Connected Hopf Algebras with Dixmier Bases and Infinite Primary Decomposition

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SUMMARY : *In this paper we show the existence of invariant primary decompositions in the categories of modules and rings over a Hopf algebra of Dixmier type.*

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Let H be a Hopf algebra over a field $\mathbb{K}$ and R a commutative $\mathbb{K}$-algebra with an H-module structure. Let $I \subseteq R$ be an ideal. Then I is called **invariant** if

$$\mathrm{H}(I) \subseteq I.$$

Assume that I has a (possibly infinite) primary decomposition

$$I = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \mathfrak{q}_3 \cap \cdots.$$

In this paper we show that under certain assumptions on H an invariant ideal with a primary decomposition has an **invariant primary decomposition**, i.e.,

$$I = \mathfrak{q}_1' \cap \mathfrak{q}_2' \cap \mathfrak{q}_3' \cap \cdots,$$

where the $\mathfrak{q}_i'$'s and $Rad\,(\mathfrak{q}_i')$'s are invariant ideals for all i.

In the special case where $\mathrm{H} = P$ is the mod-p-Steenrod algebra and R is an unstable noetherian $\mathbb{K}$-algebra over P the existence of invariant primary decompositions was established in [6]. This was extended to unstable noetherian $R \odot P$-modules in [4] (see also [2]). This was further generalized to *non*noetherian unstable $R \odot P$-modules (R still noetherian) in the sense that if an unstable module admits a *finite* primary decomposition then it admits an invariant (still finite) primary decomposition, see [5].

In [9] arbitrary pointed Hopf algebras are considered. It is shown that in the categories of commutative noetherian $\mathbb{K}$-algebras R and noetherian (H, R)-modules invariant primary decompositions exist.

Finally, [10] deals with pointed Hopf algebras over a field $\mathbb{K}$ of characteristic zero and *non*-commutative noetherian rings R over it. It is shown that the nilradical of R as well as all minimal primes are invariant.

In this paper we come back to the study of commutative rings R and modules M, but we drop any finiteness assumption. In particular, neither R nor M need to be noetherian. We assume that H is a Hopf algebra of Dixmier type. We show that if an invariant ideal (or a module) admits a (possibly infinite) primary decomposition, then it admits an invariant primary decomposition. In Section 1 we define Hopf algebras of Dixmier type, and present various examples. In Section 2 we introduce the J_D-functor that turns arbitrary ideals (or modules) into invariant ones. We proceed with the verification of several properties of J_D which leads to the proof of the desired result in Theorem 2.12. Finally in Section 3 we translate these results into the context of modules, and obtain the existence of invariant primary decompositions in Theorem 3.1.

§1. Dixmier Bases for Hopf Algebras

In [10] it is shown that every connected Hopf algebra over a field of characteristic zero is a quotient of a Hopf algebra with Dixmier basis. In this section we extend this result to positive characteristic.

We recall the definition of Dixmier basis, see [10].

DEFINITION: *Let* H *be a Hopf algebra over a field* $\mathbb{K}$. *Denote by* Δ *the comultiplication. We say that the subset* $D \subseteq \mathrm{H}$ *is a* **Dixmier basis** *for* H, *if*

(1) *D is a $\mathbb{K}$-linear basis for* H.
(2) *D is well ordered by some ordering “<”.*

(3) *There exists a multiplication*

$$D \times D \longrightarrow D, \ (d, t) \longmapsto d \odot t$$

such that

$$(\text{⌘}) \qquad \Delta(d \odot t) = \lambda\, d \otimes t + \sum_{d' < d} d' \otimes t'' + \sum_{t' < t} d'' \otimes t'$$

for some non zero scalar $\lambda \in \mathbb{K}$.

We call the property (⌘) *the* **Dixmier Property**.

The following example is taken from [10], Example 4.

EXAMPLE 1.1 : Let H = $\mathbb{K}[t]$ the algebra of polynomial in one variable t over a field $\mathbb{K}$. The comultiplication is given by

$$\Delta(t) = t \otimes 1 + 1 \otimes t.$$

If the field $\mathbb{K}$ has characteristic zero, then we can choose

$$D = \{1, t, t^2, \ldots\}$$

as a Dixmier basis with multiplication

$$t^i \odot t^j = t^{i+j}.$$

The set D is ordered in the obvious way: $t^i < t^j$ if and only if $i < j$. If the characteristic of $\mathbb{K}$ is $p > 0$, then $\mathbb{K}[t]$ admits no Dixmier basis as we see next. Since D is a linear basis it must contain $\mu_i t^i + M_i$, $\mu_i \in \mathbb{K} \setminus 0$ and some $M_i \in \mathbb{K}[t]$, for all $i \in \mathbb{N}_0$. Thus

$$t^i \odot t^{p-i} = \sum_{k=1}^{n} \lambda_k t^{m_k}$$

for some $m_k \in \mathbb{N}_0$ and $\lambda_k \in \mathbb{K}$. Then

$$\Delta(t^i \odot t^{p-i}) = \Delta(\sum_{k=1}^{n} \lambda_k t^{m_k}) = \sum_{k=1}^{n} \lambda_k \sum_{j=0}^{m_k} \binom{m_k}{j} t^j \otimes t^{m_k - j}.$$

Since $t^i \otimes t^{p-i}$ must be a nontrivial summand in the sum on the right, we have that $m_k = p$ for certain k. However this gives

$$\Delta(t^i \odot t^{p-i}) = \Delta(t^p + \sum_{k=1, x_k \neq p}^{n} \lambda_k t^{m_k}) = 1 \otimes t^p + t^p \otimes 1 + \Delta(\sum_{k=1, m_k \neq p}^{n} \lambda_k t^{m_k}).$$

Thus $t^i \otimes t^{p-i}$ does not occur as a nontrivial summand in $\Delta(t^i \otimes t^{p-i})$, and hence in positive characteristic $\mathbb{K}[t]$ does not admit a Dixmier basis.

We can see this also in the following way:

EXAMPLE 1.2 : Consider the truncated polynomial algebra

$$R = \mathbb{K}1 + \mathbb{K}x + \mathbb{K}x^2 + \cdots + \mathbb{K}x^{p-1}$$

over a field of characteristic p. Then the Hopf algebra H = $\mathbb{K}[t]$ acts on R via

$$t(x) = 1.$$

The nil radical of R is

$$Nil\,(R) = (x).$$

If H had a Dixmier basis then the nil radical of R would be invariant under the action of H, see Theorem 4.3 in [10]. However $1 \notin Nil\,(R)$, cf. Examples 3 and 4 in [10].

More generally we cite the following result.

THEOREM 1.3: *If* H *is a connected Hopf algebra over a field of characteristic zero, then* H *is a quotient of a Hopf algebra with Dixmier basis.*

PROOF: Theorem 12 in [10]. ☯

We need a similar result for Hopf algebras over fields of positive characteristic. For this we start with the following construction.

Let $\mathbb{K}$ be a field of any characteristic. Denote by

$$\mathrm{H}_\infty = \mathbb{K} < h_1,\ h_2,\ h_3,\ \ldots >$$

the free Hopf algebra on the h_i's over $\mathbb{K}$ with comultiplication given by

$$\Delta(h_k) = \sum_{i=0}^{k} h_i \otimes h_{k-i},$$

where $h_0 = 1$. Note that this is a cocommutative Hopf algebra. We want to show that the set D consisting of all monomials in the h_i's is a Dixmier basis for H_∞. Obviously D is a $\mathbb{K}$-linear basis for H. Next we need to define an order on D.

DEFINITION: *Let*

$$d = h_{i_1} \cdots h_{i_k} \in D$$

be a monomial. The **special degree** *of d is defined by*

$$\mathrm{spdeg}(d) = i_1 + \cdots + i_k.$$

We denote the **length** *of d by*

$$l(d) = k.$$

With the help of these two degrees associated to a monomial $d \in D$ we define a well order on D as follows.

DEFINITION: *Let d and d' be elements in D. We say that $d < d'$ if one of the following statements is true:*

(1) $\mathrm{spdeg}(d) < \mathrm{spdeg}(d')$ *or*
(2) $\mathrm{spdeg}(d) = \mathrm{spdeg}(d')$ *and* $l(d) < l(d')$ *or*
(3) $\mathrm{spdeg}(d) = \mathrm{spdeg}(d')$ *and* $l(d) = l(d')$ *and* $d <_{\mathrm{lex}} d'$. [1]

LEMMA 1.4: *The set D of all monomials is well ordered by "$<$".*

PROOF: It is obvious that any two elements in D are comparable. To show that any nonempty subset has a least element, pick a chain

$$d_0 > d_1 > d_2 > \cdots$$

with $d_i \in D$ for all i. Since the special degree $\mathrm{spdeg}(d_0)$ is finite there are only finitely many d_i's in the chain of smaller special degree. Thus without loss of generality we can assume that

$$\mathrm{spdeg}(d_i) = \mathrm{spdeg}(d_j) \quad \forall\ i,\ j.$$

Similary, the length of d_0 is finite, and so without loss of generality we assume that

$$l(d_i) = l(d_j) \quad \forall\ i, j.$$

[1] "$<_{\mathrm{lex}}$" denotes the lexicographic order.

Thus

$$d_i >_{\text{lex}} d_{i+1} \quad \forall\ i.$$

Since the lexicographic order turns the set of monomials into a well ordered set, we are done. ☯

DEFINITION: *Let d and t be elements in D with $l(d) \leq l(t)$. We assume without loss of generality that*

$$d = h_{i_1} \cdots h_{i_k}$$

and

$$t = h_{m_1} \cdots h_{m_p} h_{j_1} \cdots h_{j_k}.$$

We define a multiplication as follows

$$d \odot t = h_{m_1} \cdots h_{m_p} h_{i_1+j_1} \cdots h_{i_k+j_k}.$$

If $l(d) > l(t)$ then we define

$$d \odot t = t \odot d.$$

Note that $d \odot t \in D$.

PROPOSITION 1.5 (Dixmier Basis): *With the preceding notation D is a Dixmier basis for H_∞.*

PROOF: By definition D is a $\mathbb{K}$-linear basis for H_∞. By Lemma 1.4 we know that "<" defines a well ordering on D. Thus we need to show that the Dixmier property (⌘) holds.

CASE $l(d) \leq l(t)$: We find

$$\Delta(d \odot t) = \sum_{\alpha_1=0}^{m_1} \cdots \sum_{\alpha_p=0}^{m_p} \sum_{\alpha_{p+1}=0}^{i_1+j_1} \cdots \sum_{\alpha_{p+k}=0}^{i_k+j_k} h_{\beta_1} \cdots h_{\beta_{p+k}} \otimes h_{\alpha_1} \cdots h_{\alpha_{p+k}},$$

where $\beta_r = m_r - \alpha_r$ for $r \leq p$ and $\beta_{p+r} = i_r + j_r - \alpha_r$. Let

$$a \otimes b = h_{\beta_1} \cdots h_{\beta_{p+k}} \otimes h_{\alpha_1} \cdots h_{\alpha_{p+k}}$$

be a summand of $\Delta(d \odot t)$. We note that

$$\operatorname{spdeg}(d \odot t) = \operatorname{spdeg}(d) + \operatorname{spdeg}(t) = \operatorname{spdeg}(a) + \operatorname{spdeg}(b).$$

Furthermore,

$$(\star) \qquad l(d \odot t) = l(t) \geq l(b).$$

We proceed by proof by contradiction. To this end assume that $a \geq d$ and $b \geq t$. Thus by $(\star)$ we obtain that $l(b) = l(t)$. Moreover, since $b \geq t$ we obtain that

$$b = h_{\alpha_1} \cdots h_{\alpha_{p+k}} \geq_{\text{lex}} t = h_{m_1} \cdots h_{m_p} h_{j_1} \cdots h_{j_k}.$$

Hence

$$\beta_r \geq m_r \quad \forall\ r = 1, \ldots, p.$$

Therefore, $\beta_r = m_r$ for $r = 1, \ldots, p$, and thus $\alpha_r = 0$ for $r = 1, \ldots, p$. Therefore d and a have the same length.

Next assume that $b > t$. Then there exists an index x such that

$$\beta_{p+x} > j_x \quad \text{and} \quad \beta_y = j_y$$

for $y = 1, \ldots, x-1$. Hence $\alpha_y = i_y$ and $\alpha_x < i_x$, and thus $a < d$. This contradicts our assumption. Therefore $t = b$. Hence $d = a$.

Finally observe that the case $t = b$ occur exactly once in the above sum.

CASE $l(d) > l(t)$: This follows immediately from the first case, because H_∞ is cocommutative.☯

We summarize these results in the following proposition.

PROPOSITION 1.6: *Let H_∞ be the free $\mathbb{K}$-algebra on countably many generators $h_0 = 1,\ h_1,\ h_2, \dots$ with an Hopf algebra structure given by*

$$\Delta(h_k) = \sum_{i=0}^{k} h_i \otimes h_{k-i}.$$

Let D be the linear basis containing all monomials in the h_i's. Then D is a Dixmier basis for H_∞.

PROOF: ☯

DEFINITION: *We call an Hopf algebra* H *that is a quotient of a Hopf algebra* H_D *with Dixmier basis a* **Hopf algebra of Dixmier type**.

EXAMPLE 1.7 : Denote by P the Steenrod algebra of reduced powers over a finite field $\mathbb{F}_q$ of order q. It is the free associative $\mathbb{F}_q$-algebra generated by the reduced powers $\mathcal{P}^0 = \mathrm{id},\ \mathcal{P}^1,\ \mathcal{P}^2,\ \dots$ modulo the Adem-Wu relations

$$\mathcal{P}^i\mathcal{P}^j = \sum_{k=0}^{[i/q]}(1)^{i+qk}\binom{(q-1)(j-k)-1}{i-qk}\mathcal{P}^{i+j-k}\mathcal{P}^k, \quad \text{whenever } i,\ j>0 \text{ and } i<qj.$$

The Steenrod algebra has an $\mathbb{F}_q$-linear basis D consisting of the so-called admissible monomials

$$\mathcal{P}^I \overset{\text{def}}{=} \mathcal{P}^{i_1}\cdots\mathcal{P}^{i_k} \quad \text{with} \quad i_s \geq q\, i_{s+1} \ \ \forall\ s = 1, \dots, k,$$

see, e.g., Proposition 2.1 in [8]. We define an order on D as we did in the case of H_∞ by replacing the special degree by the moment $m(\mathcal{P}^I)$:

$$m(\mathcal{P}^I) = \sum_{s=1}^{k} s\, i_s.$$

We define a multiplication $\odot$ on D in the following way:

$$\mathcal{P}^I \odot \mathcal{P}^J = \mathcal{P}^{i_1+j_1}\cdots\mathcal{P}^{i_k+j_k}\mathcal{P}^{j_{k+1}}\cdots\mathcal{P}^{j_l},$$

where $\mathcal{P}^J = \mathcal{P}^{j_1}\cdots\mathcal{P}^{j_l}$ and $l \geq k$. If $k \geq l$ we set

$$\mathcal{P}^I \odot \mathcal{P}^J = \mathcal{P}^J \odot \mathcal{P}^I.$$

We find

$$\begin{aligned}
\Delta(\mathcal{P}^I \odot \mathcal{P}^J) &= \Delta(\mathcal{P}^{i_1+j_1}\cdots\mathcal{P}^{i_k+j_k}\mathcal{P}^{j_{k+1}}\cdots\mathcal{P}^{j_l}) \\
&= \sum_{\alpha_1=0}^{i_1+j_1}\cdots\sum_{\alpha_l=0}^{j_l}\mathcal{P}^{\alpha_1}\cdots\mathcal{P}^{\alpha_l}\otimes\mathcal{P}^{i_1+j_1-\alpha_1}\cdots\mathcal{P}^{j_l-\alpha_l} \\
&= \mathcal{P}^I\otimes\mathcal{P}^J + \sum_{\alpha_1=0,\neq i_1}^{i_1+j_1}\cdots\sum_{\alpha_k=0,\neq i_k}^{i_k+j_k}\sum_{\alpha_{k+1}=1}^{i_1+j_1}\cdots\sum_{\alpha_l=1}^{j_l}\mathcal{P}^{\alpha_1}\cdots\mathcal{P}^{\alpha_l}\otimes\mathcal{P}^{i_1+j_1-\alpha_1}\cdots\mathcal{P}^{j_l-\alpha_l}.
\end{aligned}$$

If the first component of one of the summands has moment

$$m(\mathcal{P}^{\alpha_1}\cdots\mathcal{P}^{\alpha_l}) = \sum_{s=1}^{l} s\alpha_s \le \sum_{s=1}^{k} s\, i_s = m(\mathcal{P}^I)$$

then it can be written as a sum of admissible monomials of smaller moment, see, e.g., Proposition 2.1 in [8]. If its moment is larger than the moment of $\mathcal{P}^I$, then the moment of the second component

$$m(\mathcal{P}^{i_1+j_1-\alpha_1}\cdots\mathcal{P}^{j_l-\alpha_l}) = \sum_{s=1}^{l} s(i_s + j_s - \alpha_s) \le \sum_{s=1}^{l} s j_s = m(\mathcal{P}^J).$$

Thus in this case the second component can be written as a sum of admissible monomials of smaller moment. Therefore, our product on D satisfies the Dixmier Property.

§2. Primary Decomposition of Ideals

Let H_D be a Hopf algebra with Dixmier basis D. Let

$$\varphi : \mathrm{H}_D \longrightarrow \mathrm{H}$$

be a projection of Hopf algebras. Thus, the Hopf algebra H has Dixmier type. We note that any commutative $\mathbb{K}$-algebra R over H can be consider as an H_D-module via φ. Thus without loss of generality we assume that $\mathrm{H} = \mathrm{H}_D$ in the following.

Recall that an ideal $I \subseteq R$ is called **invariant** if

$$\mathrm{H}(I) \subseteq I.$$

Assume that I has a (possibly infinite) primary decomposition

$$I = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \mathfrak{q}_3 \cap .....$$

In this section we are going to show that I has an **invariant primary decomposition**, i.e., there exist invariant primary ideals $\mathfrak{q}_1'$, $\mathfrak{q}_2'$, ... such that

$$I = \mathfrak{q}_1' \cap \mathfrak{q}_2' \cap \mathfrak{q}_3' \cap .....$$

Furthermore we show that the prime ideals associated [2] to this decomposition, $\mathfrak{p}_i = Rad\,(\mathfrak{q}_i)$, are also invariant.

We need some preliminary results for that and start with the following definition, cf. Chapter 9 in [7] and Section 4 in [10].

DEFINITION: *Let $I \subseteq R$ be an ideal. Denote by*

$$J_\infty(I) \subseteq I$$

the maximal invariant subideal of I. We define

$$J_D(I) = \{r \in I \mid d_i(r) \in I \ \forall \ i \in \mathbb{N}_0\},$$

where $d_0 = 1 \le d_1 \le d_2 \le ....$ form a Dixmier basis D for H.

[2] Since we are dealing with infinite primary decompositions, the prime ideals are not determined by I.

PROPOSITION 2.1: *With the above notation*

$$J_D(I) = J_\infty(I),$$

for any ideal $I \subseteq R$.

PROOF: Let $r \in J_\infty(I)$. Then

$$d_i(r) \in J_\infty(I) \subseteq I$$

for all $i \in \mathbb{N}_0$. Hence $r \in J_D(I)$.

Conversely, since D is a linear basis for H, the set $J_D(I)$ is invariant. Thus

$$J_D(I) \subseteq J_\infty(I)$$

by maximality of $J_\infty(I)$. ☯

REMARK: Note that the preceding result means in particular that $J_D(I)$ is an ideal.

DEFINITION: *We define a map from R into the power series ring $R[[\xi]]$ in one transcendental variable ξ as follows.*

$$\mathrm{H}(\xi) : R \longrightarrow R[[\xi]],\ r \longmapsto \sum_{i\geq 0} d_i(r)\xi^i.$$

Note that, by construction this map is an algebra homomorphism.

We obtain the following characterization of $J_D(I)$, cf. Landweber's result in the context of unstable actions of the Steenrod algebra on a noetherian ring, see Lemma 9.1.1 [7].

PROPOSITION 2.2: *Consider the canonical inclusion $R \hookrightarrow R[[\xi]], r \longmapsto r$. Then we have that*

$$J_D(I) = \mathrm{H}(\xi)^{-1}\left(I[[\xi]]\right).$$

PROOF: Let $r \in J_D(I)$. Then, by Proposition 2.1, we have that

$$d_i(r) \in J_D(I) \subseteq I$$

for all $i \in \mathbb{N}_0$. Hence

$$\mathrm{H}(\xi)(r) \in I[[\xi]],$$

and thus

$$r \in \mathrm{H}(\xi)^{-1}\left(I[[\xi]]\right).$$

Conversely, for any element $r \in \mathrm{H}(\xi)^{-1}\left(I[[\xi]]\right)$, we have that

$$\mathrm{H}(\xi)(r) \in I[[\xi]].$$

This means that

$$d_i(r) \in I$$

for all $i \in \mathbb{N}_0$. Thus $r \in J_D(I)$ as desired. ☯

We need the following technical result:

LEMMA 2.3: *Let $R \hookrightarrow R[[\xi]]$ be the canonical inclusion. If $\mathfrak{p} \subseteq R$ is a prime ideal, then so is $\mathfrak{p}[[\xi]] \subseteq R[[\xi]]$.*

PROOF: See Exercise 5 (v) in Chapter of [1]. ☯

PROPOSITION 2.4: *Let $\mathfrak{p} \subseteq R$ be a prime ideal, then so is $J_D(\mathfrak{p})$.*

PROOF: If $I \subseteq R$ is prime, then so is $I[[\xi]] \subseteq R[[\xi]]$ by Lemma 2.3. Hence, also

$$J_D(I) = \mathrm{H}(\xi)^{-1}\left(I[[\xi]]\right) \subseteq R$$

is prime. ☯

COROLLARY 2.5: *Let $I \subseteq R$ be an invariant ideal. Then all minimal prime ideals $I \subseteq \mathfrak{p} \subseteq R$ containing I are invariant.*

PROOF: Consider the canonincal projection

$$\varphi : R \longrightarrow R/I.$$

The minimal prime ideals $\mathfrak{p} \subseteq I$ project down to the minimal prime ideals $\overline{\mathfrak{p}} \subseteq R/I$. They are invariant by the preceding Proposition 2.4. Thus the ideals $\mathfrak{p}$ are also invariant. ☯

PROPOSITION 2.6: *Let $I \subseteq R$ be a radical ideal. Then*

$$J_D(I) = Rad\,(J_D(I)).$$

PROOF: The inclusion "$\subset$" is obvious. In order to show the reverse inclusion take an element

$$a \in Rad\,(J_D(I)).$$

Then there exists some power $n \in \mathbb{N}$ such that

$$a^n \in J_D(I).$$

Hence $d_i(a^n) \in J_D(I)$ for all elements in the Dixmier basis $d_i \in D$. Assume that $a \notin J_D(I)$. Then there exists a minimal element $d \in D$ such that

$$d(a) \notin I.$$

We observe that

$$d^{\odot n}(a^n) = d(a)^n + \sum d_{i_1}(a) \cdots d_{i_n}(a),$$

for some $i_1, \ldots, i_n \in \mathbb{N}_0$. Note that for every summand of the sum on the right we have that

$$d_{i_j} < d$$

for at least one index i_j. Thus

$$\sum d_{i_1}(a) \cdots d_{i_n}(a) \in I$$

by minimality of d. Since $a^n \in J_D(I)$ we have that

$$d^{\odot n}(a^n) \in J_D(I) \subseteq I.$$

Therefore

$$d(a)^n \in I$$

and thus

$$d(a) \in I.$$

This contradicts our assumption, and concludes the proof. ☯

PROPOSITION 2.7: *Let $I \subseteq R$ be an ideal. Then*

$$J_D(Rad\,(I)) = Rad\,(J_D(I)).$$

PROOF: Since

$$I \subseteq Rad\,(I)$$

we have that

$$J_D(I) \subseteq J_D(Rad\,(I)).$$

Therefore

$$Rad\,(J_D(I)) \subseteq Rad\,(J_D(Rad\,(I))) = J_D(Rad\,(I))$$

by Proposition 2.6. To prove the converse inclusion observe that

$$J_D(Rad\,(I)) \subseteq Rad\,(J_D(Rad\,(I))) = J_D(Rad\,(I))$$

by Proposition 2.6. ☯

PROPOSITION 2.8: *Let $\mathfrak{q} \subseteq R$ be a primary ideal such that $Rad\,(\mathfrak{q}) = \mathfrak{p}$ is invariant. Then $J_D(\mathfrak{q})$ is also $\mathfrak{p}$-primary.*

PROOF: Let $\mathfrak{q} \subseteq R$ be a $\mathfrak{q}$-primary ideal. Let $a, b \in R$ such that

$$ab \in J_D(\mathfrak{q}).$$

Assume without loss of generality that $b \notin Rad\,(J_D(\mathfrak{q})) = \mathfrak{p}$. We need to show that $a \in J_D(\mathfrak{q})$.

First note that $ab \in \mathfrak{q}$ and $b \notin \mathfrak{p} = J_D(\mathfrak{p})$ implies that $a \in \mathfrak{q}$ because $\mathfrak{q}$ is primary by assumption. We proceed by contradiction. Let $d \in D$ be minimal such that $d(a) \notin \mathfrak{q}$. Then

$$(d \odot 1)(ab) = d(a)b + \sum_{d' < d} d'(a)b \in \mathfrak{q}.$$

The sum on the right is by choice of d in $\mathfrak{q}$. Therefore

$$d(a)b \in \mathfrak{q}$$

and hence $d(a) \in \mathfrak{q}$ because b is not in its radical. This is a contradiction and completes the proof. ☯

In order to be able to prove that $J_D(\mathfrak{q})$ is primary whenever $\mathfrak{q}$ is we need to go back to the construction of $J_D(—)$.

Consider the canonical inclusion

$$R \hookrightarrow R[[\xi]].$$

Let $I \subseteq R$ be any ideal, and $I[[\xi]]$ its image. Set

$$I_i = \{f(\xi) = \sum_{j=0}^{\infty} a_j \xi^j \mid a_0, \dots, a_i \in I\}.$$

By construction we have that I_i is an ideal in $R[[\xi]]$, $i \in \mathbb{N}_0$ and

$$I[[\xi]] = \bigcap_{i=0}^{\infty} I_i.$$

LEMMA 2.9: *Let $\mathfrak{q} \subseteq R$ be a $\mathfrak{p}$-primary ideal. Then $\mathfrak{q}_i$ is $\mathfrak{p}[[\xi]]$-primary.*

PROOF: We start by showing that $Rad(\mathfrak{q}_i) = \mathfrak{p}[[\xi]]$ for all i. Let $f \in Rad(\mathfrak{q}_i)$. Then there exists an $n \in \mathbb{N}$ such that

$$f^n \in \mathfrak{q}[[\xi]] \subseteq \mathfrak{p}[[\xi]].$$

Hence $f \in \mathfrak{p}[[\xi]]$ by Lemma 2.3.

Conversely, if $f \in \mathfrak{p}[[\xi]]$ then for every $i \in \mathbb{N}_0$ there exists an $n_i \in \mathbb{N}$ such that

$$f^{n_i} \in \mathfrak{q}_i.$$

Thus $f \in Rad(\mathfrak{q}_i)$ as claimed.

Therefore the radical of $\mathfrak{q}_i$ is $\mathfrak{p}[[\xi]]$ for all $i \in \mathbb{N}_0$.

In order to show that $\mathfrak{q}_i \subseteq R[[\xi]]$ is $\mathfrak{q}[[\xi]]$-primary, take $f, h \in \mathbb{R}[[\xi]]$ such that their product $fh \in \mathfrak{q}_i$. We assume that f is not contained in $\mathfrak{p}[[\xi]]$. Set

$$f = \sum_{j=0}^{\infty} a_j \xi^j \quad \text{and} \quad h = \sum_{k=0}^{\infty} b_k \xi^k.$$

Then

$$fh = \sum_{j=0}^{\infty} (\sum_{k=0}^{j} a_{j-k} b_k) \xi^j.$$

By assumption

$$(*) \qquad \sum_{k=0}^{j} a_{j-k} b_k \in \mathfrak{q} \quad \forall\, j = 0, \ldots, i.$$

Since $f \notin \mathfrak{p}[[\xi]]$ there exists a minimal j such that $a_j \notin \mathfrak{p}$. Without loss of generality $j = 0$. Then it follows inductively by $(*)$ that $h^n \in \mathfrak{q}_i$ for some $n \in \mathbb{N}$, since $\mathfrak{q}$ is $\mathfrak{p}$-primary. ☯

At this point we need a technical lemma. The following result has been proven in Lemma 1.1 in [5] in the context of modules over the Steenrod algebra.

LEMMA 2.10: *The functor J_D commutes with arbitrary intersections:*

$$J_D(\bigcap_i I_i) = \bigcap_i J_D(I_i)$$

PROOF: By definition

$$J_D(\bigcap_i I_i) \subseteq \bigcap_i I_i$$

is the largest invariant subideal. Since

$$\bigcap_i J_D(I_i) \subseteq \bigcap_i I_i$$

is also invariant we find that

$$\bigcap_i J_D(I_i) \subseteq J_D(\bigcap_i I_i).$$

To prove the reverse inclusion let

$$r \in \bigcap_i J_D(I_i).$$

Then

$$r \in J_D(I_i) \subseteq I_i$$

for all i. Hence $d_j(r) \in J_D(I_i) \subseteq I_i$ for all i and j. Thus

$$r \in J_D(\bigcap_i I_i)$$

as claimed. ☯

PROPOSITION 2.11: *Let $\mathfrak{q} \subseteq R$ be $\mathfrak{p}$-primary. Then $J_D(\mathfrak{q})$ admits an invariant primary composition such that $J_D(\mathfrak{p})$ is its only associate prime ideal.*

PROOF: By Proposition 2.2 and the preceding Lemma 2.9

$$\begin{aligned} J_D(\mathfrak{q}) &= H(\xi)^{-1}(\mathfrak{q}[[\xi]]) \\ &= H(\xi)^{-1}(\bigcap_i \mathfrak{q}_i) \\ &= \bigcap_i H(\xi)^{-1}(\mathfrak{q}_i). \end{aligned}$$

Thus $J_D(\mathfrak{q})$ has a primary decomposition consisting of the primary ideals $H(\xi)^{-1}(\mathfrak{q}_i) \subseteq R$. Their radical is given by

$$Rad\,(H(\xi)^{-1}(\mathfrak{q}_i)) = H(\xi)^{-1} Rad\,(\mathfrak{q}_i) = H(\xi)^{-1}(\mathfrak{p}[[\xi]]) = J_D(\mathfrak{p}).$$

Thus by Proposition 2.8 $J_D(H(\xi)^{-1}(\mathfrak{q}_i))$ is $J_D(\mathfrak{p})$-primary. Therefore we obtain an invariant primary decomposition

$$J_D(\mathfrak{q}) = \bigcap_i J_D(H(\xi)^{-1}(\mathfrak{q}_i))$$

with invariant radical $J_D(\mathfrak{p})$. ☯

Combing the preceding results we obtain the following theorem.

THEOREM 2.12: *Let* H *be a Hopf algebra of Dixmier type over a field $\mathbb{K}$, let R be a commutative $\mathbb{K}$-algebra with an* H*-module structure. Let $I \subseteq R$ be an invariant ideal. Assume has a (possibly infinite) primary decomposition. Then I has an invariant primary decomposition.*

PROOF: Let

$$I = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \mathfrak{q}_3 \cap \cdots$$

be a primary decomposition of the invariant ideal $I \subseteq R$. Then

$$(⌘) \qquad I = J_D(I) = J_D(\mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \mathfrak{q}_3 \cap \cdots) = J_D(\mathfrak{q}_1) \cap J_D(\mathfrak{q}_2) \cap J_D(\mathfrak{q}_3) \cap \cdots$$

by Lemma 2.10. By Proposition 2.11 the ideals $J_D(\mathfrak{q}_i)$ have invariant primary decompositions with one associated prime ideal $J_D(\mathfrak{p})$, for all $i \in \mathbb{N}$. ☯

REMARK: Note that it is not clear whether the invariant primary decomposition (⌘) is irredundant or irreducible even if the orginal one was.

EXAMPLE 2.13 : Let R be a graded connected commutative algebra over a finite field $\mathbb{F}$. Let R be an algebra over the Steenrod algebra. Then every ideal invariant under the Steenrod algebra action which admits a primary decomposition admits an invariant primary decomposition. Note that this is independent on whether the P-action is unstable or not.

§3. Primary Decomposition of Modules

In this section we translate the preceding results to R-modules M, where R as well as M admit an action of an Hopf algebra H of Dixmier type, and the two actions are compatible.

Let $N \subseteq M$ be an R-submodule of M. Assume that N admits a (possibly infinite) primary decomposition

$$N = Q_1 \cap Q_2 \cap Q_3 \cap \ldots.$$

We define the functor J_D on the category of modules exactly as we did for ideals, see Section 2. By Lemma 2.10 we obtain

$$(\star) \qquad J_D(N) = J_D(Q_1) \cap J_D(Q_2) \cap J_D(Q_3) \cap \ldots.$$

By definition $J_D(N) \subseteq M$ is an invariant H-module. We claim that the $J_D(Q_i)$'s admit invariant primary decompositions.

Since $Q_i \subseteq M$ is a primary module, the ideal

$$\mathfrak{q}_i = (Q_i : M) \subseteq R$$

is primary. Thus by Proposition 2.11

$$J_D(\mathfrak{q}_i) = J_D(Q_i : M)$$

admits an invariant primary decomposition

$$J_D(\mathfrak{q}_i) = \cap_j \mathfrak{q}_{ij}$$

such that $Rad\,(\mathfrak{q}_{ij}) = J_D(\mathfrak{p}_i)$ for all j. We note that

$$J_D(Q_i : M) = (J_D(Q_i) : M)$$

by Lemma 1.4. in [4].[3]

We have that

$$J_D(Q_i) = \cap_j \mathfrak{q}_{ij} M$$

by definition of $\mathfrak{q}_{ij}$. By construction, the submodules $\mathfrak{q}_{ij} M \subseteq M$ are primary and their radicals are $J_D(\mathfrak{p}_i)$ for all j. Since $\mathfrak{q}_{ij} \subseteq R$ is an invariant ideal, we have that $\mathfrak{q}_{ij} M \subseteq M$ is an invariant submodule. Thus we have proven the following result:

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[3] Since this reference deals with the special case of unstable modules over the Steenrod algebra we add the proof:
Since $J_D(Q) \subseteq Q$ we have that

$$(J_D(Q) : M) \subseteq (Q : M).$$

Next we show that the ideal $(J_D(Q) : M) \subseteq R$ is invariant. To this end let $r \in (J_D(Q) : M)$. Then

$$d_i(rM) \subseteq J_D(Q) \quad \forall \ d_i \in D.$$

Let d_i be minimal such that $d_i(r) \notin (J_D(Q) : M)$. Then we obtain by the Dixmier property

$$d_i(r)M = d_i(r)d_0(M) = \lambda^{-1}\left((d_i \odot d_0)(rM) - \sum_{j<i} d_j(r)d_k(M)\right).$$

By minimality of d_i the right hand side of this equation is in $J_D(Q)$, hence so is the left hand side, i.e.,

$$d_i(r) \in (J_D(Q) : M) \quad \forall \ i \in \mathbb{N}_0.$$

This is a contradiction. Thus it follows that

$$(J_D(Q) : M) \subseteq J_D(Q : M)$$

THEOREM 3.1: *Let* H *be a Hopf algebra of Dixmier type over a field* $\mathbb{K}$*, let* R *be a commutative* $\mathbb{K}$*-algebra and* M *be an* R*-module. Assume that* R *admits an action of* H*, and* M *is an* (H, R)*-module. Let* $N \subseteq M$ *be an* (H, R)*-submodule of* M*. Assume that* N *admits a (possibly infinite) primary decomposition*

$$N = Q_1 \cap Q_2 \cap Q_3 \cap \ldots.$$

Then N *admits an invariant primary decomposition.*

PROOF: ☯

EXAMPLE 3.2 : Let P be the mod-p-Steenrod algebra and M any $P \odot R$-module. If $N \subseteq M$ has a primary decomposition, then $J_D(N)$ has one. In particular if N is an invariant submodule with a primary decomposition, then N has an invariant primary decomposition with invariant associated prime ideals. Note that it is not necessary to assume that the action is unstable.

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by maximality of $J_D(Q)$. To show the reverse inclusion, take an element $r \in J_D(Q : M)$. Then by the Dixmier property

$$d_i(rM) = (d_i \odot d_0)(rM) = d_i(r)M + \sum_{j<i} d_j(r)d_k(M).$$

By induction we can assume that the right hand side is in Q, thus so is the left hand side, and we are done.

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