subset M_{*-1}(\kappa(y_0))$.□

Remark 1.40 Thus if M_* satisfies Axiom **(B5)** one gets an exact sequence

$$0 \to M_{*-1}(\mathcal{O}_{y_0,z}) \to M_*(F)/M_*(X) \to \oplus_{y\in X^{(1)}-\{y_0\}} H^1_y(X;M_*)$$

If k is infinite, Lemma 1.39 shows that it is in fact a short exact sequence. We don't know whether this is still true over a finite field.□

Lemma 1.41 *Assume that M_* satisfies all the previous Axioms of this section, including* **(B4)**, **(B5)**. *Assume the field k is infinite.*

1) Let X be a local smooth k-scheme of dimension 2, with field of functions F and closed point z, let $y_0 \in X^{(1)}$ be such that $\overline{y_0}$ is smooth over k. Choose a uniformizing element π of $\mathcal{O}_{X,y_0}$. Then the homomorphism $M_{*-1}(\kappa(y_0)) \cong H^1_{y_0}(X;M) \stackrel{\partial^{y_0}_z}{\to} H^2_z(X;M)$ induces an isomorphism

$$\Theta_{y_0,\pi} : M_{*-1}(\kappa(y_0))/M_{*-1}(\mathcal{O}_{y_0,z}) = H^1_z(\overline{y_0}; M_{*-1}) \cong H^2_z(X;M)$$

2) Assume $f : X' \to X$ is an étale morphisms between smooth local k-schemes of dimension 2, with closed points respectfully z' and z and with the same residue field $\kappa(z) = \kappa(z')$. Then the induced morphism $H^2_z(X;M_*) \to H^2_{z'}(X';M_*)$ is an isomorphism. In particular, M_* satisfies Axiom **(A5) (ii)**.

Proof. 1) We know from the previous Remark that the sequence $0 \to M_{*-1}(\mathcal{O}_{y_0}) \to M_*(F)/M_*(X) \to \oplus_{y\in X^{(1)}-\{y_0\}} H^1_y(X;M_*) \to 0$ is a short exact sequence. By the definition of $H^2_z(X;M)$ given by the short exact sequence (1.5), this provides a short exact sequence of the form

$$0 \to M_{*-1}(\mathcal{O}_{y_0,z}) \to M_{*-1}(\kappa(y_0)) \to H^2_z(X;M) \to 0$$

and produces the required isomorphism $\Theta_{y_0,\pi}$.

2) Choose $y_0 \in X^{(1)}$ such that $\overline{y_0}$ is smooth over k and a uniformizing element $\pi \in \mathcal{O}_{X,y_0}$. Clearly the pull back of y_0 to X' is still a smooth divisor denoted by y_0', and the image of π is a uniformizing element for $\mathcal{O}_{y_0'}$. Then the following diagram clearly commutes

$$\begin{array}{ccc} H^1_{z'}(\overline{y_0'}; M_{*-1}) & \stackrel{\Theta_{y_0',\pi'}}{\to} & H^2_{z'}(X';M) \\ \uparrow & & \uparrow \\ H^1_z(\overline{y_0}; M_{*-1}) & \stackrel{\Theta_{y,\pi}}{\to} & H^2_z(X;M_*) \end{array}$$

Thus all the morphisms in this diagram are isomorphisms.□

Definition 1.42 *Let M_* be a functor $\mathcal{F}_k \to \mathcal{A}b_*$ endowed with data* **(D4) (i)**, **(D4) (ii)** *and* **(D4) (iii)***; we will be saying that M_* is a* $\mathbb{Z}$-graded $\mathbb{A}^1_k$-module *if it satisfies moreover the Axioms* **(B0)**, **(B1)**, **(B2)**, **(B3)**, **(HA)**, **(B4)** *and if $M_*|_{k(T)}$ satisfies* **(B5)**.

Theorem 1.43 *Let M_* be a $\mathbb{Z}$-graded $\mathbb{A}^1_k$-module. Then endowed with the s_v's constructed in Lemma 1.33, for each n, M_n is an unramified $\mathcal{F}_k$-set in the* *sense of Definition 1.9. By Lemma 1.12 it thus defines an unramified sheaf of abelian groups on Sm_k. This unramified sheaf of abelian groups satisfies Axioms* **(A2')**, **(A6)** *and its base change to any infinite field $F \in \mathcal{F}_k$ satisfies* **(A5)**.

Corollary 1.44 *Let M_* be a $\mathbb{Z}$-graded $\mathbb{A}^1_k$-module. Then for each n, M_n is a strongly $\mathbb{A}^1$-invariant sheaf.*

Proof. If k is infinite this follows from the previous Theorem and Theorem 1.26. If k is finite this is proven in Theorem A.8.□

Proof of Theorem 1.43. The previous results (Lemmas 1.32 and 1.34) have already established that M_n is an unramified sheaf of abelian groups on $\tilde{Sm}_k$, satisfying all the Axioms for unramified sheaves on Sm_k except Axiom **(A4)**. Axiom **(A2')** is proven in Corollary 1.37. Axiom **(A5)(i)** is clear and Axiom **(A5)(ii)** holds if k is infinite by Lemma 1.41. Axiom **(A6)** holds by Corollary 1.38. The only remaining point is Axiom (**A4**). But by Remark 1.17 to prove (**A4**) in general it is sufficient to treat the case k is infinite. We assume from now on in this proof that k is infinite.

We start by checking the first part of Axiom (**A4**). Let $X = Spec(A)$ be a local smooth k-scheme of dimension 2 with closed point z and function field F. Let $y_0 \in X^{(1)}$ be such that $\overline{y_0}$ is smooth over k. Choose a pair (π_0, π_1) of generators for the maximal ideal of A, such that π_0 defines y_0. Clearly $\overline{\pi_1} \in \mathcal{O}(\overline{y_0})$ is a uniformizing element for $z \in \mathcal{O}(\overline{y_0})$.

We consider the complex (1.5) of X with coefficients in M_* and the induced commutative square:

$$\begin{array}{ccc} M_*(F) & \stackrel{\Sigma_{y\in X^{(1)}-\{y_0\}}\partial_y}{\longrightarrow} & \oplus_{y\in X^{(1)}-\{y_0\}} H^1_y(X;M_*) \\ \downarrow \partial_{y_0} & & \downarrow -\Sigma_{y\in X^{(1)}-\{y_0\}}\partial^y_z \\ H^1_{y_0}(X;M_*) & \stackrel{\partial^{y_0}_z}{\longrightarrow} & H^2_z(X;M_*) \end{array}$$

We put this square at the top of the commutative square

$$\begin{array}{ccc} H^1_{y_0}(X;M_*) & \stackrel{\partial^{y_0}_z}{\longrightarrow} & H^2_z(X;M_*) \\ \downarrow \wr & & \downarrow \wr \\ M_{*-1}(\kappa(y_0)) & \stackrel{\partial^{\overline{\pi_1}}_z}{\longrightarrow} & M_{*-2}(\kappa(z)) \end{array}$$

where $H^1_{y_0}(X;M_*) \stackrel{\sim}{\to} M_{*-1}(\kappa(y_0))$ is the inverse to the canonical isomorphism θ_{π_0} induced by π_0, and where $H^2_z(X;M_*) \stackrel{\sim}{\to} M_{*-2}(\kappa(z))$ is obtained by composing the inverse to the isomorphism Θ_{y_0,π_0} obtained by the previous lemma and $\theta_{\overline{\pi_1}}$.

Now we add on the left top corner the morphism $M_{*-1}(\mathcal{O}_{X,y_0}) \to M_*(F)$, $\alpha \mapsto [\pi_0]\,\alpha$. We thus get a commutative square of the form:

$$\begin{array}{ccccc} M_{*-1}(\mathcal{O}_{\pi_0}) & \stackrel{[\pi_0].-}{\to} & M_*(F) & \stackrel{\Sigma_{y\in X^{(1)}-\{y_0\}}\partial_y}{\longrightarrow} & \oplus_{y\in X^{(1)}-\{y_0\}} H^1_y(X;M_*) \\ & & \downarrow \partial^{\pi_0}_{y_0} & & \downarrow \\ & & M_{*-1}(\kappa(y_0)) & \stackrel{\partial^{\overline{\pi_1}}_z}{\longrightarrow} & M_{*-2}(\kappa(z)) \end{array} \tag{1.8}$$

As for $y \neq y_0$, π_0 is unit in $\mathcal{O}_{X,y}$ we see that if $\alpha \in \cap_{y\in X^{(1)}} M_*(\mathcal{O}_y)$ the image of α through the composition $M_{*-1}(\mathcal{O}_{y_0}) \stackrel{[\pi_0]-}{\to} M_*(F) \stackrel{\Sigma_{y\in X^{(1)}-\{y_0\}}\partial_y}{\longrightarrow} \oplus_{y\in X^{(1)}-\{y_0\}} H^1_y(X;M_*)$ is zero. By the commutativity of the above diagram this shows that the image of such an α through $s_{y_0} = \partial^{\pi_0}_{y_0}([y_0].-)$ lies in the kernel of $\partial^{\overline{\pi_1}}_z$. But this kernel is $M_{*-1}(\mathcal{O}_{\overline{y_0},z})$ and this proves the first part of Axiom (**A4**) (for M_{*-1} thus) for M_*.

Now we prove the second part of Axiom (**A4**). Assume that $\kappa(z)$ is separable over k. Let $y_1 \in X^{(1)}$ be such that $\overline{y_1}$ is smooth over k and different from $\overline{y_0}$. Clearly the intersection $\overline{y_0} \cap \overline{y_1}$ is the point z as a closed subset. If $\overline{y_0}$ and $\overline{y_1}$ do not intersect transversally, we may choose a $y_2 \in X^{(1)}$ which will intersects transversally both $\overline{y_0}$ and $\overline{y_1}$. Thus we may this way reduce to the case, that $\overline{y_0}$ and $\overline{y_1}$ do intersect transversally.

Choose $\pi_1 \in A$ which defines $\overline{y_1}$. Clearly (π_0, π_1) generate the maximal ideal of A. Now we want to prove that the two morphisms $\cap_{y\in X^{(1)}} M_*(\mathcal{O}_y) \to M_{*-2}(\kappa(z))$ obtained by using y_0 is the same as the one obtained by using y_1.

We contemplate the complex (1.5) for X and expand the equation $\partial\circ\partial = 0$ for the elements of the form $[\pi_0][\pi_1]\alpha$ with $\alpha \in \cap_{y\in X^{(1)}} M_*(\mathcal{O}_y)$. From our axioms it follows that if $y \neq y_0$ and $y \neq y_1$ then $\partial_y([\pi_0][\pi_1]\alpha) = 0$. Now $\partial^{\pi_1}_{y_1}([\pi_0][\pi_1]\alpha)$ is $[\overline{\pi_0}]s_{y_1}(\alpha) \in M_{*-1}(\kappa(y_1)) \stackrel{\theta_{y_1}}{\cong} H^1_{y_1}(X;M_*)$ and $\partial^{\pi_0}_{y_0}([\pi_0][\pi_1]\alpha)$ is (using Axiom **(B0)**) $- < -1 > [\overline{\pi_1}]s_{y_0}(\alpha) \in M_{*-1}(\kappa(y_0)) \stackrel{\theta_{y_0}}{\cong} H^1_{y_0}(X;M_*)$. Now we compute the last boundary morphism and find that the sum

$$\Theta_{y_1,\pi_1} \circ \theta_{\overline{\pi_0}}(s^{\overline{\pi_0}}_z \circ s_{y_1}(\alpha)) + \Theta_{y_0,\pi_0} \circ \theta_{\overline{\pi_1}}(- < -1 > s^{\overline{\pi_1}}_z \circ s_{y_0}(\alpha)) = 0$$

vanishes in $H^2_z(X;M)$ (as $\partial\circ\partial=0$). Lemma 1.45 below exactly yields, from this, the required equality $s_z\circ s_{y_1}(\alpha)=s_z\circ s_{y_0}(\alpha)$.□

Lemma 1.45 *Assume that M_* is as above. Assume the field k is infinite. Let $X=Spec(A)$ be a local smooth k-scheme of dimension 2, with field of functions F and closed point z. Let (π_0,π_1) be elements of A generating the maximal ideal of A and let $y_0\in X^{(1)}$ the divisor of X corresponding to π_0 and $y_1\in X^{(1)}$ that corresponding to π_0. Assume both are smooth over k. Then the composed isomorphism*

$$M_{*-2}(\kappa(v))\overset{\theta_{\overline{\pi_1}}}{\cong} H^1_z(\overline{y_0};M_{*-1})\overset{\Theta_{y_0,\pi_0}}{\cong} H^2_z(X;M)$$

is equal to $<-1>$ times the isomorphism

$$M_{*-2}(\kappa(v))\overset{\theta_{\overline{\pi_0}}}{\cong} H^1_z(\overline{y_1};M_{*-1})\overset{\Theta_{y_1,\pi_1}}{\cong} H^2_z(X;M)$$

Proof. We first observe that if $f:X'\to X$ is an étale morphism, with X' smooth local of dimension two, with closed point z' having the same residue field as z, and if y'_0 and y'_1 denote respectfully the pull-back of y_0 and y_1, then the elements (π_0,π_1) of $A'=\mathcal{O}(X')$ satisfy the same conditions. Clearly, by the previous Lemma, the assertion is true for X if and only if it is true for X', because the θ_π's and $\Theta_{y,\pi}$'s are compatible. Now there is a Nisnevich neighborhood of z: $\Omega\to X$ and an étale morphism $\Omega\to(\mathbb{A}^2_{\kappa(z)})_{(0,0)}$ which is also an étale neighborhood and such that (π_0,π_1) corresponds to the coordinates (T_0,T_1). In this way we reduce to the case $X=(\mathbb{A}^2_{\kappa(z)})_{(0,0)}$ and $(\pi_0,\pi_1)=(T_0,T_1)$.

Now one reapplies exactly the same computation as in the proof of the Theorem to elements of the form $[T_0][T_1](\alpha|_{F(T_0,T_1)})\in M_*(F(T_0,T_1))$ with $\alpha\in M_{*-2}(F)$. Now the point is that using our axioms $s^{\overline{T_0}}_{(0,0)}\circ s_{Y_1}(\alpha|_{F(T_0,T_1)})=s^{\overline{T_0}}_{(0,0)}(\alpha|_{F(T_0)})=\alpha$ and the same holds for the other term. We thus get from the proof the equality, for each $\alpha\in M_{*-2}(F)$

$$\Theta_{Y_1,T_1}\circ\theta_{\overline{T_0}}(\alpha)=\Theta_{Y_0,T_0}\circ\theta_{\overline{T_1}}(<-1>\alpha)$$

which proves our claim.□

Let M_* be a $\mathbb{Z}$-graded $\mathbb{A}^1_k$-module. Observe that for any discrete valuation v on $F \in \mathcal{F}_k$ the image of $(\mathcal{O}_v)^\times \times M_{(*-1)}(\mathcal{O}_v) \to M_*(F)$, $(u,\alpha) \mapsto [u]\alpha$ lies in $M_*(\mathcal{O}_v)$. This produces for each $n \in \mathbb{Z}$ a morphism of sheaves on Sm_k: $\mathbb{G}_m \times M_{(*-1)} \to M_*$.

Lemma 1.46 *The previous morphism of sheaves induces for any n, an isomorphism $(M_n)_{-1} \cong M_{(n-1)}$.*

Proof. This easily follows from the short exact sequence

$$0 \to M_n(F) = M_n(\mathbb{A}^1_F) \to M_n(\mathbb{G}_m|_F) \xrightarrow{\partial^T_{D_0}} M_{n-1}(F) \to 0$$

given by Axiom **(HA) (i)**.□

Remark 1.47 Conversely assume k is perfect. Given a $\mathbb{Z}$-graded abelian sheaf M_* on Sm_k, consisting of strongly $\mathbb{A}^1$-invariant sheaves, together with isomorphisms $(M_n)_{-1} \cong M_{(n-1)}$, then using our result in Section 3, one may show that evaluation on fields yields a functor $\mathcal{F}_k \to \mathcal{A}b_*$ to $\mathbb{Z}$-graded abelian groups together with Data **(D4) (i)**, **(D4) (ii)** and **(D4) (iii)** satisfying Axioms **(B0)**, **(B1)**, **(B2)**, **(B3)**, **(HA)**, **(B4)** and **(B5)**. In this way we get an equivalence of categories.

Remark 1.48 By the results of Section 3) any strongly $\mathbb{A}^1$-invariant sheaf is strictly $\mathbb{A}^1$-invariant. The category $\mathcal{HM}_k$ of homotopy modules over k (see also [11]) consisting of $\mathbb{Z}$-graded strictly $\mathbb{A}^1$-invariant abelian sheaves M_* on Sm_k, together with isomorphisms $(M_n)_{-1} \cong M_{(n-1)}$, is the heart of the *homotopy t-structure* on the stable $\mathbb{A}^1$-homotopy category of $\mathbb{P}^1$-spectra over k. this is proven in [31, 30] over a perfect field k.□

Remark 1.49 Our approach can be used also to analyze Rost cycle modules [44], at least over a perfect field k. Let $\mathcal{M}_k$ be the full subcategory of $\mathbb{Z}$-graded $\mathbb{A}^1_k$-modules (or equivalently of the category $\mathcal{HM}_k$ introduced in the previous remark) be the full subcategory consisting of those M_* satisfying $< u >= 1$ for each $u \in F^\times$. Those M_* have a trivial $\mathbb{Z}[F^\times]$-module structure. Observe that in that case the residue morphisms ∂^π_v become canonical (independent of π). Then Rost's Axioms implies the existence of an obvious forgetful functor from his category of cycle modules over k to $\mathcal{M}_k$. This can be shown to be an equivalence of category (using for instance [11] or by direct inspection using our construction of transfers in [38]. This means that in the concept of cycle module, one may forget the transfers (but should keep trike of consequences like Axioms **(B4)** and **(B5)**).

One gets back Gersten complexes from [32, 9] and canonical transfers by [38]. It is clear that Rost's complexes are isomorphic (canonically) to the associated Gersten complexes.

In general (relaxing the assumption that the $\mathbb{Z}[F^\times]$-module structure is trivial, one needs some work to prove that the Gersten (or Cousin) complex from [9] for M_* is indeed explicitly constructed like in Rost using residues, normalization process and transfers. This is of course conjectured to be true (and known in some case like [46]).□

2 Unramified Milnor-Witt K-theories

Our aim in this section is to compute (or describe), for any integer $n > 0$, the free strongly $\mathbb{A}^1$-invariant sheaf, which we denote by $\mathbb{Z}_{st-\mathbb{A}^1}(n)$ on the n-th smash power of $\mathbb{G}_m$. As we will prove in Section 3 that any strongly $\mathbb{A}^1$-invariant sheaf of abelian groups is also strictly $\mathbb{A}^1$-invariant, this is also the free strictly $\mathbb{A}^1$-invariant sheaf on $(\mathbb{G}_m)^{\wedge n}$. We will make a free use of the previous section.

2.1 Milnor-Witt K-theory of fields

The following definition was found in collaboration with Mike Hopkins:

Definition 2.1 *Let F be a commutative field. The Milnor-Witt K-theory of F is the graded associative ring $K^{MW}_*(F)$ generated by the symbols $[u]$, for each unit $u \in F^\times$, of degree $+1$, and one symbol η of degree -1 subject to the following relations:*

1 *(Steinberg relation) For each $a \in F^\times - \{1\}$: $[a].[1-a] = 0$*

2 *For each pair $(a,b) \in (F^\times)^2$: $[ab] = [a] + [b] + \eta.[a].[b]$*

3 *For each $u \in F^\times$: $[u].\eta = \eta.[u]$*

4 *Set $h := \eta.[-1] + 2$. Then $\eta \; . \; h = 0$*

These Milnor-Witt K-theory groups were introduced by the author in a different (and more complicated) way, until the previous presentation was found with Mike Hopkins. The advantage of this presentation was made clear in our computations of the stable $\pi_0^{\mathbb{A}^1}$ in [31, 30] as the relations all have very natural explanations in the stable $\mathbb{A}^1$-homotopical world. To perform these computations in the unstable world and also to produce unramified Milnor-Witt K-theory sheaves in a completely elementary way, over any field (any characteristic) we will need to use an "unstable" variant of that presentation in Lemma 2.4.

Remark 2.2 The quotient ring $K_*^{MW}(F)/\eta$ is the Milnor K-theory $K_*^M(F)$ of F defined in [28]: indeed if η is killed, the symbol $[u]$ becomes additive. Observe precisely that η controls the failure of $u \mapsto [u]$ to be additive in Milnor-Witt K-theory.

With all this in mind, it is natural to introduce the Witt K-theory of F as the quotient $K_*^W(F) := K_*^{MW}(F)/h$. It was studied in [34] and will also be used in our computations below. In *loc. cit.* it was proven that the non-negative part is the quotient of the ring $Tens_{W(F)}(I(F))$ by the Steinberg relation $<< u >> . << 1-u >>$. This can be shown to still hold in characteristic 2.

Proceeding along the same line, it is easy to prove that the non-negative part $K_{\geq 0}^{MW}(F)$ is isomorphic to the quotient of the ring $Tens_{K_0^{MW}(F)}(K_1^{MW}(F))$ by the Steinberg relation $[u].[1-u]$. This is related to our old definition of $K_*^{MW}(F)$.□

We will need at some point a presentation of the group of weight n Milnor-Witt K-theory. The following one will suffice for our purpose. One may give some simpler presentation but we won't use it:

Definition 2.3 *Let F be a commutative field. Let n be an integer. We let $\tilde{K}_n^{MW}(F)$ denote the abelian group generated by the symbols of the form $[\eta^m, u_1, \dots, u_r]$ with $m \in \mathbb{N}$, $r \in \mathbb{N}$, and $n = r - m$, and with the u_i's unit in F, and subject to the following relations:*

1_n *(Steinberg relation) $[\eta^m, u_1, \dots, u_r] = 0$ if $u_i + u_{i+1} = 1$, for some i.*

2_n *For each pair $(a,b) \in (F^\times)^2$ and each i: $[\eta^m, \dots, u_{i-1}, ab, u_{i+1}, \dots] = [\eta^m, \dots, u_{i-1}, a, u_{i+1}, \dots] + [\eta^m, \dots, u_{i-1}, b, u_{i+1}, \dots]$
$+[\eta^{m+1}, \dots, u_{i-1}, a, b, u_{i+1}, \dots]$.*

4_n *For each i, $[\eta^{m+2}, \ldots, u_{i-1}, -1, u_{i+1}, \ldots] + 2[\eta^{m+1}, \ldots, u_{i-1}, u_{i+1}, \ldots] = 0$*

The following lemma is straightforward:

Lemma 2.4 *For any field F, any integer n, the correspondence $[\eta^m, u_1, \ldots, u_n] \mapsto \eta^m[u_1] \ldots [u_n]$ induces an isomorphism*

$$\tilde{K}_n^{MW}(F) \cong K_n^{MW}(F)$$

Proof. The proof consists in expressing the possible relations between elements of degree n. That is to say the element of degree n in the two-sided ideal generated by the relations of Milnor-Witt K-theory, except the number 3, which is encoded in our choices. We left the details to the reader.□

Now we establish some elementary but useful facts. For any unit $a \in F^\times$, we set $< a >= 1 + \eta[a] \in K_0^{MW}(F)$. Observe then that $h = 1+ < -1 >$.

Lemma 2.5 *Let $(a, b) \in (F^\times)^2$ be units in F. We have the followings formulas:*

1) $[ab] = [a]+ < a > .[b] = [a]. < b > +[b]$;
2) $< ab > = < a > . < b >$; $K_0^{MW}(F)$ is central in $K_*^{MW}(F)$;
3) $< 1 >= 1$ in $K_0^{MW}(F)$ and $[1] = 0$ in $K_1^{MW}(F)$;
4) $< a >$ is a unit in $K_0^{MW}(F)$ whose inverse is $< a^{-1} >$;
5) $[\frac{a}{b}] = [a]- < \frac{a}{b} > .[b]$. In particular one has: $[a^{-1}] = - < a^{-1} > .[a]$.

Proof. 1) is obvious. One obtains the first relation of 2) by applying η to relation **2** and using relation **3**. By 1) we have for any a and b: $< a > .[b] = [b]. < a >$ thus the elements $< a >$ are central.

Multiplying relation **4** by [1] (on the left) implies that $(< 1 > -1).(< -1 > +1) = 0$ (observe that $h = 1+ < -1 >$). Using **2** this implies that $< 1 >= 1$. By 1) we have now $[1] = [1]+ < 1 > .[1] = [1] + 1.[1] = [1] + [1]$; thus $[1] = 0$. 4) follows clearly from 2) and 3). 5) is an easy consequence of 1) 2) 3) and 4).□

Lemma 2.6 *1) For each $n \geq 1$, the group $K_n^{MW}(F)$ is generated by the products of the form $[u_1]. \ldots .[u_n]$, with the $u_i \in F^\times$.*

2) For each $n \leq 0$, the group $K_n^{MW}(F)$ is generated by the products of the form $\eta^n. < u >$, with $u \in F^\times$. In particular the product with η: $K_n^{MW}(F) \to K_{n-1}^{MW}(F)$ is always surjective if $n \leq 0$.

Proof. An obvious observation is that the group $K_n^{MW}(F)$ is generated by the products of the form $\eta^m.[u_1].\dots.[u_\ell]$ with $m \geq 0$, $\ell \geq 0$, $\ell - m = n$ and with the u_i's units. The relation **2** can be rewritten $\eta.[a].[b] = [ab] - [a] - [b]$. This easily implies the result using the fact that $< 1 >= 1$.□

Remember that $h = 1+ < -1 >$. Set $\epsilon := - < -1 > \in K_0^{MW}(F)$. Observe then that relation **4** in Milnor-Witt K-theory can also be rewritten $\epsilon.\eta = \eta$.

Lemma 2.7 *1) For $a \in F^\times$ one has: $[a].[-a] = 0$ and $< a > + < -a >= h$;*
2) For $a \in F^\times$ one has: $[a].[a] = [a].[-1] = \epsilon[a][-1] = [-1].[a] = \epsilon[-1][a]$;
3) For $a \in F^\times$ and $b \in F^\times$ one has $[a].[b] = \epsilon.[b].[a]$;
4) For $a \in F^\times$ one has $< a^2 >= 1$.

Corollary 2.8 *The graded $K_0^{MW}(F)$-algebra $K_*^{MW}(F)$ is ϵ-graded commutative: for any element $\alpha \in K_n^{MW}(F)$ and any element $\beta \in K_m^{MW}(F)$ one has*

$$\alpha.\beta = (\epsilon)^{n.m}\beta.\alpha$$

Proof. It suffices to check this formula on the set of multiplicative generators $F^\times \amalg \{\eta\}$: for products of the form $[a].[b]$ this is 3) of the previous Lemma. For products of the form $[a].\eta$ or $\eta.\eta$, this follows from the relation **3** and relation **4** (reading $\epsilon.\eta = \eta$) in Milnor-Witt K-theory.□

Proof of Lemma 2.7. We adapt [28]. Start from the equality (for $a \neq 1$) $-a = \frac{1-a}{1-a^{-1}}$. Then $[-a] = [1-a] - < -a > .[1-a^{-1}]$. Thus

$$[a].[-a] = [a][1-a] - < -a > .[a].[1-a^{-1}] = 0 - < -a > .[a].[1-a^{-1}] =$$

$$< -a >< a > [a^{-1}][1-a^{-1}] = 0$$

by **1** and 1) of lemma 2.5. The second relation follows from this by applying η^2 and expanding.

As $[-a] = [-1]+ < -1 > [a]$ we get

$$0 = [a].[-1]+ < -1 > [a][a]$$

so that $[a].[a] = - < -1 > [a].[-1] = [a].[-1]$ because $0 = [1] = [-1]+ < -1 > [-1]$. Using $[-a][a] = 0$ we find $[a][a] = - < -1 > [-1][a] = [-1][a]$.

Finally expanding

$$0 = [ab].[-ab] = ([a]+ < a > .[b])([-a]+ < -a > [b])$$

gives

$$0 =< a > ([b][-a]+ < -1 > [a][b])+ < -1 > [-1][b]$$

as $[-a] = [a]+ < a > [-1]$ we get

$$0 =< a > ([b][a]+ < -1 > [a][b]) + [b][-1]+ < -1 > [-1][b]$$

the last term is 0 by 3) so that we get the third claim.

The fourth one is obtained by expanding $[a^2] = 2[a] + \eta[a][a]$; now due to point 2) we have $[a^2] = 2[a] + \eta[-1][a] = (2 + \eta[-1])[a] = h[a]$. Applying η we thus get 0.□

Let us denote (in any characteristic) by $GW(F)$ the Grothendieck-Witt ring of isomorphism classes of non-degenerate symmetric bilinear forms [29]: this is the group completion of the commutative monoid of isomorphism classes of non-degenerate symmetric bilinear forms for the direct sum.

For $u \in F^\times$, we denote by $< u >\in GW(F)$ the form on the vector space of rank one F given by $F^2 \to F$, $(x, y) \mapsto uxy$. By the results of *loc. cit.*, these $< u >$ generate $GW(F)$ as a group. The following Lemma is (essentially) [29, Lemma (1.1) Chap. IV]:

Lemma 2.9 *[29] The group $GW(F)$ is generated by the elements $< u >$, $u \in F^\times$, and the following relations give a presentation of $GW(F)$:*

(i) $< u(v^2) >=< u >$;

(ii) $< u > + < -u >= 1+ < -1 >$;

(iii) $< u > + < v >=< u + v > + < (u + v)uv >$ *if* $(u + v) \neq 0$.

When $char(F) \neq 2$ the first two relations imply the third one and one obtains the standard presentation of the Grothendieck-Witt ring $GW(F)$, see [45,]. If $char(F) = 2$ the third relation becomes $2(< u > -1) = 0$.

We observe that the subgroup (h) of $GW(F)$ generated by the hyperbolic plan $h = 1+ < -1 >$ is actually an ideal (use the relation **(ii)**). We let $W(F)$ be the quotient (both as a group or as a ring) $GW(F)/(h)$ and let $W(F) \to \mathbb{Z}/2$ be the corresponding mod 2 rank homomorphism; $W(F)$ is the Witt ring of F [29], and [45] in characteristic $\neq 2$. Observe that the following commutative square of commutative rings

$$\begin{array}{ccc} GW(F) & \rightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ W(F) & \rightarrow & \mathbb{Z}/2 \end{array} \tag{2.1}$$

is cartesian. The kernel of the mod 2 rank homomorphism $W(F) \to \mathbb{Z}/2$ is denoted by $I(F)$ and is called the fundamental ideal of $W(F)$.

It follows from our previous results that $u \mapsto < u > \in K_0^{MW}(F)$ satisfies all the relations defining the Grothendieck-Witt ring. Only the last one requires a comment. As the symbol $< u >$ is multiplicative in u, we may reduce to the case $u + v = 1$ by dividing by $< u + v >$ if necessary. In that case, this follows from the Steinberg relation to which one applies η^2. We thus get a ring epimorphism (surjectivity follows from Lemma 2.6)

$$\phi_0 : GW(F) \twoheadrightarrow K_0^{MW}(F)$$

For $n > 0$ the multiplication by $\eta^n : K_0^{MW}(F) \to K_{-n}^{MW}(F)$ kills h (because $h.\eta = 0$ and thus we get an epimorphism:

$$\phi_{-n} : W(F) \twoheadrightarrow K_{-n}^{MW}(F)$$

Lemma 2.10 *For each field F, each $n \geq 0$ the homomorphism ϕ_{-n} is an isomorphism.*

Proof. Following [3], let us define by $J^n(F)$ the fiber product $I^n(F) \times_{i^n(F)} K_n^M(F)$, where we use the Milnor epimorphism $s_n : K_n^M(F)/2 \twoheadrightarrow i^n(F)$, with $i^n(F) := I^n(F)/I^{(n+1)}(F)$. For $n \leq 0$, $I^n(F)$ is understood to be $W(F)$. Now altogether the $J^*(F)$ form a graded ring and we denote by $\eta \in J^{-1}(F) = W(F)$ the element $1 \in W(F)$. For any $u \in F^\times$, denote by $[u] \in J^1(F) \subset I(F) \times F^\times$ the pair $(< u > -1, u)$. Then the four relations hold in $J^*(F)$ which produces an epimorphism $K_*^{MW}(F) \twoheadrightarrow J^*(F)$. For $n > 0$ the composition of epimorphisms $W(F) \to K_{-n}^{MW}(F) \to J^{-n}(F) = W(F)$ is the identity. For $n = 0$ the composition $GW(F) \to K_0^{MW}(F) \to J^0(F) = GW(F)$ is also the identity. The Lemma is proven.□

Corollary 2.11 *The canonical morphism of graded rings $K_*^{MW}(F) \to W(F)[\eta, \eta^{-1}]$ induced by $[u] \mapsto \eta^{-1}(< u > -1)$ induces an isomorphism $K_*^{MW}(F)[\eta^{-1}] = W(F)[\eta, \eta^{-1}]$.*

Remark 2.12 For any F let $I^*(F)$ denote the graded ring consisting of the powers of the fundamental ideal $I(F) \subset W(F)$. We let $\eta \in I^{-1}(F) = W(F)$ be the generator. Then the product with η acts as the inclusions $I^n(F) \subset I^{n-1}(F)$. We let $[u] =< u > -1 \in I(F)$ be the opposite to the Pfister form $<< u >>= 1- < u >$. Then these symbol satisfy the relations of Milnor-Witt K-theory [34] and the image of h is zero. We obtain in this way an epimorphism $K^W_*(F) \twoheadrightarrow I^*(F)$, $[u] \mapsto < u > -1 = - << u >>$. This ring $I^*(F)$ is exactly the image of the morphism $K^{MW}_*(F) \to W(F)[\eta, \eta^{-1}]$ considered in the Corollary above.

We have proven that this is always an isomorphism in degree ≤ 0. In fact this remains true in degree 1, see Corollary 2.47 for a stronger version. In fact it was proven in [34] (using [**?**] and Voevodsky's proof of the Milnor conjectures) that

$$K^W_*(F) \twoheadrightarrow I^*(F) \tag{2.2}$$

is an isomorphism in characteristic $\neq 2$. Using Kato's proof of the analogues of those conjectures in characteristic 2 [21] we may extend this result for any field F.

From that we may also deduce (as in [34]) that the obvious epimorphism

$$K^W_*(F) \twoheadrightarrow J^*(F) \tag{2.3}$$

is always an isomorphism.□

Here is a very particular case of the last statement, but completely elementary:

Proposition 2.13 *Let F be a field for which any unit is a square. Then the epimorphism*

$$K^{MW}_*(F) \to K^M_*(F)$$

is an isomorphism in degrees ≥ 0, and the epimorphism

$$K^{MW}_*(F) \to K^W_*(F)$$

is an isomorphism in degrees < 0. In fact $I^n(F) = 0$ for $n > 0$ and $I^n(F) = W(F) = \mathbb{Z}/2$ for $n \leq 0$. In particular the epimorphisms (eq:kwi) and (eq:kmwj) are isomorphisms.

Proof. The first observation is that $< -1 >= 1$ and thus $2\eta = 0$ (fourth relation in Milnor-Witt K-theory). Now using Lemma 2.14 below we see that for any unit $a \in F^\times$, $\eta[a^2] = 2\eta[a] = 0$, thus as any unit b is a square, we get that for any $b \in F^\times$, $\eta[b] = 0$. This proves that the second relation of Milnor-Witt K-theory gives for units (a, b) in F: $[ab] = [a] + [b] + \eta[a][b] = [a] + [b]$. The proposition now follows easily from these observations.□

Lemma 2.14 *Let $a \in F^\times$ and let $n \in \mathbb{Z}$ be an integer. Then the following formula holds in $K_1^{MW}(F)$:*

$$[a^n] = n_\epsilon [a]$$

where for $n \geq 0$, we $n_\epsilon \in K_0^{MW}(F)$ is defined as follows

$$n_\epsilon = \sum_{i=1}^{n} < (-1)^{(i-1)} >$$

(and satisfies for $n > 0$ the relation $n_\epsilon =< -1 > (n-1)_\epsilon + 1$) and where for $n \leq 0$, $n_\epsilon := - < -1 > (-n)_\epsilon$.□

Proof. The proof is quite straightforward by induction: one expands $[a^n] = [a^{n-1}] + [a] + \eta[a^{n-1}][a]$ as well as $[a^{-1}] = - < a > [a] = -([a] + \eta[a][a])$.□

2.2 Unramified Milnor-Witt K-theories

In this section we will define for each $n \in \mathbb{Z}$ an explicit sheaf $\underline{\mathbf{K}}_n^{MW}$ on Sm_k called *unramified* Milnor-Witt K-theory in weight n, whose sections on any field $F \in \mathcal{F}_k$ is the group $K_n^{MW}(F)$. In the next section we will prove that for $n > 0$ this sheaf $\underline{\mathbf{K}}_n^{MW}$ is the free strongly $\mathbb{A}^1$-invariant sheaf generated by $(\mathbb{G}_m)^{\wedge n}$.

Residue homomorphisms. Recall from [28], that for any discrete valuation v on a field F, with valuation ring $\mathcal{O}_v \subset F$, and residue field $\kappa(v)$, one can define a unique homomorphism (of graded groups)

$$\partial_v : K_*^M(F) \to K_{*-1}^M(\kappa(v))$$

called "residue" homomorphism, such that

$$\partial_v(\{\pi\}\{u_2\} \dots \{u_n\}) = \{\overline{u_2}\} \dots \{\overline{u_n}\}$$

for any uniformizing element π and units $u_i \in \mathcal{O}_v^\times$, and where $\overline{u}$ denotes the image of $u \in \mathcal{O}_v \cap F^\times$ in $\kappa(v)$.

In the same way, given a uniformizing element π, one has:

Theorem 2.15 *There exists one and only one morphism of graded groups*

$$\partial_v^\pi : K_*^{MW}(F) \to K_{*-1}^{MW}(\kappa(v))$$

which commutes to product by η and satisfying the formulas:

$$\partial_v^\pi([\pi][u_2]\dots[u_n]) = [\overline{u_2}]\dots[\overline{u_n}]$$

and

$$\partial_v^\pi([u_1][u_2]\dots[u_n]) = 0$$

for any units u_1, ..., u_n of $\mathcal{O}_v$.

Proof. Uniqueness follows from the following Lemma as well as the formulas $[a][a] = [a][-1]$, $[ab] = [a] + [b] + \eta[a][b]$ and $[a^{-1}] = - < a > [a] = -([a] + \eta[a][a])$. The existence follows from Lemma 2.16 below.□

To define the residue morphism ∂_v^π we use the method of Serre [28]. Let ξ be a variable of degree 1 which we adjoin to $K_*^{MW}(\kappa(v))$ with the relation $\xi^2 = \xi[-1]$; we denote by $K_*^{MW}(\kappa(v))[\xi]$ the graded ring so obtained.

Lemma 2.16 *Let v be a discrete valuation on a field F, with valuation ring $\mathcal{O}_v\mathbb{F}$ and let π be a uniformizing element of v. The map*

$$\mathbb{Z} \times \mathcal{O}_v^\times = F^\times \to K_*^{MW}(\kappa(v))[\xi]$$

$$(\pi^n.u) \mapsto \Theta_\pi(\pi^n.u) := [\overline{u}] + (n_\epsilon < \overline{u} >).\xi$$

and $\eta \mapsto \eta$ satisfies the relation of Milnor-Witt K-theory and induce a morphism of graded rings:

$$\Theta_\pi : K_*^{MW}(F) \to K_*^{MW}(\kappa(v))[\xi]$$

Proof. We first prove the first relation of Milnor-Witt K-theory. Let $\pi^n.u \in F^\times$ with u in $\mathcal{O}_v^\times$. We want to prove $\Theta_\pi(\pi^n.u)\Theta_\pi(1 - \pi^n.u) = 0$ in $K_*^{MW}(\kappa(v))[\xi]$. If $n > 0$, then $1 - \pi^n.u$ is in $\mathcal{O}_v^\times$ and clearly by definition

$\Theta_\pi(1-\pi^n.u) = 0$. If $n = 0$, then write $1 - u = \pi^m.v$ with v a unit in $\mathcal{O}_v$. If $m > 0$ the symmetric reasoning allows to conclude. If $m = 0$, then $\Theta_\pi(u) = [\overline{u}]$ and $\Theta_\pi(1-u) = [1-\overline{u}]$ in which case the result is also clear.

It remains to consider the case $n < 0$. Then $\Theta_\pi(\pi^n.u) = [\overline{u}] + (n_\epsilon < \overline{u} >)\xi$. Moreover we write $(1-\pi^n.u)$ as $\pi^n(-u)(1-\pi^{-n}u^{-1})$ and we observe that $(-u)(1-\pi^{-n}u^{-1})$ is a unit on $\mathcal{O}_v$ so that $\Theta_\pi(1-\pi^n.u) = [-\overline{u}] + n_\epsilon < -\overline{u} > \xi$. Expanding $\Theta_\pi(\pi^n.u)\Theta_\pi(1-\pi^n.u)$ we find $[\overline{u}][-\overline{u}] + n_\epsilon < \overline{u} > \xi[-\overline{u}] + n_\epsilon < -\overline{u} > [\overline{u}][\xi] + (n_\epsilon)^2 < -1 > \xi^2$. We observe that $[\overline{u}][-\overline{u}] = 0$ and that $(n_\epsilon)^2 < -1 > \xi^2 = (n_\epsilon)^2[-1] < -1 > \xi = n_\epsilon < -1 > \xi[-1]$ because $(n_\epsilon)^2[-1] = n_\epsilon[-1]$ (this follows from Lemma 2.14 : $(n_\epsilon)^2[-1] = n_\epsilon[(-1)^n] = [(-1)^{n^2}] = [(-1)^n]$ as $n^2 - n$ is even). Thus $\Theta_\pi(\pi^n.u)\Theta_\pi(1-\pi^n.u) = n_\epsilon\{??\}\xi$ where the expression $\{??\}$ is

$$< -\overline{u} > ([\overline{u}] - [-\overline{u}]) + < -1 > [-1]$$

But $[\overline{u}] - [-\overline{u}] = [\overline{u}] - [\overline{u}] - [-1] - \eta[\overline{u}][-1] = - < \overline{u} > [-1]$ thus $< -\overline{u} > ([\overline{u}] - [-\overline{u}]) = - < -1 > [-1]$, proving the result.

We now check relation 2 of Milnor-Witt K-theory. Expanding we find that the coefficient which doesn't involve ξ is 0 and the coefficient of ξ is

$$n_\epsilon < \overline{u} > + m_\epsilon < \overline{v} > - n_\epsilon < -\overline{u} > (< \overline{v} > -1) + m_\epsilon < \overline{v} > (< u > -1)$$

$$+ n_\epsilon m_\epsilon < \overline{uv} > (< -1 > -1)$$

A careful computation (using $< \overline{u} > + < -\overline{u} > = < 1 > + < -1 > = < \overline{uv} > + < -\overline{uv} >$ yields that this term is

$$n_\epsilon + m_\epsilon - n_\epsilon m_\epsilon + < -1 > n_\epsilon m_\epsilon$$

which is shown to be $(n+m)_\epsilon$. The last two relations of the Milnor-Witt K-theory are very easy to check.□

We now proceed as in [28], we set for any $\alpha \in K_n^{MW}(F)$:

$$\Theta_\pi(\alpha) := s_v^\pi(\alpha) + \partial_v^\pi(\alpha).\xi$$

The homomorphism ∂_v^π so defined is easily checked to have the required properties. Moreover $s_v^\pi : K_*^{MW}(F) \to K_*^{MW}(\kappa(v))$ is clearly a morphism of rings, and as such is the unique one mapping η to η and $\pi^n u$ to $[\overline{u}]$.

Proposition 2.17 *We keep the previous notations and assumptions. For any $\alpha \in K_*^{MW}(F)$:*

1) $\partial_v^\pi([-\pi].\alpha) = < -1 > s_v^\pi(\alpha)$;

2) $\partial_v^\pi([u].\alpha) = [\overline{u}]\partial_v^\pi(\alpha)$ for any $u \in \mathcal{O}_v^\times$.

3) $\partial_v^\pi(< u > .\alpha) = < \overline{u} > \partial_v^\pi(\alpha)$ for any $u \in \mathcal{O}_v^\times$.

Proof. We observe that, for $n \geq 1$, $K_n^{MW}(F)$ is generated as group by elements of the form $\eta^m[\pi][u_2]\ldots[u_{n+m}]$ or of the form $\eta^m[u_1][u_2]\ldots[u_{n+m}]$, with the u_i's units of $\mathcal{O}_v$ and with $n+m \geq 1$. Thus it suffices to check the formula on these elements. This is quite straightforward.□

Remark 2.18 A heuristic but useful explanation of this "trick" of Serre is the following. $Spec(F)$ is the open complement in $Spec(\mathcal{O}_v)$ of the closed point $Spec(\kappa(v))$. If one had a tubular neighborhood for that close immersion, there should be a morphism $E(\nu_v) - \{0\} \to Spec(F)$ of the complement of the zero section of the normal bundle to $Spec(F)$; the map θ_π is the map induced in cohomology by this "hypothetical" morphism. Observe that choosing π corresponds to trivializing ν_v, in which case $E(\nu_v) - \{0\}$ becomes $(\mathbb{G}_m)_{Spec(\kappa(v))}$. Then the ring $K_*^{MW}(\kappa(v))[\xi]$ is just the ring of sections of K_*^{MW} on $(\mathbb{G}_m)_{Spec(\kappa(v))}$. The "funny" relation $\xi^2 = \xi[-1]$ which is true for any element in $K_*^{MW}(F)$, can also be explained by the fact that the reduced diagonal $(\mathbb{G}_m)_{Spec(\kappa(v))} \to (\mathbb{G}_m)^{\wedge 2}_{Spec(\kappa(v))}$ is equal to the multiplication by $[-1]$.□

Lemma 2.19 *For any fields extension $E \subset F$ and for any discrete valuation on F which restricts to a discrete valuation w on E with ramification index e. Let π be a uniformizing element of v and ρ a uniformizing element of w. Write it $\rho = u\pi^e$ with $u \in \mathcal{O}_v^\times$. Then for each $\alpha \in K_*^{MW}(E)$ one has*

$$\partial_v^\pi(\alpha|_F) = e_\epsilon < \overline{u} > (\partial_w^\rho(\alpha))|_{\kappa(v)}$$

Proof. We just observe that the square (of rings)

$$\begin{array}{ccc} K_*^{MW}(F) & \stackrel{\Theta_\pi}{\to} & K_*^{MW}(\kappa(v))[\xi] \\ \uparrow & & \uparrow \Psi \\ K_*^{MW}(E) & \stackrel{\Theta_\rho}{\to} & K_*^{MW}(\kappa(w))[\xi] \end{array}$$

where Ψ is the ring homomorphism defined by $[a] \mapsto [a|_F]$ for $a \in \kappa(v)$ and $\xi \mapsto [\overline{u}] + e_\epsilon < \overline{u} > \xi$ is commutative. It is sufficient to check the commutativity in degree 1. This is not hard.□

Using the residue homomorphism and the previous Lemma one may define for any discrete valuation v on F the subgroup $\underline{\mathbf{K}}_n^{MW}(\mathcal{O}_v) \subset K_n^{MW}(F)$ as the kernel of ∂_v^π. From our previous Lemma (applied to $E = F$, $e = 1$), it is clear that the kernel doesn't depend on π, only on v. We define $H_v^1(\mathcal{O}_v; \underline{\mathbf{K}}_n^{MW})$ as the quotient group $K_n^{MW}(F)/K_n^{\mathcal{R}}(\mathcal{O}_v)$. Once we choose a uniformizing element π we get of course a canonical isomorphism $K_n^{MW}(\kappa(v)) = H_v^1(\mathcal{O}_v; \underline{\mathbf{K}}_n^{MW})$.

Remark 2.20 One very important feature of residue homomorphisms is that in the case of Milnor K-theory, these residues homomorphisms don't depend on the choice of π, only on the valuation, but in the case of Milnor-Witt K-theory, they do depend on the choice of π: for $u \in \mathcal{O}^\times$, as one has $\partial_v^\pi([u.\pi]) = \partial_v^\pi([\pi]) + \eta.[\overline{u}] = 1 + \eta.[\overline{u}]$.

This property of independence of the residue morphisms on the choice of π is a general fact (in fact equivalent) for the $\mathbb{Z}$-graded unramified sheaves M_* considered above for which the $\mathbb{Z}[F^\times/F^{\times 2}]$-structure is trivial, like Milnor K-theory. These are called "oriented": in the spirit of Remark 2.18.

Remark 2.21 To make the residue homomorphisms "canonical" (see [3, 4, 46] for instance), one defines for a field κ and a one dimensional κ-vector space L, twisted Milnor-Witt K-theory groups: $K_*^{MW}(\kappa; L) = K_*^{MW}(\kappa) \otimes_{\mathbb{Z}[\kappa^\times]} \mathbb{Z}[L - \{0\}]$, where the group ring $\mathbb{Z}[\kappa^\times]$ acts through $u \mapsto < u >$ on $K_*^{MW}(\kappa)$ and through multiplication on $\mathbb{Z}[L - \{0\}]$. The canonical residue homomorphism is of the following form

$$\partial_v : K_*^{MW}(F) \to K_{*-1}^{MW}(\kappa(v); m_v/(m_v)^2)$$

with $\partial_v([\pi].[u_2] \dots [u_n]) = [\overline{u_2}] \dots [\overline{u_n}] \otimes \overline{\pi}$, where $m_v/(m_v)^2$ is the cotangent space at v (a one dimensional $\kappa(v)$-vector space).$\square$

The following result and its proof follow closely Bass-Tate [5]:

Theorem 2.22 *Let v be a discrete valuation ring on a field F. Then the subring*

$$\underline{\mathbf{K}}_*^{MW}(\mathcal{O}_v) \subset K_*^{MW}(F)$$

is as a ring generated by the elements η and $[u] \in K_1^{MW}(F)$, with $u \in \mathcal{O}_v^\times$ a unit of $\mathcal{O}_v$.

Consequently, the group $\underline{\mathbf{K}}_n^{MW}(\mathcal{O}_v)$ is generated by symbols $[u_1] \dots [u_n]$ with the u_i's in $\mathcal{O}_v^\times$ for $n \geq 1$ and by the symbols $\eta^{-n} < u >$ with the u's in $\mathcal{O}_v^\times$ for $n \leq 0$

Proof. The last statement follows from the first one as in Lemma 2.6.

We consider the quotient graded abelian group Q_* of $K_*^{MW}(F)$ by the subring A_* generated by the elements and $\eta \in K_{-1}^{MW}(F)$ and $[u] \in K_1^{MW}(F)$, with $u \in \mathcal{O}_v^\times$ a unit of $\mathcal{O}_v$. We choose a uniformizing element π. The valuation morphism induces an epimorphism $Q_* \to K_{*-1}^{MW}(\kappa(v))$. It clearly suffices to check that this is an isomorphism. We will produce an epimorphism $K_{*-1}^{MW}(\kappa(v)) \to Q_*$ and show that the composition $K_{*-1}^{MW}(\kappa(v)) \to Q_* \to K_{*-1}^{MW}(\kappa(v))$ is the identity.

We construct a $K_*^{MW}(\kappa(v))$-module structure on $Q_*(F)$. Denote by $\mathcal{E}_*$ the graded ring of endomorphisms of the graded abelian group $Q_*(F)$. First the element η still acts on Q_* and yields an element $\eta \in \mathcal{E}_{-1}$. Let $a \in \kappa(v)^\times$ be a unit in $\kappa(v)$. Choose a lifting $\tilde{\alpha} \in \mathcal{O}_v^\times$. Then multiplication by $\tilde{\alpha}$ clearly induces a morphism of degree $+1$, $Q_* \to Q_{*+1}$. We first claim that it doesn't depend on the choice of $\tilde{\alpha}$. Let $\tilde{\alpha}' = \beta\tilde{\alpha}$ be another lifting so that $u \in \mathcal{O}_v^\times$ is congruent to 1 mod π. Expending $[\tilde{\alpha}'] = [\tilde{\alpha}] + [\beta] + \eta[\tilde{\alpha}][\beta]$ we see that it is sufficient to check that for any $a \in F^\times$, the product $[\beta][a]$ lies in the subring A_*. Write $a = \pi^n.u$ with $u \in \mathcal{O}_v^\times$. Then expending $[\pi^n.u]$ we end up to checking the property for the product $[\beta][\pi^n]$, and using Lemma 2.14 we may even assume $n = 1$. Write $beta = 1 - \pi^n.v$, with $n > 0$ and $v \in \mathcal{O}_v^\times$.

Thus we have to prove that the products of the above form $[1 - \pi^n.v][\pi]$ are in A_*. For $n = 1$, the Steinberg relation yields $[1 - \pi.v][\pi.v] = 0$. Expending $[\pi.v] = [\pi](1 + \eta[v]) + [v]$, implies $[1 - \pi.v][\pi](1 + \eta[v])$ is in A_*. But by Lemma 2.7, $1 + \eta[v] =< v >$ is a unit of A_*, with inverse itself. Thus $[1 - \pi.v][\pi] \in A_*$. Now if $n \geq 2$, $1 - \pi^n.v = (1 - \pi) + \pi(1 - \pi^{n-1}v) = (1 - \pi)(1 + \pi(\frac{1-\pi^{n-1}}{1-\pi})) = (1 - \pi)(1 - \pi w)$, with $w \in \mathcal{O}_v^\times$. Expending, we get $[1 - \pi^n.v][\pi] = [1 - \pi][\pi] + [1 - \pi w][\pi] + \eta[1 - \pi][1 - \pi w][\pi] = [1 - \pi w][\pi]$. Thus the result holds in general.

We thus define this way elements $[u] \in \mathcal{E}_1$. We now claim these elements (together with η) satisfy the four relations in Milnor-Witt K-theory: this is very easy to check, by the very definitions. Thus we get this way a $K_*^{MW}(\kappa(v))$-module structure on Q_*. Pick up the element $[\pi] \in Q_1 = K_1^{MW}(F)/A_1$. Its image through ∂_v^π is the generator of $K_*^{MW}(\kappa(v))$ and the homomorphism $K_{*-1}^{MW}(\kappa(v)) \to Q_*, \alpha \mapsto \alpha.[\pi]$ provides a section of $\partial_v^\pi : Q_* \to K_{*-1}^{MW}(\kappa(v))$. This is clear from our definitions.

It suffices now to check that $K_{*-1}^{MW}(\kappa(v)) \to Q_*$ is onto. Using the fact that any element of F can be written $\pi^n u$ for some unit $u \in \mathcal{O}_v^\times$, we see that $K_*^{MW}(F)$ is generated as a group by elements of the form $\eta^m[\pi][u_2]\ldots[u_n]$ or

$\eta^m[u_1]\dots[u_n]$, with the u_i's in $\mathcal{O}_v^\times$. But the latter are in A_* and the former are clearly, modulo A_*, in the image of $K_{*-1}^{MW}(\kappa(v)) \to Q_*$.□

Remark 2.23 In fact one may also prove as in *loc. cit.* the fact that the morphism Θ_π defined in the Lemma 2.16 is onto and its kernel is the ideal generated by η and the elements $[u] \in K_1^{MW}(F)$ with $u \in \mathcal{O}_v^\times$ a unit of $\mathcal{O}_v$ congruent to 1 modulo π. We will not give the details here, we do not use these results.□

Theorem 2.24 *For any field F the following diagram is a (split) short exact sequence of $K_*^{MW}(F)$-modules:*

$$0 \to K_n^{MW}(F) \to K_n^{MW}(F(T)) \stackrel{\Sigma\partial^P_{(P)}}{\longrightarrow} \oplus_P K_{n-1}^{MW}(F[T]/P) \to 0$$

(where P runs over the set of unitary irreducible polynomials of $F[T]$).

Proof. It it is again very much inspired from [28]. We first observe that the morphism $K_*^{MW}(F) \to K_*^{MW}(F(T))$ is a split monomorphism; from our previous computations we see that $K_*^{MW}(F(T)) \stackrel{\partial^T_{(T)}([T]\cup -)}{\longrightarrow} K_*^{MW}(F)$ provides a retraction.

Now we define a filtration on $K_*^{MW}(F(T))$ by sub-rings L_d's

$$L_0 = K_*^{MW}(F) \subset L_1 \subset \dots \subset L_d \subset \dots \subset K_*^{MW}(F(T))$$

such that L_d is exactly the sub-ring generated by $\eta \in K_{-1}^{MW}(F(T))$ and all the elements $[P] \in K_1^{MW}(F(T))$ with $P \in F[T] - \{0\}$ of degree less or equal to d. Thus L_0 is indeed $K_*^{MW}(F) \subset K_*^{MW}(F(T))$. Observe that $\bigcup_d L_d = K_*^{MW}(F(T))$. Observe that each L_d is actually a sub $K_*^{MW}(F)$-algebra.

Also observe that using the relation $[a.b] = [a]+[b]+\eta[a][b]$ that if $[a] \in L_d$ and $[b] \in L_d$ then so are $[ab]$ and $[\frac{a}{b}]$. As a consequence, we see that for $n \geq 1$, $L_d(K_n^{MW}(F(T)))$ is the sub-group generated by symbols $[a_1]\dots[a_n]$ such that each a_i itself is a fraction which involves only polynomials of degree $\leq d$. In degree ≤ 0, we see in the same way that $L_d(K_n^{MW}(F(T)))$ is the sub-group generated by symbols $< a > \eta^n$ with a a fraction which involves only polynomials of degree $\leq d$.

It is also clear that for $n \geq 1$, $L_d(K_n^{MW}(F(T)))$ is generated as a group by elements of the form $\eta^m[a_1]\dots[a_{n+m}]$ with the a_i of degree $\leq d$.

Lemma 2.25 *1) For $n \geq 1$, $L_d(K_n^{MW}(F(T)))$ is generated by the elements of $L_{(d-1)}(K_n^{MW}(F(T)))$ and elements of the form $\eta^m[a_1]\dots[a_{n+m}]$ with a_1 of degree d and the a_i's, $i \geq 2$ of degree $\leq (d-1)$.*

2) Let $P \in F[T]$ be a unitary polynomial of degree $d > 0$. Let G_1, ..., G_i be be polynomials of degrees $\leq (d-1)$. Finally let G be the rest of the Euclidean division of $\Pi_{j\in\{1,\dots,i\}}G_j$ by P, so that G has degree $\leq (d-1)$. Then one has in the quotient group $K_2^{MW}(F(T))/L_{d-1}$ the equality

$$[P][G_1 \dots G_i] = [P][G]$$

Proof. 1) We proceed as in Milnor's paper. Let f_1 and f_2 be polynomials of degree d. We may write $f_2 = -af_1 + g$, with $a \in F^{\times}$ a unit and g of degree $\leq (d-1)$. If $g = 0$, the we have $[f_1][f_2] = [f_1][a(-f_1)] = [f_1][a]$ (using the relation $[f_1, -f_1] = 0$). If $g \neq 0$ then as in *loc. cit.* we get $1 = \frac{af_1}{g} + \frac{f_2}{g}$ and the Steinberg relation yields $[\frac{af_1}{g}][\frac{f_2}{g}] = 0$. Expanding with η we get: $([f_1] - [\frac{g}{a}] - \eta[\frac{g}{a}][\frac{af_1}{g}])[\frac{f_2}{g}] = 0$, which readily implies (still in $K_2^{MW}(F(T))$):

$$([f_1] - [\frac{g}{a}])[\frac{f_2}{g}] = 0$$

But expanding the right factor now yields

$$([f_1] - [\frac{g}{a}])([f_2] - [g] - \eta[g][\frac{f_2}{g}]) = 0$$

which implies (using again the previous vanishing):

$$([f_1] - [\frac{g}{a}])([f_2] - [g]) = 0$$

We see that $[f_1][f_2]$ can be expressed as a sum of symbols in which at most one of the factor as degree d, the other being of smaller degree. An easy induction proves 1).

2) We first establish the case $i = 2$. We start with the Euclidean division $G_1G_2 = PQ + G$. We get from this the equality $1 = \frac{G}{G_1.G_2} + \frac{PQ}{G_1.G_2}$ which gives $[\frac{PQ}{G_1.G_2}][\frac{G}{G_1.G_2}] = 0$. We expand the left term as $[\frac{PQ}{G_1.G_2}] =< \frac{Q}{G_1.G_2} > [P] + [\frac{Q}{G_1.G_2}]$. We thus obtain $[P][\frac{G}{G_1.G_2}] = - < \frac{Q}{G_1.G_2} > [\frac{Q}{G_1.G_2}][\frac{G}{G_1.G_2}]$ but the right

hand side is clearly in $L_{(d-1)}$ (observe Q has degree $\leq (d-1)$) thus $[P][\frac{G}{G_1.G_2}] \in L_{(d-1)} \subset K_2^{MW}(F(T))$. Now $[\frac{G}{G_1G_2}] = [G] - [G_1G_2] - \eta[G_1G_2][\frac{G}{G_1G_2}]$. Thus $[P][\frac{G}{G_1.G_2}] = [P][G] - [P][G_1G_2] + <-1> \eta[G_1G_2][P][\frac{G}{G_1G_2}]$. This shows that modulo $L_{(d-1)}$, $[P][G] - [P][G_1G_2]$ is zero, as required.

For the case $i \geq 3$ we proceed by induction. Let $\Pi_{j\in\{2,\ldots,i\}} G_j = P.Q + G'$ be the Euclidean division of $\Pi_{j\in\{2,\ldots,i\}} G_j$ by P with G' of degree $\leq (d-1)$. Then the rest G of the Euclidean division by P of $G_1 \ldots G_i$ is the same as the rest of the Euclidean division of G_1G' by P. Now $[P][G_1 \ldots G_i] = [P][G_1] + [P][G_2 \ldots G_i] + \eta[P][G_2 \ldots G_i][G_1]$. By the inductive assumption this is equal, in $K_2^{MW}(F(T))/L_{d-1}$, to $[P][G_1] + [P][G'] + \eta[P][G'][G_1] = [P][G'G_1]$. By the case 2 previously proven we thus get in $K_2^{MW}(F(T))/L_{d-1}$,

$$[P][G_1 \ldots G_i] = [P][G_1G'] = [P][G]$$

which proves our claim.□

Now we continue the proof of Theorem 2.24 following Milnor's proof of [28, Theorem 2.3]. Let $d \geq 1$ be an integer and let $P \in F[T]$ be a unitary irreducible polynomial of degree d. We denote by $\mathcal{K}_P \subset L_d/L_{(d-1)}$ the sub-graded group generated by elements of the form $\eta^m[P][G_1] \ldots [G_n]$ with the G_i of degree $(d-1)$. For any polynomial G of degree $\leq (d-1)$, the multiplication by $\epsilon[G]$ induces a morphism:

$$\epsilon[G].: \mathcal{K}_P \to \mathcal{K}_P$$

$$\eta^m[P][G_1] \ldots [G_n] \mapsto \epsilon[G]\eta^m[P][G_1] \ldots [G_n] = \eta^m[P][G][G_1] \ldots [G_n]$$

of degree $+1$. Let $\mathcal{E}_P$ be the graded associative ring of graded endomorphisms of $\mathcal{K}_P$. We claim that the map $(F[T]/P)^\times \to (\mathcal{E}_P)_1, \overline{(G)} \mapsto \epsilon[G]$. (where G has degree $\leq (d-1)$) and the element $\eta \in (\mathcal{E}_P)_{-1}$ (corresponding to the multiplication by η) satisfy the four relations of the Milnor-Witt K-theory. Let us check the Steinberg relation. Let $G \in F[T]$ be of degree $\leq (d-1)$. Then so is $1-G$ and the relation $(\epsilon[G].) \circ (\epsilon[1-G].) = 0 \in \mathcal{E}_P$ is clear. Let us check relation **2**. We let H_1 and H_2 be polynomials of degree $\leq (d-1)$. Let G be the rest of division of H_1H_2 by P. By definition $\epsilon[\overline{(H_1)}\overline{(H_2)}]$. is $\epsilon[\overline{(G)}]$.. But by the part 2) of the Lemma we have (in $\mathcal{K}_P \subset K_m^{MW}(F(T))/L_{(d-1)}$):

$$\epsilon[\overline{(G)}].(\eta^m[P][G_1] \ldots [G_n]) = \eta^m[P][G][G_1] \ldots [G_n] = \eta^m[P][H_1H_2][G_1] \ldots [G_n]$$

which easily implies the claim. The last two relations are easy to check.

We thus obtain a morphism of graded ring $K_*^{MW}(F[T]/P) \to \mathcal{E}_P$. By letting $K_*^{MW}(F[T]/P)$ act on $[P] \in L_d/L_{(d-1)} \subset K_1^{MW}(F(T))/L_{(d-1)}$ we obtain a graded homomorphism

$$K_*^{MW}(F[T]/P) \to \mathcal{K}_P \subset L_d/L_{(d-1)}$$

which is clearly an epimorphism. By the first part of the Lemma, we see that the induced homomorphism

$$\oplus_P K_*^{MW}(F[T]/P) \to L_d/L_{(d-1)} \tag{2.4}$$

is an epimorphism. Now using our definitions, one checks as in [28] that for P of degree d, the residue morphism ∂^P vanishes on $L_{(d-1)}$ and that moreover the composition

$$\oplus_P K_*^{MW}(F[T]/P) \twoheadrightarrow L_d(K_n^{MW}(F(T)))/L_{(d-1)}(K_n^{MW}(F(T))) \stackrel{\Sigma_P \partial^P}{\longrightarrow} \oplus_P K_*^{MW}(F[T]/P)$$

is the identity. As in *loc. cit.* this implies the Theorem, with the observation that the quotients L_d/L_{d-1} are $K_*^{MW}(F)$-modules and the residues maps are morphisms of $K_*^{MW}(F)$-modules.□

Remark 2.26 We observe that the previous Theorem in negative degrees is exactly [28, Theorem 5.3].

Now we fixe a base field k. We will make constant use of the results of Section 1.3. We endow the functor $F \mapsto K_*^{MW}(F)$, $\mathcal{F}_k \to \mathcal{A}b_*$ with Data **(D4) (i)**, **(D4) (ii)** and **(D4) (iii)**. The datum **(D4) (i)** comes from the $K_0^{MW}(F) = GW(F)$-module structure on each $K_n^{MW}(F)$ and the datum **(D4) (ii)** comes from the product $F^\times \times K_n^{MW}(F) \to K_{(n+1)}^{MW}(F)$. The residue homomorphisms ∂_v^π gives the Data **(D4) (iii)**. We observe of course that these Data are extended from the prime field of k.

Axioms **(B0)**, **(B1)** and **(B2)** are clear from our previous results. The Axiom **(B3)** follows at once from Lemma 2.19.

Axiom **(HA) (ii)** is clear. The Theorem 2.24 establishes Axiom **(HA) (i)**.

For any discrete valuation v on $F \in \mathcal{F}_k$, and any uniformizing element π, define morphisms of the form $\partial_z^y : K_n^{\mathcal{R}}(\kappa(y)) \to K_{n-1}^{\mathcal{R}}(\kappa(z))$ for any $y \in$

$(\mathbb{A}^1_F)^{(1)}$ and $z \in (\mathbb{A}^1_{\kappa(v)})^{(1)}$ fitting in the following diagram:

$$\begin{array}{ccccccccc} 0 & \to & K_*^{MW}(F) & \to & K_*^{MW}(F(T)) & \to & \oplus_{y\in(\mathbb{A}^1_F)^{(1)}} K_{*-1}^{MW}(\kappa(y)) & \to & 0 \\ & & \downarrow \partial_v^\pi & & \downarrow \partial_{v[T]}^\pi & & \downarrow \Sigma_{y,z}\partial_z^{\pi,y} & & \\ 0 & \to & K_{*-1}^{MW}(\kappa(v)) & \to & K_{n-1}^{MW}(\kappa(v)(T)) & \to & \oplus_{z\in\mathbb{A}^1_{\kappa(v)}} K_{*-2}^{\mathcal{R}}(\kappa(v)) & \to & 0 \end{array} \tag{2.5}$$

The following Theorem establishes Axiom **(B4)**.

Theorem 2.27 *Let v be a discrete valuation on $F \in \mathcal{F}_k$, let π be a uniformizing element. Let $P \in \mathcal{O}_v[T]$ be an irreducible primitive polynomial, and $Q \in \kappa(v)[T]$ be an irreducible unitary polynomial.*

(i) If the closed point $Q \in \mathbb{A}^1_{\kappa(v)} \subset \mathbb{A}^1_{\mathcal{O}_v}$ is not in the divisor D_P then the morphism $\partial_Q^{\pi,P}$ is zero.

(ii) If Q is in $D_P \subset \mathbb{A}^1_{\mathcal{O}_v}$ and if the local ring $\mathcal{O}_{D_P,Q}$ is a discrete valuation ring with π as uniformizing element then

$$\partial_Q^{\pi,P} = - < -\frac{\overline{P'}}{Q'} > \partial_Q^Q$$

Proof. Let $d \in \mathbb{N}$ be an integer. We will say that Axiom **(B4)** holds in degree $\leq d$ if for any field $F\mathcal{F}_k$, any irreducible primitive polynomial $P \in \mathcal{O}_v[T]$ of degree $\leq d$, any unitary irreducible $Q \in \kappa(v)[T]$ then: if Q doesn't lie in the divisor D_P, the homomorphism ∂_Q^P is 0 on $K_*^{MW}(F[T]/P)$ and if Q lies in D_P and that the local ring $\mathcal{O}_{\overline{y},z}$ is a discrete valuation ring with π as uniformizing element , then the homomorphism ∂_Q^P is equal to $-\partial_Q^\pi$.

We use Remark 1.17 to reduce to the case the base field k is infinite.

We now proceed by induction on d to prove that Axiom **(B4)** holds in degree $\leq d$ for any d. For $d = 0$ this is trivial. For $d = 1$ this is very easy.

We will use:

Lemma 2.28 *Assume $\kappa(v)$ is infinite. Let P be a primitive irreducible polynomial of degree d in $F[T]$. Let Q be a unitary irreducible polynomial in $\kappa(v)[T]$.*

Assume either that $\overline{P}$ is prime to Q, or that Q divides $\overline{P}$ and that the local ring $\mathcal{O}_{D_P,Q}$ is a discrete valuation ring with uniformizing element π.

Then the elements of the form $\eta^m[\overline{G_1}]\dots[\overline{G_n}]$, where all the G_i's are irreducible elements in $\mathcal{O}_v[T]$ of degree $< d$, such that, either G_1 is equal to π or $\overline{G_1}$ is prime to Q, and for any $i \geq 2$, $\overline{G_i}$ is prime to Q, generate $K_*^{MW}(F[T]/P)$ as a group.

Proof. First the symbols of the form $\eta^m[\overline{G_1}]\dots[\overline{G_n}]$ with the G_i irreducible elements of degree $< d$ of $\mathcal{O}_v[T]$ clearly generate the Milnor-Witt K-theory of $f[T]/P$ as a group.

1) We first assume that $\overline{P}$ is prime to Q. It suffices to check that those element above are expressible in terms of symbols of the form of the Lemma. Pick up one such $\eta^m[\overline{G_1}]\dots[\overline{G_n}]$. Assume that there exists i such that $\overline{G_i}$ is divisible by Q (otherwise there is nothing to prove), for instance G_1.

As the field $\kappa(v)$ is infinite, there is an $\alpha \in \mathcal{O}_v$ such that $G_1(\alpha)$ is a unit in $\mathcal{O}_v^\times$. Then there exists a unit u in $\mathcal{O}_v^\times$ and an integer v (actually the valuation of $P(\alpha)$ at π) such that $P + u\pi^v G$ is divisible by $T - \alpha$ in $\mathcal{O}_v[T]$. Write $P + u\pi^v G_1 = (T-\alpha)H_1$. Observe that Q which divides $\overline{G_1}$ and is prime to $\overline{P}$ must be prime to both $T - \overline{\alpha}$ and $\overline{H_1}$.

Observe that $\frac{(T-\alpha)}{u\pi^v}H_1 = \frac{P}{u\pi^v} + G_1$ is the Euclidean division of $\frac{(T-\alpha)}{u\pi^v}H_1$ by P. By Lemma 2.25 one has in $K_*^{MW}(F(T))$, modulo L_{d-1}

$$\eta^m[P][G_1][G_2]\dots[G_n] = \eta^m[P][\frac{(T-\alpha)}{u\pi^v}H_1][G_2]\dots[G_n]$$

Because $\partial_{D_P}^P$ vanishes on L_{d-1}, applying $\partial_{D_P}^P$ to the previous congruence yields the equality in $K_*^{MW}(F[T]/P)$

$$\eta^m[\overline{G_1}]\dots[\overline{G_n}] = \eta^m[\frac{(T-\alpha)}{u\pi^v}\overline{H_1}][G_2]\dots[\overline{G_n}]$$

Expanding $[\frac{(T-\alpha)}{u\pi^v}\overline{H_1}]$ as $[\frac{(T-\alpha)}{u\pi^v}] + [\overline{H_1}] + \eta[\frac{(T-\alpha)}{u\pi^v}][\overline{H_1}]$ shows that we may strictly reduce the number of G_i's whose mod π reduction is divisible by Q. This proves our first claim (using the relation $[\pi][\pi] = [\pi][-1]$ we may indeed assume that only G_1 is maybe equal to π).

2) Now assume that Q divides $\overline{P}$ and that the local ring $\mathcal{O}_{D_P,Q}$ is a discrete valuation ring with uniformizing element π. By our assumption, any

non-zero element in the discrete valuation ring $\mathcal{O}_{D_P,Q} = (\mathcal{O}_v[T]/P)_Q$ can be written as

$$\pi^v \frac{\overline{R}}{\overline{S}}$$

with R and S polynomials in $\mathcal{O}_v[T]$ of degree $< d$ whose mod π reduction in $\kappa(v)[T]$ is prime to Q. From this, it follows easily that the symbols of the form $\eta^m[\overline{G_1}]\dots[\overline{G_n}]$, with the G_i's being either a polynomial in $\mathcal{O}_v[T]$ of degree $< d$ whose mod π reduction in $\kappa(v)[T]$ is prime to Q, either equal to π.

The Lemma is proven.□

Now let $d > 0$ and assume the claim is proven in degrees $< d$, for all fields. Let P be a primitive irreducible polynomial of degree d in $\mathcal{O}_v[T]$. Let Q be a unitary irreducible polynomial in $\kappa(v)[T]$.

Under our inductive assumption, we may compute $\partial_Q^{\pi,P}(\eta^m[G_1]\dots[\overline{G_n}])$ for any sequence G_1, .., G_n as in the Lemma.

Indeed, the symbol $\eta^m[P][G_1]\dots[\overline{G_n}] \in K_{n-m}^{MW}$ has residue at P the symbol $\eta^m[\overline{G_1}]\dots[\overline{G_n}]$. All its other potentially non trivial residues concern irreducible polynomials of degree $< d$. By the (proof of) Theorem 2.24, we know that there exists an $\alpha \in L_{d-1}(K_{n-m}^{MW}(F(T))$ such that

$$\eta^m[P][G_1]\dots[\overline{G_n}] + \alpha$$

has only one non vanishing residue, which is at P, and which equals $\eta^m[\overline{G_1}]\dots[\overline{G_n}]$.

Then clearly the support of α (which means the set of points of codimension one in $\mathbb{A}^1_F$ where α has a non trivial residue) consists of the divisors defined by the G_i's (P doesn't appear). But those doesn't contain Q.

Using the commutative diagram which defines the ∂_Q^P's, we may compute $\partial_Q^{\pi,P}(\eta^m[\overline{G_1}]\dots[\overline{G_n}])$ as

$$\partial_Q^Q(\partial_v^\pi(\eta^m[P][G_1]\dots[G_n]+\alpha)) = \partial_Q^Q(\partial_v^\pi(\eta^m[P][G_1]\dots[G_n])+\sum_i \partial_Q^{\pi,G_i}(\partial_{D_{G_i}}^{G_i}(\alpha))$$

By our inductive assumption, $\sum_i \partial_Q^{\pi,G_i}(\partial_{D_{G_i}}^{G_i}(\alpha)) = 0$ because the supports G_i do not contain Q.

We then have two cases:

1) G_1 is not π. Then

$$\partial_v^\pi(\eta^m[P][G_1]\dots[G_n]) = 0$$

as every element lies in $\mathcal{O}_{v[T]}^\times$. Thus in that case, $\partial_Q^{\pi,P}(\eta^m[\overline{G_1}]\dots[\overline{G_n}]) = 0$ which is compatible with our claim.

2) $G_1 = \pi$. Then

$$\begin{aligned}\partial_v^\pi(\eta^m[P][\pi][G_2]\dots[G_n]) &= - < -1 > \partial_v^\pi(\eta^m[\pi][P][G_2]\dots[G_n]) \\ &= - < -1 > \eta^m[\overline{P}][\overline{G_2}]\dots[\overline{G_n}]\end{aligned}$$

Applying ∂_Q^Q yields 0 if $\overline{P}$ is prime to Q, as all the terms are units. If $\overline{P} = QR$, then R is a unit in $(\mathbb{A}^1_{\kappa v})_Q$ by our assumptions. Expending $[QR] = [Q] + [R] + \eta[Q][R]$, we get

$$\begin{aligned}\partial_Q^{\pi,P}(\eta^m[\overline{G_1}]\dots[\overline{G_n}]) &= - < -1 > \eta^m([\overline{G_2}]\dots[\overline{G_n}] + \eta[\overline{R}][\overline{G_2}]\dots[\overline{G_n}]) \\ &= - < -\overline{R} > \eta^m[\overline{G_2}]\dots[\overline{G_n}]\end{aligned}$$

It remains to observe that $\overline{R} = \frac{\overline{P'}}{Q'}$.

By the previous Lemma the symbols we used generate $K_*^{MW}(F[T]/P)$. Thus the previous computations prove the Theorem.□

Now we want to prove Axiom **(B5)**. Let X be a local smooth k-scheme of dimension 2, with field of functions F and closed point z, let $y_0 \in X^{(1)}$ be such that $\overline{y_0}$ is smooth over k. Choose a uniformizing element π of $\mathcal{O}_{X,y_0}$. Denote by $\mathcal{K}_n(X; y_0)$ the kernel of the map

$$K_n^{MW}(F) \overset{\Sigma_{y\in X^{(1)}-\{y_0\}}\partial_y}{\longrightarrow} \oplus_{y\in X^{(1)}-\{y_0\}} H^1_y(X; \underline{\mathbf{K}}_n^{MW}) \tag{2.6}$$

By definition $\underline{\mathbf{K}}_n^{MW}(X) \subset \mathcal{K}_n(X; y_0)$. The morphism $\partial_{y_0}^\pi : K_n^{MW}(F) \to K_{n-1}^{MW}(\kappa(y_0))$ induces an injective homomorphism $\mathcal{K}_n(X; y_0)/\underline{\mathbf{K}}_n^{MW}(X) \subset K_{n-1}^{MW}(\kappa(y_0))$.

We first observe:

Lemma 2.29 *Assume k is infinite. Keep the previous notations and assumptions. Then $\underline{\mathbf{K}}_{n-1}^{MW}(\mathcal{O}_{y_0}) \subset \mathcal{K}_n(X; y_0)/\underline{\mathbf{K}}_n^{MW}(X) \subset K_{n-1}^{MW}(\kappa(y_0))$.*

Proof. As k is infinite, we may apply Gabber's lemma to y_0, and in this way, we see (by an easy diagram chase) that we can reduce to the case $X = (\mathbb{A}^1_U)_z$ where U is a smooth local k-scheme of dimension 1. As Theorem 2.27 implies Axiom **(B4)**, we know by **??** that the following complex

$$0 \to \underline{\mathbf{K}}_n^{MW}(X) \to K_n^{MW}(F) \stackrel{\Sigma_{y\in X^{(1)}}\partial_y}{\longrightarrow} \oplus_{y\in X^{(1)}} H^1_y(X;\underline{\mathbf{K}}_n^{MW}) \to H^2_z(X;\underline{\mathbf{K}}_n^{MW}) \to 0$$

Moreover, we know from there that for $\overline{y}_0$ smooth, the morphism $H^1_y(X;\underline{\mathbf{K}}_n^{MW}) \to H^2_z(X;\underline{\mathbf{K}}_n^{MW})$ can be "interpreted" as the residue map. Its kernel is thus $\underline{\mathbf{K}}_{n-1}^{MW}(\mathcal{O}_{y_0}) \subset K_{n-1}^{MW}(\kappa(y_0)) \cong H^1_y(X;\underline{\mathbf{K}}_n^{MW})$. The exactness of the previous complex implies then at once that in that case

$$\mathcal{K}_n(X;y_0)/\underline{\mathbf{K}}_n^{MW}(X) = \underline{\mathbf{K}}_{n-1}^{MW}(\mathcal{O}_{y_0})$$

proving the statement.□

Our last objective is now to show that in fact $\underline{\mathbf{K}}_{n-1}^{MW}(\mathcal{O}_{y_0}) = \mathcal{K}_n(X;y_0)/\underline{\mathbf{K}}_n^{MW}(X) \subset K_{n-1}^{MW}(\kappa(y_0))$. To do this we observe that by Lemma 1.39, for k infinite, the morphism (2.6) above is an epimorphism. Thus the previous statement is equivalent to the fact that the diagram

$$0 \to \underline{\mathbf{K}}_{n-1}^{MW}(\mathcal{O}_{y_0}) \to K_n^{MW}(F)/\underline{\mathbf{K}}_n^{MW}(X) \stackrel{\Sigma_{y\in X^{(1)}-\{y_0\}}\partial_y}{\longrightarrow} \oplus_{y\in X^{(1)}-\{y_0\}} H^1_y(X;\underline{\mathbf{K}}_n^{MW}) \to 0$$

is a short exact sequence or in other words that the epimorphism

$$\Phi_n(X;y_0) : K_n^{MW}(F)/\underline{\mathbf{K}}_n^{MW}(X)+\underline{\mathbf{K}}_{n-1}^{MW}(\mathcal{O}_{y_0}) \stackrel{\Sigma_{y\in X^{(1)}-\{y_0\}}\partial_y}{\longrightarrow} \oplus_{y\in X^{(1)}-\{y_0\}} H^1_y(X;\underline{\mathbf{K}}_n^{MW}) \qquad (2.7)$$

is an isomorphism. We also observe that the group $K_n^{MW}(F)/\underline{\mathbf{K}}_n^{MW}(X) + \underline{\mathbf{K}}_{n-1}^{MW}(\mathcal{O}_{y_0})$ doesn't depend actually on the choice of a local parametrization of $\overline{y_0}$.

Theorem 2.30 *Assume k is infinite. Let X be a local smooth k-scheme of dimension 2, with field of functions F and closed point z, let $y_0 \in X^{(1)}$ be such that $\overline{y_0}$ is smooth over k. Then the epimorphism $\Phi_n(X;y_0)$(2.7) is an isomorphism.*

Proof. We know from Axiom **(B1)** (that is to say Theorem 2.27) and Lemma **??** that the assertion is true for X a localization of $\mathbb{A}^1_U$ at some codimension 2 point, where U is a smooth local k-scheme of dimension 1.

Lemma 2.31 *Given any element $\alpha \in K_n^{MW}(F)$, write it as $\alpha = \sum_i \alpha_i$, where the α_i's are pure symbols. Let $Y \subset X$ be the union of the hypersurfaces defined by each factor of each pure symbol α_i. Let $X \to \mathbb{A}^1_U$ be an étale morphism with U smooth local of dimension 1, with field of functions E, such that $Y \to \mathbb{A}^1_U$ is a closed immersion. Then for each i there exists a pure symbol $\beta_i \in K_n^{MW}(E(T))$ which maps to α_i modulo $\underline{\mathbf{K}}_n^{MW}(X) \subset K_n^{MW}(F)$.*

As a consequence, if $\partial_y(\alpha) \neq 0$ in $H^1_y(X; \underline{\mathbf{K}}_n^{MW})$ for some $y \in X^{(1)}$ then $y \in Y$ and $\partial_y(\alpha) = \partial_y(\beta) =\in H^1_y(X; \underline{\mathbf{K}}_n^{MW}) = H^1_y(\mathbb{A}^1_U; \underline{\mathbf{K}}_n^{MW})$.

Proof. Let us denote by π_j the irreducible elements in the factorial ring $\mathcal{O}(U)[T]$ corresponding to the irreducible components of $Y \subset \mathbb{A}^1_U$. Each $\alpha_i = [\alpha_i^1] \ldots [\alpha_i^n]$ is a pure symbol in which each term α_i^s decomposes as a product $\alpha_i^s = u_i^s {\alpha'}_i^s$ of a unit u_i^s in $\mathcal{O}(X)^\times$ and a product ${\alpha'}_i^s$ of π_j's (this follows from our choices and the factoriality property of $A := \mathcal{O}(X)$. Thus α'_i is in the image of $K_n^{MW}(E(T)) \to K_n^{MW}(F)$. Now by construction, $A/(\Pi\pi_j) = B/(\Pi\pi_j)$, where $B = \mathcal{O}(U)[T]$. Thus one may choose unit v_i^s in $B^\times$ with $w_i^s := \frac{u_i^s}{v_i^s} \equiv 1[\Pi\pi_j]$.

Now set $\beta_i^s = v_i^s {\alpha'}_i^s$, $\beta_i := [\beta_i^1] \ldots [\beta_i^n]$. Then we claim that β_i maps to α_i modulo $\underline{\mathbf{K}}_n^{MW}(X) \subset K_n^{MW}(F)$. In other words, we claim that $[\alpha_i^1] \ldots [\alpha_i^n] - [\beta_i^1] \ldots [\beta_i^n]$ lies in $\underline{\mathbf{K}}_n^{MW}(X)$ which means that each of its residue at any point of codimension one in X vanishes. Clearly, by construction the only non-zero residues can only occur at each π_j.

We end up in showing the following: given elements $\beta^s \in A - \{0\}$, $s \in \{1, \ldots, n\}$ and $w^s \in A^\times$ which is congruent to 1 modulo each irreducible element π which divides one of the β^s, then for each such π, $\partial^\pi([\beta^1] \ldots [\beta^n]) = \partial^\pi([w^1\beta^1] \ldots [w^n\beta^n])$. We expand $[w^1\beta^1] \ldots [w^n\beta^n]$ as $[w^1][w^2\beta^2] \ldots [w^n\beta^n] + [\beta^1][w^2\beta^2] \ldots [w^n\beta^n] + \eta[w^1][\beta^1][w^2\beta^2] \ldots [w^n\beta^n]$. Now using Proposition 2.17 and the fact that $\overline{w^i}^\pi = 1$, we immediately get $\partial^\pi([w^1\beta^1] \ldots [w^n\beta^n]) = \partial^\pi([\beta^1][w^2\beta^2] \ldots [w^n\beta^n])$ which gives the result. An easy induction gives the result. This proof can obviously be adapted for pure symbols of the form $\eta^n[\alpha]$. □

Now the theorem follows easily from the Lemma. Let $\overline{\alpha} \in K_n^{MW}(F)/\underline{\mathbf{K}}_n^{MW}(X) + \underline{\mathbf{K}}_{n-1}^{MW}(\mathcal{O}_{y_0})$ be in the kernel of $\Phi_n(X; y_0)$. Assume $\alpha \in K_n^{MW}(F)$ represents $\overline{\alpha}$. By Gabber's Lemma there exists an étale morphism $X \to \mathbb{A}^1_U$ with U smooth local of dimension 1, with field of functions E, such that $Y \cup \overline{y_0} \to \mathbb{A}^1_U$ is a closed immersion, where Y is obtained by writing α as a sum of pure symbols α_i's. By the previous Lemma, we may find β_i in $K_n^{MW}(E(T))$ mapping to α modulo $\underline{\mathbf{K}}_n^{MW}(X)$ yo α_i. Let β be the sum of the β_i's. Then $\overline{\beta} \in K_n^{MW}(E(T))/\underline{\mathbf{K}}_n^{MW}((\mathbb{A}^1_U)_z) + \underline{\mathbf{K}}_{n-1}^{MW}(\mathcal{O}_{y_0})$ is also in the kernel of our morphism $\Phi_n((\mathbb{A}^1_U)_z; y_0)$. Thus $\overline{\beta} = 0$ and so $\overline{\alpha} = 0$.□

Unramified $K^{\mathcal{R}}$-theories. We will now slightly generalize our construction by allowing some "admissible" relations in $K_*^{MW}(F)$. An admissible set of relations $\mathcal{R}$ is the datum for each $F \in \mathcal{F}_k$ of a graded ideal $\mathcal{R}_*(F) \subset K_*^{MW}(F)$ with the following properties:

(1) For any extension $E \subset F$ in $\mathcal{F}_k$, $\mathcal{R}_*(E)$ is mapped into $\mathcal{R}_*(F)$;

(2) For any discrete valuation v on $F \in \mathcal{F}_k$, any uniformizing element π, $\partial_v^\pi(\mathcal{R}_*(F)) \subset \mathcal{R}_*(\kappa(v))$;

(3) For any $F \in \mathcal{F}_k$ the following sequence is a short exact sequence:

$$0 \to \mathcal{R}_*(F) \to \mathcal{R}_*(F(T)) \overset{\sum_P \partial^P_{D_P}}{\longrightarrow} \oplus_P \mathcal{R}_{*-1}(F[t]/P) \to 0 \quad \square$$

The third one is clearly usually more difficult to check.

Given an admissible relation $\mathcal{R}$, for each $F \in \mathcal{F}_k$ we simply denote by $K_*^{\mathcal{R}}(F)$ the quotient graded ring $K_*^{MW}(F)/\mathcal{R}_*(F)$. The property (1) above means that we get this way a functor

$$\mathcal{F}_k \to \mathcal{A}b_*$$

This functor is moreover clearly endowed with data **(D4) (i)** and **(D4) (ii)** coming from the K_*^{MW}-algebra structure. The property (2) defines the data **(D4) (iii)**. The axioms **(B0)**, **(B1)**, **(B2)**, **(B3)** are immediate consequences from those for K_*^{MW}. Property (3) implies axiom **(HA) (i)**. Axiom **(HA) (ii)** is clear. Axioms **(B4)** and **(B5)** are also consequences from the corresponding axioms just established for K_*^{MW}. We thus get as in Theorem 1.43 a $\mathbb{Z}$-graded strongly $\mathbb{A}^1$-invariant sheaf, denoted by $\underline{\mathbf{K}}_*^{\mathcal{R}}$ with isomorphisms $(\underline{\mathbf{K}}_n^{\mathcal{R}})_{-1} \cong \underline{\mathbf{K}}_{n-1}^{\mathcal{R}}$. There is obviously a structure of $\mathbb{Z}$-graded sheaf of algebras over $\underline{\mathbf{K}}_*^{MW}$.

Lemma 2.32 *Let $R_* \subset K_*^{MW}(k)$ be a graded ideal. For any $F \in \mathcal{F}_k$, denote by $\mathcal{R}_*(F) := R_*.K_*^{MW}(F)$ the ideal generated by R_*. Then $\mathcal{R}_*(F)$ is an admissible relation on K_*^{MW}. We denote the quotient simply by $K_*^{MW}(F)/R_*$.*

Proof. Properties (1) and (2) are easy to check. We claim that the property (3) also hold: this follows from Theorem 2.24 which states that the morphisms and maps are $K_*^{MW}(F)$-module morphisms.□

Example 2.33 For instance we may take an integer n and $R_* = (n) \subset K_*^{MW}(k)$; we obtain mod n Milnor-Witt unramified sheaves. For $R_* = (\eta)$ the ideal generated by η, this yields Milnor K-theory. For $R_* = (n, \eta)$ this yields mod n Milnor K-theory. For $\mathcal{R} = (h)$, this yields Witt K-theory, for $\mathcal{R} = (\eta, \ell)$ this yields mod ℓ Milnor K-theory.□

Example 2.34 Let $\mathcal{R}_*^I(F)$ be the kernel of the epimorphism $K_*^{MW}(F) \twoheadrightarrow I^*(F)$, $[u] \mapsto < u > -1 = - << u >>$ described in [34], see also Remark 2.12. Recall from the Remark 2.12 that $K_*^{MW}(F)[\eta^{-1}] = W(F)[\eta, \eta^{-1}]$ and that $I^*(F)$ is the image of $K_*^{MW}(F) \to W(F)[\eta, \eta^{-1}]$. Now by our previous results the morphism $K_*^{MW}(F) \to W(F)[\eta, \eta^{-1}]$ induces a morphism of $\mathbb{Z}$-graded $\mathbb{A}^1_k$-modules. We now conclude using the Lemma 2.35 below.□

Let $\phi : M_* \to N_*$ be a morphism (in the obvious sense) of $\mathbb{Z}$-graded $\mathbb{A}^1_k$-modules. Denote for each $F \in \mathcal{F}_k$ by $Im(\phi)_*(F)$ (resp. $Ker(\phi)_*(F)$) the image (resp. the kernel) of $\phi(F) : M_*(F) \to N_*(F)$. This extends easily to a functor $\mathcal{F}_k \to \mathcal{A}b_*$. The data **(D4) (i)**, **(D4) (ii)** and **(D4) (iii)** on both M_* and N_* clearly induce data of the same nature on $Im(\phi)_*$ and $Ker(\phi)_*$.

Lemma 2.35 *Let $\phi : M_* \to N_*$ be a morphism of $\mathbb{Z}$-graded $\mathbb{A}^1_k$-modules. Then $Im(\phi)_*$ and $Ker(\phi)_*$ are $\mathbb{Z}$-graded $\mathbb{A}^1_k$-modules.*

Proof. The only difficulty is to check axiom **(HA) (i)**. It is in fact very easy to check using the axioms **(HA) (i)** and **(HA) (ii)** for M_* and N_*. Indeed **(HA) (ii)** provides a splitting of the short exact sequences of **(HA) (i)** for M_* and N_* which are compatible. One get the axiom **(HA) (i)** for $Im(\phi)_*$ and $Ker(\phi)_*$ using the snake lemma.□

2.3 Milnor-Witt K-theory and strongly $\mathbb{A}^1$-invariant sheaves

In this section, k is again any commutative field. Fix a natural number $n \geq 1$. Recall from [39] that $(\mathbb{G}_m)^{\wedge n}$ denotes the n-th smash power of the pointed space $\mathbb{G}_m$. We first construct a canonical morphism of pointed spaces

$$\sigma_n : (\mathbb{G}_m)^{\wedge n} \to \underline{\mathbf{K}}_n^{MW}$$

$(\mathbb{G}_m)^{\wedge n}$ is *a priori* the associated sheaf to the naive presheaf $\Theta_n : X \mapsto (\mathcal{O}^\times(X))^{\wedge n}$ but in fact:

Lemma 2.36 *The presheaf $\Theta_n : X \mapsto (\mathcal{O}(X)^\times)^{\wedge n}$ is an unramified sheaf.*

Proof. It is as a presheaf clearly unramified in the sense of our definition 1.1 thus automatically a sheaf in the Zariski topology. One way further check it is a sheaf in the Nisnevich topology as well by checking Axiom **(A1)**. Each time we use the following easy observation. Let E_α be a family of pointed subsets in a pointed set E. Then $\cap_\alpha (E_\alpha)^{\wedge n} = (\cap_\alpha E_\alpha)^{\wedge n}$ inside $E^{\wedge n}$.□

Fix an irreducible $X \in Sm_k$ with function field F. There is a tautological symbol map $(\mathcal{O}(X)^\times)^{\wedge n} \subset (F^\times)^{\wedge n} \to K_n^{MW}(F)$ that takes a symbol $(u_1, \ldots, u_n) \in (\mathcal{O}(X)^\times)^{\wedge n}$ to the corresponding symbol in $[u_1]\ldots[u_n] \in K_n^{MW}(F)$. But clearly this symbol $[u_1]\ldots[u_n] \in K_n^{MW}(F)$ lies in $\underline{\mathbf{K}}_n^{MW}(X)$, that is to say each of its residues at points of codimension 1 in X is 0. This follows at once from the definitions and elementary formulas for the residues.

This defines a morphism of sheaves on $\tilde{Sm}_k$. Now to show that this extends to a morphism of sheaves on Sm_k, using the equivalence of categories of Theorem 1.11 (and its proof) we end up to show that our symbol maps commutes to restriction maps s_v, which is also clear from the elementary formulas we proved in Milnor-Witt K-theory. In this way we have obtained our canonical symbol map

$$\sigma_n : (\mathbb{G}_m)^{\wedge n} \to \underline{\mathbf{K}}_n^{MW}$$

From the previous section we know that $\underline{\mathbf{K}}_n^{MW}$ is a strongly $\mathbb{A}^1$-invariant sheaf.

Our aim in this section is to prove:

Theorem 2.37 *Let $n \geq 1$. The morphism σ_n is the universal morphism from $(\mathbb{G}_m)^{\wedge n}$ to a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups. In other words, given a morphism of pointed sheaves $\phi : (\mathbb{G}_m)^{\wedge n} \to M$, with M a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups, then there exists a unique morphism of sheaves of abelian groups $\Phi : \underline{\mathbf{K}}_n^{MW} \to M$ such that $\Phi \circ \sigma_n = \phi$.*

Remark 2.38 The statement is wrong if we release the assumption that M is a sheaf of abelian groups. The free strongly $\mathbb{A}^1$-invariant sheaf of groups generated by $\mathbb{G}_m$ will be seen in **??** to be non commutative. For $n = 2$, it is a sheaf of abelian groups. For $n > 2$ it is not known to us.

The statement is also clearly false for $n = 0$: $(\mathbb{G}_m)^{\wedge 0}$ is just $Spec(k)_+$, that is to say $Spec(k)$ with a base point added, and the free strongly $\mathbb{A}^1$-invariant generated by $Spec(k)_+$ is $\mathbb{Z}$, not $\underline{\mathbf{K}}_0^{MW}$. To see a analogous presentation of $\underline{\mathbf{K}}_0^{MW}$ see Theorem 2.46 below.□

Roughly, the idea of the proof is to first use Lemma 2.4 to show that $\phi : (\mathbb{G}_m)^{\wedge n} \to M$ induces on fields $F \in \mathcal{F}_k$ a morphism $K_n^{MW}(F) \to M(F)$ and then to use our work on unramified sheaves in section 1.1 to observe this induces a morphism of sheaves.

Theorem 2.39 *Let M be a strongly $\mathbb{A}^1$-invariant sheaf, let $n \geq 1$ be an integer, and let $\phi : (\mathbb{G}_m)^{\wedge n} \to M$ be a morphism of pointed sheaves. For any field $F \in \mathcal{F}_k$, there is unique morphism*

$$\Phi(F) : K_n^{MW}(F) \to M(F)$$

such that for any $(u_1, \ldots, u_n) \in (F^\times)^n$, $\Phi_n(F)([u_1, \ldots, u_n]) = \phi(u_1, \ldots, u_n)$.

Preliminaries. We will freely use some notions and some elementary results from [39].

Let M be a sheaf of groups on Sm_k. Recall that we denote by M_{-1} the sheaf $M^{(\mathbb{G}_m)}$, and for $n \geq 0$, by M_{-n} the n-th iteration of this construction. To say that M is strongly $\mathbb{A}^1$-invariant is equivalent to the fact that $K(M, 1)$ is $\mathbb{A}^1$-local [39]. Indeed from *loc. cit.*, for any pointed space $\mathcal{X}$, we have $Hom_{\mathcal{H}_\bullet(k)}(\mathcal{X}; K(M, 1)) \cong H^1(\mathcal{X}; M)$ and $Hom_{\mathcal{H}_\bullet(k)}(\Sigma(\mathcal{X}); K(M, 1)) \cong \tilde{M}(X))$. Here we denote for M is (a strongly $\mathbb{A}^1$-invariant) sheaf of abelian groups and $\mathcal{X}$ a pointed space by $\tilde{M}(\mathcal{X})$ the kernel of the evaluation at the base point of $M(\mathcal{X}) \to M(k)$, so that $M(\mathcal{X})$ splits as $M(k) \oplus \tilde{M}(\mathcal{X})$.

We also observe that because M is assumed abelian, the map (from "pointed to base point free classes")

$$Hom_{\mathcal{H}_\bullet(k)}(\Sigma(\mathcal{X}); K(M, 1)) \to Hom_{\mathcal{H}(k)}(\Sigma(\mathcal{X}); K(M, 1))$$

is a bijection.

From Lemma 1.31 and its proof we know that in that case, $R\mathbf{Hom}_\bullet(\mathbb{G}_m; K(M,1))$ is canonically isomorphic to $K(M_{-1},1)$ and that M_{-1} is also strongly $\mathbb{A}^1$-invariant. We also know that $R\Omega_s(K(M,1) \cong M$.

As a consequence, for a strongly $\mathbb{A}^1$-invariant sheaf of abelian group M, the evaluation map

$$Hom_{\mathcal{H}_\bullet(k)}(\Sigma((\mathbb{G}_m)^{\wedge n}), K(M,1)) \to M_{-n}(k)$$

is an isomorphism of abelian groups.

Now for $\mathcal{X}$ and $\mathcal{Y}$ pointed spaces, the cofibration sequence $\mathcal{X} \vee \mathcal{Y} \to \mathcal{X} \times \mathcal{Y} \to \mathcal{X} \wedge \mathcal{Y}$ splits after applying the suspension functor Σ. Indeed, as $\Sigma(\mathcal{X} \times \mathcal{Y})$ is a co-group object in $\mathcal{H}_\bullet(k)$ the (ordered) sum of the two morphism $\Sigma(\mathcal{X} \times \mathcal{Y}) \to \Sigma(\mathcal{X}) \vee \Sigma(\mathcal{Y}) = \Sigma(\mathcal{X} \vee \mathcal{Y})$ gives a left inverse to $\Sigma(\mathcal{X}) \vee \Sigma(\mathcal{Y}) \to \Sigma(\mathcal{X} \times \mathcal{Y})$. This left inverse determines an $\mathcal{H}_\bullet(k)$-isomorphism $\Sigma(\mathcal{X}) \vee \Sigma(\mathcal{Y}) \vee \Sigma(\mathcal{X} \wedge \mathcal{Y}) \cong \Sigma(\mathcal{X} \times \mathcal{Y})$.

We thus get canonical isomorphisms:

$$\tilde{M}(\mathcal{X} \times \mathcal{Y}) = \tilde{M}(\mathcal{X}) \oplus \tilde{M}(\mathcal{Y}) \oplus \tilde{M}(\mathcal{X} \wedge \mathcal{Y})$$

and analogously

$$H^1(\mathcal{X} \times \mathcal{Y}; M) = H^1(\mathcal{X}; M) \oplus H^1(\mathcal{Y}; M) \oplus H^1(\mathcal{X} \wedge \mathcal{Y}; M)$$

As a consequence, the product $\mu : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ on $\mathbb{G}_m$ induces in $\mathcal{H}_\bullet(k)$ a morphism $\Sigma(\mathbb{G}_m \times \mathbb{G}_m) \to \mathbb{Z}(\mathbb{G}_m)$ which using the above splitting decomposes as

$$\Sigma(\mu) = \langle Id_{\Sigma(\mathbb{G}_m)}, d_{\Sigma(\mathbb{G}_m)}, \eta \rangle : \Sigma(\mathbb{G}_m) \vee \Sigma(\mathbb{G}_m) \vee \Sigma((\mathbb{G}_m)^{\wedge 2}) \to \Sigma(\mathbb{G}_m)$$

The morphism $\Sigma((\mathbb{G}_m)^{\wedge 2}) \to \Sigma(\mathbb{G}_m)$ so defined is denoted η. It can be shown to be isomorphic in $\mathcal{H}_\bullet(k)$ to the Hopf map $\mathbb{A}^2 - \{0\} \to \mathbb{P}^1$.

Let M be a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups. We will denote by

$$\eta : M_{-2} \to M_{-1}$$

the morphism of strongly $\mathbb{A}^1$-invariant sheaves of abelian groups induced by η.

In the same way let $\Psi : \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m) \cong \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m)$ be the twist morphism and for M a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups, we still denote by

$$\Psi : M_{-2} \to M_{-2}$$

the morphism of strongly $\mathbb{A}^1$-invariant sheaves of abelian groups induced by Ψ.

Lemma 2.40 *Let M be a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups. Then the morphisms $\eta \circ \Psi$ and η*

$$M_{-2} \to M_{-1}$$

are equal.

Proof. This is a direct consequence of the fact that μ is commutative.□

As a consequence, for any $m \geq 1$, the morphisms of the form

$$M_{-m-1} \to M_{-1}$$

obtained by composing m times morphisms induced by η doesn't depend on the chosen ordering. We thus simply denote by $\eta^m : M_{-m-1} \to M_{-1}$ this canonical morphism.

Proof of Theorem 2.39 By Lemma 2.6 1), the uniqueness is clear. By a base change argument analogous to [32, Corollary 5.2.7], we may reduce to the case $F = k$.

¿From now on we fix a morphism of pointed sheaves $\phi : (\mathbb{G}_m)^{\wedge n} \to M$, with M a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups. We first observe that ϕ determines and is determined by the $\mathcal{H}_\bullet(k)$-morphism $\phi : \Sigma((\mathbb{G}_m)^{\wedge n}) \to K(M,1)$, or equivalently by the associated element $\phi \in M_{-n}(k)$.

For any symbol $(u_1, \ldots, u_r) \in (k^\times)^r$, $r \in \mathbb{N}$, we let $S^0 \to (\mathbb{G}_m)^{\wedge r}$ be the (ordered) smash-product of the morphisms $[u_i] : S^0 \to \mathbb{G}_m$ determined by u_i. For any integer $m \geq 0$ such that $r = n + m$, we denote by $[\eta^m, u_1, \ldots, u_r] \in M(k) \cong Hom_{\mathcal{H}_\bullet(k)}(\Sigma(S^0), K(M,1))$ the composition

$$\eta^m \circ \Sigma([u_1, \ldots, u_n]) : \Sigma(S^0) \to \Sigma((\mathbb{G}_m)^{\wedge r}) \stackrel{\eta^m}{\to} \Sigma((\mathbb{G}_m)^{\wedge n}) \stackrel{\phi}{\to} K(M,1)$$

The theorem now follows from the following:

Lemma 2.41 *The previous assignment* $(m, u_1, \ldots, u_r) \mapsto [\eta^m, u_1, \ldots, u_r] \in M(k)$ *satisfies the relations of Definition 2.3 and as a consequence induce a morphism*

$$\Phi(k) : K_n^{MW}(k) \to M(k)$$

Proof. The proof of the Steinberg relation $\mathbf{1_n}$ will use the following stronger result by P. Hu and I. Kriz:

Lemma 2.42 *(Hu-Kriz [19]) The canonical morphism of pointed sheaves* $(\mathbb{A}^1 - \{0,1\})_+ \to \mathbb{G}_m \wedge \mathbb{G}_m$, $x \mapsto (x, 1-x)$ *induces a trivial morphism* $\tilde{\Sigma}(\mathbb{A}^1 - \{0,1\}) \to \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m)$ *(where* $\tilde{\Sigma}$ *means unreduced suspension*[4]*) in* $\mathcal{H}_\bullet(k)$.

For any $a \in k^\times - \{1\}$ the suspension of the morphism of the form $[a, 1-a] : S^0 \to (\mathbb{G}_m)^{\wedge 2}$ factors in $\mathcal{H}_\bullet(k))$ through $\tilde{\Sigma}(\mathbb{A}^1 - \{0,1\}) \to \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m)$ as the morphism $Spec(k) \to \mathbb{G}_m \wedge \mathbb{G}_m$ factors itself through $\mathbb{A}^1 - \{0,1\}$. This clearly implies the Steinberg relation in our context as the morphism of the form $\Sigma([u_i, 1-u_i]) : \Sigma(S^0) \to \Sigma((\mathbb{G}_m)^{\wedge 2})$ appears as a factor in the morphism which defines the symbol $[\eta^m, u_1, \ldots, u_r]$, with $u_i + u_{i+1} = 1$, in $M(k)$.

Now, to check the relation $\mathbf{2}_n$, we observe that the pointed morphism $[ab] : S^0 \to \mathbb{G}_m$ factors as $S^0 \xrightarrow{[a][b]} \mathbb{G}_m \times \mathbb{G}_m \xrightarrow{\mu} \mathbb{G}_m$. Taking the suspension and using the above splitting which defines η, yields that

$$\Sigma([ab]) = \Sigma([a]) \vee \Sigma([b]) \vee \eta([a][b]) : \Sigma(S^0) \to \Sigma(\mathbb{G}_m)$$

in the group $Hom_{\mathcal{H}_\bullet(k)}(\Sigma(S^0), \Sigma(\mathbb{G}_m))$ whose law is denoted by $\vee$. This clearly implies relation $\mathbf{2}_n$.

Now we come to check the relation $\mathbf{4}_n$. For any $a \in k^\times$, the morphism $a : \mathbb{G}_m \to \mathbb{G}_m$ given by multiplication by a is not pointed (unless $a = 1$). However the pointed morphism $a_+ : (\mathbb{G}_m)_+ \to \mathbb{G}_m$ induces after suspension $\Sigma(a_+) : S^1 \vee \Sigma(\mathbb{G}_m) \cong \Sigma((\mathbb{G}_m)_+) \to \Sigma(\mathbb{G}_m)$. We denote by $< a >: \Sigma(\mathbb{G}_m) \to \Sigma(\mathbb{G}_m)$ the morphism in $\mathcal{H}_\bullet(k)$ induced on the factor $\Sigma(\mathbb{G}_m)$. We need:

Lemma 2.43 *1) For any* $a \in k^\times$*, the morphism* $M_{-1} \to M_{-1}$ *induced by* $< a >: \Sigma(\mathbb{G}_m) \to \Sigma(\mathbb{G}_m)$ *is equal to* $Id + \eta \circ [a]$.

[4] observe that if $k = \mathbb{F}_2$, $\mathbb{A}^1 - \{0,1\}$ has no rational point

2) The twist morphism $\Psi \in Hom_{\mathcal{H}_\bullet(k)}(\Sigma(\mathbb{G}_m \wedge \mathbb{G}_m), \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m))$ and the inverse, for the group structure, of $Id_{\mathbb{G}_m} \wedge < -1 > \cong < -1 > \wedge Id_{\mathbb{G}_m}$ have the same image in the set $Hom_{\mathcal{H}(k)}(\Sigma(\mathbb{G}_m \wedge \mathbb{G}_m), \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m))$.

Remark 2.44 In fact the map

$$Hom_{\mathcal{H}_\bullet(k)}(\Sigma(\mathbb{G}_m \wedge \mathbb{G}_m), \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m)) \to Hom_{\mathcal{H}(k)}(\Sigma(\mathbb{G}_m \wedge \mathbb{G}_m), \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m))$$

is a bijection. Indeed we know that $\Sigma(\mathbb{G}_m \wedge \mathbb{G}_m))$ is $\mathbb{A}^1$-equivalent to $\mathbb{A}^2 - \{0\}$ and also to SL_2 because the morphism $SL_2 \to \mathbb{A}^2 - \{0\}$ (forgetting the second column) is an $\mathbb{A}^1$-weak equivalence. As SL_2 is a group scheme, the classical argument shows that this space is $\mathbb{A}^1$-simple. Thus for any pointed space $\mathcal{X}$, the action of $\pi_1^{\mathbb{A}^1}(SL_2)(k)$ on $Hom_{\mathcal{H}_\bullet(k)}(\mathcal{X}, SL_2)$ is trivial. We conclude because as usual, for any pointed spaces $\mathcal{X}$ and $\mathcal{Y}$, with $\mathcal{Y}$ $\mathbb{A}^1$-connected, the map $Hom_{\mathcal{H}_\bullet(k)}(\mathcal{X}, \mathcal{Y}) \to Hom_{\mathcal{H}(k)}(\mathcal{X}, \mathcal{Y})$ is the quotient by the action of the group $\pi_1^{\mathbb{A}^1}(\mathcal{Y})(k)$.

Proof. 1) The morphism $a : \mathbb{G}_m \to \mathbb{G}_m$ is equal to the composition $\mathbb{G}_m \stackrel{[a]\times Id}{\to} \mathbb{G}_m \times \mathbb{G}_m \stackrel{\mu}{\to} \mathbb{G}_m$. Taking the suspension, the previous splittings give easily the result.

2) Through the $\mathcal{H}_\bullet(k)$-isomorphism $\Sigma(\mathbb{G}_m \wedge \mathbb{G}_m) \cong \mathbb{A}^2 - \{0\}$, the twist morphism becomes the opposite of the permutation isomorphism $(x, y) \mapsto (y, x)$. This follows easily from the definition of this isomorphism using the Mayer-Vietoris square

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{G}_m & \subset & \mathbb{A}^1 \times \mathbb{G}_m \\ \cap & & \cap \\ \mathbb{G}_m \times \mathbb{A}^1 & \subset & \mathbb{A}^2 - \{0\} \end{array}$$

and the fact that our automorphism on $\mathbb{A}^2 - \{0\}$ permutes the top right and bottom left corner.

Consider the action of $GL_2(k)$ on $\mathbb{A}^2 - \{0\}$. As any matrix in $SL_2(k)$ is a product of elementary matrices, the associated automorphism $\mathbb{A}^2 - \{0\} \cong \mathbb{A}^2 - \{0\}$ is the identity in $\mathcal{H}(k)$. As the permutation matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is congruent to $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ modulo $SL_2(k)$, we get the result.$\square$

Proof of Theorem 2.37 By Lemma 2.45 below, we know that for any smooth irreducible X with function field F, the restriction map $M(X) \subset M(F)$ is injective.

As $\underline{\mathbf{K}}_n^{MW}$ is unramified, the Remark 1.15 of section 1.1 shows that to produce a morphism of sheaves $\Phi : \underline{\mathbf{K}}_n^{MW} \to M$ it is sufficient to prove that for any discrete valuation v on $F \in \mathcal{F}_k$ the morphism $\Phi(F) : K_n^{MW}(F) \to M(F)$ maps $\underline{\mathbf{K}}_n^{MW}(\mathcal{O}_v)$ into $M(\mathcal{O}_v)$ and in case the residue field $\kappa(v)$ is separable, that some square is commutative (see Remark 1.15).

But by Theorem 2.22, we know that the subgroup $\underline{\mathbf{K}}_n^{MW}(\mathcal{O}_v)$ of $K_n^{MW}(F)$ is the one generated by symbols of the form $[u_1, \ldots, u_n]$, with the $u_i \in \mathcal{O}_v^\times$. The claim is now trivial: for any such symbol there is a smooth model X of $\mathcal{O}_v$ and a morphism $X \to (\mathbb{G}_m)^{\wedge n}$ which induces $[u_1, \ldots, u_n]$ when composed with $(\mathbb{G}_m)^{\wedge n} \to \underline{\mathbf{K}}_n^{MW}$. But know composition with $\phi : (\mathbb{G}_m)^{\wedge n} \to M$ gives an element of $M(X)$ which lies in $M(\mathcal{O}_v) \subset M(F)$ which is by definition the image of $[u_1, \ldots, u_n]$ through $\Phi(F)$. A similar argument applies to check the commutativity of the square of the Remark 1.15: one may choose then X so that there is a closed irreducible $Y \subset X$ of codimension 1, with $\mathcal{O}_{X,\eta_Y} = \mathcal{O}_v \subset F$. Then the restriction of $\Phi([u_1, \ldots, u_n]) \subset M(\mathcal{O}_v)$ is just induced by the composition $Y \to X \to (\mathbb{G}_m)^{\wedge n} \to M$, and this is also compatible with the s_v in Milnor-Witt K-theory.□

Lemma 2.45 *Let M be an $\mathbb{A}^1$-invariant sheaf of pointed sets on Sm_k. Then for any smooth irreducible X with function field F, the kernel of the restriction map $M(X) \subset M(F)$ is trivial.*

In case M is a sheaf of groups, we see that the restriction map $M(X) \to M(F)$ is injective.

Proof. This follows from [32, Lemma 6.1.4] which states that $L_{\mathbb{A}^1}(X/U)$ is always 0-connected for U non-empty dense in X. Now the kernel of $M(X) \to M(U)$ is covered by $Hom_{\mathcal{H}_\bullet(k)}(X/U, M)$, which is trivial as M is his own π_0 and $L_{\mathbb{A}^1}(X/U)$ is 0-connected.□

We know deal with $\underline{\mathbf{K}}_0^{MW}$. We observe that there is a canonical morphism of sheaves of sets $\mathbb{G}_m/2 \to \underline{\mathbf{K}}_0^{MW}$, $U \mapsto < U >$, where $\mathbb{G}_m/2$ means the cokernel in the category of sheaves of abelian groups of $\mathbb{G}_m \overset{2}{\to} \mathbb{G}_m$.

Theorem 2.46 *The canonical morphism of sheaves $\mathbb{G}_m/2 \to \underline{\mathbf{K}}_0^{MW}$ is the universal morphism of sheaves of sets to a strongly $\mathbb{A}^1$-invariant sheaf of* *abelian groups. In other words* $\underline{\mathbf{K}}_0^{MW}$ *is the free strongly* $\mathbb{A}^1$*-invariant sheaf on the space* $\mathbb{G}_m/2$.

Proof. Let M be a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups. Denote by $\mathbb{Z}[\mathcal{S}]$ the free sheaf of abelian groups on a sheaf of sets $\mathcal{S}$. When $\mathcal{S}$ is pointed, then the latter sheaf splits canonically as $\mathbb{Z}[\mathcal{S}] = \mathbb{Z} \oplus \mathbb{Z}(\mathcal{S})$ where $\mathbb{Z}(\mathcal{S})$ is the free sheaf of abelian groups on the pointed sheaf of sets $\mathcal{S}$, meaning the quotient $\mathbb{Z}[\mathcal{S}]/\mathbb{Z}[*]$ (where $* \to \mathcal{S}$ is the base point). Now a morphism of sheaves of sets $\mathbb{G}_m/2 \to M$ is the same as a morphism of sheaves of abelian groups $\mathbb{Z}[\mathbb{G}_m] = \mathbb{Z} \oplus \mathbb{Z}(\mathbb{G}_m) \to M$. By the Theorem 2.37 a morphism $\mathbb{Z}(\mathbb{G}_m) \to M$ is the same as a morphism $\underline{\mathbf{K}}_1^{MW} \to M$.

Thus to give a morphism of sheaves of sets $\mathbb{G}_m/2 \to M$ is the same as to give a morphism of sheaves of abelian groups $\mathbb{Z} \oplus \underline{\mathbf{K}}_1^{MW} \to M$ together with extra conditions. One of this conditions is clearly that the composition $\mathbb{Z} \oplus \underline{\mathbf{K}}_1^{MW} \stackrel{[2]}{\to} \mathbb{Z} \oplus \underline{\mathbf{K}}_1^{MW} \to M$ is equal to $\mathbb{Z} \oplus \underline{\mathbf{K}}_1^{MW} \stackrel{[*]}{\to} \mathbb{Z} \oplus \underline{\mathbf{K}}_1^{MW} \to M$. Here $[*]$ is represented by the matrix $\begin{pmatrix} Id_{\mathbb{Z}} & 0 \\ 0 & 0 \end{pmatrix}$ and $[2]$ by the matrix $\begin{pmatrix} Id_{\mathbb{Z}} & 0 \\ 0 & [2]_1 \end{pmatrix}$. The morphism $[2]_1 : \underline{\mathbf{K}}_1^{MW} \to \underline{\mathbf{K}}_1^{MW}$ is the one induced by the square map on $\mathbb{G}_m$. From Lemma 2.14, we know that this map is the multiplication by $2_\epsilon = h$. recall that we set $\underline{\mathbf{K}}_1^W := \underline{\mathbf{K}}_1^{MW}/h$. Thus any morphism of sheaves of sets $\mathbb{G}_m/2 \to M$ determines a canonical morphism $\mathbb{Z} \oplus \underline{\mathbf{K}}_1^W \to M$. Moreover the morphism $\mathbb{Z}[\mathbb{G}_m] \to \mathbb{Z} \oplus \underline{\mathbf{K}}_1^W$ factors through $\mathbb{Z}[\mathbb{G}_m] \to \mathbb{Z}[\mathbb{G}_m/2]$; this morphism is induced by the map $U \mapsto (1, < U >)$.

We have thus proven that given any morphism $\phi : \mathbb{Z}[\mathbb{G}_m/2] \to M$, there exists a unique morphism $\mathbb{Z} \oplus \underline{\mathbf{K}}_1^W \to M$ such that the composition $\mathbb{Z}[\mathbb{G}_m/2] \to \mathbb{Z} \oplus \underline{\mathbf{K}}_1^W \to M$ is ϕ. As $\mathbb{Z} \oplus \underline{\mathbf{K}}_1^W$ is a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups, it is the free one on $\mathbb{G}_m/2$.

Our claim is now that the canonical morphism $i : \mathbb{Z} \oplus K_1^W \to \underline{\mathbf{K}}_0^{MW}$ is an isomorphism.

We know proceed closely to proof of Theorem 2.37. We first observe that for any $F \in \mathcal{F}_k$, the canonical map $\mathbb{Z}[F^\times/2] \to \mathbb{Z} \oplus K_1^W(F)$ factors through $\mathbb{Z}[F^\times/2] \twoheadrightarrow K_0^{MW}(F)$. This is indeed very simple to check using the presentation of $K_0^{MW}(F)$ given in Lemma 2.9. We denote by

$j(F) : K_0^{MW}(F) \to \mathbb{Z} \oplus K_1^W(F)$ the morphism so obtained.

Using Theorem 2.22 and the same argument as in the end of the proof of Theorem 2.37 we see that the $j(F)$'s actually come from a morphism of sheaves $j : \underline{\mathbf{K}}_0^{MW} \to \mathbb{Z} \oplus \underline{\mathbf{K}}_1^W$. It is easy to check on $F \in \mathcal{F}_k$ that i and j are inverse morphisms to each other.□

The following corollary is immediate from the Theorem and its proof:

Corollary 2.47 *The canonical morphism*

$$K_1^W(F) \to I(F)$$

is an isomorphism.

3 $\mathbb{A}^1$-homotopy sheaves and $\mathbb{A}^1$-homology sheaves

In this section we assume the reader is conformable with [39]. We will freely use the basic notions and some of the results.

3.1 Strongly $\mathbb{A}^1$-invariance of the sheaves $\pi_n^{\mathbb{A}^1}$, $n \geq 1$

Our aim in this section is to prove:

Theorem 3.1 *For any pointed space $\mathcal{B}$, its $\mathbb{A}^1$-fundamental sheaf of groups $\pi_1^{\mathbb{A}^1}(\mathcal{B})$ is strongly $\mathbb{A}^1$-invariant.*

To prove this theorem, we will "directly" observe that the sheaf $\mathcal{G} := \pi_1^{\mathbb{A}^1}(\mathcal{B})$ is unramified and satisfies the assumption of Theorem 1.22 of [**?**].

The previous Theorem is equivalent to the following:

Theorem 3.2 *Let $\mathcal{B}$ be a pointed simplicial presheaf of sets on Sm_k which satisfies the B.G. property in the Nisnevich topology and the $\mathbb{A}^1$-invariance property (see [39]). Then the associated sheaf of groups to the presheaf $U \mapsto \pi_1(\mathcal{B}(U))$ is strongly $\mathbb{A}^1$-invariant.*

Proof. The simplicial fibrant resolution $L_{\mathbb{A}^1}(\mathcal{X})$ of the $\mathbb{A}^1$-localization of a pointed space $\mathcal{X}$ satisfies the assumptions of Theorem 3.2. This proves one implication. By the results of [39] any pointed simplicial presheaf of sets $\mathcal{B}$ satisfying the assumptions of Theorem 3.2 is simplicially equivalent to the fibrant resolution of the $\mathbb{A}^1$-localization of the associated sheaf to $\mathcal{B}$.□

We observe the following immediate corollary:

Corollary 3.3 *For any pointed space $\mathcal{B}$, and any integer $n \geq 1$, the $\mathbb{A}^1$-homotopy sheaf of groups $\pi_n^{\mathbb{A}^1}(\mathcal{B})$ is strongly $\mathbb{A}^1$-invariant.*

Proof. Apply the Theorem to the $(n-1)$-th iterated simplicial loop space $\Omega_s^{(n-1)}(\mathcal{B})$ of $\mathcal{B}$, which is still $\mathbb{A}^1$-local.□

We now start the proof of Theorem 3.1 with some remarks and preliminaries. We observe first that we may assume $\mathcal{B}$ is $\mathbb{A}^1$-local and, by the following lemma, we may assume further that $\mathcal{B}$ is 0-connected:

Lemma 3.4 *Given a pointed $\mathbb{A}^1$-local space $\mathcal{B}$, the connected component of the base point $\mathcal{B}^{(0)}$ is also $\mathbb{A}^1$-local and the morphism*

$$\pi_1^{\mathbb{A}^1}(\mathcal{B}^{(0)}) \to \pi_1^{\mathbb{A}^1}(\mathcal{B})$$

is an isomorphism.

Proof. Indeed, by [39] the $\mathbb{A}^1$-localization of a 0-connected space is still 0-connected; thus the obvious morphism $L_{\mathbb{A}^1}(\mathcal{B}^{(0)}) \to \mathcal{B}$ induced by $\mathcal{B}^{(0)} \to \mathcal{B}$ and the fact that $\mathcal{B}$ is $\mathbb{A}^1$-connected, induces $L_{\mathbb{A}^1}(\mathcal{B}^{(0)}) \to \mathcal{B}^{(0)}$, providing a left inverse to $\mathcal{B}^{(0)} \to L_{\mathbb{A}^1}(\mathcal{B}^{(0)})$. Thus $\mathcal{B}^{(0)}$ is a retract in $\mathcal{H}(k)$ of the $\mathbb{A}^1$-local space $L_{\mathbb{A}^1}(\mathcal{B}^{(0)})$ so is also $\mathbb{A}^1$-local.□

¿From now on, $\mathcal{B}$ is a fixed $\mathbb{A}^1$-connected and $\mathbb{A}^1$-local space. For an open immersion $U \subset X$ and any $n \geq 0$ we set

$$\Pi_n(X, U) := [S^n \wedge (X/U), \mathcal{B}]_{\mathcal{H}_\bullet(k)} = \pi_n(\mathcal{B}(X/U))$$

where S^n denotes the simplicial n-sphere. For $n = 0$ these are just pointed sets, for $n = 1$ these are groups and for $i \geq 2$ these are abelian groups. In fact in the proof below we will only use the case $n = 0$ and $n = 1$. We may extend these definitions to an open immersion $U \subset X$ between essentially smooth k-schemes, by passing to the (co)limit.

The following is our main technical Lemma, and will be proven following the lines of [9, Key Lemma], using Gabber's presentation Lemma:

Lemma 3.5 *Assume k is infinite. Let X be a smooth k-scheme, $S \subset X$ be a finite set of points and $Z \subset X$ be a closed subscheme of codimension $d > 0$. Then there exists an open subscheme $\Omega \subset X$ containing S and a closed subscheme $Z' \subset \Omega$, of codimension $d-1$, containing $Z_\Omega := Z \cap \Omega$ and such that the map of pointed sheaves*

$$\Omega/(\Omega - Z') \to \Omega/(\Omega - Z_\Omega)$$

is the trivial map in $\mathcal{H}_\bullet(k)$.

Proof. By Gabber's geometric presentation Lemma of *loc. cit.* there exists an open neighborhood Ω of S, and an étale morphism $\phi : \Omega \to \mathbb{A}^1_V$ with V some open subset in some affine space over k such that $Z_\Omega := Z \cap \Omega \to \mathbb{A}^1_V$ is a closed immersion, $\phi^{-1}(Z_\Omega) = Z_\Omega$ and $Z_\Omega \to V$ is a finite morphism. Let F denotes the image of Z_Ω in V. Then set $Z' := \phi^{-1}(\mathbb{A}^1_F)$. Observe that $dim(F) = dim(Z)$ thus $codim(Z') = d-1$. Because we work in the Nisnevich topology, the morphism of sheaves

$$\Omega/(\Omega - Z_\Omega) \to \mathbb{A}^1_V/(\mathbb{A}^1_V - Z_\Omega)$$

is an isomorphism. The commutative square

$$\begin{array}{ccc} \Omega/(\Omega - Z') & \to & \Omega/(\Omega - Z_\Omega) \\ \downarrow & & \downarrow \wr \\ \mathbb{A}^1_V/(\mathbb{A}^1_V - \mathbb{A}^1_F) & \to & \mathbb{A}^1_V/(\mathbb{A}^1_V - Z_\Omega) \end{array}$$

implies that it suffices to show that the map of pointed sheaves

$$\mathbb{A}^1_V/(\mathbb{A}^1_V - \mathbb{A}^1_F) \to \mathbb{A}^1_V/(\mathbb{A}^1_V - Z_\Omega)$$

is the trivial map in $\mathcal{H}_\bullet(k)$. Now because $Z \to F$ is finite, the composition $Z \to \mathbb{A}^1_F \subset \mathbb{P}^1_F$ is still a closed immersion, which has thus empty intersection with the section at infinity $s_\infty : V \to \mathbb{P}^1_V$. By the Mayer-Vietoris property the morphism $\mathbb{A}^1_V/(\mathbb{A}^1_V - Z_\Omega) \to \mathbb{P}^1_V/(\mathbb{P}^1_V - Z_\Omega)$ is an isomorphism of pointed sheaves. It suffices thus to check that

$$\mathbb{A}^1_V/(\mathbb{A}^1_V - \mathbb{A}^1_F) \to \mathbb{P}^1_V/(\mathbb{P}^1_V - Z_\Omega)$$

is the trivial map in $\mathcal{H}_\bullet(k)$. But clearly the morphism $s_0 : V/(V-F) \to \mathbb{A}^1_V/(\mathbb{A}^1_V - \mathbb{A}^1_F)$ induced by the zero section is an $\mathbb{A}^1$-weak equivalence. As the composition $s_0 : V/(V-F) \to \mathbb{A}^1_V/(\mathbb{A}^1_V - \mathbb{A}^1_F) \to \mathbb{P}^1_V/(\mathbb{P}^1_V - Z_\Omega)$ is $\mathbb{A}^1$-homotopic (by the obvious $\mathbb{A}^1$-homotopy which relates the zero section to the section at infinity) to the section at infinity $s_\infty : V/(V-F) \to \mathbb{P}^1_V/(\mathbb{P}^1_V - Z_\Omega)$ we get the result because as noted previously s_∞ is disjoint from Z_Ω and thus $s_\infty : V/(V-F) \to \mathbb{P}^1_V/(\mathbb{P}^1_V - Z_\Omega)$ is equal to the point.□.

Corollary 3.6 *Assume k is infinite. Let X be a smooth (or essentially smooth) k-scheme, $S \in X$ be a finite set of points and $Z \subset X$ be a closed subscheme of codimension $d > 0$. Then there exists an open subscheme $\Omega \subset X$ containing S and a closed subscheme $Z' \subset \Omega$, of codimension $d-1$, containing $Z_\Omega := Z \cap \Omega$ and such that for any $n \in \mathbb{N}$ the map*

$$\Pi_n(\Omega, \Omega - Z_\Omega) \to \Pi_n(\Omega, \Omega - Z')$$

is the trivial map.

In particular, observe that if Z has codimension 1 and X is irreducible, Z' must be Ω. Thus for any $n \in \mathbb{N}$ the map

$$\Pi_n(\Omega, \Omega - Z_\Omega) \to \Pi_n(\Omega)$$

is the trivial map.

Proof. For X smooth this is an immediate consequence of the Lemma. In case X is an essentially smooth k-scheme, we get the result by an obvious passage to the colimit, using standard results on limit of schemes [16].□

Fix an essentially smooth k-scheme X. For any flag of open subschemes of the form $V \subset U \subset X$ one has the following homotopy exact sequence (which could be continued on the left):

$$\begin{array}{c} \cdots \to \Pi_1(X,U) \to \Pi_1(X,V) \to \Pi_1(U,V) \to \\ \Pi_0(X,U) \to \Pi_0(X,V) \to \Pi_0(U,V) \end{array} \tag{3.1}$$

where the exactness at $\Pi_0(X,V)$ is the exactness in the sense of pointed sets, and at $\Pi_0(X,U)$ we observe that there is an action of the group $\Pi_1(X,U)$ on the set $\Pi_0(X,U)$ and the exactness is in the usual sense. The exactness everywhere else is as diagram of groups.

We now assume that X is the localization of a smooth k-scheme at a point x. We still denote by x the close point in X. For any flag $\mathcal{F}$: $Z^2 \subset Z^1 \subset X$ of closed reduced subschemes, with Z^i of codimension at least i, we set $U_i = X - Z^i$ so that we get a corresponding flag of open subschemes $U_1 \subset U_2 \subset X$. The set $\mathcal{F}$ of such flags is ordered by increasing inclusion (of closed subschemes). Given a flag as above and applying the above observation with $U = U_1$ and $V = \emptyset$ we get an exact sequence:

$$\cdots \to \Pi_1(X,U_1) \to \Pi_1(X) \to \Pi_1(U_1) \to \Pi_0(X,U_1) \to \Pi_0(X) \to \Pi_0(U_1)$$

By the corollary above, applied to X, to $S = \{x\}$, and to the closed subset Z^1, we see that Ω must be X itself and thus that the maps (for any n)

$$\Pi_n(X,U_1) \to \Pi_n(X)$$

are trivial. We thus get a short exact sequence

$$1 \to \Pi_1(X) \to \Pi_1(U_1) \to \Pi_0(X,U_1) \to * \tag{3.2}$$

and a map of pointed sets $\Pi_0(X) \to \Pi_0(U_1)$ which has trivial kernel.

Passing to the right filtering colimit on flags we get a short exact sequence

$$1 \to \Pi_1(X) \to \Pi_1(F) \to colim_{\mathcal{F}}\Pi_0(X,U_1) \to * \tag{3.3}$$

and a pointed map with trivial kernel $\Pi_0(X) \to \Pi_0(F)$, where we denote by F the field of functions of X. But now we observe that $\mathcal{B}$ being 0-connected we have $\Pi_0(F) = *$, and thus $\Pi_0(X) = *$.

To understand a bit further the short exact sequence (3.3) we now consider for each flag $\mathcal{F}$ as above the part of the exact sequence obtained above for the flag of open subschemes $U_1 \subset U_2 \subset X$:

$$\to \Pi_0(X,U_2) \to \Pi_0(X,U_1) \to \Pi_0(U_2,U_1) \tag{3.4}$$

By the Corollary 3.6 applied to X, $S = \{x\}$ and to the closed subset $Z^2 \subset X$, we see that Ω must be X and that there exists $Z' \subset X$ of codimension 1, containing Z such that

$$\Pi_0(X, U_2) \to \Pi_0(X, X - Z')$$

is the trivial map. Define the flag $\mathcal{F}' : Z'^2 \subset Z'^1 \subset X$ by setting $Z'^2 = Z^2$ and $Z'^1 = Z1 \cup Z'$ we see that the map

$$colim_{\mathcal{F}} \Pi_0(X, U_2) \to colim_{\mathcal{F}} \Pi_0(X, U_1)$$

is trivial. Thus we conclude that

$$colim_{\mathcal{F}} \Pi_0(X, U_1) \to colim_{\mathcal{F}} \Pi_0(U_2, U_1) \tag{3.5}$$

has trivial kernel. However using now the exact sequence involving the flags of open subsets of the form $\emptyset \subset U_1 \subset U_2$ we see that there is a natural action of $\Pi_1(F)$ on $colim_{\mathcal{F}} \Pi_0(U_2, U_1)$ which makes the map (3.5) $\Pi_1(F)$-equivariant. As the source $colim_{\mathcal{F}} \Pi_0(X, U_1)$ is one orbit under $\Pi_1(F)$ by (3.3), the equivariant map (3.5) which has trivial kernel must be injective. We thus have proven that if k is an infinite field and X is a smooth local k-scheme with function field F. The natural sequence:

$$1 \to \Pi_1(X) \to \Pi_1(F) \Rightarrow colim_{\mathcal{F}} \Pi_0(U_2, U_1)$$

(the double arrow refereing to an action) is exact.

An interesting example is the case where X is the localization at a point x of codimension 1. The set $colim_{\mathcal{F}} \Pi_0(U_2, U_1)$ reduces to the $\Pi_1(F)$-set $\Pi_0(X, X - \{x\})$ because there is only one non-empty closed subset of codimension > 0, the closed point itself. Moreover by the exact sequence (3.2) shows that the action of $\Pi_1(F)$ on $\Pi_0(Y, U - \{y\})$ is transitive and the latter set can be identified with the quotient $\Pi_1(F)/\Pi_1(X)$; in that case we simply denote this set by $H^1_y(X; \Pi_1)$.

We observe that any étale morphism $X' \to X$ between smooth local k-schemes induces a morphism of corresponding associated exact sequences

$$\begin{array}{ccccccc} 1 & \to & \Pi_1(X) & \to & \Pi_1(F) & \Rightarrow & colim_{\mathcal{F}} \Pi_0(U_2, U_1) \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \to & \Pi_1(X') & \to & \Pi_1(F') & \Rightarrow & colim_{\mathcal{F}'} \Pi_0(U'_2, U'_1) \end{array}$$

When $X' \to X$ runs over the set of localizations at points of codimension one in X we get a $\Pi_1(F)$-equivariant map

$$colim_{\mathcal{F}}\Pi_0(U_2, U_1) \to \Pi_{y\in X^{(1)}} H^1_y(\Pi_1)$$

Lemma 3.7 *(compare [9, Lemma 1.2.1]) The above map is injective and its image is the weak product, yielding a bijection:*

$$colim_{\mathcal{F}}\Pi_0(U_2, U_1) \cong \Pi'_{y\in X^{(1)}} H^1_y(\Pi_1)$$

Corollary 3.8 *Assume k is infinite.*

1) let X be a smooth local k-scheme with function field F. Then the natural sequence:

$$1 \to \Pi_1(X) \to \Pi_1(F) \Rightarrow \Pi'_{y\in X^{(1)}} H^1_y(X; \Pi_1)$$

is exact.

2) the Zariski sheaf associated with $X \mapsto \Pi_1(X)$ is a sheaf in the Nisnevich topology and coincides with $\pi_1^{\mathbb{A}^1}(\mathcal{B})(F)$, which is thus unramified.

Proof. 1) is clear. Let's prove the 2). Let's denote by $\mathcal{G}$ the sheaf $(\Pi_1)_{Zar}$. Observe that for X local $\mathcal{G}(X) = \Pi_1(X)$. 1) implies that for any k-smooth X irreducible with function field F the natural sequence:

$$1 \to \mathcal{G}(X) \to \mathcal{G}(F) \Rightarrow \Pi'_{y\in X^{(1)}} H^1_y(X; \mathcal{G})$$

is exact.

For X of dimension 1 with closed point y, the exact sequence 3.3 yields a bijection $H^1_y(X; \Pi_1) = H^1_y(X; \mathcal{G}) = H^1_{Nis}(X, X - \{y\}; \pi_1(\mathcal{B}))$.

If $V \to X$ is an étale morphism between local k-smooth schemes of dimension 1, with closed points y' and y respectfully, and with same residue fields $\kappa(y) = \kappa(y')$, the map

$$H^1_{Nis}(X, X - \{y\}; \pi_1(\mathcal{B})) \to H^1_{Nis}(V, V - \{y'\}; \pi_1(\mathcal{B})) \qquad (3.6)$$

is thus bijective.

It follows that the correspondence $X \mapsto \Pi'_{y\in X^{(1)}} H^1_y(X; \mathcal{G})$ is a sheaf in the Nisnevich topology on $\tilde{Sm}_k$.

Using our above exact sequence this implies easily that $X \mapsto \mathcal{G}(X)$ is a sheaf in the Nisnevich topology. The same exact sequence applied to the henselization X of a k-smooth local scheme implies that the obvious morphism $\mathcal{G}(X) \to \pi_1^{\mathbb{A}^1}(\mathcal{B})(X)$ is a bijection. Thus the morphism $\mathcal{G} \to \pi_1^{\mathbb{A}^1}(\mathcal{B})$ is an isomorphism of sheaves of groups in the Nisnevich topology.$\square$

The sheaf $\pi_1(\mathcal{B})$ is strongly $\mathbb{A}^1$-invariant.

We now want to use the results of Section 1.2 to prove that $\mathcal{G} = \pi_1(\mathcal{B})$ is strongly $\mathbb{A}^1$-invariant.

We still denote by $\mathcal{G}$ the Nisnevich sheaf $\pi_1(\mathcal{B})$. By the previous corollary, for any smooth local k-scheme X, one has $\mathcal{G}(X) = \pi_1^{\mathbb{A}^1}(\mathcal{B})(X) = \Pi_1(X)$.

In view of Theorem 1.26 the following result implies Theorem 3.1 over an infinite field. Theorem A.7 deduce Theorem 3.1 over any finite field.

Theorem 3.9 *Assume k is infinite. The unramified sheaf of groups $\mathcal{G}$ satisfies the Axioms* **(A2')**, **(A5)** *and* **(A6)** *of Theorem 1.26. In particular $\mathcal{G}$ is strongly $\mathbb{A}^1$-invariant.*

Proof. We first prove Axiom **(A5)**. Axiom **(A5) (i)** follows at once from the fact proven above that (3.6) is a bijection. From that fact we see that

$$1 \to \mathcal{G}(X) \to \mathcal{G}(F) \rightrightarrows \Pi'_{y \in X^{(1)}} H^1_y(X; \mathcal{G})$$

defines on the category of smooth k-schemes of dimension ≤ 1 a short exact sequence of Zariski and Nisnevich sheaves. As the right hand side is flasque in the Nisnevich topology, we get for any smooth k-scheme V of dimension ≤ 1 a bijection

$$H^1_{Zar}(V; \mathcal{G}) = H^1_{Nis}(V; \mathcal{G}) = {}_{\mathcal{G}(F)} \backslash \Pi'_{y \in X^{(1)}} H^1_y(X; \mathcal{G})$$

For X a smooth local k-scheme of dimension 2 with closed point z and $V = X - \{z\}$ (which is of dimension 1), we get $H^1_{Nis}(V; \mathcal{G}) = H^2_z(X; \mathcal{G})$. Proceeding as in the proof of Lemma 1.24 we get Axiom **(A5) (ii)**.

Now we prove Axiom **(A2')**. We recall from Lemma 3.7 that the map

$$colim_{\mathcal{F}}\Pi_0(U_2,U_1) \cong \Pi'_{y\in X^{(1)}}H^1_y(X;\mathcal{G})$$

is a bijection for any smooth k-scheme X.

Let $z \in X^{(2)}$. Denote by X_z the localization of X at z and by $V_z = X_z - \{z\}$. We have just proven that $H^1_{Zar}(V_z;\mathcal{G}) = H^1_{Nis}(V_z;\mathcal{G}) = H^2_z(X;\mathcal{G})$. The middle term is also equal to $\Pi_0(V_z) = [(V_z)_+,\mathcal{B}]_{\mathcal{H}_\bullet(k)}$ because $\mathcal{B}$ is connected with $\pi_1(\mathcal{B}) = \mathcal{G}$ and V_z is smooth of dimension 1.

Now for a fixed flag $\mathcal{F}$ in X, by definition, the composition $\Pi_0(U_2,U_1) \to H^2_z(X;\mathcal{G})$ is trivial if $z \in U_2$ and is the composition of the map $\Pi_0(U_2,U_1) \to \Pi_0(U_2)$ and of the map $\Pi_0(U_2) \to \Pi_0(V_z) = H^2_z(X;\mathcal{G})$. Thus given an element of $\Pi'_{y\in X^{(1)}}H^1_y(X;\mathcal{G})$ which comes from $\Pi_0(U_2,U_1)$, its boundary to $H^2_z(X;\mathcal{G})$ at points z of codimension 2 are trivial except maybe for those z not in U_2: but there are only finitely many of those, which establishes Axiom **(A2')**.

We now prove Axiom **(A6)**. Using the Lemma 3.10 below, we see by that for any field $F \in \mathcal{F}_k$, the map $[\Sigma((\mathbb{A}^1_F)_+),\mathcal{B}]_{\mathcal{H}_\bullet(k)} \to [\Sigma((\mathbb{A}^1_F)_+),B(\mathcal{G})]_{\mathcal{H}_\bullet(k)} = \mathcal{G}(\mathbb{A}^1_F)$ is onto. As $\mathcal{B}$ is $\mathbb{A}^1$-local, $[\Sigma((\mathbb{A}^1_F)_+),\mathcal{B}]_{\mathcal{H}_\bullet(k)} = [\Sigma(SpecF_+),\mathcal{B}]_{\mathcal{H}_\bullet(k)} = \mathcal{G}(F)$ and this shows that the map $\mathcal{G}(F) \to \mathcal{G}(\mathbb{A}^1_F)$ is onto. Thus it is an isomorphism as any F-rational point of $\mathbb{A}^1_F$ provides a left inverse. By part 2) of Lemma 1.16 this implies that $\mathcal{G}$ is $\mathbb{A}^1$-invariant.

By 2) of Lemma 3.10 we see that for any (essentially) smooth k-scheme X of dimension ≤ 1, the map $[(\mathbb{A}^1_X)+,\mathcal{B}]_{\mathcal{H}_\bullet(k)} \to [(\mathbb{A}^1_X)_+,B(\mathcal{G})]_{\mathcal{H}_\bullet(k)} = H^1_{Nis}(\mathbb{A}^1_X;\mathcal{G})$ is onto. As $\mathcal{B}$ is 0-connected and $\mathbb{A}^1$-local, this shows that if moreover X is a local scheme $H^1_{Nis}(\mathbb{A}^1_X;\mathcal{G}) = *$.

As we know that $\mathcal{G}$ satisfies **(A5)**, Lemma 1.24 implies that $H^1_{Zar}(\mathbb{A}^1_X;\mathcal{G}) = *$. By Remark 1.22 we conclude that $C^*(\mathbb{A}^1_X;\mathcal{G})$ is exact, the axiom **(A6)** is proven, and the Theorem as well.□

Lemma 3.10 *1) For any smooth k-scheme X of dimension ≤ 1 the map*

$$Hom_{\mathcal{H}_{s,\bullet}(k)}(\Sigma(X_+),\mathcal{B}) \to Hom_{\mathcal{H}_{s,\bullet}(k)}(\Sigma(X_+),B(\mathcal{G})) = \mathcal{G}(X)$$

is surjective.

2) For any smooth k-scheme X of dimension ≤ 2 the map

$$Hom_{\mathcal{H}_{s,\bullet}(k)}(X_+,\mathcal{B}) \to Hom_{\mathcal{H}_{s,\bullet}(k)}(X_+,B(\mathcal{G})) = H^1_{Nis}(X;\mathcal{G})$$

is surjective and injective if $dim(X) \leq 1$.

Proof. This is proven using the Postnikov tower $\{P^n(\mathcal{B})\}_{n\in\mathbb{N}}$ of $\mathcal{B}$, see [39] for instance, together with standard obstruction theory, see [36, Appendix B].□

$\mathbb{G}_m$-loop spaces

Theorem 3.11 *For any pointed $\mathbb{A}^1$-local space $\mathcal{B}$ which is 0-connected, so is the function space $R\mathbf{Hom}_\bullet(\mathbb{G}_m,\mathcal{B})$ and for any integer $n>0$, the canonical morphism*

$$\pi_n^{\mathbb{A}^1}(R\mathbf{Hom}_\bullet(\mathbb{G}_m,\mathcal{B}))\to(\pi_n^{\mathbb{A}^1}(\mathcal{B}))_{-1}$$

is an isomorphism.

In particular, by induction on $i\geq 0$, one gets an isomorphism for any $n>0$

$$[S^n\wedge(\mathbb{G}_m)^{\wedge i},\mathcal{B}]_{\mathcal{H}_\bullet(k)}\cong\pi_n^{\mathbb{A}^1}(\mathcal{B})_{-i}(k)$$

Proof. The fact that $R\mathbf{Hom}_\bullet(\mathbb{G}_m,\mathcal{B})$ is $\mathbb{A}^1$-connected is proven as follows. We know from [32] that to show that a space $\mathcal{Z}$ is $\mathbb{A}^1$-connected, it suffices to show that the sets $[(Spec(F)_+;\mathcal{Z})$ are trivial for any $F\in\mathcal{F}_k$. Base an easy base change argument we may reduce to $F=k$. $\mathbb{G}_m$ having dimension one, we conclude from the Lemma 3.12 below and an obvious obstruction theory argument using Lemma 3.10.

The canonical morphism of the statement is induced by the natural transformation of presheaves of groups "evaluation on the n-th homotopy sheaves"

$$[S^n\wedge\mathbb{G}_m\wedge(U_+),\mathcal{B}]_{\mathcal{H}_\bullet(k)}\to\pi_n(\mathcal{B})_{-1}(U)$$

Observe that the associated sheaf to the presheaf on the left is exactly $\pi_n^{\mathbb{A}^1}(R\mathbf{Hom}_\bullet(\mathbb{G}_m,\mathcal{B}))$.

Now by Lemma 1.31 and Corollary 3.3 both sheaves involved in the morphism are strongly $\mathbb{A}^1$-invariant. To check it is an isomorphism it is sufficient to check that it is an isomorphism on each $F\in\mathcal{F}_k$.

It is also clear that the morphism in degree n applied to $R\Omega_s^1(\mathcal{B})$ is the morphism in degree $n+1$ corresponding to $\mathcal{B}$. Thus by induction, it is sufficient to treat the case $n=1$.

By an easy base change argument we may assume $F=k$ is the base field.

Using again Lemma 3.10 we easily get the result from Lemma 3.12.□

Lemma 3.12 *Let $\mathcal{G}$ be a strongly $\mathbb{A}^1$-invariant sheaf of groups. Then $H^1(\mathbb{G}_m;\mathcal{G})$ is trivial.*

Proof. For k infinite, we use the results of section 1.2. For k finite we use the results of the Appendix. We know from there that H^1 is always computed using the explicit complex $C^*(-;\mathcal{G})$. Thus we reduce to proving the fact that the action of $\mathcal{G}(k(T))$ on $\Pi'_{y\in(\mathbb{G}_m)^{(1)}} H^1_y(\mathbb{G}_m;\mathcal{G})$ is transitive. But this follows at once from the fact that the action of $\mathcal{G}(k(T))$ on $\Pi'_{y\in(\mathbb{A}^1)^{(1)}} H^1_y(\mathbb{A}^1;\mathcal{G})$ is transitive (because $H^1(\mathbb{A}^1;\mathcal{G})$ is trivial) and the fact that the epimorphism $\Pi'_{y\in(\mathbb{G}_m)^{(1)}} H^1_y(\mathbb{G}_m;\mathcal{G})$ is an obvious quotient of $\Pi'_{y\in(\mathbb{A}^1)^{(1)}} H^1_y(\mathbb{A}^1;\mathcal{G})$ as a $\mathcal{G}(k(T))$-set.□

3.2 $\mathbb{A}^1$-derived category and Eilenberg-MacLane spaces

The derived category. Let us denote by $\mathcal{A}b(k)$ the abelian category of sheaves of abelian groups on Sm_S in the Nisnevich topology. Let $\mathcal{C}_*(\mathcal{A}b(k))$ be the category of chain complexes[5] in $\mathcal{A}b(k)$.

The derived category of $\mathcal{A}b(k)$ is the category $D(\mathcal{A}b(k))$ obtained from $\mathcal{C}_*(\mathcal{A}b(k))$ by inverting the class Qis of quasi-isomorphisms between chain complexes. There are several ways to describe this category. The closest to the intuition coming from standard homological algebra [14] is the following.

Definition 3.13 *1) A morphism of chain complexes $C_* \to D_*$ in $\mathcal{C}_*(\mathcal{A}b(k))$ is said to be a* cofibration *if it is a monomorphism. It is called a* trivial cofibration *if it is furthermore a quasi-isomorphism.*

2) A chain complex K_* is said to be fibrant *if for any trivial cofibration $i : C_* \to D_*$ and any morphism $f : C_* \to K_*$, there exists a morphism $g : D_* \to K_*$ such that $g \circ i = f$.*

The following "fundamental lemma of homological algebra" seems to be due to Joyal [20] in the more general context of chain complexes in a Grothendieck abelian category [14]. One can find a proof in the case of abelian category of sheaves in [18]. In fact in both cases one endows the

[5] with differential of degree -1

category $\mathcal{C}_*(\mathcal{A}b(k))$ with a structure of model category and apply the homotopical of Quillen [43].

Lemma 3.14 *1) For any chain complex $D_* \in \mathcal{C}_*(\mathcal{A}b(k))$ there exists a functorial trivial cofibration $D_* \to D_*^f$ to a fibrant complex.*

2) A quasi-isomorphism between fibrant complexes is a homotopy equivalence.

3) If D_* is a fibrant chain complex, then for any chain complex C_* the natural map

$$\pi(C_*, D_*) \to Hom_{D(\mathcal{A}b(k))}(C_*, D_*)$$

is an isomorphism.

Here we denote by $\pi(C_*, D_*)$ the group of homotopy classes of morphisms of chain complexes in the usual sense. Thus to compute the group $Hom_{D(\mathcal{A}b(k))}(C_*, D_*)$ for any chain complexes C_* and D_*, one just chooses a quasi-isomorphism $D_* \to D_*^f$ to a fibrant complex (also called a *fibrant resolution*) and then one uses the chain of isomorphisms

$$\pi(C_*, D_*^f) \cong Hom_{D(\mathcal{A}b(k))}(C_*, D_*^f) \cong Hom_{D(\mathcal{A}b(k))}(C_*, D_*)$$

The main use we will make of this property is a "concrete" description of internal derived Hom-complex $R\underline{Hom}(C_*, D_*)$: it is given by the naive internal Hom-complex $\underline{Hom}(C_*, D_*^f)$, for C_* a chain complex which sections on any smooth k-scheme are torsion free abelian groups (to simplify). Indeed, it is clear that $\underline{Hom}(C_*, D_*^f)$ is fibrant; using part 2 of the above Lemma and and obvious adjunction formula for homotopies of morphisms of chain complexes we get that this functor $D(\mathcal{A}b(k)) \to D(\mathcal{A}b(k))$, $D_* \mapsto \underline{Hom}(C_*, D_*^f)$ is the right adjoint to the functor $D(\mathcal{A}b(k)) \to D(\mathcal{A}b(k))$, $B_* \mapsto B_* \otimes C_*$.

The $\mathbb{A}^1$-derived category. The following definition was mentioned in [32, Remark 9] and is directly inspired from [39, 49]:

Definition 3.15 *1) A chain complex $D_* \in C_*(\mathcal{A}b(k))$ is called* $\mathbb{A}^1$-local *if and only if for any $C_* \in C_*(\mathcal{A}b(k))$, the projection $C_* \otimes \mathbb{Z}(\mathbb{A}^1) \to C_*$ induces a bijection :*

$$Hom_{D(\mathcal{A}b(k))}(C_*, D_*) \to Hom_{D(\mathcal{A}b(k))}(C_* \otimes \mathbb{Z}(\mathbb{A}^1), D_*)$$

We will denote by $D_{\mathbb{A}^1-loc}(\mathcal{A}b(k)) \subset D(\mathcal{A}b(k))$ the full subcategory consisting of $\mathbb{A}^1$-local complexes.

*2) A morphism $f : C_* \to D_*$ in $C_*(\mathcal{A}b(k))$ is called an $\mathbb{A}^1$-*quasi isomorphism *if and only if for any $\mathbb{A}^1$-local chain complex E_*, the morphism :*

$$Hom_{D(\mathcal{A}b(k))}(D_*, E_*) \to Hom_{D(\mathcal{A}b(k))}(C_*, E_*)$$

is bijective. We will denote by $\mathbb{A}^1$-Qis the class of $\mathbb{A}^1$-quasi isomorphisms.

3) The $\mathbb{A}^1$-derived category $D_{\mathbb{A}^1}(\mathcal{A}b(k))$ is the category obtained by inverting the all the $\mathbb{A}^1$-quasi isomorphisms.

All the relevant properties we need are consequences of the following:

Lemma 3.16 *[39, 32] There exists a functor $L_{\mathbb{A}^1} : C_*(\mathcal{A}b(k)) \to C_*(\mathcal{A}b(k))$, called the $\mathbb{A}^1$-localization functor, together with a natural transformation*

$$\theta : Id \to L_{\mathbb{A}^1}$$

such that for any chain complex C_*, $\theta_{C_*} : C_* \to L_{\mathbb{A}^1}(C_*)$ is an $\mathbb{A}^1$-quasi isomorphism whose target is an $\mathbb{A}^1$-local fibrant chain complex.

It is standard to deduce:

Corollary 3.17 *The functor $L_{\mathbb{A}^1} : C_*(\mathcal{A}b(k)) \to C_*(\mathcal{A}b(k))$ induces a functor*

$$D(\mathcal{A}b(k)) \to D_{\mathbb{A}^1-loc}(\mathcal{A}b(k))$$

which is left adjoint to the inclusion $D_{\mathbb{A}^1-loc}(\mathcal{A}b(k)) \subset D(\mathcal{A}b(k))$, and which induces an equivalence of categories

$$D_{\mathbb{A}^1}(\mathcal{A}b(k)) \to D_{\mathbb{A}^1-loc}(\mathcal{A}b(k))$$

Proof of Lemma 3.16. We proceed as in [32]. We fix once for all a functorial fibrant resolution $C_* \to C_*^f$. Let C_* be a chain complex. We let $L_{\mathbb{A}^1}^{(1)}(C_*)$ be the cone in $\mathcal{C}_*(\mathcal{A}b(k))$ of the obvious morphism

$$ev_1 : \underline{Hom}(\mathbb{Z}_\bullet(\mathbb{A}^1), C_*^f) \to C_*^f$$

We let $C_* \to L^{(1)}_{\mathbb{A}^1}(C_*)$ denote the obvious morphism. Define by induction on $n \geq 0$, $L^{(n)}_{\mathbb{A}^1} := L^{(1)}_{\mathbb{A}^1} \circ L^{(n-1)}_{\mathbb{A}^1}$. We have natural morphisms, for any chain complex C_*, $L^{(n-1)}_{\mathbb{A}^1}(C_*) \to L^{(n)}_{\mathbb{A}^1}(C_*)$ and we set

$$L^{\infty}_{\mathbb{A}^1}(C_*) = colim_{n \in \mathbb{N}} L^{(n)}_{\mathbb{A}^1}(C_*)$$

As in [32, Theorem 4.2.1] we have:

Proposition 3.18 *For any chain complex C_* the complex $L^{\infty}_{\mathbb{A}^1}(C_*)$ is $\mathbb{A}^1$-local and the morphism*

$$C_* \to L^{\infty}_{\mathbb{A}^1}(C_*)$$

is an $\mathbb{A}^1$-quasi isomorphism.

This proves Lemma 3.16. In the sequel we set $L_{\mathbb{A}^1}(C_*) := L^{\infty}_{\mathbb{A}^1}(C_*)^f$: this is the $\mathbb{A}^1$-localization of C_*.

Remark 3.19 It should be noted that we have used implicitly the fact that we are working with the Nisnevich topology, as well as the B.G.-property from [39]: for a general topology on a site together with an interval in the sense of [39], the analogue localization functor would require more "iterations", indexed by some well chosen big enough ordinal number.

The (analogue of the) stable $\mathbb{A}^1$-connectivity theorem of [32] in $D(\mathcal{A}b(k))$ is the following:

Theorem 3.20 *Let C_* be a (-1)-connected chain complex. Then its $\mathbb{A}^1$-localization $L_{\mathbb{A}^1}(C_*)$ is still (-1)-connected.*

The proof is exactly the same as the case of S^1-spectra treated in [32]. Following the same procedure as in *loc. cit.*, this implies that for an $\mathbb{A}^1$-local chain complex C_* each of its truncations $\tau_{\geq n}(C_*)$ is still $\mathbb{A}^1$-local and thus each of its homology sheaves are automatically strictly $\mathbb{A}^1$-invariant. This endows the triangulated category $D(\mathcal{A}b(k))$ with a natural non degenerated t-structure [6] analogous to the homotopy t-structure of Voevodsky on $DM(k)$. The heart of that t-structure on $D(\mathcal{A}b(k))$ is precisely the category $\mathcal{A}b_{\mathbb{A}^1}(k)$ of strictly $\mathbb{A}^1$-invariant sheaves.

An easy consequence is:

Corollary 3.21 *The category $\mathcal{A}b_{\mathbb{A}^1}(k)$ of strictly $\mathbb{A}^1$-invariant sheaves is abelian, and the inclusion functor $\mathcal{A}b_{\mathbb{A}^1}(k) \subset \mathcal{A}b(k)$ is exact.*

Chain complexes and Eilenberg-MacLane spaces. Recall from [39], that for any simplicial sheaf of sets $\mathcal{X}$ we denote by $C_*(\mathcal{X})$ the (normalized) chain complex in $C_*(\mathcal{A}b(k))$ associated to the free simplicial sheaf of abelian groups $\mathbb{Z}(\mathcal{X})$ on $\mathcal{X}$. This construction defines a functor

$$C_* : \Delta^{op}Shv_{Nis}(Sm_k) \to C_*(\mathcal{A}b(k))$$

which is well known (see [39, 25] for instance) to have a right adjoint

$$K : C_*(\mathcal{A}b(k)) \to \Delta^{op}Shv_{Nis}(Sm_k)$$

called the Eilenberg-MacLane space functor.

For an abelian sheaf $M \in \mathcal{A}b(k)$ and an integer n we define the pointed simplicial sheaf $K(M,n)$ (see [39, page 56]) by applying K to the shifted complex $M[n]$, of the complex M placed in degree 0. If $n < 0$, the space $K(M,n)$ is a point. If $n \geq 0$ then $K(M,n)$ has only one non-trivial homotopy sheaf which is the n-th and which is canonically isomorphic to M. More generally, for a chain complex C_*, the space KC_* has for n-th homotopy sheaf 0 for $n < 0$, and the n-th homology sheaf $H_n(C_*)$ for $n \geq 0$.

It is clear that $C_* : \Delta^{op}Shv_{Nis}(Sm_k) \to C_*(\mathcal{A}b(k))$ sends simplicial weak equivalences to quasi-isomorphisms and $K : C_*(\mathcal{A}b(k)) \to \Delta^{op}Shv_{Nis}(Sm_k)$ maps quasi-isomorphisms to simplicial weak equivalences. If C_* is fibrant, it follows that $K(C_*)$ is simplicially fibrant. Thus the two functors induce a pair of adjoint functors

$$C_* : \mathcal{H}_s(k) \to D(\mathcal{A}b(k))$$

and

$$K : D(\mathcal{A}b(k)) \to \mathcal{H}_s(k)$$

As a consequence it is clear that if C_* is an $\mathbb{A}^1$-local complex, the space $K(C_*)$ is an $\mathbb{A}^1$-local space. Thus $C_* : \mathcal{H}_s(k) \to D(\mathcal{A}b(k))$ maps $\mathbb{A}^1$-weak equivalences to $\mathbb{A}^1$-quasi isomorphisms and induces a functor

$$C_*^{\mathbb{A}^1} : \mathcal{H}(k) \to D_{\mathbb{A}^1}(\mathcal{A}b(k))$$

which in concrete terms, maps a space $\mathcal{X}$ to the $\mathbb{A}^1$-localization of $C_*(\mathcal{X})$. We denote the latter by $C_*^{\mathbb{A}^1}(\mathcal{X})$ and call it the $\mathbb{A}^1$-chain complex of $\mathcal{X}$.

The functor $C_*^{\mathbb{A}^1} : \mathcal{H}(k) \to D_{\mathbb{A}^1}(\mathcal{A}b(k))$ admits as right adjoint the functor $K^{\mathbb{A}^1} : D_{\mathbb{A}^1}(\mathcal{A}b(k)) \to \mathcal{H}(k)$ induced by $C_* \mapsto K(L_{\mathbb{A}^1}(C_*))$. We observe that for an $\mathbb{A}^1$-local complex C_*, the space $K(C_*)$ is automatically $\mathbb{A}^1$-local and thus simplicially equivalent to the space $K^{\mathbb{A}^1}(C_*)$.

We will need the following:

Proposition 3.22 *Let C_* be a 0-connected chain complex in $\mathcal{C}_*(\mathcal{A}b(k))$. Then the following conditions are equivalent:*
(i) the space $K(C_*)$ is $\mathbb{A}^1$-local.
(ii) the chain complex C_* is $\mathbb{A}^1$-local.

Proof. For each complex C_* we simply denote by $(C_*)^{(\mathbb{A}^1)}$ the function complex $\underline{Hom}(\mathbb{Z}_\bullet(\mathbb{A}^1), C_*^f)$. And we let $(C_*)^{(\mathbb{A}^1)}_{\geq 0}$ denote the non negative part of $(C_*)^{(\mathbb{A}^1)}$. It is clear that the tautological $\mathbb{A}^1$-homotopy $(C_*)^{(\mathbb{A}^1)} \otimes \mathbb{Z}(\mathbb{A}^1) \to (C_*)^{(\mathbb{A}^1)}$ between the Identity and the 0-morphism, induces an $\mathbb{A}^1$-homotopy $(C_*)^{(\mathbb{A}^1)}_{\geq 0} \otimes \mathbb{Z}(\mathbb{A}^1) \to (C_*)^{(\mathbb{A}^1)}_{\geq 0}$ as well. Thus $(C_*)^{(\mathbb{A}^1)}_{\geq 0}$ is $\mathbb{A}^1$-contractible. We consider the morphism of "evaluation at one" $(C_*)^{(\mathbb{A}^1)}_{\geq 0} \to C_*$. And we set $U_{\mathbb{A}^1}(C_*) := cone((C_*)^{(\mathbb{A}^1)}_{\geq 0} \to C_*)$. For each $n > 0$ we let $U^{(n)}_{\mathbb{A}^1}$ denote the n-iteration of that functor. We then denote by $U^{\infty}_{\mathbb{A}^1}(C_*)$ the colimit of the diagram

$$C_* \to U_{\mathbb{A}^1}(C_*) \to \dots \to U^{(n)}_{\mathbb{A}^1}(C_*) \to \dots$$

Some observations:

(1) By the very construction, for any $n \geq 1$, there is a canonical morphism

$$U^{(n)}_{\mathbb{A}^1}(C_*) \to L^{(n)}_{\mathbb{A}^1}(C_*)$$

which induces an isomorphism on each homology sheaves in dimension ≥ 1. When C_* is 0-connected, it is exactly the truncation $L^{(n)}_{\mathbb{A}^1}(C_*)_{\geq 1}$: this is one the main point here!

(2) each morphism $U^{(n)}_{\mathbb{A}^1}(C_*) \to U^{(n+1)}_{\mathbb{A}^1}(C_*)$ is an $\mathbb{A}^1$-quasi-isomorphism because $(C_*)^{(\mathbb{A}^1)}_{\geq 0}$ was shown above to be $\mathbb{A}^1$-contractible. Moreover:

Lemma 3.23 *For any C_* the morphism of simplicial sheaves*

$$K(U^{(n)}_{\mathbb{A}^1}(C_*)) \to K(U^{(n+1)}_{\mathbb{A}^1}(C_*))$$

is an $\mathbb{A}^1$*-weak equivalence of spaces.*

As a consequence

$$K(C_*) \to K(U_{\mathbb{A}^1}^{(\infty)}(C_*))$$

is an $\mathbb{A}^1$*-weak equivalence of spaces.*

Proof. Indeed, this is a principal $K((C_*)_{\geq 0}^{(\mathbb{A}^1)})$-principal fibration by construction. Thus $K(U_{\mathbb{A}^1}^{(n+1)}(C_*))$ is simplicially weakly equivalent to the Borel construction of $K(U_{\mathbb{A}^1}^{(n)}(C_*))$ with respect to the action of the group $K((C_*)_{\geq 0}^{(\mathbb{A}^1)})$. But now the Borel construction

$$E(K((C_*)_{\geq 0}^{(\mathbb{A}^1)})) \times_{K((C_*)_{\geq 0}^{(\mathbb{A}^1)})} K(U_{\mathbb{A}^1}^{(n)}(C_*))$$

is filtered by the skeleton of $E(K((C_*)^{(\mathbb{A}^1)}$. The first filtration is $K(U_{\mathbb{A}^1}^{(n)}(C_*))$ and the others are of the form $(K((C_*)_{\geq 0}^{(\mathbb{A}^1)}))^{\wedge i} \wedge S^i \wedge (K(U_{\mathbb{A}^1}^{(n)}(C_*))_+)$ with $i > 0$ which is thus $\mathbb{A}^1$-weakly contractible.□

If C_* is 0-connected, by property (1) the colimit $U_{\mathbb{A}^1}^{(\infty)}(C_*) \to L_{\mathbb{A}^1}(C_*)$ of the morphisms $U_{\mathbb{A}^1}^{(n)}(C_*) \to L_{\mathbb{A}^1}^{(n)}(C_*)$ is isomorphic in $D(\mathcal{A}b(k))$ to the truncation

$$L_{\mathbb{A}^1}(C_*)_{\geq 1} \to L_{\mathbb{A}^1}(C_*)$$

By the connectivity Theorem 3.20, the previous morphism is a quasi-isomorphism. Recall that the space $K(L_{\mathbb{A}^1}(C_*))$ is $\mathbb{A}^1$-local. The following obvious corollary of what we have done easily implies the Proposition 3.22.□

Corollary 3.24 *For any* 0*-connected* C_**, the morphism of simplicial sheaves*

$$K(C_*) \to K(U_{\mathbb{A}^1}(C_*)) \cong K(L_{\mathbb{A}^1}(C_*)_{\geq 1}) \cong K(L_{\mathbb{A}^1}(C_*))$$

is an $\mathbb{A}^1$*-weak equivalence of* 0*-connected spaces to an* $\mathbb{A}^1$*-local space. It is thus isomorphic to the* $\mathbb{A}^1$*-localization of the source.*

We now deduce the following important property, which is one of the main tool in this paper:

Theorem 3.25 *A strongly* $\mathbb{A}^1$*-invariant sheaf of abelian groups is strictly* $\mathbb{A}^1$*-invariant.*

Proof. Indeed, M is strongly $\mathbb{A}^1$-invariant means that $K(M,1)$ is $\mathbb{A}^1$-local. By the Proposition 3.22, it follows that the chain complex $M[1]$ is $\mathbb{A}^1$-local, which means that M is strictly $\mathbb{A}^1$-invariant.□

Remark 3.26 The previous result can be used to simplify some proofs in [48]. We may also suppress in most results the assumption of perfectness of the base field.□

Remark 3.27 It could be possible to prove the Theorem without mentioning spaces and working only with complexes by introducing the $\mathbb{A}^1$-homotopy category of non-negative chain complexes which is equivalent to that of simplicial abelian sheaves by the Dold-Kan correspondence [25]. The proof would be exactly along the same lines. However, our method of proof yields slightly more: for instance Corollary 3.24 will be used below in a non trivial way.

The following consequence is one of our main structural result:

Theorem 3.28 *Let $\mathcal{X}$ be a pointed 0-connected space. Then $\mathcal{X}$ is $\mathbb{A}^1$-local if and only if $\pi_1(\mathcal{X})$ is strongly $\mathbb{A}^1$-invariant and if, for $n \geq 2$, $\pi_n(\mathcal{X})$ is strictly $\mathbb{A}^1$-invariant.*

This clearly follows from Corollary 3.3 and Theorem 3.25.

3.3 The Hurewicz theorem and some of its consequences

The following definition was made in [32]:

Definition 3.29 *Let $\mathcal{X}$ be a space and $n \in \mathbb{Z}$ be an integer. We let $H_n^{\mathbb{A}^1}(\mathcal{X})$ denote the n-th homology sheaf of the $\mathbb{A}^1$-chain complex $C_*^{\mathbb{A}^1}(\mathcal{X})$ of $\mathcal{X}$, and call it the n-th homology sheaf of $\mathcal{X}$.*

If $\mathcal{X}$ is pointed, we set $\tilde{H}_n^{\mathbb{A}^1}(\mathcal{X}) = Ker(H_n^{\mathbb{A}^1}(\mathcal{X}) \to H_n^{\mathbb{A}^1}(Spec(k))$ and call it the n-th reduced homology sheaf of $\mathcal{X}$. As $H_n^{\mathbb{A}^1}(Spec(k)) = 0$ for $n \neq 0$ and $\mathbb{Z}$ for $n = 0$, this means that as graded abelian sheaves

$$H_*^{\mathbb{A}^1}(\mathcal{X}) = \mathbb{Z} \oplus \tilde{H}_*^{\mathbb{A}^1}(\mathcal{X})$$

Remark 3.30 We observe that the $\mathbb{A}^1$-localization functor commutes to the suspension in $D(\mathcal{A}b(k))$. As an immediate consequence, we see that there exists a canonical suspension isomorphism for any pointed space $\mathcal{X}$ and any integer $n \in \mathbb{Z}$:

$$\tilde{H}_n^{\mathbb{A}^1}(\mathcal{X}) \cong \tilde{H}_{n+1}^{\mathbb{A}^1}(\Sigma(\mathcal{X}))$$

Using the $\mathbb{A}^1$-connectivity Theorem 3.20 and its consequences, we get

Corollary 3.31 *The* $\mathbb{A}^1$*-homology sheaves* $H_n^{\mathbb{A}^1}(\mathcal{X})$ *of a space* $\mathcal{X}$ *vanish for* $n < 0$ *and are are strictly* $\mathbb{A}^1$*-invariant sheaves for* $n \geq 0$*.*

Remark 3.32 We conjectured in [32] that this result should still hold over a general base; J. Ayoub produced in [1] a counter-example over a base of dimension 2. The case of a base dimension 1 is still open.□

Remark 3.33 In classical topology, one easily computes the whole homology of the sphere S^n: $H_i(S^n) = 0$ for $i > n$. In the $\mathbb{A}^1$-homotopy world, the analogue of this vanishing in big dimensions is unfortunately highly non-trivial and unknown. It is natural to make the:

Conjecture 3.34 *Let* X *be a smooth quasi-projective* k*-scheme of dimension* d*. Then* $H_n^{\mathbb{A}^1}(X) = 0$ *for* $n > 2d$ *and in fact if moreover* X *is affine then* $H_n^{\mathbb{A}^1}(X) = 0$ *for* $n > d$*.*

That would implies that the $\mathbb{A}^1$-homology of $(\mathbb{P}^1)^{\wedge n}$ vanishes in degrees $> 2n$. This is in fact a stronger version of the vanishing conjecture of Beilinson-Soulé. It was also formulated in [32].

Computations of higher $\mathbb{A}^1$-homotopy or $\mathbb{A}^1$-homology sheaves seem rather difficult in general. In fact, given a space, we now "understand" its first non-trivial $\mathbb{A}^1$-homotopy sheaf, but we do not know at the moment any "non-trivial" example where one can compute the next non-trivial $\mathbb{A}^1$-homotopy sheaf without using deep results like Milnor or Bloch-Kato conjectures.□

Using the adjunction between the functors C_* and K it is clear that for a fixed pointed space $\mathcal{X}$ the adjunction morphism

$$\mathcal{X} \to K(C_*(\mathcal{X}))$$

induces a morphism, for each $n \in \mathbb{Z}$

$$\pi_n^{\mathbb{A}^1}(\mathcal{X}) \to H_n^{\mathbb{A}^1}(\mathcal{X})$$

which we call the *Hurewicz morphism*.

The following two theorems form the weak form of our Hurewicz theorem:

Theorem 3.35 *Let $\mathcal{X}$ be a pointed 0-$\mathbb{A}^1$-connected space. Then the Hurewicz morphism*

$$\pi_1^{\mathbb{A}^1}(\mathcal{X}) \to H_1^{\mathbb{A}^1}(\mathcal{X})$$

is the initial morphism from the sheaf of groups $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ to a strictly $\mathbb{A}^1$-invariant sheaf (of abelian groups). This means that given a strictly $\mathbb{A}^1$-invariant sheaf M and a morphism of sheaves of groups

$$\pi_1^{\mathbb{A}^1}(\mathcal{X}) \to M$$

it factors uniquely through $\pi_1^{\mathbb{A}^1}(\mathcal{X}) \to H_1^{\mathbb{A}^1}(\mathcal{X})$.

Proof. Let M be a strictly $\mathbb{A}^1$-invariant sheaf. The group of morphisms of sheaves $Hom_{\mathcal{G}r}(\pi_1^{\mathbb{A}^1}(\mathcal{X}), M)$ is equal to the group of simplicial homotopy classes $Hom_{\mathcal{H}_s(k)}(L_{\mathbb{A}^1}(\mathcal{X}), K(M,1))$ which, because $K(M,1)$ is $\mathbb{A}^1$-local, is also $Hom_{\mathcal{H}(k)}(\mathcal{X}, K(M,1))$; by our above adjunction, this is also $Hom_{D_{\mathbb{A}^1}(\mathcal{A}b(k))}(C_*^{\mathbb{A}^1}(\mathcal{X}), K(M,1))$, and the latter is exactly $Hom_{\mathcal{A}b(k)}(H_1^{\mathbb{A}^1}(\mathcal{X}), M)$ because $C_*^{\mathbb{A}^1}(\mathcal{X})$ is 0-connected.□

Remark 3.36 It is not yet known, though expected, that the Hurewicz morphism is an epimorphism in degree one and that its kernel is always the commutator subgroup.

Theorem 3.37 *Let $n > 1$ be an integer and let $\mathcal{X}$ be a pointed $(n-1)$-$\mathbb{A}^1$-connected space. Then*

$$H_i^{\mathbb{A}^1}(\mathcal{X}) = 0$$

for each $i \in \{0, \ldots, n-1\}$ and the Hurewicz morphism

$$\pi_n^{\mathbb{A}^1}(\mathcal{X}) \to H_n^{\mathbb{A}^1}(\mathcal{X})$$

is an isomorphism between strictly $\mathbb{A}^1$-invariant sheaves.

Proof. Apply the same argument as in the previous theorem, using $K(M,n)$, and the fact from Theorem 3.28 that the $\mathbb{A}^1$-homotopy sheaves $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ are strictly $\mathbb{A}^1$-invariant.□

The following immediate consequence is the unstable $\mathbb{A}^1$-connectivity theorem:

Theorem 3.38 *Let $n > 0$ be an integer and let $\mathcal{X}$ be a pointed $(n-1)$-connected space. Then its $\mathbb{A}^1$-localization is still $(n-1)$-connected.*

For any sheaf of sets F on Sm_k, let us denote by $\mathbb{Z}_{\mathbb{A}^1}(F)$ the strictly $\mathbb{A}^1$-invariant sheaf

$$\mathbb{Z}_{\mathbb{A}^1}(F) := H_0^{\mathbb{A}^1}(F)$$

where F is considered as a space in the right hand side. This strictly $\mathbb{A}^1$-invariant sheaf is the free strictly $\mathbb{A}^1$-invariant sheaf generated by in the following sense: for any strictly $\mathbb{A}^1$-invariant sheaf M the natural map

$$Hom_{\mathcal{A}b(k)}(\mathbb{Z}_{\mathbb{A}^1}(F), M) \to Hom_{Shv(Sm_k)}(F, M)$$

is clearly a bijection.

If F is pointed, we denote by $\mathbb{Z}_{\mathbb{A}^1,\bullet}(F)$ the reduced homology sheaf $\tilde{H}_0^{\mathbb{A}^1}(F)$.

Our previous results and proofs immediately yield:

Corollary 3.39 *For any integer $n \geq 2$ and any pointed sheaf of sets F the canonical morphism*

$$\pi_n^{\mathbb{A}^1}(\Sigma^n(F)) \to H_n^{\mathbb{A}^1}(\Sigma^n(F)) \cong \mathbb{Z}_{\mathbb{A}^1,\bullet}(F)$$

is an isomorphism.

The last isomorphism is the suspension isomorphism from Remark 3.30.

Now by Theorem 3.25, the free strictly $\mathbb{A}^1$-invariant sheaf generated by a (pointed) sheaf F is the same sheaf as the free strongly $\mathbb{A}^1$-invariant sheaf of abelian groups generated by the same (pointed) sheaf. Our main computation in Theorem 2.37 thus yields the following analogue of Theorem 1 which was announced as Theorem 19 in the introduction:

Theorem 3.40 *For $n \geq 2$ one has canonical isomorphisms of strictly $\mathbb{A}^1$-invariant sheaves*

$$\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}) \cong \pi_n^{\mathbb{A}^1}((\mathbb{P}^1)^{\wedge n}) \cong \mathbb{Z}_{\mathbb{A}^1,\bullet}((\mathbb{G}_m)^{\wedge n}) \cong \underline{\mathbf{K}}_n^{MW}$$

Remark 3.41 Observe that the previous computation of $\pi_1^{\mathbb{A}^1}(\mathbb{A}^2 - \{0\})$ requires a slightly more subtle argument, as it concerns the $\mathbb{A}^1$-fundamental group. The morphism $SL_2 \to \mathbb{A}^2 - \{0\}$ being an $\mathbb{A}^1$-weak equivalence, we know *a priori* that $\pi_1^{\mathbb{A}^1}(\mathbb{A}^2 - \{0\})$ is a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups, as is the $\mathbb{A}^1$-fundamental group of any group (or h-group) as usual. The free strongly $\mathbb{A}^1$-invariant sheaf of groups on $\mathbb{G}_m \wedge \mathbb{G}_m$ is commutative and it is thus $\underline{\mathbf{K}}_2^{MW}$.

We postpone the computation of $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ to section **??**.$\square$

Remark 3.42 For any $n \geq 0$ we let S^n denote $(S^1)^{\wedge n}$. We observe that $\mathbb{A}^n - \{0\}$ is canonically isomorphic in $\mathcal{H}_\bullet(k)$ to $S^{n-1} \wedge (\mathbb{G}_m)^{\wedge n}$ and $(\mathbb{P}^1)^{\wedge n}$ is canonically isomorphic to $S^n \wedge (\mathbb{G}_m)^{\wedge n}$, see [39, § Spheres, suspensions and Thom spaces p. 110]. It is thus natural for any $n \geq 0$ and any $i \geq 0$ to study the "sphere" of the form $S^n \wedge (\mathbb{G}_m)^{\wedge i}$.

The Hurewicz Theorem implies that it is at least $n-1$ connected and if $n \geq 2$, provides a canonical isomorphism

$$\pi_n^{\mathbb{A}^1}(S^n \wedge (\mathbb{G}_m)^{\wedge i}) \cong \underline{\mathbf{K}}_i^{MW}$$

for $i \geq 1$ and $\pi_n^{\mathbb{A}^1}(S^n) = \mathbb{Z}$ for $i = 0$ (and $n \geq 1$).

In case $n = 0$ our sphere is just a smash-power of $\mathbb{G}_m$ which is itself $\mathbb{A}^1$-invariant.

For $n = 1$ the question is harder and we only get, by the Hurewicz Theorem, a canonical epimorphism $\pi_1^{\mathbb{A}^1}(S^1 \wedge (\mathbb{G}_m)^{\wedge i}) \twoheadrightarrow \underline{\mathbf{K}}_i^{MW}$. This epimorphism has a non trivial kernel for $i = 1$ (see the computation of $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ in Section **??**). We have just observed in the previous remark that this epimorphism is an isomorphism for $i = 2$. We don't know $\pi_1^{\mathbb{A}^1}(S^1 \wedge (\mathbb{G}_m)^{\wedge i})$ for $i > 2$.$\square$

Corollary 3.43 *Let $(n, i) \in \mathbb{N}^2$ and $(m, j) \in \mathbb{N}^2$ be pairs of integers. For $n \geq 2$ we have a canonical isomorphism:*

$$Hom_{\mathcal{H}_\bullet(k)}(S^m \wedge (\mathbb{G}_m)^{\wedge j}, S^n \wedge (\mathbb{G}_m)^{\wedge i}) \cong \begin{cases} 0 \text{ if } m < n \\ K_{i-j}^{MW}(k) \text{ if } m = n \text{ and } i > 0 \\ 0 \text{ if } m = n\ ,\ j > 0 \text{ and } i = 0 \\ \mathbb{Z} \text{ if } m = n \text{ and } j = i = 0 \end{cases}$$

Proof. This follows immediately from our previous computation, from Theorem 3.11 and Remark **??** which clearly implies that the product induces isomorphisms $(\underline{\mathbf{K}}_n^{MW})_{-1} \cong \underline{\mathbf{K}}_{n-1}^{MW}$.□

$\mathbb{A}^1$**-fibration sequences and applications** In this paragraph we give some natural consequences of the (weak) Hurewicz Theorem and of our structure result for $\mathbb{A}^1$-homotopy sheaves Theorem 3.28.

We first recall some terminology.

Definition 3.44 *1) A simplicial fibration sequence between spaces*

$$\Gamma \to \mathcal{X} \to \mathcal{Y}$$

with $\mathcal{Y}$ pointed, is a diagram such that the composition of the two morphisms is the trivial one and such that the induced morphism from Γ to the simplicial homotopy fiber of $\mathcal{X} \to \mathcal{Y}$ is a simplicial weak equivalence.

2) An $\mathbb{A}^1$-fibration sequence between spaces

$$\Gamma \to \mathcal{X} \to \mathcal{Y}$$

with $\mathcal{Y}$ pointed, is a diagram such that the composition of the two morphisms is the trivial one and such that the induced diagram between $\mathbb{A}^1$-localizations

$$L_{\mathbb{A}^1}(\Gamma) \to L_{\mathbb{A}^1}(\mathcal{X}) \to L_{\mathbb{A}^1}(\mathcal{Y})$$

is a simplicial fibration sequence.

A basic problem is that it is not true in general that a simplicial fibration sequence is an $\mathbb{A}^1$-fibration sequence. For instance, let $\mathcal{X}$ be a fibrant pointed space, denote by $\mathcal{P}(\mathcal{X})$ the pointed space $\underline{Hom}_\bullet(\Delta^1, \mathcal{X})$ of pointed paths $\Delta^1 \to \mathcal{X}$ in $\mathcal{X}$ so that we have a simplicial fibration sequence

$$\Omega^1(\mathcal{X}) \to \mathcal{P}(\mathcal{X}) \to \mathcal{X}$$

whose fiber $\Omega^1(\mathcal{X}) := \underline{Hom}_\bullet(S^1, \mathcal{X})$ is the simplicial loop space of $\mathcal{X}$ (with $S^1 = \Delta^1/\partial\Delta^1$ is the simplicial circle). The following observation is an immediate consequence of our definitions, the fact that if $\mathcal{X}$ is $\mathbb{A}^1$-fibrant so is $\Omega^1(\mathcal{X})$, and the fact that an $\mathbb{A}^1$-weak equivalence between $\mathbb{A}^1$-local space is a simplicial weak equivalence:

Lemma 3.45 *Let $\mathcal{X}$ be a simplicially fibrant pointed space. The paths simplicial fibration sequence $\Omega^1(\mathcal{X}) \to \mathcal{P}(\mathcal{X}) \to \mathcal{X}$ above is an $\mathbb{A}^1$-fibration sequence if and only if the canonical morphism*

$$L_{\mathbb{A}^1}(\Omega^1(\mathcal{X})) \to \Omega^1(L_{\mathbb{A}^1}(\mathcal{X}))$$

is a simplicial weak equivalence.

We now observe:

Theorem 3.46 *Let $\mathcal{X}$ be a (simplicially) fibrant pointed space. Then the canonical morphism*

$$L_{\mathbb{A}^1}(\Omega^1(\mathcal{X})) \to \Omega^1(L_{\mathbb{A}^1}(\mathcal{X}))$$

is a simplicial weak equivalence if and only if the sheaf of groups $\pi_0^{\mathbb{A}^1}(\Omega^1(\mathcal{X})) = \pi_0(L_{\mathbb{A}^1}(\Omega^1(\mathcal{X})))$ is strongly $\mathbb{A}^1$-invariant.

Proof. From Theorem 3.1 the condition is clearly necessary. To prove the converse we may clearly assume $\mathcal{X}$ is 0-connected (and fibrant). In that case the inclusion of $\mathcal{X}^{(0)} \subset \mathcal{X}$ of the sub-space consisting of "simplices whose vertices are the base point" is a simplicial weak equivalence: use [25] and stalks to check it. Using the Kan model $G(\mathcal{X}^{(0)})$ for the simplicial loop space on a pointed 0-reduced Kan simplicial set (*loc. cit.* for instance) one obtain a canonical morphism $\mathcal{X}^{(0)} \to B(G(\mathcal{X}^{(0)}))$ which is clearly also a simplicial weak equivalence (by checking on stalks). Thus this defines in the simplicial homotopy category $\mathcal{H}_{s,\bullet}(k)$ a canonical pointed isomorphism between $\mathcal{X}$ and $B(G(\mathcal{X}^{(0)}))$ and in particular a canonical pointed isomorphism between $\Omega^1(\mathcal{X})$ and $G(\mathcal{X}^{(0)})$. Now we observe that by Lemma 3.47 below, we may choose $L_{\mathbb{A}^1}$ so that $L_{\mathbb{A}^1}$ maps groups to groups. Thus $G(\mathcal{X}^{(0)}) \to L_{\mathbb{A}^1}(G(\mathcal{X}^{(0)}))$ is an $\mathbb{A}^1$-weak equivalence between simplicial sheaves of groups. By Lemma 3.48 we see that

$$\mathcal{X} \cong B(G(\mathcal{X}^{(0)})) \to B(L_{\mathbb{A}^1}(G(\mathcal{X}^{(0)})))$$

is always an $\mathbb{A}^1$-weak equivalence. Now assuming that $\pi_0(L_{\mathbb{A}^1}(\Omega^1(\mathcal{X}))) \cong \pi_1(B(L_{\mathbb{A}^1}(G(\mathcal{X}^{(0)}))))$ is strongly $\mathbb{A}^1$-invariant, and the higher homotopy sheaves of $B(L_{\mathbb{A}^1}(G(\mathcal{X}^{(0)})))$ are strictly $\mathbb{A}^1$-invariant, we see using Theorem 3.28 that the space $B(L_{\mathbb{A}^1}(G(\mathcal{X}^{(0)})))$ is $\mathbb{A}^1$-local. It is thus the $\mathbb{A}^1$-localization of $\mathcal{X}$.$\square$

Recall from [39] that an $\mathbb{A}^1$-resolution functor is a pair (Ex, θ) consisting of a functor $Ex : \Delta^{op}Shv(Sm_k) \to \Delta^{op}Shv(Sm_k)$ and a natural transformation $\theta : Id \to Ex$ such that for any space $\mathcal{X}$, $Ex(\mathcal{X})$ is fibrant and $\mathbb{A}^1$-local, and $\theta(\mathcal{X}) : \mathcal{X} \to Ex(\mathcal{X})$ is an $\mathbb{A}^1$-weak equivalence.

Lemma 3.47 *[39] There exists an $\mathbb{A}^1$-resolution functor (Ex, θ) which commutes to any finite products.*

Proof. Combine [39, Theorem 1.66 page 69] with the construction of the explicit I-resolution functor given page 92 of *loc. cit.*.□

Recall that a principal fibration $G - \mathcal{X} \to \mathcal{Y}$ with simplicial group G is the same thing as a G-torsor over $\mathcal{Y}$.

Lemma 3.48 *Let*

$$\begin{array}{ccccc} G & - & \mathcal{X} & \to & \mathcal{Y} \\ \downarrow & & \downarrow & & \downarrow \\ G' & - & \mathcal{X}' & \to & \mathcal{Y}' \end{array}$$

be a commutative diagram of spaces in which the horizontal lines are principal fibrations with simplicial groups G and G'. Assume the vertical morphism (of simplicial groups) $G \to G'$ and the morphism of spaces $\mathcal{X} \to \mathcal{X}'$ are both $\mathbb{A}^1$-weak equivalence. Then

$$\mathcal{Y} \to \mathcal{Y}'$$

is an $\mathbb{A}^1$-weak equivalence.

Proof. Given a simplicial sheaf of groups G we use the model $E(G)$ of simplicially contractible space on which G acts freely given by the diagonal of the simplicial space $n \mapsto E(G_n)$ where $E(G)$ for a simplicially constant sheaf of group is the usual model (see [39, page 128] for instance). We may as well consider it a the diagonal of the simplicial space $m \mapsto G^{m+1}$, the action of G being the diagonal one. For any G-space $\mathcal{X}$ we introduce the Borel construction

$$EG \times_G \mathcal{X}$$

where G acts diagonally on $E(G) \times \mathcal{X}$. If the action of G is free on $\mathcal{X}$, the morphism $EG \times_G \mathcal{X} \to {}_G\backslash\mathcal{X}$ is a simplicial weak equivalence. Thus in the statement we may replace $\mathcal{Y}$ by $EG \times_G \mathcal{X}$ and $\mathcal{Y}'$ by $EG' \times_{G'} \mathcal{X}'$ respectfully. Now from our recollection above, $EG \times_G \mathcal{X}$ is the diagonal space of the simplicial space $m \mapsto G^{m+1} \times_G \mathcal{X}$; it thus simplicially equivalent to its homotopy colimit (see [7] and [39, page 54]). The Lemma thus follows from Lemma 2.12 page 73 of *loc. cit.* and the fact that for any m the morphism

$$G^{m+1} \times_G \mathcal{X} \to (G')^{m+1} \times_{G'} \mathcal{X}'$$

are $\mathbb{A}^1$-weak-equivalences. This is easy to prove by observing that the G-space G^{m+1} is functorially G-isomorph to $G \times (G^m)$ with action given on the left factor only. Thus the spaces $G^{m+1} \times_G \mathcal{X}$ are separately (not taking the simplicial structure into account) isomorph to $G^m \times \mathcal{X}$.□

Definition 3.49 *1) A* homotopy principal G-fibration

$$G - \mathcal{X} \to \mathcal{Y}$$

with simplicial group G consists of a G-space $\mathcal{X}$ and a G-equivariant morphism $\mathcal{X} \to \mathcal{Y}$ (with trivial action on $\mathcal{Y}$) such that the obvious morphism

$$EG \times_G \mathcal{X} \to \mathcal{Y}$$

is a simplicial weak equivalence.

2) Let $G - \mathcal{X} \to \mathcal{Y}$ be a (homotopy) G-principal fibration with structure group G. We say that it is an $\mathbb{A}^1$-homotopy G-principal fibration if the diagram

$$L_{\mathbb{A}^1}(\mathcal{X}) \to L_{\mathbb{A}^1}(\mathcal{Y})$$

is a homotopy principal fibration with structure group $L_{\mathbb{A}^1}(G)$.

In the previous statement, we used an $\mathbb{A}^1$-localization functor which commutes to finite product (such a functor exists by Lemma 3.47).

Theorem 3.50 *Let $G - \mathcal{X} \to \mathcal{Y}$ be a (homotopy) principal fibration with structure group G such that $\pi_0^{\mathbb{A}^1}(G)$ is strongly $\mathbb{A}^1$-invariant. Then it is an $\mathbb{A}^1$-homotopy G-principal fibration.*

Proof. We contemplate the obvious commutative diagram of spaces:

$$\begin{array}{ccccc} G & - & \mathcal{X} & \to & \mathcal{Y} \\ || & & \uparrow \wr & & \uparrow \wr \\ G & - & E(G) \times \mathcal{X} & \to & E(G) \times_G \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ L_{\mathbb{A}^1}(G) & - & E(L_{\mathbb{A}^1}(G)) \times L_{\mathbb{A}^1}(\mathcal{X}) & \to & E(L_{\mathbb{A}^1}(G)) \times_{L_{\mathbb{A}^1}(G)} L_{\mathbb{A}^1}(\mathcal{X}) \end{array}$$

where the upper vertical arrows are simplicial weak equivalences. By Lemma 3.48 the right bottom vertical arrow is an $\mathbb{A}^1$-weak equivalence. By the very definition, to prove the claim we only have to show now that the obvious morphism $E(L_{\mathbb{A}^1}(G)) \times_{L_{\mathbb{A}^1}(G)} L_{\mathbb{A}^1}(\mathcal{X}) \to L_{\mathbb{A}^1}(\mathcal{Y})$ is a simplicial weak equivalence.

As $E(G) \times_G \mathcal{X} \to E(L_{\mathbb{A}^1}(G)) \times_{L_{\mathbb{A}^1}(G)} L_{\mathbb{A}^1}(\mathcal{X})$ is an $\mathbb{A}^1$-weak equivalence, we clearly only have to show that the space $E(L_{\mathbb{A}^1}(G)) \times_{L_{\mathbb{A}^1}(G)} L_{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-local. But it fits, by construction, into a simplicial fibration sequence of the form

$$L_{\mathbb{A}^1}(\mathcal{X}) \to E(L_{\mathbb{A}^1}(G)) \times_{L_{\mathbb{A}^1}(G)} L_{\mathbb{A}^1}(\mathcal{X}) \to B(L_{\mathbb{A}^1}(G))$$

As $\pi_0^{\mathbb{A}^1}(G)$ is strongly $\mathbb{A}^1$-invariant the 0-connected space $B(L_{\mathbb{A}^1}(G))$ is $\mathbb{A}^1$-local by Theorem 3.28. This easily implies the claim using the Lemma **??** above.□

Lemma 3.51 *Let* $\Gamma \to \mathcal{X} \to \mathcal{Y}$ *be a simplicial fibration sequence with* $\mathcal{Y}$ *pointed and* 0*-connected. If* Γ *and* $\mathcal{Y}$ *are* $\mathbb{A}^1$*-connected, then so is* $\mathcal{X}$*.*

Proof. We use the commutative diagram of spaces

$$\begin{array}{ccccc} \Gamma & \to & \mathcal{X} & \to & \mathcal{Y} \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma^{\mathbb{A}^1} & \to & \mathcal{X}^{\mathbb{A}^1} & \to & \mathcal{Y}^{\mathbb{A}^1} \end{array}$$

where the horizontal rows are both simplicial fibration sequences (we denote here by $\mathcal{Z}^{\mathbb{A}^1}$ the right simplicially derived functor $R\underline{Hom}(\mathbb{A}^1, \mathcal{Z})$, see [39]). We must prove that the middle vertical arrow is a simplicial weak equivalence knowing that both left and right vertical arrows are. But using stalks we reduce easily to the corresponding case for simplicial sets, which is well-known.□

Example 3.52 1) For instance any SL_n-torsors, $n \geq 2$, satisfy the property of the Theorem because $\pi_0^{\mathbb{A}^1}(SL_n) = *$: this follows from the fact that over a field $F \in \mathcal{F}_k$, any element of $SL_n(F)$ is a product of elementary matrices, which shows that over $\pi_0^{\mathbb{A}^1}(SL_n)(F) = *$. From [30] this implies the claim.

2) Any GL_n-torsors, for $n \geq 1$, satisfy this condition as well as $\pi_0^{\mathbb{A}^1}(GL_n) = \mathbb{G}_m$ is strictly $\mathbb{A}^1$-invariant. This equality follows from the previous statement.

3) This is also the case for finite groups or abelian varieties: as these are flasque as sheaves, the H^1_{Nis} is trivial.

4) In fact we do not know any example of smooth algebraic group G over k whose $\pi_0^{\mathbb{A}^1}$ is strongly $\mathbb{A}^1$-invariant.$\square$

Theorem 3.53 *Let $\Gamma \to \mathcal{X} \to \mathcal{Y}$ be a simplicial fibration sequence with $\mathcal{Y}$ pointed and 0-connected. Assume that the sheaf of groups $\pi_0^{\mathbb{A}^1}(\Omega^1(\mathcal{Y})) = \pi_0(L_{\mathbb{A}^1}(\Omega^1(\mathcal{Y})))$ is strongly $\mathbb{A}^1$-invariant. Then $\Gamma \to \mathcal{X} \to \mathcal{Y}$ is also an $\mathbb{A}^1$-fibration sequence.*

Proof. This theorem is an easy reformulation of the previous one (using a little bit its proof) by considering a simplicial group G with a simplicial weak equivalence $\mathcal{Y} \cong B(G)$.$\square$

We observe that the assumptions of the Theorem are fulfilled if $\mathcal{Y}$ is simplicially 1-connected, or if it is 0-connected and if $\pi_1(\mathcal{Y})$ itself is strongly $\mathbb{A}^1$-invariant. This follows from the following Lemma applied to $\Omega^1(\mathcal{Y})$.

Lemma 3.54 *Let $\mathcal{X}$ be a space. Assume its sheaf $\pi_0(\mathcal{X})$ is $\mathbb{A}^1$-invariant. Then the morphism $\pi_0(\mathcal{X}) \to \pi_0^{\mathbb{A}^1}(\mathcal{X}) = \pi_0(L_{\mathbb{A}^1}(\mathcal{X}))$ is an isomorphism.*

Proof. This Lemma follows from the fact that $\pi_0(\mathcal{X}) \to \pi_0^{\mathbb{A}^1}(\mathcal{X})$ is always an epimorphism [39, Corollary 3.22 page 94] and the fact that as a space the $\mathbb{A}^1$-invariant sheaf $\pi_0(\mathcal{X})$ is $\mathbb{A}^1$-local. This produces a factorization of the identity of $\pi_0(\mathcal{X})$ as $\pi_0(\mathcal{X}) \to \pi_0^{\mathbb{A}^1}(\mathcal{X}) = \pi_0(L_{\mathbb{A}^1}(\mathcal{X})) \to \pi_0(\mathcal{X})$ which clearly implies the result.$\square$

The relative $\mathbb{A}^1$-connectivity theorem.

Definition 3.55 *A morphism of spaces $\mathcal{X} \to \mathcal{Y}$ is said to be n-connected for some integer $n \geq 0$ if each stalk of that morphism (at any point of any smooth k-scheme) is n-connected in the usual sense.*

When the spaces are pointed and $\mathcal{Y}$ is 0-connected this is equivalent to the fact that the simplicial homotopy fiber of the morphism is n-connected.

The relative $\mathbb{A}^1$-connectivity theorem refers to:

Theorem 3.56 *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism with $\mathcal{Y}$ pointed and 0-connected. Assume that the sheaf of groups $\pi_0^{\mathbb{A}^1}(\Omega^1(\mathcal{Y})) = \pi_0(L_{\mathbb{A}^1}(\Omega^1(\mathcal{Y})))$ is strongly $\mathbb{A}^1$-invariant (for instance if $\mathcal{Y}$ is simplicially 1-connected, or if $\pi_1(\mathcal{Y})$ itself is strongly $\mathbb{A}^1$-invariant). Let $n \geq 1$ be an integer and assume f is $(n-1)$-connected, then so is the morphism*

$$L_{\mathbb{A}^1}(\mathcal{X}) \to L_{\mathbb{A}^1}(\mathcal{Y})$$

Proof. Let $\Gamma \to \mathcal{X}$ be the homotopy fiber. By Theorem 3.53 above the diagram $L_{\mathbb{A}^1}(\Gamma) \to L_{\mathbb{A}^1}(\mathcal{X}) \to L_{\mathbb{A}^1}(\mathcal{Y})$ is a simplicial fibration sequence. Our connectivity assumption is that $\pi_i(\Gamma) = 0$ for $i \in \{0, \ldots, n-1\}$. By the unstable $\mathbb{A}^1$-connectivity Theorem 3.38, the space $L_{\mathbb{A}^1}(\Gamma)$ is also $(n-1)$-connected. Thus so is $L_{\mathbb{A}^1}(\mathcal{X}) \to L_{\mathbb{A}^1}(\mathcal{Y})$.□

The strong form of the Hurewicz theorem. This refers to the following classical improvement of the weak Hurewicz Theorem:

Theorem 3.57 *Let $n > 1$ be an integer and let $\mathcal{X}$ be a pointed $(n-1)$-$\mathbb{A}^1$-connected space. Then $H_i^{\mathbb{A}^1}(\mathcal{X}) = 0$ for each $i \in \{0, \ldots, n-1\}$, the Hurewicz morphism $\pi_n^{\mathbb{A}^1}(\mathcal{X}) \to H_n^{\mathbb{A}^1}(\mathcal{X})$ is an isomorphism, and moreover the Hurewicz morphism*

$$\pi_{n+1}^{\mathbb{A}^1}(\mathcal{X}) \to H_{n+1}^{\mathbb{A}^1}(\mathcal{X})$$

is an epimorphism of sheaves.

Proof. We may assume $\mathcal{X}$ fibrant and $\mathbb{A}^1$-local. Consider the canonical morphism $\mathcal{X} \to K(C_*(\mathcal{X}))$ and let us denote by Γ its simplicial homotopy fiber. The classical Hurewicz Theorem for simplicial homotopy tells us that Γ is simplicially n-connected (just compute on the stalks).

Now as $K(C_*(\mathcal{X}))$ is 1-connected the Theorem 3.56 above tells us that the morphism $\mathcal{X} = L_{\mathbb{A}^1}(\mathcal{X}) \to L_{\mathbb{A}^1}(K(C_*(\mathcal{X})))$ is still n-connected. But as $K(L_{\mathbb{A}^1}(C_*(\mathcal{X}))) \to L_{\mathbb{A}^1}(K(C_*(\mathcal{X})))$ is a simplicial weak equivalence by Corollary 3.24 we conclude that $\mathcal{X} \to K(C_*^{\mathbb{A}^1}(\mathcal{X}))$ is n-connected, which gives exactly the strong form of Hurewicz Theorem.□

Remark 3.58 For $n = 1$ if one assumes that $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ is abelian (thus strictly $\mathbb{A}^1$-invariant) the Theorem remains true.

A stability result. Recall that for a fibrant space $\mathcal{X}$ and an integer n the space $P^{(n)}(\mathcal{X})$ denotes the n-th stage of the Postnikov tower for $\mathcal{X}$ [39, page 55]. If $\mathcal{X}$ is pointed, we denote by $\mathcal{X}^{(n+1)} \to \mathcal{X}$ the homotopy fiber at the point of $\mathcal{X} \to P^{(n)}(\mathcal{X})$. The space $\mathcal{X}^{(n+1)}$ is of course n-connected. There exists by functoriality a canonical morphism $\mathcal{X}^{(n)} \to (L_{\mathbb{A}^1}(\mathcal{X}))^{(n)}$. As the target is clearly $\mathbb{A}^1$-local, we thus get a canonical morphism of pointed spaces

$$L_{\mathbb{A}^1}(\mathcal{X}^{(n)}) \to (L_{\mathbb{A}^1}(\mathcal{X}))^{(n)}$$

Theorem 3.59 *Let $\mathcal{X}$ be a pointed connected space. Assume $n > 0$ is an integer such that the sheaf $\pi_1(\mathcal{X})$ is strongly $\mathbb{A}^1$-invariant and for each $1 < i \leq n$, the sheaf $\pi_i(\mathcal{X})$ is strictly $\mathbb{A}^1$-invariant. Then for each $i \leq n+1$ the above morphism $L_{\mathbb{A}^1}(\mathcal{X}^{(i)}) \to (L_{\mathbb{A}^1}(\mathcal{X}))^{(i)}$ is a simplicial weak equivalence.*

We obtain immediately the following:

Corollary 3.60 *Let $\mathcal{X}$ be a pointed connected space. Assume $n > 0$ is an integer such that the sheaf $\pi_1(\mathcal{X})$ is strongly $\mathbb{A}^1$-invariant and for each $1 < i \leq n$, the sheaf $\pi_i(\mathcal{X})$ is strictly $\mathbb{A}^1$-invariant. Then for $i \leq n$ the morphism*

$$\pi_i(\mathcal{X}) \to \pi_i^{\mathbb{A}^1}(\mathcal{X}) = \pi_i(L_{\mathbb{A}^1}(\mathcal{X}))$$

is an isomorphism and the morphism

$$\pi_{n+1}(\mathcal{X}) \to \pi_{n+1}^{\mathbb{A}^1}(\mathcal{X}) = \pi_{n+1}(L_{\mathbb{A}^1}(\mathcal{X}))$$

is the universal morphism from $\pi_{n+1}(\mathcal{X})$ to a strictly $\mathbb{A}^1$-invariant sheaf.

Proof. We proceed by induction on n. Assume the statement of the Theorem is proven for $n-1$. We apply Theorem 3.53 to the simplicial fibration sequence $\mathcal{X}^{(n+1)} \to \mathcal{X} \to P^{(n)}(\mathcal{X})$; $P^{(n)}(\mathcal{X})$ satisfies indeed the assumptions. Thus we get a simplicial fibration sequence

$$L_{\mathbb{A}^1}(\mathcal{X}^{(n+1)}) \to L_{\mathbb{A}^1}(\mathcal{X}) \to L_{\mathbb{A}^1}(P^{(n)}(\mathcal{X}))$$

Then we observe that by induction and the Corollary 3.60 above that the morphism $P^{(n)}(\mathcal{X}) \to P^{(n)}(L_{\mathbb{A}^1}(\mathcal{X}))$ is a simplicial weak equivalence. Thus $P^{(n)}(\mathcal{X}) \cong L_{\mathbb{A}^1}(P^{(n)}(\mathcal{X})) \cong P^{(n)}(L_{\mathbb{A}^1}(\mathcal{X}))$. These two facts imply the claim.$\square$

The $\mathbb{A}^1$-simplicial suspension Theorem.

Theorem 3.61 *Let $\mathcal{X}$ be a pointed space and let $n \geq 2$ be an integer. If $\mathcal{X}$ is $(n-1)$-$\mathbb{A}^1$-connected space the canonical morphism*

$$L_{\mathbb{A}^1}(\mathcal{X}) \to \Omega^1(L_{\mathbb{A}^1}(\Sigma^1(\mathcal{X})))$$

is $2(n-1)$-($\mathbb{A}^1$)-connected.

Proof. We first observe that the classical suspension Theorem implies that for any simplicially $(n-1)$-connected space $\mathcal{Y}$ the canonical morphism

$$\mathcal{Y} \to \Omega^1(\Sigma^1(\mathcal{Y}))$$

is simplicially $2(n-1)$-connected. Thus the theorem follows from: We apply this to the space $\mathcal{Y} = L_{\mathbb{A}^1}(\mathcal{X})$ itself, which is simplicially $(n-1)$-connected. Thus the morphism $L_{\mathbb{A}^1}(\mathcal{X}) \to \Omega^1(\Sigma^1(L_{\mathbb{A}^1}(\mathcal{X})))$ is simplicially $2(n-1)$-connected. This implies in particular that the suspension morphisms $\pi_i^{\mathbb{A}^1}(\mathcal{X}) \to \pi_{i+1}(\Sigma_s^1(L_{\mathbb{A}^1}(\mathcal{X})))$ are isomorphisms for $i \leq 2(n-1)$ and an epimorphism for $i = 2n-1$.

¿From Theorem 3.59 and its corollary, this implies that $\Sigma^1(\mathcal{X}) \to L_{\mathbb{A}^1}(\Sigma^1(\mathcal{X}))$ induces an isomorphism on π_i for $i \leq 2n-1$ and that the morphism $\pi_{2n}(\Sigma^1(\mathcal{X})) \to \pi_{2n}(L_{\mathbb{A}^1}(\Sigma^1(\mathcal{X})))$ is the universal morphism to a strictly $\mathbb{A}^1$-invariant sheaf. Thus it follows formally that $\pi_{2n-1}^{\mathbb{A}^1}(\mathcal{X}) \to \pi_{2n}(L_{\mathbb{A}^1}(\Sigma_s^1(\mathcal{X})))$ is a categorical epimorphism in the category of strictly $\mathbb{A}^1$-invariant sheaves. As by Corollary 3.21 this category is an abelian category for which the inclusion into $\mathcal{A}b(k)$ is exact, it follows that the morphism $\pi_{2n-1}^{\mathbb{A}^1}(\mathcal{X}) \to \pi_{2n}(L_{\mathbb{A}^1}(\Sigma_s^1(\mathcal{X})))$ is actually an epimorphism of sheaves. Thus the morphism $\Sigma^1(\mathcal{X}) \to L_{\mathbb{A}^1}(\Sigma^1(\mathcal{X}))$ is a $(2n-1)$-simplicial weak-equivalence. The morphism

$$\Omega^1(\Sigma^1(\mathcal{X})) \to \Omega^1(L_{\mathbb{A}^1}(\Sigma(\mathcal{X})))$$

is thus a $2(n-1)$-simplicial weak-equivalence. The composition

$$L_{\mathbb{A}^1}(\mathcal{X}) \to \Omega^1(\Sigma^1(L_{\mathbb{A}^1}(\mathcal{X}))) \to \Omega^1(L_{\mathbb{A}^1}(\Sigma(\mathcal{X})))$$

is thus also simplicially $2(n-1)$-connected.□

4 $\mathbb{A}^1$-coverings, $\pi_1^{\mathbb{A}^1}(\mathbb{P}^n)$ and $\pi_1^{\mathbb{A}^1}(SL_n)$

4.1 $\mathbb{A}^1$-coverings, universal $\mathbb{A}^1$-covering and $\pi_1^{\mathbb{A}^1}$

Definition 4.1 *1) A* simplicial covering *$\mathcal{Y} \to \mathcal{X}$ is a morphism of spaces which has the unique right lifting property with respect to simplicially trivial* *cofibrations. This means that given any commutative square of spaces*

$$\begin{array}{ccc} \mathcal{A} & \to & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{B} & \to & \mathcal{X} \end{array}$$

in which $\mathcal{A} \to \mathcal{B}$ is an simplicially trivial cofibration, there exists one and exactly one morphism $\mathcal{B} \to \mathcal{Y}$ which let the whole diagram commutative.

2) An $\mathbb{A}^1$-covering $\mathcal{Y} \to \mathcal{X}$ *is a morphism of spaces which has the unique right lifting property with respect to $\mathbb{A}^1$-trivial cofibrations*[6]*.*

Lemma 4.2 *A morphism $\mathcal{Y} \to \mathcal{X}$ is a simplicial (resp $\mathbb{A}^1$-) covering if and only if it has the unique right lifting property with respect to any simplicial (resp $\mathbb{A}^1$-) weak equivalence.*

Proof. It suffices to prove that coverings have the unique lifting property with respect to weak-equivalences (both in the simplicial and in the $\mathbb{A}^1$-structure). Pick up a commutative square as in the definition with $\mathcal{A} \to \mathcal{B}$ a weak-equivalence. Factor it as a trivial cofibration $\mathcal{A} \to \mathcal{C}$ and a trivial fibration $\mathcal{C} \to \mathcal{B}$. In this way we clearly reduce to the case $\pi : \mathcal{A} \to \mathcal{B}$ is a trivial fibration. Uniqueness is clear as trivial fibrations are epimorphisms of spaces. Let's prove the existence statement. For both structures the spaces are cofibrant thus one gets a section $i : \mathcal{B} \to \mathcal{A}$ which is of course a trivial cofibration. Now we claim that $f : \mathcal{A} \to \mathcal{Y}$ composed with $i \circ \pi : \mathcal{A} \to \mathcal{A}$ is equal to f. This follows from the unique lifting property applied to i. Thus $f \circ i : \mathcal{B} \to \mathcal{Y}$ is a solution and we are done.□

Remark 4.3 A morphism $Y \to X$ in Sm_k, with X irreducible, is a simplicial covering if and only if Y is a disjoint union of copies of X mapping identically to X.

We will see below that $\mathbb{G}_m$-torsor are examples of $\mathbb{A}^1$-coverings. It could be the case that a morphism in Sm_k is an $\mathbb{A}^1$-covering if and only if it has the right lifting property with respect to only the 0-sections morphisms of the form $U \to \mathbb{A}^1 \times U$, for $U \in Sm_k$.□

[6] remember [39] that this means both a monomorphism and an $\mathbb{A}^1$-weak equivalence

The simplicial theory.

Lemma 4.4 *If $\mathcal{Y} \to \mathcal{X}$ is a simplicial covering for each $x \in X \in Sm_k$ the morphism of simplicial sets $\mathcal{Y}_x \to \mathcal{X}_x$ is a covering of simplicial sets.*

Proof. For $i \in \{0, \dots, n\}$ we let as usual $\Lambda^{n,i} \subset \Delta^n$ be the union of all the faces of Δ^n but the i-th. The inclusion $\Lambda^{n,i} \subset \Delta^n$ is then a simplicial equivalence (of simplicial sets). Now for any $U \in Sm_k$ and any inclusion $\Lambda^{n,i} \subset \Delta^n$ as above, we apply the definition of simplicial covering to $\Lambda^{n,i} \times U \subset \Delta^n \times U$. When U runs over the set of Nisnevich neighborhoods of $x \in X$, this easily implies that $\mathcal{Y}_x \to \mathcal{X}_x$ has the right lifting property with respect to the $\Lambda^{n,i} \subset \Delta^n$, proving our claim.□

For any pointed simplicially connected space $\mathcal{Z}$ there exists a canonical morphism in $\mathcal{H}_{s,\bullet}(k)$ of the form $\mathcal{Z} \to BG$, where G is the fundamental group sheaf $\pi_1(\mathcal{Z})$; this relies on the Postnikov tower [39] for instance. Using now Prop. 1.15 p.130 of *loc. cit.* one gets a canonical isomorphism class $\tilde{\mathcal{Z}} \to \mathcal{Z}$ of G-torsor. Choosing one representative, we may point it by lifting the base point of $\mathcal{Z}$. Now this pointed G-torsor is canonical up to isomorphism. To prove this we first observe that $\tilde{\mathcal{Z}}$ is simplicially 1-connected. Now we claim that any pointed simplicially 1-connected covering $\mathcal{Z}' \to \mathcal{Z}$ over $\mathcal{Z}$ is canonically isomorphic to this one.

Indeed, one first observe that the composition $\mathcal{Z}' \to \mathcal{Z} \to BG \to \mathcal{B}G$ (where $\mathcal{B}G$ means a simplicially fibrant resolution of BG) is homotopically trivial. This follows from the fact that $\mathcal{Z}'$ is 1-connected.

Now let $\mathcal{E}G \to \mathcal{B}G$ be the universal covering of $\mathcal{B}G$ (given by Prop. 1.15 p.130 of *loc. cit.*). Clearly this is also a simplicial fibration, thus $\mathcal{E}G$ is simplicially fibrant. Thus we get the existence of a lifting $\mathcal{Z}' \to \mathcal{E}G$. Now clearly the commutative square

$$\begin{array}{ccc} \mathcal{Z}' & \to & \mathcal{E}G \\ \downarrow & & \downarrow \\ \mathcal{Z} & \to & \mathcal{B}G \end{array}$$

Using the Lemma above, we see that this this square is cartesian on each stalk (by the classical theory), thus cartesian. This proves precisely that $\mathcal{Z}'$ as a covering is isomorphic to $\tilde{\mathcal{Z}}$. But then as a pointed covering, it is canonically isomorphic to $\tilde{\mathcal{Z}} \to \mathcal{Z}$ because the automorphism group of the pointed covering $\tilde{\mathcal{Z}} \to \mathcal{Z}$ is clearly trivial.

Now given any pointed simplicial covering $\mathcal{Z}' \to \mathcal{Z}$ one may consider the connected component of the base point $\mathcal{Z}'^{(0)}$ of $\mathcal{Z}'$. Clearly $\mathcal{Z}'^{(0)} \to \mathcal{Z}$ is still a pointed (simplicial) covering. Now the universal covering (constructed above) of $\mathcal{Z}'^{(0)}$ is clearly also the universal covering of $\mathcal{Z}$. One thus get a unique isomorphism from the pointed universal covering of $\mathcal{Z}$ to that of $\mathcal{Z}'^{(0)}$. The composition $\tilde{\mathcal{Z}} \to \mathcal{Z}'$ is clearly the unique morphism of pointed coverings (use stalks). Thus $\tilde{\mathcal{Z}} \to \mathcal{Z}$ is the universal object in the category of pointed coverings of $\mathcal{Z}$.□

The $\mathbb{A}^1$-theory.

We want to prove the analogue statement in the case of $\mathbb{A}^1$-coverings. We observe that as any simplicially trivial cofibration is an $\mathbb{A}^1$-trivial cofibration, an $\mathbb{A}^1$-covering is in particular a simplicial covering.

Before that we first establish the following Lemma which will provide us with our two basic examples of $\mathbb{A}^1$-coverings.

Lemma 4.5 *1) A G-torsor $\mathcal{Y} \to \mathcal{X}$ with G a strongly $\mathbb{A}^1$-invariant sheaf is an $\mathbb{A}^1$-covering.*

2) Any G-torsor $\mathcal{Y} \to \mathcal{X}$ in the étale topology, with G a finite étale k-group of order prime to the characteristic, is an $\mathbb{A}^1$-covering.

Proof. 1) Recall from [39, Prop. 1.15 p. 130] that the set $H^1(\mathcal{X};G)$ of isomorphism classes (denoted by $P(\mathcal{X};G)$ in *loc. cit.*) of G-torsors over a space $\mathcal{X}$ is in one-to-one correspondence with $[\mathcal{X}, BG]_{\mathcal{H}_s(k)}$ (observe we used the simplicial homotopy category). By the assumption on G, BG is $\mathbb{A}^1$-local. Thus we get now a one-to-one correspondence $H^1(\mathcal{X};G) \cong [\mathcal{X}, BG]_{\mathcal{H}(k)}$. Now let us choose a commutative square like in the definition, with the right vertical morphism a G-torsor. This implies that the pull-back of this G-torsor to $\mathcal{B}$ is trivial when restricted to $\mathcal{A} \subset \mathcal{B}$. By the property just recalled, we get that $H^1(\mathcal{B};G) \to H^1(\mathcal{A};G)$ is a bijection, thus the G-torsor over $\mathcal{B}$ itself is trivial. This fact proves the existence of a section $s : \mathcal{B} \to Y$ of $Y \to X$.

The composition $s \circ (\mathcal{A} \subset \mathcal{B}) : \mathcal{A} \to \mathcal{Y}$ may not be equal to the given top morphism $s_0 : \mathcal{A} \to \mathcal{Y}$ in the square. But then there exists a morphism $g : \mathcal{A} \to G$ with $s = g.s_0$ (by one of the properties of torsors).

But as G is $\mathbb{A}^1$-invariant the restriction map $G(\mathcal{B}) \to G(\mathcal{A})$ is an isomorphism. Let $\tilde{g} : \mathcal{B} \to G$ be the extension of g. Then clearly $\tilde{g}^{-1}.s : \mathcal{B} \to \mathcal{Y}$ is still a section of the torsor, but now moreover its restriction to $\mathcal{A} \subset \mathcal{B}$ is equal to s_0. We have proven the existence of an $s : \mathcal{B} \to \mathcal{Y}$ which makes the diagram commutative. The uniqueness follows from the previous reasoning as the restriction map $G(\mathcal{B}) \to G(\mathcal{A})$ is an isomorphism.

2) Recall from [39, Prop. 3.1 p. 137] that the étale classifying space $B_{et}(G) = R\pi_*(BG)$ is $\mathbb{A}^1$-local. Here $\pi : (Sm_k)_{et} \to (Sm_k)_{Nis}$ is the canonical morphism of sites. But then for any space $\mathcal{X}$, the set $[\mathcal{X}, B_{et}(G)]_{\mathcal{H}(k)} \cong [\mathcal{X}, B_{et}(G)]_{\mathcal{H}_s(k)}$ is by adjunction (see *loc. cit.* § Functoriality p. 61]) in natural bijection with $Hom_{\mathcal{H}_s(Sm_k)_{et}}(\pi^*(\mathcal{X}), BG) \cong H^1_{et}(\mathcal{X}; G)$.

This proves also in that case that the restriction map $H^1_{et}(\mathcal{B}; G) \to H^1_{et}(\mathcal{A}; G)$ is a bijection. We know moreover that G is $\mathbb{A}^1$-invariant as space, thus $G(\mathcal{B}) \to G(\mathcal{A})$ is also an isomorphism. The same reasoning as previously yields the result.□

Example 4.6 1) Any $\mathbb{G}_m$-torsor $\mathcal{Y} \to \mathcal{X}$ is an $\mathbb{A}^1$-covering. Thus any line bundle yields a $\mathbb{A}^1$-covering. In particular, a connected smooth projective k-variety of dimension ≥ 1 as always non trivial $\mathbb{A}^1$-coverings!

2) Any finite étale Galois covering $Y \to X$ between smooth k-varieties whose Galois group has order prime to $char(k)$ is an $\mathbb{A}^1$-covering. More generally, one could show that any finite étale covering between smooth k-varieties which can be covered by a surjective étale Galois covering $Z \to X$ with group a finite étale k-group G of order prime to $char(k)$ is an $\mathbb{A}^1$-covering. In characteristic 0, for instance, any finite étale covering is an $\mathbb{A}^1$-covering.□

Lemma 4.7 *1) Any pull-back of an $\mathbb{A}^1$-covering is an $\mathbb{A}^1$-covering.*

2) The composition of two $\mathbb{A}^1$-coverings is a $\mathbb{A}^1$-covering.

3) Any $\mathbb{A}^1$-covering is an $\mathbb{A}^1$-fibration in the sense of [39].

4) A morphism $\mathcal{Y}_1 \to \mathcal{Y}_2$ of $\mathbb{A}^1$-coverings $\mathcal{Y}_i \to \mathcal{X}$ which is an $\mathbb{A}^1$-weak equivalence is an isomorphism.

Proof. Only the last statement requires an argument. It follows from Lemma 4.2: applying it to $\mathcal{Y}_1 \to \mathcal{Y}_2$ one first get a retraction $\mathcal{Y}_2 \to \mathcal{Y}_1$ and to check that this retraction composed with $\mathcal{Y}_1 \to \mathcal{Y}_2$ is the identity of $\mathcal{Y}_2$ one uses once more the Lemma 4.2.$\square$

We now come to the main result of this section:

Theorem 4.8 *Any pointed $\mathbb{A}^1$-connected space $\mathcal{X}$ admits a universal pointed $\mathbb{A}^1$-covering $\tilde{\mathcal{X}} \to \mathcal{X}$ in the category of pointed coverings of $\mathcal{X}$. It is (up to unique isomorphism) the unique pointed $\mathbb{A}^1$-covering whose source is $\mathbb{A}^1$-simply connected. It is a $\pi_1^{\mathbb{A}^1}(\mathcal{X})$-torsor over $\mathcal{X}$ and the canonical morphism*

$$\pi_1^{\mathbb{A}^1}(\mathcal{X}) \to Aut_{\mathcal{X}}(\tilde{\mathcal{X}})$$

is an isomorphism.

Proof. Let $\mathcal{X}$ be a pointed $\mathbb{A}^1$-connected space. Let $\mathcal{X} \to L_{\mathbb{A}^1}(\mathcal{X})$ be its $\mathbb{A}^1$-localization. Let $\tilde{\mathcal{X}}_{\mathbb{A}^1}$ be the universal covering of $L_{\mathbb{A}^1}(\mathcal{X})$ in the simplicial meaning. It is a $\pi_1^{\mathbb{A}^1}(\mathcal{X})$-torsor by construction. From Lemma 4.5 $\tilde{\mathcal{X}}_{\mathbb{A}^1} \to L_{\mathbb{A}^1}(\mathcal{X})$ is thus also an $\mathbb{A}^1$-covering (as $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ is strongly $\mathbb{A}^1$-invariant. Let $\tilde{\mathcal{X}} \to \mathcal{X}$ be its pull back to $\mathcal{X}$. This is a pointed $\pi_1^{\mathbb{A}^1}(\mathcal{X})$-torsor and still a pointed $\mathbb{A}^1$-covering. We claim it is the universal pointed $\mathbb{A}^1$-covering of $\mathcal{X}$.

Next we observe that $\tilde{\mathcal{X}}$ is $\mathbb{A}^1$-simply connected. This follows from the left properness property of the $\mathbb{A}^1$-model category structure on the category of spaces [39] that $\tilde{\mathcal{X}} \to \tilde{\mathcal{X}}_{\mathbb{A}^1}$ is an $\mathbb{A}^1$-weak equivalence.

Now we prove the universal property. Let $\mathcal{Y} \to \mathcal{X}$ be a pointed $\mathbb{A}^1$-covering. Let $\mathcal{Y}^{(0)} \subset \mathcal{Y}$ the inverse image of (the image of) the base point in $\pi_0^{\mathbb{A}^1}(\mathcal{Y})$. We claim (like in the above simplicial case) that $\mathcal{Y}^{(0)} \to \mathcal{X}$ is still an $\mathbb{A}^1$-covering. It follows easily from the fact that an $\mathbb{A}^1$-trivial cofibration induces an isomorphism on $\pi_0^{\mathbb{A}^1}$. In this way we reduce to proving the universal property for pointed $\mathbb{A}^1$-coverings $\mathcal{Y} \to \mathcal{X}$ with $\mathcal{Y}$ also $\mathbb{A}^1$-connected.

By Lemma 4.9 below there exists a cartesian square of pointed spaces

$$\begin{array}{ccc} \mathcal{Y} & \to & \mathcal{Y}' \\ \downarrow & & \downarrow \\ \mathcal{X} & \to & L_{\mathbb{A}^1}(\mathcal{X}) \end{array}$$

with $\mathcal{Y}' \to L_{\mathbb{A}^1}(\mathcal{X})$ a pointed $\mathbb{A}^1$-covering of $L_{\mathbb{A}^1}(\mathcal{X})$. By the above theory of simplicial coverings, there exists a unique morphism of pointed coverings

$$\begin{array}{ccc} \tilde{\mathcal{X}}_{\mathbb{A}^1} & \to & \mathcal{Y}' \\ \downarrow & & \downarrow \\ L_{\mathbb{A}^1}(\mathcal{X}) & = & L_{\mathbb{A}^1}(\mathcal{X}) \end{array}$$

Pulling-back this morphism to $\mathcal{X}$ yields a pointed morphism of $\mathbb{A}^1$-coverings

$$\begin{array}{ccc} \tilde{\mathcal{X}} & \to & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{X} & = & \mathcal{X} \end{array}$$

Now it suffices to check that there is only one such morphism. Let f_1 and f_2 be morphisms $\tilde{\mathcal{X}} \to \mathcal{Y}$ of pointed $\mathbb{A}^1$-coverings of $\mathcal{X}$. We want to prove they are equal. We again apply Lemma 4.9 below to each f_i and get a cartesian square of pointed spaces

$$\begin{array}{ccc} \tilde{\mathcal{X}} & \stackrel{\tilde{f}_i}{\to} & \tilde{\mathcal{X}}_i' \\ \downarrow & & \downarrow \\ \mathcal{Y} & \to & \mathcal{Y}' \end{array}$$

in which $\tilde{\mathcal{X}} \to \tilde{\mathcal{X}}_i$ is an $\mathbb{A}^1$-weak equivalence. As a consequence the pointed $\mathbb{A}^1$-coverings $\tilde{\mathcal{X}}_i' \to \mathcal{Y}'$ to the $\mathbb{A}^1$-local space $\mathcal{Y}'$ are simply $\mathbb{A}^1$-connected and are thus both the simplicial universal pointed covering of $\mathcal{Y}'$ (and of $L_{\mathbb{A}^1}(\mathcal{X})$): let $\phi : \tilde{\mathcal{X}}_1' \cong \tilde{\mathcal{X}}_2'$ be the canonical isomorphism of pointed coverings. To check $f_1 = f_2$, it clearly suffices to check that $\phi \circ \tilde{f}_1 = \tilde{f}_2$. But there exists $\psi : \tilde{\mathcal{X}} \to \pi_1^{\mathbb{A}^1}(\mathcal{X})$ such that $\tilde{f}_2 = \psi.(\phi \circ \tilde{f}_1)$. But as $\tilde{\mathcal{X}}$ is $\mathbb{A}^1$-connected, ψ is constant, *i.e.* factor as $\tilde{\mathcal{X}} \to * \to \pi_1^{\mathbb{A}^1}(\mathcal{X})$. But as all the morphisms are pointed, that constant $* \to \pi_1^{\mathbb{A}^1}(\mathcal{X})$ must be the neutral element so that $\phi \circ \tilde{f}_1 = \tilde{f}_2$.

We observe that if $\mathcal{Y} \to \mathcal{X}$ is a pointed $\mathbb{A}^1$-covering with $\mathcal{Y}$ simply $\mathbb{A}^1$-connected, the unique morphism $\tilde{\mathcal{X}} \to \mathcal{Y}$ is an $\mathbb{A}^1$-weak equivalence and thus an isomorphism by Lemma 4.7 4).

Finally it only remains to prove the statement concerning the morphism

$$\pi_1^{\mathbb{A}^1}(\mathcal{X}) \to Aut_{\mathcal{X}}(\tilde{\mathcal{X}})$$

Here the right hand side means the sheaf of groups on Sm_k which to U associates the group of automorphisms $Aut_{\mathcal{X}}(\tilde{\mathcal{X}})(U)$ of the covering $\tilde{\mathcal{X}} \times U \to \mathcal{X} \times U$. We observe that if two automorphisms $\phi_i \in Aut_{\mathcal{X}}(\tilde{\mathcal{X}})(U)$, $i \in \{1, 2\}$, coincide on the base-point section $U \to \tilde{\mathcal{X}} \times U$ then $\phi_1 = \phi_2$. Indeed as $\tilde{\mathcal{X}} \times U \to \mathcal{X} \times U$ is a $\pi_1^{\mathbb{A}^1}(\mathcal{X})$-torsor, there is $\alpha : \tilde{\mathcal{X}} \times U \to \pi_1^{\mathbb{A}^1}(\mathcal{X})$ with $\phi_2 = \alpha.\phi_1$. But $\pi_0^{\mathbb{A}^1}(\tilde{\mathcal{X}} \times U) = \pi_0^{\mathbb{A}^1}(U)$ and α factors through $\pi_0^{\mathbb{A}^1}(U) \to \pi_1^{\mathbb{A}^1}(\mathcal{X})$. As the composition of α with the base-point section $U \to \tilde{\mathcal{X}} \times U$ is the neutral element, we conclude that α is the neutral element and $\phi_1 = \phi_2$.

This first shows that the above morphism is a monomorphism. Let $\phi \in Aut_{\mathcal{X}}(\tilde{\mathcal{X}})(U)$. Composing ϕ with the base-point section $U \to \tilde{\mathcal{X}} \times U$ we get $\psi \in \pi_1^{\mathbb{A}^1}(\mathcal{X})(U)$. But the automorphisms ϕ and ψ coincide by construction on the base-point section. Thus they are equal and our morphism is also onto. The Theorem is proven.□

Lemma 4.9 *Let $\mathcal{Y} \to \mathcal{X}$ be a pointed $\mathbb{A}^1$-covering between pointed $\mathbb{A}^1$-conected spaces. Then for any $\mathbb{A}^1$-weak equivalence $\mathcal{X} \to \mathcal{X}'$ any there exists a cartesian square of spaces*

$$\begin{array}{ccc} \mathcal{Y} & \to & \mathcal{Y}' \\ \downarrow & & \downarrow \\ \mathcal{X} & \to & \mathcal{X}' \end{array}$$

in which the right vertical morphism is an $\mathbb{A}^1$-covering (and thus the top horizontal morphism an $\mathbb{A}^1$-weak equivalence).

Proof. Let $\mathcal{X}' \to L_{\mathbb{A}^1}(\mathcal{X}')$ be the $\mathbb{A}^1$-localization of $\mathcal{X}'$. As by construction, $L_{\mathbb{A}^1}(-)$ is a functor on spaces we get a commutative square

$$\begin{array}{ccc} \mathcal{Y} & \to & L_{\mathbb{A}^1}(\mathcal{Y}) \\ \downarrow & & \downarrow \\ \mathcal{X} & \to & L_{\mathbb{A}^1}(\mathcal{X}') \end{array}$$

in which the horizontal arrows are $\mathbb{A}^1$-weak equivalences. As the left vertical arrow is an $\mathbb{A}^1$-fibration (by Lemma 4.7) with $\mathbb{A}^1$-homotopy fiber equal to the fiber $\Gamma \subset \mathcal{Y}$, which is an $\mathbb{A}^1$-invariant sheaf, thus is $\mathbb{A}^1$-local, the $\mathbb{A}^1$-homotopy fiber of the pointed morphism $L_{\mathbb{A}^1}(\mathcal{Y}) \to L_{\mathbb{A}^1}(\mathcal{X}')$ is $\mathbb{A}^1$-equivalent to the previous one (because the square is obviously $\mathbb{A}^1$-homotopy cartesian).

As both $L_{\mathbb{A}^1}(\mathcal{Y})$ and $L_{\mathbb{A}^1}(\mathcal{X}')$ are $\mathbb{A}^1$-fibrant and (simplicially) connected, this means (using the theory of simplicial coverings for $L_{\mathbb{A}^1}(\mathcal{X}')$) that there exists a commutative square

$$\begin{array}{ccc} L_{\mathbb{A}^1}(\mathcal{Y}) & \to & \mathcal{Y}' \\ \downarrow & & \downarrow \\ L_{\mathbb{A}^1}(\mathcal{X}') & = & L_{\mathbb{A}^1}(\mathcal{X}') \end{array}$$

in which $\mathcal{Y}' \to L_{\mathbb{A}^1}(\mathcal{X}')$ is an ($\mathbb{A}^1$-)covering and $L_{\mathbb{A}^1}(\mathcal{Y}) \to \mathcal{Y}'$ an ($\mathbb{A}^1$-)weak equivalence.

This $\mathbb{A}^1$-homotopy cartesian square induces a commutative square

$$\begin{array}{ccc} \mathcal{Y} & \to & \mathcal{Y}" \\ \downarrow & & \downarrow \\ \mathcal{X} & = & \mathcal{X} \end{array} \tag{4.1}$$

in which both vertical morphisms are $\mathbb{A}^1$-coverings and the top horizontal morphism is an $\mathbb{A}^1$-weak equivalence (by the properness of the $\mathbb{A}^1$-model structure [39]), where $\mathcal{Y}"$ is the fiber product $\mathcal{Y}' \times_{L_{\mathbb{A}^1}(\mathcal{X}')} \mathcal{X}$. By Lemma 4.7 $\mathcal{Y} \to \mathcal{Y}"$ is an isomorphism. This finishes our proof as $\mathcal{Y}"$ is clearly the pull-back of an $\mathbb{A}^1$-covering of $\mathcal{X}'$ because $\mathcal{X} \to L_{\mathbb{A}^1}(\mathcal{X}')$ factor through $\mathcal{X} \to \mathcal{X}'$.□

Remark 4.10 Let us denote by $Cov_{\mathbb{A}^1}(\mathcal{X})$ the category of $\mathbb{A}^1$-coverings of a fixed pointed $\mathbb{A}^1$-connected space $\mathcal{X}$. The fiber Γ_{x_0} of an $\mathbb{A}^1$-covering $\mathcal{Y} \to \mathcal{X}$ over the base point x_0 is clearly an $\mathbb{A}^1$-invariant sheaf of sets. One may define a natural right action of $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ on $\Gamma_{x_0}(\mathcal{Y} \to \mathcal{X})$ and it can be shown that the induced functor Γ_{x_0} from $Cov_{\mathbb{A}^1}(\mathcal{X})$ to the category of $\mathbb{A}^1$-invariant sheaves with a right action of $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ is an equivalence of categories.

When $\mathcal{X}$ is an arbitrary space, this correspondence can be extended to an equivalence between the category $Cov_{\mathbb{A}^1}(\mathcal{X})$ and some category of "functor-sheaves" defined on the fundamental $\mathbb{A}^1$-groupoid of $\mathcal{X}$.□

We end this section by mentionning the (easy version of the) Van-Kampen Theorem.

Remark 4.11 The trick to deduce this kind of results is to observe that for any pointed connected space $\mathcal{X}$, the map $[\mathcal{X}, BG]_{\mathcal{H}_{s,\bullet}(k)} \to Hom_{\mathcal{G}r}(\pi_1(\mathcal{X}), G)$ is a bijection. This follows as usual by considering the functoriality of the Postnikov tower [39]. But then if G is a strongly $\mathbb{A}^1$-invariant sheaf, we get in the same way:

$$[\mathcal{X}, BG]_{\mathcal{H}_\bullet(k)} \to Hom_{\mathcal{G}r_{\mathbb{A}^1}}(\pi_1(\mathcal{X}), G)$$

where $\mathcal{G}r_{\mathbb{A}^1}$ denotes the category of strongly $\mathbb{A}^1$-invariant sheaves of groups. It follows at once that the inclusion $\mathcal{G}r_{\mathbb{A}^1} \subset \mathcal{G}r$ admits a left adjoint $G \mapsto G_{\mathbb{A}^1}$, with $G_{\mathbb{A}^1} := \pi_1^{\mathbb{A}^1}(BG) = \pi_1(L_{\mathbb{A}^1}(BG))$. As a consequence, $\mathcal{G}r_{\mathbb{A}^1}$ admits all colimits. For instance we get the existence of sums denoted by $*^{\mathbb{A}^1}$ in $\mathcal{G}r_{\mathbb{A}^1}$: if G_i is a family of strongly $\mathbb{A}^1$-invariant sheaves, their sum $*_i^{\mathbb{A}^1} G_i$ is $(*_i G_i)_{\mathbb{A}^1}$ where $*$ means the usual sum in $\mathcal{G}r$.

Theorem 4.12 *Let X be an $\mathbb{A}^1$-connected pointed smooth scheme. Let $\{U_i\}_i$ be a an open covering of X by $\mathbb{A}^1$-connected open subschemes which contains the base point. Assume furthermore that each intersection $U_i \cap U_j$ is still $\mathbb{A}^1$-connected. Then for any strongly $\mathbb{A}^1$-invariant sheaf of groups G, the following diagram*

$$*_{i,j}^{\mathbb{A}^1} \pi_1^{\mathbb{A}^1}(U_i \cap U_j) \rightrightarrows *_i^{\mathbb{A}^1} \pi_1^{\mathbb{A}^1}(U_i) \to \pi_1^{\mathbb{A}^1}(X) \to *$$

is right exact in $\mathcal{G}r_{\mathbb{A}^1}$.

Proof. We let $\check{C}(\mathcal{U})$ the simplicial space associated to the covering U_i of X (the Čech object of the covering). By definition, $\check{C}(\mathcal{U}) \to X$ is a simplicial weak equivalence. Thus Remark 4.11, it follows that for any $G \in \mathcal{G}r_{\mathbb{A}^1}$

$$Hom_{\mathcal{G}r_{\mathbb{A}^1}}(\pi_1(X), G) = [\check{C}(\mathcal{U}), BG]_{\mathcal{H}_{s,\bullet}(k)}$$

Now the usual skeletal filtration of $\check{C}(\mathcal{U})$ easily yields the fact that the obvious diagram (of sets)

$$Hom_{\mathcal{G}r}(\pi_1^{\mathbb{A}^1}(\check{C}(\mathcal{U})), G) \to \Pi_i Hom_{\mathcal{G}r}(\pi_1^{\mathbb{A}^1}(U_i), G) \rightrightarrows \Pi_{i,j} Hom_{\mathcal{G}r}(\pi_1^{\mathbb{A}^1}(U_i \cap U_j), G)$$

is exact. Putting all these together we obtain our claim.□

4.2 Basic computation: $\pi_1^{\mathbb{A}^1}(\mathbb{P}^n)$ and $\pi_1^{\mathbb{A}^1}(SL_n)$ for $n \geq 2$

The following is the easiest application of the preceding results:

Theorem 4.13 *For $n \geq 2$ the canonical $\mathbb{G}_m$-torsor*

$$\mathbb{G}_m - (\mathbb{A}^{n+1} - \{0\}) \to \mathbb{P}^n$$

is the universal $\mathbb{A}^1$-covering of $\mathbb{P}^n$. This defines a canonical isomorphism $\pi_1^{\mathbb{A}^1}(\mathbb{P}^n) \cong \mathbb{G}_m$.

Proof. For $n \geq 2$, the pointed space $\mathbb{A}^{n+1} - \{0\}$ is $\mathbb{A}^1$-simply connected by Theorem 3.38. We now conclude by Theorem 4.8.□

For $n = 1$, $\mathbb{A}^2 - \{0\}$ is no longer 1-$\mathbb{A}^1$-connected. We now compute $\pi_1^{\mathbb{A}^1}(\mathbb{A}^2 - \{0\})$. As $SL_2 \to \mathbb{A}^2 - \{0\}$ is an $\mathbb{A}^1$-weak equivalence, $\pi_1^{\mathbb{A}^1}(\mathbb{A}^2 - \{0\}) \cong \pi_1^{\mathbb{A}^1}(SL_2)$. Now, the $\mathbb{A}^1$-fundamental sheaf of groups $\pi_1^{\mathbb{A}^1}(G)$ of a group-space G is always a sheaf of abelian groups by the classical argument. Here we mean by "group-space" a group object in the category of spaces, that is to say a simplicial sheaf of groups on Sm_k.

By the Hurewicz Theorem and Theorem 3.25 we get canonical isomorphisms $\pi_1^{\mathbb{A}^1}(SL_2) = H_1^{\mathbb{A}^1}(SL_2) = H_1^{\mathbb{A}^1}(\mathbb{A}^2 - \{0\}) = \underline{\mathbf{K}}_2^{MW}$.

Finally the classical argument also yields:

Lemma 4.14 *Let G be a group-space which is $\mathbb{A}^1$-connected. Then there exists a unique group structure on the pointed space $\tilde{G}$ for which the $\mathbb{A}^1$-covering $\tilde{G} \to G$ is an (epi-)morphism of group-spaces. The kernel is central and canonically isomorphic to $\pi_1^{\mathbb{A}^1}(\tilde{G})$.*

Altogether we have obtained:

Theorem 4.15 *The universal $\mathbb{A}^1$-covering of SL_2 given by Theorem 4.8 admits a group structure and we get in this way a central extension of sheaves of groups*

$$0 \to \underline{\mathbf{K}}_2^{MW} \to \tilde{SL}_2 \to SL_2 \to 1$$

Remark 4.16 Over an infinite field, this extension is also a central extension in the Zariski topology by the Theorem 1.26. In fact the results of the Appendix show that it is always the case.

This central extension can be constructed in the following way:

Lemma 4.17 *Let $B(SL_2)$ denote the simplicial classifying space of SL_2. Then there exists a unique $\mathcal{H}_{s,\bullet}(k)$-morphism*

$$e_2 : B(SL_2) \to K(\underline{\mathbf{K}}_2^{MW}, 2)$$

which composed with $\Sigma(SL_2) \subset B(SL_2)$ gives the canonical cohomology class $\Sigma(SL_2) \cong \Sigma(\mathbb{A}^2 - \{0\}) \to K(\underline{\mathbf{K}}_2^{MW}, 2)$.

The central extension of SL_2 associated with this element of $H^2(SL_2; \underline{\mathbf{K}}_2^{MW})$ is canonically isomorphic to the central extension of Theorem 4.15.

Proof. We use the skeletal filtration F_s of the classifying space BG; it has the property that (simplicially) $F_s/F_{s-1} \cong \Sigma^s(G^{\wedge s})$. Clearly now, using the long exact sequences in cohomology with coefficients in $\underline{\mathbf{K}}_2^{MW}$ one sees that the restriction:

$$H^2(B(SL_2); \underline{\mathbf{K}}_2^{MW}) \to H^2(F_1; \underline{\mathbf{K}}_2^{MW}) = H^2(\Sigma(SL_2); \underline{\mathbf{K}}_2^{MW})$$

is an isomorphism.

Now it is well-known that such an element in $H^2(B(SL_2); \underline{\mathbf{K}}_2^{MW})$ corresponds to a central extension of sheaves as above: just take the pointed simplicial homotopy fiber Γ of (a representative of) the previous morphism $B(SL_2) \to K(\underline{\mathbf{K}}_2^{MW}, 2)$. Using the long exact homotopy sequence of simplicial homotopy sheaves of this fibration yields the required central extension:

$$0 \to \underline{\mathbf{K}}_2^{MW} \to \pi_1(\Gamma) \to SL_2 \to 0$$

To check it is the universal $\mathbb{A}^1$-covering for SL_2, just observe that the map $B(SL_2) \to K(\underline{\mathbf{K}}_2^{MW}, 2)$ is onto on $\pi_2^{\mathbb{A}^1}$ as the map $\Sigma(SL_2) \cong \Sigma(\mathbb{A}^2 - \{0\}) \to K(\underline{\mathbf{K}}_2^{MW}, 2)$ is already onto (actually an isomorphism) on $\pi_2^{\mathbb{A}^1}$. Now by Theorem 3.50 the simplicial homotopy fiber sequence is also an $\mathbb{A}^1$-fiber homotopy sequence. Then the long exact homotopy sequence in $\mathbb{A}^1$-homotopy sheaves this time shows that Γ is simply 2-$\mathbb{A}^1$-connected. Thus the group-object $\pi_1(\Gamma)$ is simply $\mathbb{A}^1$-connected thus is canonically isomorphic to $\tilde{SL}_2$.□

Remark 4.18 1) As a $\underline{\mathbf{K}}_2^{MW}$-torsor (forgetting the group structure) $\tilde{SL}_2$ can easily be described as follows. We use the morphism $SL_2 \to \mathbb{A}^2 - \{0\}$. It is thus sufficient to describe a $\underline{\mathbf{K}}_2^{MW}$-torsor over $\mathbb{A}^2 - \{0\}$. We use the open covering of $\mathbb{A}^2 - \{0\}$ by the two obvious open subsets $\mathbb{G}_m \times \mathbb{A}^1$ and $\mathbb{A}^1 \times \mathbb{G}_m$. Their intersection is exactly $\mathbb{G}_m \times \mathbb{G}_m$. The tautological symbol $\mathbb{G}_m \times \mathbb{G}_m \to \underline{\mathbf{K}}_2^{MW}$ (see Section 2.3) defines a 1-cocycle on $\mathbb{A}^2 - \{0\}$ with values in $\underline{\mathbf{K}}_2^{MW}$ and thus an $\underline{\mathbf{K}}_2^{MW}$-torsor. The pull-back of this torsor to SL_2 is $\tilde{SL}_2$. It suffices to check that it is simply $\mathbb{A}^1$-connected. This follows in the same way as in the previous proof from the fact the $\mathcal{H}_\bullet(k)$-morphism $\mathbb{A}^2 - \{0\} \to K(\underline{\mathbf{K}}_2^{MW}, 1)$ induced by the previous 1-cocycle is an isomorphism on $\pi_1^{\mathbb{A}^1}$.

2) For any SL_2-torsor ξ over a smooth scheme X (or equivalently a rank two vector bundle ξ over X with a trivialization of $\Lambda^2(\xi)$) the composition of the $\mathcal{H}_{s,\bullet}(k)$-morphisms $X \to B(SL_2)$ classifying ξ and of $e_2 : B(SL_2) \to K(\underline{\mathbf{K}}_2^{MW}, 2)$ defines an element $e_2(\xi) \in H^2(X; \underline{\mathbf{K}}_2^{MW})$; this can be shown to coincide with the Euler class of ξ defined in [3], see [36].□

The computation of $\pi_1^{\mathbb{A}^1}(SL_n)$, $n \geq 3$

We first observe:

Lemma 4.19 *1) For $n \geq 3$, the inclusion $SL_n \subset SL_{n+1}$ induces an isomorphism*

$$\pi_1^{\mathbb{A}^1}(SL_n) \cong \pi_1^{\mathbb{A}^1}(SL_{n+1})$$

2) The inclusion $SL_2 \subset SL_3$ induces an epimorphism

$$\pi_1^{\mathbb{A}^1}(SL_2) \twoheadrightarrow \pi_1^{\mathbb{A}^1}(SL_3)$$

Proof. We denote by $SL'_n \subset SL_{n+1}$ the subgroup formed by the matrices of the form

$$\begin{pmatrix} 1 & 0 \dots 0 \\ ? & \\ \vdots & M \\ ? & \end{pmatrix}$$

with $M \in SL_n$. Observe that the group homomorphism $SL'_n \to SL_n$ is an $\mathbb{A}^1$-weak equivalence: indeed the inclusion $SL_n \subset SL'_n$ shows SL'_n is the semi-direct product of SL_n and $\mathbb{A}^n$ so that as a space SL'_n is the product $\mathbb{A}^n \times SL_n$.

The group SL'_n is the isotropy subgroup of $(1, 0, \ldots, 0)$ under the right action of SL_{n+1} on $\mathbb{A}^{n+1} - \{0\}$. The following diagram

$$SL'_n - SL_{n+1} \to \mathbb{A}^{n+1} - \{0\} \tag{4.2}$$

is thus an SL'_n-Zariski torsor over $\mathbb{A}^{n+1} - \{0\}$, where the map $SL_{n+1} \to \mathbb{A}^{n+1} - \{0\}$ assigns to a matrix its first horizontal line.

By Theorem 3.53, and our computations, the simplicial fibration sequence (4.2) is still an $\mathbb{A}^1$-fibration sequence. The associated long exact sequence of $\mathbb{A}^1$-homotopy sheaves, together with the fact that $\mathbb{A}^{n+1} - \{0\}$ is $(n-1)$-$\mathbb{A}^1$-connected and that $SL_n \subset SL'_n$ an $\mathbb{A}^1$-weak equivalence implies the claim.□

Now we may state the following result which implies the point 2) of Theorem 22:

Theorem 4.20 *The canonical isomorphism $\pi_1^{\mathbb{A}^1}(SL_2) \cong \underline{\mathbf{K}}_2^{MW}$ induces through the inclusions $SL_2 \to SL_n$, $n \geq 3$, an isomorphism*

$$\underline{\mathbf{K}}_2^M = \underline{\mathbf{K}}_2^{MW}/\eta \cong \pi_1^{\mathbb{A}^1}(SL_n) = \pi_1^{\mathbb{A}^1}(SL_\infty) = \pi_1^{\mathbb{A}^1}(GL_\infty)$$

Remark 4.21 Let $\mathbb{A}^3 - \{0\} \to B(SL'_2)$ be the morphism in $\mathcal{H}_{s,\bullet}(k)$ which classifies the SL'_2-torsor (4.2) over $\mathbb{A}^3 - \{0\}$. Applying $\pi_2^{\mathbb{A}^1}$ yields a morphism:

$$\underline{\mathbf{K}}_3^{MW} = \pi_2^{\mathbb{A}^1}(\mathbb{A}^3 - \{0\}) \to \pi_2^{\mathbb{A}^1}(B(SL'_2)) \cong \pi_2^{\mathbb{A}^1}(B(SL_2)) = \pi_1^{\mathbb{A}^1}(SL_2) = \underline{\mathbf{K}}_2^{MW}$$

This morphism can be shown in fact to be the Hopf morphism η in Milnor-Witt K-theory sheaves. The proof we give below only gives that this morphism is η up to multiplication by a unit in $W(k)$.□

Remark 4.22 We will use in the proof the "second Chern class morphism", a canonical $\mathcal{H}_\bullet(k)$-morphism

$$c_2 : B(GL_\infty) \to K(\underline{\mathbf{K}}_2^M, 2)$$

more generally the n-th Chern class morphism $c_n : B(GL_\infty) \to K(\underline{\mathbf{K}}_n^M, n)$ is defined as follows: in $\mathcal{H}_\bullet(k)$, $B(GL_\infty)$ is canonically isomorphic to the infinite grassmanian $\mathbb{G}r_\infty$ [39]. This space is the filtering colimit of the finite Grassmanian $\mathbb{G}r_{m,i}$ [*loc. cit.* p. 138]. But clearly, $[\mathbb{G}r_{m,i}, K(\underline{\mathbf{K}}_n^M, n)]_{\mathcal{H}_\bullet(k)}$ is the cohomology group $H^n(\mathbb{G}r_{m,i}; \underline{\mathbf{K}}_n^M, n)$. This group is isomorphic to the n-th Chow group $CH^n(\mathbb{G}r_{m,i})$ by Rost [44], and we let $c_n \in [\mathbb{G}r_{m,i}, K(K_n^M, n)]_{\mathcal{H}_\bullet(k)}$ denote the n-th Chern class of the tautological rank m vector over bundle on $\mathbb{G}r_{m,i}$ [15]. As the Chow groups of the Grassmanians stabilize *loc. cit.*, the Milnor exact sequence gives a canonical element $c_n \in [B(GL_\infty), K(\underline{\mathbf{K}}_n^M, n)]_{\mathcal{H}_\bullet(k)}$.

Form this definition it is easy to check that c_2 is the unique morphism $B(GL_\infty) \to K(\underline{\mathbf{K}}_2^M, 2)$ whose composite with $\Sigma(GL_2) \to B(GL_2) \to B(GL_\infty) \to K(\underline{\mathbf{K}}_2^M, 2))$ is the canonical composition $\Sigma(GL_2) \to \Sigma(\mathbb{A}^2 - \{0\}) \to K(\underline{\mathbf{K}}_2^{MW}, 2)) \to K(\underline{\mathbf{K}}_2^M, 2))$.□

Proof of Theorem 4.20. Lemma 4.19 implies that we only have to show that the epimorphism

$$\pi : \underline{\mathbf{K}}_2^{MW} = \pi_1^{\mathbb{A}^1}(SL_2) \twoheadrightarrow \pi_1^{\mathbb{A}^1}(SL_3)$$

has exactly has kernel the image $\mathcal{I}(\eta) \subset \underline{\mathbf{K}}_2^{MW}$ of $\eta : \underline{\mathbf{K}}_3^{MW} \to \underline{\mathbf{K}}_2^{MW}$.

The long exact sequence of homotopy sheaves of the $\mathbb{A}^1$-fibration sequence 4.2 $SL_2' - SL_3 \to \mathbb{A}^3 - \{0\}$ and the $\mathbb{A}^1$-weak equivalence $SL_2' \to SL_2$ provides an exact sequence

$$\underline{\mathbf{K}}_3^{MW} = \pi_1^{\mathbb{A}^1}(\mathbb{A}^3 - \{0\}) \to \underline{\mathbf{K}}_2^{MW} = \pi_1^{\mathbb{A}^1}(SL_2) \twoheadrightarrow \pi_1^{\mathbb{A}^1}(SL_3) \to 0$$

But from Remark **??** and the fact that $\underline{\mathbf{K}}_n^{MW}$ is the free strongly $\mathbb{A}^1$-invariant sheaf on $\mathbb{G}_m$ we get that the obvious morphism

$$Hom_{\mathcal{A}b(k)}(\underline{\mathbf{K}}_3^{MW}, \underline{\mathbf{K}}_2^{MW}) \to \underline{\mathbf{K}}_{-1}^{MW}(k) = W(k)$$

is an isomorphism. Thus this means that the connecting homomorphism $\underline{\mathbf{K}}_3^{MW} \to \underline{\mathbf{K}}_2^{MW}$ is a multiple of η. This proves the inclusion $Ker(\pi) \subset \mathcal{I}(\eta)$. Now the morphism $\pi_1^{\mathbb{A}^1}(SL_2) \twoheadrightarrow \pi_1^{\mathbb{A}^1}(SL_3) \to \underline{\mathbf{K}}_2^M$ induced by the second Chern class (*cf* remark 4.22) is the obvious projection $\underline{\mathbf{K}}_2^{MW} \to \underline{\mathbf{K}}_2^{MW}/\eta = \underline{\mathbf{K}}_2^M$. This shows the converse inclusion.□

4.3 The computation of $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$

We recall from [39] that there is a canonical $\mathcal{H}_\bullet(k)$-isomorphism $\mathbb{P}^1 \cong \Sigma(\mathbb{G}_m)$ induced by the covering of $\mathbb{P}^1$ by its two standard $\mathbb{A}^1$'s intersecting to $\mathbb{G}_m$.

Thus to compute $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ is the same thing as to compute $\pi_1^{\mathbb{A}^1}(\Sigma(\mathbb{G}_m))$.

Let us denote by $Shv_\bullet$ the category of sheaves of pointed sets on Sm_k. For any $\mathcal{S} \in Shv_\bullet$, we denote by $\theta_{\mathcal{S}} : \mathcal{S} \to \pi_1^{\mathbb{A}^1}(\Sigma(\mathcal{S}))$ the canonical $Shv_\bullet$-morphism obtained by composing $\mathcal{S} \to \pi_1(\Sigma(\mathcal{S}))$ and $\pi_1(\Sigma(\mathcal{S})) \to \pi_1^{\mathbb{A}^1}(\Sigma(\mathcal{S}))$.

Lemma 4.23 *The morphism $\mathcal{S}$ induces for any strongly $\mathbb{A}^1$-invariant sheaf of groups G a bijection $Hom_{\mathcal{G}r}(\pi_1^{\mathbb{A}^1}(\Sigma(\mathcal{S})), G) \cong Hom_{Shv_\bullet}(\mathcal{S}, G)$.*

Proof. As the classifying space BG is $\mathbb{A}^1$-local the map $[\Sigma(\mathcal{S}), BG]_{\mathcal{H}_{s,\bullet}(k)} \to [\Sigma(\mathcal{S}), BG]_{\mathcal{H}_\bullet(k)}$ is a bijection.

Now the obvious map $[\Sigma(\mathcal{S}), BG]_{\mathcal{H}_{s,\bullet}(k)} \to Hom_{\mathcal{G}r}(\pi_1^{\mathbb{A}^1}(\Sigma(\mathcal{S})), G)$ given by the functor π_1 is bijective, see Remark 4.11.

The classical adjunction $[\Sigma(\mathcal{S}), BG]_{\mathcal{H}_{s,\bullet}(k)} \cong [\mathcal{S}, \Omega^1(BG)]_{\mathcal{H}_{s,\bullet}(k)}$ and the canonical $\mathcal{H}_{s,\bullet}(k)$-isomorphism $G \cong \Omega^1(BG)$ are checked to provide the required bijection.$\square$

The previous result can be expressed by saying that the sheaf of groups $F_{\mathbb{A}^1}(\mathcal{S}) := \pi_1^{\mathbb{A}^1}(\Sigma(\mathcal{S}))$ is the "free strongly $\mathbb{A}^1$-invariant" sheaf of groups on the pointed sheaf of sets $\mathcal{S}$. In the sequel we will simply denote, for $n \geq 1$, by $F_{\mathbb{A}^1}(n)$ the sheaf $\pi_1^{\mathbb{A}^1}(\Sigma((\mathbb{G}_m)^{\wedge n}))$.

We have proven in section 4.2 that $F_{\mathbb{A}^1}(2)$ is abelian and (thus) isomorphic to $\underline{\mathbf{K}}_2^{MW}$. Our aim is to describe $F_{\mathbb{A}^1}(1) = \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$.

The Hopf map of a sheaf of group.

Recall that for two pointed spaces $\mathcal{X}$ and $\mathcal{Y}$ we let $\mathcal{X} * \mathcal{Y}$ denote the *reduced join* of $\mathcal{X}$ and $\mathcal{Y}$, that is to say the quotient of $\Delta^1 \times \mathcal{X} \times \mathcal{Y}$ by the relations $(0, x, y) = (0, x, y')$, $(1, x, y) = (1, x', y)$ and $(t, x_0, y_0) = (t, x_0, y_0)$ where x_0 (resp. y_0) is the base point of $\mathcal{X}$ (resp. $\mathcal{Y}$). It is a homotopy colimit of the diagram of pointed spaces

$$\begin{array}{ccc} \mathcal{X} & & \\ \uparrow & & \\ \mathcal{X} \times \mathcal{Y} & \to & \mathcal{Y} \end{array}$$

Example 4.24 $\mathbb{A}^2 - \{0\}$ has canonically the $\mathbb{A}^1$-homotopy type of $\mathbb{G}_m * \mathbb{G}_m$: use the classical covering of $\mathbb{A}^2 - \{0\}$ by $\mathbb{G}_m \times \mathbb{A}^1$ and $\mathbb{A}^1 \times \mathbb{G}_m$ with intersection $\mathbb{G}_m \times \mathbb{G}_m$.□

The join $\mathcal{X} * (point)$ of $\mathcal{X}$ and the point is called the cone of $\mathcal{X}$ and is denoted by $C(\mathcal{X})$. It is the smash product $\Delta^1 \wedge \mathcal{X}$ with Δ^1 pointed by 1. we let $\mathcal{X} \subset C(\mathcal{X})$ denote the canonical inclusion. The quotient is obviously $\Sigma(\mathcal{X})$. The "anticone" $C'(\mathcal{X})$ is the the smash product $\Delta^1 \wedge \mathcal{X}$ with Δ^1 pointed by 0.

The join obviously contains the wedge $C(\mathcal{X}) \vee C'(\mathcal{Y})$. Clearly the quotient $(\mathcal{X} * \mathcal{Y})/(\mathcal{X} \vee \mathcal{Y})$ is $\Sigma(\mathcal{X} \times \mathcal{Y})$ and the quotient $(\mathcal{X} * \mathcal{Y})/(C(\mathcal{X}) \vee C'(\mathcal{Y}))$ is $\Sigma(\mathcal{X} \wedge \mathcal{Y})$.

The morphism of pointed spaces $\mathcal{X} * \mathcal{Y} \to \Sigma(\mathcal{X} \wedge \mathcal{Y})$ is thus a simplicial weak-equivalence. The diagram of pointed spaces

$$\begin{array}{ccc} \Sigma(\mathcal{X} \times \mathcal{Y}) & & \Sigma(\mathcal{X} \times \mathcal{Y}) \\ \uparrow & & \downarrow \\ \mathcal{X} * \mathcal{Y} & \stackrel{\sim}{\to} & \Sigma(\mathcal{X} \wedge \mathcal{Y}) \end{array}$$

defines a canonical $\mathcal{H}_{s,\bullet}(k)$-morphism

$$\omega_{\mathcal{X},\mathcal{Y}} : \Sigma(\mathcal{X} \times \mathcal{Y}) \to \Sigma(\mathcal{X} \times \mathcal{Y})$$

The following result is classical:

Lemma 4.25 *The $\mathcal{H}_{s,\bullet}(k)$-morphism $\omega_{\mathcal{X},\mathcal{Y}}$ is (for the co-h-group structure on $\Sigma(\mathcal{X} \times \mathcal{Y})$) equal in $\mathcal{H}_{s,\bullet}(k)$ to $(\pi_1)^{-1}.Id_{\Sigma(\mathcal{X}\times\mathcal{Y})}.(\pi_2)^{-1}$, where π_1 is the obvious composition $\Sigma(\mathcal{X} \times \mathcal{Y}) \to \Sigma(\mathcal{X}) \to \Sigma(\mathcal{X} \times \mathcal{Y})$ and π_2 is defined the same way using $\mathcal{Y}$.*

Proof. To prove this, the idea is to construct an explicit model for the map $\Sigma(\mathcal{X} \times \mathcal{Y}) \to \mathcal{X} * \mathcal{Y}$. One may use as model for $\Sigma(\mathcal{X} \times \mathcal{Y})$ the amalgamate sum Σ of $C(\mathcal{X} \times \mathcal{Y})$, $\Delta^1\mathcal{X} \times \mathcal{Y}$ and $C'(\mathcal{X} \times \mathcal{Y})$ obviously glued together. Collapsing $C(\mathcal{X} \times \mathcal{Y}) \vee C'(\mathcal{X} \times \mathcal{Y})$ in that space gives exactly $\Sigma(\mathcal{X} \times \mathcal{Y})$ thus $\Sigma \to \Sigma(\mathcal{X} \times \mathcal{Y})$ is a simplicial weak equivalence. Now there is an obvious map $\Sigma \to \mathcal{X} * \mathcal{Y}$ given by the obvious inclusions of the cones and the canonical projection on the middle. It then remains to understand the composition $\Sigma \to \mathcal{X} * \mathcal{Y} \to \Sigma(\mathcal{X} \times \mathcal{Y})$. This is easily analyzed and yields the result.□

Now let G be a sheaf of groups. We consider the pointed map

$$\mu'_G : G \times G \to G \quad , \quad (g, h) \mapsto g^{-1}.h$$

This morphism induces a morphism $\Delta^1 \times G \times G \to \Delta \times G$ which is easily seen to induce a canonical morphism

$$\eta_G : G * G \to \Sigma(G)$$

which is called the (geometric) Hopf map of G.

We will still denote by $\eta_G : \Sigma(G \wedge G) \to \Sigma(G)$ the canonical $\mathcal{H}_\bullet(k)$-morphism obtained as the composition of the geometric Hopf map and the inverse to $G * G \to \Sigma(G \wedge G)$.

Example 4.26 Example 4.24 implies that the Hopf fibration $\mathbb{A}^2 - \{0\} \to \mathbb{P}^1$ is canonically $\mathbb{A}^1$-equivalent to the geometric Hopf map $\eta_{\mathbb{G}_m}$

$$\Sigma(\mathbb{G}_m \wedge \mathbb{G}_m) \to \Sigma(\mathbb{G}_m)$$

We observe that G acts diagonally on $G * G$ and that the geometric Hopf map $\eta_G : G * G \to \Sigma(G)$ is a G-torsor. It is well known that the classifying map $\Sigma(G) \to BG$ for this G torsor is the canonical one [27]. By Theorem 3.53 if $\pi_0^{\mathbb{A}^1}(G)$ is a strongly $\mathbb{A}^1$-invariant sheaf, the simplicial fibration

$$G * G \to \Sigma(G) \to BG \tag{4.3}$$

is also an $\mathbb{A}^1$-fibration sequence.

Remark 4.27 Examples are given by $G = SL_n$ and $G = GL_n$ for any $n \geq 1$. In fact we do not know any connected smooth algebraic k-group which doesn't satisfy this assumption.$\square$

The following result is an immediate consequence of Lemma 4.25:

Corollary 4.28 *For any sheaf of groups G, the composition*

$$\Sigma(G \times G) \to \Sigma(G \wedge G) \stackrel{\eta_G}{\to} \Sigma(G)$$

is equal in $[\Sigma(G \times G), \Sigma(G)]_{\mathcal{H}_{s,\bullet}(k)}$ (for the usual group structure) to

$$(\Sigma(\chi_1))^{-1}.\Sigma(\mu').(\Sigma(pr_2))^{-1}$$

where χ_1 is the obvious composition $G \times G \stackrel{pr_1}{\to} G \stackrel{g \mapsto g^{-1}}{\longrightarrow} G \stackrel{Id_G \times *}{\to} G \times G$ and pr_2 is the composition $G \times G \stackrel{pr_2}{\to} G \stackrel{* \times Id_G}{\to} G \times G$.

We now specialize to $G = \mathbb{G}_m$. From what we have just done, the fibration sequence (4.3) $\mathbb{G}_m * \mathbb{G}_m \to \Sigma(\mathbb{G}_m) \to B\mathbb{G}_m$ is $\mathbb{A}^1$-equivalent to

$$\mathbb{A}^2 - \{0\} \to \mathbb{P}^1 \to \mathbb{P}^\infty$$

As the spaces $\Sigma(\mathbb{G}_m) \cong \mathbb{P}^1$ and $B(\mathbb{G}_m) \cong \mathbb{P}^\infty$ are $\mathbb{A}^1$-connected, the long exact sequence of homotopy sheaves induces at once a short exact sequence of sheaves of groups

$$1 \to \underline{\mathbf{K}}_2^{MW} \to F_{\mathbb{A}^1}(1) \to \mathbb{G}_m \to 1 \tag{4.4}$$

We simply denote by $\theta : \mathbb{G}_m \to F_{\mathbb{A}^1}(1)$ the section $\theta_{\mathbb{G}_m}$. As the sheaf of pointed sets $F_{\mathbb{A}^1}(1)$ is the product $\underline{\mathbf{K}}_2^{MW} \times \mathbb{G}_m$ (using θ), the following result entirely describes the group structure on $F_{\mathbb{A}^1}(1)$ and thus the sheaf of groups $F_{\mathbb{A}^1}(1)$:

Theorem 4.29 *1) The morphism of sheaves of sets*

$$\mathbb{G}_m \times \mathbb{G}_m \to \underline{\mathbf{K}}_2^{MW}$$

induced by the morphism $(U, V) \mapsto \theta(U^{-1})^{-1}\theta(U^{-1}V)\theta(V)^{-1}$ *is equal to the symbol* $(U, V) \mapsto [U][V]$.

2) The short exact sequence (4.4):

$$1 \to \underline{\mathbf{K}}_2^{MW} \to F_{\mathbb{A}^1}(1) \to \mathbb{G}_m \to 1$$

is a central extension.

Proof. 1) follows directly from the definitions and the Corollary 4.28.

2) For two units U and V in some $F \in \mathcal{F}_k$ the calculation in 1) easily yields $\theta(U)\theta(V)^{-1} = -[U][-V]\theta(U^{-1}V)^{-1}$ and $\theta(U)^{-1}\theta(V) = [U^{-1}][-V]\theta(U^{-1}V)$.

Now we want to check that the action by conjugation of $\mathbb{G}_m$ on $\underline{\mathbf{K}}_2^{MW}$ (through θ) is trivial. It clearly suffices to check it on fields. For units U, V and W in some field $F \in \mathcal{F}_k$, we get (using the previous formulas):

$$\theta(W)([U][V])\theta(W)^{-1} = (-[W][-U^{-1}]+[UW][-U^{-1}.V]-[WV][-V])\theta(W^{-1})^{-1}.\theta(W)^{-1}$$

Now applying 1) to $U = W = V$ yields (as θ is pointed) $\theta(W^{-1})^{-1}.\theta(W)^{-1} = [W][W]$.

Now the claim follows from the easily checked equality in $\underline{\mathbf{K}}_2^{MW}(F)$

$$-[W][-U^{-1}] + [UW][-U^{-1}.V] - [WV][-V] + [W][W] = [U][V]$$

which finally yields $\theta(W)([U][V])\theta(W)^{-1} = [U][V]$.□

Remark 4.30 Though it is the more "geometric" way to describe $F_{\mathbb{A}^1}(1)$ it is not the most natural. We denote by $F(\mathcal{S})$ the free sheaf of groups on the pointed sheaf of sets $\mathcal{S}$. This is also the sheaf $\pi_1(\Sigma(\mathcal{S}))$. Its stalks are the free groups generated by the pointed stalks of $\mathcal{S}$.

For a sheaf of group G let $c_G : F(G) \twoheadrightarrow G$ be the canonical epimorphism induced by the identity of G, which admits θ_G as a section (in $Shv_\bullet$). Consider the $Shv_\bullet$-morphism $\theta^{(2)} : G^{\wedge 2} \to F(G)$ given by $(U, V) \mapsto \theta_G(U).\theta_G(U).\theta_G(UV)^{-1}$ This morphism induces a morphism $F(G^{\wedge 2}) \to F(G)$.

A classical result of group theory, a proof of which can be found in [10, Theorem 4.6] gives that the diagram

$$1 \to F(G^{\wedge 2}) \to F(G) \to G \to 1$$

is a short exact sequence of sheaves of groups. If G is strongly $\mathbb{A}^1$-invariant, we deduce the compatible short exact sequence of strongly $\mathbb{A}^1$-invariant

$$1 \to F_{\mathbb{A}^1}(G^{\wedge 2}) \to F_{\mathbb{A}^1}(G) \to G \to 1$$

But now $\theta_G(U).\theta_G(U).\theta_G(UV)^{-1}$ is the tautological symbol $G^2 \to F_{\mathbb{A}^1}(G^{\wedge 2})$. In the case of $\mathbb{G}_m$ this implies (in a easier way) that the extension

$$1 \to \underline{\mathbf{K}}_2^{MW} = F_{\mathbb{A}^1}(\mathbb{G}_m^{\wedge 2}) \to F_{\mathbb{A}^1}(1) \to \mathbb{G}_m \to 1$$

is central. It is of course isomorphic to (4.4) but not equal as an extension! Indeed, as a consequence of the Theorem, we get for the extension (4.4) the formula $\theta(U)\theta(V) =< -1 > [U][V]\theta(UV)$, but for the previous extension one has by construction $\theta(U)\theta(V) = [U][V]\theta(UV)$.

Remark 4.31 As a consequence we also see that the sheaf $F_{\mathbb{A}^1}(1)$ is never abelian. Indeed the formula $\theta(U)\theta(V) =< -1 > [U][V]\theta(UV)$ implies

$$\theta(U)\theta(V)\theta(U)^{-1} = h([U][V])\theta(V)$$

Now given any field k one can show that there always exists such an F and such units with $h([U][V]) \neq 0 \in \underline{\mathbf{K}}_2^{MW}(F)$. Take $F = k(U,V)$ to be the field of rational fraction in U and V over k. The composition of the residues morphisms ∂^U and $partial^V$: $K_2^{MW}(k(U,V) \to K_0^{MW}(k)$ commutes to multiplication by h. As the image of the symbol $[U][V]$ is one, the claim follows by observing that $h \neq 0 \in k_0^{MW}(k)$.□

Endomorphisms of $F_{\mathbb{A}^1}(1) = \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$.

We want to understand the monoid of endomorphisms $\mathcal{E}nd(F_{\mathbb{A}^1}(1))$ of the sheaf of groups $F_{\mathbb{A}^1}(1)$. As $F_{\mathbb{A}^1}(1)$ is the free strongly $\mathbb{A}^1$-invariant sheaf on the pointed sheaf $\mathbb{G}_m$, we see that as a set $\mathcal{E}nd(F_{\mathbb{A}^1}(1)) = Hom_{Shv_\bullet}(\mathbb{G}_m, F_{\mathbb{A}^1}(1))$. By definition the latter set is $F_{\mathbb{A}^1}(1)_{-1}(k)$. As a consequence we observe that there is a natural group structure on $\mathcal{E}nd(F_{\mathbb{A}^1}(1))$.

Remark 4.32 It follows from our results that the obvious map

$$[\mathbb{P}^1, \mathbb{P}^1]_{\mathcal{H}_\bullet(k)} \to \mathcal{E}nd(F_{\mathbb{A}^1}(1))$$

is a bijection. The above group structure comes of course from the natural group structure on $[\mathbb{P}^1, \mathbb{P}^1]_{\mathcal{H}_\bullet(k)} = [\Sigma(\mathbb{G}_m), \mathbb{P}^1]_{\mathcal{H}_\bullet(k)}$.

The functor $\mathcal{G} \mapsto (\mathcal{G})_{-1}$ is exact in the following sense:

Lemma 4.33 *For any short exact sequence $1 \to \mathcal{K} \to \mathcal{G} \to \mathcal{H} \to 1$ of strongly $\mathbb{A}^1$-invariant sheaves yields, the diagram of*

$$1 \to (\mathcal{K})_{-1} \to (\mathcal{G})_{-1} \to (\mathcal{H})_{-1} \to 1$$

is still a short exact sequence of strongly $\mathbb{A}^1$-invariant sheaves.

Proof. We already know from Lemma 1.31 that the sheaves are strongly $\mathbb{A}^1$-invariant sheaves. The only problem is in fact to check that the morphism $\mathcal{G}_{-1} \to \mathcal{H}_{-1}$ is still an epimorphism. For any $x \in X \in Sm_k$, let X_x be the localization of X at x. We claim that the morphism $\mathcal{G}(\mathbb{G}_m \times X_x) \to \mathcal{H}(\mathbb{G}_m \times X_x)$ is an epimorphism of groups. Using the very definition of the functor $\mathcal{G} \mapsto (\mathcal{G})_{-1}$ this claim easily implies the result.

Now a element in $\mathcal{H}(\mathbb{G}_m \times X_x)$ comes from an element $\alpha \in \mathcal{H}(\mathbb{G}_m \times U)$, for some open neighborhood U of x. Pulling back the short exact sequence to $\mathbb{G}_m \times U$ yields a $\mathcal{K}$-torsor on $\mathbb{G}_m \times U$. But clearly $H^1(\mathbb{G}_m \times X_x; \mathcal{G})$ is trivial by our results of section 1.2. This means that up to shrinking a bit U the $\mathcal{K}$ torsor is trivial. But this means exactly that there a $\beta \in \mathcal{G}(\mathbb{G}_m \times U)$ lifting α. The Lemma is proven.□

Applying this to the short exact sequence (4.4) $1 \to \underline{\mathbf{K}}_2^{MW} \to F_{\mathbb{A}^1}(1) \to \mathbb{G}_m \to 1$, which is a central extension by Theorem 4.29, obviously yields a central extension as well:

$$0 \to (\underline{\mathbf{K}}_2^{MW})_{-1} \to (F_{\mathbb{A}^1}(1))_{-1} \to (\mathbb{G}_m)_{-1} \to 1$$

But now observe that $\mathbb{Z} = (\mathbb{G}_m)_{-1}$ so that the epimorphism $(F_{\mathbb{A}^1}(1))_{-1} \to (\mathbb{G}_m)_{-1}$ admits a canonical section sending 1 to the identity.

Corollary 4.34 *The sheaf of groups $(F_{\mathbb{A}^1}(1))_{-1}$ is abelian and is canonically isomorphic to $\mathbb{Z} \oplus \underline{\mathbf{K}}_1^{MW}$.*

Proof. The only remaining point is to observe from remark **??** that the products induce an isomorphism $\underline{\mathbf{K}}_1^{MW} \cong (\underline{\mathbf{K}}_2^{MW})_{-1}$.□

We let $\rho : \mathbb{G}_m \to (F_{\mathbb{A}^1}(1))_{-1} = \mathbb{Z} \oplus \underline{\mathbf{K}}_1^{MW}$ be the morphism of sheaves which maps U to $(1, [U])$. Observe it is not a morphism of sheaves of groups.

Theorem 4.35 *Endowed with the previous abelian group structure and the composition of morphisms $\mathcal{E}nd(F_{\mathbb{A}^1}(1)) \cong [\mathbb{P}^1, \mathbb{P}^1]_{\mathcal{H}_\bullet(k)}$ is an associative ring. $\rho(k)$ induces a group homomorphism $k^\times \to \mathcal{E}nd(F_{\mathbb{A}^1}(1))^\times$ to the group of units and the induced ring homomorphism*

$$\Theta(k) : \mathbb{Z}[k^\times] \to \mathcal{E}nd(F_{\mathbb{A}^1}(1))$$

is onto. As a consequence, $\mathcal{E}nd(F_{\mathbb{A}^1}(1))$ is a commutative ring.

Proof. Let $\mathbb{Z}(k^\times)$ be the free abelian group on $k^\times$ with the relation the symbol $1 \in k^\times$ equals 0. It is clear that $\mathbb{Z}[k^\times]$ splits as $\mathbb{Z} \oplus \mathbb{Z}(k^\times)$ in a compatible way to the splitting of Corollary 4.34 so that $\Theta(k)$ decomposes as the identity of $\mathbb{Z}$ plus the obvious symbol $\mathbb{Z}(k^\times) \to \underline{\mathbf{K}}_1^{MW}(k)$. But then we know from Lemma 2.6 that this is an epimorphism.□

The canonical morphism

$$[\mathbb{P}^1,\mathbb{P}^1]_{\mathcal{H}_\bullet(k)} = \mathcal{E}nd(F_{\mathbb{A}^1}(1)) \to K_0^{MW}(k) = \mathcal{E}nd(\underline{\mathbf{K}}_1^{MW})$$

given by the "Brouwer degree" (which means evaluation of $\mathbb{A}^1$-homology in degree 1) is thus an epimorphism as $\mathbb{Z}[k^\times] \to K_0^{MW}(k)$ is onto.

To understand this a bit further, we use Theorem 2.46 and its corollary which show that $\underline{\mathbf{K}}_1^W \to \underline{\mathbf{I}}$ is an isomorphism.

In fact $K_0^{MW}(k)$ splits canonically as $\mathbb{Z} \oplus I(k)$ as an abelian group, and moreover this decomposition is compatible through the above epimorphism to that of Corollary 4.34. This means that the kernel of

$$[\mathbb{P}^1,\mathbb{P}^1]_{\mathcal{H}_\bullet(k)} = [\Sigma(\mathbb{G}_m),\Sigma(\mathbb{G}_m)]_{\mathcal{H}_\bullet(k)} \twoheadrightarrow K_0^{MW}(k)$$

is isomorphic to the kernel of $K_1^{MW}(k) \twoheadrightarrow I(k)$. As $K_1^{MW}(k) \stackrel{h}{\to} K_1^{MW}(k) \to I(k) \to 0$ is always an exact sequence by Theorem 2.46 and its corollary, and as $K_1^{MW}(k) \stackrel{h}{\to} K_1^{MW}(k)$ factors through $K_1^{MW}(k) \twoheadrightarrow K_1^{MW}(k)/\eta = k^\times$ we get an exact sequence of the form

$$k^\times \to K_1^{MW}(k) \to I(k) \to 0$$

where the map $k^\times \to K_1^{MW}(k)$ arises from multiplication by h. Clearly -1 is mapped to 0 in $K_1^{MW}(k)$ because $h.[-1] = [-1]+ < -1 > [-1] = [(-1)(-1)] = [1] = 0$. Moreover the composition $k^\times \to K_1^{MW}(k) \to k^\times$ is the squaring map. Thus $(k^\times)/_{(\pm 1)} \to K_1^{MW}(k)$ is injective and equal to the kernel. We thus get altogether:

Theorem 4.36 *The diagram previously constructed*

$$0 \to (k^\times)/_{(\pm 1)} \to [\mathbb{P}^1,\mathbb{P}^1]_{\mathcal{H}_\bullet(k)} \to GW(k) \to 0$$

is a short exact sequence of abelian groups.

Remark 4.37 J. Lannes has observed that as a ring, $\tilde{GW}(k) := [\mathbb{P}^1,\mathbb{P}^1]_{\mathcal{H}_\bullet(k)} = \mathcal{E}nd(F_{\mathbb{A}^1}(1))$ is the Grothendieck ring of isomorphism classes of symmetric inner product spaces over k with a given diagonal basis, where an isomorphism between two such objects is a linear isomorphism preserving the inner product and with determinant 1 in the given basis. It fits in the following cartesian square of rings

$$\begin{array}{ccc} \tilde{GW}(k) & \to & \mathbb{Z}\oplus k^\times \\ \downarrow & & \downarrow \\ GW(k) & \to & \mathbb{Z}\oplus (k^\times/2) \end{array}$$

The bottom horizontal map is the rank plus the determinant. The group structure on the right hand side groups is the obvious one. The product structure is given by $(n,U).(m,V) = (nm, U^m.V^n)$.

C. Cazanave in a work in preparation, has attacked a proof of the previous two results by a different method using the approach of Barge and Lannes on Bott periodicity for orthogonal algebraic K-theory [2]. This allows him to construct an invariant $[\mathbb{P}^1,\mathbb{P}^1]_{\mathcal{H}_\bullet(k)} \to \tilde{GW}(k)$.□

Remark 4.38 We may turn $\Theta(k)$ into a morphism of sheaves of abelian groups $\Theta : \mathbb{Z}[\mathbb{G}_m] \to (F_{\mathbb{A}^1}(1))_{-1} \cong \mathbb{Z}\oplus \underline{\mathbf{K}}_1^{MW}$ induced by ρ. Here $\mathbb{Z}[\mathcal{S}]$ means the free sheaf of abelian groups generated by the sheaf of sets $\mathcal{S}$. $\Theta : \mathbb{Z}[\mathbb{G}_m] \to (F_{\mathbb{A}^1}(1))_{-1} \cong \mathbb{Z}\oplus \underline{\mathbf{K}}_1^{MW}$ is then the universal morphism of sheaves of abelian groups to a strictly $\mathbb{A}^1$-invariant sheaf. As a consequence the target is also a sheaf of commutative rings: it is the $\mathbb{A}^1$-group ring on $\mathbb{G}_m$.□

Free homotopy classes $[\mathbb{P}^1,\mathbb{P}^1]_{\mathcal{H}(k)}$**.** By Remark 2.44 to understand the set $[\mathbb{P}^1,\mathbb{P}^1]_{\mathcal{H}(k)}$ we have to understand the action of $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)(k)$ on $[\mathbb{P}^1,\mathbb{P}^1]_{\mathcal{H}_\bullet(k)}$ and to compute the quotient.

Clearly, as $[\mathbb{P}^1,\mathbb{P}^1]_{\mathcal{H}_\bullet(k)} \cong [\mathbb{P}^1, B(F_{\mathbb{A}^1}(1))]_{\mathcal{H}_\bullet(k)} \cong \mathcal{E}nd(F_{\mathbb{A}^1}(1))$, this action is given on the right hand side by the action by conjugation of $F_{\mathbb{A}^1}(1)$ on the target. Now the abelian group structure comes from the source and thus this action is an action of the group $F_{\mathbb{A}^1}(1)$ on the abelian group $\mathcal{E}nd(F_{\mathbb{A}^1}(1))$. As $K_2^{MW}(k)\subset F_{\mathbb{A}^1}(1)$ is central this action factors through an action of $k^\times$ on the abelian group $\mathcal{E}nd(F_{\mathbb{A}^1}(1))$.

Lemma 4.39 *The action of $k^\times$ on $\mathcal{E}nd(F_{\mathbb{A}^1}(1)) = K_1^{MW}(k)\oplus\mathbb{Z}$ is given as follows. For any $u\in k^\times$ and any $(v,n)\in k^\times\times\mathbb{Z}$ one has in $K_1^{MW}(k)\oplus\mathbb{Z}$*

$$c_u([v],n) = ([v]-nh[u],n)$$

Proof. To find the action of $k^\times$ by conjugation on $\mathcal{E}nd(F_{\mathbb{A}^1}(1)) = (F_{\mathbb{A}^1})_{-1}(k)$ we observe that by Remark 4.31 we understand this action on $F_{\mathbb{A}^1}(1)$.

We may explicit this action on $F_{\mathbb{A}^1}(k(T)$ and observe that the isomorphism $\mathcal{E}nd(F_{\mathbb{A}^1}(1)) = K_1^{MW}(k) \oplus \mathbb{Z} = (F_{\mathbb{A}^1})_{-1}(k)$ is obtained by cup-product by T on the left $[T]\cup : K_1^{MW}(k) \oplus \mathbb{Z} \to F_{\mathbb{A}^1}(k(T))$. Thus for $(v,n) \in k^\times \times \mathbb{Z}$ the corresponding element $[T] \cup ([v], n)$ in $F_{\mathbb{A}^1}(k(T))$ is $[T][v].\theta(T)^n$.

Now by the formula in Remark 4.31 we get for any $u \in k^x$, and any $(v,n) \in k^\times \times \mathbb{Z}$:

$$c_u([T][v].\theta(T)^n) = [T][v].(h[u][T].\theta(T))^n$$

$$= ([T][v] + nh[u][T])\theta(T)^n = [T]([v] - nh[u])\theta(T)^n$$

This implies the Lemma.□

Corollary 4.40 *Assume that for each $n \geq 1$ the map $k^\times \to k^\times$, $u \mapsto u^n$ is onto. Then the surjective map*

$$[\mathbb{P}^1, \mathbb{P}^1]_{\mathcal{H}(k)} \to [\mathbb{P}^1, \mathbb{P}^\infty]_{\mathcal{H}(k)} = \mathbb{Z}$$

has trivial fibers over any integer $n \neq 0$ and its fiber over 0 is exactly $K_1^{MW}(k) = [\mathbb{P}^1, \mathbb{A}^2 - \{0\}]_{\mathcal{H}(k)}$.

Proof. First if every unit is a square, by Proposition 2.13, we know that $K_1^{MW}(k) \to k^\times$ is an isomorphism. On the set of pairs $(v,n) \in K_1^{MW}(k) \times \mathbb{Z} = \mathcal{E}nd(F_{\mathbb{A}^1}(1))$ the action of $u \in k^\times$ is thus given as

$$c_u([v], n) = ([vu^{-2n}], n)$$

But for $n \neq 0$, any unit w can be written vu^{-2n} for some u and v by assumption on k. Thus the results.□

A Unramified and strongly $\mathbb{A}^1$-invariant sheaves over finite fields

Theorem A.1 *Any strongly $\mathbb{A}^1$-invariant sheaf of groups on Sm_k is unramified.*

Proof. Applying 3.8 to $\mathcal{B} = B\mathcal{G}$, we already know this result over an infinite field. It suffices thus to do the case k is perfect which includes finite fields. We now assume k is perfect.

As $\mathcal{G}$ is strongly $\mathbb{A}^1$-invariant the classifying space BG is $\mathbb{A}^1$-local in the sense of [39], moreover using *loc. cit.* we know that for any smooth scheme X, the group of morphism $Hom_{\mathcal{H}_\bullet(k)}(\Sigma(X_+); BG)$ is naturally isomorphic to $\mathcal{G}(X)$. Moreover the pointed set $Hom_{\mathcal{H}_\bullet(k)}(X_+; BG)$ is naturally isomorphic to $H^1_{Nis}(X; BG)$.

The property **(0)** of unramified sheaves is clear. To prove properties **(1)** and **(2)** we proceed as follows. Let $X \in Sm_k$ and let $Z \subset X$ be a closed subset having everywhere codimension $\geq i$, and let $U \subset X$ be the open complement. Endow Z with the reduced induced structure.

Assume first that Z is smooth and has trivial normal bundle in X. The cofibration sequence $X_+ \to X/U \to \Sigma_s(U_+) \to \Sigma_s(X_+) \to \Sigma_s(X/U)$, the result recalled above and the purity Theorem [39] yield an exact sequence

$$1 \to \mathcal{G}(\mathbb{A}^i_Z/(\mathbb{A}^i - \{0\})_Z) \to \mathcal{G}(X) \to \mathcal{G}(U) \to H^1_{Nis}(\mathbb{A}^i_Z/(\mathbb{A}^i - \{0\})_Z); \mathcal{G})$$

If $i \geq 1$, $\mathcal{G}(\mathbb{A}^i_Z/(\mathbb{A}^i - \{0\})_Z)$ is trivial because $\mathcal{G}$ is $\mathbb{A}^1$-invariant and $\mathcal{G}(Z) = \mathcal{G}(\mathbb{A}^i_Z) \to \mathcal{G}((\mathbb{A}^i - \{0\})_Z)$ is injective as the section $(1, 0, \dots, 0)$ provides a left inverse. If $i \geq 2$, then $(\mathbb{A}^i_Z)/(\mathbb{A}^i - \{0\})_Z = (\mathbb{A}^i/(\mathbb{A}^i - \{0\})) \wedge (Z_+)$ is $\mathbb{A}^1$-equivalent to a double simplicial suspension, and thus the set $H^1_{Nis}(\mathbb{A}^i_Z/(\mathbb{A}^i - \{0\})_Z); \mathcal{G})$ vanishes.

Because k is perfect, there is an increasing flag of closed subschemes $\subset Z^n \subset Z^{n-1} \subset \dots \subset Z$ with $Z^m - Z^{m+1}$ smooth over k of codimension $\geq i$ in X and we may assume further that the normal bundle of $Z^m - Z^{m+1}$ in $X - Z^{m+1}$ is trivial. Using the results above, altogether this implies that $\mathcal{G}$ is unramified (see also Remark 1.4).□

The following result is useful to deduce results for finite fields from result for infinite fields. The analogue over an infinite field is trivial (compare [32, Lemme 6.4.8]):

Lemma A.2 *Let k be a field. Let $\Omega \subset \mathbb{A}^1$ be a dense open subset. Then the pointed morphism $(\mathbb{A}^1)_+ \to \mathbb{A}^1/\Omega$ is trivial in $\mathcal{H}_\bullet(k)$. As a consequence, $\Sigma(\Omega_+) \to \Sigma((\mathbb{A}^1)_+) \cong \Sigma(Spec(k)_+)$ admits a section in $\mathcal{H}_\bullet(k)$.*

Proof. By Mayer-Vietoris we get an isomorphism of pointed sheaves of sets $\mathbb{A}^1/\Omega = \vee_x \mathbb{A}^1/(\mathbb{A}^1 - \{x\})$, where x runs over the closed points in $\mathbb{A}^1 - \Omega$. But in $\mathcal{H}_{Zar,\bullet}(k)$, $\mathbb{A}^1/(\mathbb{A}^1 - \{x\})$ is the simplicial suspension of $Spec(\kappa(x)_+)$. By Corollary 3.22 p. 95 [39], the $\mathbb{A}^1$-localization of a suspension is 0-connected. Thus $\vee_x \mathbb{A}^1/(\mathbb{A}^1 - \{x\})$, which is a simplicial suspension is $\mathbb{A}^1$-connected and any morphism in $\mathcal{H}_\bullet(k)$ from $Spec(k)_+$ to an $\mathbb{A}^1$-connected pointed space is trivial.□

Corollary A.3 *Let k be any field. Let $\mathcal{B}$ be an $\mathbb{A}^1$-local pointed space. Let $f : Spec(k(T)) \to Spec(k)$ be the canonical morphism. Then for any $X \in Sm_k$ and any $n \in \mathbb{N}$, the pointed map $Hom_{\mathcal{H}_\bullet(k)}(\Sigma^n(X_+); \mathcal{B}) \to Hom_{\mathcal{H}_\bullet(k(T))}(\Sigma^n((X|_{k(T)})_+); f^*(\mathcal{B}))$ has trivial kernel. For $n \geq 1$, being a group homomorphism, it is injective.*

Proof. Using the obvious unstable analogues of [32, Corollary 5.2.7] we see that the map

$$colim_\Omega Hom_{\mathcal{H}_\bullet(k)}(\Sigma^n(\Omega \times X_+); \mathcal{B}) \to Hom_{\mathcal{H}_\bullet(k(T))}(\Sigma^n((X|_{k(T)})_+); f^*(\mathcal{B})) \ )$$

is a bijection for any $n \in \mathbb{N}$, where Ω runs over the ordered set (for inclusion) of open dense subsets of $\mathbb{A}^1$.

Now assume $\alpha \in Hom_{\mathcal{H}_\bullet(k)}(\Sigma^n(X_+); \mathcal{B})$ becomes trivial on $k(T)$. Then by the above formula it must be trivial in $Hom_{\mathcal{H}_\bullet(k)}(\Sigma^n((\Omega \times X)_+); \mathcal{B})$ for some open non-empty Ω in $\mathbb{A}^1$. This means, using the cofibration sequence $(\Omega_+)\wedge(X_+) \to (\mathbb{A}^1_+)\wedge(X_+) \to (\Omega/\mathbb{A}^1)\wedge(X_+)$ and observing that $(\Omega\times X)_+ = (\Omega_+)\wedge(X_+)$, that α is the restriction of an element β in $Hom_{\mathcal{H}_\bullet(k)}((\Omega/\mathbb{A}^1)\wedge \Sigma^n(X_+); \mathcal{B})$. But Lemma A.2 implies that the restriction of this element to $Hom_{\mathcal{H}_\bullet(k)}(\Sigma^n(X_+); \mathcal{B}) = Hom_{\mathcal{H}_\bullet(k)}((\mathbb{A}^1_+) \wedge \Sigma^n(X_+); \mathcal{B})$, which is α, is trivial.□

Corollary A.4 *Let k be a field. Let $\mathcal{B}$ be an $\mathbb{A}^1$-local pointed space. For any field $F \in \mathcal{F}_k$, the map $Hom_{\mathcal{H}_\bullet(k)}(\Sigma^n(Spec(F)_+); \mathcal{B}) \to Hom_{\mathcal{H}_\bullet(k)}(\Sigma^n((Spec(F(T))_+); \mathcal{B})$ has trivial kernel. For $n \geq 1$, being a group homomorphism, it is injective.*

Consequently, if $\mathcal{B}_{k(T)}$ is n-$\mathbb{A}^1$-connected for some n, $\mathcal{B}$ is n-$\mathbb{A}^1$-connected.

Proof. By an easy base change argument, the map $Hom_{\mathcal{H}_\bullet(k)}(\Sigma^n(Spec(F)_+); \mathcal{B}) \to Hom_{\mathcal{H}_\bullet(F)}(\Sigma^n(Spec(F)_+); \mathcal{B}|F)$ is a bijection and in the same way,

$Hom_{\mathcal{H}_\bullet(k)}(\Sigma^n((Spec(F(T))_+);\mathcal{B}) \cong Hom_{\mathcal{H}_\bullet(F)}(\Sigma^n((Spec(F(T))_+);\mathcal{B}_F)$ in the latter we consider $F(T)$ as an obvious extension of F.

Now by Corollary A.3, we get the first statement. The last statement is proven as follows. If $\mathcal{B}_{k(T)}$ is n-$\mathbb{A}^1$-connected, then for any $F \in \mathcal{F}_k$ and any i the same base change argument yields that $Hom_{\mathcal{H}_\bullet(k)}(\Sigma^i((Spec(F(T))_+);\mathcal{B}) \cong Hom_{\mathcal{H}_\bullet(k)}(\Sigma^i((Spec(F(T))_+);\mathcal{B}|_{k(T)})$ where $F(T)$ is obviously considered as a separable extension of $k(T)$. If $i \leq n$ it follows from that that $Hom_{\mathcal{H}_\bullet(k)}(\Sigma^i((Spec(F(T))_+);\mathcal{B})$ is trivial. By the first statement we deduce that $Hom_{\mathcal{H}_\bullet(k)}(\Sigma^i((Spec(F)_+);\mathcal{B})$ is also trivial.

We conclude with [32, Lemma 6.1.3] that $\mathcal{B}$ is n-$\mathbb{A}^1$-connected.□

Corollary A.5 *Let M be a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups on a perfect field k. Set for any $X \in Sm_k$ and any $y \in X^{(1)}$ set $H^1_y(X;M) := colim_{y\in U}[U/(U-\overline{y}), BM]_{\mathcal{H}_\bullet(k)}$.*

Then for any $F \in \mathcal{F}_k$, the obvious diagram

$$0 \to M(F) \to M(F(T)) \to \oplus_{P\in(\mathbb{A}^1_F)^{(1)}} H^1_P(\mathbb{A}^1_F;M) \to 0$$

is a short exact sequence.

Proof. Using our definitions the construction of that diagram and its exactness follow from Lemma A.2. Indeed the pointed morphism $(\mathbb{A}^1_F)_+ \to \mathbb{A}^1_F/\Omega$ is trivial in $\mathcal{H}_\bullet(F)$, with Ω an open subset of $\mathbb{A}^1_F$. As a consequence, for any $F \in \mathcal{F}_k$, the cofibration sequence $(\mathbb{A}^1_F/\Omega) \to \Sigma(\Omega_+) \to \Sigma((\mathbb{A}^1_F)_+)$ is split.

Taking morphisms to (the base change to F) BM and letting Ω decrease we get a (colimit of split) exact sequence

$$0 \to M(F) \to M(F(T)) \to colim_\Omega[(\mathbb{A}^1_F/\Omega),(BM)|F]_{\mathcal{H}_\bullet(F)} \to 0$$

But by an easy adjunction, and convenient choices of k-smooth models of F and Ω we easily obtain an isomorphism $colim_\Omega[(\mathbb{A}^1_F/\Omega),(BM)|F]_{\mathcal{H}_\bullet(F)} \to \oplus_{P\in(\mathbb{A}^1_F)^{(1)}} H^1_P(\mathbb{A}^1_F;M)$.□

Corollary A.6 *Let k be a field. Let $\mathcal{G}$ be sheaf of groups on Sm_k. Then there exists a universal morphism $\mathcal{G} \to \mathcal{G}_{\mathbb{A}1}$ from $\mathcal{G}$ to strongly $\mathbb{A}^1$-invariant sheaf. If $\mathcal{G}$ is abelian, so is $\mathcal{G}_{\mathbb{A}1}$. Moreover, for any field $F \in \mathcal{F}_k$, the morphism $\mathcal{G}|_F \to \mathcal{G}_{\mathbb{A}1}|_F$ is still the universal morphism from $\mathcal{G}|_F$ to strongly $\mathbb{A}^1$-invariant sheaf.*

Proof. Let $B\mathcal{G} \to L_{\mathbb{A}^1}(B\mathcal{G})$ be the $\mathbb{A}^1$-localization of the pointed space $B\mathcal{G}$. From [39, Corollary 1.24 p. 104 & Proposition 2.8 p. 108], the inverse image functor f^*, where $f : Spec(k(T)) \to Spec(k)$ is the structure morphism, is exact, preserves $\mathbb{A}^1$-weak equivalences and $\mathbb{A}^1$-local objects.

Let's denote by $\mathcal{G}'$ the sheaf $\pi_1^{\mathbb{A}^1}(L_{\mathbb{A}^1}(B\mathcal{G}))$. By Theorem 1.26 $\mathcal{G}'|k(T)$ is a strongly $\mathbb{A}^1$-invariant sheaf and $B(f^*\mathcal{G}) \cong f^*(B\mathcal{G}) \to f^*(L_{\mathbb{A}^1}(B\mathcal{G}))$ is the $\mathbb{A}^1$-localization of $B(f^*\mathcal{G})$.

Now let's denote by $\mathcal{G}$" the $\pi_1^{\mathbb{A}^1}(L_{\mathbb{A}^1}(B\mathcal{G}'))$. One has a canonical morphism $\mathcal{G} \to \mathcal{G}' \to \mathcal{G}$" and any morphism from $\mathcal{G}$ to a strongly $\mathbb{A}^1$-invariant sheaf H factors uniquely through $\mathcal{G} \to \mathcal{G}$".

It only remains to show $\mathcal{G}$" is strongly $\mathbb{A}^1$-invariant. By Corollary A.3 applied to $\mathcal{B} = L_{\mathbb{A}^1}(B\mathcal{G}')$, we see that for any $X \in Sm_k$ and any $n \geq 2$, the group $Hom_{\mathcal{H}_\bullet(k)}(\Sigma^n(X_+); L_{\mathbb{A}^1}(B\mathcal{G}))$ is trivial, as it injects into $Hom_{\mathcal{H}_\bullet(k(T))}(\Sigma^n(X_{k(T)+}); B(\mathcal{G}|_{k(T)}))$ which is trivial. This shows that the higher homotopy sheaves of $L_{\mathbb{A}^1}(B\mathcal{G}')$ are trivial, except π_1 (as we know from [39] that the $\mathbb{A}^1$-localization of a connected space is connected). Thus, $L_{\mathbb{A}^1}(B\mathcal{G}')$ is canonically isomorphic to $B(\mathcal{G}')$. Thus $\mathcal{G}$" is strongly $\mathbb{A}^1$-invariant.

To prove that if $\mathcal{G}$ is abelian so is $\mathcal{G}_{\mathbb{A}1}$ we conclude by using the fact [39] that there exists an $\mathbb{A}^1$-localization functor which commutes to finite products.

The last statement is clear by our construction.□

Theorem A.7 *Let k be a field and $\mathcal{B}$ be a pointed $\mathbb{A}^1$-connected space. Then $\pi_1^{\mathbb{A}^1}(\mathcal{B})$ is strongly $\mathbb{A}^1$-invariant.*

Proof. By Theorem 3.9 we know this result for any infinite field. We might thus assume k is finite but we will only use the fact that for any field F the field $F(T)$ is infinite! Let us denote by $\mathcal{G}$ the sheaf $\pi_1^{\mathbb{A}^1}(\mathcal{B})$.

We consider the canonical morphism $\mathcal{G} \to \mathcal{G}_{\mathbb{A}^1}$. Theorem 3.9 and Corollary A.6 implies that this morphism induces an isomorphism on any infinite $F \in \mathcal{K}_k$.

Now for any $F \in \mathcal{F}_k$, the morphisms $\mathcal{G}(F) \to \mathcal{G}(F(T))$ and $\mathcal{G}_{\mathbb{A}^1}(F) \to \mathcal{G}_{\mathbb{A}^1}(F(T))$ are monomorphism by Corollary A.4.

Thus for any $F \in \mathcal{F}_k$, the morphism $\mathcal{G}(F) \to \mathcal{G}_{\mathbb{A}^1}(F)$ is a monomorphism.

We now want to prove now that $\mathcal{G}(F) \to \mathcal{G}_{\mathbb{A}^1}(F)$ is surjective for any $F \in \mathcal{F}_k$.

Consider the simplicial homotopy fiber Γ of $\mathcal{B} \to B\mathcal{G}_{\mathbb{A}^1}$. It is an $\mathbb{A}^1$-local space as both $\mathcal{B}$ and $B\mathcal{G}_{\mathbb{A}^1}$ are. But clearly $\Gamma|_{k(T)}$ is $\mathbb{A}^1$-connected. By Corollary A.4, Γ is $\mathbb{A}^1$-connected. The long exact homotopy sequence of $\Gamma \to \mathcal{B} \to B\mathcal{G}_{\mathbb{A}^1}$ easily implies that $\mathcal{G}(F) \to \mathcal{G}_{\mathbb{A}^1}(F$ is onto for any $F \in \mathcal{F}_k$ and that $\pi_1^{\mathbb{A}^1}(\Gamma) = *$.

This implies that $\mathcal{G} \to \mathcal{G}_{\mathbb{A}^1}$ is an isomorphism.□

We finish the Appendix by proving:

Theorem A.8 *Assume k is a finite field. Let M_* be a $\mathbb{Z}$-graded $\mathbb{A}^1_k$-module. Then for each n, the unramified sheaf M_n obtained by Theorem 1.43 is strongly $\mathbb{A}^1$-invariant.*

Proof. From what has been done in Theorem 1.43 and right before, we know that each M_n defines an unramified sheaf of abelian groups which satisfies Axioms **(A2')**, **(A6)**. It also satisfies Axiom **(A5)** over infinite fields.

As in the proof of the previous Theorem, we consider the universal morphism $M_n \to (M_n)_{\mathbb{A}^1}$ to a strongly $\mathbb{A}^1$-invariant sheaf of groups. The latter is abelian by Corollary A.6.

The axiom **(HA)(i)** implies that for any $F \in \mathcal{F}_k$, the morphism $M_*(F) \to M_*(F(T))$ is injective. As in the proof of the previous Theorem, we get that the morphism $M_n \to (M_n)_{\mathbb{A}^1}$ is a monomorphism of sheaves.

Now we claim that as M_* satisfies Axiom **(HA)**, there is a canonical commutative diagram for any $F \in \mathcal{F}_k$

$$\begin{array}{ccccccc}
0 \to & M_n(F) & \to & M(F(T)) & \to & \oplus_{P\in(\mathbb{A}^1_F)^{(1)}} H^1_P(\mathbb{A}^1_F; M_n) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & (M_n)_{\mathbb{A}^1}(F) & \to & (M_n)_{\mathbb{A}^1}(F(T)) & \to & \oplus_{P\in(\mathbb{A}^1_F)^{(1)}} H^1_P(\mathbb{A}^1_F; (M_n)_{\mathbb{A}^1}) & \to 0
\end{array}$$

The bottom exact sequence being given by Corollary A.5. To check the commutativity, one may extends to an infinite field, where $M_n \to (M_n)_{\mathbb{A}^1}$ is an isomorphism. The point is then to check that the construction from A.5 gives back the exact sequence of Axiom **(HA)**.

Now the middle vertical arrow is an isomorphism by assumption. Thus it is clearly sufficient to prove that the morphisms $H^1_P(\mathbb{A}^1_F; M_n) \to H^1_P(\mathbb{A}^1_F; (M_n)_{\mathbb{A}^1})$ are monomorphisms. But as P is a tautological uniformizing element of its associated valuation we get on the left side an isomorphism $M_{n-1}(F[T]/P) \cong$

$H^1_P(\mathbb{A}^1_F; M_n)$ and using homotopy purity [39], we get on the left side an isomorphism $((M_n)_{\mathbb{A}^1})_{-1}(F[T]/P) \cong H^1_P(\mathbb{A}^1_F; (M_n)_{\mathbb{A}^1})$. These isomorphisms are easily checked to be compatible (base change to an infinite field) so that we reduce to proving that for each P, the induced morphism $M_{n-1}(F[T]/P) \to ((M_n)_{\mathbb{A}^1})_{-1}(F[T]/P)$ is injective. But this follows from the Lemma below.□

Lemma A.9 *Let $M \to N$ be a be monomorphism of sheaves of abelian groups on Sm_k. Then $M_{-1} \to N_{-1}$ is still a monomorphism.*

Proof. As the sheaves are abelian, the group $M(\mathbb{G}_m \times X)$ canonically splits as the direct sum $M(X) \oplus M_{-1}(X)$. Thus for any such X, the monomorphism $M(\mathbb{G}_m \times X) \to N(\mathbb{G}_m \times X)$ splits accordingly as a sum of monomorphisms.□

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Fabien Morel
Mathematisches Institut der LMU
Theresienstr. 39
D-80 333 Muenchen
morel@mathematik.uni-muenchen.de