SUPPORT VARIETIES FOR THE STEENROD ALGEBRA

DANIEL K. NAKANO AND JOHN H. PALMIERI

1 Introduction

In recent years the spectrum of the cohomology ring has been a fundamental object of interest in the study of modular group cohomology. Quillen [**Q1-2**] proved that this variety can be expressed as a disjoint union of locally closed varieties involving elementary abelian subgroups. Avrunin and Scott [**AS**] later generalized Quillen's result for varieties associated to modules defined by Carlson [**C1**]. For restricted Lie algebra cohomology no such stratification theorem exists. Friedlander and Parshall [**FP1-2**] have shown that the cohomological varieties in this context can be realized as subvarieties of the given Lie algebra when viewed as an affine space.

Fix a prime p and let A be the mod p Steenrod algebra. The cohomology of A, $\mathrm{Ext}^*_A(\mathbf{F}_p, \mathbf{F}_p)$, is an object of much interest in algebraic topology. One way to study the cohomology ring is to approximate A by finite-dimensional sub-Hopf algebras and then try to compute their cohomology rings. By a theorem of Wilkerson [**W**], the cohomology of any of these finite-dimensional Hopf algebras is a finitely generated commutative $\mathbf{F}_p$-algebra; hence one can get a great deal of information (as in the case of groups and Lie algebras) by studying the spectrum of this ring. One of the goals of this paper will be to show how the aforementioned results for finite groups and restricted Lie algebras can be used to study the cohomological varieties for finite-dimensional sub-Hopf algebras of the Steenrod algebra.

Given a finite-dimensional sub-Hopf algebra B of A, let $|B|_{\mathbf{F}_p}$ be the spectrum of $\mathrm{Ext}^*_B(\mathbf{F}_p, \mathbf{F}_p)$. For a finite-dimensional B-module M, the

Date. 16 December 1994.

1991 *Mathematics Subject Classification.* Primary 55S10, 16W30, 18G15; Secondary 17B56, 20J06.

Research of the first author partially supported by NSF grant DMS-9206284.

Research of the second author partially supported by NSF grant DMS-9407459.

support variety of B, denoted by $|B|_M$, is the spectrum of $\mathrm{Ext}_B^*(\mathbf{F}_p, \mathbf{F}_p)/J_B(M)$, where $J_B(M)$ is the annihilator ideal of $\mathrm{Ext}_B^*(M, M)$ via the map

$$- \otimes M : \mathrm{Ext}_B^*(\mathbf{F}_p, \mathbf{F}_p) \to \mathrm{Ext}_B^*(M, M).$$

In this paper we identify these varieties for all B and M; hence we determine, up to nilpotence, the ring $\mathrm{Ext}_B^*(\mathbf{F}_p, \mathbf{F}_p)/J_B(M)$. Since $\mathrm{Ext}_B^*(M, M)$ is finitely generated as a module over this ring, this more or less identifies $\mathrm{Ext}_B^*(M, M)$ modulo nilpotence.

To state our results, we need to review a few facts about the Steenrod algebra; see Section 5 for more details. Fix a prime p and let $k = \overline{\mathbf{F}}_p$—we work over the algebraic closure of $\mathbf{F}_p$ for technical reasons. Let P be the Hopf algebra dual to $k[\xi_1, \xi_2, \xi_3, \ldots]$, $\deg(\xi_n) = 2^n - 1$ if $p = 2$, $\deg(\xi_n) = 2p^n - 2$ if p odd, with diagonal given by

$$\Delta(\xi_n) = \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \xi_i$$

(where $\xi_0 = 1$). For p odd, let A be the Hopf algebra dual to $k[\xi_1, \xi_2, \xi_3, \ldots] \otimes E[\tau_0, \tau_1, \tau_2, \ldots]$, $\deg(\tau_n) = 2p^n - 1$, with diagonal given by

$$\begin{aligned} \Delta(\xi_n) &= \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \xi_i, \\ \Delta(\tau_n) &= \tau_n \otimes 1 + \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \tau_i. \end{aligned}$$

Except for the fact that we are working over $\overline{\mathbf{F}}_p$ rather than $\mathbf{F}_p$, for p odd A is the mod p Steenrod algebra and P is the mod p reduced powers, while for $p = 2$, P is the mod 2 Steenrod algebra. Let the *Milnor basis* for either P or A be the dual to the monomial basis, let P_t^s be the Milnor basis element dual to $\xi_t^{p^s}$, and let Q_n be the Milnor basis element dual to τ_n. Recall that if B is a sub-Hopf algebra of P (resp., A), then B is generated as an algebra by the P_t^s's (resp., the P_t^s's and Q_n's) that it contains. If $s < t$, then $(P_t^s)^p = 0$, and for all n, $(Q_n)^2 = 0$, so we refer to the P_t^s's with $s < t$ and the Q_n's contained in B as the *differentials* in B, and we let $D(B) = \{P_t^s \in B,\ s < t\} \cup \{Q_n \in B\}$. Also, so-called "quasi-elementary" Hopf algebras play an important role here; see Definition 2.3 for the general definition, and Proposition 5.2 for a description of the quasi-elementary sub-Hopf algebras of P and A. The following is our main theorem.

Theorem 1.1. *Let B be a finite-dimensional sub-Hopf algebra of P or A, and let M be a finitely generated B-module. Let $\mathbf{C}_B$ be the collection of quasi-elementary sub-Hopf algebras of B.*

(a) *If Q and Q' are quasi-elementary sub-Hopf algebras of B with $Q' \leq Q$, then $|Q'|_M \hookrightarrow |Q|_M$. There is an inseparable isogeny* (*in the sense of Benson* [**Ben**, p. 172])

$$\bigcup_{Q \in \mathbf{C}_B} |Q|_M \to |B|_M,$$

where the union is taken with respect to these inclusions.

(b) *Fix $Q \in \mathbf{C}_B$, and let $L_Q = \operatorname{Span}_k D(Q)$. Then $x^p = 0$ for all $x \in L_Q$; furthermore, there is a bijective map $\Phi_Q : |Q|_k \to L_Q$ so that*

$$\Phi_Q(|Q|_M) = \{x \in L_Q : M|_{\langle x \rangle} \text{ is not free}\}.$$

The maps Φ_Q are compatible with the inclusions of part (a). Hence $|B|_k$ is the union of affine subvarieties, one for each $Q \in \mathbf{C}_B$.

Here $\langle x \rangle$ denotes the algebra generated by x. Note that we can also express the freeness condition in terms of homology groups: if for some $x \in B$ we have $x^n = 0$, then for any B-module M we define

$$H(M, x) = \frac{\ker(x : M \to M)}{\operatorname{im}(x^{n-1} : M \to M)}.$$

It is easy to check that $M|_{\langle x \rangle}$ is free if and only if $H(M, x) = 0$. We also have the following results.

Theorem 1.2. *Let Q be a finite-dimensional quasi-elementary sub-Hopf algebra of P or A and let M, N be Q-modules. The map $\Phi : |Q|_k \to L_Q$ has the following properties:*

(a) $\Phi(|Q|_{M \oplus N}) = \Phi(|Q|_M) \cup \Phi(|Q|_N)$.
(b) $\Phi(|Q|_{M \otimes N}) = \Phi(|Q|_M) \cap \Phi(|Q|_N)$.

Theorem 1.3. *Suppose B is a finite-dimensional sub-Hopf algebra of P or A. The following numbers are equal.*

(a) *the Krull dimension of* $\operatorname{Ext}^*_B(k, k)$
(b) $\dim |B|_k$
(c) $\max_{Q \in \mathbf{C}_B} \operatorname{Krull} \dim H^*(Q, k)$
(d) $\max_{Q \in \mathbf{C}_B} \# D(Q)$

The equality of (a) and (d) for $p = 2$ is originally due to Wilkerson [**W**].

Theorem 1.4. *Let B be a finite-dimensional sub-Hopf algebra of P or A and let $\hat{\mathbf{C}}_B$ denote the collection of maximal quasi-elementary* *sub-Hopf algebras of B. Then*

$$\operatorname{depth}(\operatorname{Ext}_B^*(k,k)) \leq \min_{Q\in\hat{\mathbf{C}}_B} \#D(Q).$$

One can compute these numbers easily for particular examples; for instance, one sees that the Krull dimension of $\operatorname{Ext}_{P_n}^*(k,k)$ is quadratic in n [**W**, 6.6], while the depth is at most linear. Consequently, the cohomology ring $\operatorname{Ext}_{P_n}^*(k,k)$ is far from being Cohen-Macaulay for n large. Here P_n is the sub-Hopf algebra of P generated by $\{P_t^s : s+t \leq n+1\}$. Wilkerson gives the formula for the Krull dimension when $p=2$; at odd primes, the Krull dimension is one more than at $p=2$. Theorem 1.4 implies that $\operatorname{depth}(\operatorname{Ext}_{P_n}^*(k,k)) \leq n+1$ for $p=2$, and $\operatorname{depth}(\operatorname{Ext}_{P_n}^*(k,k)) \leq 2n$ for p odd.

This next result proves a conjecture of Margolis [**Mar**, p. 344].

Corollary 1.5. *Let B be a sub-Hopf algebra of P or A and let M, N be finitely generated B-modules. Given a differential $x \in D(B)$, then $H(M\otimes N, x) = 0$ if and only if $H(M,x)=0$ or $H(N,x)=0$.*

Part (a) of the following result is due to Adams and Margolis [**AM1**] for $p=2$, Moore and Peterson [**MP**] for odd primes; it is a simple corollary of our other results. Let $c_B(M)$ be the *complexity* of M as an B-module—see Section 2 for a definition.

Corollary 1.6. *Let B be a finite-dimensional sub-Hopf algebra of P or A and M a finitely generated graded B-module.*

(a) *M is free if and only if $H(M,x)=0$ for all $x\in D(B)$.*
(b) *$c_B(M) \leq \sum_{x\in D(B)} c_{\langle x\rangle}(M)$.*

We can also use properties of the mod 2 Steenrod algebra to prove the following refinement of Theorem 1.1.

Theorem 1.7. *Let $p=2$. Suppose B is a finite-dimensional sub-Hopf algebra of P and M is a finitely generated B-module. Let $V = \operatorname{Span}_k D(B)$. Then there is an injective map $\Phi : |B|_k \to V$ so that*

$$\Phi(|B|_M) = \{x \in V : x^2 = 0,\ H(M,x)\neq 0\}.$$

Our proof of this result is too unwieldy to generalize easily to odd primes, but it does check out in a number of examples; hence we have the following conjecture. Indeed, there may be a valid generalization for any finite-dimensional graded connected cocommutative Hopf algebra, using the termination, at some finite stage, of the Ivanovskiĭ spectral sequence [**BS**].

Conjecture 1.8. *Suppose p is odd, B is a finite-dimensional sub-Hopf algebra of P or A, and M is a finitely generated B-module. Let $V = \operatorname{Span}_k D(B)$. Then there is an injective map $\Phi : |B|_k \to V$ so that*

$$\Phi(|B|_M) = \{x \in V : x^p = 0,\ H(M, x) \neq 0\}.$$

The paper is organized as follows. In Section 2 we discuss basic facts about support varieties for finite-dimensional Hopf algebras and recall an analogue of the Quillen stratification theorem involving quasi-elementary Hopf algebras. In Section 3 we extend this last result to non-trivial modules, à la Avrunin and Scott. In Section 4 we use results on the cohomology of restricted Lie algebras to construct the map Φ in Theorem 1.1(b). Finally, in Section 5 we apply the results in the previous sections to finite-dimensional sub-Hopf algebras of the Steenrod algebra.

Throughout this paper we have tried to make our arguments apply to a fairly general family of Hopf algebras; Sections 2–4 start with an assumption on the Hopf algebras under consideration, and Section 5 consists primarily of verifying these assumptions for sub-Hopf algebras of the Steenrod algebra. It is our hope that this approach makes the paper accessible in other contexts. The authors would like to thank the Department of Mathematics at the University of Wisconsin, Madison, for its hospitality during the completion of this work.

2 Varieties for modules

Let B be a finite-dimensional cocommutative Hopf algebra over an algebraically closed field k of characteristic $p > 0$; given a B-module M, let $H^n(B, M) = \operatorname{Ext}^n_B(k, M)$ denote the nth cohomology group of B with coefficients in M. We will make the following assumption on the cohomology ring.

Assumption 2.1. $\operatorname{Ext}^*_B(k, k)$ *is a finitely generated k-algebra, and for M and N finite-dimensional B-modules,* $\operatorname{Ext}^*_B(M, N)$ *is finitely generated as a module over* $\operatorname{Ext}^*_B(k, k)$.

This has been proved in a number of different contexts—for group algebras by Evens [**E**], for restricted enveloping algebras by Friedlander and Parshall [**FP2**], and for graded connected Hopf algebras by Wilkerson [**W**] and Bajer and Sadofsky [**BS**].

Let $\mathcal{V} = \{V_t : t \in \mathbb{N}\}$ be a sequence of finite-dimensional vector spaces over k. The *rate of growth* of $\mathcal{V}$, denoted by $r(\mathcal{V})$, is the smallest integer $s \geq 0$ such that for all $t \in \mathbb{N}$, we have $\dim_k V_t \leq K \cdot t^{s-1}$ for some constant $K \geq 0$. Let N be a B-module and let

$$... \to P_2 \to P_1 \to P_0 \to N \to 0 \tag{2.1}$$

be the minimal projective resolution of N. The *complexity* of N [**Alp**], $c_B(N)$, is by definition $r(\{P_t : t \in \mathbb{N}\})$.

The finite generation of cohomology (Assumption 2.1) allows us to define varieties associated to the cohomology ring. If $\mathrm{char}(k) = p$ is odd, we let $R_B = \mathrm{Ext}_B^{2*}(k,k)$; if $p = 2$, we let $R_B = \mathrm{Ext}_B^*(k,k)$. Given a B-module M, let $J_B(M) \subseteq R_B$ be the annihilator ideal of $\mathrm{Ext}_B^*(M,M)$. The support variety associated to M, $|B|_M$, is defined to be the affine homogeneous variety $\mathrm{Maxspec}(R_B/J_B(M))$. The following proposition summarizes the general properties about the complexity of a module and the dimension of the support variety. The proof of this result for group algebras given in [**C2**] can be easily adapted to this more general situation.

Theorem 2.2. *Let B be a finite-dimensional cocommutative Hopf algebra satisfying Assumption 2.1, and let $\{S_j : j = 1,..,r\}$ be the complete set of non-isomorphic simple B-modules. Then the following numbers are equal:*

(1) $c_B(M)$
(2) $\dim |B|_M$
(3) $r(\{\mathrm{Ext}_B^t(M,M) : t \in \mathbb{N}\})$
(4) $r(\{\mathrm{Ext}_B^t(M, \oplus_{i=1}^r S_i) : t \in \mathbb{N}\})$

We remark that since B is a finite-dimensional cocommutative Hopf algebra, then B is a self-injective algebra [**LS**, p. 85]. This fact implies that $c_B(M) = 0$ if and only if M is a projective B-module.

As in the case for group algebras it is straightforward to verify the equality

$$|B|_{M\oplus N} = |B|_M \cup |B|_N$$

for B-modules M and N. In order to get more detailed information about the varieties $|B|_M$ (e.g., an analogue of the Quillen stratification theorem, or a proof that $|B|_{M\otimes N} = |B|_M \cap |B|_N$), we need to work with "quasi-elementary" Hopf algebras. These algebras act as replacements for elementary abelian p-groups.

From this point on, the Hopf algebras will be graded connected, in addition to finite-dimensional and cocommutative. As noted above, Assumption 2.1 holds in this context. Furthermore, the only simple module is the trivial module k. In this case the variety $|B|_M$ can be defined to be $\mathrm{Maxspec}(R_B/I_B(M))$ where $I_B(M) \subseteq R_B$ is the annihilator ideal of $\mathrm{Ext}^*_B(k, M)$ via the action by Yoneda composition.

We write $C \leq B$ if C is a sub-Hopf algebra of B and $C \triangleleft B$ if C is a normal sub-Hopf algebra of B. If $C \triangleleft B$ then the quotient Hopf algebra will be denoted by $B//C$. For $C \leq B$ let $\mathrm{res}_{B,C}$ be the restriction map $\mathrm{Ext}^*_B(k,k) \to \mathrm{Ext}^*_C(k,k)$ and $\mathrm{res}^*_{B,C} : |B|_k \to |C|_k$ be the corresponding map on varieties. By convention we will use May's indexing [**May**] for Steenrod operations acting on Hopf algebra cohomology. The following definition first appeared in [**P**], with inspiration from work of Wilkerson.

Definition 2.3. Let Q be a graded connected cocommutative Hopf algebra over k; with $\mathrm{char}(k) = p$. Q is *quasi-elementary* if and only if no product

$$\begin{array}{ll} \prod w & \text{if } p = 2, \\ \prod u \prod \beta\widetilde{\mathcal{P}}^0(v) & \text{if } p \text{ odd,} \end{array}$$

is nilpotent. Here, w ranges over any subset of $\mathrm{Ext}^1_Q(k,k)) \setminus \{0\}$, u ranges over any subset of $\mathrm{Ext}^{1,\mathrm{odd}}_Q(k,k) \setminus \{0\}$, and v ranges over any subset of $\mathrm{Ext}^{1,\mathrm{even}}_Q(k,k)) \setminus \{0\}$.

For our purposes, the actual definition is not important; we will instead draw mainly on the following theorem. See also Proposition 5.2 for a classification of the quasi-elementary sub-Hopf algebras of the Steenrod algebra.

Theorem 2.4 ([**HS**], [**P**]). *Let B be a finite dimensional graded connected cocommutative Hopf algebra over a field k of characteristic $p > 0$.*

(a) *Suppose C is a sub-Hopf algebra of B. For every element $y \in \mathrm{Ext}^*_C(k,k)$, there is an integer n so that $y^{p^n} \in \mathrm{im}(\mathrm{res}_{B,C})$.*
(b) *Let M be a finitely generated B-module and fix $z \in \mathrm{Ext}^*_B(M,M)$. Then z is nilpotent if and only if $\mathrm{res}_{B,Q}(z)$ is nilpotent for every quasi-elementary sub-Hopf algebra Q of B.*
(c) *Let $\mathbf{C}_B$ be the category with objects the sub-Hopf algebras of quasi-elementary sub-Hopf algebras of B, and with morphisms given by inclusions. Then the natural map*

$$\mathrm{Ext}^*_B(k,k) \to \varprojlim_{Q\in\mathbf{C}_B} \mathrm{Ext}^*_Q(k,k)$$

is a finite map, where every element in the kernel is nilpotent, and for every y in the range, there is an n so that y^{p^n} is in the image. Equivalently, the natural map

$$\varinjlim_{Q\in\mathbf{C}_B} |Q|_k \to |B|_k$$

is an inseparable isogeny.

Given Q a quasi-elementary sub-Hopf algebra of B, define

$$|Q|_k^+ = |Q|_k \setminus \bigcup_{Q'<Q} \operatorname{res}^*_{Q,Q'} |Q'|_k,$$

and for any B-module M, define $|Q|_M^+ = |Q|_k^+ \cap |Q|_M$. The following result is an immediate corollary of Theorem 2.4.

Corollary 2.5. *The variety $|B|_k$ is a disjoint union of subvarieties $\operatorname{res}^*_{B,Q} |Q|_k^+$, one for each quasi-elementary $Q \leq B$. Furthermore, for each Q the map $\operatorname{res}^*_{B,Q} : |Q|_k \to |B|_k$ induces an inseparable isogeny*

$$|Q|_k^+ \to \operatorname{res}^*_{B,Q} |Q|_k^+.$$

3 Stratification theorem for modules

Let B be a finite-dimensional graded connected cocommutative Hopf algebra over an algebraically closed field k of characteristic $p > 0$ (hence, Assumption 2.1 holds). Also, let $\mathbf{C}_B$ denote the category of quasi-elementary sub-Hopf algebras of B and their sub-Hopf algebras. In this section we present an Avrunin-Scott theorem for B-modules, Theorem 3.4. Before proceeding further we need to make the following assumption. This assumption holds for group algebras (in which case quasi-elementary is the same as elementary abelian) and for sub-Hopf algebras of the Steenrod algebra (see Section 5).

Assumption 3.1. *If Q is a quasi-elementary sub-Hopf algebra of B and $Q' \leq Q$, then Q' is quasi-elementary. In this case, the induced map $\operatorname{res}^*_{Q,Q'} : |Q'|_k \to |Q|_k$ is injective.*

We will now proceed to prove the stratification theorem for modules. The method of proof will follow the presentation of the analogous group theoretic result found in [**Ben**, Chapter 5]). For the sake of clarity to the reader we have included details of some of the proofs along with the logical progression of the relevant theorems. The first result provides a decomposition of the variety $|B|_M$ into a union of closed subvarieties.

Proposition 3.2. *For any finitely generated B-module M, we have*

$$|B|_M = \bigcup_{Q \in \mathbf{C}_B} \operatorname{res}^*_{B,Q} |Q|_M.$$

Proof. This follows from Theorem 2.4(b), just as in [**Ben**, Thm. 5.7.4]. □

Lemma 3.3. (a) *Suppose* $C \triangleleft B$ *with* $B//C = k[x]/(x^2)$*. Let* $v \in \operatorname{Ext}^1_B(k,k)$ *be the image of a non-zero element in* $\operatorname{Ext}^1_{B//C}(k,k)$*. Fix* $z \in \operatorname{Ext}^*_C(k,k)$*; if* $\operatorname{res}_{B,C}(z) \in J_C(M)$*, then some power of* z *lies in the ideal generated by* $J_B(M)$ *and* v*.*

(b) *Let* p *be odd and* $C \triangleleft B$ *with* $B//C = k[x]/(x^p)$*. Let* $v \in \operatorname{Ext}^1_B(k,k)$ *be the image of a non-zero element in* $\operatorname{Ext}^1_{B//C}(k,k)$*. Fix* $z \in \operatorname{Ext}^*_C(k,k)$*; if* $\operatorname{res}_{B,C}(z) \in J_C(M)$*, then some power of* z *lies in the ideal generated by* $J_B(M)$ *and* $\beta\widetilde{\mathcal{P}}^0(v)$*.*

(c) *Let* $Q' \leq Q$ *be quasi-elementary sub-Hopf algebras of* B *and* M *be a* Q*-module. Then (identifying* $|Q'|_k$ *with its image under* $|Q'|_k \hookrightarrow |Q|_k$*) we have*

$$|Q'|_M = |Q'|_k \cap |Q|_M.$$

Proof. The first two assertions, (a) and (b), can be proved as in [**Ben**, Lem. 5.7.6] by applying [**P**, Lemma 3.2]. Part (c) follows by using parts (a)–(b) and [**HS**, Lemma A.11], just as in [**Ben**, Prop. 5.7.7]. □

The next result provides a stratification of the variety $|B|_M$ into a disjoint union of locally closed subvarieties.

Theorem 3.4. *Let* M *be a finitely generated* B*-module.*

(a) *The variety* $|B|_M$ *is a disjoint union of subvarieties*

$$|B|_M = \coprod_{Q \in \mathbf{C}_B} \operatorname{res}^*_{B,Q}(|Q|^+_M). \tag{3.1}$$

Furthermore, the map $\operatorname{res}^*_{B,Q}$ *induces an inseparable isogeny*

$$|Q|^+_M \to \operatorname{res}^*_{B,Q}(|Q|^+_M).$$

(b) *The natural map* $\varinjlim_{Q \in \mathbf{C}_B} |Q|_M \to |B|_M$ *is an inseparable isogeny.*

Proof. By Lemma 3.3(c), we see that

$$|Q|^+_M = |Q|_M \setminus \bigcup_{Q' < Q} \operatorname{res}^*_{Q,Q'} |Q'|_M,$$

so $|Q|_M$ is a disjoint union of the $|Q'|^+_M$'s, for $Q' \leq Q$. From Proposition 3.2, then, we have

$$|B|_M = \bigcup_{Q \in \mathbf{C}_B} \operatorname{res}^*_{B,Q} |Q|^+_M.$$

By Theorem 2.5, the varieties $\operatorname{res}^*_{B,Q} |Q|^+_M$ are disjoint and we have the desired inseparable isogeny. The second assertion follows immediately from part (a). □

Corollary 3.5. *Let* C *be a sub-Hopf algebra of* B *and let* M *be a finitely generated* B*-module. Then* $|C|_M = (\operatorname{res}^*_{B,C})^{-1}|B|_M$*.*

Proof. Let Q be a quasi-elementary sub-Hopf algebra of C. The map $\mathrm{res}^*_{B,C}$ maps $\mathrm{res}^*_{C,Q}(|Q|^+_M)$ into $\mathrm{res}^*_{B,Q}(|Q|^+_M)$, where both varieties are isomorphic to $|Q|^+_M$ by Theorem 3.4. Hence $(\mathrm{res}^*_{B,C})^{-1}(\mathrm{res}^*_{B,Q}\,|Q|^+_M) = \mathrm{res}^*_{C,Q}\,|Q|^+_M$. The result now follows by using (3.1). □

We can use Corollary 3.5 to prove that the support variety of a tensor product of modules is the intersection of the support varieties.

Theorem 3.6. *Let M_1 and M_2 be finitely generated B-modules. Then*

$$|B|_{M_1 \otimes M_2} = |B|_{M_1} \cap |B|_{M_2}.$$

Proof. First observe that the Künneth formula for Ext over a tensor product of Hopf algebras yields a natural isomorphism $|B \otimes B|_k \cong |B|_k \times |B|_k$. By restricting this isomorphism we have $|B \otimes B|_{M_1 \otimes M_2} \cong |B|_{M_1} \times |B|_{M_2}$. The algebra B injects into $B \otimes B$ via the coproduct map, and the B-module structure on $M_1 \otimes M_2$ is given by restricting the $B \otimes B$-module structure on $M_1 \otimes M_2$. Hence, if $f = \mathrm{res}^*_{B\otimes B,B}$ then

$$|B|_{M_1 \otimes M_2} = f^{-1}(|B \otimes B|_{M_1 \otimes M_2}) = f^{-1}(|B|_{M_1} \times |B|_{M_2}) = |B|_{M_1} \cap |B|_{M_2},$$

by Corollary 3.5. □

4 Rank varieties

In the previous section it was shown that for a B-module M the computation of $|B|_M$ reduces to computing $|Q|_M$ for all quasi-elementary sub-Hopf algebras Q of B. Our goal will be to provide concrete realizations of $|Q|_M$ in an affine space by using results on the cohomology of restricted Lie algebras. For L a restricted Lie algebra, let $V(L)$ be the restricted universal enveloping algebra of L; for any vector space Z, let $E(Z)$ be the exterior algebra on Z. In addition to Assumptions 2.1 and 3.1, we have the following.

Assumption 4.1. *For each $Q \in \mathbf{C}_B$ we have $Q \cong Q_0 \otimes E_0$ as Hopf algebras with the following properties.*

(a) *There exists a restricted Lie subalgebra of Q denoted by L_Q, such that $Q_0 \cong V(L_Q)$ as algebras.*
(b) *There exists a sub-vector space of Q denoted by Z_Q, such that $E_0 \cong E(Z_Q)$ as algebras.*

Moreover, the choice of (L_Q, Z_Q) is natural with respect to inclusions (i.e. if $Q' \leq Q$ then $L_{Q'} \leq L_Q$ and $Z_{Q'} \subseteq Z_Q$).

It should be mentioned that the restricted Lie algebras given in Assumption 4.1 may not consist of primitive elements in Q; thus, the ordinary coalgebra structure on $V(L_Q)$ given by $x \overset{\Delta}{\mapsto} x \otimes 1 + 1 \otimes x$ for $x \in L_Q$ need not coincide with the coalgebra structure on Q. The following results provide an explicit description of the variety $|Q|_M$.

Theorem 4.2. *Suppose Q is a quasi-elementary sub-Hopf algebra of B with L_Q and Z_Q as in Assumption 4.1. Then there exists a finite map $\Phi : |Q|_k \to L_Q \oplus Z_Q$ so that for any finitely generated Q-module M, we have the following:*

(a) $\Phi(|Q|_M) = \{x \in L_Q \oplus Z_Q : \ x^p = 0, \ M|_{\langle x \rangle} \text{ is not free}\}$;
(b) $c_Q(M) = \dim |Q|_M = \dim \Phi(|Q|_M)$.

Proof. Let $V(L_Q)$ denote the restricted enveloping algebra of L_Q with the usual coalgebra structure and let $L_Q^\sharp$ denote the dual of L_Q. For any $V(L_Q)$-module M, it was shown in [**FP1**, (1.4)] there exists a finite map $\sigma^\bullet : S^*(L_Q^\sharp) \to H^*(V(L_Q), k)$ which induces a map $\sigma : |L_Q|_k \to L_Q$ such that

$$\Phi(|L_Q|_M) = \{x \in L_Q : \ x^{[p]} = 0, \ M|_{\langle x \rangle} \text{ is not free}\}.$$

Here, $|L_Q|_M = \operatorname{Maxspec}(R_{V(L_Q)}/I_{V(L_Q)}(M))$. A simple calculation shows that there is an isomorphism $\delta^\bullet : S^*(Z_Q^\sharp) \to H^*(E(Z_Q), k)$, and one can show (for example, by using [**Mil**, 2.2 and 2.3] or by imitating the argument in [**Ben**, Thm. 5.8.3]) that the map on varieties $\delta : |E(Z_Q)| \to Z_Q$ has the property that

$$\delta(|E(Z_Q)|_M) = \{x \in Z_Q : \ x^2 = 0, \ M|_{\langle x \rangle} \text{ is not free}\}.$$

Since $Q \cong Q_0 \otimes E_0$ as Hopf algebras with $Q_0 = V(L_Q)$ and $E_0 = E(Z_Q)$ as algebras we can use $\sigma^\bullet$ and $\delta^\bullet$ to get a finite map

(4.1)
$$\Phi^\bullet : S^*((L_Q \oplus Z_Q)^\sharp) \to H^*(Q_0, k) \otimes H^*(E_0, k) \cong H^*(Q, k).$$

This induces the map $\Phi = (\sigma, \delta) : |Q|_k \to L_Q \oplus Z_Q$. The first assertion is now proven by using the fact that $|Q|_M \cong |Q_0|_M \times |E_0|_M$. The second assertion follows by Theorem 2.2 and the fact that Φ is finite. □

Proposition 4.3. *Suppose $Q' \leq Q$, with associated Lie algebras $L_{Q'} \leq L_Q$ and vector spaces $Z_{Q'} \subseteq Z_Q$. Let M and N be finitely generated Q-modules. Then*

(a) $\Phi(|Q'|_M) = \Phi(|Q|_M) \cap (L_{Q'} \oplus Z_{Q'})$.
(b) $\Phi(|Q|_{M \oplus N}) = \Phi(|Q|_M) \cup \Phi(|Q|_N)$.
(c) $\Phi(|Q|_{M \otimes N}) = \Phi(|Q|_M) \cap \Phi(|Q|_N)$, *if $\Phi : |Q|_k \to L_Q \oplus Z_Q$ is an isomorphism.*

Proof. The first part (a) follows immediately from Theorem 4.2. The second assertion (b) is true because $|Q|_{M\oplus N} \cong |Q|_M \cup |Q|_N$. Part (c) holds because of Theorem 3.6 and the fact that $|Q|_M$ is a subvariety of $|Q|_k$. □

We will show in Lemma 5.3 that for any quasi-elementary sub-Hopf algebra Q of the Steenrod algebra, $\Phi : |Q|_k \to L_Q \oplus Z_Q$ is an isomorphism. Now we show that any closed homogeneous subvariety of $|B|_k$ is realizable as $|B|_M$ for some finitely generated module M.

Theorem 4.4. *Let B be a finite-dimensional graded cocommutative Hopf algebra satisfying Assumptions 3.1 and 4.1, and suppose that for each $Q \in \mathbf{C}_B$, the map $\Phi : |Q|_k \to L_Q \oplus Z_Q$ is an isomorphism.*

(a) *Any closed homogeneous variety of $|B|_k$ is equal to $|B|_M$ for some finite-dimensional B-module M.*
(b) *If M is a finite-dimensional indecomposable B-module then the corresponding projective variety* $\operatorname{Proj}(|B|_M)$ *is connected.*

Proof. First we describe the support varieties for particular modules, the L_ζ's; from this, we will be able to prove part (a) of the theorem. For $\zeta \in H^n(B,k) = \operatorname{Hom}_B(\Omega^n(k),k)$, let L_ζ be the kernel of the map $\zeta : \Omega^n(k) \to k$. Here, $\Omega^1(k)$ is the kernel of the map $P_0 \to k$ and $\Omega^n(k)$ is the kernel of the map $P_{n-1} \to P_{n-2}$ for $n \geq 2$ in (2.1) for $N = k$. Also, let $\langle\zeta\rangle \subseteq |B|_k$ be the set of all maximal ideals in $H^*(B,k)$ containing ζ. We claim that $|B|_{L_\zeta} = \langle\zeta\rangle$.

In order to prove this, it suffices to assume that $B = Q$ is a quasi-elementary Hopf algebra—see the remarks preceding [**Ben**, Prop. 5.9.1] and Theorem 3.4. We have $\Phi^\bullet : S^*((L_Q \oplus Z_Q)^\sharp) \to H^*(Q,k)$ (4.1). Let ζ be such that there is an $F \in S^*((L_Q \oplus Z_Q)^\sharp)$ with $\Phi^\bullet(F) = \zeta$. By extending the results in [**FP2**, (4.2)] it follows that $\Phi(|Q|_{L_\zeta})$ is $Z(F)$ where $Z(F)$ is the zero set of F. Now observe that $\Phi(\langle\zeta\rangle) \subseteq Z(F)$ by definition of Φ. If $m \in |Q|_{L_\zeta}$, then $F \in (\Phi^\bullet)^{-1}(m)$ because $\Phi(|Q|_{L_\zeta}) = Z(F)$; therefore, $\zeta \in m$ and $|Q|_{L_\zeta} \subseteq Z(F)$. Applying Φ it follows that $Z(F) = \Phi(|Q|_{L_\zeta}) \subseteq \Phi(\langle\zeta\rangle)$, thus $\Phi(\langle\zeta\rangle) = Z(F)$ and $\Phi(|Q|_{L_\zeta}) = \Phi(\langle\zeta\rangle)$. Since Φ is an isomorphism, we have $|Q|_{L_\zeta} = \langle\zeta\rangle$.

Now assume W is a closed homogeneous subvariety of $|B|_k$ and let $I(W) = (\zeta_1, \zeta_2, \ldots \zeta_s) \subseteq H^*(B,k)$ be the corresponding ideal ($\Phi^\bullet$ is finite so we may choose ζ_j such that there exists F_j which maps to ζ_j under $\Phi^\bullet$). Let $M = L_{\zeta_1} \otimes L_{\zeta_2} \otimes \ldots \otimes L_{\zeta_s}$. Then by Theorem 3.6

$$|B|_M = \cap_{i=1}^s |B|_{L_\zeta} = \cap_{i=1}^s \langle\zeta_i\rangle = W.$$

The second assertion (b) follows by the same argument given in [**Ben**, Thm. 5.12.1] with (a) and Theorem 3.6. □

5 Applications to the Steenrod algebra

In this section we apply the results in the previous three sections to finite sub-Hopf algebras of the mod p Steenrod algebra (for all p) and the mod p reduced powers (for p odd). Recall that we defined Hopf algebras P and A in Section 1.

The sub-Hopf algebras of P and A have been classified; for example, any sub-Hopf algebra B of A has the form

$$B = \Big(k[\xi_1, \xi_2, \xi_3, \dots] \otimes E[\tau_0, \tau_1, \tau_2, \dots]/(\xi_1^{p^{n_1}}, \xi_2^{p^{n_2}}, \dots; \tau_0^{e_0}, \tau_1^{e_1}, \dots)\Big)^*$$

for exponents $n_i \in \{0, 1, 2, \dots\} \cup \{\infty\}$ and $e_i \in \{0, 1\}$ satisfying certain conditions (see [**AD**] and [**AM2**]). The sequence of exponents $(n_1, n_2, \dots; e_0, e_1, \dots)$ is called the *profile function* for B and we write $B = A(n_1, n_2, \dots; e_0, e_1, \dots)$. The analogous results hold for sub-Hopf algebras of P, and we use similar notation.

We define sub-Hopf algebras $E(m), Q(m) \leq P$ and $R(m) \leq A$:

$$E(m) = P(\underbrace{0, 0, \dots, 0, 0}_{m}, m+1, m+1, m+1, \dots),\ m \geq 0,$$

$$Q(m) = P(\underbrace{0, 0, \dots, 0}_{m-2}, 1, m, m, m, \dots),\ m \geq 2,$$

$$R(0) = A(0, 0, 0, \dots; 0, 0, 0, \dots),$$

$$R(1) = A(1, 1, 1, \dots; 1, 0, 0, \dots),$$

$$R(m) = A(\underbrace{0, 0, \dots, 0}_{m-2}, 1, m, m, m, \dots; \underbrace{1, 1, \dots, 1}_{m}, 0, 0, 0, \dots),\ m \geq 2.$$

We will show that a sub-Hopf algebra Q of P is quasi-elementary if and only if $Q \leq E(m)$ for some m when $p = 2$, or $Q \leq Q(m)$ for some m when p is odd. Similarly, $Q \leq A$ is quasi-elementary if and only if $Q \leq R(m)$ for some $m \geq 0$. First we compute the cohomology of these Hopf algebras.

Lemma 5.1. (a) *For $p = 2$, any $Q \leq E(m)$ is an exterior algebra on the P_t^s's that it contains; e.g.,*

$$E(m) = E(P_t^s : t \geq m+1, s \leq m).$$

Hence $\mathrm{Ext}_Q^*(k, k)$ is a polynomial algebra, with one generator h_{ts} for each $P_t^s \in Q$.

(b) *For p odd, any $Q \leq Q(m)$ is isomorphic as an algebra to the restricted universal enveloping algebra on the P_t^s's that it contains. Furthermore, $\mathrm{Ext}_Q^*(k, k)$ is of the form $P \otimes T$, where P is a polynomial algebra with one generator b_{ts} for each $P_t^s \in Q$,* *and T consists of nilpotent elements (T is a subquotient of the exterior algebra $E(h_{ts} : P^s_t \in Q)$).*

(c) *For p odd, any $R \leq R(m)$ is isomorphic as a Hopf algebra to the tensor product of an exterior algebra on the Q_n's in R with the algebra generated by the P^s_t's in R. The latter, considered as a sub-Hopf algebra of P, is a sub-Hopf algebra of $Q(m)$. Hence $\operatorname{Ext}^*_R(k,k)$ is of the form $P' \otimes P \otimes T$, where P' is a polynomial algebra with one generator v_n for each $Q_n \in R$, and P and T are as in part (b).*

Proof. For part (a), it is well-known that $E(m)$ (and hence any sub-Hopf algebra $Q \leq E(m)$) is exterior; see [**Mar**], for example. The cohomology computation is standard. Part (c) follows from part (b) and an easy computation with Milnor basis elements.

We prove part (b) for $Q = Q(m)$; the reader can easily do the more general case. Note that the collection of P^s_t's under consideration is

$$S = \{P^0_{m-1}\} \cup \{P^u_v : v \geq m, m-1 \geq u \geq 0\}.$$

These all satisfy $(P^s_t)^p = 0$; also, the only non-zero commutator relations between them are the following: for all $v \geq m$, we have

$$[P^0_{m-1}, P^{m-1}_v] = P^0_{m+v-1}.$$

Let L be the vector space spanned by S. Then L is a graded restricted Lie algebra contained in $Q(m)$, and L generates $Q(m)$ as an algebra. Therefore, there is an induced epimorphism of algebras

$$V(L) \twoheadrightarrow Q(m).$$

By comparing dimensions, we see that this map must be an isomorphism.

To compute the cohomology of $Q(m)$, we let $D = P(\underbrace{0,\dots,0}_{m-1}, m-1, m-1, m-1, \dots)$, and we use the Cartan-Eilenberg spectral sequence associated to the central extension

$$D \to Q(m) \to Q(m)//D.$$

Both D and $Q(m)//D$ are polynomial algebras truncated at height p, so we can compute their cohomology to the get the E_2-term:

$$E_2 \cong k[b_{ts} : P^s_t \in Q(m)] \otimes E(h_{ts} : P^s_t \in Q(m)).$$

Just as in the computation in [**W**, 6.3], d_2 is determined by the family of differentials (for all $t \geq m$)

$$d_2(h_{t0}) = h_{m-1,0} h_{t-m+1,m-1},$$

and the spectral sequence collapses at E_3. □

We can now show that the sub-Hopf algebras of $E(m)$, $Q(m)$, and $R(m)$ are precisely the quasi-elementary sub-Hopf algebras of P for $p = 2$, P for p odd, and of A, respectively.

Proposition 5.2. (a) *For $p = 2$, a sub-Hopf algebra Q of P is quasi-elementary if and only if $Q \leq E(m)$ for some m.*
(b) *For p odd, a sub-Hopf algebra Q of P is quasi-elementary if and only if $Q \leq Q(m)$ for some m.*
(c) *For p odd, a sub-Hopf algebra Q of A is quasi-elementary if and only if $Q \leq R(m)$ for some m.*

Proof. Part (a) follows from [**W**, 6.4], so we assume that p is odd. For part (b), if $Q \leq Q(m)$, then by the computation in Lemma 5.1, $\mathrm{Ext}^1_Q(k,k)$ is spanned as a vector space by a subset of $\{h_{ts} : P^s_t \in Q\}$. For each $h_{ts} \in \mathrm{Ext}^*_Q(k,k)$, we have $\beta\widetilde{\mathcal{P}}^0(h_{ts}) = b_{ts}$, and no product of b_{ts}'s is nilpotent. Hence Q is quasi-elementary. Conversely, one can apply the arguments in [**W**, 6.4] to show that if Q is not a proper sub-Hopf algebra of any $Q(m)$, then Q is not quasi-elementary.

For part (c), if $Q \leq R(m)$, then Q is quasi-elementary by the previous part and Lemma 5.1. Conversely, assume that Q is quasi-elementary. Suppose n is the smallest integer so that $Q_n \in Q$. Then $P^s_t \in Q$ only if $s < n$—otherwise, if t is the smallest integer so that $P^n_t \in Q$, then in Q we will have $Q_{n+t} = [Q_n, P^n_t]$. In $\mathrm{Ext}^*_Q(k,k)$, then, $h_{tn} = [\xi_t^{p^n}]$ and $v_n = [\tau_n]$ are permanent cycles, and we have the relation $h_{tn}v_n = 0$. Applying $\beta\mathcal{P}^{p^{n+t}}$ (*not* $\beta\widetilde{\mathcal{P}}^{p^{n+t}}$) gives $b_{tn}v_n^p = 0$, so Q could not be quasi-elementary.

Hence as Hopf algebras, Q splits as a tensor product as in the statement of Lemma 5.1(c): $Q = E \otimes Q$, with E exterior on a collection of Q_n's, and $Q' \leq P$. Now, $Q \leq A$ is quasi-elementary if and only if $Q' \leq P$ is, so apply part (b). □

Recall that Assumption 2.1 holds for all finite-dimensional graded connected cocommutative Hopf algebras. For the finite-dimensional quasi-elementary sub-Hopf algebras of P and A, Assumption 3.1 follows from Proposition 5.2 and our cohomology calculations. Furthermore, Assumption 4.1 is the content of Lemma 5.1. We record the following lemma, for later use.

Lemma 5.3. *If Q is a finite-dimensional quasi-elementary sub-Hopf algebra of P or of A, then the map Φ of Theorem 4.2 gives an isomorphism between $|Q|_k$ and L_Q, for $Q \leq P$, or between $|Q|_k$ and $L_Q \oplus Z_Q$, for $Q \leq A$.*

Proof. By our cohomology calculations, the variety $|Q|_k$ is affine of dimension $\#D(Q)$. According to Theorem 4.2, $\dim |Q|_k = \dim \Phi(|Q|_k) = \dim_k L_Q$; hence, $\Phi(|Q|_k) \cong L_Q$ and the result follows. □

We now present the proofs of the non-trivial parts of the results in Section 1.

Proof of Theorem 1.1. Part (a) is a combination of Assumption 3.1 and Corollary 3.5. Part (b) follows from Theorem 4.2 and Lemma 5.3. □

Proof of Theorem 1.2(b). This follows by Proposition 4.3 since Φ is an isomorphism. □

Proof of Theorem 1.4. The irreducible components of the variety $|B|_k$ are the varieties $|Q|_k$, one for each $Q \in \hat{\mathbf{C}}_B$, so the depth of $\mathrm{Ext}_B^*(k,k)$ is bounded above by $\min_{Q\in\hat{\mathbf{C}}_B} \dim |Q|_k$. □

Proof of Corollary 1.5. This follows from Theorem 1.1(c) and Theorem 1.2(b). □

Proof of Corollary 1.6. Let $T = G_m$ be the one-dimensional algebraic group over k. If $t = (a)$, $a \in k - \{0\}$, then T acts as a group of automorphisms on B by letting $t.b = a^{\deg\ b} b$. In this setting the category of graded B-modules is equivalent to the category of B-modules which admit an additional T-structure whose module map $B \otimes M \to M$ is T-equivariant. Furthermore, $\Phi(|Q|_M)$ is a T-stable variety for any graded B-module (see [**FP1**], [**FP2**]).

For part (a) if M is a free B-module, then $\Phi(|Q|_M) = 0$ for all $Q \in \mathbf{C}_B$; hence by Theorem 1.1(c), $H(M,x) = 0$ for all $x \in D(B)$. Conversely, suppose M is not free as a B-module; then $\Phi(|Q|_M) \neq \{0\}$ for some $Q \in \mathbf{C}_B$. Since M is graded the variety $\Phi(|Q|_M)$ is T-stable. Since the differentials in B have distinct degrees, it follows that there exists a differential $x \in \Phi(|Q|_M)$ (see [**FP1**, Prop. 3.4]), and thus $H(M,x) \neq 0$. Part (b) can be proved by using the same argument given in [**N**, Thm. 2.2] (for G_1T-modules). □

Proof of Theorem 1.7. We know that for each quasi-elementary $Q \leq A$ there is a bijection

$$\Phi_Q : |Q|_M \to \{x \in V : x \in Q, x^2 = 0, H(M,x) \neq 0\}.$$

Furthermore, these maps are natural with respect to inclusions $Q' \leq Q$ of quasi-elementary Hopf algebras. Hence we want to show that $x \in V$ satisfies $x^2 = 0$ if and only if $x \in V \cap Q$ for some quasi-elementary $Q \leq A$.

Let $x = \sum_{P_t^s \in A, s<t} h_{t,s} P_t^s$, and suppose $x^2 = 0$. Since $(P_t^s)^2 = 0$ if $s < t$, we have

$$x^2 = \sum_{P_t^s, P_v^u} h_{t,s} h_{v,u} [P_t^s, P_v^u].$$

Certainly if each commutator is zero (i.e., if all of the P_t^s's with nonzero coefficients are in the same quasi-elementary $Q \leq A$), then $x^2 = 0$. We want to prove the converse.

Assume x cannot be written as $\sum_{P_t^s \in Q} h_{t,s} P_t^s$ for any quasi-elementary $Q \leq A$; we want to show that $x^2 \neq 0$. Look at the collection of nonzero commutators of P_t^s's with nonzero coefficients in x, and consider the smallest dimension of such. If there is only one commutator there, then x^2 can't be zero. We claim that there can't be more than one commutator of the same (minimal) degree. Suppose $[P_{t_1}^{s_1}, P_{v_1}^{u_1}]$ and $[P_{t_2}^{s_2}, P_{v_2}^{u_2}]$ are nonzero and in the same minimal degree, with $s_i < t_i$, $u_i < v_i$ for $i = 1, 2$.

We need the following two facts:

- For $p = 2$, the degree of P_t^s is $2^{s+t} - 2^s$.
- Consider $x = P_t^s$ and $y = P_v^u$ with $s < t$ and $u < v$, and assume that $t \leq v$. Then x and y commute if and only if $u < t$.

Now we compute: since the commutators are nonzero, we have $s_i < t_i \leq u_i < v_i$ for each i. Since the degrees are equal, we have

$$2^{u_1+v_1} + 2^{s_1+t_1} + 2^{s_2} + 2^{u_2} = 2^{u_2+v_2} + 2^{s_2+t_2} + 2^{s_1} + 2^{u_1}.$$

From this, we see that $s_1 = s_2$ and $u_1 + v_1 = u_2 + v_2$, and hence either $t_1 = t_2$, $u_1 = u_2$, and $v_1 = v_2$, or (without loss of generality) $u_1 = s_1 + t_1 < u_2 = s_2 + t_2$. But then $v_2 > t_1$ and $u_2 \geq t_1$, so $[P_{t_1}^{s_1}, P_{v_2}^{u_2}] \neq 0$, and is in a smaller degree. □

We conclude with some examples. Let P_n be the sub-Hopf algebra of P with profile function $(n+1, n, n-1, \dots, 2, 1, 0, 0, 0, \dots)$. We say that a map $\rho : H_1 \to H_2$ of commutative k-algebras is an *F-isomorphism* if H_2 is finitely generated as a module over H_1, the kernel of ρ consists of nilpotent elements, and for every $y \in H_2$, for some n we have $y^{p^n} \in \operatorname{im} \rho$.

Example 5.4. (a) For all p, $P_0 = k[P_1^0]/((P_1^0)^p)$, so $|P_0|_k$ is affine 1-space, spanned by P_1^0.

(b) For all p, $|P_1|_k$ is affine 2-space, spanned by P_1^0 and P_2^0. This means that $H^*(P_1, k)$ is F-isomorphic to a rank 2 polynomial algebra. For instance, when $p = 2$, $H^*(P_1, k)$ has a subalgebra $k[h_{10}, h_{20}^4]$ over which $H^*(P_1, k)$ is finitely generated.

(c) When $p = 2$, then $|P_2|_k$ is

$$\operatorname{Span}\{P_1^0, P_2^0, P_3^0\} \cup \operatorname{Span}\{P_2^0, P_3^0, P_2^1\} \subseteq \operatorname{Span}\{P_1^0, P_2^0, P_3^0, P_2^1\}.$$

I.e., $H^*(P_2, k)$ is F-isomorphic to $k[h_{10}, h_{20}, h_{30}, h_{21}]/(h_{10}h_{21})$.

(d) When p is odd, then $|P_2|_k$ is the above affine 4-space, so $H^*(P_2, k)$ is F-isomorphic to $k[b_{10}, b_{20}, b_{30}, b_{21}]$.

(e) Suppose p is odd, and let $A_1 = A(1, 0, 0, \ldots; 0, 0, 1, 1, 1, \ldots)$. Then

$$|A_1|_k = \mathrm{Span}\{Q_0, Q_1\} \cup \mathrm{Span}\{P_1^0, Q_1\},$$

so $H^*(P_1, k)$ is F-isomorphic to $k[v_0, v_1, b_{10}]/(v_0 b_{10})$.

References

[**AM1**] J. F. Adams and H. R. Margolis, Modules over the Steenrod algebra, *Topology* **10** (1971), 271–282.

[**AM2**] J. F. Adams and H. R. Margolis, Sub-Hopf algebras of the Steenrod algebra, *Math. Proc. Cambridge Philos. Soc.* **76** (1974), 45–52.

[**Alp**] J. L. Alperin, Periodicity in groups, *Illinois J. Math.* **21** (1977), 776–783.

[**AD**] D. W. Anderson and D. M. Davis, A vanishing theorem in homological algebra, *Comment. Math. Helv.* **48** (1973), 318–327.

[**AS**] G. S. Avrunin and L. L. Scott, Quillen stratification for modules, *Invent. Math.* **66** (1982), 277–286.

[**BS**] A. Bajer and H. Sadofsky, Cohomology of finite-dimensional connected cocommutative Hopf algebras, *J. Pure and Applied Algebra* **94 (2)** (1994), 115–126.

[**Ben**] D. J. Benson, *Representations and cohomology, Volume II*, Cambridge University Press, 1991.

[**C1**] J. F. Carlson, The varieties and cohomology ring of a module, *J. Algebra* **85** (1983), 104–143.

[**C2**] J. F. Carlson, *Module varieties and cohomology rings of finite groups*, Vorlesungen aus dem Fachberick der Universität Essen, 1985.

[**E**] L. Evens, The cohomology ring of a finite group, *Trans. Amer. Math. Soc.* **101** (1961), 224–239.

[**FP1**] E. M. Friedlander and B. J. Parshall, Support varieties for restricted Lie algebras, *Invent. Math.* **86** (1986), 553–562.

[**FP2**] E. M. Friedlander and B. J. Parshall, Geometry of p-unipotent Lie algebras, *J. Algebra* **109** (1987), 25–45.

[**HS**] M. J. Hopkins and J. H. Smith, Nilpotence and stable homotopy theory II, preprint.

[**LS**] R. G. Larson and M. E. Sweedler, An associative orthogonal bilinear form for Hopf algebras, *Amer. J. Math.* **91** (1969), 75–94.

[**Mar**] H. R. Margolis, *Spectra and the Steenrod algebra*, North-Holland, 1983.

[**May**] J. P. May, A general algebraic approach to Steenrod operations, *The Steenrod Algebra and its Applications* (F. P. Peterson, ed.), 1970, Springer Lecture Notes in Mathematics 168, pp. 153–231.

[**Mil**] H. R. Miller, A localization theorem in homological algebra, *Math. Proc. Cambridge Philos. Soc.* **84** (1978), 73–84.

[**MP**] J. C. Moore and F. P. Peterson, Modules over the Steenrod algebra, *Topology*, **11** (1972), 387–395.

[**N**] D. K. Nakano, A bound on the complexity for G_1T modules, to appear, *Proc. Amer. Math. Soc.*.

[**P**] J. H. Palmieri, A note on the cohomology of finite dimensional cocommutative Hopf algebras, in preparation.

[**Q1**] D. G. Quillen, A cohomological criterion for p-nilpotence, *J. Pure and Applied Algebra* **1** (1971), 361–372.

[**Q2**] D. G. Quillen, The spectrum of an equivariant cohomology ring I and II, *Ann. of Math.* **94** (1971), 549–602.

[**W**] C. Wilkerson, The cohomology algebras of finite dimensional Hopf algebras, *Trans. Amer. Math. Soc.* **264** (1981), 137–150.

Department of Mathematics, Northwestern University, Evanston, IL. 60201 U.S.A.

E-mail address: nakano@math.nwu.edu

Department of Mathematics, M.I.T., Cambridge, MA 02139 U.S.A

E-mail address: palmieri@math.mit.edu