THE INTEGRAL COHOMOLOGY RING OF $E_8/T^1{\cdot}E_7$

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Abstract. We determine the integral cohomology ring of the homogeneous space $E_8/T^1{\cdot}E_7$ by the Borel presentation and a method due to Toda. Then using the Gysin exact sequence associated with the circle bundle $S^1 \to E_8/E_7 \to E_8/T^1{\cdot}E_7$, we also determine the integral cohomology ring of E_8/E_7.

1. Introduction

Let G be a compact connected Lie group and H a centralizer of a toral subgroup. Then the homogeneous space G/H is called a *generalized flag manifold* and plays an important role in the modern mathematics such as algebraic topology, differential geometry and algebraic geometry. In fact, G/H admits a complex structure, Kähler structure and symplectic structure. In algebraic topology, it is a classical problem to determine the integral cohomology ring $H^*(G/H;\mathbb{Z})$ of G/H. Since the Chow ring of G/H, $A(G/H)$ is canonically isomorphic to $H^*(G/H;\mathbb{Z})$ ([10]), the determination of $H^*(G/H;\mathbb{Z})$ is of fundamental importance.

There are two descriptions of the cohomology ring $H^*(G/H;\mathbb{Z})$; One is the Borel presentation which uses the invariants of Weyl groups and the spectral sequence technique ([2]). The other is the Schubert presentation which uses the so called *Schubert calculus* ([1]). Using the Borel presentation of rational cohomology and the results on mod p cohomology $H^*(G;\mathbb{Z}/p\mathbb{Z})$ for all primes p, Toda initiated the research for computing the integral cohomology ring of G/H in [17]. Along the lines of his idea, the integral cohomology rings of various flag manifolds have been computed explicitly([18], [19], [12], [11], [15]). On the other hand, recently H. Duan developed extensively a multiplicative formula of Schubert classes which is a generalization of the Littlewood-Richerdson rule of Grassmann manifold ([8]). Furthermore, H. Duan

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and X. Zhao also computed the integral cohomology rings of the above flag manifolds independently of Toda's method ([9]). Until recently none of these two methods have been successful in computing the integral cohomology rings of homogeneous spaces of the exceptional Lie group E_8. The exceptional Lie group E_8 contains a closed connected subgroup $T^1\cdot E_7$ whose local type is $T^1\times E_7$, where T^1 is a certain one dimensional torus (see [12], §2). It is obtained as a centralizer of a certain one dimensional torus (see 2.1). Hence the homogeneous space $E_8/T^1{\cdot}E_7$ is a generalized flag manifold. In this paper, using the above method due to Borel and Toda, we compute the integral cohomology ring of $E_8/T^1\cdot E_7$ explicitly.

The motivation of the current work is to study the cohomology of the irreducible symmetric space $EIX=E_8/S^3{\cdot}E_7$ (see [12], §1), as well as the integral cohomology ring of the full flag manifold E_8/T, where T is a maximal torus in E_8, and the Chow ring of the corresponding complex algebraic group ([13], [14]). In fact, we will use Theorem 3.8 to compute $H^*(E_8/T;\mathbb{Z})$, which is the only remaining case among G/T's for G simple, in our forthcoming paper([16]). Moreover, the homogeneous space $E_8/T^1{\cdot}E_7$ is a generating variety of E_8 in the sense of Bott ([5]) and its integral cohomology ring is needed to compute the Pontrjagin ring $H_*(\Omega E_8;\mathbb{Z})$, where ΩE_8 denotes the based loop space of E_8.

The paper is organized as follows: In §2, we compute the rings of invariants of the Weyl groups of E_8 and $T^1{\cdot}E_7$. In §3, using these results and the Borel presentation of rational cohomology ring, we compute the rational cohomology ring of $E_8/T^1\cdot E_7$. Then, by investigating the integral cohomology ring of E_8/T in low degrees, we determine the integral cohomology ring of $E_8/T^1\cdot E_7$ explicitly. Furthermore, using the Gysin exact sequence associated with the circle bundle $S^1\longrightarrow E_8/E_7\longrightarrow E_8/T^1\cdot E_7$, we also determine the integral cohomology ring of E_8/E_7.

Throughout this paper, $\sigma_i(x_1,\dots,x_n)$ denotes the i-th elementary symmetric function in variables $x_1,\dots,x_n$.

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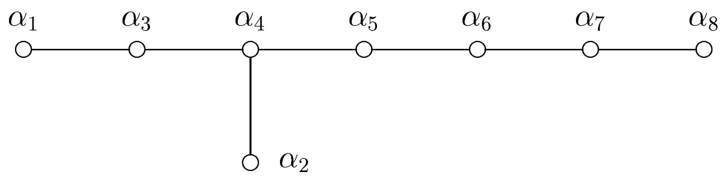
2. The rational invariant subalgebras of the Weyl groups

2.1. **Notations.** Let T be a maximal torus of E_8. According to [7], the Dynkin diagram of E_8 is as follows:



where α_i's are the simple roots. As usual we may regard each root as an element of $H^1(T;\mathbb{Z}) \cong H^2(BT;\mathbb{Z})$.

Let C_8 be the centralizer of a one dimensional torus determined by $\alpha_i = 0$ $(i \neq 8)$. Then as shown in ([12], §2),

$$C_8 = T^1 \cdot E_7 \quad \text{and} \quad T^1 \cap E_7 \cong \mathbb{Z}/2\mathbb{Z}.$$

The Weyl groups $W(\cdot)$ of E_8, C_8 are respectively given as follows:

$$W(E_8) = \langle s_i \ (1 \leq i \leq 8) \rangle, \quad W(C_8) = \langle s_i \ (i \neq 8) \rangle,$$

where s_i denotes the simple reflection corresponding to the simple root α_i.

Let $\{\omega_i\}_{1 \leq i \leq 8}$ be the fundamental weights corresponding to the system of the simple roots $\{\alpha_i\}_{1 \leq i \leq 8}$. We also regard each weight as an elememt of $H^2(BT;\mathbb{Z})$ and then $\{\omega_i\}_{1 \leq i \leq 8}$ forms a basis of $H^2(BT;\mathbb{Z})$. The action of s_i's on $\{\omega_i\}_{1 \leq i \leq 8}$ is given as follows:

$$s_i(\omega_k) = \begin{cases} \omega_i - \displaystyle\sum_{j=1}^{8} \frac{2(\alpha_i|\alpha_j)}{(\alpha_j|\alpha_j)}\omega_j & \text{if} \quad k = i, \\ \omega_k & \text{if} \quad k \neq i. \end{cases}$$

Next we define the elements

$$\begin{aligned}
t_8 &= \omega_8, \\
t_7 &= s_8(t_8) = \omega_7 - \omega_8, \\
t_6 &= s_7(t_7) = \omega_6 - \omega_7, \\
t_5 &= s_6(t_6) = \omega_5 - \omega_6, \\
t_4 &= s_5(t_5) = \omega_4 - \omega_5, \\
t_3 &= s_4(t_4) = \omega_2 + \omega_3 - \omega_4, \\
t_2 &= s_3(t_3) = \omega_1 + \omega_2 - \omega_3, \\
t_1 &= s_1(t_2) = -\omega_1 + \omega_2, \\
c_i &= \sigma_i(t_1, \ldots, t_8), \\
t &= \omega_2 = \frac{1}{3}c_1.
\end{aligned}$$

Then t and t_i's span $H^2(BT;\mathbb{Z})$, since each ω_i is an integral linear combination of t and t_i's and we have the following isomorphism:

$$H^*(BT;\mathbb{Z}) = \mathbb{Z}[t_1,\ldots,t_8,t]/(c_1 - 3t).$$

Furthermore, the action of s_i's on $\{t_i\}_{1\le i\le 8}$ and t is given by TABLE 1, where blanks indicate the trivial action.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
t_1	t_2	$t-t_2-t_3$						
t_2	t_1	$t-t_1-t_3$	t_3					
t_3		$t-t_1-t_2$	t_2	t_4				
t_4				t_3	t_5			
t_5					t_4	t_6		
t_6						t_5	t_7	
t_7							t_6	t_8
t_8								t_7
t		$2t-t_1-t_2-t_3$						

TABLE 1

Putting

$$\begin{aligned} u &= t_8, \\ \tau &= t - \frac{3}{2}u, \\ \tau_i &= t_i - \frac{1}{2}u \ (1 \le i \le 7), \end{aligned}$$

we have

$$H^*(BT;\mathbb{Q}) = \mathbb{Q}[u,\tau,\tau_1,\ldots\tau_7]/(\bar{c}_1 - 3\tau)$$

for $\bar{c}_1 = \tau_1 + \cdots + \tau_7$. The action of s_i's on $\{\tau_i\}_{1\le i\le 7}$ and τ is given by TABLE 2, where blanks also indicate the trivial action.

Since $E_7 \cap T = T'$ is a maximal torus of E_7, we have a commutative diagram of natural maps:

$$(2.1) \qquad \begin{array}{ccccc} E_7/T' & \xrightarrow{\sim} & C_8/T & \xrightarrow{i} & E_8/T \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \iota_0} \\ BT' & \xrightarrow{g} & BT & \xrightarrow{=} & BT. \end{array}$$

Since E_8 is 2-connected, $\iota_0^* : H^2(BT;\mathbb{Z}) \xrightarrow{\sim} H^2(E_8/T;\mathbb{Z})$. Under this isomorphism, we denote the ι_0^*-images of t_i $(1 \le i \le 8)$, t by the same letters. Thus we have the generators t_i $(1 \le i \le 8)$, $t \in H^2(E_8/T;\mathbb{Z})$

	s_1	s_2	s_3	s_4	s_5	s_6	s_7
τ_1	τ_2	$\tau-\tau_2-\tau_3$					
τ_2	τ_1	$\tau-\tau_1-\tau_3$	τ_3				
τ_3		$\tau-\tau_1-\tau_2$	τ_2	τ_4			
τ_4				τ_3	τ_5		
τ_5					τ_4	τ_6	
τ_6						τ_5	τ_7
τ_7							τ_6
τ		$-\tau+\tau_4+\tau_5+\tau_6+\tau_7$					
u							

TABLE 2

with a relation $c_1 = 3t$. We donote the generators t_i $(1 \le i \le 7), x$ in [19] by t'_i $(1 \le i \le 7)$, t'. Then, by a similar arguments to that in ([19], §1), we have

$$g^*(t_i) = t'_i \ (1 \le i \le 7),\ g^*(t_8) = 0,\ g^*(t) = t'. \tag{2.2}$$

2.2. **Invariant subalgebra of $W(C_8)$.** We recall the rational invariant forms for E_7 given in [19]. We put

$$x'_i = 2t'_i - t' \ (1 \le i \le 7), \quad x'_8 = t'.$$

Then the set

$$S' = \{\pm\,(x'_i + x'_j)\ (1 \le i < j \le 8)\} \subset H^2(BT';\mathbb{Q})$$

is invariant under the action of $W(E_7)$. Thus we have $W(E_7)$-invariant forms

$$I'_n = \sum_{y \in S'} y^n \in H^{2n}(BT';\mathbb{Q})^{W(E_7)}. \tag{2.3}$$

Then direct computation using the same method as in ([19], §2) yields the following results:

(2.4)

$$\begin{aligned}
I'_2 &= -2^5 \cdot 3(c'_2 - 4t'^2),\\
I'_6 &\equiv 2^8 \cdot 3^2({c'_3}^2 + 8c'_6) \quad \bmod (t', \mathfrak{a}_6'),\\
I'_8 &\equiv 2^{12} \cdot 5(2{c'_4}^2 - 3c'_3c'_5) \quad \bmod (t', \mathfrak{a}_8'),\\
I'_{10} &\equiv 2^{12} \cdot 3^2 \cdot 5 \cdot 7({c'_5}^2 - 4c'_3c'_7) \quad \bmod (t', \mathfrak{a}_{10}'),\\
I'_{12} &\equiv 2^{15} \cdot 3^2 \cdot 5(-54{c'_6}^2 + 18c'_5c'_7 - c'_3c'_4c'_5) \quad \bmod (t', \mathfrak{a}_{12}'),\\
I'_{14} &\equiv 2^{16} \cdot 3 \cdot 7 \cdot 11 \cdot 29(2{c'_7}^2 + 2c'_3c'_4c'_7 - c'_3c'_5c'_6) \quad \bmod (t', \mathfrak{a}_{14}'),\\
I'_{18} &\equiv 2^{21} \cdot 5 \cdot 1229(-126c'_5c'_6c'_7 - 5c'_3c'_4c'_5c'_6) \quad \bmod (t', \mathfrak{a}_{18}'),
\end{aligned}$$

where $c'_i = \sigma_i(t'_1, \ldots, t'_7)$ and $\mathfrak{a}_i{}'$ denotes the ideal of $H^*(BT'; \mathbb{Q})$ generated by I'_j for $j < i$ with $j \in \{2, 6, 8, 10, 12, 14, 18\}$. We also recall the following result:

Proposition 2.1 ([19], Lemma 2.1)**.** *The rational invariant subalgebra of the Weyl group $W(E_7)$ is given as follows*:

$$H^*(BT'; \mathbb{Q})^{W(E_7)} = \mathbb{Q}[I'_2, I'_6, I'_8, I'_{10}, I'_{12}, I'_{14}, I'_{18}].$$

TABLE 2 shows that the action of $W(C_8)$ on $\tau, \tau_1, \ldots, \tau_7$ is the same as that of $W(E_7)$ on $t', t'_1, \ldots, t'_7$. Therefore we have

Lemma 2.2. *If we represent*

$$I'_n = \psi_n(t', t'_1, \ldots, t'_7) \in H^{2n}(BT'; \mathbb{Q})^{W(E_7)},$$

the rational invariant subalgebra of the Weyl group $W(C_8)$ is given as follows:

$$H^*(BT; \mathbb{Q})^{W(C_8)} = \mathbb{Q}[u, J_2, J_6, J_8, J_{10}, J_{12}, J_{14}, J_{18}],$$

where

$$J_n = \psi_n(\tau, \tau_1, \ldots, \tau_7) \in H^{2n}(BT; \mathbb{Q})^{W(C_8)}.$$

2.3. **Invariant subalgebra of $W(E_8)$.** We put

$$\xi_i = 2t_i - \frac{2}{3}t \ (1 \le i \le 8), \quad \xi_9 = -\frac{2}{3}t.$$

Then we see from TABLE 1 that the set

$$S = \{\pm\,(\xi_i - \xi_j)\ (1 \le i < j \le 9),\ \pm\,(\xi_i + \xi_j + \xi_k)\ (1 \le i < j < k \le 9)\}$$

is invariant under the action of $W(E_8)$. In fact, S is an orbit of $2\omega_8$ under the action of $W(E_8)$. Thus we have $W(E_8)$-invariant forms

$$I_n = \sum_{y \in S} y^n \in H^{2n}(BT; \mathbb{Q})^{W(E_8)}.$$

Let us compute I_n's in the following way; We put

$$s_n = {\xi_1}^n + \cdots + {\xi_9}^n, \quad d_n = \sigma_n(\xi_1, \ldots, \xi_9).$$

Then s_n's and d_n's are related to each other by the Newton formula:

$$(2.5) \qquad s_n = \sum_{i=1}^{n-1} (-1)^{i-1} s_{n-i} d_i + (-1)^{n-1} n d_n \ \ (d_n = 0 \ \text{ for } \ n > 9).$$

Note that

$$s_0 = 9, \quad s_1 = d_1 = \xi_1 + \cdots + \xi_9 = 0.$$

Then

$$
\begin{aligned}
\sum_n \frac{I_n}{n!} &= \sum_{i<j} e^{\xi_i-\xi_j} + \sum_{i<j} e^{-\xi_i+\xi_j} \sum_{i<j<k} e^{\xi_i+\xi_j+\xi_k} + \sum_{i<j<k} e^{-\xi_i-\xi_j-\xi_k} \\
&= \left(\sum_i e^{\xi_i}\right)\left(\sum_j e^{-\xi_j}\right) - 9 + \frac{1}{3}\left(\sum_i e^{3\xi_i} + \sum_i e^{-3\xi_i}\right) \\
&- \frac{1}{2}\left\{\left(\sum_i e^{\xi_i}\right)\left(\sum_i e^{2\xi_i}\right) + \left(\sum_i e^{-\xi_i}\right)\left(\sum_i e^{-2\xi_i}\right)\right\} \\
&+ \frac{1}{6}\left\{\left(\sum_i e^{\xi_i}\right)^3 + \left(\sum_i e^{-\xi_i}\right)^3\right\} \\
&= \left(\sum_n \frac{s_n}{n!}\right)\left(\sum_m \frac{(-1)^m s_m}{m!}\right) - 9 + \frac{1}{3}\left(\sum_n \frac{3^n s_n}{n!} + \sum_n \frac{(-1)^n 3^n s_n}{n!}\right) \\
&- \frac{1}{2}\left\{\left(\sum_n \frac{s_n}{n!}\right)\left(\sum_m \frac{2^m s_m}{m!}\right) + \left(\sum_n \frac{(-1)^n s_n}{n!}\right)\left(\sum_m \frac{(-1)^m 2^m s_m}{m!}\right)\right\} \\
&+ \frac{1}{6}\left\{\left(\sum_n \frac{s_n}{n!}\right)^3 + \left(\sum_n \frac{(-1)^n s_n}{n!}\right)^3\right\}.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
I_n &= \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} s_i s_{n-i} + 2\cdot 3^{n-1} s_n \\
&- \sum_{i=0}^{n} \binom{n}{i} 2^{n-i} s_i s_{n-i} + \frac{1}{3}\sum_{i=0}^{n}\sum_{j=0}^{n-i} \binom{n}{i}\binom{n-i}{j} s_i s_j s_{n-i-j}
\end{aligned}
\tag{2.6}
$$

for n even. On the other hand, since $\xi_i = 2t_i - \frac{2}{3}t$ $(1 \le i \le 8)$ and $\xi_9 = -\frac{2}{3}t$, we have

$$
\begin{aligned}
\sum_{n=0}^{9} d_n = \prod_{i=1}^{9}(1+\xi_i) &= \left(1-\frac{2}{3}t\right)\prod_{i=1}^{8}\left(1-\frac{2}{3}t+2t_i\right) \\
&= \left(1-\frac{2}{3}t\right)\sum_{i=0}^{8}\left(1-\frac{2}{3}t\right)^{8-i} 2^i c_i
\end{aligned}
$$

and therefore

$$(2.7)\qquad d_n = \sum_{i=0}^{n} \left\{ \binom{8-i}{n-i} + \binom{8-i}{n-i-1} \right\} \left(-\frac{2}{3}t\right)^{n-i} 2^i c_i$$

for $1 \le n \le 9$. Using (2.6), (2.5) and (2.7), we can compute I_n's explicitly and obtain the following results:

Lemma 2.3.

(i) $I_2 = -2^5 \cdot 3 \cdot 5(c_2 - 4t^2)$,

(ii) $I_8 \equiv 2^{14} \cdot 3 \cdot 5\{-18c_8 - 3c_3c_5 + 2c_4^2 + t(12c_7 - 3c_3c_4) + t^2(-6c_6 + 3c_3^2) + 12t^3c_5 + 2t^4c_4 - 12t^5c_3 + 14t^8\} \mod \tilde{\mathfrak{a}}_8$,

(iii) $I_{12} \equiv 2^{18} \cdot 3^5 \cdot 7 \left(c_6^2 - \frac{5}{3}c_5c_7 + \frac{5}{54}c_3c_4c_5 - \frac{1}{6}c_3^2c_6 + \frac{1}{24}c_3^4\right) \mod (t, c_8, \tilde{\mathfrak{a}}_{12})$,

(iv) $I_{14} \equiv 2^{20} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \left(c_7^2 - \frac{1}{2}c_3c_5c_6 + \frac{1}{3}c_3c_4c_7 + \frac{1}{6}c_4c_5^2\right) \mod (t, c_8, \tilde{\mathfrak{a}}_{14})$,

(v) $I_{18} \equiv -2^{23} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \left(c_3^6 - 7c_3^4c_6 + \frac{29}{9}c_3^3c_4c_5 + 182c_3^2c_5c_7 + 75c_3c_5^3 - \frac{476}{3}c_3c_4c_5c_6 - 24c_5c_6c_7\right) \mod (t, c_8, \tilde{\mathfrak{a}}_{18})$,

(vi) $I_{20} \equiv 2^{27} \cdot 3^3 \cdot 5^2 \cdot 11 \cdot 17 \cdot 41 \left(\frac{1}{144}c_5^4 - \frac{1}{18}c_3c_5^2c_7 - \frac{1}{54}c_3^2c_4c_5^2 - \frac{1}{27}c_3^3c_4c_7 + \frac{1}{18}c_3^3c_5c_6\right) \mod (t, c_8, \tilde{\mathfrak{a}}_{20})$,

(vii) $I_{24} \equiv 2^{32} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 19 \cdot 199 \left(\frac{31}{8640}c_3^5c_4c_5 + \frac{1}{480}c_3^4c_5c_7 + \frac{337}{25920}c_3^3c_5^3 - \frac{71}{4320}c_3^3c_4c_5c_6 + \frac{31}{240}c_3^2c_5c_6c_7 + \frac{31}{480}c_3c_5^3c_6 - \frac{22}{135}c_3c_4c_5^2c_7 - \frac{1}{120}c_4c_5^4\right) \mod (t, c_8, \tilde{\mathfrak{a}}_{20})$,

(viii) $I_{30} \equiv 2^{38} \cdot 3^4 \cdot 5^5 \cdot 7 \cdot 11 \cdot 13 \cdot 61 \left(-\frac{599}{51480}c_3^5c_4c_5c_6 + \frac{47}{34560}c_3^5c_5^3 + \frac{1519}{25920}c_3^4c_5c_6c_7 + \frac{6293}{7290}c_3^3c_4c_5^2c_7 - \frac{32537}{25920}c_3^3c_5^3c_6 + \frac{189919}{466560}c_3^2c_4c_5^4 + \frac{2012}{1215}c_3c_4c_5^2c_6c_7 - \frac{16693}{25920}c_3c_5^4c_7 - \frac{223}{6480}c_4c_5^4c_6 - \frac{1}{1728}c_5^6\right) \mod (t, c_8, \tilde{\mathfrak{a}}_{20})$, *where* $c_i = \sigma_i(t_1, \ldots, t_8)$ *and* $\tilde{\mathfrak{a}}_i$ *denotes the ideal of* $H^*(BT;\mathbb{Q})$ *generated by* I_j *for* $j < i$ *with* $j \in \{2, 8, 12, 14, 18, 20, 24, 30\}$.

Next consider the following elements of $H^*(BT;\mathbb{Q})$:

(2.8)

$$\begin{aligned}
\tilde{I}_{20} &= 9u^{20} + 45u^{14}v + 12u^{12}w + 60u^8v^2 + 30u^4vw + 10u^2v^3 + 3w^2, \\
\tilde{I}_{24} &= 11u^{24} + 60u^{18}v + 21u^{14}w + 105u^{12}v^2 + 60u^8vw + 60u^6v^3 + 9u^4w^2 \\
&\quad +30u^2v^2w + 5v^4, \\
\tilde{I}_{30} &= -9u^{30} - 24u^{24}v - 12u^{20}w + 36u^{14}vw - 40u^{12}v^3 - 12u^{10}w^2 + 120u^8v^2w \\
&\quad -140u^6v^4 + 24u^4vw^2 - 40u^2v^3w - 16v^5 - 8w^3,
\end{aligned}$$

where

$$(2.9) \qquad \begin{aligned}
u &= t_8, \\
v &= \frac{1}{46080}J_6 - \frac{273}{640}u^6, \\
w &= \frac{1}{15482880}J_{10} - \frac{55}{24}u^4v - \frac{666919}{645120}u^{10}.
\end{aligned}$$

Remark 2.4. *We have chosen the elements* u, v, w *so that they become the generators of the integral cohomology ring of* $E_8/T^1 \cdot E_7$ *(see* 3.3*).*

In order to find the relations among J_n's, I_n's and $\tilde{I}_{20}, \tilde{I}_{24}, \tilde{I}_{30}$, we consider the ring of invariants:

$$A = H^*(BT;\mathbb{Q})^{\langle s_1, s_3, \ldots, s_7\rangle}.$$

Then A is a subalgebra of $H^*(BT;\mathbb{Q})$ containing $H^*(BT;\mathbb{Q})^{W(C_8)}$. More explicitly, we have

$$(2.10) \qquad A = \mathbb{Q}[u, c_1, c_2, \ldots, c_7].$$

In fact, we can show (2.10) as follows; Putting

$$\tilde{c}_i = \sigma_i(t_1, \ldots, t_7),$$

we have

$$c_n = \tilde{c}_n + u\tilde{c}_{n-1} \quad (1 \leq n \leq 8),$$

since

$$\sum_{n=0}^{8} c_n = \prod_{i=1}^{8}(1 + t_i) = (1 + u)\prod_{i=1}^{7}(1 + t_i) = (1 + u)\sum_{n=0}^{7}\tilde{c}_n.$$

Conversely, one can express

$$\tilde{c}_n = c_n - uc_{n-1} + u^2c_{n-2} - \cdots + (-1)^n u^n \quad (1 \leq n \leq 7).$$

In particular, the following relation holds:

$$(2.11) \qquad c_8 = uc_7 - u^2c_6 + u^3c_5 - u^4c_4 + u^5c_3 - u^6c_2 + u^7c_1 - u^8.$$

Therefore, by TABLE 1, we have

$$\begin{aligned} A &= H^*(BT;\mathbb{Q})^{\langle s_1, s_3, \ldots, s_7\rangle} \\ &= \mathbb{Q}[t_1, t_2, \ldots, t_7, u]^{\langle s_1, s_3, \ldots, s_7\rangle} \\ &= \mathbb{Q}[u, \tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_7] \\ &= \mathbb{Q}[u, c_1, c_2, \ldots, c_8]/(c_8 - uc_7 + u^2c_6 - \cdots + u^8) \\ &= \mathbb{Q}[u, c_1, c_2, \ldots, c_7], \end{aligned}$$

which has shown (2.10).

Denote by

$$\mathfrak{a}_i \subset A \text{ (resp. } \mathfrak{b}_i \subset H^*(BT;\mathbb{Q})^{W(C_8)}),$$

the ideal of A (resp. of $H^*(BT;\mathbb{Q})^{W(C_8)}$) generated by I_j's for $j < i$ where $j \in \{2, 8, 12, 14, 18, 20, 24, 30\}$.

The remainder of this section is devoted to proving the next lemma:

Lemma 2.5. *In* $H^*(BT;\mathbb{Q})^{W(C_8)} = \mathbb{Q}[u, J_2, J_6, J_8, J_{10}, J_{12}, J_{14}, J_{18}]$, *we have*

(i) $I_2 = 5J_2 + 120u^5$, $\quad I_8 = 2^2 \cdot 3J_8 + (decomp.)$,

$$I_{12} = -2^2 \cdot 7J_{12} + (decomp.), \qquad I_{14} = \frac{2^3 \cdot 3 \cdot 5}{29} J_{14} + (decomp.),$$

$$I_{18} = -\frac{2^4 \cdot 3 \cdot 5^2 \cdot 7 \cdot 13}{1229} J_{18} + (decomp.),$$

where $(decomp.)$ *means decomposable elements.*

(ii) $I_{20} \equiv 2^{27} \cdot 3^2 \cdot 5^2 \cdot 11 \cdot 17 \cdot 41 \tilde{I}_{20} \mod \mathfrak{b}_{20}$,

$$I_{24} \equiv 2^{32} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 199 \tilde{I}_{24} \mod \mathfrak{b}_{20},$$

$$I_{30} \equiv 2^{35} \cdot 3^4 \cdot 5^5 \cdot 7 \cdot 11 \cdot 13 \cdot 61 \tilde{I}_{30} \mod \mathfrak{b}_{20}.$$

Suppose Lemma 2.5 for the moment. Then we have the following:

Lemma 2.6. *The rational invariant subalgebra of the Weyl group* $W(E_8)$ *is given as follows*:

$$H^*(BT;\mathbb{Q})^{W(E_8)} = \mathbb{Q}[I_2, I_8, I_{12}, I_{14}, I_{18}, I_{20}, I_{24}, I_{30}].$$

Proof. By Lemma 2.5, I_i is not a polynomial of I_j's for $j < i$ where $i = 2, 8, 12, 14, 18, 20, 24, 30$. Since

$$H^*(BT;\mathbb{Q})^{W(E_8)} \cong H^*(BE_8;\mathbb{Q}) = \mathbb{Q}[y_4, y_{16}, y_{24}, y_{28}, y_{36}, y_{40}, y_{48}, y_{60}]$$

with $\deg(y_i) = i$, we have the required result. □

Proof of Lemma 2.5 (i). Since $\tau_i = t_i - \frac{1}{2}u$ $(1 \le i \le 7)$ and $\tau = t - \frac{3}{2}u$, we have

$$\tau_i \equiv t_i \ (1 \le i \le 7), \ \tau \equiv t \mod (u).$$

Therefore, putting

$$\bar{c}_i = \sigma_i(\tau_1, \ldots, \tau_7),$$

we obtain

$$c_n \equiv \bar{c}_n \ (1 \le n \le 7), \ c_8 \equiv 0 \mod (u).$$

Then, in view of (2.4) and Lemma 2.2, we have

(2.12)

$$\begin{aligned}
J_2 &= -2^5 \cdot 3(\bar{c}_2 - 4\tau^2) \\
&\equiv -2^5 \cdot 3c_2 \mod (t, u), \\
J_6 &\equiv 2^8 \cdot 3^2(\bar{c}_3^2 + 8\bar{c}_6) \mod (\tau, \bar{\mathfrak{a}}_6) \\
&\equiv 2^8 \cdot 3^2(c_3^2 + 8c_6) \mod (t, u, \bar{\mathfrak{a}}_6), \\
J_8 &\equiv 2^{12} \cdot 5(2\bar{c}_4^2 - 3\bar{c}_3\bar{c}_5) \mod (\tau, \bar{\mathfrak{a}}_8) \\
&\equiv 2^{12} \cdot 5(2c_4^2 - 3c_3c_5) \mod (t, u, \bar{\mathfrak{a}}_8), \\
J_{10} &\equiv 2^{12} \cdot 3^2 \cdot 5 \cdot 7(\bar{c}_5^2 - 4\bar{c}_3\bar{c}_7) \mod (\tau, \bar{\mathfrak{a}}_{10}) \\
&\equiv 2^{12} \cdot 3^2 \cdot 5 \cdot 7(c_5^2 - 4c_3c_7) \mod (t, u, \bar{\mathfrak{a}}_{10}), \\
J_{12} &\equiv 2^{15} \cdot 3^2 \cdot 5(-54\bar{c}_6^2 + 18\bar{c}_5\bar{c}_7 - \bar{c}_3\bar{c}_4\bar{c}_5) \mod (\tau, \bar{\mathfrak{a}}_{12}) \\
&\equiv 2^{15} \cdot 3^2 \cdot 5(-54c_6^2 + 18c_5c_7 - c_3c_4c_5) \mod (t, u, \bar{\mathfrak{a}}_{12}), \\
J_{14} &\equiv 2^{16} \cdot 3 \cdot 7 \cdot 11 \cdot 29(2\bar{c}_7^2 + 2\bar{c}_3\bar{c}_4\bar{c}_7 - \bar{c}_3\bar{c}_5\bar{c}_6) \mod (\tau, \bar{\mathfrak{a}}_{14}) \\
&\equiv 2^{16} \cdot 3 \cdot 7 \cdot 11 \cdot 29(2c_7^2 + 2c_3c_4c_7 - c_3c_5c_6) \mod (t, u, \bar{\mathfrak{a}}_{14}), \\
J_{18} &\equiv 2^{21} \cdot 5 \cdot 1229(-126\bar{c}_5\bar{c}_6\bar{c}_7 - 5\bar{c}_3\bar{c}_4\bar{c}_5\bar{c}_6) \mod (\tau, \bar{\mathfrak{a}}_{18}) \\
&\equiv 2^{21} \cdot 5 \cdot 1229(-126c_5c_6c_7 - 5c_3c_4c_5c_6) \mod (t, u, \bar{\mathfrak{a}}_{18}),
\end{aligned}$$

where $\bar{\mathfrak{a}}_i$ denotes the ideal of $H^*(BT;\mathbb{Q})^{W(C_8)}$ generated by J_j's for $j < i$ with $j \in \{2, 6, 8, 10, 12, 14, 18\}$.

Now we prove the last formula of (i): For degree reasons, in $H^*(BT;\mathbb{Q})^{W(C_8)} = \mathbb{Q}[u, J_2, J_6, J_8, J_{10}, J_{12}, J_{14}, J_{18}]$, we can put

(2.13) $$I_{18} = \alpha_{18}J_{18} + (\text{decomp.})$$

for some $\alpha_{18} \in \mathbb{Q}$. On the other hand, by using (2.10) and (2.12), we have

$$\begin{aligned}
A/(t,u,\bar{\mathfrak{a}}_{18}) &= A/(t,u,J_2,J_6,J_8,J_{10},J_{12},J_{14}) \\
&= \mathbb{Q}[u,c_1,c_2,c_3,c_4,c_5,c_6,c_7] \\
&\qquad \Big/ \begin{pmatrix} t,\ u,\ c_2,\ c_3^2+8c_6, c_4^2-\frac{3}{2}c_3c_5,\ c_5^2-4c_3c_7, \\ c_6^2-\frac{1}{3}c_5c_7+\frac{1}{54}c_3c_4c_5,\ c_7^2+c_3c_4c_7-\frac{1}{2}c_3c_5c_6 \end{pmatrix} \\
&= \mathbb{Q}[c_3,c_4,c_5,c_6,c_7] \\
&\qquad \Big/ \begin{pmatrix} c_3^2+8c_6,\ c_4^2-\frac{3}{2}c_3c_5,\ c_5^2-4c_3c_7, \\ c_6^2-\frac{1}{3}c_5c_7+\frac{1}{54}c_3c_4c_5,\ c_7^2+c_3c_4c_7-\frac{1}{2}c_3c_5c_6 \end{pmatrix}.
\end{aligned}$$

We consider (2.13) in the ring $A/(t,u,\bar{\mathfrak{a}}_{18})$. Then, by Lemma 2.3 (v) and (2.12), we have

$$\begin{aligned}
I_{18} &\equiv -2^{25}\cdot 3\cdot 5^3\cdot 7\cdot 13(-126c_5c_6c_7-5c_3c_4c_5c_6), \\
J_{18} &\equiv 2^{21}\cdot 5\cdot 1229(-126c_5c_6c_7-5c_3c_4c_5c_6).
\end{aligned}$$

Therefore $\alpha_{18} = -\dfrac{2^4\cdot 3\cdot 5^2\cdot 7\cdot 13}{1229}$. Similar tedious computation gives the other formulas. □

Before proceeding the proof of Lemma 2.5 (ii), we need the following lemma:

Lemma 2.7. *Explicit forms of $W(C_8)$-invariants J_6, J_{10} are given as follows*:

$$\begin{aligned}
J_6 = {} & 2^8\cdot 3\{24\bar{c}_6+3\bar{c}_3^2-4\bar{c}_2\bar{c}_4-2\bar{c}_2^3+(-12\bar{c}_5-6\bar{c}_2\bar{c}_3)\tau+(31\bar{c}_2^2+16\bar{c}_4)\tau^2 \\
& +12\bar{c}_3\tau^3-136\bar{c}_2\tau^4+188\tau^6\}, \\
J_{10} = {} & 2^{12}\cdot 3\{105\bar{c}_5^2-420\bar{c}_3\bar{c}_7+90\bar{c}_2\bar{c}_3\bar{c}_5-60\bar{c}_2\bar{c}_4^2+300\bar{c}_2^2\bar{c}_6+15\bar{c}_2^2\bar{c}_3^2 \\
& -20\bar{c}_2^3\bar{c}_4-2\bar{c}_2^5+(-30\bar{c}_2^3\bar{c}_3-330\bar{c}_2^2\bar{c}_5+480\bar{c}_2\bar{c}_7+90\bar{c}_2\bar{c}_3\bar{c}_4-210\bar{c}_4\bar{c}_5)\tau \\
& +(270\bar{c}_2^2\bar{c}_4-210\bar{c}_2\bar{c}_3^2-150\bar{c}_3\bar{c}_5-2220\bar{c}_2\bar{c}_6+75\bar{c}_2^4+345\bar{c}_4^2)\tau^2 \\
& +(480\bar{c}_2{}^2\bar{c}_3-570\bar{c}_3\bar{c}_4+2070\bar{c}_2\bar{c}_5-660\bar{c}_7)\tau^3 \\
& +(-1050\bar{c}_2\bar{c}_4+4080\bar{c}_6-950\bar{c}_2^3+705\bar{c}_3^2)\tau^4+(-2250\bar{c}_2\bar{c}_3-3420\bar{c}_5)\tau^5 \\
& +(1580\bar{c}_4+5165\bar{c}_2^2)\tau^6+2820\bar{c}_3\tau^7-12360\bar{c}_2\tau^8+10868\tau^{10}\},
\end{aligned}$$

where $\bar{c}_i = \sigma_i(\tau_1,\ldots,\tau_7)$.

Proof. Since J_n has the same expression as I'_n replacing t', c'_i with $\tau, \bar{c}_i$ (Lemma 2.2), we have to compute the $W(E_7)$-invariant forms I'_6, I'_{10} explicitly. But this can be done from the data in ([19], §2). $\square$

We can rewrite J_6, J_{10} in terms of t, u, c_i $(2 \le i \le 7)$; Since $u = t_8$ and $\tau_i = t_i - \frac{1}{2}u$ $(1 \le i \le 7)$, we have

$$\left(1+\frac{1}{2}u\right)\sum_{n=0}^{7}\bar{c}_n = \left(1+\frac{1}{2}u\right)\prod_{i=1}^{7}(1+\tau_i) = \left(1+\frac{1}{2}u\right)\prod_{i=1}^{7}\left(1-\frac{1}{2}u+t_i\right)$$
$$= \prod_{i=1}^{8}\left(1-\frac{1}{2}u+t_i\right) = \sum_{i=0}^{8}\left(1-\frac{1}{2}u\right)^{8-i} c_i$$

and hence

$$\bar{c}_n + \frac{1}{2}u\bar{c}_{n-1} = \sum_{i=0}^{n}\binom{8-i}{n-i}\left(-\frac{1}{2}\right)^{n-i} c_i u^{n-i} \quad (1 \le n \le 7).$$

From this, we obtain

(2.14)

$$\begin{aligned}
\bar{c}_1 &= 3t - \frac{9}{2}u,\\
\bar{c}_2 &= c_2 - 12tu + \frac{37}{4}u^2,\\
\bar{c}_3 &= c_3 - \frac{7}{2}c_2u + \frac{87}{4}tu^2 - \frac{93}{8}u^3,\\
\bar{c}_4 &= c_4 - 3c_3u + \frac{11}{2}c_2u^2 - 24tu^3 + \frac{163}{16}u^4,\\
\bar{c}_5 &= c_5 - \frac{5}{2}c_4u + 4c_3u^2 - \frac{21}{4}c_2u^3 + \frac{297}{16}tu^4 - \frac{219}{32}u^5,\\
\bar{c}_6 &= c_6 - 2c_5u + \frac{11}{4}c_4u^2 - \frac{13}{4}c_3u^3 + \frac{57}{16}c_2u^4 - \frac{45}{4}tu^5 + \frac{247}{64}u^6,\\
\bar{c}_7 &= c_7 - \frac{3}{2}c_6u + \frac{7}{4}c_5u^2 - \frac{15}{8}c_4u^3 + \frac{31}{16}c_3u^4 - \frac{63}{32}c_2u^5 + \frac{381}{64}tu^6 - \frac{255}{128}u^7.
\end{aligned}$$

On the other hand, by Lemma 2.3 (ii) and (2.11), we have

$$\begin{aligned}
I_8 &\equiv 2^{14}\cdot 3\cdot 5(2c_4^2 - 3c_3c_5) \mod (t, c_8, \mathfrak{a}_8),\\
c_8 &= uc_7 - u^2c_6 + u^3c_5 - u^4c_4 + u^5c_3 - u^6c_2 + u^7c_1 - u^8\\
&\equiv uc_7 - u^2c_6 + u^3c_5 - u^4c_4 + u^5c_3 - u^8 \mod (t, \mathfrak{a}_8),
\end{aligned}$$

and hence

$$
\begin{aligned}
c_4^2 &\equiv \frac{3}{2}c_3c_5 \mod (t, c_8, \mathfrak{a}_{12}),\\
u^8 &\equiv uc_7 - u^2c_6 + u^3c_5 - u^4c_4 + u^5c_3 \mod (t, c_8, \mathfrak{a}_8).
\end{aligned}
$$

Therefore, by (2.9) and Lemma 2.7, we obtain
(2.15)

$$
\begin{aligned}
v &= \frac{1}{46080}J_6 - \frac{273}{640}u^6\\
&\equiv \frac{2}{5}c_6 + \frac{1}{20}c_3^2 - \frac{1}{2}c_5u + \frac{1}{3}c_4u^2 - \frac{1}{2}c_3u^3 \mod (t, \mathfrak{a}_8),\\
w &= \frac{1}{15482880}J_{10} - \frac{55}{24}u^4v - \frac{666919}{645120}u^{10}\\
&\equiv \frac{1}{12}c_5^2 - \frac{1}{3}c_3c_7 + \left(\frac{1}{2}c_3c_6 - \frac{1}{6}c_4c_5\right)u - \frac{1}{6}c_3c_5u^2 + \left(-c_7 + \frac{1}{3}c_3c_4\right)u^3\\
&\quad -\frac{1}{2}c_3^2u^4 + \frac{1}{3}c_4u^6 + \frac{1}{2}c_3u^7 \mod (t, c_8, \mathfrak{a}_{12}).
\end{aligned}
$$

Under these preparations, we will prove Lemma 2.5 (ii).

Proof of Lemma 2.5 (ii). First note that

$$
\begin{aligned}
H^*(BT;\mathbb{Q})^{W(C_8)} &= \mathbb{Q}[u, J_2, J_6, J_8, J_{10}, J_{12}, J_{14}, J_{18}]\\
&= \mathbb{Q}[u, I_2, v, I_8, w, I_{12}, I_{14}, I_{18}]
\end{aligned}
$$

by (i). Since $I_{20} \in H^*(BT;\mathbb{Q})^{W(E_8)} \subset H^*(BT;\mathbb{Q})^{W(C_8)}$, we can put

$$
\begin{aligned}
(*)\quad I_{20} \equiv\ & 2^{27}\cdot 3^3\cdot 5^2\cdot 11\cdot 17\cdot 41(\lambda_1u^{20} + \lambda_2u^{14}v + \lambda_3u^8v^2 + \lambda_4u^2v^3\\
&+ \lambda_5u^{10}w + \lambda_6u^4vw + \lambda_7w^2) \mod \mathfrak{b}_{20}
\end{aligned}
$$

for some $\lambda_i \in \mathbb{Q}$.

In order to determine the coefficients λ_i, we need the following lemma, which is directly verified by making use of (2.10) and Lemma 2.3.

Lemma 2.8.

$$
\begin{aligned}
A/(t, c_8, \mathfrak{a}_{20}) =\ & A/(t, I_2, c_8, I_8, I_{12}, I_{14}, I_{18})\\
=\ & \mathbb{Q}[u, c_3, c_4, c_5, c_6, c_7]/J,
\end{aligned}
$$

where J is the ideal generated by

$$
\begin{aligned}
&uc_7 - u^2c_6 + u^3c_5 - u^4c_4 + u^5c_3 - u^8,\\
&c_4^2 - \frac{3}{2}c_3c_5,\\
&c_6^2 - \frac{5}{3}c_5c_7 + \frac{5}{54}c_3c_4c_5 - \frac{1}{6}c_3^2c_6 + \frac{1}{24}c_3^4,\\
&c_7^2 - \frac{1}{2}c_3c_5c_6 + \frac{1}{3}c_3c_4c_7 + \frac{1}{6}c_4c_5^2,\\
&c_3^6 - 7c_3^4c_6 + \frac{29}{9}c_3^3c_4c_5 + 182c_3^2c_5c_7 + 75c_3c_5^3 - \frac{476}{3}c_3c_4c_5c_6 - 24c_5c_6c_7.
\end{aligned}
$$

In particular, $A/(t, c_8, \mathfrak{a}_{20})$ has a basis $\{u^ic_3^jc_4^kc_5^lc_6^mc_7^n \ (0 \le i \le 7,\ 0 \le j \le 5,\ 0 \le l,\ 0 \le k, m, n \le 1)\}$ as a $\mathbb{Q}$-vector space.

Now we consider the relation $(*)$ in the ring $A/(t, c_8, \mathfrak{a}_{20})$. By Lemma 2.3 (vi), we have

$$
I_{20} \equiv 2^{27}\cdot 3^3\cdot 5^2\cdot 11\cdot 17\cdot 41\left(\frac{1}{144}c_5^4 - \frac{1}{18}c_3c_5^2c_7 - \frac{1}{54}c_3^2c_4c_5^2 - \frac{1}{27}c_3^3c_4c_7 + \frac{1}{18}c_3^3c_5c_6\right).
$$

On the other hand, using (2.15) and Lemma 2.8, we can rewrite each monomial in the right hand side of $(*)$. For example, we have

$$
\begin{aligned}
w^2 \equiv\ &\frac{1}{144}c_5^4 - \frac{1}{18}c_3c_5^2c_7 - \frac{1}{54}c_3^2c_4c_5^2 + \frac{1}{18}c_3^3c_5c_6 - \frac{1}{27}c_3^3c_4c_7\\
&- \frac{1}{12}u^7c_3^2c_7 + \frac{1}{12}u^7c_3c_5^2 + \frac{1}{24}u^6c_3^3c_5 + \frac{1}{3}u^6c_3c_5c_6 - \frac{5}{9}u^6c_3c_4c_7\\
&+ \cdots.
\end{aligned}
$$

Then using the second half of Lemma 2.8, the coefficients in $(*)$ are obtained as follows:

$$
\lambda_1 = 3,\ \lambda_2 = 15,\ \lambda_3 = 20,\ \lambda_4 = \frac{10}{3},\ \lambda_5 = 4,\ \lambda_6 = 10,\ \lambda_7 = 1.
$$

Thus we have obtained

$$
\begin{aligned}
I_{20} &\equiv 2^{27}\cdot 3^3\cdot 5^2\cdot 11\cdot 17\cdot 41\Big(3u^{20} + 15u^{14}v + 20u^8v^2 + \frac{10}{3}u^2v^3 + 4u^{10}w\\
&\quad + 10u^4vw + w^2\Big)\\
&\equiv 2^{27}\cdot 3^2\cdot 5^2\cdot 11\cdot 17\cdot 41(9u^{20} + 45u^{14}v + 60u^8v^2 + 10u^2v^3 + 12u^{10}w\\
&\quad + 30u^4vw + 3w^2)\\
&\equiv 2^{27}\cdot 3^2\cdot 5^2\cdot 11\cdot 17\cdot 41\tilde{I}_{20} \qquad \bmod \mathfrak{b}_{20}.
\end{aligned}
$$

Putting

$$(**)\quad I_{24} \equiv 2^{32}\cdot 3^3\cdot 5^2\cdot 7\cdot 11\cdot 19\cdot 199\,\big(\mu_1 u^{24}+\mu_2 u^{18}v+\mu_3 u^{12}v^2+\mu_4 u^6 v^3 + \mu_5 v^4+\mu_6 u^{14}w+\mu_7 u^8 vw+\mu_8 u^2v^2w+\mu_9 u^4w^2\big) \mod \mathfrak{b}_{20}$$

for some $\mu_i \in \mathbb{Q}$, we will proceed quite similarly. By Lemma 2.3 (vii), we have

$$I_{24} \equiv 2^{32}\cdot 3^3\cdot 5^2\cdot 7\cdot 11\cdot 19\cdot 199\left(\frac{31}{8640}c_3^5c_4c_5+\frac{1}{480}c_3^4c_5c_7+\frac{337}{25920}c_3^3c_5^3 -\frac{71}{4320}c_3^3c_4c_5c_6+\frac{31}{240}c_3^3c_5c_6c_7+\frac{31}{480}c_3c_5^3c_6-\frac{22}{135}c_3c_4c_5^2c_7-\frac{1}{120}c_4c_5^4\right).$$

On the other hand, in $A/(t, c_8, \mathfrak{a}_{20})$, we have, for example,

$$v^4 \equiv \frac{31}{8640}c_3^5c_4c_5+\frac{1}{480}c_3^4c_5c_7+\frac{337}{25920}c_3^3c_5^3-\frac{71}{4320}c_3^3c_4c_5c_6+\frac{31}{240}c_3^3c_5c_6c_7 +\frac{31}{480}c_3c_5^3c_6-\frac{22}{135}c_3c_4c_5^2c_7-\frac{1}{120}c_4c_5^4+\frac{11}{16}u^7c_3^4c_5-u^7c_3^2c_5c_6 -\frac{11}{18}u^7c_3c_4c_5^2+\frac{9}{160}u^6c_3^4c_6-\frac{619}{480}u^6c_3^3c_4c_5+\cdots.$$

Then using the second half of Lemma 2.8, the coefficients in $(**)$ are obtained as follows:

$$\mu_1=\frac{11}{5},\ \mu_2=12,\ \mu_3=21,\ \mu_4=12,\ \mu_5=1,\ \mu_6=\frac{21}{5},\ \mu_7=12,\ \mu_8=6,\ \mu_9=\frac{9}{5}.$$

Thus we have obtained

$$\begin{aligned} I_{24} &\equiv 2^{32}\cdot 3^3\cdot 5^2\cdot 7\cdot 11\cdot 19\cdot 199\left(\frac{11}{5}u^{24}+12u^{18}v+21u^{12}v^2+12u^6v^3 +v^4+\frac{21}{5}u^{14}w+12u^8vw+6u^2v^2w+\frac{9}{5}u^4w^2\right)\\ &\equiv 2^{32}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 19\cdot 199(11u^{24}+60u^{18}v+105u^{12}v^2+60u^6v^3 + 5v^4+21u^{14}w+60u^8vw+30u^2v^2w+9u^4w^2)\\ &\equiv 2^{32}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 19\cdot 199\tilde{I}_{24} \mod \mathfrak{b}_{20}. \end{aligned}$$

Finally, we can also put

$$(***)\quad I_{30} \equiv 2^{38}\cdot 3^4\cdot 5^5\cdot 7\cdot 11\cdot 13\cdot 61(\nu_1u^{30}+\nu_2u^{24}v+\nu_3u^{18}v^2+\nu_4u^{12}v^3 + \nu_5u^6v^4+\nu_6v^5+\nu_7u^{20}w+\nu_8u^{14}vw+\nu_9u^8v^2w+\nu_{10}u^2v^3w + \nu_{11}u^{10}w^2+\nu_{12}u^{14}vw^2+\nu_{13}w^3) \mod \mathfrak{b}_{20}$$

for some $\nu_i \in \mathbb{Q}$. Then, by Lemma 2.3 (viii), we have

$$I_{30} \equiv 2^{38}\cdot 3^4\cdot 5^5\cdot 7\cdot 11\cdot 13\cdot 61\left(-\frac{599}{51840}c_3^5c_4c_5c_6+\frac{47}{34560}c_3^5c_5^3+\frac{1519}{25920}c_3^4c_5c_6c_7\right.$$
$$+\frac{6293}{7290}c_3^3c_4c_5^2c_7-\frac{32537}{25920}c_3^3c_5^3c_6+\frac{189919}{466560}c_3^2c_4c_5^4+\frac{2012}{1215}c_3c_4c_5^2c_6c_7-\frac{16693}{25920}c_3c_5^4c_7$$
$$\left.-\frac{223}{6480}c_4c_5^4c_6-\frac{1}{1728}c_5^6\right).$$

On the other hand, in $A/(t, c_8, \mathfrak{a}_{20})$, we have, for example,

$$v^5 \equiv \frac{31}{11520}c_3^5c_4c_5c_6-\frac{47}{69120}c_3^5c_5^3+\frac{1}{640}c_3^4c_5c_6c_7+\frac{1993}{5760}c_3^3c_5^3c_6-\frac{91}{360}c_3^3c_4c_5^2c_7$$
$$-\frac{1279}{11520}c_3^2c_4c_5^4-\frac{49}{135}c_3c_4c_5^2c_6c_7+\frac{7}{1440}c_4c_5^4c_6+\frac{299}{1920}c_3c_5^4c_7+\cdots,$$
$$w^3 \equiv \frac{1}{162}c_3^5c_4c_5c_6-\frac{5}{81}c_3^4c_5c_6c_7+\frac{365}{648}c_3^3c_5^3c_6-\frac{1043}{2916}c_3^3c_4c_5^2c_7-\frac{1079}{5832}c_3^2c_4c_5^4$$
$$-\frac{226}{243}c_3c_4c_5^2c_6c_7+\frac{2}{81}c_4c_5^4c_6+\frac{431}{1296}c_3c_5^4c_7+\frac{1}{1728}c_5^6+\cdots.$$

Then using the second half of Lemma 2.8, the coefficients in $(***)$ are obtained as follows:

$$\nu_1=-\frac{9}{8},\ \nu_2=-3,\ \nu_3=0,\ \nu_4=-5,\ \nu_5=-\frac{35}{2},\ \nu_6=-2,\ \nu_7=-\frac{3}{2}$$
$$\nu_8=\frac{9}{2},\ \nu_9=15,\ \nu_{10}=-5,\ \nu_{11}=-\frac{3}{2},\ \nu_{12}=3,\ \nu_{13}=-1.$$

Thus we have obtained

$$I_{30} \equiv 2^{38}\cdot 3^4\cdot 5^2\cdot 7\cdot 11\cdot 13\cdot 61\left(-\frac{9}{8}u^{30}-3u^{24}v-5u^{12}v^3-\frac{35}{2}u^6v^4-2v^5\right.$$
$$\left.-\frac{3}{2}u^{20}w+\frac{9}{2}u^{14}vw+15u^8v^2w-5u^2v^3w-\frac{3}{2}u^{10}w^2+3u^4vw^2-w^3\right)$$
$$\equiv 2^{35}\cdot 3^4\cdot 5^2\cdot 7\cdot 11\cdot 13\cdot 61(-9u^{30}-24u^{24}v-40u^{12}v^3-140u^6v^4-16v^5$$
$$-12u^{20}w+36u^{14}vw+120u^8v^2w-40u^2v^3w-12u^{10}w^2+24u^4vw^2-8w^3)$$
$$\equiv 2^{35}\cdot 3^4\cdot 5^2\cdot 7\cdot 11\cdot 13\cdot 61\tilde{I}_{30} \mod \mathfrak{b}_{20}.$$

Consequently, we have established Lemma 2.5. □

3. Cohomology of $E_8/T^1{\cdot}E_7$

3.1. **Rational cohomology ring of $E_8/T^1{\cdot}E_7$.** With the above results, we will compute the rational cohomology ring of E_8/C_8. First of all we review the classical results of Borel ([2]); Let G be a compact connected Lie group, H a closed connected subgroup of G of maximal rank and T a commnon maximal torus. Consider the fibration

$$G/H \xrightarrow{\iota} BH \xrightarrow{\rho} BG.$$

Since $H^*(BG;\mathbb{Q})$ is a polynomial ring generated by elements of even degrees and $H^*(G/H;\mathbb{Q})$ has vanishing odd dimensional part (Hirsch formula [2]), the rational cohomology spectral sequence for this fibration collapses. In particular, we have the following description of the rational cohomology ring of G/H:

$$\begin{aligned} H^*(G/H;\mathbb{Q}) &\overset{\iota^*}{\underset{\sim}{\longleftarrow}} H^*(BH;\mathbb{Q})/(\rho^* H^+(BG;\mathbb{Q})) \\ &\cong H^*(BT;\mathbb{Q})^{W(H)}/(H^+(BT;\mathbb{Q})^{W(G)}), \end{aligned}$$

where $H^+ = \oplus_{i>0} H^i$ and ( ) means the ideal generated by the elements in parenthesis.

We apply this result to the fibration:

$$E_8/C_8 \xrightarrow{\iota} BC_8 \xrightarrow{\rho} BE_8.$$

Then using Lemmas 2.2, 2.6, 2.5 and (2.9), we have

$$\begin{aligned} H^*(E_8/C_8;\mathbb{Q}) &\cong H^*(BT;\mathbb{Q})^{W(C_8)}/(H^+(BT;\mathbb{Q})^{W(E_8)}) \\ &\cong \mathbb{Q}[u, J_2, J_6, J_8, J_{10}, J_{12}, J_{14}, J_{18}]/(I_2, I_8, I_{12}, I_{14}, I_{18}, I_{20}, I_{24}, I_{30}) \\ &\cong \mathbb{Q}[u, I_2, v, I_8, w, I_{12}, I_{14}, I_{18}]/(I_2, I_8, I_{12}, I_{14}, I_{18}, I_{20}, I_{24}, I_{30}) \\ &\cong \mathbb{Q}[u, v, w]/(\tilde{I}_{20}, \tilde{I}_{24}, \tilde{I}_{30}). \end{aligned}$$

Thus we have obtained

Lemma 3.1. *The rational cohomology ring of* $E_8/T^1 \cdot E_7$ *is given as follows*:

$$H^*(E_8/T^1 \cdot E_7;\mathbb{Q}) = \mathbb{Q}[u, v, w]/(\tilde{I}_{20}, \tilde{I}_{24}, \tilde{I}_{30}),$$

where $\deg u = 2,\ \deg v = 12,\ \deg w = 20$ *and* $\tilde{I}_{20}, \tilde{I}_{24}$ *and* $\tilde{I}_{30}$ *are given by* (2.8).

3.2. **Integral cohomology ring of E_8/T in low degrees.** Consider the fibration

$$E_7/T' \cong C_8/T \xrightarrow{i} E_8/T \xrightarrow{p} E_8/C_8.$$

Since $H^*(E_8/C_8;\mathbb{Z})$ and $H^*(E_7/T';\mathbb{Z})$ have no torsion and vanishing odd dimensional part by Bott [4], the Serre spectral sequence for the above fibration collapses and the following sequence

$$\mathbb{Z} \to H^*(E_8/C_8;\mathbb{Z}) \xrightarrow{p^*} H^*(E_8/T;\mathbb{Z}) \xrightarrow{i^*} H^*(C_8/T;\mathbb{Z}) \cong H^*(E_7/T';\mathbb{Z}) \to \mathbb{Z}$$

is co-exact; that is,

p^* is injective, i^* is surjective and
$\operatorname{Ker} i^* = (p^* H^+(E_8/C_8;\mathbb{Z}))$, the ideal generated by $p^*H^+(E_8/C_8;\mathbb{Z})$.

Therefore we will obtain some information about the generators of $H^*(E_8/C_8;\mathbb{Z})$ by considering $\operatorname{Ker} i^*$. In order to investigate $\operatorname{Ker} i^*$, we will determine $H^*(E_8/T;\mathbb{Z})$ up to degrees ≤ 36. First we need a simple lemma.

Lemma 3.2. *For the elements t and $c_i = \sigma_i(t_1, \ldots, t_8) \in H^*(BT;\mathbb{Z})$, we have*

$$\begin{aligned}
\text{(i)}\quad & Sq^2(c_2) \equiv c_3 + tc_2, \\
& Sq^4(c_3) \equiv c_5 + tc_4 + c_2c_3, \\
& Sq^8(c_5 + tc_4) \equiv tc_8 + c_2c_7 + c_3c_6 + c_4c_5 + tc_4^2 + t^2c_7 + t^3c_6 + t^2c_2c_5 \\
& \qquad + t^2c_3c_4, \\
& Sq^{14}(c_8 + c_4^2 + t^2c_6 + t^4c_4 + t^8) \equiv (c_8 + t^2c_6 + t^4c_4 + t^8)(c_7 + tc_6) \mod 2. \\
\text{(ii)}\quad & \mathcal{P}^1(c_2 - t^2) \equiv c_4 + c_2^2 + t^4, \\
& \mathcal{P}^3(c_4 - t^4) \equiv c_5^2 + 2c_4c_6 + 2c_3c_7 + 2c_2c_8 + c_3^2c_4 + c_2c_4^2 + c_2^2c_6 + 2c_2c_3c_5 \\
& \qquad + 2t^{10} \mod 3. \\
\text{(iii)}\quad & \mathcal{P}^1(c_2 + t^2) \equiv c_6 + 2c_3^2 + 4c_2c_4 + 2c_2^3 + 2tc_5 + tc_2c_3 + 4t^2c_4 + 4t^2c_2^2 \\
& \qquad + 3t^3c_3 + t^4c_2 + 2t^6 \mod 5.
\end{aligned}$$

Proof. (i) follows immediately from the Wu formula: $Sq^{2i-2}(c_i) \equiv \sum_{j=0}^{i-1} c_{2i-1-j}c_j$ and $c_1 = 3t \equiv t \mod 2$.

(ii) Put

$$p_i = t_1^i + \cdots + t_8^i \quad \text{(power sum).}$$

Then p_i's and c_i's are related to each other by the Newton formula:

$$p_n = \sum_{i=1}^{n-1} (-1)^{i-1} p_{n-i}c_i + (-1)^{n-1} nc_n.$$

In particular, considering with mod 3 coefficients, we have

$$\begin{aligned}
p_1 &\equiv c_1 \equiv 0, \\
p_2 &\equiv c_2,
\end{aligned}$$

$$
\begin{aligned}
p_4 &\equiv 2c_4 + 2c_2^2,\\
p_{10} &\equiv 2c_5^2 + c_4c_6 + c_3c_7 + c_2c_8 + 2c_3^2c_4 + 2c_2c_4^2 + 2c_2^2c_6 + c_2^3c_4 + c_2c_3c_5\\
&\quad + c_2^5.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\mathcal{P}^1(p_2) &\equiv \mathcal{P}^1\left(\sum_i t_i^2\right) \equiv \sum_i \mathcal{P}^1(t_i^2) \equiv \sum_i 2t_i\mathcal{P}^1(t_i) \equiv \sum_i 2t_i \cdot t_i^3\\
&\equiv \sum_i 2t_i^4 \equiv 2p_4,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{P}^3(p_4) &\equiv \mathcal{P}^3\left(\sum_i t_i^4\right) \equiv \sum_i \mathcal{P}^3(t_i^4) \equiv \sum_i (2\mathcal{P}^3(t_i^2)t_i^2 + 2\mathcal{P}^2(t_i^2)\mathcal{P}^1(t_i^2))\\
&\equiv \sum_i (2t_i^6 \cdot 2t_i^4) \equiv \sum_i t_i^{10} \equiv p_{10}.
\end{aligned}
$$

Using these facts, we have easily the required results.

(iii) Similar computation yields the required results. □

Lemma 3.3. *The integral cohomology ring of E_8/T for degrees ≤ 36 is given as follows*:

$$
\begin{aligned}
H^*(E_8/T;\mathbb{Z}) = \mathbb{Z}[&t_1,\ldots,t_8,t,\gamma_3,\gamma_4,\gamma_5,\gamma_6,\gamma_9,\gamma_{10},\gamma_{15}]\\
&/(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6,\rho_8,\rho_9,\rho_{10},\rho_{12},\rho_{14},\rho_{15},\rho_{18}),
\end{aligned}
$$

where $t_1,\ldots,t_8,t \in H^2$ are as in §2, $\gamma_i \in H^{2i}$ $(i = 3,4,5,6,9,10,15)$ and

$$
\begin{aligned}
\rho_1 &= c_1 - 3t,\\
\rho_2 &= c_2 - 4t^2,\\
\rho_3 &= c_3 - 2\gamma_3,\\
\rho_4 &= c_4 + 2t^4 - 3\gamma_4,\\
\rho_5 &= c_5 - 3t\gamma_4 + 2t^2\gamma_3 - 2\gamma_5,\\
\rho_6 &= c_6 - 2{\gamma_3}^2 - t\gamma_5 + t^2\gamma_4 - 2t^6 - 5\gamma_6,\\
\rho_8 &= -3c_8 + 3{\gamma_4}^2 - 2\gamma_3\gamma_5 + t(2c_7 - 6\gamma_3\gamma_4) + t^2(2{\gamma_3}^2 - 5\gamma_6) + 3t^3\gamma_5\\
&\quad + 4t^4\gamma_4 - 6t^5\gamma_3 + t^8,\\
\rho_9 &= 2c_6\gamma_3 + tc_8 + t^2c_7 - 3t^3c_6 - 2\gamma_9,\\
\rho_{10} &= {\gamma_5}^2 - 2c_7\gamma_3 - t^2c_8 + 3t^3c_7 - 3\gamma_{10},
\end{aligned}
$$

$$\rho_{12} = 15{\gamma_6}^2 + 2\gamma_3\gamma_4\gamma_5 - 2c_7\gamma_5 + 2{\gamma_3}^4 + 10{\gamma_3}^2\gamma_6 - 3c_8\gamma_4 - 2{\gamma_4}^3$$
$$+ t(c_8\gamma_3 - 2{\gamma_3}^2\gamma_5 + 4c_7\gamma_4 + 6\gamma_3{\gamma_4}^2) + t^2(3\gamma_{10} - 25\gamma_4\gamma_6 - c_7\gamma_3 - 16{\gamma_3}^2\gamma_4)$$
$$+ t^3(25\gamma_3\gamma_6 - 3\gamma_4\gamma_5 + 10{\gamma_3}^3) + t^4(3c_8 + 3\gamma_3\gamma_5 + 5{\gamma_4}^2) + t^5(-3c_7 - 5\gamma_3\gamma_4)$$
$$+ 4t^6{\gamma_3}^2 - 7t^8\gamma_4 + 4t^9\gamma_3,$$
$$\rho_{14} = c_7^2 - 3c_8\gamma_6 + 6\gamma_4\gamma_{10} - 4c_8{\gamma_3}^2 + 6c_7\gamma_3\gamma_4 - 6{\gamma_3}^2{\gamma_4}^2 - 12{\gamma_4}^2\gamma_6 - 2\gamma_3\gamma_5\gamma_6$$
$$+ t(24\gamma_3\gamma_4\gamma_6 - 8c_7{\gamma_3}^2 - 8c_7\gamma_6 + 4c_8\gamma_5 - 6\gamma_3\gamma_{10} + 12{\gamma_3}^3\gamma_4)$$
$$+ t^2(-2\gamma_3\gamma_4\gamma_5 + 6{\gamma_4}^3 + 2{\gamma_3}^2\gamma_6 + 20{\gamma_6}^2 - 4{\gamma_3}^4 - c_7\gamma_5)$$
$$+ t^3(-12\gamma_3{\gamma_4}^2 + 8c_8\gamma_3 - 5c_7\gamma_4 + 3\gamma_5\gamma_6) + t^4(3\gamma_{10} - 26\gamma_4\gamma_6 + 6c_7\gamma_3 - 4{\gamma_3}^2\gamma_4)$$
$$+ t^5(24\gamma_3\gamma_6 + 3\gamma_4\gamma_5 + 12{\gamma_3}^3) + t^6(-6c_8 + 2{\gamma_4}^2) - 4t^7c_7 + t^8(6\gamma_6 - 6{\gamma_3}^2)$$
$$- 6t^{10}\gamma_4 + 12t^{11}\gamma_3 - 2t^{14},$$
$$\rho_{15} = (c_8 - t^2c_6 + 2t^3\gamma_5 + 3t^4\gamma_4 - t^8)(c_7 - 3tc_6) - 2({\gamma_3}^2 + c_6)(\gamma_9 - c_6\gamma_3) - 2\gamma_{15},$$
$$\rho_{18} = -\gamma_9^2 - 27c_8\gamma_{10} - 18\gamma_4^2\gamma_{10} + 4\gamma_3^3\gamma_9 + 10\gamma_3\gamma_6\gamma_9 + 6\gamma_3\gamma_5\gamma_{10} - 6\gamma_3\gamma_4\gamma_5\gamma_6$$
$$- 18c_7\gamma_3\gamma_4^2 + 9c_8\gamma_4\gamma_6 + 3c_8\gamma_3^2\gamma_4 + 18\gamma_3^2\gamma_4^3 + 36\gamma_4^3\gamma_6 + 6c_7^2\gamma_4 + 6c_7\gamma_3^2\gamma_5$$
$$- 6\gamma_3^3\gamma_4\gamma_5 + 6c_7\gamma_5\gamma_6 - 4\gamma_3^6 - 30\gamma_6^3 - 26\gamma_3^4\gamma_6 - 55\gamma_3^2\gamma_6^2 - 27c_7c_8\gamma_3$$
$$+ t(2\gamma_3\gamma_5\gamma_9 - 72c_7\gamma_4\gamma_6 + 24c_8\gamma_4\gamma_5 + 12c_7\gamma_3^2\gamma_4 + 12c_7\gamma_{10} + c_8\gamma_9 + 6c_7^2\gamma_3$$
$$- 8c_8\gamma_3\gamma_6 + 36\gamma_3\gamma_4\gamma_{10} - 108\gamma_3\gamma_4^2\gamma_6 - 4\gamma_3^2\gamma_5\gamma_6 - 5c_8\gamma_3^3 + 2\gamma_3^4\gamma_5 - 54\gamma_3^3\gamma_4^2)$$
$$+ t^2(88\gamma_3^4\gamma_4 - 7c_8^2 + c_7\gamma_9 - 45\gamma_3^2\gamma_{10} - 2\gamma_3\gamma_4\gamma_9 - 10c_8\gamma_3\gamma_5 + 28c_7\gamma_3\gamma_6 - 9c_7\gamma_4\gamma_5$$
$$+ 225\gamma_4\gamma_6^2 - 18\gamma_4^4 - 27c_8\gamma_4^2 + 283\gamma_3^2\gamma_4\gamma_6 - 39\gamma_6\gamma_{10} + 12\gamma_3\gamma_4^2\gamma_5 - 19c_7\gamma_3^3)$$
$$+ t^3(-9\gamma_5\gamma_{10} - 150\gamma_3\gamma_6^2 - 165\gamma_3^3\gamma_6 - 6\gamma_3^2\gamma_9 - 10c_7\gamma_3\gamma_5 + 63c_7\gamma_4^2 + 43c_7c_8$$
$$+ 9\gamma_4\gamma_5\gamma_6 - \gamma_3^2\gamma_4\gamma_5 + 54\gamma_3\gamma_4^3 - 15\gamma_6\gamma_9 + 46c_8\gamma_3\gamma_4 - 42\gamma_3^5)$$
$$+ t^4(-9c_8\gamma_6 + 6\gamma_4\gamma_{10} - 16c_7^2 - 103\gamma_3^2\gamma_4^2 - 3\gamma_5\gamma_9 + 36\gamma_3\gamma_5\gamma_6 - 135\gamma_4^2\gamma_6$$
$$- 119c_7\gamma_3\gamma_4 + 9\gamma_3^3\gamma_5 - 19c_8\gamma_3^2)$$
$$+ t^5(39c_7\gamma_6 - 18\gamma_4^2\gamma_5 + 117c_7\gamma_3^2 + 195\gamma_3\gamma_4\gamma_6 - 12c_8\gamma_5 + 3\gamma_4\gamma_9 + 36\gamma_3\gamma_{10} + 87\gamma_3^3\gamma_4)$$
$$+ t^6(33\gamma_4^3 + 4\gamma_3\gamma_9 + 18c_7\gamma_5 + 3\gamma_3\gamma_4\gamma_5 - 32\gamma_3^4 - 62\gamma_3^2\gamma_6)$$
$$+ t^7(-87\gamma_3\gamma_4^2 - 16\gamma_3^2\gamma_5 - 6c_7\gamma_4 - 45\gamma_5\gamma_6 + 4c_8\gamma_3)$$
$$+ t^8(-39\gamma_{10} + 115\gamma_3^2\gamma_4 - 77c_7\gamma_3 + 81\gamma_4\gamma_6)$$
$$+ t^9(-6\gamma_9 + 18\gamma_4\gamma_5 - 30\gamma_3^3 - 57\gamma_3\gamma_6)$$
$$+ t^{10}(-9c_8 - 27\gamma_4^2 + 9\gamma_3\gamma_5)$$
$$+ t^{11}(33\gamma_3\gamma_4 + 48c_7)$$
$$- 34t^{12}\gamma_3^2 - 18t^{13}\gamma_5 + 9t^{14}\gamma_4 + 12t^{15}\gamma_3 - 6t^{18}.$$

Proof. According to Toda ([17], Proposition 3.2), one can give the general description of $H^*(E_8/T;\mathbb{Z})$ as follows:

$$H^*(E_8/T;\mathbb{Z}) = \mathbb{Z}[t_1,\dots,t_8,t,\gamma_3,\gamma_4,\gamma_5,\gamma_6,\gamma_9,\gamma_{10},\gamma_{15}] \\ /(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6,\rho_8,\rho_9,\rho_{10},\rho_{12},\rho_{14},\rho_{15},\rho_{18},\rho_{20},\rho_{24},\rho_{30}),$$

where $t_1,\dots,t_8,t \in H^2$ as above and

$$\begin{aligned} \rho_1 &= c_1 - 3t, \\ \rho_i &= \delta_i - 2\gamma_i \ (i = 3,5,9,15), \\ \rho_i &= \delta_i - 3\gamma_i \ (i = 4,10), \\ \rho_6 &= \delta_6 - 5\gamma_6. \end{aligned}$$

Here δ_i $(i = 3,4,5,6,9,10,15)$ is an arbitrary element satisfying

$$\delta_3 \equiv Sq^2(\rho_2),\ \delta_5 \equiv Sq^4(\delta_3),\ \delta_9 \equiv Sq^8(\delta_5),\ \delta_{15} \equiv Sq^{14}(\rho_8) \mod 2,$$
$$\delta_4 \equiv \mathcal{P}^1(\rho_2),\ \delta_{10} \equiv \mathcal{P}^3(\delta_4) \mod 3,$$
$$\delta_6 \equiv \mathcal{P}^1(\rho_2) \mod 5.$$

Other relation ρ_j $(j = 2,8,12,14,18,20,24,30)$ is determined by the maximality of the integer n_j in

$$n_j \cdot \rho_j \equiv \iota_0^*(I_j) \mod (\rho_i; i < j). \tag{3.1}$$

Now let us determine the generators and the relations explicitly;

(1) In view of Lemma 2.3 (i) and (3.1), we can take

$$\rho_2 = c_2 - 4t^2.$$

(2) By Lemma 3.2 (i), we have

$$\delta_3 \equiv Sq^2(\rho_2) \equiv Sq^2(c_2) \equiv c_3 + tc_2 \equiv c_3 \mod (2,\rho_1,\rho_2)$$

and we can take $\delta_3 = c_3$ so that

$$\rho_3 = c_3 - 2\gamma_3.$$

(3) By Lemma 3.2 (ii), we have

$$\delta_4 \equiv \mathcal{P}^1(\rho_2) \equiv \mathcal{P}^1(c_2 - 4t^2) \equiv c_4 + c_2^2 + t^4 \equiv c_4 + 2t^4 \mod (3,\rho_1,\rho_2)$$

and we can take $\delta_4 = c_4 + 2t^4$ so that

$$\rho_4 = c_4 + 2t^4 - 3\gamma_4.$$

(4) By Lemma 3.2 (i), we have

$$\begin{aligned} \delta_5 &\equiv Sq^4(\delta_3) \equiv Sq^4(c_3) \equiv c_5 + tc_4 + c_2c_3 \equiv c_5 + tc_4 \mod (2,\rho_1,\rho_2,\rho_3,\rho_4) \\ &\equiv c_5 - 3t\gamma_4 + 2t^2\gamma_3 \mod (2,\rho_1,\rho_2,\rho_3,\rho_4) \end{aligned}$$

and we can take $\delta_5 = c_5 - 3t\gamma_4 + 2t^2\gamma_3$ so that

$$\rho_5 = c_5 - 3t\gamma_4 + 2t^2\gamma_3 - 2\gamma_5.$$

(5) By Lemma 3.2 (iii), we have

$$\begin{aligned}\delta_6 &\equiv \mathcal{P}^1(\rho_2) \equiv \mathcal{P}^1(c_2-4t^2) \equiv c_6+2c_3^2+4c_2c_4+2c_2^3+2tc_5+tc_2c_3\\ &\quad +4t^2c_4+4t^2c_2^2+3t^3c_3+t^4c_2+2t^6 \mod (5,\rho_1)\\ &\equiv c_6+3\gamma_3^2+4t\gamma_5+t^2\gamma_4+3t^6 \mod (5,\rho_1,\rho_2,\rho_3,\rho_4,\rho_5)\\ &\equiv c_6-2\gamma_3^2-t\gamma_5+t^2\gamma_4-2t^6 \mod (5,\rho_1,\rho_2,\rho_3,\rho_4,\rho_5)\end{aligned}$$

and we can take $\delta_6 = c_6-2\gamma_3^2-t\gamma_5+t^2\gamma_4-2t^6$ so that

$$\rho_6 = c_6-2\gamma_3^2-t\gamma_5+t^2\gamma_4-2t^6-5\gamma_6.$$

(6) By Lemma 2.3 (ii), we have

$$\begin{aligned}I_8 &\equiv 2^{14}\cdot 3\cdot 5\{-18c_8-3c_3c_5+2c_4^2+t(12c_7-3c_3c_4)+t^2(-6c_6+3c_3^2)\\ &\quad +12t^3c_5+2t^4c_4-12t^5c_3+14t^8\} \mod (I_2)\\ &\equiv 2^{15}\cdot 3^2\cdot 5\{-3c_8+3\gamma_4^2-2\gamma_3\gamma_5+t(2c_7-6\gamma_3\gamma_4)+t^2(2\gamma_3^2-5\gamma_6)\\ &\quad +3t^3\gamma_5+4t^4\gamma_4-6t^5\gamma_3+t^8\} \mod (\rho_i; i<8).\end{aligned}$$

Hence, by (3.1), we have

$$2^{15}\cdot 3^2\cdot 5\,\rho_8 \equiv \iota_0^*(I_8) \mod (\rho_i; i<8)$$

and it follows the form of ρ_8.
(7) By Lemma 3.2 (i) , we have

$$\begin{aligned}\delta_9 &\equiv Sq^8(\delta_5) \equiv Sq^8(c_5+tc_4) \equiv tc_8+c_2c_7+c_3c_6+c_4c_5+tc_4^2+t^2c_7\\ &\quad +t^3c_6+t^2c_2c_5+t^2c_3c_4 \mod (2,\rho_1)\\ &\equiv tc_8+t^2c_7+t^3c_6 \mod (2,\rho_i; i<9)\\ &\equiv tc_8+t^2c_7-3t^3c_6+2c_6\gamma_3 \mod (2,\rho_i; i<9)\end{aligned}$$

and we can take $\delta_9 = tc_8+t^2c_7-3t^3c_6+2c_6\gamma_3$ so that

$$\rho_9 = tc_8+t^2c_7-3t^3c_6+2c_6\gamma_3-2\gamma_9.$$

(8) By Lemma 3.2 (ii), we have

$$\begin{aligned}\delta_{10} &\equiv \mathcal{P}^3(\delta_4) \equiv \mathcal{P}^3(c_4+2t^4) \equiv c_5^2+2c_4c_6+2c_3c_7+2c_2c_8+c_3^2c_4\\ &\quad +c_2c_4^2+c_2^2c_6+2c_2c_3c_5+2t^{10} \mod (3,\rho_1)\\ &\equiv \gamma_5^2+c_7\gamma_3+2t^2c_8 \mod (3,\rho_i; i<10)\\ &\equiv {\gamma_5}^2-2c_7\gamma_3+3t^3c_7-t^2c_8 \mod (3,\rho_i; i<10)\end{aligned}$$

and we can take $\delta_{10} = \gamma_5^2-2c_7\gamma_3+3t^3c_7-t^2c_8$ so that

$$\rho_{10} = \gamma_5^2-2c_7\gamma_3+3t^3c_7-t^2c_8.$$

(9) By (2.6), (2.5) and (2.7), we obtain

$$I_{12} \equiv 2^{18}\cdot 3^4\cdot 5\cdot 7\left\{\frac{3}{5}{c_6}^2 - c_5c_7 - c_4c_8 + \frac{1}{6}c_3c_4c_5 - \frac{2}{27}c_4^3 - \frac{1}{10}c_3^2c_6 + \frac{1}{40}c_3^4\right.$$
$$+ t\left(\frac{1}{2}c_3c_8 + \frac{7}{3}c_4c_7 - \frac{3}{5}c_5c_6 - \frac{1}{5}c_3^2c_5 + \frac{1}{6}c_3c_4^2\right)$$
$$+ t^2\left(\frac{2}{5}c_5^2 - \frac{5}{2}c_3c_7 - \frac{2}{3}c_4c_6 - \frac{1}{6}c_3^2c_4\right)$$
$$+ t^3\left(\frac{19}{10}c_3c_6 - \frac{2}{3}c_4c_5 - \frac{1}{5}c_3^3\right) + t^4\left(-\frac{1}{9}c_4^2 + \frac{19}{30}c_3c_5\right) + t^5\left(\frac{14}{3}c_7 + \frac{1}{2}c_3c_4\right)$$
$$\left. + t^6\left(-\frac{56}{15}c_6 + \frac{23}{30}c_3^2\right) - \frac{2}{15}t^7c_5 - \frac{5}{9}t^8c_4 - \frac{22}{15}t^9c_3 + \frac{154}{135}t^{12}\right\} \mod (I_2)$$

$$\equiv 2^{18}\cdot 3^4\cdot 5\cdot 7\{15{\gamma_6}^2 + 2\gamma_3\gamma_4\gamma_5 - 2c_7\gamma_5 + 2{\gamma_3}^4 + 10{\gamma_3}^2\gamma_6 - 3c_8\gamma_4 - 2{\gamma_4}^3$$
$$+ t(c_8\gamma_3 - 2{\gamma_3}^2\gamma_5 + 4c_7\gamma_4 + 6\gamma_3{\gamma_4}^2) + t^2(3\gamma_{10} - 25\gamma_4\gamma_6 - c_7\gamma_3 - 16{\gamma_3}^2\gamma_4)$$
$$+ t^3(25\gamma_3\gamma_6 - 3\gamma_4\gamma_5 + 10{\gamma_3}^3) + t^4(3c_8 + 3\gamma_3\gamma_5 + 5{\gamma_4}^2) + t^5(-3c_7 - 5\gamma_3\gamma_4)$$
$$+ 4t^6{\gamma_3}^2 - 7t^8\gamma_4 + 4t^9\gamma_3\} \mod (\rho_i; i < 12).$$

Hence, by (3.1), we have

$$2^{18}\cdot 3^4\cdot 5\cdot 7\ \rho_{12} \equiv \iota_0^*(I_{12}) \mod (\rho_i; i < 12)$$

and it follows the form of ρ_{12}. Quite similarly, we have

$$2^{20}\cdot 3^2\cdot 5^2\cdot 7\cdot 11\ \rho_{14} \equiv \iota_0^*(I_{14}) \mod (\rho_i; i < 14),$$
$$2^{26}\cdot 3^3\cdot 5^2\cdot 7\cdot 13\ \rho_{18} \equiv \iota_0^*(I_{18}) \mod (\rho_i; i < 18)$$

and it follows the forms of ρ_{14} and ρ_{18}.

(10) Finally, we will determine the relation ρ_{15}. Since

$$\rho_8 = -3c_8 + 3{\gamma_4}^2 - 2\gamma_3\gamma_5 + t(2c_7 - 6\gamma_3\gamma_4) + t^2(2{\gamma_3}^2 - 5\gamma_6) + 3t^3\gamma_5$$
$$+ 4t^4\gamma_4 - 6t^5\gamma_3 + t^8$$
$$\equiv c_8 + c_4^2 + t^2c_6 + t^4c_4 + t^8 \mod (2, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6),$$

we have, by Lemma 3.2 (i),

$$\delta_{15} \equiv Sq^{14}(\rho_8)$$
$$\equiv Sq^{14}(c_8 + c_4^2 + t^2c_6 + t^4c_4 + t^8)$$
$$\equiv (c_8 + t^2c_6 + t^4c_4 + t^8)(c_7 + tc_6) \mod (2, \rho_1)$$
$$\equiv (c_8 - t^2c_6 + 2t^3\gamma_5 + 3t^4\gamma_4 - t^8)(c_7 - tc_6) - 2(c_6 + \gamma_3^2)(\gamma_9 - c_6\gamma_3)$$
$$\mod (2, \rho_i; i < 15)$$

and we can take

$$\delta_{15} = (c_8 - t^2 c_6 + 2t^3\gamma_5 + 3t^4\gamma_4 - t^8)(c_7 - tc_6) - 2(c_6 + \gamma_3^2)(\gamma_9 - c_6\gamma_3)$$

so that

$$\rho_{15} = (c_8 - t^2 c_6 + 2t^3\gamma_5 + 3t^4\gamma_4 - t^8)(c_7 - tc_6) - 2(c_6 + \gamma_3^2)(\gamma_9 - c_6\gamma_3) - 2\gamma_{15}.$$

Consequently, we have established the lemma. □

In order to determine $\operatorname{Ker} i^*$, we need the result on $H^*(E_7/T';\mathbb{Z})$ in ([15], Theorem 5.9), which can be restated as follows:

Theorem 3.4. *The integral cohomology ring of E_7/T' is given as follows*:

$$\begin{aligned} H^*(E_7/T';\mathbb{Z}) = \mathbb{Z}[t'_1,\ldots,t'_7,t',\gamma'_3,\gamma'_4,\gamma'_5,\gamma'_9] \\ /(\rho'_1,\rho'_2,\rho'_3,\rho'_4,\rho'_5,\rho'_6,\rho'_8,\rho'_9,\rho'_{10},\rho'_{12},\rho'_{14},\rho'_{18}), \end{aligned}$$

where $t'_1,\ldots,t'_7,t' \in H^2$ are as in §2, $\gamma'_i \in H^{2i}$ $(i = 3,4,5,9)$ and

$$\begin{aligned}
\rho'_1 &= c'_1 - 3t', \\
\rho'_2 &= c'_2 - 4t'^2, \\
\rho'_3 &= c'_3 - 2\gamma'_3, \\
\rho'_4 &= c'_4 + 2t'^4 - 3\gamma'_4, \\
\rho'_5 &= c'_5 - 3t'\gamma'_4 + 2t'^2\gamma'_3 - 2\gamma'_5, \\
\rho'_6 &= {\gamma'_3}^2 + 2c'_6 - 2t'\gamma_5 - 3t'^2\gamma'_4 + t'^6, \\
\rho'_8 &= 3{\gamma'_4}^2 - 2\gamma'_3\gamma'_5 + t'(2c'_7 - 6\gamma'_3\gamma'_4) - 9t'^2c'_6 + 12t'^3\gamma'_5 + 15t'^4\gamma'_4 - 6t'^5\gamma'_3 - t'^8, \\
\rho'_9 &= 2c'_6\gamma'_3 + t'^2c'_7 - 3t'^3c'_6 - 2\gamma'_9, \\
\rho'_{10} &= {\gamma'_5}^2 - 2c'_7\gamma'_3 + 3t'^3c'_7, \\
\rho'_{12} &= 3c'^2_6 - 2{\gamma'_4}^3 - 2c'_7\gamma'_5 + 2\gamma'_3\gamma'_4\gamma'_5 + t'(4c'_7\gamma'_4 - 2c'_6\gamma'_5 + 6\gamma'_3{\gamma'_4}^2) \\
&\quad + t'^2(-3c'_7\gamma'_3 + 3c'_6\gamma'_4) + t'^3(-12\gamma'_4\gamma'_5 + 5c'_6\gamma'_3) + t'^4(-2\gamma'_3\gamma'_5 - 15{\gamma'_4}^2) \\
&\quad - 10t'^6c'_6 + 12t'^7\gamma'_5 + 19t'^8\gamma'_4 - 6t'^9\gamma'_3 - 2t'^{12}, \\
\rho'_{14} &= {c'_7}^2 + 6c'_7\gamma'_3\gamma'_4 - 2c'_6\gamma'_3\gamma'_5 - t'^2c'_7\gamma'_5 + t'^3(-9c'_7\gamma'_4 + 3c'_6\gamma'_5) - 6t'^4c'_7\gamma'_3 \\
&\quad + 9t'^7c'_7, \\
\rho'_{18} &= -{\gamma'_9}^2 + 2c'_6c'_7\gamma'_5 + 6c'_7\gamma'_3{\gamma'_4}^2 - 2{c'_7}^2\gamma'_4 - 2c'_6\gamma'_3\gamma'_4\gamma'_5 + 2c'_6\gamma'_3\gamma'_9 \\
&\quad + t'(-6{c'_7}^2\gamma'_3 + 24c'_6c'_7\gamma'_4) + t'^2(-25c'_7\gamma'_4\gamma'_5 + c'_7\gamma'_9 - 18c'_6c'_7\gamma'_3) \\
&\quad + t'^3(-45c'_7{\gamma'_4}^2 + 20c'_7\gamma'_3\gamma'_5 + 3c'_6\gamma'_4\gamma'_5 - 3c'_6\gamma'_9)
\end{aligned}$$

$$+ t'^4(11c_7'^2 + 2c_6'\gamma_3'\gamma_5' + 48c_7'\gamma_3'\gamma_4') + 51t'^5c_6'c_7' - 53t'^6c_7'\gamma_5'$$
$$+ t'^7(-69c_7'\gamma_4' - 3c_6'\gamma_5') + 16t'^8c_7'\gamma_3' + 15t'^{11}c_7'.$$

Remark 3.5. *Using the result in* [15], *we expressed the relations* ρ'_{12}, ρ'_{14} *and* ρ'_{18} *in terms of the generators* $t'_1, \dots, t'_7, t', \gamma'_3, \gamma'_4, \gamma'_5, \gamma'_9$.

Corollary 3.6. *For the induced homomorphism*

$$i^* : H^*(E_8/T;\mathbb{Z}) \longrightarrow H^*(C_8/T;\mathbb{Z}) \cong H^*(E_7/T';\mathbb{Z}),$$

we obtain that

$$\operatorname{Ker} i^* = (u, \tilde{\gamma}_6, \gamma_{10}, \gamma_{15}),$$

where $\tilde{\gamma}_6 = 2\gamma_6 + \gamma_3^2 - t^2\gamma_4 + t^6$.

Proof. By (2.2), we have

$$i^*(t_i) = t'_i \ (1 \le i \le 7),\ i^*(t_8) = 0,\ i^*(t) = t' \tag{3.2}$$

and therefore

$$i^*(c_n) = c'_n \ (1 \le n \le 7),\ i^*(c_8) = 0. \tag{3.3}$$

Then it is verified directly that

$$\begin{aligned} i^*(\gamma_i) &= \gamma'_i \ (i = 3,4,5,9), \\ i^*(\gamma_6) &= c'_6 - t'\gamma_5{}' - t'^2\gamma_4{}', \\ i^*(\gamma_{10}) &= 0, \\ i^*(\gamma_{15}) &= 0. \end{aligned} \tag{3.4}$$

Now we put

$$I = (u, \tilde{\gamma}_6, \gamma_{10}, \gamma_{15}),$$

the ideal of $H^*(E_8/T;\mathbb{Z})$ generated by the elements in the parenthesis. Using (3.2), (3.4) and Theorem 3.4, we see that I is contained in $\operatorname{Ker} i^*$. Hence there is an induced map

$$H^*(E_8/T;\mathbb{Z})/I \longrightarrow H^*(C_8/T;\mathbb{Z}) \cong H^*(E_7/T';\mathbb{Z}).$$

Then, by Lemma 3.3, we have

$$\begin{aligned} H^*(E_8/T;\mathbb{Z})/I = {} & \mathbb{Z}[t_1, \dots, t_8, t, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_9, \gamma_{10}, \gamma_{15}] \\ & /(u, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \tilde{\gamma}_6, \rho_8, \rho_9, \rho_{10}, \gamma_{10}, \rho_{12}, \rho_{14}, \rho_{15}, \gamma_{15}, \rho_{18}) \\ = {} & \mathbb{Z}[t_1, \dots, t_7, t, \gamma_3, \gamma_4, \gamma_5, \gamma_9] \\ & /(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \tilde{\gamma}_6, \rho_8, \rho_9, \rho_{10}, \rho_{12}, \rho_{14}, \rho_{18}) \end{aligned}$$

for degrees ≤ 36. Then it follows from Lemma 3.3, Theorem 3.4, (3.2), (3.3) and (3.4) that

$$\begin{aligned} & i^*(\rho_i) \equiv \rho'_i \ (i = 1,2,3,4,5,8,9,10,12,14,18), \\ & i^*(\tilde{\gamma}_6) \equiv \rho'_6. \end{aligned}$$

Therefore this map induces an isomorphism and the assertion follows.

□

3.3. **Generators of $H^*(E_8/T^1\cdot E_7;\mathbb{Z})$.** From Corollary 3.6, we see that $H^*(E_8/C_8;\mathbb{Z})$ is generated by some four elements $\tilde{u}\in H^2$, $\tilde{v}\in H^{12}$, $\tilde{w}\in H^{20}$ and $\tilde{x}\in H^{30}$ such that

$$(\tilde{u},\tilde{v},\tilde{w},\tilde{x})=(u,\tilde{\gamma}_6,\gamma_{10},\gamma_{15}) \tag{3.5}$$

as ideals. So our next task is to describe these generators in the ring $H^*(E_8/T;\mathbb{Z})$. Hereafter we identify $H^*(E_8/C_8;\mathbb{Z})$ with the subalgebra $\operatorname{Im}p^*$ of $H^*(E_8/T;\mathbb{Z})$.

Firstly, by Lemma 2.7 and (2.14), we have

(3.6)

$$\begin{aligned}
\tilde{J}_6 &= \frac{1}{2^{10}\cdot 3^2\cdot 5}J_6\\
&\equiv \frac{2}{5}c_6+\frac{1}{20}c_3^2+c_5\left(-\frac{1}{5}t-\frac{1}{2}u\right)+c_4\left(\frac{1}{2}tu+\frac{1}{3}u^2\right)\\
&+c_3\left(-\frac{1}{5}t^3-\frac{1}{2}t^2u-\frac{1}{2}u^3\right)+\frac{1}{5}t^6+t^5u-\frac{1}{3}t^4u^2+t^3u^3+t^2u^4-tu^5\\
&+\frac{273}{640}u^6 \mod (I_2);
\end{aligned}$$

(3.7)

$$\begin{aligned}
\tilde{J}_{10} &= \frac{1}{2^{14}\cdot 3^3\cdot 5\cdot 7}J_{10}\\
&\equiv \frac{1}{12}c_5^2-\frac{1}{3}c_3c_7+\frac{1}{2}c_3c_6u+c_4c_5\left(-\frac{1}{6}t-\frac{1}{6}u\right)+c_3c_5\left(\frac{1}{6}t^2-\frac{23}{84}u^2\right)\\
&+c_4^2\left(\frac{1}{12}t^2+\frac{1}{6}tu+\frac{1}{14}u^2\right)+c_3^2\left(\frac{1}{12}t^4+\frac{23}{84}t^2u^2-\frac{37}{96}u^4\right)\\
&+c_3c_4\left(-\frac{1}{6}t^3-\frac{1}{6}t^2u-\frac{23}{84}tu^2+\frac{1}{3}u^3\right)\\
&+c_7\left(t^3+\frac{1}{6}t^2u-\frac{17}{42}tu^2+\frac{5}{14}u^3\right)+c_6\left(-\frac{3}{2}t^3u-\frac{3}{14}t^2u^2+\frac{1}{2}tu^3-\frac{37}{84}u^4\right)\\
&+c_5\left(-\frac{1}{3}t^5+\frac{1}{6}t^4u+\frac{23}{21}t^3u^2-\frac{1}{3}t^2u^3+\frac{1}{24}tu^4+\frac{71}{336}u^5\right)\\
&+c_4\left(\frac{1}{3}t^6+\frac{1}{6}t^5u+\frac{17}{42}t^4u^2-\frac{2}{3}t^3u^3-\frac{1}{3}t^2u^4+\frac{5}{16}tu^5-\frac{131}{504}u^6\right)\\
&+c_3\left(-\frac{1}{3}t^7+\frac{1}{6}t^6u-\frac{23}{21}t^5u^2-\frac{1}{3}t^4u^3+\frac{37}{24}t^3u^4+\frac{73}{48}t^2u^5-\frac{3}{2}tu^6+\frac{239}{336}u^7\right)
\end{aligned}$$

$$+\frac{1}{3}t^{10}-\frac{1}{3}t^9u+\frac{7}{6}t^8u^2+\frac{2}{3}t^7u^3-\frac{7}{8}t^6u^4-\frac{35}{8}t^5u^5+\frac{233}{72}t^4u^6+\frac{7}{24}t^3u^7$$
$$-\frac{129}{56}t^2u^8+\frac{215}{168}tu^9-\frac{208601}{645120}u^{10} \mod (I_2).$$

On the other hand, by Lemma 3.3, we have

$$\begin{aligned} c_3 &= 2\gamma_3, \\ c_4 &= 3\gamma_4 - 2t^4, \\ c_5 &= 2\gamma_5 + 3t\gamma_4 - 2t^2\gamma_3, \\ c_6 &= 5\gamma_6 + 2{\gamma_3}^2 + t\gamma_5 - t^2\gamma_4 + 2t^6 \end{aligned} \tag{3.8}$$

in $H^*(E_8/T;\mathbb{Z}) \hookrightarrow H^*(E_8/T;\mathbb{Q})$. Substituting (3.8) into (3.6), we have

$$\begin{aligned} \tilde{J}_6 &= \frac{1}{2^{10}\cdot 3^2\cdot 5}J_6 \\ &= 2\gamma_6 + {\gamma_3}^2 - u\gamma_5 + \gamma_4(-t^2+u^2) - u^3\gamma_3 + t^6 - t^4u^2 \\ &\quad + t^3u^3 + t^2u^4 - tu^5 + \frac{273}{640}u^6. \end{aligned} \tag{3.9}$$

Similarly, substituting (3.8) into (3.7) and using the relations ρ_8, ρ_9 and ρ_{10}, we have

(3.10)

$$\begin{aligned} \tilde{J}_{10} &= \frac{1}{2^{14}\cdot 3^3\cdot 5\cdot 7}J_{10} \\ &= \gamma_{10} + u\gamma_9 - u^3c_7 - u\gamma_4\gamma_5 + 2u^2{\gamma_4}^2 - 2u^2\gamma_3\gamma_5 + \gamma_3\gamma_4(-6tu^2+2u^3) \\ &\quad + {\gamma_3}^2\left(2t^2u^2 + 2tu^3 + \frac{7}{24}u^4\right) + \gamma_6\left(-5t^2u^2 + 5tu^3 + \frac{55}{12}u^4\right) \\ &\quad + \gamma_5\left(t^4u + 3t^3u^2 + t^2u^3 - \frac{55}{24}u^5\right) \\ &\quad + \gamma_4\left(6t^4u^2 - 3t^3u^3 - \frac{103}{24}t^2u^4 - tu^5 + \frac{79}{24}u^6\right) \\ &\quad + \gamma_3\left(-6t^5u^2 - 2t^4u^3 + 4t^3u^4 + 6t^2u^5 - 4tu^6 - \frac{31}{24}u^7\right) \\ &\quad + 4t^7u^3 + \frac{55}{24}t^6u^4 - 6t^5u^5 - \frac{7}{24}t^4u^6 + \frac{79}{24}t^3u^7 + \frac{31}{24}t^2u^8 - \frac{55}{24}tu^9 \\ &\quad + \frac{666919}{645120}u^{10}. \end{aligned}$$

Now let us determine our generators $\tilde{u}, \tilde{v}, \tilde{w}$ and $\tilde{x}$; Obviously, we can take $u = t_8$ as our generator $\tilde{u}$.

Next, since $H^*(E_8/C_8;\mathbb{Q})$ is generated by u, J_6 and J_{10} (see 3.1), we can put

$$\tilde{v} = \alpha\tilde{J}_6 + \beta u^6$$

for some $\alpha, \beta \in \mathbb{Q}$. On the other hand, by (3.5), we can express

$$\tilde{v} = \tilde{\gamma}_6 + f$$

for some element $f \in (u) \cap H^{12}(E_8/T;\mathbb{Z})$. Then using (3.9), we have

$$2\gamma_6 + {\gamma_3}^2 - t^2\gamma_4 + t^6 + f = \alpha(2\gamma_6 + {\gamma_3}^2 - t^2\gamma_4 + t^6)$$
$$+ \alpha(-u\gamma_5 + u^2\gamma_4 - u^3\gamma_3 - t^4u^2 + t^3u^3 + t^2u^4 - tu^4) + \left(\frac{273}{640}\alpha + \beta\right) u^6,$$

and we can take $\alpha = 1,\ \beta = -\dfrac{273}{640}$. Thus we see that

(3.11)

$$\begin{aligned} v &= \frac{1}{2^{10}\cdot 3^2\cdot 5} J_6 - \frac{273}{640}u^6 \\ &= 2\gamma_6 + {\gamma_3}^2 - u\gamma_5 + \gamma_4(-t^2 + u^2) - u^3\gamma_3 + t^6 - t^4u^2 + t^3u^3 + t^2u^4 - tu^5 \end{aligned}$$

can be chosen as our generator $\tilde{v}$.

Similarly, we can put

$$\tilde{w} = \lambda\tilde{J}_{10} + \mu u^4 v + \nu u^{10}$$

for some $\lambda, \mu, \nu \in \mathbb{Q}$. On the other hand, by (3.5), we can express

$$\tilde{w} = \gamma_{10} + g$$

for some element $g \in (u, \tilde{\gamma}_6) \cap H^{20}(E_8/T;\mathbb{Z})$. Then using (3.10), we can take $\lambda = 1$ and hence

$$\begin{aligned} \gamma_{10} + g = {} & \gamma_{10} + u\gamma_9 - u^3c_7 - u\gamma_4\gamma_5 + 2u^2{\gamma_4}^2 - 2u^2\gamma_3\gamma_5 + \gamma_3\gamma_4(-6tu^2 + 2u^3) \\ & + {\gamma_3}^2\left\{2t^2u^2 + 2tu^3 + \left(\mu + \frac{7}{24}\right)u^4\right\} \\ & + \gamma_6\left\{-5t^2u^2 + 5tu^3 + \left(2\mu + \frac{55}{12}\right)u^4\right\} \\ & + \gamma_5\left\{t^4u + 3t^3u^2 + t^2u^3 + \left(-\mu - \frac{55}{24}\right)u^5\right\} \\ & + \gamma_4\left\{6t^4u^2 - 3t^3u^3 + \left(-\mu - \frac{103}{24}\right)t^2u^4 - tu^5 + \left(\mu + \frac{79}{24}\right)u^6\right\} \\ & + \gamma_3\left\{-6t^5u^2 - 2t^4u^3 + 4t^3u^4 + 6t^2u^5 - 4tu^6 + \left(-\mu - \frac{31}{24}\right)u^7\right\} \end{aligned}$$

$$+ 4t^7u^3 + \left(\mu + \frac{55}{24}\right) t^6u^4 - 6t^5u^5 + \left(-\mu - \frac{7}{24}\right) t^4u^6 + \left(\mu + \frac{79}{24}\right) t^3u^7$$
$$+ \left(\mu + \frac{31}{24}\right) t^2u^8 + \left(-\mu - \frac{55}{24}\right) tu^9 + \left(\nu + \frac{666919}{645120}\right) u^{10},$$

and we can take $\mu = -\frac{55}{24}$, $\nu = -\frac{666919}{645120}$. Thus we see that

(3.12)

$$\begin{aligned} w &= \frac{1}{2^{14}\cdot 3^3\cdot 5\cdot 7}J_{10} - \frac{55}{24}u^4v - \frac{666919}{645120}u^{10} \\ &= \gamma_{10} + u\gamma_9 - u^3c_7 - u\gamma_4\gamma_5 + 2u^2{\gamma_4}^2 - 2u^2\gamma_3\gamma_5 + \gamma_3\gamma_4(-6tu^2+2u^3) \\ &\quad +{\gamma_3}^2(2t^2u^2+2tu^3-2u^4) + \gamma_6(-5t^2u^2+5tu^3) + \gamma_5(t^4u+3t^3u^2+t^2u^3) \\ &\quad +\gamma_4(6t^4u^2-3t^3u^3-2t^2u^4-tu^5+u^6) \\ &\quad +\gamma_3(-6t^5u^2-2t^4u^3+4t^3u^4+6t^2u^5-4tu^6+u^7) \\ &\quad +4t^7u^3-6t^5u^5+2t^4u^6+t^3u^7-t^2u^8 \end{aligned}$$

can be chosen as our generator $\tilde{w}$.

Finally, we have to find an element x of degree 30 such that $x \equiv \gamma_{15}$ mod (u, v, w) in $H^*(E_8/T;\mathbb{Z})$. Consider the element

$$x = \frac{1}{2}u^{15} \text{ in } H^{30}(E_8/T;\mathbb{Q}).$$

In fact, it can be shown that x is an integral cohomology class and is contained in $\operatorname{Im} p^*$. Furthermore, we can check directly that

$$x \equiv \gamma_{15} \mod (u, v, w)$$

(see appendix). Hence the element x can be chosen as our generator $\tilde{x}$.

Remark 3.7. *We can describe the element x explicitly in the ring $H^*(E_8/T;\mathbb{Z})$. But it is too lengthy to write down here, so we will give the explicit form of x in the appendix.*

3.4. **Integral cohomology ring of $E_8/T^1\cdot E_7$.** Using the element x, we can rewrite $\tilde{I}_{30}$:

$$\begin{aligned} \tilde{I}_{30} = &\ -9u^{30} - 24u^{24}v - 12u^{20}w + 36u^{14}vw - 40u^{12}v^3 - 12u^{10}w^2 + 120u^8v^2w \\ &- 140u^6v^4 + 24u^4vw^2 - 40u^2v^3w - 16v^5 - 8w^3 \\ = &\ -36x^2 - 48u^9vx - 24u^5wx + 36u^{14}vw - 40u^{12}v^3 - 12u^{10}w^2 + 120u^8v^2w \\ &- 140u^6v^4 + 24u^4vw^2 - 40u^2v^3w - 16v^5 - 8w^3 \\ = &\ 4(-9x^2 - 12u^9vx - 6u^5wx + 9u^{14}vw - 10u^{12}v^3 - 3u^{10}w^2 + 30u^8v^2w \\ &- 35u^6v^4 + 6u^4vw^2 - 10u^2v^3w - 4v^5 - 2w^3). \end{aligned}$$

Therefore, in view of Lemma 3.1, we obtain the following main result:

Theorem 3.8. *The integral cohomology ring of* $E_8/T^1\cdot E_7$ *is given as follows*:

$$H^*(E_8/T^1\cdot E_7;\mathbb{Z}) = \mathbb{Z}[u,v,w,x]/(r_{15},r_{20},r_{24},r_{30}),$$

where $\deg u = 2, \deg v = 12, \deg w = 20, \deg x = 30$ *and*

$$\begin{aligned}
r_{15} &= u^{15} - 2x,\\
r_{20} &= 9u^{20} + 45u^{14}v + 12u^{12}w + 60u^8v^2 + 30u^4vw + 10u^2v^3 + 3w^2,\\
r_{24} &= 11u^{24} + 60u^{18}v + 21u^{14}w + 105u^{12}v^2 + 60u^8vw + 60u^6v^3 + 9u^4w^2\\
&\quad + 30u^2v^2w + 5v^4,\\
r_{30} &= -9x^2 - 12u^9vx - 6u^5wx + 9u^{14}vw - 10u^{12}v^3 - 3u^{10}w^2 + 30u^8v^2w\\
&\quad - 35u^6v^4 + 6u^4vw^2 - 10u^2v^3w - 4v^5 - 2w^3.
\end{aligned}$$

In order to determine the integral cohomology ring of E_8/E_7, we consider the Gysin exact sequence associated with the following circle bundle

$$S^1 \longrightarrow E_8/E_7 \xrightarrow{\pi} E_8/T^1\cdot E_7, \tag{3.13}$$

where π is the natural projection. In this case, it reduces to the following short exact sequeces:
(3.14)

$$\begin{aligned}
0 \longrightarrow H^{\mathrm{odd}}(E_8/E_7;\mathbb{Z}) &\longrightarrow H^*(E_8/T^1\cdot E_7;\mathbb{Z})\\
&\xrightarrow{\times u} H^*(E_8/T^1\cdot E_7;\mathbb{Z}) \xrightarrow{\pi^*} H^{\mathrm{even}}(E_8/E_7;\mathbb{Z}) \longrightarrow 0,
\end{aligned}$$

where $H^{\mathrm{even}} = \oplus_{i\geq 0}H^{2i}$ and $H^{\mathrm{odd}} = \oplus_{i\geq 0}H^{2i+1}$.

From the exactness of (3.14), it follow that $H^{\mathrm{even}}(E_8/E_7;\mathbb{Z})$ is isomorphic to $H^*(E_8/T^1\cdot E_7;\mathbb{Z})/(u)$. Define the elements z_i $(i = 12, 20, 30)$ of $H^*(E_8/E_7;\mathbb{Z})$ as follows:

$$z_{12} = \pi^*(v),\ z_{20} = \pi^*(w),\ z_{30} = \pi^*(x).$$

Then, by Theorem 3.8, we obtain

$$\begin{aligned}
H^{\mathrm{even}}(E_8/E_7;\mathbb{Z}) &= \mathbb{Z}[z_{12},z_{20},z_{30}]/(2z_{30}, 3{z_{20}}^2, 5{z_{12}}^4, 4{z_{12}}^5 + 2{z_{20}}^3 + 9{z_{30}}^2).\\
&= \mathbb{Z}[z_{12},z_{20},z_{30}]/(2z_{30}, 3{z_{20}}^2, 5{z_{12}}^4, {z_{12}}^5 + {z_{20}}^3 + {z_{30}}^2).
\end{aligned}$$

By Poincaré duality there exist elements $z_i \in H^i(E_8/E_7;\mathbb{Z})$ $(i = 59, 71, 79, 83, 91, 95, 103, 115)$ such that

$$z_{12}^3z_{20}z_{59} = z_{12}^2z_{20}z_{71} = z_{12}^3z_{79} = z_{12}z_{20}z_{83} = z_{12}^2z_{91} = z_{20}z_{95} = z_{12}z_{103} = z_{115}.$$

Then it is not hard to show that

$$
\begin{aligned}
z_{71} &= z_{12} z_{59}, \\
z_{79} &= z_{20} z_{59}, \\
z_{83} &= z_{12}^2 z_{59}, \\
z_{91} &= z_{12} z_{20} z_{59}, \\
z_{103} &= z_{12}^2 z_{20} z_{59}, \\
z_{115} &= z_{12}^3 z_{20} z_{59}.
\end{aligned}
$$

Summing up the results, we obtain the following:

Corollary 3.9. *The structure of* $H^*(E_8/E_7;\mathbb{Z})$ *is given by the following table*:

nontrivial $H^k(E_8/E_7;\mathbb{Z})$	basis elements
$H^0 = \mathbb{Z}$	1
$H^{12} = \mathbb{Z}$	z_{12}
$H^{20} = \mathbb{Z}$	z_{20}
$H^{24} = \mathbb{Z}$	${z_{12}}^2$
$H^{30} = \mathbb{Z}_2$	z_{30}
$H^{32} = \mathbb{Z}$	$z_{12} z_{20}$
$H^{36} = \mathbb{Z}$	${z_{12}}^3$
$H^{40} = \mathbb{Z}_3$	${z_{20}}^2$
$H^{42} = \mathbb{Z}_2$	$z_{12} z_{30}$
$H^{44} = \mathbb{Z}$	${z_{12}}^2 z_{20}$
$H^{48} = \mathbb{Z}_5$	${z_{12}}^4$
$H^{50} = \mathbb{Z}_2$	$z_{20} z_{30}$
$H^{52} = \mathbb{Z}_3$	$z_{12} {z_{20}}^2$
$H^{54} = \mathbb{Z}_2$	${z_{12}}^2 z_{30}$
$H^{56} = \mathbb{Z}$	${z_{12}}^3 z_{20}$
$H^{59} = \mathbb{Z}$	z_{59}
$H^{62} = \mathbb{Z}_2$	$z_{12} z_{20} z_{30}$
$H^{64} = \mathbb{Z}_3$	$z_{12}^2 z_{20}^2$
$H^{66} = \mathbb{Z}_2$	$z_{12}^3 z_{30}$
$H^{68} = \mathbb{Z}_5$	$z_{12}^4 z_{20}$
$H^{71} = \mathbb{Z}$	$z_{12} z_{59}$
$H^{74} = \mathbb{Z}_2$	$z_{12}^2 z_{20} z_{30}$
$H^{76} = \mathbb{Z}_3$	$z_{12}^3 z_{20}^2$
$H^{79} = \mathbb{Z}$	$z_{20} z_{59}$
$H^{83} = \mathbb{Z}$	${z_{12}}^2 z_{59}$

nontrivial $H^k(E_8/E_7;\mathbb{Z})$	basis elements
$H^{86}=\mathbb{Z}_2$	$z_{12}^3 z_{20} z_{30}$
$H^{91}=\mathbb{Z}$	$z_{12} z_{20} z_{59}$
$H^{95}=\mathbb{Z}$	${z_{12}}^3 z_{59}$
$H^{103}=\mathbb{Z}$	${z_{12}}^2 z_{20} z_{59}$
$H^{115}=\mathbb{Z}$	${z_{12}}^3 z_{20} z_{59}$

4. APPENDIX

In 3.3, we defined the element x as a rational cohomology class given by

$$x=\frac{1}{2}u^{15} \quad \text{in} \quad H^{30}(E_8/T;\mathbb{Q}).$$

We need to show that x is in fact an integral cohomology class. By Lemma 3.3, the following relations hold in $H^*(E_8/T\ \mathbb{Z})$:

$$\begin{aligned}
c_1 &= 3t,\\
c_2 &= 4t^2,\\
c_3 &= 2\gamma_3,\\
c_4 &= 3\gamma_4-2t^4,\\
c_5 &= 2\gamma_5+3t\gamma_4-2t^2\gamma_3,\\
c_6 &= 5\gamma_6+2\gamma_3^2+t\gamma_5-t^2\gamma_4+2t^6.
\end{aligned} \tag{4.1}$$

Note that, by (2.11) and (4.1), the following relation holds:
(4.2)

$$\begin{aligned}
c_8 &= uc_7-5u^2\gamma_6-2u^2\gamma_3^2+(-tu^2+2u^3)\gamma_5+(t^2u^2+3tu^3-3u^4)\gamma_4\\
&\quad +(-2t^2u^3+2u^5)\gamma_3-2t^6u^2+2t^4u^4-4t^2u^6+3tu^7-u^8.
\end{aligned}$$

Using (4.2), we can rewrite the higher relations $\rho_8,\rho_9,\rho_{10},\rho_{12},\rho_{14}$ and ρ_{15}. For example,

$$\begin{aligned}
\rho_8 &= -3c_8+3\gamma_4^2-2\gamma_3\gamma_5+t(2c_7-6\gamma_3\gamma_4)+t^2(2\gamma_3^2-5\gamma_6)+3t^3\gamma_5\\
&\quad +4t^4\gamma_4-6t^5\gamma_3+t^8\\
&= 3\gamma_4^2-2\gamma_3\gamma_5+(2t-3u)c_7-6t\gamma_3\gamma_4+(-5t^2+15u^6)\gamma_6\\
&\quad +(2t^2+6u^2)\gamma_3^2+(3t^3+3tu^2-6u^3)\gamma_5+(4t^4-3t^2u^2-9tu^3+9u^4)\gamma_4\\
&\quad +(-6t^5+6t^2u^3-6u^5)\gamma_3+t^8+6t^6u^2-6t^4u^4+12t^2u^6-9tu^7+3u^8.
\end{aligned}$$

Using the relations ρ_i $(i = 1,2,3,4,5,6,8,9,10,12,14,15)$, we rewrite the element $x = \dfrac{1}{2}u^{15}$ as follows:

$$
\begin{aligned}
x &= \frac{1}{2}u^{15} \\
&= \frac{1}{2}\left\{u^{15} - \rho_{15} + u\rho_{14} - u^3\rho_{12} + (t^4u - t^2u^3)\rho_{10}\right. \\
&+(\gamma_6 + t\gamma_5 + u\gamma_5 + u^2\gamma_4 + u^3\gamma_3 + t^6 + t^4u^2 + t^3u^3 + t^2u^4 + tu^5 + u^6)\rho_9 \\
&\left.-39u^7\rho_8\right\} + (-tu\gamma_3 + 2t^3u^2 - 5tu^4)\rho_{10} \\
&= \gamma_{15} - 20\gamma_3{\gamma_6}^2 + 3{\gamma_3}^2\gamma_9 - 23{\gamma_3}^3\gamma_6 - 6{\gamma_3}^5 + 4\gamma_6\gamma_9 \\
&+ 3u\gamma_4\gamma_{10} - u\gamma_5\gamma_9 - 3u{\gamma_3}^2{\gamma_4}^2 + 3uc_7\gamma_3\gamma_4 - 6u{\gamma_4}^2\gamma_6 + (-3t + 2u){\gamma_3}^3\gamma_5 \\
&+ (-4t + 4u)\gamma_3\gamma_5\gamma_6 \\
&+ (-t^2 - u^2)\gamma_4\gamma_9 + (t^2 + tu - u^2)c_7{\gamma_3}^2 + (9t^2 + 12tu + 5u^2)\gamma_3\gamma_4\gamma_6 \\
&+ (5t^2 + 6tu + 2u^2){\gamma_3}^3\gamma_4 + (3t^2 + 4tu + u^2)c_7\gamma_6 \\
&+ (-6t^3 - 2t^2u - 6tu^2 + 5u^3){\gamma_3}^4 - u^3\gamma_3\gamma_9 + (3t^2u + u^3){\gamma_4}^3 + (2t^2u + 3tu^2)c_7\gamma_5 \\
&+ (-45t^3 + 10t^2u - 40tu^2){\gamma_6}^2 + (t^3 - 2t^2u + tu^2 - u^3)\gamma_3\gamma_4\gamma_5 \\
&+ (-33t^3 + t^2u - 31tu^2 + 13u^3){\gamma_3}^2\gamma_6 \\
&+ (-2t^4 - 4t^3u - 3tu^3 + 3u^4)c_7\gamma_4 + (-9t^4 - 6t^3u - 18t^2u^2 + 5tu^3 - 3u^4)\gamma_5\gamma_6 \\
&+ (-3t^4 - 3t^3u - 7t^2u^2 + 5tu^3 - 4u^4){\gamma_3}^2\gamma_5 + (-t^4 - 6t^3u - t^2u^2 - 3tu^3)\gamma_3{\gamma_4}^2 \\
&+ (-3t^4u - 6t^3u^2 + 3t^2u^3 + 15tu^4)\gamma_{10} + (-3t^4u + t^3u^2 + 5t^2u^3 + 10tu^4 - u^5)c_7\gamma_3 \\
&+ (15t^5 - 2t^4u + 3t^3u^2 + 14t^2u^3 - 16tu^4 + 3u^5){\gamma_3}^2\gamma_4 \\
&+ (39t^5 - 13t^4u + 8t^3u^2 + 35t^2u^3 - 31tu^4 - 3u^5)\gamma_4\gamma_6 \\
&+ (t^6 - t^4u^2 - t^3u^3 - t^2u^4 - tu^5 - u^6)\gamma_9 \\
&+ (-13t^6 + 12t^5u + 5t^4u^2 - 56t^3u^3 + 8t^2u^4 + 21tu^5 + 2u^6)\gamma_3\gamma_6 \\
&+ (6t^6 + 3t^5u + 2t^4u^2 + 7t^3u^3 + t^2u^4 - 8tu^5 + 3u^6)\gamma_4\gamma_5 \\
&+ (-8t^6 + 6t^5u + 2t^4u^2 - 22t^3u^3 + 6t^2u^4 + 8tu^5 - 2u^6){\gamma_3}^3 \\
&+ (-6t^7 + t^6u - 7t^4u^3 + 5t^3u^4 + 3t^2u^5 + 3tu^6 - 63u^7){\gamma_4}^2 \\
&+ (-t^7 + 2t^6u + t^5u^2 - 11t^4u^3 + 6t^3u^4 + 5t^2u^5 + 6tu^6 + 39u^7)\gamma_3\gamma_5 \\
&+ (2t^8 + 6t^7u + 3t^6u^2 - 4t^5u^3 - 15t^4u^4 + 6t^3u^5 + 3t^2u^6 - 40tu^7 + 59u^8)c_7 \\
&+ (3t^8 + t^6u^2 + 11t^5u^3 + 14t^4u^4 - 20t^3t^5 - 4t^2u^6 + 118tu^7 + 3u^8)\gamma_3\gamma_4 \\
&+ (-48t^9 + 3t^8u - 41t^7u^2 + 18t^6u^3 + 16t^5u^4 - 13t^4u^5 - 67t^3u^6 + 125t^2u^7 \\
&- 15tu^8 - 291u^9)\gamma_6
\end{aligned}
$$

$$+ (-18t^9 - 3t^8u - 16t^7u^2 + 10t^6u^3 - 4t^5u^4 - 8t^4u^5 - 16t^3u^6 - 23t^2u^7 - 10tu^8$$
$$- 115u^9){\gamma_3}^2$$
$$+ (-6t^{10} - 3t^9u - 9t^8u^2 + 5t^7u^3 - 5t^6u^4 - 14t^4u^6 - 52t^3u^7 + 6t^2u^8 - 60tu^9$$
$$+ 117u^{10})\gamma_5$$
$$+ (18t^{11} - 3t^{10}u + 5t^9u^2 + 11t^8u^3 - 28t^7u^4 + 8t^6u^5 + 20t^5u^6 - 64t^4u^7 - 15t^3u^8$$
$$+ 54t^2u^9 + 178tu^{10} - 177u^{11})\gamma_4$$
$$+ (-2t^{12} + 6t^{11}u + 2t^{10}u^2 - 20t^9u^3 + 11t^8u^4 + 22t^7u^5 - 8t^6u^6 + 83t^5u^7$$
$$+ 15t^4u^8 + 5t^3u^9 - 116t^2u^{10} + tu^{11} + 117u^{12})\gamma_3$$
$$- 12t^{15} - t^{14}u - 10t^{13}u^2 + 6t^{12}u^3 + 7t^{11}u^4 - 13t^{10}u^5 - 31t^9u^6 + 9t^8u^7 - t^7u^8$$
$$- 118t^6u^9 - 18t^5u^{10} + 131t^4u^{11} - 6t^3u^{12} - 233t^2u^{13} + 175tu^{14} - 58u^{15},$$

which has shown that x is an integral cohomology class.

Next, we have to show that

$$x \mod (u, v, w) \equiv \gamma_{15}.$$

By (3.11) and (3.12), we have

$$v \equiv 2\gamma_6 + \gamma_3^2 - t^2\gamma_4 + t^6 \mod (u),$$
$$w \equiv \gamma_{10} \mod (u, v).$$

Therefore, in the ring $H^*(E_8/T;\mathbb{Z})/(u, v, w)$, the following relations hold:

$$\begin{aligned} u &= 0, \\ \gamma_3^2 &= -2\gamma_6 + t^2\gamma_4 - t^6, \\ \gamma_{10} &= 0. \end{aligned} \tag{4.3}$$

On the other hand, we determined the ring $H^*(E_8/T;\mathbb{Z})$ up to degrees $\leqq 36$ (Lemma 3.3). Taking (4.3) into account, we can show directly that $x = \gamma_{15}$ in the ring $H^*(E_8/T;\mathbb{Z})/(u, v, w)$.

References

[1] I.N. Bernstein, I.M. Gelfand and S.I. Gelfand, *Schubert cells and the cohomology of the spaces G/P*, L.M.S. Lecture Notes vol.69 Cambridge Univ. Press, 1982, 115-140.

[2] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. **57** (1953), 115-207.

[3] A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces*, Amer. J. Math. **80** (1958), 458-538.

[4] R. Bott, *An application of the Morse theory to the topology of Lie groups*, Bull. Soc. Math. France **84** (1956), 251-282.

[5] R. Bott, *The space of loops on a Lie group*, Michigan Math. J. **5** (1958), 35-61

[6] R. Bott and H. Samelson, *Application of the theory of Morse to the symmetric spaces*, Amer. J. Math. **80** (1958), 964-1029.
[7] N. Bourbaki, *Groupes et Algèbre de Lie* IV − VI, Masson, Paris, 1968.
[8] H. Duan, *Multiplicative rule of Schubert classes*, Invent. Math. **159** (2005), 407-436 .
[9] H. Duan and X. Zhao, *The Chow rings of generalized Grassmannians*, available on arXiv: math.AG/0511332.
[10] A. Grothendieck, *Torsion homologique et sections rationnelles*, Séminaire Claude Chevalley, 3 (1958), exposé 5, 1-29.
[11] K. Ishitoya, *Integral cohomology ring of the symmetric space EII*, J. Math. Kyoto Univ. **17** (1977), 375-397.
[12] K. Ishitoya and H. Toda, *On the cohomology of irreducible symmetric spaces of exceptional type*, J. Math. Kyoto Univ. **17** (1977), 225-243.
[13] V. Kac, *Torsion in cohomology of compact Lie groups and Chow ring of reductive algebraic groups*, Invent. Math. **80** (1985), 69-79.
[14] R. Marlin, *Anneaux de Chow des groupes algebriques $SO(n), Spin(n), G_2$ et F_4*, Publications Math. d'Orsay, 95-7419 (1974).
[15] M. Nakagawa, *The integral cohomology ring of E_7/T*, J. Math. Kyoto Univ. **41** (2001), 303-321.
[16] M. Nakagawa, *The integral cohomology ring of E_8/T*, In preparation.
[17] H. Toda, *On the cohomology ring of some homogeneous spaces*, J. Math. Kyoto Univ. **15** (1975), 185-199.
[18] H. Toda and T. Watanabe, *The integral cohomology ring of F_4/T and E_6/T*, J. Math. Kyoto Univ. **14** (1974), 257-286.
[19] T. Watanabe, *The integral cohomology ring of the symmetric space $EVII$*, J. Math. Kyoto Univ. **15** (1975), 363-385.

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