Rings of Generalized and Stable Invariants of Pseudoreflections and Pseudoreflection Groups

Frank Neumann, Mara D. Neusel and Larry Smith

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mathematisches institut der universität göttingen

bunsenstrasse 3 - 5

d 37073 göttingen

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SUMMARY : *Let* $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ *be a representation of a finite group* G *over the field* $\mathbb{F}$ *and* $\mathbb{F}[V]$ *the space of polynomial functions on* $V = \mathbb{F}^n$. *We associate to* G *an ideal* $J_\infty(G) \subset \mathbb{F}[V]$ *called the ideal of stable invariants of* G *(or more precisely* ϱ*). If* $\mathcal{S} \subset \mathrm{GL}(n, \mathbb{F})$ *is a set of pseudoreflections we associate to* $\mathcal{S}$ *an ideal* $I(\mathcal{S}) \subset \mathbb{F}[V]$ *called the ideal of generalized invariants of* $\mathcal{S}$. *When* G *is a pseudoreflection group we investigate* $I(\mathcal{S})$ *for various choices of* $\mathcal{S} \subset G$ *and the relation between* $J_\infty(G)$ *and* $I(\mathcal{S})$. *To a representation* $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ *of a finite group, respectively to a set* $\mathcal{S} \subset \mathrm{GL}(n, \mathbb{F})$ *of pseudoreflections, we also associate a ring* $\overline{\mathrm{gr}}_{J_\infty(G)}$, *respectively* $\overline{\mathrm{gr}}_{I(\mathcal{S})}$. *We show that* $\overline{\mathrm{gr}}_{I(\mathcal{S})}$ *is always a polynomial algebra over* $\mathbb{F}$, *and whenever* $\varrho(G)$ *is generated by semisimple pseudoreflections* $\mathcal{S}$ *that* $\overline{\mathrm{gr}}_{J_\infty} \cong \overline{\mathrm{gr}}_{I(\mathcal{S})}$.

Let $\mathbb{F}$ be a field and $V = \mathbb{F}^n$ an n-dimensional vector space over $\mathbb{F}$. Denote by $\mathbb{F}[V]$ the algebra of polynomial functions on V. We regard $\mathbb{F}[V]$ as a graded algebra over $\mathbb{F}$ with homogeneous component of degree d, $\mathbb{F}[V]_d$, the homogeneous polynomials of degree d. If G is a finite group and $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ is a faithful representation we denote by $\mathbb{F}[V]^G$ the ring of invariants of G. (As a general reference for invariant theory and many of the notations we use we refer to [11].)

A linear automorphism $s : V \longrightarrow V$ is called a **pseudoreflection** if

(i) $s \neq 1$,

(ii) s has finite order,

(iii) s leaves a codimension one subspace, called the **hyperplane of** s, fixed.

If $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ is a faithful representation and $\varrho(G) \subseteq \mathrm{GL}(n, \mathbb{F})$ is generated by pseudoreflections then we refer[1] to G as a **pseudoreflection group**. If $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ is a finite pseudoreflection group and $|G|$ is relatively prime to the characteristic of $\mathbb{F}$, then by [10], [3], [12] $\mathbb{F}[V]^G = \mathbb{F}[f_1, \dots, f_n]$ is a polynomial algebra. The polynomials $f_1, \dots, f_n \in \mathbb{F}[V]$ form a regular sequence. If the characteristic of $\mathbb{F}$ divides $|G|$ (the order of G) this may no longer be true. For example, the Weyl group $W(\mathbf{F}_4)$ of the root system of type $\mathbf{F}_4$ has a faithful representation over the Galois field of p elements as a pseudoreflection group. For $p = 3$ the ring of invariants is not a polynomial algebra (see e.g. [11] § 7.4 and § 10.3 example 1, and [17]).

The purpose of this note is to examine finite pseudoreflection groups in a modular setting, i.e., where the order of the group is divisible by the characteristic of the field. We introduce the **ring of generalized invariants** associated to a set of pseudoreflections $\mathcal{S} \subset \mathrm{GL}(n, \mathbb{F})$ and the **ring of stable invariants** associated to a pseudoreflection group and develop their properties. In further notes we compute these rings for Coxeter and Shephard groups.

We next review our results.

If s is a pseudoreflection and $\ell_s \in \mathbb{F}[V]$ a linear form with

$$\ker(\ell_s) = V^s = \{v \in V \mid s(v) = v\},$$

then for each $f \in \mathbb{F}[V]$

$$sf - f = \Delta_s(f)\ell_s$$

in $\mathbb{F}[V]$ for a unique $\Delta_s(f) \in \mathbb{F}[V]$ (see e.g. [3], [4] or [11] § 7.1). Note that

$$\Delta_s(f) = 0 \iff sf = f.$$

[1] This is an abuse of terminology. Actually it is ϱ that is a **pseudoreflection representation**. See [11] § 5.6 example 1 for a group with two quite different pseudoreflection representations.

If $\mathcal{S} \subset \mathrm{GL}(V)$ is a collection of pseudoreflections then the **ideal of generalized invariants** $I(\mathcal{S})$ is defined by [4] [6]

$$I(\mathcal{S}) = \left\{ f \in \mathbb{F}[V] \mid \deg(f) > 0 \text{ and } \Delta_{s_1} \cdots \Delta_{s_{\deg(f)}}(f) = 0 \; \forall \, s_1, \ldots, s_{\deg(f)} \in \mathcal{S} \right\},$$

where $\Delta_{s_1} \cdots \Delta_{s_{\deg(f)}}$ denotes the composition of the operators $\Delta_{s_1}, \ldots, \Delta_{s_{\deg(f)}}$. (N.b. If $s \in \mathrm{GL}(V)$ is a pseudoreflection then ℓ_s, and hence also Δ_s are well defined up to a nonzero scalar. Hence $I(\mathcal{S})$ depends only on $\mathcal{S}$ and not the choice of ℓ_s for $s \in \mathcal{S}$.) Following Kac and Peterson [6] we show that $I(\mathcal{S})$ is generated by a regular sequence $h_1, \ldots, h_n \in \mathbb{F}[V]$.

If A is a graded connected commutative algebra over a field $\mathbb{F}$ and $I \subset A$ a proper ideal, then we may associate to the filtration of A

$$A \supset I \supseteq I^2 \supseteq \cdots \supseteq I^m \supseteq \cdots$$

by the powers of the ideal I, (we use the convention that $I^0 = A$) the associated graded algebra, which is a bigraded algebra defined by

$$\mathrm{gr}_I(A)_{m,\, n-m} = \left(I^m / I^{m+1} \right)_n$$

for $n, m \in \mathbb{N}$. By totalizing we obtain a positively graded commutative algebra over $\mathbb{F}$ containing A/I is a subalgebra, and $\overline{\mathrm{gr}}_I(A) = \mathbb{F} \otimes_{A/I} \mathrm{gr}_I(A)$ is a graded *connected* commutative algebra over $\mathbb{F}$.

Starting with the ideal of stable invariants $I(\mathcal{S})$ of $\mathcal{S} \subset \mathrm{GL}(n, \mathbb{F})$ we obtain the **ring of generalized invariants** $\overline{\mathrm{gr}}_{I(\mathcal{S})}(\mathbb{F}[V])$. The ring of generalized invariants is shown to be a polynomial algebra.

If $\mathcal{S} \subset \mathrm{GL}(n, \mathbb{F})$ is a collection of pseudoreflections the subgroup of $\mathrm{GL}(n, \mathbb{F})$ generated by $\mathcal{S}$ is denoted by $G(\mathcal{S})$. The ideal $I(\mathcal{S})$ of generalized invariants of $\mathcal{S}$ contains the $G(\mathcal{S})$-invariant polynomials of positive degree, and hence the ideal $\left(\overline{\mathbb{F}[V]^G}\right)$ that they generate. If $G(\mathcal{S})$ is a finite group of order prime to the characteristic of the field $\mathbb{F}$, then $\mathbb{F}[V]^{G(\mathcal{S})} = \mathbb{F}[f_1, \ldots, f_n]$ and $I(\mathcal{S}) = (f_1, \ldots, f_n)$. Thus our results may be seen as generalizations of the theorem of Shephard-Todd ([10] theorem 5.1, [11] theorem 7.4.1) to a modular setting.

Actually one would like to associate a ring of generalized invariants to an arbitrary pseudoreflection group rather then to a collection of pseudoreflections. We show by examples that the ideal $I(\mathcal{S})$ does not fulfill this goal. If however the pseudoreflection group G is generated by pseudoreflections of order relatively prime to the characteristic of $\mathbb{F}$ this may be achieved in the following way.

Let A be a graded connected algebra over a field $\mathbb{F}$. Denote by $\overline{A} \subset A$ the augmentation ideal of A, i.e. the ideal of A generated by the elements of positive degree. If G is a group and $\alpha : G \hookrightarrow \mathrm{Aut}(A)$ a faithful representation of G as a group of algebra automorphisms of A, we denote by $A^G = \{ a \in A \mid ga = a \; \forall \, g \in G \}$ the subalgebra of A left invariant by G. The algebra A^G is also graded and connected. We define the **algebra of coinvariants**

of G by $A_G := \mathbb{F} \otimes_{A^G} A$, where A is regarded as an A^G module via the inclusion and $\mathbb{F}$ via the augmentation. The algebra of coinvariants may also be viewed as A/I where I is the ideal generated by $\{a \in A^G \mid \deg(a) > 0\}$. The ideal I is stable under the action of G on A so there is induced an action of G on A_G. It is therefore possible to repeat the above constructions forming $(A_G)^G$, $A_{GG} := (A_G)_G$ etc. By iteration we obtain a sequence of algebra epimorphisms

$$A \longrightarrow A_G \longrightarrow A_{GG} \longrightarrow \ \cdots \ \longrightarrow A_{G\cdots G} \longrightarrow \cdots .$$

Denote by J_m the kernel of the map $A \longrightarrow A_{G \underset{\leftarrow m \rightarrow}{\cdots} G}$. These ideals form an ascending chain

$$(0) = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_m \subseteq \cdots \subseteq A$$

in A. The ideal $J_\infty := \bigcup J_m \subset A$ is called the **ideal of stable invariants**. It is characterized by the following conditions:

(i) $J_\infty \subset A$ is stable under the G-action,

(ii) $(A/J_\infty)^G \cong \mathbb{F}$ (i.e. G acts fixedpoint freely on $(\overline{A/J_\infty})$,

(iii) if $I \subset A$ is an ideal such that $(A/I)^G \cong \mathbb{F}$ then $J_\infty \subseteq I$. The ring $\overline{\mathrm{gr}}_{J_\infty(G)}$ is **the ring of stable invariants of** G.

If $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ is a representation of a finite group G then G acts on $\mathbb{F}[V]$ and there is the ideal of stable invariants $J_\infty(G) \subset \mathbb{F}[V]$. If G is a pseudoreflection group and $|G|$ is relatively prime to the characteristic of $\mathbb{F}$ then Chevalley [3] has shown that $\mathbb{F}[V]_G$ is the regular representation of G. In particular $(\mathbb{F}[V]_G)^G \cong \mathbb{F}$ and hence $J_\infty(G) = J_1(G) = (\overline{\mathbb{F}[V]^G})$ is the ideal of $\mathbb{F}[V]$ generated by the G-invariants of positive degree. If however $|G| = 0 \in \mathbb{F}$ then $J_\infty(G)$ can be distinctly larger that $J_1(G)$. If $\mathcal{S} \subset \mathrm{GL}(n, \mathbb{F})$ is a set of pseudoreflections we examine the relationship between $I(\mathcal{S})$, $J_\infty(G(\mathcal{S}))$ and $\mathbb{F}[V]^{G(\mathcal{S})}$ and clarify some remarks of Kac and Peterson in [6] § 3.

We have included a rather large number of examples to illustrate the theory and to prevent false interpertations, and as such, they are an integral part of our work.

The results of this note have applications to p-compact groups [5], which are a homotopical generalization of a compact Lie group. These investigations grew out of a study of torsion phenomena in the cohomology of compact Lie groups and their classifying spaces, with an eye towards their generalization to p-compact groups, in the Topologie-Oberseminar in Göttingen in the winter semester of 1993/94. We wish to thank the other members of the seminar for their active participation. One of us, M.D.N., would like to thank the Schweizer National Fond for financial support during the preparation of this manuscript.

§ 1. The Ideal of Stable Invariants

Let A be a positively graded connected algebra over a field $\mathbb{F}$ and G a group of algebra automorphisms of A. Denote by $A^G \subseteq A$ the subalgebra of G-invariant elements and set $A_G := \mathbb{F} \otimes_{A^G} A$, where A is regarded as an A^G-module via the inclusion and $\mathbb{F}$ via the augmentation. A_G is called **the algebra of coinvariants**. The kernel of the quotient map $A \longrightarrow A_G$ is stable under the action of the group G on A and hence A_G inherits a G-action from A. We may therefore iterate the construction of the coinvariants and inductively define A_{G^m} for $m \in \mathbb{N}$ by $A_{G^m} = (A_{G^{m-1}})_G$. (Of course A_{G^0} is just A and A_{G^1} is A_G.) Denote by $J_m \subset A$ the kernel of the natural map $A \longrightarrow A_{G^m}$. The ideals $J_m,\ m \in \mathbb{N}$ form an ascending chain

$$(0) = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_m \subseteq \cdots \subset A$$

and we set $J_\infty = \bigcup_{m \in \mathbb{N}} J_m$, which is called the **ideal of stable invariants**. Alternatively, the ideals $J_m,\ m \in \mathbb{N}$ may be defined inductively[2] by

$$J_m = \begin{cases} (0) & \text{for } m = 0 \\ (\{a \in A \mid ga - a \in J_{m-1}\}) & \text{for } m > 0\,. \end{cases}$$

If B is a graded connected algebra over $\mathbb{F}$ on which G acts, we say that G acts **fixed point freely on** B if $B^G = \mathbb{F}$, in otherwords, if the action of G on the homogeneous component B_m of B of degree m has only $0 \in B_m$ as a fixed point for all $m > 0$. The action of G on $A/J_\infty =: A_{G^\infty}$ is fixed point free. The following lemma shows how this property characterizes J_∞.

Lemma 1.1 : *Let A and B be positively graded commutative connected algebras over a field $\mathbb{F}$ on which the group G acts by algebra automorphisms. Assume the action of G on B is fixed point free. The quotient map $q : A \longrightarrow A_{G^\infty}$ has the following universal property: for any algebra homomorphism $\varphi : A \longrightarrow B$ commuting with the G-action, there exists a unique algebra homomorphism $\tilde{\varphi} : A_{G^\infty} \longrightarrow B$ making the following diagram*

$$\begin{array}{ccc} A & \xrightarrow{\ q\ } & A_{G^\infty} \\ {\scriptstyle\varphi}\big\downarrow & \swarrow {\scriptstyle\tilde{\varphi}} & \\ B & & \end{array}$$

commute. □

Let $\varrho : G \hookrightarrow \mathrm{GL}(n,\ \mathbb{F})$ be a representation of a finite group. The totalization (see [11] chapter 4) of the algebra of coinvariants $\mathbb{F}[V]_G$ is finite dimensional and generated by the homogeneous component $(\mathbb{F}[V]_G)_1$ of degree 1, which is the image of V^* under the quotient map $\mathbb{F}[V] \longrightarrow \mathbb{F}[V]_G$. Since the dimensions of the homogeneous components $(\mathbb{F}[V]_{G^i})_1$ decrease with i the algebras $\mathbb{F}[V]_{G^i}$ become isomorphic for $i \geq n$ and the chain of ideals

$$(0) = J_0 \subseteq J_1(\mathbb{F}[V]) \subseteq J_2(\mathbb{F}[V]) \subseteq \cdots \subseteq J_m(\mathbb{F}[V]) \subseteq \cdots \subset \mathbb{F}[V]$$

[2] If $X \subset A$ then (X) denotes the ideal of A generated by X.

stabalizes after at most n steps.

The totalization of any of the quotient algebras $\mathbb{F}[V]/J_i(G)$ is finite dimensional, so the radical of the ideal $J_i(G)$ is the maximal ideal of $\mathbb{F}[V]$. Hence each of the ideals $J_i(G) \subset \mathbb{F}[V],\ i \in \mathbb{N}$ is primary.

Example 1.2 : Let p be an odd prime and $\mathbb{F}$ a field of characteristic p. The matrices

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}(2,\ \mathbb{F})$$

generate a dihedral group of order $2p$. We recall [11] § 5.6 example 1 that

$$\mathbb{F}[x,\ y]^{D_{2p}} \cong \mathbb{F}_p[y,\ (xy^{p-1} - x^p)^2]$$

where $x,\ y \in V^* = \mathrm{Hom}(V,\ \mathbb{F})$ is the dual of the canonical basis for $\mathbb{F}^2$. Therefore

$$\mathbb{F}[x,\ y]_{D_{2p}} \cong \frac{\mathbb{F}[x,\ y]}{(y, (xy^{p-1} - x^p)^2)} \cong \frac{\mathbb{F}[\overline{x}]}{(\overline{x}^{2p})}$$

where $\overline{x} \in \mathbb{F}[x,\ y]_{D_{2p}}$ is the residue class of x. The action of D_{2p} on x, y is given by the transposed matrices so

$$S(x) = x + y \qquad T(x) = -x$$
$$S(y) = y \qquad T(y) = y\,.$$

From these formulae it follows that

$$(\mathbb{F}[x,\ y]_{D_{2p}})^{D_{2p}} \cong \frac{\mathbb{F}[\overline{x}^2]}{(\overline{x}^{2p})}$$

and hence

$$\mathbb{F}[x,\ y]_{D_{2p}D_{2p}} \cong \frac{\mathbb{F}[\overline{x}]}{(\overline{x}^2)}\,.$$

The action of D_{2p} on $\mathbb{F}[x,\ y]_{D_{2p}D_{2p}}$ is fixed point free, so $J_2 = \cdots = J_\infty = (x^2,\ y) \subset \mathbb{F}[x,\ y]$ is the ideal of stable invariants.

The ideal of stable invariants is an aspect of the modular case as the following result shows.

Proposition 1.3 : *Let $\varrho : G \hookrightarrow \mathrm{GL}(n,\ \mathbb{F})$ be a representation of a finite group. Let $H < G$ be a subgroup with $|H| \in \mathbb{F}^\times$. Then $J_n(G)$, for $n = 1,\ 2, \ldots, \infty$, is generated by elements from $\overline{\mathbb{F}[V]^H}$. In particular, if $|G| \in \mathbb{F}^\times$ then $J_1(G) = J_2(G) = \cdots = J_\infty(G)$.*

Proof : By definition $J_1(G)$ is generated by $\overline{\mathbb{F}[V]^G}$ and $\mathbb{F}[V]^G \subseteq \mathbb{F}[V]^H$ so the result is established for $n = 1$. Proceeding inductively we suppose that $J_n(G)$ is generated by elements of $\mathbb{F}[V]^H$ of positive degree. Let $f \in J_{n+1}(G)$. Then $gf - f \in J_n(G)$ for all $g \in G$, say

$$gf - f = F_g,\ \ F_g \in J_n(G)\,.$$

Summing these equations (see [11] § 2.4 for the definition of the transfer Tr^H and the projection operator π^H) over all $g \in H$ gives

$$\mathrm{Tr}^H(f) - |H| \cdot f = \sum_{h \in H} F_h\,,$$

so solving for f we find

$$(*) \qquad f = \pi^H(f) + F$$

where $\pi^H(f) \in \mathbb{F}[V]^H$ and $F \in J_n(G)$. If $h_1, \ldots, h_m \in \overline{\mathbb{F}[V]^H}$ generate $J_n(G)$, then we may find $f_1, \ldots, f_k \in J_{n+1}(G)$ such that $h_1, \ldots, h_m, f_1, \ldots, f_k$ generate $J_{n+1}(G)$. Equation $(*)$ then shows that $h_1, \ldots, h_m, \pi^H f_1, \ldots, \pi^H f_k$ also generate $J_{n+1}(G)$, completing the induction step and the proof. □

Corollary 1.4 : *Let $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of a finite group over the field $\mathbb{F}$. Then*

$$J_n(G) \subseteq \bigcap_{H \le G \ , \ |H| \in \mathbb{F}^\times} J(H) \,. \quad \square$$

For example if G is a group of order $p^a q^b$ and $\mathbb{F}$ is a field of characteristic p, then $J_n(G)$ is contained in the intersection of $J(H)$ where H ranges over all the q-Sylow supgroups of G.

Corollary 1.5 : *Let $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of a finite group. Suppose $|G| \in \mathbb{F}^\times$. Then $J_\infty(G)$ is generated by a regular sequence if and only if $\varrho(G)$ is generated by pseudoreflections.*

Proof : By 1.3 $J_\infty(G) = (\overline{\mathbb{F}[V]^G})$, so there is a regular sequence $f_1, \ldots, f_n \in \mathbb{F}[V]^G$ generating $(\overline{\mathbb{F}[V]^G})$. We claim that $f_1, \ldots, f_n$ generate $\mathbb{F}[V]^G$ as an algebra. To prove this let $F \in \mathbb{F}[V]^G$ have positive degree. We may proceed inductively and suppose that for all $H \in \mathbb{F}[V]^G$ with $\deg(H) < \deg(F)$ that H belongs to the subalgebra generated by $f_1, \ldots, f_n$. Since $F \in (\overline{\mathbb{F}[V]^G})$ there are $H_1, \ldots, H_n \in \mathbb{F}[V]$ such that

$$F = \sum_{i=1}^{n} H_i f_i \,.$$

Apply π^G to this equation to obtain

$$F = \pi^G(F) = \sum_{i=1}^{n} \pi^G(H_i) f_i \,.$$

The classes $\pi^G(H_i)$ for $i = 1, \ldots, n$ belong to $\mathbb{F}[V]^G$ and have strictly smaller degree than F. Therefore the righthand side of the preceding equation belongs to the subalgebra generated by $f_1, \ldots, f_n$ completing the inductive step.

Since a regular sequence is algebraically independent the result follows from the theorem of Shephard-Todd [10] [3] [12]. □

The following result is useful when discussing Coxeter groups [7].

COROLLARY 1.6 : *Let $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of a finite group. Assume that $J_\infty(G) = (f_1, \ldots, f_n)$, where $f_1, \ldots, f_n \in \mathbb{F}[V]$ is a regular sequence. If $H < G$ is a subgroup with $|H| \in \mathbb{F}^\times$ then $|H|$ divides $\prod_{i=1}^{n} \deg(f_i)$.*

PROOF : By 1.3 we may find elements $h_1, \ldots, h_n \in \overline{\mathbb{F}[V]^H}$ which are a minimal generating set for $J_\infty(G)$. Then

$$\mathrm{Tot}\left(\frac{\mathbb{F}[V]}{(h_1, \ldots, h_n)}\right) = \mathrm{Tot}\left(\mathbb{F}[V]/J_\infty(G)\right)$$

is finite dimensional, so by the Noether Normalization Theorem ([11] theorem 5.3.3) $h_1, \ldots, h_n \in \mathbb{F}[V]^H$ is a system of parameters. Hence by Macaulay's theorem ([11] theorem 6.7.7) $h_1, \ldots, h_n$ is a regular sequence in $\mathbb{F}[V]$, and by Solomon's theorem ([11] theorem 6.8.1) the result follows. □

COROLLARY 1.7 : *Let $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of a finite group G. Assume $J_\infty(G) = (f_1, \ldots, f_n)$, where $f_1, \ldots, f_n \in \mathbb{F}[V]$ is a regular sequence. Let $\nu_p(k)$ denote the power of p in the integer k, i.e. $k = p^{\nu_p(k)}\ell$ with $(p, \ell) = 1$. If $\mathbb{F}$ has characteristic p then $|G|/p^{\nu_p(|G|)}$ divides $\prod_{i=1}^{n} \deg(f_i)$.*

PROOF : Apply 1.6 to all p'-Sylow subgroups of G, with $p' \neq p$, and use that $\mathrm{l.c.m.}\left\{|\mathrm{Syl}_{p'}(G)|\right\} = \frac{|G|}{p^{\nu_p(|G|)}}$. □

For p-groups the stable invariants are also uninteresting as the following result shows.

PROPOSITION 1.8 : *Let A be a positively graded, connected, commutative algebra of finite type over a field of characteristic $p \neq 0$. If P is a finite p-group of automorphisms of A then $J_\infty(P) = (\overline{A}) \subset A$. (Or put another way, $A_{P^\infty} = \mathbb{F}$.)*

PROOF : If M is a nontrivial finite dimensional P-module then the class equation (see e.g. [11] pp.85) shows

$$|M| = |M^P| + \sum_i |P : P_i|$$

where the sum ranges over the nontrivial isotropy subgroups of P on M. Hence $|M^P| \equiv 0 \mod p$ so $M^P \neq \{0\}$.

Therefore if the homogeneous component $[A_{P^m}]_k$ of degree k of A_{P^m} is not zero, then $\dim([A_{P^m}]_k) > \dim([A_{P^{m+1}}]_k)$. Since $\dim(A_k)$ is finite it follows that for large enough m and any $k \in \mathbb{N}$ that $[A_{P^m}]_k = 0$ and the result follows. □

If $\varrho : P \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ is a representation of a finite p-group over a field of characteristic p then the smallest integer k for which $J_k(P) = (V^*)$ is the socle length of V^* regarded as a module over the group ring $\mathbb{F}(P)$ (see e.g. [2] chapter 1).

PROPOSITION 1.9 : *Let $\varrho : G \hookrightarrow \mathrm{GL}(n,\ \mathbb{F})$ be a representation of a finite group. Assume that $J_\infty(G) = (f_1, \ldots,\ f_n) \subset \mathbb{F}[V]$ is generated by n elements. Then $f_1, \ldots,\ f_n$ is a regular sequence.*

PROOF : Since $\mathbb{F}[V]/J_\infty(G)$ is a quotient of $\mathbb{F}[V]_G$ it follows that $\mathrm{Tot}(\mathbb{F}[V]/J_\infty(G))$ is finite dimensional. Hence $f_1, \ldots,\ f_n \in \mathbb{F}[V]$ is a system of parameters by the Noether Normalization Theorem ([11] theorem 5.3.3). Hence by Macaulay's theorem ([11] theorem 6.7.7) it follows that $f_1, \ldots,\ f_n \in \mathbb{F}[V]$ is a regular sequence. □

As with the case of rings of invariants, the stable invariants of the full general linear group $\mathrm{GL}(n,\ \mathbb{F})$ over a finite field $\mathbb{F}$ are universal. Here is a computation of these universal stable invariants in a special case.

EXAMPLE 1.10 : Consider the group $\mathrm{GL}(2,\ \mathbb{F}_2)$. The group $\mathrm{GL}(2,\ \mathbb{F}_2)$ has order 6 (abstractly it is isomorphic to Σ_3, the symmetric group on 3 letters) and the elements are

$$\begin{array}{cccccccc} T & = & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & A & = & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & B & = & \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ I & = & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & TA & = & \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} & AT & = & \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array},$$

so $T,\ A,\ B$ generate $\mathrm{GL}(2,\ \mathbb{F}_2)$. The action of $\mathrm{GL}(2,\ \mathbb{F}_2)$ is therefore determined by the three involutive pseudoreflections (in this case transvections) $T,\ A,\ B$, and is given by the schema:

$$\begin{array}{ccc} T & A & B \\ x \mapsto y & x \mapsto x+y & x \mapsto x \\ y \mapsto x & y \mapsto y & y \mapsto x+y\,. \end{array}$$

The invariants of this group are [11] section 8.1

$$\mathbb{F}_2[x,\ y]^{\mathrm{GL}(2,\ \mathbb{F}_2)} = \mathbb{F}_2[\mathbf{d}_0,\ \mathbf{d}_1]\,,$$

where $\mathbf{d}_0,\ \mathbf{d}_1$ are the Dickson polynomials

$$\begin{array}{lcll} \mathbf{d}_0 & = & x^2y + xy^2, & \deg(\mathbf{d}_0) = 3 \\ \mathbf{d}_1 & = & x^2 + xy + y^2, & \deg(\mathbf{d}_1) = 2\,. \end{array}$$

The coinvariants $\mathbb{F}_2[x,\ y]_{\mathrm{GL}(2,\ \mathbb{F}_2)}$ are a Poincaré duality algebra (see e.g.[11] § 6.5) of dimension 3. One way to visualize this algebra is with the aid of the following diagram (see [11] section 1.2)

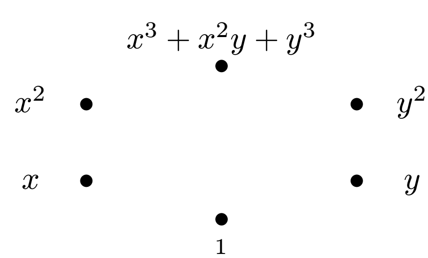

Diagram 1.1 : $\mathbb{F}[x,\ y]_{\mathrm{GL}(2,\ \mathbb{F}_2)}$

where the nodes on a horizontal level indicate basis elements of degree equal to the height of the node above the node labeled 1, which has degree 0. The polynomial $h = x^3 + x^2y + y^3 \in \mathbb{F}_2[x,\ y]_{\mathrm{GL}(2,\ \mathbb{F}_2)}$ must be invariant under the action of $\mathrm{GL}(2,\ \mathbb{F}_2)$ on $\mathbb{F}_2[x,\ y]_{\mathrm{GL}(2,\ \mathbb{F}_2)}$ because the group $\mathrm{GL}(2,\ \mathbb{F}_2)$ has no nontrivial 1-dimensional representations over $\mathbb{F}_2$. The following formulas (details left to the reader)

$$\begin{aligned} h - Th &= \mathbf{d}_0 \in J_1 \\ h - Ah &= \mathbf{d}_0 \in J_1 \\ h - Bh &= \mathbf{d}_0 \in J_1 \end{aligned}$$

indeed verify that $h \in J_2$.

In fact $(\mathbf{d}_0,\ \mathbf{d}_1,\ h) = J_2 = J_3 = \cdots = J_\infty$. To see this note that the quotient map

$$\mathbb{F}_2[x,\ y] \longrightarrow \mathbb{F}_2[x,\ y]_{\mathrm{GL}(2,\ \mathbb{F}_2)}$$

is an isomorphism in degree 1. Hence there no fixed points in $\mathbb{F}_2[x,\ y]_{\mathrm{GL}(2,\ \mathbb{F}_2)}$ of degree 1, since there are none in $\mathbb{F}_2[x,\ y]$ of degree 1. The Frobenous map $\zeta : \ell \mapsto \ell^2$ is a $\mathrm{GL}(2,\ \mathbb{F}_2)$-equivariant isomorphism

$$\left(\mathbb{F}_2[x,\ y]_{\mathrm{GL}(2,\ \mathbb{F}_2)}\right)_1 \xrightarrow{\zeta} \left(\mathbb{F}_2[x,\ y]_{\mathrm{GL}(2,\ \mathbb{F}_2)}\right)_2$$

and therefore $\left(\mathbb{F}_2[x,\ y]_{\mathrm{GL}(2,\ \mathbb{F}_2)}\right)_2$ contains no fixed points. This shows that $J_2 = J_3 = \cdots = J_\infty$.

The polynomial $h \in \mathbb{F}_2[x,\ y]$ is left invariant by the matrix $C = TA$, and the subalgebra of $\mathbb{F}_2[x,\ y]$ generated by $h,\ \mathbf{d}_0,\ \mathbf{d}_1$ is the ring of invariants of the cyclic subgroup $\mathbb{Z}/3 \subset \mathrm{GL}(2,\ \mathbb{F}_2)$ generated by the matrix

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \in \mathrm{GL}(2,\ \mathbb{F}_2)\,.$$

This may be verified by using Molien's theorem to compute the Poincaré series of $\mathbb{F}_2[x,\ y]^{\mathbb{Z}/3}$ ([11] section 4.3) and the theorem of Hochster and Eagon ([11] section 6.7). The first step is to choose a characteristic zero lift [3] of the representation $\varrho : \mathbb{Z}/3 \hookrightarrow \mathrm{GL}(2,\ \mathbb{F}_2)$. This is accomplished by the representation $\tilde{\varrho} : \mathbb{Z}/3 \hookrightarrow \mathrm{GL}(2,\ \mathbb{Z})$ generated by the matrix

$$\tilde{C} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \in \mathrm{GL}(2,\ \mathbb{Z})\,.$$

Molien's theorem ([11] theorem 4.2) yields the following formula for the Poincaré series of

[3] The operation of $\mathrm{GL}(2, \mathbb{F}_2)$ on $\mathbb{F}_2^2$ permutes the nonzero elements of $\mathbb{F}_2^2$ cyclically. This suggests the characteristic zero lift is the cyclic permutation of the third complex roots of unity in the hexagonal lattice in $\mathbb{C}$. For another example of this sort see the discussion of the mod 3 representation of the quaternion group in [11] section 4.3 example 1.

$\mathbb{F}_2[x,\ y]^{\mathbb{Z}/3}$:

$$P(\mathbb{F}_2[x,\ y]^{\mathbb{Z}/3},\ t) = \frac{1}{3}\left[\frac{1}{\det\begin{bmatrix}1-t & 0\\ 0 & 1-t\end{bmatrix}} + \frac{1}{\det\begin{bmatrix}1 & t\\ -t & 1+t\end{bmatrix}} + \frac{1}{\det\begin{bmatrix}1+t & -t\\ t & 1\end{bmatrix}}\right]$$
$$= \frac{1}{3}\left[\frac{1}{(1-t)^2} + \frac{1}{1+t+t^2} + \frac{1}{1+t+t^2}\right]$$
$$\vdots$$
$$= \frac{1+t^3}{(1-t^2)(1-t^3)}.$$

The ring $\mathbb{F}_2[x,\ y]^{\mathbb{Z}/3}$ is Cohen-Macaulay and $\mathbf{d}_0,\ \mathbf{d}_1 \in \mathbb{F}_2[x,\ y]$ are a system of parameters. Hence $\mathbb{F}_2[x,\ y]^{\mathbb{Z}/2}$ is a free finitely generated module over $\mathbb{F}_2[\mathbf{d}_0,\ \mathbf{d}_1]$. The Poincaré series of $\mathbb{F}_2[\mathbf{d}_0, \mathbf{d}_1] \oplus \mathbb{F}_2[\mathbf{d}_0,\ \mathbf{d}_1] \cdot h$ is also $\frac{1+t^3}{(1-t^2)(1-t^3)}$, so the map

$$\mathbb{F}_2[\mathbf{d}_0, \mathbf{d}_1] \oplus \mathbb{F}_2[\mathbf{d}_0,\ \mathbf{d}_1] \cdot h \longrightarrow \mathbb{F}_2[x,\ y]^{\mathbb{Z}/3}$$

is a surjection, and hence an isomorphism. Therefore $\mathbb{F}_2[x,\ y]^{\mathbb{Z}/3}$ is the subalgebra of $\mathbb{F}_2[x,\ y]$ generated by the stable invariants $\mathbf{d}_0,\ \mathbf{d}_1,\ h$ of $\mathrm{GL}(2,\ \mathbb{F}_2)$.

The stable invariants of the finite general linear groups are the subject of an ulterior investigation.

Another important new feature of invariant theory over Galois fields is the Steenrod algebra (see e.g. [14], [11] chapters 10 and 11), which operates on the ring of invariants. The existence of these operations impose significant restrictions on the subalgebras of $\mathbb{F}_q[V]$ which can be rings of invariants. See e.g. [1], [11] chapter 10, [16], and [19].

PROPOSITION 1.11 : *Let $\varrho : G \longrightarrow \mathrm{GL}(n,\ \mathbb{F}_q)$ be a representation of a finite group G. Then the ideals $J_i(G) \subset \mathbb{F}_q[V]$ are closed under the action of the Steenrod algebra for $i = 1,\ 2, , \dots, \infty$. Hence $\mathbb{F}[V]/J_i(G)$ is an unstable algebras over the Steenrod algebra for $i \in \mathbb{N}$.*

PROOF : This is immediate from the Cartan formula. □

§ 2. The Ideal of Generalized Invariants

Let $s \in \mathrm{GL}(n,\ \mathbb{F})$ be a pseudoreflection and $\ell_s \in \mathbb{F}[V]$ a linear form with $\ker(\ell_s) = H_s = \{v \in V \mid sv = v\}$. The linear form ℓ_s depends on s only up to a nonzero scalar. If $f \in \mathbb{F}[V]$ has positive degree then

$$(s-1)(f) = \ell_s \cdot \Delta_s(f)$$

for a unique $\Delta_s(f) \in \mathbb{F}[V]$ (see e.g. [11] § 7.1 and the references there). If f has degree k then $\Delta_s(f)$ has degree $k-1$, and $\Delta_s(f) = 0$ if and only if $sf = f$. For $a \in \mathbb{F}[V]$ of degree 0 set $\Delta_s(a) = 0$.

The operator $\Delta_s : \mathbb{F}[V] \longrightarrow \mathbb{F}[V]$ is linear and satisfies the following twisted derivation formula:

$$\Delta_s(fh) = \Delta_s(f) \cdot h + s(f) \cdot \Delta_s(h)$$

(see e.g. [4]). Hence inductively one obtains

$$\Delta_s(f^k) = \Delta_s(f) \left(f^{k-1} + f^{k-2} \cdot s(f) + \cdots + s(f)^{k-1} \right)$$

for any $k \in \mathbb{N}$.

The polynomial $\Delta_s(f)$ depends on the choice of ℓ_s. However, ℓ_s is well defined up to a nonzero scalar, so $\Delta_s(f)$ depends on s only up to a nonzero scalar also. If $s_1, \ldots, s_m \in \mathrm{GL}(n, \mathbb{F})$ are pseudoreflections, and $f \in \mathbb{F}[V]$ with $\deg(f) < m$, then $\Delta_{s_1} \cdots \Delta_{s_m}(f) = 0$ for degree reasons. Therefore the longest composition $\Delta_{s_1} \cdots \Delta_{s_m}$ that can evaluate nontrivially on f occurs when $m = \deg(f)$. In this case $\Delta_{s_1} \cdots \Delta_{s_m}(f)$ has degree zero so may be identified with a field element.

If $\mathcal{S} \subset \mathrm{GL}(n, \mathbb{F})$ is a set of pseudoreflections a polynomial $f \in \mathbb{F}[V]$ of positive degree is called an **$\mathcal{S}$-generalized invariant** if

$$\Delta_{s_1} \cdots \Delta_{s_{\deg(f)}}(f) = 0$$

for all $\deg(f)$-tuples $s_1, \ldots, s_{\deg(f)} \in \mathcal{S}$. The set of all generalized invariants of $\mathcal{S}$ is denoted by $I(\mathcal{S})$. Note that $I(\mathcal{S})$ is independent of the choices $\ell_s,\ s \in \mathcal{S}$, and hence depends only on $\mathcal{S} \subset \mathrm{GL}(n, \mathbb{F})$.

We begin by reviewing some results of Kac and Peterson. Since their work uses other terminology and notations, and has appeared only in abridged form [6] we include proofs of our own in the hopes of improving the readibility of this note.

Lemma 2.1 (Demazure, Kac and Peterson): *Let $\mathcal{S} \subset \mathrm{GL}(n, \mathbb{F})$ be a set of pseudoreflections. Then $I(\mathcal{S}) \subset \mathbb{F}[V]$ is an ideal.*

Proof : Let $h \in I(\mathcal{S})$ have degree k and $f \in \mathbb{F}[V]$ have degree l. We show by double induction on k and l that $f \cdot h \in I(\mathcal{S})$.

For $l = 0$ there is nothing to prove. For $k = 1$ and l arbitrary we have

$$\begin{aligned} \Delta_{s_1} \cdots \Delta_{s_{l+1}}(f \cdot h) &= \Delta_{s_1} \cdots \Delta_{s_l} \left[\Delta_{s_{l+1}}(f) \cdot h + s_{l+1}(f) \cdot \Delta_{s_{l+1}}(h) \right] \\ &= \Delta_{s_1} \cdots \Delta_{s_l}(\Delta_{s_{l+1}}(f) \cdot h) \end{aligned}$$

since $\Delta_s(h) = 0\ \forall\ s \in \mathcal{S}$ as $h \in I(\mathcal{S})$ and $\deg(h) = 1$. But $\Delta_{s_{l+1}}(f)$ has degree $l-1$ and $h \in I(\mathcal{S})$, so by induction on l it follows that $\Delta_{s_{l+1}}(f) \cdot h \in I(\mathcal{S})$. Since $\Delta_{s_{l+1}}(f) \cdot h$ has degree l it follows that $\Delta_{s_1} \cdots \Delta_{s_l}(\Delta_{s_{l+1}}(f) \cdot h) = 0$ as required.

Proceeding inductively over k we have

$$\Delta_{s_1} \cdots \Delta_{s_{k+l}}(f \cdot h) = \Delta_{s_1} \cdots \Delta_{s_{k+l-1}} \left[\Delta_{s_{k+l}}(f) \cdot h + s_{k+l}(f) \cdot \Delta_{s_{k+l}}(h)\right].$$

Note that $\Delta_{s_{k+l}}(h) \in I(\mathcal{S})$. Since $\deg(\Delta_{s_{k+l}}(h)) = k-1$ it follows from the induction hypothesis that $s_{k+l}(f) \cdot \Delta_{s_{k+l}}(h) \in I(\mathcal{S})$. The polynomial $\Delta_{s_{k+l}}(f)$ has degree $l-1$ so by induction on l it follows that $\Delta_{s_{k+l}}(f) \cdot h \in I(\mathcal{S})$. Hence $\Delta_{s_{k+l}}(f) \cdot h + s_{k+l}(f) \cdot \Delta_{s_{k+l}}(h)$ belongs to $I(\mathcal{S})$ and has degree $k+l-1$. Hence

$$\Delta_{s_1} \cdots \Delta_{s_{k+l-1}} \left[\Delta_{s_{k+l}}(f) \cdot h + s_{k+l}(f) \cdot \Delta_{s_{k+l}}(h)\right] = 0$$

completing the double induction. □

Lemma 2.2 : *Let $\mathcal{S} \subset \mathrm{GL}(n,\ \mathbb{F})$ be a set of pseudoreflections. Then $f \in I(\mathcal{S})$ if and only if $\Delta_s(f) \in I(\mathcal{S})$ for all $s \in \mathcal{S}$.*

Proof : Suppose $\Delta_s(f) \in I(\mathcal{S})$ for all $s \in \mathcal{S}$. Let $d = \deg(f)$ and $s_1, \ldots, s_d \in \mathcal{S}$. Then

$$\Delta_{s_1} \cdots \Delta_{s_d}(f) = \Delta_{s_1} \cdots \Delta_{s_{d-1}}(\Delta_{s_d}(f)) = 0$$

as $\Delta_{s_d}(f) \in I(\mathcal{S})$ and $\deg(\Delta_{s_d}(f)) = d-1$. The converse is equally straightforward. □

For a set of pseudoreflections $\mathcal{S} \subset \mathrm{GL}(n,\ \mathbb{F})$ the group generated by $\mathcal{S}$ is denoted by $G(\mathcal{S})$. It is of course a subgroup of $\mathrm{GL}(n,\ \mathbb{F})$ generated by pseudoreflections. If $f \in \mathbb{F}[V]^{G(\mathcal{S})}$ has positive degree, then $\Delta_s(f) = 0$ for all $s \in \mathcal{S}$ and hence $f \in I(\mathcal{S})$. Therefore by 2.1 $I(\mathcal{S})$ contains the ideal $(\overline{\mathbb{F}[V]^{G(\mathcal{S})}})$ of $\mathbb{F}[V]$ generated by the elements of $\mathbb{F}[V]^{G(\mathcal{S})}$ of positive degree.

Lemma 2.3 : *Let $\mathcal{S} \subset \mathrm{GL}(n,\ \mathbb{F})$ be a collection of pseudoreflections, then $I(\mathcal{S})$ is stable under the action of $G(\mathcal{S})$ on $\mathbb{F}[V]$. Hence $G(\mathcal{S})$ operates on $\mathbb{F}[V]/I(\mathcal{S})$.*

Proof : Let $f \in I(\mathcal{S})$. Since $\mathcal{S}$ generates $G(\mathcal{S})$ it suffices to show that $sf \in I(\mathcal{S})$ for all $s \in \mathcal{S}$. If $s \in \mathcal{S}$ we have

$$sf = f + \ell_s \cdot \Delta_s(f).$$

By 2.2 $\Delta_s(f) \in I(\mathcal{S})$ and hence by 2.1 $\ell_s \cdot \Delta_s(f) \in I(\mathcal{S})$. By hypothesis $f \in I(\mathcal{S})$ so it follows from the preceding equation that $sf \in I(\mathcal{S})$. □

Observe for $f \in I(\mathcal{S})$ that $\Delta_s(f) \in I(\mathcal{S})$ for all $s \in \mathcal{S}$. If $\mathcal{T} \subset \mathcal{S}$ then $I(\mathcal{S}) \subseteq I(\mathcal{T})$ so if $\mathcal{S} = \mathcal{S}' \cup \mathcal{S}''$ then $I(\mathcal{S}) \subseteq I(\mathcal{S}') \cap I(\mathcal{S}'')$. The following lemmas are an aid in computing examples.

Lemma 2.4 : *Let $\mathcal{S} \subset \mathrm{GL}(n,\ \mathbb{F})$ be a set of pseudoreflections. Suppose that $s,\ s^k \in \mathcal{S}$ for some integer $1 < k < |s|$. Then $I(\mathcal{S}) = I(\mathcal{S} \setminus \{s^k\})$.*

Proof : Since $\mathcal{S} \setminus \{s^k\} \subset \mathcal{S}$ we have $I(\mathcal{S}) \subset I(\mathcal{S} \setminus \{s^k\})$. To establish the reverse inclusion we note that $H_{s^k} = H_s$ so we may choose $\ell_{s^k} = \ell_s$. Hence we have

$$\begin{aligned}
\Delta_{s^k}(f) &= \frac{s^k(f) - f}{\ell_s} \\
&= \frac{s^k(f) - s^{k-1}(f)}{\ell_s} + \frac{s^{k-1}(f) - s^{k-2}(f)}{\ell_s} + \cdots + \frac{s(f) - f}{\ell_s} \\
&= s^{k-1}\frac{sf - f}{\ell_s} + s^{k-2}\frac{sf - f}{\ell_s} + \cdots + \frac{sf - f}{\ell_s} \\
&= s^{k-1} \cdot \Delta_s(f) + s^{k-2} \cdot \Delta_s(f) + \cdots + \Delta_s(f) \\
&= (s^{k-1} + s^{k-2} + \cdots + 1)\Delta_s(f)\,.
\end{aligned}$$

Let $f \in I(\mathcal{S} \setminus \{s^k\})$. By 2.2 it suffices to show $\Delta_{s^k}(f) \in I(\mathcal{S})$. From the preceding formula we have

$$\Delta_{s^k}(f) = (1 + s + \cdots + s^{k-1})\Delta_s(f)\,.$$

Since $s \in \mathcal{S} \setminus \{s^k\}$ a second application of 2.2 shows $\Delta_s(f) \in I(\mathcal{S})$. From 2.3 it follows that $(1 + s + \cdots + s^{k-1})\Delta_s(f) \in I(\mathcal{S})$ as required. □

Lemma 2.5 : *Let $s_1, \ldots, s_k \in \mathrm{GL}(n, \mathbb{F})$ be pseudoreflections such that $s_1 \cdots s_j$ is a pseudoreflection for $j = 1, \ldots, k$. If $\ell_{s_i} \in \mathbb{F}[V]$, $i = 1, \ldots, k$ are linear forms with $\ker(\ell_{s_i}) = H_{s_i}$ then*

$$\Delta_{s_1 \cdots s_k}(f) = \frac{1}{\ell_{s_1 \cdots s_k}} \sum_{i=1}^{k} \Big(s_1 \cdots s_{i-1}(\ell_{s_i})\Big)\Big(s_1 \cdots s_{i-1}(\Delta_{s_i}(f))\Big)$$

for all $f \in \mathbb{F}[V]$.

Proof : By induction on k. For $k = 2$ we have

$$\begin{aligned}
\Delta_{s_1 s_2}(f) &= \frac{1}{\ell_{s_1 s_2}}\Big((s_1 s_2)(f) - f\Big) = \frac{1}{\ell_{s_1 s_2}}\Big((s_1 s_2)(f) - s_1(f) + s_1(f) - f\Big) \\
&= \frac{1}{\ell_{s_1 s_2}}\Big(s_1(\ell_{s_2}) s_1(\Delta_{s_2}(f)) + \ell_{s_1}\Delta_{s_1}(f)\Big)\,.
\end{aligned}$$

For $k > 3$ we obtain from the inductive hypothesis

$$\begin{aligned}
&\Delta_{s_1 \cdots s_k}(f) = \Delta_{s_1 \cdots s_{k-1} \cdot s_k}(f) \\
&= \frac{1}{\ell_{s_1 \cdots s_{k-1} \cdot s_k}}\Big(\ell_{s_1 \cdots s_{k-1}}\Delta_{s_1 \cdots s_{k-1}}(f) + \Big(s_1 \cdots s_{k-1}(\ell_{s_k})\Big)\Big(s_1 \cdots s_{k-1}(\Delta_{s_k}(f))\Big)\Big) \\
&= \frac{1}{\ell_{s_1 \cdots s_k}}\Big[\Big(\sum_{i=1}^{k-1}\Big(s_1 \cdots s_{i-1}(\ell_{s_i})\Big)\Big(s_1 \cdots s_{i-1}(\Delta_{s_i}(f))\Big)\Big) + \Big(s_1 \cdots s_{k-1}(\ell_{s_k})\Big)\Big(s_1 \cdots s_{k-1}(\Delta_{s_k}(f))\Big)\Big] \\
&= \frac{1}{\ell_{s_1, \ldots, s_k}} \sum_{i=1}^{k}\Big(s_1 \cdots s_{i-1}(\ell_{s_i})\Big)\Big(s_1 \cdots s_{i-1}(\Delta_{s_i}(f))\Big)\,,
\end{aligned}$$

and the result follows. □

LEMMA 2.6 : *Let $\mathcal{S}'$, $\mathcal{S}'' \subset \mathrm{GL}(n, \mathbb{F})$ be sets of pseudoreflections with $G(\mathcal{S}') = G(\mathcal{S}'')$. Suppose that $G(\mathcal{S}') \setminus \{1\}$ consists only of pseudoreflections. Set $\mathcal{S} = \mathcal{S}' \cup \mathcal{S}''$. Then $I(\mathcal{S}) = I(\mathcal{S}') \cap I(\mathcal{S}'')$.*

PROOF : The inclusion $I(\mathcal{S}) \subseteq I(\mathcal{S}') \cap I(\mathcal{S}'')$ is elementary. Let $f \in I(\mathcal{S}') \cap I(\mathcal{S}'')$. We prove $f \in I(\mathcal{S})$ by induction over $\deg(f)$ thereby establishing the reverse inclusion $I(\mathcal{S}) \supseteq I(\mathcal{S}') \cap I(\mathcal{S}'')$, and hence the lemma.

If $\deg(f) = 1$ then $f \in I(\mathcal{S})$ if and only if $\Delta_s(f) = 0$ for all $s \in \mathcal{S}$. This means however that $\Delta_s(f) = 0$ for all $s \in \mathcal{S}'$ and for all $s \in \mathcal{S}''$, i.e., $f \in I(\mathcal{S}') \cap I(\mathcal{S}'')$.

If $\deg(f) > 1$ then

$$\begin{aligned} \Delta_{s'_1 \cdots s'_{\deg(f)}}(f) &= 0 \qquad \forall\, s'_1, \ldots, s'_{\deg(f)} \in \mathcal{S}' \\ \Delta_{s''_1 \cdots s''_{\deg(f)}}(f) &= 0 \qquad \forall\, s''_1, \ldots, s''_{\deg(f)} \in \mathcal{S}'' \end{aligned}$$

and we must show

$$\Delta_{s_1 \cdots s_{\deg(f)}}(f) = 0 \qquad \forall\, s_1, \ldots, s_{\deg(f)} \in \mathcal{S} = \mathcal{S}' \cup \mathcal{S}''\,.$$

Since $s_{\deg(f)} \in G(\mathcal{S}') = G(\mathcal{S}'')$ we have formulas

$$s_{\deg(f)} = \begin{cases} t'_1 \cdots t'_k & \text{for } t'_1, \ldots, t'_k \in \mathcal{S}' \\ t''_1 \cdots t''_l & \text{for } t''_1, \ldots, t''_l \in \mathcal{S}''. \end{cases}$$

Therefore by lemma 2.5 we have

$$\Delta_{s_{\deg(f)}}(f) = \begin{cases} \Delta_{t'_1 \cdots t'_k}(f) \; = \sum_{i=1}^{k} \Big(t'_1 \cdots t'_{i-1}(\ell_{t'_i}) \Big) \Big(t'_1 \cdots t'_{i-1}(\Delta_{t'_i}(f)) \Big) & \text{for } t'_1, \ldots, t'_k \in \mathcal{S}' \\ \Delta_{t''_1 \cdots t''_l}(f) = \sum_{i=1}^{l} \Big(t''_1 \cdots t''_{i-1}(\ell_{t''_i}) \Big) \Big(t''_1 \cdots t''_{i-1}(\Delta_{t''_i}(f)0 \Big) & \text{for } t''_1, \ldots, t''_l \in \mathcal{S}'' \end{cases}$$

which yields

$$\Delta s_{\deg(f)}(f) \begin{cases} \in I(\mathcal{S}') \\ \in I(\mathcal{S}''), \end{cases}$$

which implies $\Delta_s(f) \in I(\mathcal{S}') \cap I(\mathcal{S}'')$ for all $s \in \mathcal{S}' \cup \mathcal{S}''$. Applying the induction hypothesis then yields $f \in I(\mathcal{S})$ completing the inductive step. □

For a discussion of the subgroups G of $\mathrm{GL}(n, \mathbb{F})$ such that every nonidentity element is a pseudoreflection see [11] chapter 8 section 2, 8.2.1, 8.2.11 and 8.2.17.

The initial goal was to associate to a pseudoreflection group $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ an ideal of generalized invariants. Associated to G are however several different natural sets of pseudoreflections, such as, in the notations of [11] chapter 7 for example:

$$\begin{aligned} s(G) &= \{s \in G \mid \varrho(s) \text{ is a pseudoreflection}\} \\ s_{\Delta}(G) &= \{s \in G \mid \varrho(s) \text{ is a diagonalizable pseudoreflection}\} \\ s_{\not\Delta}(G) &= \{s \in G \mid \varrho(s) \text{ is a transvection}\}\,. \end{aligned}$$

For a Coxeter group there is also the set of defining generating reflections. These sets can have different ideals of generalized invariants, whose significance depends on the situation, as the following examples show. (See however 3.4.)

EXAMPLE 2.7 : Consider the matrices

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}(2,\ \mathbb{F})$$

where $\mathbb{F}$ is a field of characteristic $p \neq 0,\ 2$. Direct computation shows that

$$S^p = I = T^2, \quad TST^{-1} = S^{-1}$$

so that the subgroup of $\mathrm{GL}(2,\ \mathbb{F})$ generated by S and T is a dihedral group D_{2p} of order $2p$. One has

$$\begin{aligned} s(D_{2p}) &= \{S^i,\ T,\ TS^i \mid i = 1, \dots, p-1\} \\ s_{\Delta}(D_{2p}) &= \{T,\ TS^i \mid i = 1, \dots, p-1\} \\ s_{\not\Delta}(D_{2p}) &= \{S^i \mid i = 1, \dots, p-1\}\,. \end{aligned}$$

The sets $s(D_{2p})$ and $s_{\Delta}(D_{2p})$ both generate D_{2p}. By lemma 2.4

$$\begin{aligned} I(s(D_{2p})) &= I(\{S,\ T,\ TS^i\}) \\ I(s_{\not\Delta}(D_{2p}) &= I(\{S\})\,. \end{aligned}$$

Denote by $x,\ y \in V^* \subset \mathbb{F}[V]$ the linear forms dual to the standard basis. The fixed hyperplanes of the pseudoreflections in D_{2p} are

$$\begin{aligned} H_{S^i} &= \left\{ \begin{bmatrix} b \\ 0 \end{bmatrix} \in \mathbb{F}^2 \right\} = \ker(y) \\ H_T &= \left\{ \begin{bmatrix} 0 \\ b \end{bmatrix} \in \mathbb{F}^2 \right\} = \ker(x) \\ H_{TS^i} &= \left\{ \begin{bmatrix} -ia \\ 2a \end{bmatrix} \in \mathbb{F}^2 \right\} = \ker(2x + iy)\,. \end{aligned}$$

This gives the following formulae

$$\begin{aligned} \Delta_{S^i}(f) &= \frac{S^i f - f}{y} \\ \Delta_T(f) &= \frac{Tf - f}{x} \\ \Delta_{TS^i}(f) &= \frac{TS^i f - f}{2x + iy}\,. \end{aligned}$$

The action of the twisted derivations on x and y are given by

$$\begin{array}{ll} \Delta_{S^i}(x) = i & \Delta_{S^i}(y) = 0 \\ \Delta_T(x) = -2 & \Delta_T(y) = 0 \\ \Delta_{TS^i}(x) = -1 & \Delta_{TS^i}(y) = 0 \end{array}$$

and therefore y belongs to all three of the ideals $I(s(D_{2p})),\ I(s_{\Delta}(D_{2p}))$ and $I(s_{\not\Delta}(D_{2p}))$.

Next note that

$$\underbrace{\Delta_S \cdots \Delta_S}_{\leftarrow\, i\, \rightarrow}(x^i) = \begin{cases} i! \neq 0 & \text{for } i < p \\ 0 & \text{for } i = p \end{cases}$$

so that $x^p \in I(\{S\}) = I(s_{\not\Delta}(D_{2p}))$, and therefore

$$I(s_{\not\Delta}(D_{2p})) = (y,\ x^p)\,.$$

Likewise the formula

$$\Delta_{TS^i}(x^2) = iy$$

implies

$$\begin{aligned} \Delta_{S^j}\Delta_{TS^i}(x^2) &= \Delta_{S^j}(iy) = 0 \\ \Delta_T\Delta_{TS^i}(x^2) &= \Delta_T(iy) = 0 \\ \Delta_{TS^j}\Delta_{TS^i}(x^2) &= \Delta_{TS^j}(iy) = 0 \end{aligned}$$

so $x^2 \in I(s_\Delta(D_{2p}))$. Hence

$$I(s_\Delta(D_{2p})) = (y,\ x^2)\,.$$

Finally since $I(s(D_{2p})) \subset I(s_\Delta(D_{2p})) \cap I(s_{\not\Delta}(D_{2p}))$ it follows that

$$I(s(D_{2p})) = (y,\ x^{2p})\,.$$

Hence the ideals $I(s(D_{2p}))$, $I(s_\Delta(D_{2p}))$ and $I(s_{\not\Delta}(D_{2p}))$ are all distinct.

Example 2.8 : Let p be an odd prime and $\mathbb{F}$ a field of characteristic p. Consider the matrices

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\quad T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},\quad D = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}(2,\ \mathbb{F})\,.$$

From example 2.7 we have that the subgroup generated by S and T is the dihedral group D_{2p} of order $2p$. Since $D = TS$ it follows that D_{2p} is also generated by T and D. The matrices S and T are pseudoreflections of orders p and 2 respectively. Direct computation shows $D^2 = I$ and hence the matrix D is a pseudoreflection of order 2.

Set $\mathcal{S} = \{S,\ T\}$, $\mathcal{D} = \{T,\ D\}$. Then $\mathcal{S},\ \mathcal{D} \subset \mathrm{GL}(2,\ \mathbb{F})$ are sets of pseudoreflections and $G(\mathcal{S}) = D_{2p} = G(\mathcal{D})$.

Denote by $x,\ y \in V^* \subset \mathbb{F}[V]$ the linear forms dual to the standard basis of $V = \mathbb{F}^2$. The fixed hyperplanes of S, T and D are

$$\begin{aligned} H_S &= \left\{ \begin{bmatrix} b \\ 0 \end{bmatrix} \in \mathbb{F}^2 \right\} = \ker(y) \\ H_T &= \left\{ \begin{bmatrix} 0 \\ b \end{bmatrix} \in \mathbb{F}^2 \right\} = \ker(x) \\ H_D &= \left\{ \begin{bmatrix} -a \\ 2a \end{bmatrix} \in \mathbb{F}^2 \right\} = \ker(2x + y)\,. \end{aligned}$$

This gives the following formulae

$$\Delta_S(f) = \frac{Sf - f}{y}$$
$$\Delta_T(f) = \frac{Tf - f}{x}$$
$$\Delta_D(f) = \frac{Df - f}{2x + y}\,.$$

The action of the twisted derivations on x and y are given by

$$\begin{aligned} \Delta_S(x) &= 1 & \Delta_S(y) &= 0 \\ \Delta_T(x) &= -2 & \Delta_T(y) &= 0 \\ \Delta_D(x) &= -1 & \Delta_D(y) &= 0 \end{aligned}$$

so as in example 2.7 we have

$$\underset{\leftarrow\, i\, \rightarrow}{\Delta_S \cdots \Delta_S}(x^i) = \begin{cases} i! \neq 0 & \text{for } i < p \\ 0 & \text{for } i = p \end{cases}$$

and $x^p \in I(\{S\})$ is the lowest power of x in $I(\mathcal{S})$ so $I(\mathcal{S}) = (y,\ x^p)$.

By contrast

$$\Delta_T(x^2) = (x + T(x))\Delta_T(x) = (x - x)(-2) = 0$$
$$\Delta_D(x^2) = (x + D(x))\Delta_D(x) = (x - x - y)(-1) = y\,.$$

Hence

$$\Delta_T\Delta_T(x^2) = 0$$
$$\Delta_T\Delta_D(x^2) = \Delta_T(y) = 0$$
$$\Delta_D\Delta_T(x^2) = 0$$
$$\Delta_D\Delta_D(x^2) = \Delta_D(y) = 0\,,$$

and therefore $x^2 \in I(\mathcal{D})$. Since $x \neq I(\mathcal{D})$ it therefore follows that $I(\mathcal{D}) = (y,\ x^2)$. Hence $I(\mathcal{S}) \neq I(\mathcal{D})$.

These examples show that for a collection $\mathcal{T} \subset \mathrm{GL}(n,\ \mathbb{F})$ of pseudoreflections, the ideal of generalized invariants $I(\mathcal{T})$ is **not** an invariant of $G(\mathcal{T})$. From section 1 example 1.2 we have that $J_\infty(D_{2p}) = (x^2,\ y) = I(\mathcal{D})$. This is a special case of the more general result 3.4 below. The ideals of generalized invariants computed so far are in each case generated by a regular sequence of maximal length. This too is no accident as we show next.

THEOREM 2.9 (Kac and Peterson): *Let $\mathcal{S} \subset \mathrm{GL}(n,\ \mathbb{F})$ be a collection of pseudoreflections. Then $I(\mathcal{S})$ is generated by a regular sequence of length n.*

The proof of this theorem rests on the following result of Vasconcelos ([18] Theorem 1.1), which we restate in the form we require.

Theorem 2.10 (Vasconcelos): *Let A be a graded connected commutative algebra over a field $\mathbb{F}$ and $I \subset A$ a homogeneous ideal of finite projective dimension. Then I is generated by a regular sequence if and only if I/I^2 is a free A/I-module.* □

Proof of Theorem 2.9 : By Hilbert's syzygy theorem ([11] theorem 6.3.1) the algebra $\mathbb{F}[V]$ has finite global dimension. Therefore by the theorem of Vasconcelos 2.10 $I := I(\mathcal{S})$ is generated by a regular sequence if and only if I/I^2 is a free $\mathbb{F}[V]/I$-module.

Choose an $\mathbb{F}$-vector space basis $\bar{f}_1, \ldots, \bar{f}_m$ for the module $QI = \mathbb{F} \otimes_{\mathbb{F}[V]} I$ of indecomposables of I. Let $f_1, \ldots, f_m \in I$ lift $\bar{f}_1, \ldots, \bar{f}_m$ to a minimal ideal basis for I. Denote by $(\mathbb{F}[V]/I) \cdot F$ the free $\mathbb{F}[V]/I$-module with generator F. Define a map

$$\varphi : \oplus_{i=1}^{m} (\mathbb{F}[V]/I) \cdot F_i \longrightarrow I/I^2$$

where $\deg(F_i) = \deg(f_i)$ for $i = 1, \ldots, m$ in the following way: for an element $(\bar{h}_1 F_1, \ldots, \bar{h}_m F_m) \in \bigoplus_{i=1}^{m} (\mathbb{F}[V]/I) \cdot F_i$ choose $h_1, \ldots, h_m \in \mathbb{F}[V]$ lifting $\bar{h}_1, \ldots, \bar{h}_m \in \mathbb{F}[V]/I$ and set

$$\varphi(\bar{h}_1 F_1, \ldots, \bar{h}_m F_m) = h_1 f_1 + \cdots h_m f_m \, ,$$

where the righthand side is to be interpreted as an element of the quotient I/I^2 of I.

We claim that φ is an isomorphism. To see this note first that φ is an epimorphism because $f_1, \ldots, f_m \in I$ generate I as an $\mathbb{F}[V]$-module, and φ is a homomorphism of $\mathbb{F}[V]$-modules. It remains to show that φ is a monomorphism.

Suppose that $\bar{h}_1 F_1 + \cdots \bar{h}_m F_m \in \ker(\varphi)$. We need to show that $\bar{h}_i = 0 \in \mathbb{F}[V]/I$ for $i = 1, \ldots, m$. Let us suppose to the contrary. By reordering we may also suppose

$$\deg(\bar{h}_1) \leq \cdots \leq \deg(\bar{h}_m) \, ,$$

and that $j \in \{1, \ldots, m\}$ is chosen so that $\bar{h}_i = 0$ for $i < j$, while $\bar{h}_j \neq 0$. Set $\deg(h_j) = d$. Choose polynomials $h_i \in \mathbb{F}[V]$ lifting $\bar{h}_i \in \mathbb{F}[V]/I$ for $i = 1, \ldots, m$. Without loss of generality we may suppose that $h_i = 0$ for $i < j$. Since $h_j \notin I$ there are pseudoreflections $s_1, \ldots, s_d \in \mathcal{S}$ such that

$$(\Delta_{s_1} \cdots \Delta_{s_d})(h_j) \neq 0 \in \mathbb{F} \, .$$

The ideals I and I^2 are invariant under the operators $\{\Delta_s \mid s \in \mathcal{S}\}$ and hence Δ_s acts also on $\mathbb{F}[V]/I$ and I/I^2 for any $s \in \mathcal{S}$. Apply $\Delta_{s_1} \cdots \Delta_{s_m}$ to the equation

$$\varphi(\bar{h}_j F_j + \cdots \bar{h}_m F_m) = 0$$

to obtain

$$\Delta_{s_1} \cdots \Delta_{s_d}(h_j f_j + \cdots + h_m f_m) = 0$$

in I/I^2. By iterating the twisted derivation formula for $\Delta_s(f \cdot h)$ we obtain

$$0 = \sum_{\deg(h_i) = d} \Delta_{s_1} \cdots \Delta_{s_d}(h_i) \cdot f_i + \sum u_i \cdot w_i \, ,$$

where $u_i \in \mathbb{F}[V]/I$ and $w_i \in I/I^2$ with $\deg(u_i),\ \deg(w_i) > 0$. Let $q : I/I^2 \longrightarrow QI$ be the quotient map. Apply q to the preceding equation to obtain

$$\sum_{\deg(h_i)=d} \Delta_{s_1} \cdots \Delta_{s_d}(h_i) \cdot \bar{f_i} = 0 \in QI\,.$$

By hypothesis $\Delta_{s_1} \cdots \Delta_{s_d}(h_j) \neq 0$ and $\deg(h_j) = d$. Therefore the preceding equation is a nontrivial linear relation between $\bar{f}_1, \ldots, \bar{f}_m$ contrary to their choice as an $\mathbb{F}$-basis for QI. Hence φ is also a monomorphism.

Let $I(\mathcal{S}) = (h_1, \ldots, h_m)$ where $h_1, \ldots, h_m \in \mathbb{F}[V]$ is a regular sequence. Since $(\overline{\mathbb{F}[V]^{G(\mathcal{S})}}) \subseteq I(\mathcal{S})$ and $\mathbb{F}[V]$ is finite over $\mathbb{F}[V]^{G(\mathcal{S})}$ it follows that $\mathbb{F}[V]/I(\mathcal{S}) = \mathbb{F}[V]/(h_1, \ldots, h_m)$ is finite dimensional. Hence $h_1, \ldots, h_m \in \mathbb{F}[V]$ is also a system of parameters, and therefore by Macaulay's theorem ([11] theorem 6.7.11) we conclude that $m = n$. □

Thus for any set of pseudoreflections $\mathcal{S}$ in $\mathrm{GL}(n,\ \mathbb{F})$ the ideal of generalized invariants $I(\mathcal{S}) \subset \mathbb{F}[V]$ is generated by a regular sequence of length $n = \dim_{\mathbb{F}}(V)$. We seek an analog of the theorem of Shephard-Todd et al. ([11] 5.5.5) relating the degrees of the polynomials in such a regular sequence to $\mathcal{S}$.

Lemma 2.11 : *Let $H \subseteq \mathrm{GL}(n,\ \mathbb{F})$ be a subgroup with $|H| \in \mathbb{F}^{\times}$. Let $h_1, \ldots, h_n \in \mathbb{F}[V]$ be a regular sequence such that the ideal $(h_1, \ldots, h_n) \subset \mathbb{F}[V]$ is stable under the action of H. Let $\pi^H : \mathbb{F}[V] \longrightarrow \mathbb{F}[V]^H$ be the averaging operator $\frac{1}{|H|} \sum_{h \in H} h$. Then*

(i) $\pi^H(h_1), \ldots, \pi^H(h_n) \in \mathbb{F}[V]$ is a regular sequence,
(ii) $\pi^H(h_1), \ldots, \pi^H(h_n) \in \mathbb{F}[V]^H$ is a regular sequence, and
(iii) $(\pi^H(h_1), \ldots, \pi^H(h_n)) = (h_1, \ldots, h_n) \subset \mathbb{F}[V]$.

Proof : Since $(h_1, \ldots, h_n) \subset \mathbb{F}[V]$ is stable under the action of H and the averaging map is an epimorphism of $\mathbb{F}[V]^H$-modules we obtain an epimorphism

$$\pi^H : \mathbb{F}[V]/(h_1, \ldots, h_n) \longrightarrow \mathbb{F}[V]^H/(\pi^H(h_1), \ldots, \pi^H(h_n))\,.$$

Since $h_1, \ldots, h_n \in \mathbb{F}[V]$ is a regular sequence $\mathbb{F}[V]/(h_1, \ldots, h_n)$ is finite dimensional, and therefore so is $\mathbb{F}[V]/(\pi^H(h_1), \ldots, \pi^H(h_n))$. The ring $\mathbb{F}[V]^H$ has Krull dimension n and therefore $\pi^H(h_1), \ldots, \pi^H(h_n) \in \mathbb{F}[V]^H$ is a system of parameters. Hence we have finite extensions

$$\mathbb{F}[\pi^H(h_1), \ldots, \pi^H(h_n)] \subseteq \mathbb{F}[V]^H \subseteq \mathbb{F}[V]$$

and therefore $\pi^H(h_1), \ldots, \pi^H(h_n)\mathbb{F}[V]$ is a system of parameters. By Macaulay's theorem ([11] theorem 6.7.7) $\pi^H(h_1), \ldots, \pi^H(h_n) \in \mathbb{F}[V]$ is a regular sequence.

The averaging operator π^H is an $\mathbb{F}[V]^H$-module splitting, and hence a fortiori an $\mathbb{F}[\pi^H(h_1), \ldots, \pi^H(h_n)]$-module splitting, to the inclusion $\mathbb{F}[V]^H \hookrightarrow \mathbb{F}[V]$ and hence $\mathbb{F}[V]^H$ is a projective, and therefore free $\mathbb{F}[\pi^H(h_1), \ldots, \pi^H(h_n)]$-module (see e.g. [11] section 6.1), so by Macaulay's theorem ([11] theorem 6.7.11) $\pi^H(h_1), \ldots, \pi^H(h_n) \in \mathbb{F}[V]^H$ is a regular sequence.

The classes $\pi^H(h_1), \ldots, \pi^H(h_n)$ belong to $(h_1, \ldots, h_n)$ so we have an epimorphism

$$q : \mathbb{F}[V]/(\pi^H(h_1), \ldots, \pi^H(h_n)) \longrightarrow \mathbb{F}[V]/(h_1, \ldots, h_n)\,.$$

The Poincaré series computation

$$\begin{aligned} P(\mathbb{F}[V]/(\pi^H(h_1), \ldots, \pi^H(h_n)),\ t) &= \prod_{i=1}^{n} \left[\frac{1 - t^{\deg(\pi^H(h_i))}}{1-t}\right] \\ &= \prod_{i=1}^{n} \left[\frac{1 - t^{\deg(h_i)}}{1-t}\right] = P(\mathbb{F}[V]/(h_1, \ldots, h_n),\ t) \end{aligned}$$

then implies that q is an isomorphism, so $(\pi^H(h_1), \ldots, \pi^H(h_n)) = (h_1, \ldots, h_n)$. □

Lemma 2.12 : *Let $H \subseteq \mathrm{GL}(n,\ \mathbb{F})$ be a subgroup with $|H| \in \mathbb{F}^\times$. Let $h_1, \ldots, h_n \in \mathbb{F}[V]$ be a regular sequence such that the ideal $(h_1, \ldots, h_n) \subset \mathbb{F}[V]$ is stable under the action of H. Then $|H|$ divides $\prod_{i=1}^{n} \deg(h_i)$.*

Proof : By 2.11 we may suppose without loss of generality that $h_1, \ldots, h_n \in \mathbb{F}[V]^H$. Since $h_1, \ldots, h_n \in \mathbb{F}[V]$ is a regular sequence it follows that $\mathbb{F}[V]$ is a free $\mathbb{F}[h_1, \ldots, h_n]$-module. The averaging map

$$\pi^H : \mathbb{F}[V] \longrightarrow \mathbb{F}[V]^H$$

is an $\mathbb{F}[V]^H$-module splitting of the inclusion $\mathbb{F}[V]^H \subseteq \mathbb{F}[V]$. Therefore $\mathbb{F}[V]^H$ is an $\mathbb{F}[h_1, \ldots, h_n]$-module direct summand of $\mathbb{F}[V]$, and hence $\mathbb{F}[V]^H$ is also a free $\mathbb{F}[h_1, \ldots, h_n]$-module (see for example [11] section 6.1). Therefore

$$\mathbb{F}[V]^H \cong \mathbb{F}[h_1, \ldots, h_n] \otimes \frac{\mathbb{F}[V]^H}{(h_1, \ldots, h_n)}\,.$$

Taking Poincaré series gives

$$\begin{aligned} &\frac{\frac{1}{|H|}}{(1-t)^n} + \text{higher terms} \\ &\qquad = P(\mathbb{F}[V]^H,\ t) = P(\mathbb{F}[h_1, \ldots, h_n],\ t) \cdot P(\mathbb{F}[V]/(h_1, \ldots, h_n),\ t) \\ &\qquad = \left[\frac{\frac{1}{\deg(h_1)\cdots\deg(h_n)}}{(1-t)^n} + \text{higher terms}\right] \cdot P(\mathbb{F}[V]/(h_1, \ldots, h_n),\ t) \end{aligned}$$

Multiplying both sides by $(1-t)^n$ and evaluating at $t = 1$ we obtain

$$\frac{1}{|H|} = \frac{1}{\deg(h_1)\cdots\deg(h_n)} \cdot \dim_{\mathbb{F}}(\mathrm{Tot}(P(\mathbb{F}[V]/(h_1, \ldots, h_n)))$$

and the result follows. □

Proposition 2.13 (Kac-Peterson): *Let $\mathcal{S} \subset \mathrm{GL}(n,\ \mathbb{F})$ be a set of pseudoreflections and $H \subseteq G(\mathcal{S})$ a subgroup with $|H| \in \mathbb{F}^{\times}$. If $h_1, \ldots, h_n \in \mathbb{F}[V]$ is a regular sequence generating the ideal of generalized invariants $I(\mathcal{S})$ then $|H|$ divides $\prod_{i=1}^{n} \deg(h_i)$.*

Proof : $I(\mathcal{S})$ is stable under the action of H by 2.3 so the result follows from 2.12.
□

Corollary 2.14 (Kac-Peterson): *Suppose that $\mathbb{F}$ is a field of characteristic p. Let $\mathcal{S} \subset \mathrm{GL}(n,\ \mathbb{F})$ be a set of pseudoreflections and $h_1, \ldots, h_n \in \mathbb{F}[V]$ a regular sequence generating the ideal of generalized invariants $I(\mathcal{S})$. Then $\prod_{i=1}^{n} \deg(h_i)$ is divisible by $\frac{|G|}{p^{\nu_p(|G|)}}$, where $\nu_p(k)$ denotes the power of p in k.*

Proof : For each prime p' different from the characteristic of $\mathbb{F}$ the order $|\mathrm{Syl}_{p'}(G(\mathcal{S}))|$ of the p'-Sylow subgroup $\mathrm{Syl}_{p'}(G(\mathcal{S}))|$ of $G(\mathcal{S})$ divides $\prod_{i=1}^{n} \deg(h_i)$ by 2.13. Hence the greatest common divisor of these numbers divides $\prod_{i=1}^{n} \deg(h_i)$ by 2.13. But the greatest common divisor of $\left\{|\mathrm{Syl}_{p'}(G(\mathcal{S}))| \mid p' \neq p\right\}$ is $\frac{|G|}{p^{\nu_p(|G|)}}$. □

When $\mathbb{F}$ is a Galois field the ideal of generalized invariants relates well to the Steenrod operations. (See [11] § 10.2 – § 10.6 for a discussion of Steenrod operations and their relevance to invariant theory.)

Proposition 2.15 : *Let $s \in \mathrm{GL}(n,\ \mathbb{F}_q)$ be a pseudoreflection, $\ell_s \in \mathbb{F}_q[V]$ a linear form with* $\ker(\ell_s) = H_s$ *and $\Delta_s : \mathbb{F}_q[V] \longrightarrow \mathbb{F}_q[V]$ an associated twisted derivation. Define the algebra homomorphism*[4]

$$P(\xi) : \mathbb{F}_q[V] \longrightarrow \mathbb{F}_q[V][[\xi]]$$

by the requirement

$$P(\xi)(f) = \sum_{i=0}^{\infty} \mathcal{P}^i(f)\xi^i .$$

Then [5]

$$\Delta_s(P(\xi)(f)) = P(\xi)\,(\Delta_s(f))\,(1 + \ell_s^{q-1}\xi)\,,$$

where Δ_s is extended to $\mathbb{F}_q[V][[\xi]]$ by $\Delta_s(\sum f_i\xi^i) = \sum \Delta_s(f_i)\xi^i$.

---

[4] If A is a ring then $A[[\xi]]$ denotes the ring of formal power series over A in the variable ξ.

[5] As usual for $q = 2$ replace $\mathcal{P}^i(f)$ by $\mathrm{Sq}^i(f)$.) To consider $P(\xi)$ as a ring homomorphism of degree 0 we give ξ the degree $(1-q)$ and the elements of V the degree 1. (Since sign conventions for commutation rules play no role in what follows we employ the grading conventions preferred by algebraists and not those of topologists.)

Proof : The group $\mathrm{GL}(n,\ \mathbb{F}_q)$ acts on $\mathbb{F}_q[V]$ and on $\mathbb{F}_q[V][[\xi]]$ by acting trivially on ξ. The map $P(\xi) : \mathbb{F}_q[V] \longrightarrow \mathbb{F}_q[V][[\xi]]$ is $\mathrm{GL}(n,\ \mathbb{F}_q)$ equivariant and an algebra homomorphism. Therefore on the one hand

$$\begin{aligned} P(\xi)(\ell_s\Delta_s(f)) &= P(\xi)(sf - f) = P(\xi)(sf) - P(\xi)(f) \\ &= s(P(\xi)(f)) - P(\xi)(f)\,, \end{aligned}$$

while on the other hand

$$\begin{aligned} P(\xi)(\ell_s\Delta_s(f)) &= \Big(P(\xi)(\ell_s)\Big)\Big(P(\xi)(\Delta_s(f))\Big) \\ &= (\ell_s + \ell_s^q\xi)P(\xi)(\Delta_s(f))\,. \end{aligned}$$

Equating gives

$$s(P(\xi)(f)) - P(\xi)(f) = (\ell_s + \ell_s^q\xi)P(\xi)(\Delta_s(f))\,.$$

Dividing by ℓ_s yields

$$\Delta_s\left(P(\xi)(f)\right) = \frac{s(P(\xi)(f)) - P(\xi)(f)}{\ell_s} = (1 + \ell_s^{q-1}\xi)P(\xi)(\Delta_s(f))$$

as required. □

Corollary 2.16 : *Let $\mathcal{S} \subset \mathrm{GL}(n,\ \mathbb{F}_q)$ be a collection of pseudoreflections. Then the ideal $I(\mathcal{S}) \subset \mathbb{F}_q[V]$ is closed under the action of the Steenrod algebra.* □

§ 3. The Relation Between Stable and Generalized Invariants

Let $\mathcal{S} \subset \mathrm{GL}(n,\ \mathbb{F})$ be a set of pseudoreflections. In this section we investigate the relations between the ideals of generalized invariants $I(\mathcal{S})$, the ring of invariants $\mathbb{F}[V]^{G(\mathcal{S})}$, and the ideal of stable invariants $J_\infty(G(\mathcal{S}))$.

Lemma 3.1 : *Let $\mathcal{S} \subset \mathrm{GL}(n,\ \mathbb{F})$ be a set of pseudoreflections and $f \in \mathbb{F}[V]$. Then $f \in \mathbb{F}[V]^{G(\mathcal{S})}$ if and only $\Delta_s(f) = 0$ for all $s \in \mathcal{S}$.*

Proof : This follows from the fact that f is invariant under the cyclic subgroup generated by s if and only if $\Delta_s(f) = 0$ and the fact that $\mathcal{S}$ generates $G(\mathcal{S})$. □

Lemma 3.2 : *Let $\mathcal{S} \subset \mathrm{GL}(n,\ \mathbb{F})$ be a set of pseudoreflections. Then $I(\mathcal{S}) \subseteq J_\infty(G(\mathcal{S}))$. More specifically, if $f \in I(\mathcal{S})$ has degree d then $f \in J_d(G(\mathcal{S}))$.*

Proof : Choose $f \in I(\mathcal{S})$ and set $d = \deg(f)$. Then

$$\Delta_{s_1} \cdots \Delta_{s_d}(f) = 0 \quad \forall\ s_1, \ldots,\ s_d \in G(\mathcal{S})\,.$$

We claim that f belongs to $J_d(G(\mathcal{S}))$. To verify this we proceed by induction on $d = \deg(f)$. If $\deg(f) = 1$ and $f \in I(\mathcal{S})$ then $\Delta_s(f) = 0$ for all $s \in \mathcal{S}$ so by 3.1 $f \in \overline{\mathbb{F}[V]}^{G(\mathcal{S})} \subset$

$J_1(G(\mathcal{S}))$. If we assume $d > 1$ and the result established for all $h \in I$ with $\deg(h) < d$, then rewriting a bit gives

$$0 = \Delta_{s_1} \cdots \Delta_{s_d}(f) = (\Delta_{s_1} \cdots \Delta_{s_{d-1}})\big(\Delta_{s_d}(f)\big),$$

so by the inductive hypothesis $\Delta_{s_d}(f) \in J_{d-1}(G(\mathcal{S}))$. Hence

$$sf - f = \ell_s \Delta_s(f) \in J_{d-1}(G(\mathcal{S}))$$

for all $s \in \mathcal{S}$. Since $\mathcal{S}$ generates $G(\mathcal{S})$ it follows that

$$gf - f \in J_{d-1}(G(\mathcal{S})) \quad \forall\, g \in G(\mathcal{S})$$

so $f \in J_d(G(\mathcal{S}))$ by definition. □

LEMMA 3.3 : *Let $\mathcal{S} \subset \mathrm{GL}(n,\ \mathbb{F})$ be a set of semisimple pseudoreflections, i.e. each $s \in \mathcal{S}$ has order relatively prime to the characteristic of $\mathbb{F}$. Then $(\overline{\mathbb{F}[V]/I(\mathcal{S})})^{G(\mathcal{S})} = J_\infty(G(\mathcal{S}))/I(\mathcal{S})$.*

PROOF : Consider the map

$$q : J_\infty(G(\mathcal{S})) \longrightarrow \mathbb{F}[V]/I(\mathcal{S}).$$

If $q(f) \in \left(\overline{\mathbb{F}[V]/I(\mathcal{S})}\right)^{G(\mathcal{S})}$ then

$$gf - f \in I(\mathcal{S}) \quad \forall\, g \in G(\mathcal{S}).$$

By lemma 3.2 $I(\mathcal{S}) \subseteq J_\infty(G(\mathcal{S}))$ so

$$gf - f \in J_\infty(G(\mathcal{S})) \quad \forall\, g \in G(\mathcal{S})$$

and hence there is an $m \in \mathbb{N}$ with

$$gf - f \in J_m(G(\mathcal{S})) \quad \forall\, g \in G(\mathcal{S}).$$

But this says that $f \in J_{m+1}(G(\mathcal{S})) \subseteq J_\infty(G(\mathcal{S}))$ so $q(J_\infty(G(\mathcal{S})))$ contains $\left(\overline{\mathbb{F}[V]/I(\mathcal{S})}\right)^{G(\mathcal{S})}$ in positive degrees.

To prove the reverse inclusion we show that $q(J_m(G(\mathcal{S}))) \subset \left(\overline{\mathbb{F}[V]/I(\mathcal{S})}\right)^{G(\mathcal{S})}$ by induction on m. By definition $J_1(G(\mathcal{S})) = (\overline{\mathbb{F}[V]^G}) \subset \mathbb{F}[V]$. Since $\mathbb{F}[V]^G \subset I(\mathcal{S})$ it follows $J_1(G(\mathcal{S})) \subseteq I(\mathcal{S})$ beginning the induction. Assume next that $f \in J_m(G(\mathcal{S}))$. Then $gf - f \in J_{m-1}(G(\mathcal{S}))$ and so by the induction hypothesis

$$q(f - gf) \in \left(\mathbb{F}[V]/I(\mathcal{S})\right)^G.$$

Hence

$$f - gf - h(f - gf) \in I(\mathcal{S}) \quad \forall\, g,\ h \in G$$

Let $s \in \mathcal{S}$ have order k. We have

$$\begin{aligned}(f - sf) - s(f - sf) &\in I(\mathcal{S}) \qquad (g = s,\ h = s)\\ (f - s^2 f) - s(f - s^2 f) &\in I(\mathcal{S}) \qquad (g = s^2,\ h = s)\\ &\vdots\\ (f - s^{k-1} f) - s(f - s^{k-1} f) &\in I(\mathcal{S}) \qquad (g = s^{k-1},\ h = s)\end{aligned}$$

and adding gives

$$kf - ksf = k(f - sf) \in I(\mathcal{S})\,.$$

Since k is invertible in $\mathbb{F}$ it follows $f - sf \in I(\mathcal{S})$ for all $s \in \mathcal{S}$. As $\mathcal{S}$ generates $G(\mathcal{S})$ it follows that $f - gf \in I(\mathcal{S})$ for all $g \in G$, i.e. $J_m(G(\mathcal{S})) \subseteq I(\mathcal{S})$ completing the induction step and hence the proof. □

THEOREM 3.4 : *Let $\mathcal{S} \subset \mathrm{GL}(n,\ \mathbb{F})$ be a set of semisimple pseudoreflections. Then $I(\mathcal{S}) = J_\infty(G(\mathcal{S}))$.*

PROOF : By 3.3 $\left(\overline{\mathbb{F}[V]/I(\mathcal{S})}\right)^{G(\mathcal{S})} = J_\infty(G(\mathcal{S}))/I(\mathcal{S})$. Let $\varphi \in (\mathbb{F}[V]/I(\mathcal{S}))^{G(\mathcal{S})}$ with $\deg(\varphi) > 0$. Choose $f \in \mathbb{F}[V]$ with $q(f) = \varphi$, where $q : \mathbb{F}[V] \longrightarrow \mathbb{F}[V]/I(\mathcal{S})$ is the quotient map. For $s \in \mathcal{S}$ let $<s>$ denote the cyclic subgroup of $G(\mathcal{S})$ generated by s. Since the order of s is invertible in $\mathbb{F}$ the projection operator

$$\pi^{<s>} = \frac{1}{|s|}\sum_{i=1}^{|s|} s^i$$

is defined. From the invariance of φ it follows that $\pi^{<s>}(\varphi) = \varphi$. This lifts to $\mathbb{F}[V]$ to give

$$(*) \qquad f - \pi^{<s>}(f) \in I(\mathcal{S})$$

Since $\pi^{<s>}(f) \in \mathbb{F}[V]^{<s>}$, i.e. $\pi^{<s>}(f)$ is $<s>$-invariant, it follows that $\Delta_s(\pi^{<s>}(f)) = 0$. Applying Δ_s to the equation $(*)$ and applying lemma 2.2 we obtain $\Delta_s(f) \in I(\mathcal{S})$. Since this holds for all $s \in \mathcal{S}$ it follows from lemma 2.2 that $f \in I(\mathcal{S})$ and hence that $\varphi = 0$. □

EXAMPLE 3.5 : Let p be an odd prime and $\mathbb{F}$ a field of characteristic p. The matrix

$$S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in \mathrm{GL}(2,\ \mathbb{F})$$

is a (nondiagonalizable) pseudoreflection of order p, so affords a representation of $\mathbb{Z}/p$. By [11] § 5.6 example 3

$$\mathbb{F}[x,\ y]^{\mathbb{Z}/p} = \mathbb{F}[x,\ y(y^{p-1} - x^{p-1})]\,.$$

By proposition 1.8 the ideal of stable invariants $J_\infty(\mathbb{Z}/p)$ is $(x,\ y)$.

To compute the ideal $I(s(\mathbb{Z}/p))$ of generalized invariants we may use 2.4 to replace $s(\mathbb{Z}/p)$ by $\{S\}$. Note that $H_S = \mathrm{Span}_{\mathbb{F}}\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \subset \mathbb{F}^2 = V$. The linear form

$$x = \ell_S : V \longrightarrow \mathbb{F} \qquad \ell_s\begin{pmatrix}u\\v\end{pmatrix} = u$$

has kernel H_s and therefore

$$\Delta_S(f) = \frac{Sf - f}{x}$$

for any $f \in \mathbb{F}[V]$. The action of S on V^* is given by

$$S(x) = x \qquad S(y) = x + y\,.$$

Hence

$$\Delta_S(x) = \frac{S(x) - x}{x} = 0$$
$$\Delta_S(y) = \frac{S(y) - y}{x} = \frac{x + y - y}{x} = 1\,.$$

Hence $y \notin I(s(\mathbb{Z}/p))$ so $I(s(\mathbb{Z}/p)) \neq J_\infty(\mathbb{Z}/p)$. The twisted derivation formula for Δ_S yields

$$\begin{aligned}\Delta_S(y^k) &= \Delta_S(y)\left(y^{k-1} + y^{k-2}S(y) + \cdots + S(y)^{k-1}\right)\\ &= y^{k-1} + y^{k-2}(x+y) + \cdots + (y+x)^{k-1}\,,\end{aligned}$$

from which it follows that

$$\underset{\leftarrow\ i\ \rightarrow}{\Delta_s \cdots \Delta_s}(y^i) = \begin{cases} i! \neq 0 & \text{for } i < p \\ 0 & \text{for } i = p.\end{cases}$$

Therefore $I(s(\mathbb{Z}/p)) = I(\{S\}) = (x,\ y^p) \neq (x,\ y) = J_\infty(\mathbb{Z}/p)$. Hence theorem 3.4 does not hold if the elements of $\mathcal{S}$ are not of order relatively prime to the characteristic of the ground field. In particular, it does not hold for the mod 2 reduction of the defining representation of a crystallographic group, i.e. a Weyl group. (See also the discussion of the dihedral group in examples 2.7 and 2.8.)

Corollary 3.6 : *Let $\mathcal{S} \subset \mathrm{GL}(n,\ \mathbb{F})$ be a set of semisimple pseudoreflections. Then $I(\mathcal{S}) = (h_1, \ldots, h_n) = J_\infty(G(\mathcal{S}))$ where $h_1, \ldots, h_n \in \mathbb{F}[V]$ is a regular sequence and moreover $\frac{|G|}{p^{\nu_p(|G|)}}$ divides $\prod_{i=1}^{n} \deg(h_i)$.*

Proof : This follows from 2.9, 3.4 and 1.6. □

Corollary 3.7 : *Let $\varrho : G \hookrightarrow \mathrm{GL}(n,\ \mathbb{F})$ be a representation of a finite group which may be generated by pseudoreflections of order relatively prime to the characteristic of $\mathbb{F}$. If $\mathcal{S}',\ \mathcal{S}'' \subset G$ are sets of semisimple pseudoreflections generating G then $I(\mathcal{S}') = I(\mathcal{S}'')$.* □

COROLLARY 3.8 : *Let $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of a finite group which may be generated by pseudoreflections of order relatively prime to the characteristic of $\mathbb{F}$. Let $h_1, \ldots, h_n \in J_\infty(G)$ be a regular sequence generating the ideal of stable invariants and let $d = \max\{\deg(h_1), \ldots, \deg(h_n)\}$. Then $J_d(G) = \cdots = J_\infty(G)$.*

PROOF : By theorem 3.4 $J_\infty(G) = I(\mathcal{S})$ for any set $\mathcal{S} \subset G$ of pseudoreflections of orders prime to the characteristic of $\mathbb{F}$ which generate G. By 3.2 $h_1, \ldots, h_n \in J_d(G(\mathcal{S}))$ and the result follows. □

COROLLARY 3.9 : *Suppose that $\mathbb{F}$ is a field of characteristic not equal to 2 and $G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ a representation of a finite group G as a Coxeter group. Let $\mathcal{S} \subset G$ be any subset of reflections of order 2 generating G. Then*

(i) $I(\mathcal{S}) = J_\infty(G)$

(ii) *G acts fixed point freely on* $\overline{\mathbb{F}[V]/I(\mathcal{S})}$.

(iii) $I(\mathcal{S}) = (h_1, \ldots, h_n) = J_\infty(G(\mathcal{S}))$ *where* $h_1, \ldots, h_n \in \mathbb{F}[V]$ *is a regular sequence and*

(iv) $\frac{|G|}{p^{\nu_p(|G|)}}$ *divides* $\prod_{i=1}^{n} \deg(h_i)$. □

§ 4. Rings of Stable and Generalized Invariants

Let $\mathcal{S} \subset \mathrm{GL}(n, \mathbb{F})$ be a set of pseudoreflections and $I(\mathcal{S})$ the ideal of generalized invariants of $\mathcal{S}$. By 2.9 $I(\mathcal{S})$ is generated by a regular sequence $h_1, \ldots, h_n \in \mathbb{F}[V]$. It is not clear what if any relation there is between the **subalgebra** $\mathbb{F}[h_1, \ldots, h_n]$ of $\mathbb{F}[V]$ generated by $h_1, \ldots, h_n$ and $\mathbb{F}[V]^{G(\mathcal{S})}$, or even if there is any invariant meaning to be assigned to $\mathbb{F}[h_1, \ldots, h_n]$. In this section we show how to associate to $\mathcal{S}$ a **ring** of generalized invariants, and a **ring** of stable invariants that have invariant meaning.

CONSTRUCTION : Let A be a graded connected commutative algebra over a field $\mathbb{F}$ and $I \subset A$ a proper ideal. Introduce the filtration of A

$$A \supset I \supseteq I^2 \supseteq \cdots \supseteq I^m \supseteq \cdots$$

by the powers of the ideal I. Denote by $\mathrm{gr}_I(A)$ the associated graded algebra. This is a bigraded algebra with

$$\mathrm{gr}_I(A)_{m,\ n-m} = \left(I^m/I^{m+1}\right)_n$$

for $n,\ m \in \mathbb{N}$, where we have used the convention that $I^0 = A$. The algebra $\mathrm{gr}_I(A)$ may be totalized to yield a positively graded commutative algebra over $\mathbb{F}$. In the sequel $\mathrm{gr}_I(A)$ denotes this totalization. Note that $A/I \subset \mathrm{gr}_I(A)$ is a subalgebra, so we may regard $\mathrm{gr}_I(A)$ as an algebra over A/I. Finally we set $\overline{\mathrm{gr}}_I(A) = \mathbb{F} \otimes_{A/I} \mathrm{gr}_I(A)$, which is a graded *connected* commutative algebra over $\mathbb{F}$.

If $a \in A$ then there is a largest integer $m(a)$ such that $a \in I^m$ but $a \notin I^{m+1}$. In this case we say that a **has filtration** $m(a)$. If $a \in A$ has filtration m and degree n then we say a **represents the element** $a + I^{m+1} \in \mathrm{gr}_I(A)_{m,n-m}$ of total degree n. If $a \in A$ has degree n and filtration m and $b \in A$ has degree k and filtration j then

$$(a + I^{m+1}) \cdot (b + I^{j+1}) = ab + I^{m+j+1} \in \mathrm{gr}_I(A)\,,$$

so the product lies in $\mathrm{gr}_I(A)_{m+j,\ n+k-(m+j)}$, and has total degree $n + k$.

Proposition 4.1 : *Let A be a graded commutative algebra over a field $\mathbb{F}$ and $I \subset A$ a proper ideal. Suppose that $I = (x_1, \ldots, x_n)$. Then $x_1, \ldots, x_n$ is a regular sequence in A if and only if $\mathrm{gr}_I(A) = A/I[\bar{x}_1, \ldots, \bar{x}_n]$ where $\bar{x}_1, \ldots, \bar{x}_n \in I/I^2 \subset \mathrm{gr}_I(A)$ denote the residue classes of $x_1, \ldots, x_n$.*

Proof : By induction on n. Consider the case $n = 1$, i.e., $x = x_1 \in A$ is a nonzero divisor of positive degree. The ideal I^m is then the principal ideal $(x^m) \subset A$ for $m > 0$. Define $\varphi : A \longrightarrow I^m$ by $\varphi(a) = a \cdot x^m$. This map satisfies $\varphi(I) \subset I^{m+1}$ and hence induces

$$\bar{\varphi} : A/I \longrightarrow I^m/I^{m+1}$$

which is an epimorphism. If $a + I \in \ker\bar{\varphi}$, then by definition, $ax^m \in (x^{m+1})$, so $ax^m = bx^{m+1}$ for some $b \in A$. Hence

$$0 = ax^m - bx^{m+1} = x^m(a - bx)\,.$$

Since $x^m \in A$ is not a zero divisor it follows $a = bx$ so $a + I = 0 \in A/I$. Hence $\bar{\varphi}$ is a monomorphism. Therefore the map

$$\Theta : (A/I)[\bar{x}] \longrightarrow \mathrm{gr}_I(A)$$

defined by

$$\Theta((a + I) \cdot \bar{x}^m) = a\bar{x}^m, \qquad m \in \mathbb{N}$$

is an isomorphism, establishing the result for $n = 1$.

Let $n \in \mathbb{N}$ and assume the result established for all ideals generated by a regular sequence of length less than n. Let $I = (x_1, \ldots, x_n)$ where $x_1, \ldots, x_n \in A$ is a regular sequence. Set $J = (x_1, \ldots, x_{n-1})$. By the induction hypothesis

$$\mathrm{gr}_J(A) = (A/J)[\bar{x}_1, \ldots, \bar{x}_{n-1}]\,.$$

The element $x_n + J \in A/J$ is a nonzero divisor, so if $K = (x_n + J) \subset \mathrm{gr}_J(A)$ is the principal ideal generated by $x_n + J$, then

$$\mathrm{gr}_I(A) \cong \mathrm{gr}_K(\mathrm{gr}_J(A)) \cong \mathrm{gr}_K\left((A/J)[\bar{x}_1, \ldots, \bar{x}_{n-1}]\right) \cong (A/I)[\bar{x}_1, \ldots, \bar{x}_n]$$

completing the induction step.

We turn next to the converse. Let $x_1, \ldots, x_n \in A$ and $(x_1, \ldots, x_n) \subset A$ the corresponding ideal. Suppose

$$\mathrm{gr}_I(A) = (A/I)[\bar{x}_1, \ldots, \bar{x}_n]$$

where $\bar{x}_1, \dots, \bar{x}_n \in I/I^2$ denote the residue classes of $x_1, \dots, x_n \in I$. We must show that $x_1, \dots, x_n \in A$ is a regular sequence.

Consider the Koszul complex (see for example [11] § 5.2)

$$\mathcal{K} = A \otimes E[u_1, \dots, u_n]$$

where $\partial(u_i) = x_i$ for $i = 1, \dots, n$. By Koszul's theorem ([11] theorem 6.2.3) we need to show that $\{\mathcal{K}, \partial\}$ is acyclic. To this end we filter $\mathcal{K}$ as follows: give A the filtration by the powers of the ideal I, give $E[u_1, \dots, u_n]$ the trivial filtration, and $\mathcal{K}$ the product filtration. In this way $\{\mathcal{K}, \partial\}$ becomes a filtered differential algebra. Let $\{E_r, d_r\}$ denote the associated spectral sequence. Since the connectivity of I^m goes to infinity with m the spectral sequence converges to $H_*(\mathcal{K}, \partial)$. From the definitions we have

$$E_0 = \mathrm{gr}_I(A) \otimes E[u_1, \dots, u_n] = (A/I)\big[\bar{x}_1, \dots, \bar{x}_n\big] \otimes E[u_1, \dots, u_n]$$

where

$$d_0(u_i) = \bar{x}_i \qquad \text{for } i = 1, \dots, n\,.$$

By Koszul's theorem ([11] theorem 6.2.3) we have $\{E_0, d_0\}$ is acyclic, and

$$(E_1)_{p,\,*} = \begin{cases} H_0(E_0, d_0) = A/I & : p = 0 \\ 0 & : \text{otherwise.} \end{cases}$$

Hence the term E_1 is concentrated on the filtration zero line, so the spectral sequence collapses to yield the isomorphism

$$H_i(\mathcal{K}, \partial) = \begin{cases} A/I & : i = 0 \\ 0 & : \text{otherwise} \end{cases}$$

as required. □

THEOREM 4.2 : *Let $\mathcal{S} \subset \mathrm{GL}(n, \mathbb{F})$ be a set of pseudoreflections and $I(\mathcal{S})$ the corresponding ideal of generalized invariants. Then*

$$\mathrm{gr}_{I(\mathcal{S})}(\mathbb{F}[V]) = (\mathbb{F}[V]/I(\mathcal{S}))\left[\bar{h}_1, \dots, \bar{h}_n\right]$$
$$\overline{\mathrm{gr}}_{I(\mathcal{S})}(\mathbb{F}[V]) = \mathbb{F}[\bar{h}_1, \dots, \bar{h}_n]\,.$$

PROOF : This is immediate from 2.9 and 4.1. □

If $\mathcal{S} \subset \mathrm{GL}(n, \mathbb{F})$ is a collection of pseudoreflections then $\mathbb{F}[V]/I(\mathcal{S})$ is a Poincaré duality algebra (by e.g. [13] proposition 3 or [11] theorem 6.5.1). The algebra $\overline{\mathrm{gr}}_{I(\mathcal{S})}(\mathbb{F}[V])$ depends only on $\mathcal{S}$ and is called the **ring of generalized invariants of $\mathcal{S}$**. By 4.2

$$\overline{\mathrm{gr}}_{I(\mathcal{S})}(\mathbb{F}[V]) = \mathbb{F}[\bar{h}_1, \dots, \bar{h}_n]$$

and although the polynomials $\bar{h}_1, \dots, \bar{h}_n$ need not be well defined their degrees are, and we call $\deg(\bar{h}_1), \dots, \deg(\bar{h}_n)$ the **generalized fundamental degrees of $\mathcal{S}$**.

Likewise for a representation $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ of a finite group G we call $\overline{\mathrm{gr}}_{J_\infty(G)}(\mathbb{F}[V])$ the **ring of stable invariants of G**. From 3.4 we obtain:

THEOREM 4.3 : *Let $\varrho : G \hookrightarrow \mathrm{GL}(n,\ \mathbb{F})$ be a representation of a finite group G. If G is generated by pseudoreflections of order relatively prime to the characteristic of $\mathbb{F}$ then*

$$\mathrm{gr}_{J_\infty(G)}(\mathbb{F}[V]) = (\mathbb{F}[V]/J_\infty(G))\left[\bar{h}_1, \ldots, \bar{h}_n\right]$$
$$\overline{\mathrm{gr}}_{J_\infty(G)}(\mathbb{F}[V]) = \mathbb{F}[\bar{h}_1, \ldots, \bar{h}_n]\,. \quad \square$$

EXAMPLE 4.4 ([11] § 8.2): Let $\mathbb{F}$ be a field of characteristic $p \neq 0$ and for $n \in \mathbb{N}$ consider the matrices

$$T_i = \begin{bmatrix} 1 & 0 & & \cdots & 0 \\ 0 & 1 & & 0\cdots & 0 \\ \vdots & & \ddots & \cdots & \\ 0 & \cdots & & 0\cdots 1 & 0 \\ 0 & \cdots & & 1\cdots 0 & 1 \end{bmatrix} \in \mathrm{GL}(n,\ \mathbb{F}) \quad i = 1, \ldots, n-1\,,$$

where the off diagonal 1 is in the i^{th} column of the last row. In other words, if $E_{i,j}$ denotes the $n \times n$ matrix with a 1 in the i^{th} row and j^{th} column, and zeros elsewhere for $i,\ j \in \{1, \ldots, n\}$, then $T_i = I + E_{n,\ i}$ for $i = 1, \ldots, n-1$. These matrices are transvections with the common direction $E_n = (0,\ 0, \ldots, 1)$. Since the matrices $T_1, \ldots, T_{n-1}$ commute with each other and all have order p, they afford a faithful representation of $E(n-1)$, an elementary abelian p-group of order p^{n-1}.

Let $\zeta \in \mathbb{F}^\times$ have order s, in other words $\zeta \in \mathbb{F}$ is a primitive s-th root of unity, and let $H(\zeta, r) \subset \mathrm{GL}(n,\ \mathbb{F})$ denote the subgroup generated by $T_1, \ldots, T_r$ and the matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \zeta \end{bmatrix} \in \mathrm{GL}(n,\ \mathbb{F})\,.$$

(This is the group denoted by $G(\zeta,\ r)^*$ in [11] § 8.2.) The groups $H(\zeta,\ r)$ are up to conjugation a complete list of subgroups of $\mathrm{GL}(n,\ \mathbb{F})$ satisfying $\dim(V_H) = n-1$, where

$$V_H = V/\mathrm{Span}_{\mathbb{F}}\{hv - v \mid h \in H,\ v \in V\}$$

is the module of covariants ([11] § 8.2) of the group $H \subset \mathrm{GL}(n,\ \mathbb{F})$. The invariant theory of the groups $H(\zeta,\ r)$ is computed in [11] proposition 8.2.13, where it is shown

$$\mathbb{F}[z_1, \ldots, z_n]^{H(\zeta,\ r)} = \mathbb{F}[z_1, \ldots, z_{n-1},\ f^s]$$

where

$$f = \prod_{\lambda \in \mathbb{F}_p} (z_n + a_r z_r + \cdots a_1 z_1) = c_{p^r}([z_i])$$

is the top Chern class ([15], [11] § 3.1 and § 3.2) of the orbit of z_n. (In the preceding product $\mathbb{F}_p < \mathbb{F}$ is the prime subfield and $z_1, \ldots, z_n \in V^*$ is the dual of the standard basis of $V = \mathbb{F}^n$.) Therefore

$$\mathbb{F}[z_1, \ldots, z_n]_{H(\zeta,\ r)} = \frac{\mathbb{F}[\bar{z}_n]}{(\bar{z}_n^{sp^r})}\,.$$

The operation of the group $H(\zeta, r)$ on V^* is given by

$$T_i(z_j) = \begin{cases} z_j & : j \neq n \\ z_n + z_i & : j = n \end{cases}$$

$$T_\zeta(z_j) = \begin{cases} z_j & : j \neq n \\ \zeta z_n & : j = n. \end{cases}$$

Therefore the action on the coinvariants is given by

$$T_i(\bar{z}_n) = \bar{z}_n \qquad i = 1, \ldots, r$$

$$T_\zeta(\bar{z}_n) = \zeta \bar{z}_n$$

where $\bar{z}_n \in \mathbb{F}[V]_{H(\zeta,\ r)}$ is the residue class of $z_n \in \mathbb{F}[V]$. From this it follows that

$$\left(\mathbb{F}[V]_{H(\zeta,\ r)}\right)^{H(\zeta,\ r)} \cong \frac{\mathbb{F}[\bar{z}_n^s]}{(\bar{z}_n^{sp^r})}$$

and hence

$$\mathbb{F}[V]_{H(\zeta,r)^2} \cong \frac{\mathbb{F}[\bar{z}_n]}{(\bar{z}_n^s)}\,.$$

Therefore

$$J_i(H(\zeta,\ r)) = \begin{cases} (z_1, \ldots,\ z_{n-1},\ z_n^{sp^r}) & : i = 1 \\ (z_1, \ldots,\ z_{n-1},\ z_n^s) & : i = 2,\ 3, \ldots, \end{cases}$$

and the ideal $J_\infty(H(\zeta,\ r))$ coincides with $J_2(H(\zeta,\ r))$, i.e.,

$$J_\infty(H(\zeta,\ r)) = (z_1, \ldots,\ z_{n-1},\ z_n^s)\,.$$

Let $D_i = T_\zeta T_i \in H(\zeta,\ r) \subset \mathrm{GL}(n,\ \mathbb{F})$ for $i = 1, \ldots, n-1$. Then direct computation shows that

$$(D_i)^k = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0\cdots & 0 \\ \vdots & \ddots & \cdots & \\ 0 & \cdots & \Phi_k(\zeta)\cdots 0 & \zeta^k \end{bmatrix} = T_\zeta^k + \Phi_k(\zeta)E_{n,\ i} \qquad i = 1, \ldots, n-1$$

where

$$\Phi_k(\zeta) = \zeta + \zeta^2 + \cdots + \zeta^k\,.$$

Since ζ is a primitive s-th root of unity

$$0 = \zeta^s - 1 = (\zeta - 1)(! + \zeta + \zeta^2 + \cdots + \zeta^{s-1})\,.$$

Therefore $\Phi_s(\zeta) = 0$ and D_i has order s for $i = 1, \ldots, n-1$. The matrix D_i fixes the codimension one subspace $\mathbb{F}^{n-1} \subset \mathbb{F}^n = V$ spanned by the first $n-1$ standard basis vectors. Hence D_i is a pseudoreflection. Therefore $H(\zeta,\ r)$ is generated by the pseudoreflections $\mathcal{D} = \{D_1, \ldots,\ D_r,\ T_\zeta\}$, all of order s, and s is relatively prime to p, which is the characteristic of $\mathbb{F}$. Hence by 3.4 $I(\mathcal{D}) = J_\infty(H(\zeta,\ r))$ and

$$\mathrm{gr}_{I(\mathcal{D})}(\mathbb{F}[V]) = \frac{\mathbb{F}[\bar{z}_n]}{(\bar{z}_n^s)}\left[\bar{z}_1, \ldots,\ \bar{z}_{n-1},\ \bar{z}_n^s\right] = \overline{\mathrm{gr}}_{J(H(\zeta,\ r))}(\mathbb{F}[V])$$

$$\overline{\mathrm{gr}}_{I(\mathcal{D})}(\mathbb{F}[V]) = \mathbb{F}[\bar{z}_1, \ldots,\ \bar{z}_{n-1},\ \bar{z}_n^s] = \overline{\mathrm{gr}}_{J(H(\zeta,\ r))}(\mathbb{F}[V])$$

Note that

$$\overline{\mathrm{gr}}_{I(\mathcal{D})}(\mathbb{F}[V]) = \mathbb{F}[\bar{z}_1, \ldots, \bar{z}_{n-1}, \ \bar{z}_n]^{\mathbb{Z}/s} = \overline{\mathrm{gr}}_{J(H(\zeta,\ r))}(\mathbb{F}[V])$$

where $\mathbb{Z}/s$ acts on $\mathbb{F}[\bar{z}_1, \ldots, \bar{z}_n]$ via the representation implemented by T_ζ.

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Frank Neumann, Larry Smith
Mathematisches Institut
Bunsenstraße 3 – 5
Universität Göttingen
D 37073 Göttingen
neumann@cfgauss.uni-math.gwdg.de
larry@cfgauss.uni-math.gwdg.de

Mara D. Neusel
Institut für Algebra und Geometrie
Postfach 4120
Universität Magdeburg
D 39016 Magdeburg
mara.neusel@mathematik.uni-magdeburg.de