The Inverse Invariant Theory Problem and Steenrod Operations

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SUMMARY : *This paper is devoted to the study of inverse invariant theory and its relationship with the $\mathcal{P}^*$–invariant prime spectrum of an unstable algebra over the Steenrod algebra. We will show that this spectrum is a chain saturated poset. Moreover we will prove the existence of Thom classes, detect a fractal of the Dickson algebra in any unstable algebra and give a counterexample to the Reverse Landweber–Stong Conjecture. Along the way to these results we will generalize the famous Adams–Wilkerson theorems to arbitrary Galois fields, have a closer look at fields and their extensions over the Steenrod algebra, and generalize some results about the unstable part of a module over the Steenrod algebra.*

Contents

Introduction

Yaşamak bir ağaç gibi tek ve hür
ve bir orman gibi kardeşçesine,
bu hasret bizim …
Nazım Hikmet

Let $\mathbb{F} := \mathbb{F}_q$ be a Galois field of characteristic p and with $q = p^s$ elements. If

$$\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$$

is a faithful representation of a finite group G of degree n over the field $\mathbb{F}$, then ϱ induces an action of the group G on the polynomial algebra

$$\mathbb{F}[V] = \mathbb{F}[x_1, \ldots, x_n]$$

in n linear indeterminates. The ring of polynomials invariant under this action, denoted by $\mathbb{F}[V]^G$, is then a graded connected[1] commutative Noetherian algebra over $\mathbb{F}$ of Krull dimension n; see [28] for an introduction to the invariant theory of finite groups.

Invariant theory of finite groups investigates the ring of polynomial invariants, $\mathbb{F}[V]^G$, for, e.g., a given group G or a family of representations; its homological, algebraical and combinatorial properties. The classical inverse question is:

Let* H^* *be a ring. When does a representation exist such that

$$\mathrm{H}^* = \mathbb{F}[V]^G\textbf{?}$$

Let's start with a list of significant properties of any ring of invariants, $\mathbb{F}[V]^G$.

[1] The terminology **connected** comes from algebraic topology. It means that in degree zero the algebra is exactly the ground field

$$\left(\mathbb{F}[V]^G\right)_{(0)} = \mathbb{F}.$$

PROPERTY 1 : By what we said so far we certainly need to assume that H^* is a graded connected commutative Noetherian algebra over $\mathbb{F}$. Denote by $\dim(\mathrm{H}^*) = n$ the Krull dimension of H^*. Let me first say some more words about the **grading convention**: It is encoded in the notation

$$\mathrm{H}^* = \left\{ \mathrm{H}^i \mid i \in \mathbb{N}_0 \right\},$$

where $\mathbb{N}_0$ is the non negative integers, i.e., using the terminology introduced by J. W. Milnor and J. C. Moore H^* is a **non negatively graded algebra**. Since sign conventions play no role in this manuscript we prefer to use the standard grading conventions of commutative algebra. If you are a topologist, you may in certain situations, want to double the degrees to bring things into conformity with the standard grading conventions of algebraic topology.

PROPERTY 2 : Secondly we need that H^* embeds integrally[2] into the polynomial ring $\mathbb{F}[V]$, since, by a classical result of Emmy Noether,

$$\mathbb{F}[V]^G \hookrightarrow \mathbb{F}[V]$$

is an integral extension for any G.

PROPERTY 3 : Thirdly we have that the full general linear group $\mathrm{GL}(n, \mathbb{F})$ is a finite group (because our ground field is finite) and provides us with a set of universal invariants, i.e.,

$$\mathscr{D}^*(n) = \mathbb{F}[\mathbf{d}_{n,0}, \ldots, \mathbf{d}_{n,n-1}] = \mathbb{F}[V]^{\mathrm{GL}(n,\ \mathbb{F})} \hookrightarrow \mathbb{F}[V]^G$$

is an integral extension for any G. The ring of polynomial invariants of $\mathrm{GL}(n, \mathbb{F})$ was calculated by L. E. Dickson in 1911, [7], and is therefore called **Dickson algebra**, compare also [28] §8.1. The generating polynomials

$$\mathbf{d}_{n,0}, \ldots, \mathbf{d}_{n,n-1}$$

are called **Dickson classes**. (One might look up their explicit description in [28] §8.1 or in the Appendix A.2.) So, since $G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ implies $\mathbb{F}[V]^{\mathrm{GL}(n,\ \mathbb{F})} \hookrightarrow \mathbb{F}[V]^G$, we need to assume that there exists an integral extension

$$\mathscr{D}^*(n) \hookrightarrow \mathrm{H}^*.$$

PROPERTY 4 : A fourth property of rings of invariants is a classical result of Emmy Noether: they are integrally closed (i.e., in their field of fractions), see [28] Proposition 1.2.4.

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[2] For two rings A^* and B^*, here and in the following "A^* embeds integrally into B^*" or "B^* contains integrally A^*" is meant to be a short way of saying that the ring extension

$$A^* \hookrightarrow B^*$$

is integral.

OBSERVATION : *Let* H^* *be a graded connected commutative Noetherian algebra over* $\mathbb{F}$. *Suppose there is an integral extensions*

$$\mathscr{D}^*(n) \hookrightarrow \mathrm{H}^* \hookrightarrow \mathbb{F}[V]$$

and let H^* *be integrally closed. Then* H^* *is a ring of invariants.*

PROOF : By what we have assumed we have the following situation

$$\begin{array}{ccccc} \mathscr{D}^*(n) & \underset{\text{integral}}{\hookrightarrow} & \mathrm{H}^* = \overline{\mathrm{H}^*} & \underset{\text{integral}}{\hookrightarrow} & \mathbb{F}[V] \\ \downarrow & & \downarrow & & \downarrow \\ FF(\mathscr{D}^*(n)) & \hookrightarrow & FF(\mathrm{H}^*) & \hookrightarrow & \mathbb{F}(V), \end{array}$$

where $FF(-)$ denotes the field of fractions functor. The field extension $FF(\mathscr{D}^*(n)) \subseteq \mathbb{F}(V)$ is Galois with Galois group $\mathrm{GL}(n,\ \mathbb{F})$. The fundamental Theorem of Galois Theory gives us a group $G \leq \mathrm{GL}(n,\ \mathbb{F})$ such that

$$FF(\mathrm{H}^*) = \mathbb{F}(V)^G \subseteq \mathbb{F}(V).$$

Since we have an integral extension

$$\mathrm{H}^* \hookrightarrow \mathbb{F}[V]^G$$

of integrally closed algebras with the same field of fractions $FF(\mathrm{H}^*) = \mathbb{F}(V)^G$, they are equal, i.e.,

$$\mathrm{H}^* = \mathbb{F}[V]^G.$$

That's all we claimed •

So, in order to solve the inverse invariant theory problem we have to characterize those H^*'s which:

PROPERTY 2 : admit an integral embedding into $\mathbb{F}[V]$, and

PROPERTY 3 : contain integrally $\mathscr{D}^*(n)$.

However, note carefully, that in the above proof it would have been enough to assume that we had a commutative diagram

$$\begin{array}{ccccc} & & \mathrm{H}^* & \underset{\text{integral}}{\hookrightarrow} & \mathbb{F}[V] \\ & & \downarrow & & \downarrow \\ \mathbb{F}(V)^{\mathrm{GL}(n,\ \mathbb{F})} = FF(\mathscr{D}^*(n)) & \hookrightarrow & FF(\mathrm{H}^*) & \hookrightarrow & \mathbb{F}(V) \end{array}$$

such that the field extension

$$\mathbb{F}(V)/FF(\mathscr{D}^*(n))$$

is Galois. So, we will characterize those H^*'s which fit into such a diagram, i.e., which

PROPERTY 2 : admit an integral embedding into $\mathbb{F}[V]$, and

PROPERTY 3′ : whose field of fractions contains the field of fractions of the Dickson algebra

$$FF(\mathscr{D}^*(n)) \hookrightarrow FF(\mathrm{H}^*) \hookrightarrow \mathbb{F}(V)$$

with $\mathbb{F}(V)/FF(\mathscr{D}^*(n))$ Galois with Galois group $\mathrm{GL}(n, \mathbb{F})$.

At this point you have to recall that we are working over a finite field. This fact gives us an additional structure: the **Steenrod algebra $\boldsymbol{\mathscr{P}}^*$**.

Recall that $\boldsymbol{\mathscr{P}}^*$ over $\mathbb{F}$ is generated[3] by the reduced power operations $\mathscr{P}^i$, $i \in \mathbb{N}_0$, which satisfy the **Adem-Wu relations**

$$\mathscr{P}^i\mathscr{P}^j = \sum_{k=0}^{[i/q]} c_{ijk}\mathscr{P}^{i+j-k}\mathscr{P}^k,$$

where by convention $\mathscr{P}^0 = 1$ and c_{ijk} are certain elements in the prime field $\mathbb{F}_p$ of $\mathbb{F}$. The exact value of the coefficients c_{ijk} can be found in Sections 10.3 and 11.1 in [28].

H^* is called[4] **an algebra over $\boldsymbol{\mathscr{P}}^*$** if it is a left $\boldsymbol{\mathscr{P}}^*$-module satisfying the **Cartan formulae**

$$\mathscr{P}^k(h'h'') = \sum_{i+j=k} \mathscr{P}^i(h')\mathscr{P}^j(h'')$$

for all $h', h'' \in \mathrm{H}^*$ and $k \in \mathbb{N}_0$. If in addition the **unstability condition**

$$\mathscr{P}^k(h) = \begin{cases} h^q & k = \deg(h) \\ 0 & k > \deg(h) \end{cases} \quad \forall\ h \in \mathrm{H}^*$$

holds then we say H^* is an **unstable algebra over the Steenrod algebra**, or simply **an unstable algebra**. So, the Steenrod algebra is a way to encode information which is hidden in the classical **Frobenius homomorphism**. If we define the so-called **giant Steenrod operation**

$$\mathcal{P}(\xi)(-) := \sum_{i\geq 0} \mathscr{P}^i(-)\xi^i$$

then H^* being an algebra over $\boldsymbol{\mathscr{P}}^*$ is equivalent to saying that the giant Steenrod operation defines an $\mathbb{F}$-algebra homomorphism

$$\mathcal{P}(\xi)(-) : \mathrm{H}^* \longrightarrow \mathrm{H}^*[[\xi]],$$

[3] In order to avoid double notation for the case $p = 2$ denote with the indulgence of topologists $\mathrm{Sq}^i := \mathscr{P}^i$ for all $i \in \mathbb{N}_0$.

[4] We assume throughout the whole paper that H^* is a graded connected commutative algebra over $\mathbb{F}$. Any further properties of H^*, like e.g. Noetherianess, is explicitely stated when needed.

where $H^*[[\xi]]$ denotes the power series in the additional variable ξ with coefficients in H^*, and $\deg(\xi) = 1 - q$. Note carefully that we the Cartan formulae are equivalent to the product rule

$$\mathcal{P}(\xi)(h_1 h_2) = \mathcal{P}(\xi)(h_1) \cdot \mathcal{P}(\xi)(h_2).$$

Moreover the unstability condition translates into demanding that the image is actually a *polynomial* in ξ

$$\mathcal{P}(\xi)(-) \in H^*[\xi]$$

with leading term $h^q \xi^{\deg(h)}$.

Coming back to our inverse invariant theory problem we notice that the action of the group G on $\mathbb{F}[V]$ commutes with raising to q-th powers. Hence any ring of invariants is an unstable algebra over the Steenrod algebra $\mathscr{P}^*$ (see [28] Chapters 10 and 11 for an introduction to the Steenrod algebra and its use in invariant theory).

We therefore work throughout the whole manuscript in the **category whose objects are unstable algebras over the Steenrod algebra $\mathscr{P}^*$ and whose morphisms are $\mathscr{P}^*$-module homomorphisms**. So, our H^* in question, will be throughout the whole manuscript an unstable algebra over the Steenrod algebra $\mathscr{P}^*$, and all maps will commute with the action of the Steenrod algebra.

Equipped with this terminology we can state the results of this manuscript.

The first goal is to show that our H^* embeds integrally into $\mathbb{F}[V]$ if and only if H^* is a Noetherian integral domain. The one statement is trivial, the other is for prime fields $\mathbb{F} = \mathbb{F}_p$ the contents of the famous Adams-Wilkerson Embedding Theorem, [1], see also Section 25 in [13]. So, we are aiming at a generalization of this theorem to arbitrary Galois fields.

To this end we need some technical preliminaries, which we will collect, following Larry Smith, in the so-called Δ-Theorem, compare [28] Theorem 10.5.4. We will introduce notions of **Δ-finiteness** and **Δ-length**, which will be central in later chapters. Moreover, we will round things out with some examples, and draw some consequences. This is the contents of Chapter 1.

In Chapter 2 we will provide a proper setup for the category of graded fields over the Steenrod algebra and generalize C. W. Wilkerson's Separable Extension Lemma, [34], or Lemma 10.5.3 in [28]. Moreover, we will introduce the notion of purely inseparable field extensions *in this category* and inseparably closed fields over $\mathscr{P}^*$. We will develop some of their properties and illustrate them with examples.

In Chapter 3 we will have a closer look at the unstable part of a **module** over $\mathcal{P}^*$ (n.b. here only the second half of the unstability condition makes sense). We generalize some results from [35], in particular the Integral Closure Theorem, see also Theorem 10.5.2 in [28].

In Chapter 4 we will define the **$\mathcal{P}^*$-inseparable closure** $\sqrt{\mathrm{H}^*}$ of an unstable algebra H^*, give a construction method, develop its main properties, and give illustrative examples. It will turn out that for integral domains our definition corresponds naturally with the notion of $\mathcal{P}^*$-inseparable closure for the associated field of fractions, which was given in Chapter 2.

This will allow us to prove the Embedding Theorem for $\mathcal{P}^*$-inseparably closed H^* in Chapter 5. We will start this chapter with the investigation of **PROPERTY 3′**. It turns out that for integral domains H^*, we can single out in $FF(\mathrm{H}^*)$ a (possibly trivial) $FF(\mathcal{D}^*(m))$, where the trancendence degree m of $FF(\mathcal{D}^*(m))$ over $\mathbb{F}$ is precisely the Δ-length of H^* defined in Chapter 1. However, the field extension

$$FF(\mathcal{D}^*(m)) \hookrightarrow FF(\mathrm{H}^*)$$

might not be algebraic. We will characterize exactly when it is. This leads in Section 5.2 to Theorem 5.2.1, a main technical ingredient, compare [28] Theorem 10.5.5, for the Embedding Theorem, which will be proved in Section 5.3.

The first important result of Chapter 6 is Theorem 6.1.1, where we prove that for an integral domain H^*, $\sqrt{\mathrm{H}^*}$ is Noetherian if and only if H^* is Noetherian. This allows us to complete the proof of the Embedding Theorem in full generality, Corollary 6.1.2. After that we will have a rest and enjoy some nice results about the $\mathcal{P}^*$-invariant prime spectrum, $Proj_{\mathcal{P}^*}(\mathrm{H}^*)$, of a Noetherian H^* in Section 6.2, which all follow quite naturally. This investigation was one motivation that started all this. We will prove that $Proj_{\mathcal{P}^*}(\mathrm{H}^*)$ forms a chain saturated sub poset of the full spectrum of homogeneous prime ideals, in particular:

[1] $Proj_{\mathcal{P}^*}(\mathrm{H}^*)$ is a finite set,

[2] for any $i = 0, \ldots, n = \dim(\mathrm{H}^*)$ there is a prime ideal $\mathfrak{p}_i \in Proj_{\mathcal{P}^*}(\mathrm{H}^*)$ of height i,

[3] for any $\mathfrak{p} \in Proj_{\mathcal{P}^*}(\mathrm{H}^*)$ there exists a saturated ascending chain of prime ideals

$$\mathfrak{p} = \mathfrak{p}_i \subsetneq \mathfrak{p}_{i+1} \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{m} \subset \mathrm{H}^*$$

starting at $\mathfrak{p}$ and ending at the maximal ideal $\mathfrak{m}$ of H^*, all of which are contained in $Proj_{\mathcal{P}^*}(\mathrm{H}^*)$, and

[4] for any $\mathfrak{p} \in Proj_{\mathcal{P}^*}(\mathrm{H}^*)$ there exists a saturated descending chain of prime ideals

$$\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_i = \mathfrak{p} \subset \mathrm{H}^*$$

starting at a minimal prime ideal of H^* and ending at $\mathfrak{p}$, all of which are contained in $Proj_{\mathcal{P}^*}(\mathrm{H}^*)$.

In Section 6.3 we will also show that for a reduced algebra H^*, i.e., $Nil(\mathrm{H}^*) = (0)$, the $\mathcal{P}^*$-inseparable closure $\sqrt{\mathrm{H}^*}$ is Noetherian if and only H^* is Noetherian.

It then takes just a tiny bit more to prove the Galois Embedding Theorem for arbitrary Galois fields, [1], see also Section 25 in [13]. This we will do in Chapter 7. We will also show what we can do if we had a diagram like

$$\begin{array}{ccc} \mathrm{H}^* & \underset{\text{integral}}{\hookrightarrow} & \mathbb{F}[V] \\ \downarrow & & \downarrow \\ FF(\mathrm{H}^*) & \underset{\text{inseparable}}{\hookrightarrow} & \mathbb{F}(V). \end{array}$$

This will generalize results by [37]. Moreover, we will show that **PROPERTY 3** (i.e. the existence of a Dickson algebra sitting integrally inside H^*) is determined by the existence of enough p-th roots in H^*. If this is not the case we can only find a fractal of the Dickson algebra [5] in H^*; this is the Little Imbedding Theorem.

Note carefully that **PROPERTY 2**, namely that,

$$\mathrm{H}^* \underset{\text{integral}}{\hookrightarrow} \mathbb{F}[V]$$

reflects faithfully the nonexistence of zero divisors, while **PROPERTY 3**

$$\mathcal{D}^*(n) \underset{\text{integral}}{\hookrightarrow} \mathrm{H}^*$$

expresses the existence of enough p-th roots. This last statement remains true for unstable algebras *with* zero divisors:

The final Chapter 8 contains additional work on unstable algebras with zero divisors and some results about $Proj_{\mathcal{P}^*}(\mathrm{H}^*)$. We prove that even if H^* has zero divisors it is an integral extension of a fractal of the Dickson

[5] A **fractal of the Dickson algebra** $\mathcal{D}^*(n)$ is a q-th power

$$\begin{aligned} \mathcal{D}^*(n)^{q^l} &= \mathbb{F}[\mathbf{d}_{n,0}^{q^l}, \ldots, \mathbf{d}_{n,n-1}^{q^l}] \\ &= \mathbb{F}[x_1^{q^l}, \ldots, x_n^{q^l}]^{\mathrm{GL}(n,\,\mathbb{F})}, \end{aligned}$$

where one can look up the verification of the last equality in [28] in the proof of Theorem 11.4.6.

algebra; this is the Big Imbedding Theorem. Finally, we prove the existence of **Thom classes**, i.e. elements which generate a $\mathscr{P}^*$-invariant principle ideal of height[6] one. In addition to this we construct a Thom class $\mathbf{t} \in \mathrm{H}^*$ which is contained in any $\mathscr{P}^*$-invariant prime ideal of positive height. From this we are able to draw additional conclusions about the $\mathscr{P}^*$-invariant prime spectrum. We will show that there is a $\mathscr{P}^*$-invariant version of Krull's Principal Ideal Theorem and its generalization, i.e., for any $\mathscr{P}^*$-invariant prime ideal $\mathfrak{p} \subset \mathrm{H}^*$ of height $i \in \{1, \dots, n = \dim(\mathrm{H}^*)\}$ there exist i elements $h_1, \dots, h_i \in \mathfrak{p}$ such that

[1] $\mathfrak{p}$ is an isolated prime ideal of $(h_1, \dots, h_i) \subset \mathrm{H}^*$,

[2] the ideals $(h_1, \dots, h_j) \subset \mathrm{H}^*$ for $j = 1, \dots, i$ are $\mathscr{P}^*$-invariant of height j.

These proofs exemplify the utility of Thom classes: they allow proofs by induction over the Krull dimension. In the last section we will look at the **Reverse Landweber-Stong Conjecture**, which had been conjectured by Larry Smith, [30]. We need some terminology: A sequence $h_1, \dots, h_k \in \mathrm{H}^*$ of elements of positive degree in H^* is called a **regular sequence** if

[1] $h_1 \in \mathrm{H}^*$ is not a zero divisor,

[2] $h_i \in \mathrm{H}^*/(h_1, \dots, h_{i-1})$ is not a zero divisor $\forall\ i = 2, \dots, k$.

Then define the **depth** (or **homological codimension**) of H^*, denoted $dp(\mathrm{H}^*)$, to be the length of the longest possible regular sequence in H^*. See [28] Chapter 6 for an introduction to the homological properties of graded connected commutative Noetherian $\mathbb{F}$-algebras.

The original **Landweber-Stong Conjecture** asserts that a ring of invariants $\mathbb{F}[V]^G$ has depth at least k, $dp(\mathrm{H}^*) \geq k$, if and only if the k *bottom* Dickson classes[7] $\mathbf{d}_{n,n-1}, \dots, \mathbf{d}_{n,n-k}$ form a regular sequence, see [17]. This conjecture was proven in 1996 by Dorra Bourguiba and Saïd Zarati, [6], using the classification of injective $\mathbb{F}[V]$-modules over $\mathscr{P}^*$ by J. Lannes and S. Zarati, see [18]. For a slightly different approach see the survey article by Larry Smith, [29].

In the aftermath Larry Smith, [30], asked whether the **Reverse Landweber-Stong Conjecture** is true, i.e., whether a ring of invariants $\mathbb{F}[V]^G$ has depth at least k if and only if the k *top* Dickson classes, $\mathbf{d}_{n,0}, \dots, \mathbf{d}_{n,k-1}$

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[6] Recall that the height of an arbitrary ideal $I \subset R$ in a commutative ring R with unity, is defined to be the minimal $k \in \mathbb{N}_0$ such that there exists a prime ideal $\mathfrak{p}$ of height k containing I

$$ht(I) := \min\{k \in \mathbb{N}_0 \mid \exists\ \mathfrak{p} \in Proj(R),\ ht(\mathfrak{p}) = k,\ I \subseteq \mathfrak{p}\}.$$

[7] Note that since we have a ring of invariants, of course, the Dickson algebra $\mathscr{D}^*(n) \hookrightarrow \mathbb{F}[V]^G$ is integrally contained in the ring of invariants.

form a regular sequence.[8] Since the Big Imbedding Theorem hands us a fractal of the Dickson algebra in any unstable algebra over the Steenrod algebra, we can formulate the Reverse Landweber-Stong Conjecture more generally as follows:

REVERSE LANDWEBER-STONG CONJECTURE : Let H^* be an unstable algebra over the Steenrod algebra. Then H^* has depth at least k if and only if high enough q-th powers of the k top Dickson classes, $\mathbf{d}_{n,}^{q^s}, \ldots, \mathbf{d}_{n,k-1}^{q^s} \in H^*$ form a regular sequence.

We wil give a counterexample to this conjecture. This emphasizes the significance of the original Landweber-Stong Conjecture and answers one of the many questions, which were the original motivation to start this investigation.

I exiled some technical stuff into the appendix.

I hope you gonna enjoy the reading of this paper and the nice results presented here.

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The last few bells and whistles were completed at the University of Minnesota, Minneapolis, in the spring of 1998 while I was a visiting assistant professor.

[8] When we take into account the $\mathcal{P}^*$-invariant prime spectrum of H^*, and in particular of the Dickson algebra $\mathcal{D}^*(n)$, then this version looks a lot more natural than the original Landweber-Stong Conjecture.

CHAPTER 1
Steenrod Operations and the Δ-Theorem

In this chapter we prepare ourselves for the journey we plan to make, i.e., we will provide the tools we will need in later chapters. In particular, we will prove the Δ-Theorem, [28] Theorem 10.5.4, in its most general form.

1.1 Steenrod Operations

Let H^* be an unstable algebra over the Steenrod algebra $\boldsymbol{\mathscr{P}}^*$. For basic facts about the Steenrod algebra we refer to [28] Chapter 10 and 11, or [27].

DEFINITION : *A map $D : \mathrm{H}^* \longrightarrow \mathrm{H}^*$ is called a* **derivation**, *if*

[1] *D acts linearly on H^*, and*

[2] *$D(h\,h') = D(h)h' + h\,D(h')$ for all $h,\ h' \in \mathrm{H}^*$.*

Recall the inductively defined elements of the Steenrod algebra

$$\mathcal{P}^{\Delta_1} := \mathcal{P}^1,$$

$$\mathcal{P}^{\Delta_i} := [\mathcal{P}^{\Delta_{i-1}}, \mathcal{P}^{q^{i-1}}],\ i \geq 2,$$

are primitive derivations,[1] see e.g. [20] Corollary 5 in §6.

Note in addition that

$$\mathcal{P}^{\Delta_0}(h) := \deg(h)h,\ \forall\ h \in \mathrm{H}^*$$

defines a derivation (that is not in $\boldsymbol{\mathscr{P}}^*$). Observe that[2]

$$\deg(\mathcal{P}^{\Delta_i}) = q^i - 1,\ \ i \geq 0.$$

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[1] These elements are the Q^i's in the notation of J. F. Adams and C. W. Wilkerson, resp. the $P^{(0,\dots,0,\ 1,\ 0,\cdots)}$'s (with the 1 in the i-th position), in J. W. Milnor's.

[2] Note again the change from the topologist's degree conventions, [20].

REMARK : The derivations defined above satisfy the following commutation rules

$$[\mathcal{P}^{\Delta_i},\ \mathcal{P}^{\Delta_j}] = \begin{cases} \mathcal{P}^{\Delta_i} & \text{for } i \neq 0 \text{ and } j = 0 \\ -\mathcal{P}^{\Delta_j} & \text{for } i = 0 \text{ and } j \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

for all $i,\ j \geq 0$, compare Lemma 10.6.4 in [28] or apply Theorem 4b of [20].

OBSERVATION 1.1.1 : *The set of all derivations $\mathcal{P}^{\Delta_i}$, $i \geq 0$, generates a left H^*-submodule of the endomorphisms of H^*, $\mathrm{End}(\mathrm{H}^*)$, which we will call $\Delta(\mathrm{H}^*)$, since, for any $a_0, \dots, a_n \in \mathrm{H}^*$, and any $i_0, \dots, i_n \geq 0$*

$$\delta := a_0\mathcal{P}^{\Delta_{i_0}} + \cdots + a_n\mathcal{P}^{\Delta_{i_n}}$$

is again a derivation.[3] *Note that an element $\delta \in \Delta(\mathrm{H}^*)$ is, by definition, zero if it evaluates on H^* to zero: $\delta(h) = 0\ \forall\ h \in \mathrm{H}^*$.*

The main goal of this section is to generalize in two ways some results of §2 and §5 of [1]:

[1] firstly, to algebras H^* over arbitrary Galois fields,

[2] and secondly by weakening the assumption of J. F. Adams and C. W. Wilkerson that H^* be an integral domain to H^* being reduced, i.e., the nil radical *Nil* $(\mathrm{H}^*) = 0$.[4]

We will summarize these results in Proposition 1.1.7.

The next convention should make life easier:

CONVENTION : Define the Steenrod operation $\mathcal{P}^i :\equiv 0$ if $i \notin \mathbb{N}_0$.

First we need a generalization of Lemma 2.2 of [1].

LEMMA 1.1.2 : *For all natural numbers $i \geq 1$ the following commutation rule holds*

$$\mathcal{P}^k\mathcal{P}^{\Delta_i} - \mathcal{P}^{\Delta_i}\mathcal{P}^k = \mathcal{P}^{\Delta_{i+1}}\mathcal{P}^{k-q^i}.$$

If $i = 0$ we have

$$\mathcal{P}^k\mathcal{P}^{\Delta_0} - \mathcal{P}^{\Delta_0}\mathcal{P}^k = k\mathcal{P}^k.$$

---

[3] Note that the H^*-sub module of the endomorphisms of H^* generated by the Steenrod derivations $\mathcal{P}^{\Delta_i}$, $i \geq 1$ is neither a sub algebra of $\mathcal{P}^*$ nor of the algebra $\mathrm{H}^* \odot \mathcal{P}^*$, see [19] Definition 2.5. In the latter it is just an $\mathbb{F}_q$-vector subspace.

[4] Note that the ground field is always assumed to be an arbitrary Galois field. Assumptions on the existence or nonexistence of nilpotent elements or zero divisors or elements of certain degrees in H^* is explicitly stated when needed. In particular, if we need to assume H^* is Noetherian we will explicitely say so.

PROOF : Since $\mathcal{P}^{\Delta_0}$ is just multiplication with the degree of h, the second statement is clear. For the first statement, note that if we had used Milnor's notation for the Steenrod operations, i.e.,

$$\mathcal{P}^{k} \triangleq P^{(k,0,\cdots)}$$

and

$$\mathcal{P}^{\Delta_i} \triangleq P^{(0,\dots,0,1,0,\cdots)},$$

where the 1 comes in the i-th position, then the proof would have been just an application of Milnor's multiplication rules for Steenrod operations, [20] Theorem 4b. The translation, back and forth, between the two notations is an exercise, that one might look up in the Appendix A.1.1 •

The second result generalizes Lemma 2.3 of [1] to arbitrary Galois fields. Note, that the proof is completely constructive, and new. Note also that we do not need to make any assumptions about the existence or non existence of zero divisors in H^*.

LEMMA 1.1.3 : *Let $h \in \mathrm{H}^*$ be an arbitrary element of degree d. Then for any $r \geq 0$ and for any $s \geq 0$ there exists an element $\mathcal{M}_{r,d} \in \mathcal{P}^*$ such that*

$$\mathcal{P}^{\Delta_{r+s}}(\mathcal{M}_{r,d}(h)) = (\mathcal{P}^{\Delta_s}(h))^{q^r}$$

where the notation $\mathcal{M}_{r,d}$ emphasizes, that $\mathcal{M}_{r,d}$ depends on r and d.

PROOF : Define inductively

$$\begin{aligned}
\mathcal{M}_{0,d} &:= \mathcal{P}^0, \\
\mathcal{M}_{1,d} &:= \mathcal{P}^{d_0-1}, \qquad \text{where } d_0 := d = \deg(h) \\
&\dots \\
\mathcal{M}_{r+1,d} &:= \mathcal{P}^{d_r-1}\mathcal{M}_{r,d}, \qquad \text{where } d_r := \deg(\mathcal{M}_{r,d}(h)).
\end{aligned}$$

Then it can be proven by induction with the help of Lemma 1.1.2 that these elements $\mathcal{M}_{r,d}$ satisfy the claimed equation. A detailed proof can be found in the Appendix A.1.2 •

REMARK : The recursion formulae for the new elements $\mathcal{M}_{r,d}$ given in the proof above lead to the following explicit description

$$\mathcal{M}_{r,d} = \mathcal{P}^{d_{r-1}-1} \cdots \mathcal{P}^{d_0-1}\mathcal{P}^0.$$

Moreover, one can prove by induction on r, that (see Appendix A.1.3)

$$d_r = q^r d_0 - q^r + 1.$$

The following series of lemmata leads to the result that, if $m+1$ derivations $\mathcal{P}^{\Delta_i}$ are linearly dependent, then any $m+1$ are. Lets start with a recollection of the definition:

DEFINITION : *Define* $D_0, \ldots, D_m \in \Delta(\mathrm{H}^*)$ *to be* **linearly dependent**, *if there exist nonzero elements* $a_0, \ldots, a_m \in \mathrm{H}^*$ *such that*

$$a_0 D_0 + \cdots + a_m D_m \equiv 0,$$

i.e., evaluating on H^* *gives zero; otherwise, call them* **linearly independent**.

The following lemma corresponds to Lemma 5.3 of [1]. Note that the assumption of the nil radical being trivial can't be removed: this is the reason why we will have to distinguish between two cases in the future.

LEMMA 1.1.4 : *Let* $Nil\,(\mathrm{H}^*) = (0)$ *and let* $r \in \mathbb{N}_0$. *If*

$$\mathcal{P}^{\Delta_{i_0+r}}, \ldots, \mathcal{P}^{\Delta_{i_m+r}}$$

are linearly dependent, then so are

$$\mathcal{P}^{\Delta_{i_0}}, \ldots, \mathcal{P}^{\Delta_{i_m}}.$$

PROOF : Without loss of generality we assume that m is minimal with the property that the derivations $\mathcal{P}^{\Delta_{i_0}}, \ldots, \mathcal{P}^{\Delta_{i_m}}$ are linearly dependent, which in turn is true if and only if

$$\det\left(\mathcal{P}^{\Delta_{i_\alpha}}(h_\beta)\right)_{\alpha,\beta=0,\ldots,m} = 0$$

for any elements $h_0, \ldots, h_m \in \mathrm{H}^*$. The "if"-part of this statement follows from the observation, that, if we expand this matrix using the first column, we get

$$a_0\mathcal{P}^{\Delta_{i_0}}(h_0) + \cdots + a_m\mathcal{P}^{\Delta_{i_m}}(h_0) = 0$$

for any element $h_0 \in \mathrm{H}^*$. So, this is a relation of linear dependence, because the coefficients $a_i \neq 0$ for $i = 0, \ldots, m$, since they are just the determinant of the appropriate sub matrices and m was chosen to be minimal. For the "only if"- part we choose a relation of linear dependence

$$a_0\mathcal{P}^{\Delta_{i_0}} + \cdots + a_m\mathcal{P}^{\Delta_{i_m}} \equiv 0.$$

Then the row vectors of our matrix

$$\begin{matrix} \left(\mathcal{P}^{\Delta_{i_0}}(h_0), \ldots, \mathcal{P}^{\Delta_{i_0}}(h_m)\right) \\ \cdots \\ \left(\mathcal{P}^{\Delta_{i_m}}(h_0), \ldots, \mathcal{P}^{\Delta_{i_m}}(h_m)\right) \end{matrix},$$

for all $h_0, \ldots, h_m \in \mathrm{H}^*$, are linearly dependent, because

$$\begin{matrix} a_0\left(\mathcal{P}^{\Delta_{i_0}}(h_0), \ldots, \mathcal{P}^{\Delta_{i_0}}(h_m)\right) \\ + \cdots + \\ a_m\left(\mathcal{P}^{\Delta_{i_m}}(h_0), \ldots, \mathcal{P}^{\Delta_{i_m}}(h_m)\right) \end{matrix} = (0, \ldots, 0),$$

hence we get

$$\det\left(\mathcal{P}^{\Delta_{i_\alpha}}(h_\beta)\right)_{\alpha,\beta=0,\ldots,m} = 0.$$

Next, Lemma 1.1.3 guarantees the existence of Steenrod powers $\mathcal{M}_{r,d_\beta}$ such that

$$(\mathcal{P}^{\Delta_{i_\alpha}}(h_\beta))^{q^r} = \mathcal{P}^{\Delta_{i_\alpha+r}}(\mathcal{M}_{r,d_\beta}(h_\beta)),$$

where $d_\beta = \deg(h_\beta)$. Therefore

$$\begin{aligned}\left(\det\left(\mathcal{P}^{\Delta_{i_\alpha}}(h_\beta)\right)_{\alpha,\beta=0,\dots,m}\right)^{q^r} &= \det\left(\mathcal{P}^{\Delta_{i_\alpha}}(h_\beta)^{q^r}\right)_{\alpha,\beta=0,\dots,m}\\ &= \det\left(\mathcal{P}^{\Delta_{i_\alpha+r}}(\mathcal{M}_{r,d_\beta}(h_\beta))\right)_{\alpha,\beta=0,\dots,m}\\ &= 0\end{aligned}$$

by assumption. Since $Nil\,(\mathrm{H}^*) = (0)$ this implies

$$\det\left(\mathcal{P}^{\Delta_{i_\alpha}}(h_\beta)\right)_{\alpha,\beta=0,\dots,m} = 0,$$

which was to be shown •

Our next lemma generalizes Lemma 5.4 of [1].

LEMMA 1.1.5: *Let $Nil\,(\mathrm{H}^*) = (0)$ and let $r \in \mathbb{N}_0$. If*

$$\mathcal{P}^{\Delta_0},\ \mathcal{P}^{\Delta_{i_1}},\dots,\ \mathcal{P}^{\Delta_{i_m}}$$

are linearly dependent, where $i_j > 0$ for $j = 1,\dots, m$, then so are

$$\mathcal{P}^{\Delta_r},\ \mathcal{P}^{\Delta_{i_1}},\dots,\ \mathcal{P}^{\Delta_{i_m}},$$

for any $r \ge 0$.

PROOF: The statement is redundant if $r = 0$. So assume that $r > 0$. Let $h_0,\dots, h_m \in \mathrm{H}^*$ such that

$$h_0\mathcal{P}^{\Delta_0} + h_1\mathcal{P}^{\Delta_{i_1}} + \cdots + h_m\mathcal{P}^{\Delta_{i_m}} = 0$$

is a nontrivial linear relation. Assume without loss of generality that h_0 is a q-th power (if not multiply the above equation by h_0^{q-1}). From the commutation rules for the derivations $\mathcal{P}^{\Delta}$'s, see page 13, we get

$$\begin{aligned}0 &= \mathcal{P}^{\Delta_r}\left(h_0\mathcal{P}^{\Delta_0} + h_1\mathcal{P}^{\Delta_{i_1}} + \cdots + h_m\mathcal{P}^{\Delta_{i_m}}\right)\\ &= \mathcal{P}^{\Delta_r}(h_0)\mathcal{P}^{\Delta_0} + \mathcal{P}^{\Delta_r}(h_1)\mathcal{P}^{\Delta_{i_1}} + \cdots + \mathcal{P}^{\Delta_r}(h_m)\mathcal{P}^{\Delta_{i_m}}\\ &\quad + h_0\mathcal{P}^{\Delta_0}\mathcal{P}^{\Delta_r} + h_1\mathcal{P}^{\Delta_{i_1}}\mathcal{P}^{\Delta_r} + \cdots + h_m\mathcal{P}^{\Delta_{i_m}}\mathcal{P}^{\Delta_r} + h_0\mathcal{P}^{\Delta_r}\\ &= h_0\mathcal{P}^{\Delta_r} + \mathcal{P}^{\Delta_r}(h_1)\mathcal{P}^{\Delta_{i_1}} + \cdots + \mathcal{P}^{\Delta_r}(h_m)\mathcal{P}^{\Delta_{i_m}},\end{aligned}$$

where the last equality holds, because

$$h_0\mathcal{P}^{\Delta_0}\mathcal{P}^{\Delta_r} + \cdots + h_m\mathcal{P}^{\Delta_{i_m}}\mathcal{P}^{\Delta_r} = \left(h_0\mathcal{P}^{\Delta_0} + \cdots + h_m\mathcal{P}^{\Delta_m}\right)\mathcal{P}^{\Delta_r} \equiv 0$$

by assumption, and $\mathcal{P}^{\Delta_r}(h_0) = 0$, since h_0 is a q-th power. Therefore

$$\mathcal{P}^{\Delta_r},\ \mathcal{P}^{\Delta_{i_1}},\dots,\ \mathcal{P}^{\Delta_{i_m}}$$

are linearly dependent •

PROPOSITION 1.1.6 : *Let Nil* $(\mathrm{H}^*) = (0)$. *Let* $\mathcal{P}^{\Delta_{i_0}}, \ldots, \mathcal{P}^{\Delta_{i_m}}$ *be linearly dependent, then so are*

$$\mathcal{P}^{\Delta_0}, \ldots, \mathcal{P}^{\Delta_m}.$$

PROOF : Denote by $\mathcal{S}$ the set of all $(m+1)$-tuples $(s_0, \ldots, s_m)$ such that

$$\mathcal{P}^{\Delta_{s_0}}, \ldots, \mathcal{P}^{\Delta_{s_m}}$$

are linearly dependent. By assumption this set in not empty. We have to show that

$$(0, \ldots, m) \in \mathcal{S}.$$

We will do this by giving an algorithm for the construction of a relation of linear dependence for $\mathcal{P}^{\Delta_0}, \ldots, \mathcal{P}^{\Delta_n}$ starting from an arbitrary $(i_0, \ldots, i_m) \in \mathcal{S}$.

By 1.1.4 we know that

$$(0,\ i_1 - i_0, \ldots, i_m - i_0) \in \mathcal{S}.$$

STEP 1 : If $(0,\ i_1 - i_0, \ldots, i_m - i_0) \neq (0, \ldots, m)$ define

$$\alpha := \min\{1, \ldots, m \mid i_\alpha - i_{\alpha-1} > 1\},$$

i.e. ,

$$(0,\ 1, \ldots, \alpha - 1, i_\alpha - i_0, \ldots, i_m - i_0) \in \mathcal{S}.$$

STEP 2 : Choosing $r = \alpha$, 1.1.5 gives after reordering that

$$(1, \ldots, \alpha,\ i_\alpha - i_0, \ldots, i_m - i_0) \in \mathcal{S}.$$

STEP 3 : Applying 1.1.4 with $r = 1$ leads to

$$(0, \ldots, \alpha - 1,\ i_\alpha - i_0 - 1, \ldots, i_m - i_0 - 1) \in \mathcal{S}.$$

If $(0, \ldots, \alpha - 1,\ i_\alpha - i_0 - 1, \ldots, i_m - i_0 - 1) = (0, \ldots, m)$ we are done. If not, either

$$(i_\alpha - i_0 - 1) - (\alpha - 1) > 1, \quad \text{or} \quad (i_\alpha - i_0 - 1) - (\alpha - 1) = 1.$$

In the first case repeat the Steps 2 and 3, in the latter the “gap” between the ascending numbers i_j has moved to the right. So, define α renewed as explained in Step 1 and repeat the procedure.

In either, case we eventually get,

$$(0, \ldots, m) \in \mathcal{S}$$

as claimed •

We summarize our results:

PROPOSITION 1.1.7: *With the above notations. If there is an $(m+1)$-tuple of linearly dependent $\mathcal{P}^{\Delta}$'s then any $m+1$ are linearly dependent.*

PROOF: Let $(i_0, \ldots, i_m) \in \mathcal{S}$. Then by Proposition 1.1.6 $(0, \ldots, m) \in \mathcal{S}$. We show from this that any $(m+1)$-tuple $(k_0, \ldots, k_m)$ belongs to $\mathcal{S}$. Define

$$r_i := k_i + (m - i).$$

We have that $(0, \ldots, m) \in \mathcal{S}$ by assumption.

STEP 1: Define $r = r_0$ and apply Lemma 1.1.5. Then we have that

$$(r_0, 1, \ldots, m) \in \mathcal{S}.$$

STEP 2: Define $r = 1$ and apply Lemma 1.1.4. Then we get

$$(r_0 - 1, 0, \ldots, m-1) \in \mathcal{S}.$$

Repeat Step 1 with $r := r_1$, and then Step 2. We get

$$(r_0 - 2, r_1 - 1, 0, \ldots, m-2) \in \mathcal{S}.$$

After doing this m times we find

$$(r_0 - m, r_1 - (m-1), \ldots, r_{m-1} - 1, 0) \in \mathcal{S}.$$

Apply Lemma 1.1.5 once more with $r = r_m$ to get

$$(k_0, \ldots, k_m) = (r_0 - m, r_1 - (m-1), \ldots, r_m) \in \mathcal{S}.$$

This gives the desired algorithm •

The following result generalizes Lemma 5.5 of [1]. We will need this later in Chapter 5.

LEMMA 1.1.8: *Let Nil* $(\mathrm{H}^*) = (0)$. *If*

$$0 \equiv \sum_{i=0}^{m} h_i \mathcal{P}^{\Delta_i} = \det \begin{bmatrix} \mathcal{P}^{\Delta_0} & \mathcal{P}^{\Delta_0}(a_1) & \cdots & \mathcal{P}^{\Delta_0}(a_m) \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{P}^{\Delta_m} & \mathcal{P}^{\Delta_m}(a_1) & \cdots & \mathcal{P}^{\Delta_m}(a_m) \end{bmatrix}$$

evaluates to zero on H^*, *for some* $a_1, \ldots, a_m \in \mathrm{H}^*$, *then*

$$h_0^{q^r} \mathcal{P}^{\Delta_r} + \cdots + h_m^{q^r} \mathcal{P}^{\Delta_{r+m}} \equiv 0$$

is zero in $\Delta(\mathrm{H}^*)$ *for any* $r \in \mathbb{N}_0$.

PROOF: From what we have proved so far we know that for any $r \in \mathbb{N}_0$

$$\mathcal{P}^{\Delta_r}, \dots, \mathcal{P}^{\Delta_{r+m}}$$

are linearly dependent as elements in $\Delta(\mathrm{H}^*)$, if $\mathcal{P}^{\Delta_0}, \dots, \mathcal{P}^{\Delta_m}$ are. Without loss of generality let m be minimal with this property. In particular, there exist elements $k_0, \dots, k_m \in \mathrm{H}^*$ such that

$$k_0\mathcal{P}^{\Delta_r} + \cdots + k_m\mathcal{P}^{\Delta_{r+m}} = 0 \in \Delta(\mathrm{H}^*).$$

We have to determine the k_i's.

First, note that Lemma 1.1.3 guarantees the existence of an element $\mathcal{M}_{r,\,d} \in \mathcal{P}^*$ depending on r and $d = \deg(h)$, such that

$$\mathcal{P}^{\Delta_{r+i}}\left(\mathcal{M}_{r,\,d}(h)\right) = \left(\mathcal{P}^{\Delta_i}(h)\right)^{q^r}.$$

Choose $a_1, \dots, a_m \in \mathrm{H}^*$ such that

$$\det\left(\mathcal{P}^{\Delta_i}(a_j)\right)_{i,\,j=1,\dots,m} \neq 0.$$

Such a choice is possible since m is minimal by assumption. Therefore, for $d_j = \deg(a_j)$,

$$\begin{aligned}\det\left(\mathcal{P}^{\Delta_{r+i}}\left(\mathcal{M}_{r,\,d_j}(a_j)\right)\right)_{i,\,j=1,\dots,m} &= \left(\det\left(\mathcal{P}^{\Delta_i}(a_j)\right)_{i,\,j=1,\dots,m}\right)^{q^r}\\ &\neq 0.\end{aligned}$$

Consider the $(m+1)\times m$ matrix

$$\begin{aligned}\mathbf{M} &:= \left(\mathcal{P}^{\Delta_{r+i}}(\mathcal{M}_{r,\,d_j}(a_j))\right)_{i=0,\dots,m,\ j=1,\dots,m}\\ &= \begin{bmatrix} \mathcal{P}^{\Delta_r}(\mathcal{M}_{r,\,d_1}(a_1)) & \cdots & \mathcal{P}^{\Delta_r}(\mathcal{M}_{r,\,d_m}(a_m)) \\ \cdots & \cdots & \cdots \\ \mathcal{P}^{\Delta_{r+m}}(\mathcal{M}_{r,\,d_1}(a_1)) & \cdots & \mathcal{P}^{\Delta_{r+m}}(\mathcal{M}_{r,\,d_m}(a_m)) \end{bmatrix}\end{aligned}$$

and denote by $\mathbf{M}_i$ the $m \times m$ sub matrix which is obtained from $\mathbf{M}$ by erasing the $i+1$-st row. From the above we have

$$\det(\mathbf{M}_i) \neq 0$$

for $i = 0, \dots, m$. Setting $k_i := \det(\mathbf{M}_i)$ it is easy to see that

$$0 = \sum_{i=0}^{m} k_i\mathcal{P}^{\Delta_{r+i}} = \det\begin{bmatrix} \mathcal{P}^{\Delta_r} & \mathcal{P}^{\Delta_r}(\mathcal{M}_{r,\,d_1}(a_1)) & \cdots & \mathcal{P}^{\Delta_r}(\mathcal{M}_{r,\,d_m}(a_m)) \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{P}^{\Delta_{r+m}} & \mathcal{P}^{\Delta_{r+m}}(\mathcal{M}_{r,\,d_1}(a_1)) & \cdots & \mathcal{P}^{\Delta_{r+m}}(\mathcal{M}_{r,\,d_m}(a_m)) \end{bmatrix}$$

(just expand the determinant using the first column). Hence

$$\sum_{i=0}^{m} k_i \mathcal{P}^{\Delta_{r+i}} = \det \begin{bmatrix} \mathcal{P}^{\Delta_r} & (\mathcal{P}^{\Delta_0}(a_1))^{q^r} & \cdots & (\mathcal{P}^{\Delta_0}(a_m))^{q^r} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{P}^{\Delta_{r+m}} & (\mathcal{P}^{\Delta_m}(a_1))^{q^r} & \cdots & (\mathcal{P}^{\Delta_m}(a_m))^{q^r} \end{bmatrix}$$

$$= \sum_{i=0}^{m} h_i^{q^r} \mathcal{P}^{\Delta_{r+i}},$$

since taking to q-th powers is additive over $\mathbb{F}_q$ •

1.2 The Δ-Theorem, Corollaries and Examples

We are now ready to prove the promised highlight of this chapter, the Δ-Theorem (compare [28] Theorem 10.5.4). In addition, we will draw some consequences and exhibit some examples.

First recall Proposition 1.1.7: if $m+1$ $\mathcal{P}^{\Delta}$'s of the H^*-module of derivations $\Delta(\mathrm{H}^*)$ are linearly dependent and if $Nil\,(\mathrm{H}^*) = (0)$ then the $m+1$ derivations of lowest degree

$$\mathcal{P}^{\Delta_0}, \ldots, \mathcal{P}^{\Delta_m}$$

are linearly dependent.[5] This motivates the following definition.

Definition : *Let H^* be an unstable algebra over the Steenrod algebra with[6] $Nil\,(\mathrm{H}^*) = (0)$ and let $\Delta(\mathrm{H}^*)$ be the H^*-module of derivations generated by $\mathcal{P}^{\Delta_i}$, $\forall\, i \in \mathbb{N}_0$. If there exists an $m \in \mathbb{N}_0$ such that some $m+1$ derivations (and hence by Proposition 1.1.7 any $m+1$ derivations) are linearly dependent, then H^* is called* **Δ-finite**. *Moreover, call the relation of linear dependence*

$$h_0\mathcal{P}^{\Delta_0} + \cdots + h_m\mathcal{P}^{\Delta_m} = 0$$

(where m is chosen to be minimal w.r.t. this property) the **Δ-relation** *of H^* and $m = m(\mathrm{H}^*)$ the* **Δ-length**.

Let's have a first example.

Example 1 : Let $\mathrm{H}^* = \mathbb{F}[x]$ be a polynomial algebra in one linear generator over $\mathbb{F} = \mathbb{F}_q$. Then H^* is Δ-finite, because

$$-x^{q-1}\mathcal{P}^{\Delta_0} + \mathcal{P}^{\Delta_1} = 0.$$

Moreover, as Lemma 1.1.8 predicts, we have

$$-(x)^{(q-1)q^i}\mathcal{P}^{\Delta_i} + \mathcal{P}^{\Delta_{i+1}} = 0.$$

[5] N.B.: This does *not* imply that $\Delta(\mathrm{H}^*)$ is finitely generated as an H^*-module.

[6] We make the assumption that H^* is reduced, because we want to use Proposition 1.1.7. However, one could define the following also without this assumption.

This leads inductively to

$$\mathcal{P}^{\Delta_i} \in \mathbb{F}[x]\mathcal{P}^{\Delta_0} \quad \forall\ i \in \mathbb{N}_0,$$

i.e.,

$$\Delta(\mathbb{F}[x]) = \mathbb{F}[x]\mathcal{P}^{\Delta_0} \subset \mathrm{End}(\mathrm{H}^*)$$

is a (even free!) $\mathbb{F}[x]$-module generated by the single derivation of lowest degree $\mathcal{P}^{\Delta_0}$. Much much later we will see that this is not an isolated example, see 8.1.3.

The following theorem shows that for Noetherian algebras the module of derivations is always finitely generated, so in particular, if we add the assumption that our algebra is reduced, it is also Δ-finite.

THEOREM 1.2.1 (Δ-Theorem): *Let* H^* *be a Noetherian unstable algebra over the Steenrod algebra* $\boldsymbol{\mathcal{P}}^*$*. Then* $\Delta(\mathrm{H}^*)$ *is finitely generated as an* H^**-module. If in addition the nil radical of* H^* *is trivial, i.e., Nil* $(\mathrm{H}^*) = (0)$*, then* H^* *is* Δ*-finite, i.e., there exist a natural number* $m \in \mathbb{N}_0$ *and nonzero elements* $h_0, \dots, h_m \in \mathrm{H}^*$ *such that*

$$h_0\mathcal{P}^{\Delta_0} + \cdots + h_m\mathcal{P}^{\Delta_m} = 0,$$

where m *is minimal with respect to this property.*

PROOF: Let $y_1, \dots, y_s$ be generators of H^* as an algebra over $\mathbb{F}_q$. Then we have a monomorphism of H^*-modules defined by

$$\begin{array}{rccc} \varphi: & \Delta(\mathrm{H}^*) & \hookrightarrow & \bigoplus_s \mathrm{H}^* \\ & \delta & \longmapsto & (\delta(y_1), \dots, \delta(y_s)). \end{array}$$

Injectivity follows from:

$$\varphi(\delta) = 0 \iff \delta(\mathrm{H}^*) \equiv 0 \iff \delta = 0,$$

by definition of $\Delta(\mathrm{H}^*)$. Hence $\Delta(\mathrm{H}^*)$ is isomorphic to its image $\varphi(\Delta(\mathrm{H}^*))$, which in turn is a submodule of a finitely generated Noetherian H^*-module, namely $\bigoplus_s \mathrm{H}^*$. Therefore $\Delta(\mathrm{H}^*)$ is also finitely generated as an H^*-module, i.e., there exists a natural number, $g = g(\mathrm{H}^*)$, such that $\Delta(\mathrm{H}^*)$ is generated by

$$\delta_1, \dots, \delta_g.$$

Without loss of generality we can assume that the δ's are $\mathcal{P}^{\Delta}$'s and we get a generating set consisting of

$$\mathcal{P}^{\Delta_{i_1}}, \dots, \mathcal{P}^{\Delta_{i_g}},$$

which in turn means that any other element $\mathcal{P}^{\Delta_i}$ in our module can be expressed as a linear combination of these, in particular, for $i \in \mathbb{N}_0$

$$\mathcal{P}^{\Delta_i} = a_1\mathcal{P}^{\Delta_{i_1}} + \cdots + a_g\mathcal{P}^{\Delta_{i_g}}$$

for some $a_1, \ldots, a_g \in H^*$. In other words: for any $i \notin \{i_1, \ldots, i_g\}$ the elements

$$\mathcal{P}^{\Delta_i},\ \mathcal{P}^{\Delta_{i_1}}, \ldots, \mathcal{P}^{\Delta_{i_g}}$$

are linearly dependent. Let m be minimal with respect to the property that $\mathcal{P}^{\Delta_{i_0}}, \ldots, \mathcal{P}^{\Delta_{i_m}}$ are linearly dependent, for some $i_0, \ldots, i_n \in \mathbb{N}_0$. Then, by Proposition 1.1.6, we get that[7]

$$\mathcal{P}^{\Delta_0}, \ldots, \mathcal{P}^{\Delta_m}$$

are linearly dependent. Hence H^* is Δ-finite and we are all set •

However, Noetherianess is not necessary for being Δ-finite. The next example illustrates this.

EXAMPLE 2 : Let $\mathbb{F} = \mathbb{F}_2$ be the field with two elements. Take the polynomial algebra in two linear generators, $\mathbb{F}[x,\ y]$, and let H^* be the sub algebra generated by $x,\ xy,\ xy^2,\ xy^3, \cdots$. Then H^* is not Noetherian. However, H^* is Δ-finite with Δ-relation

$$x(xy^2 + x^2y)\mathcal{P}^{\Delta_0} + x(x^2 + xy + y^2)\mathcal{P}^{\Delta_1} + x\mathcal{P}^{\Delta_2} = 0$$

of length 2. Note carefully, that the coefficients are multiples of the Dickson classes of $\mathbb{F}[x,\ y]$

$$x\mathbf{d}_{2,\,0} = x(xy^2 + x^2y)$$
$$x\mathbf{d}_{2,\,1} = x(x^2 + xy + y^2)$$
$$x\mathbf{d}_{2,\,2} = x,$$

where $\mathbf{d}_{n,\,n} = 1,\ \forall\ n$ by convention.

REMARK : Since, for $i \geq 0$,

$$\deg(\mathcal{P}^{\Delta_i}) = q^i - 1,$$

the homogeneity of the Δ-relation leads to

$$\deg(h_i) = q^m - q^i + \deg(h_m),$$

for any $i = 0, \ldots, m-1$ as elements of the graded algebra H^*.

REMARK : If $h_m = 1$ (or more generally a unit) then the coefficients $h_0, \ldots, h_{m-1}$ of the Δ-relation are uniquely determined by minimality.

Let's have a look at another example.

[7] It is exactly here, where our assumption that the nil radical of H^* is zero, comes into the game: Proposition 1.1.6 depends on Lemma 1.1.3, which in turn needs precisely this: the non existence of nil potent elements.

EXAMPLE 3: Take the field $\mathbb{F}_2$ of two elements and consider a polynomial algebra in two linear generators $x,\ y$ over $\mathbb{F}_2$, $\mathbb{F}[x,\ y]$. We have a chain of unstable algebras

$$\mathbb{F}_2[x^2,\ y^2] \hookrightarrow \mathbb{F}_2[x^2,\ y] \hookrightarrow \mathbb{F}_2[x,\ y].$$

The Δ-relation in the smallest of these, $\mathbb{F}_2[x^2,\ y^2]$, has length 0

$$\mathcal{P}^{\Delta_0} \equiv 0.$$

For the middle algebra, $\mathbb{F}_2[x^2,\ y]$, we get a Δ-relation of length 1

$$y\mathcal{P}^{\Delta_0} + \mathcal{P}^{\Delta_1} \equiv 0.$$

For the big algebra, $\mathbb{F}_2[x,\ y]$, our Δ-relation has length 2

$$(x^2 y + x y^2)\mathcal{P}^{\Delta_0} + (x^2 + x y + y^2)\mathcal{P}^{\Delta_1} + \mathcal{P}^{\Delta_2} \equiv 0.$$

Note that the length of the respective Δ-relations does not exceed the Krull dimension. This is true in a more general setting:

COROLLARY 1.2.2: *Let* H^* *be reduced Noetherian. Then the Krull dimension of* H^**,* $\dim(\mathrm{H}^*)$*, is at least* m*, the length of the* Δ*-relation.*

PROOF: Let $\dim(\mathrm{H}^*) = n$. Then any set of $n+1$ elements $a_0, \ldots, a_n \in \mathrm{H}^*$ is algebraically dependent. Hence the determinant of the generalized Jacobian matrix vanishes or is a zero divisor, see Theorem A.4.1 in the Appendix,

$$\det\left(\mathcal{P}^{\Delta_i}(a_j)\right)_{i,\ j=0,\ldots,\ n} = \overline{h}_0\mathcal{P}^{\Delta_0}(a_0) + \cdots + \overline{h}_n\mathcal{P}^{\Delta_n}(a_0)$$

for any $a_0 \in \mathrm{H}^*$, where we get the last equation from the Lagrange expansion using the first column. Hence for some $h \in \mathrm{H}^* \setminus \{0\}$ we get a relation of the form

$$0 = \overline{h}\det\left(\mathcal{P}^{\Delta_i}(a_j)\right)_{i,\ j=0,\ldots,\ n} = \overline{h}\left(\overline{h}_0\mathcal{P}^{\Delta_0}(a_0) + \cdots + \overline{h}_n\mathcal{P}^{\Delta_n}(a_0)\right)$$

However, if this is the trivial relation, i.e.,

$$\overline{h\,h_0} = \cdots = \overline{h\,h_n} = 0,$$

then, since

$$\overline{h}_k = \det\left(\mathcal{P}^{\Delta_i}(a_j)\right)_{i,\ j=0,\ldots,\hat{k},\ldots,\ n},$$

where we adopt the topologist's convention that $\widehat{\ }$ means that this element is omitted, the generalized Jacobian matrix

$$\overline{h}_n = \det\left(\mathcal{P}^{\Delta_i}(a_j)\right)_{i,\ j=0,\ldots,\ n-1}$$

vanishes or is a zerodivisor. So, we would have by Lagrange expansion a relation of length at most $n-1$

$$\tilde{h}\left(\tilde{h}_0\mathcal{P}^{\Delta_0}(a_0)+\cdots+\tilde{h}_{n-1}\mathcal{P}^{\Delta_{n-1}}(a_0)\right)=0,$$

for all $a_0 \in \mathrm{H}^*$. Proceeding this way we end up with a nontrivial Δ-relation

$$\begin{aligned} 0 &= \check{h}\det\left(\mathcal{P}^{\Delta_i}(a_j)\right)_{i,\ j=0,\dots,k} \\ &= \check{h}\left(\check{h}_0\mathcal{P}^{\Delta_0}(a_0)+\cdots+\check{h}_k\mathcal{P}^{\Delta_k}(a_0)\right) \end{aligned}$$

for some k , $0\le k\le n$ •

Recall the example above, $\mathbb{F}_2[x,\ y]$, and its Δ-relation

$$(x^2y+xy^2)\mathcal{P}^{\Delta_0}+(x^2+xy+y^2)\mathcal{P}^{\Delta_1}+\mathcal{P}^{\Delta_2}\equiv 0.$$

Note that the coefficients are exactly the Dickson classes

$$\begin{aligned} \mathbf{d}_{2,0} &= x^2y+xy^2 \\ \mathbf{d}_{2,1} &= x^2+xy+y^2. \end{aligned}$$

This is no accident as the next theorem shows, compare [28] Proposition 10.4.3 for the case of prime fields.

THEOREM 1.2.3: *The Δ-relation for $\mathbb{F}[x_1,\dots,x_n]$ is given by*

$$(-1)^n\mathbf{d}_{n,0}\mathcal{P}^{\Delta_0}+\cdots+(-1)\mathbf{d}_{n,n-1}\mathcal{P}^{\Delta_{n-1}}+\mathcal{P}^{\Delta_n}\equiv 0.$$

PROOF: It is enough to show that

$$(-1)^n\mathbf{d}_{n,0}\mathcal{P}^{\Delta_0}(x_i)+\cdots+(-1)\mathbf{d}_{n,n-1}\mathcal{P}^{\Delta_{n-1}}(x_i)+\mathcal{P}^{\Delta_n}(x_i)=0$$

for any algebra generator $x_i\in\mathbb{F}[x_1,\dots,x_n]$. Since our generators have degree 1 we have

$$\mathcal{P}^{\Delta_j}(x_i)=x_i^{q^j},\quad \forall\ i=1,\dots,n,\ \forall\ j\ge 0.$$

Hence

$$\det\left(\mathcal{P}^{\Delta_j}(x_i)\right)_{i,\ j}=\det\left(x_i^{q^j}\right)_{i,\ j}.$$

In [7] L. E. Dickson showed that

$$\det\left(\mathcal{P}^{\Delta_j}(x_i)\right)_{i=1,\dots,n,\ j=0,\dots,\widehat{k},\dots,n}=\mathbf{d}_{n,\ k}.$$

Therefore we get for any $l=1,\dots,n$

$$
\begin{aligned}
&(-1)^n \mathbf{d}_{n,0}\left(\mathcal{P}^{\Delta_0}(x_l)\right) + \cdots + (-1)\mathbf{d}_{n,n-1}\left(\mathcal{P}^{\Delta_{n-1}}(x_l)\right) + \left(\mathcal{P}^{\Delta_n}(x_l)\right) \\
&\quad = (-1)^n \left[\det\left(\mathcal{P}^{\Delta_j}(x_i)\right)_{i=1,\dots,n}^{j=1,\dots,n}\left(\mathcal{P}^{\Delta_0}(x_l)\right) - \cdots\right. \\
&\qquad \left.\cdots + (-1)^{n-1}\det\left(\mathcal{P}^{\Delta_j}(x_i)\right)_{i=1,\dots,n}^{j=0,\dots,n-2,\ n}\left(\mathcal{P}^{\Delta_{n-1}}(x_l)\right) + (-1)^n\left(\mathcal{P}^{\Delta_n}(x_l)\right)\right] \\
&\quad = (-1)^n \left[\det\left(x_i^{q^j}\right)_{i=1,\dots,n}^{j=1,\dots,n}\left(\mathcal{P}^{\Delta_0}(x_l)\right) - \cdots\right. \\
&\qquad \left.\cdots + (-1)^{n-1}\det\left(x_i^{q^j}\right)_{i=1,\dots,n}^{j=0,\dots,n-2,\ n}\left(\mathcal{P}^{\Delta_{n-1}}(x_l)\right) + (-1)^n\left(\mathcal{P}^{\Delta_n}(x_l)\right)\right] \\
&\quad = (-1)^n \left[\det\left(x_i^{q^j}\right)_{i=1,\dots,n}^{j=1,\dots,n} x_l - \cdots + (-1)^{n-1}\det\left(x_i^{q^j}\right)_{i=1,\dots,n}^{j=0,\dots,n-2,\ n} x_l^{q^{n-1}}\right. \\
&\qquad \left. + (-1)^n x_l^{q^n}\right] \\
&\quad = (-1)^n \det\begin{bmatrix} x_l & x_1 & \cdots & x_n \\ x_l^q & x_1^q & \cdots & x_n^q \\ \cdots & \cdots & \cdots & \cdots \\ x_l^{q^n} & x_1^{q^n} & \cdots & x_n^{q^n} \end{bmatrix}
\end{aligned}
$$

$= 0$, where we made use of Lagrange expansion with respect to the first column and the assumption that $l \in \{1, \dots, n\}$ •

Remark: Note that the Δ-relation for the Dickson algebra, $\mathcal{D}^*(n)$, has length n, because otherwise there would exist elements $a_0, \dots, a_m \in \mathcal{D}^*(n)$, $m \le n-1$, not all zero, such that

$$a_0\mathcal{P}^{\Delta_0}(\mathbf{d}_{n,i}) + \cdots + a_m\mathcal{P}^{\Delta_m}(\mathbf{d}_{n,i}) = 0$$

for any $i = 0, \dots, n-1$. However, we know how the Steenrod algebra acts on the Dickson polynomials (compare Proposition A.2.1 in the Appendix); in particular we get

$$
\begin{aligned}
0 &= a_0\mathcal{P}^{\Delta_0}(\mathbf{d}_{n,i}) + \cdots + a_m\mathcal{P}^{\Delta_m}(\mathbf{d}_{n,i}) \\
&= (-1)^{i+1}a_i\mathbf{d}_{n,0} \quad \forall\ i = 0, \dots, m.
\end{aligned}
$$

This means that all the coefficients $a_0, \dots, a_m$ must be zero, which contradicts our assumption. Hence, by uniqueness, the Δ-relation given in the above theorem is also the Δ-relation for the Dickson algebra $\mathcal{D}^*(n)$, and hence for any unstable algebra between $\mathcal{D}^*(n)$ and $\mathbb{F}[V]$. Note in Example 1 the Dickson algebra $\mathcal{D}^*(2)$ is not contained in $\mathbb{F}[x^2, y^2]$ or $\mathbb{F}[x^2, y]$.

We will need the following proposition later.[8]

[8] I agree these calculations at the high school algebra level are not really exciting; but be a bit more patient: we are almost there.

PROPOSITION 1.2.4: *Let* H^* *be* Δ*-finite. Let* $h_0, \dots, h_m$ *be the coefficients occuring in the* Δ*-relation. Then for any* $\alpha \in \mathbb{N}_0$

$$\Delta(\alpha,\ h_0, \dots,\ h_m) := \mathcal{P}^{\alpha}(h_0)\mathcal{P}^{\Delta_0} + \cdots + \mathcal{P}^{\alpha}(h_m)\mathcal{P}^{\Delta_m} + \sum_{i=0}^{m} \mathcal{P}^{\alpha - q^i}(h_i)\mathcal{P}^{\Delta_{i+1}} = 0$$

is zero as an element of the H^**-module* $\Delta(\mathrm{H}^*)$.

PROOF: The statement for $\alpha = 0$ is the contents of the Δ-Theorem, 1.2.1, i.e., we have $\Delta(0,\ h_0, \dots,\ h_m) = 0$ in $\Delta(\mathrm{H}^*)$, because[9] $\mathcal{P}^0$ is the identity. We proceed by induction on α. So, let $\alpha = 1$, then[10]

$$\begin{aligned}
0 &= \mathcal{P}^1 \Delta_0 \\
&= \sum_{i=0}^{m} \mathcal{P}^1(h_i)\mathcal{P}^{\Delta_i} + h_i \mathcal{P}^1 \mathcal{P}^{\Delta_i} \\
&= \sum_{i=0}^{m} \left(\mathcal{P}^1(h_i)\mathcal{P}^{\Delta_i} + h_i \mathcal{P}^{\Delta_i}\mathcal{P}^1\right) + h_0 \mathcal{P}^1 \\
&= \left(\sum_{i=0}^{m} \mathcal{P}^1(h_i)\mathcal{P}^{\Delta_i} + h_0\mathcal{P}^1\right) + \sum_{i=0}^{m} h_i \mathcal{P}^{\Delta_i}\mathcal{P}^1 \\
&= \Delta_1 + \Delta_0 \mathcal{P}^1,
\end{aligned}$$

with a little help from the Cartan formulae and Lemma 1.1.2. Since $\Delta_0\mathcal{P}^1 = 0$, the statement is true for $\alpha = 1$.

Let $\alpha > 1$. Then

$$\begin{aligned}
0 &= \mathcal{P}^{\alpha}\Delta_0 \\
&= \sum_{r=0}^{\alpha} \left(\mathcal{P}^r(h_0)\mathcal{P}^{r-\alpha}\mathcal{P}^{\Delta_0} + \cdots + \mathcal{P}^r(h_m)\mathcal{P}^{r-\alpha}\mathcal{P}^{\Delta_m}\right) \\
&= \sum_{r=0}^{\alpha} \left(\mathcal{P}^r(h_0)\mathcal{P}^{\Delta_0}\mathcal{P}^{\alpha-r} + \cdots + \mathcal{P}^r(h_m)\mathcal{P}^{\Delta_m}\mathcal{P}^{r-\alpha}\right) \\
&\quad + \sum_{r=0}^{\alpha} (r-\alpha)\mathcal{P}^r(h_0)\mathcal{P}^{r-\alpha} \\
&\quad + \sum_{r=0}^{\alpha} \left(\mathcal{P}^r(h_1)\mathcal{P}^{\Delta_2}\mathcal{P}^{r-\alpha-q} + \cdots + \mathcal{P}^r(h_m)\mathcal{P}^{\Delta_{m+1}}\mathcal{P}^{r-\alpha-q^m}\right)
\end{aligned}$$

---

[9] We will omit the $h_0, \dots,\ h_m$ in parentheses and write the α as a subscript (i.e. $\Delta_\alpha := \Delta(\alpha,\ h_0, \dots,\ h_m)$) in the future whenever there is no way to misunderstand.

[10] I know that with $\alpha = 0$ our induction started already. However, the case $\alpha = 1$ should convince you that you gonna need a huge piece of paper, a dozen sharp pencils and lots of stamina to do the general case.

$$
\begin{aligned}
&= \sum_{r=0}^{\alpha} \left(\mathcal{P}^r(h_0)\mathcal{P}^{\Delta_0}\mathcal{P}^{\alpha-r} + \cdots + \mathcal{P}^r(h_m)\mathcal{P}^{\Delta_m}\mathcal{P}^{\alpha-r}\right)\\
&\quad + \sum_{r=0}^{\alpha} \left((\alpha - r)\mathcal{P}^r(h_0)\mathcal{P}^{\alpha-r}\right) + \sum_{r=0}^{\alpha-q} \left(\mathcal{P}^r(h_1)\mathcal{P}^{\Delta_2}\mathcal{P}^{\alpha-r-q}\right) + \cdots\\
&\quad \cdots + \sum_{r=0}^{\alpha-q^m} \left(\mathcal{P}^r(h_m)\mathcal{P}^{\Delta_{m+1}}\mathcal{P}^{\alpha-r-q^m}\right)\\
&= \sum_{r=0}^{\alpha} \left(\mathcal{P}^r(h_0)\mathcal{P}^{\Delta_0}\mathcal{P}^{\alpha-r} + \cdots + \mathcal{P}^r(h_m)\mathcal{P}^{\Delta_m}\mathcal{P}^{\alpha-r}\right)\\
&\quad + \sum_{r=1}^{\alpha} \left(\mathcal{P}^{r-1}(h_0)\mathcal{P}^1\mathcal{P}^{\alpha-r}\right) + \sum_{r=q}^{\alpha} \left(\mathcal{P}^{r-q}(h_1)\mathcal{P}^{\Delta_2}\mathcal{P}^{\alpha-r}\right) + \cdots\\
&\quad \cdots + \sum_{r=q^m}^{\alpha} \left(\mathcal{P}^{r-q^m}(h_m)\mathcal{P}^{\Delta_{m+1}}\mathcal{P}^{\alpha-r}\right)\\
&= \Delta_\alpha + \Delta_{\alpha-1}\mathcal{P}^1 + \cdots + \Delta_0\mathcal{P}^\alpha
\end{aligned}
$$

where we made extensive use of the Cartan formulae, Lemma 1.1.2, and the following Adem-Wu relation:

$$\mathcal{P}^1\mathcal{P}^{\alpha-r} = (\alpha - r + 1)\mathcal{P}^{\alpha-r+1}.$$

By the induction hypothesis $\Delta_0 = \cdots = \Delta_{\alpha-1} = 0$, hence

$$\Delta_\alpha = 0,$$

as we have claimed •

CHAPTER 2

Some Field Theory over the Steenrod Algebra

[redacted]ng to set up the ca[redacted]
[redacted]a in a proper way. We al[redacted]
[redacted]d to prove the Embedding T[redacted]
[redacted]n Lemma due to Clarence W.Wi[redacted]
[redacted].3, to the context of actions of the [redacted]
[redacted]r that we will have a look at the muc[redacted]
[redacted]parable field extensions. We will give a pr[redacted]
[redacted]rability and inseparably closed in our category. [redacted]
[redacted]*nstruct* the inseparable closure of a field over $\mathscr{P}^*$, prov[redacted]
[redacted]c properties, and describe some illustrative examples.

[redacted].1 Graded Fields over the Steenrod Algebra

[redacted]t us start with a description of the category we are working in, na[redacted]
[redacted]**category of graded fields over the Steenrod algebra** $\mathscr{P}^*$. A **gra**[redacted]
[redacted]K* of characteristic p is a graded connected commutative algeb[redacted]
[redacted], without zero divisors

$$\mathbb{K}^* = \{\mathbb{K}^i \mid i \in \quad \},$$

[redacted]homogeneous element is invertible. Note that there are also
[redacted]h negative degree and note also that this definition implies
[redacted]neous elements (which one could formally construct) need
[redacted], but they are *by definition* also not elements of $\mathbb{K}^*$. So,
[redacted]ading and take the totalization [1], we have only a *ring*,
[redacted]erence between the totalization of a graded field $\mathbb{K}^*$
[redacted]ions.

[redacted]tand the ring $\mathrm{Tot}(\mathbb{K}^*) = \oplus_i \mathbb{K}^i$, compare [28] §4.1.

Let $\mathbb{L}^*/\mathbb{K}^*$ be an extension of graded fields, and consider polynomials

$$p(X) \in \mathbb{K}^*[X]$$

in one variable with coefficients in $\mathbb{K}^*$. Let $p(X)$ have degree d as polynomial in X, i.e.,

$$p(X) = k_d x^d + k_{d-1} X^{d-1} + \cdots + k_1 X + k_0,$$

with $k_d \neq 0$. Obviously,

$$p(l) = k_d l^d + \cdots + k_1 l + k_0 \notin \mathbb{L}^*$$

for an arbitrary $l \in \mathbb{L}^*$, because this expression may not be homogeneous. This means evaluating a polynomial $p(X)$ at $l \in \mathbb{L}^*$ does not lead to an element of $\mathbb{L}^*$ unless

$$\deg(k_i) + i \deg(l)$$

is a constant for all $i = 0, \ldots, d$. Call a polynomial that satisfies this condition $\deg(l)$-**graded**. We can therefore speak of the roots l of a $\deg(l)$-graded polynomial $p(X)$ and hence the following definitions make sense.

DEFINITION : *Let $\mathbb{L}^*/\mathbb{K}^*$ be an extension of graded fields, and $l \in \mathbb{L}^*$. A monic irreducible polynomial in $\mathbb{K}^*[X]$ is called* ***-minimal polynomial of** *l, written*

$$\operatorname{minpol}_{l\in\mathbb{L}^*/\mathbb{K}^*}(X) \in \mathbb{K}^*[X]$$

if $\operatorname{minpol}_{l\in\mathbb{L}^*/\mathbb{K}^*}(X)$ *is* $\deg(l)$*-graded and*

$$\operatorname{minpol}_{l\in\mathbb{L}^*/\mathbb{K}^*}(l) = 0 \in \mathbb{L}^*.$$

(Note that as in the ungraded case, $\operatorname{minpol}_{l\in\mathbb{L}^*/\mathbb{K}^*}(X)$ *is the unique monic polynomial of minimal degree in X with root l.) We say that* $\operatorname{minpol}_{l\in\mathbb{L}^*/\mathbb{K}^*}(X)$ *is* ***-separable***, resp.* ***-inseparable** *if the derivative*

$$\frac{\mathrm{d}}{\mathrm{d}X}\operatorname{minpol}_{l\in\mathbb{L}^*/\mathbb{K}^*}(X)$$

does not vanish, resp. vanishes. We can then define $l \in \mathbb{L}^*/\mathbb{K}^*$ to be an ***-algebraic element over** $\mathbb{K}^*$ *if there exists a *-minimal polynomial for l; otherwise call l* ***-transcendental***. We call an *-algebraic l* ***-separable***, resp.* ***-inseparable over** $\mathbb{K}^*$ *if the *-minimal polynomial is *-separable, resp. *-inseparable. Finally, call the field extension $\mathbb{L}^*/\mathbb{K}^*$* ***-algebraic***, if any $l \in \mathbb{L}^*/\mathbb{K}^*$ is *-algebraic over $\mathbb{K}^*$; otherwise call it a* ***-transcendental** *field extension. If we obtain $\mathbb{L}^*$ from $\mathbb{K}^*$ by adjoining elements of some trancendence set over $\mathbb{K}^*$, then the $\mathbb{L}^*/\mathbb{K}^*$ is called a* ***-purely trancendental** *field extension.*[2] *Call an *-algebraic extension* ***-separable** *if any $l \in \mathbb{L}^*/\mathbb{K}^*$ is *-separable; otherwise call it an* ***-inseparable** *field extension. A *-inseparable*

[2] Compare [38], Chapter II, Section 12, for the classical terminology.

field extension is called ***-purely inseparable** *if every element* $l \in \mathbb{L}^*$ *is *-inseparable over* $\mathbb{K}^*$.[3]

Given this terminology we can talk about **graded fields** $\mathbb{K}^*$ **over the Steenrod algebra** $\mathcal{P}^*$ (or $\mathcal{P}^*$**-graded fields** for short): just replace the word "algebra" by "field" in the definition in the introduction. However, note carefully that the second part of the unstability condition doesn't make sense anymore, since for a non zero element $0 \neq k \in \mathbb{K}^n$ of positive degree n we have $k^{-1} \in \mathbb{K}^{-n}$. So, the second part of the unstability condition would imply that $k^{-1} = \mathcal{P}^0(k^{-1}) = 0$, which is nonsense.

2.2 Separable Extensions

We begin with the general version of the Separable Extension Lemma. The techniques used for the existence proof are essentially the same as the original ones used by C. W. Wilkerson, while those used for the proof of the uniqueness are as simple as they are new.

PROPOSITION 2.2.1 (Separable Extension Lemma, General Version): *Let* $\mathbb{L}^*/\mathbb{K}^*$ *be a *-separable extension of a field* $\mathbb{K}^*$ *of characteristic* p. *Moreover let the Steenrod algebra* $\mathcal{P}^*$ *over* $\mathbb{F}_q$ *act on* $\mathbb{K}^*$. *Then there exists a unique extension* [4] *of this action to* $\mathbb{L}^*$.

PROOF : Denote by

$$\mathcal{P}^i_{\mathbb{K}^*} : \mathbb{K}^* \longrightarrow \mathbb{K}^*$$

the i-th reduced Steenrod power on $\mathbb{K}^*$. First we want to show that there exists an extension of this

$$\mathcal{P}^i_{\mathbb{L}^*} : \mathbb{L}^* \longrightarrow \mathbb{L}^*$$

to a self map of $\mathbb{L}^*$. Since the field extension is *-separable any element $l \in \mathbb{L}^*$ has a *-separable minimal polynomial over $\mathbb{K}^*$

$$\mathrm{minpol}_{l\in\mathbb{L}^*/\mathbb{K}^*}(X) = \sum_{j=0}^{m} k_j X^j \in \mathbb{K}^*[X].$$

We proceed by induction on i in order to construct the $\mathcal{P}^i_{\mathbb{L}^*}$'s.

For $i = 0$ choose $\mathcal{P}^0_{\mathbb{L}^*}$ to be the identity operator. So let $i > 0$. By the above

---

[3] See [38], Chapter II, Section 5, for the classical terminology.

[4] In other words, a *-separable field extension, where the little field admits an action of the Steenrod algebra, is automatically a separable field extension in the category of fields over the Steenrod algebra, i.e., is a $\mathcal{P}^*$-separable field extension. However, be prepared! The notion of *-inseparability can't be translated that easily to fields over the Steenrod algebra.

observation we have

$$0 = \sum_{j=0}^{m} k_j l^j$$

for suitable $k_j \in \mathbb{K}^*$. Then, formally we have, (since $0 \in \mathbb{K}^*$)

$$\begin{aligned}
0 &= \mathcal{P}^i_{\mathbb{K}^*}\left(\sum_{j=0}^{m} k_j l^j\right) \\
&= \sum_{j=0}^{m}\left(\sum_{r+s=i} \mathcal{P}^r_{\mathbb{K}^*}(k_j)\mathcal{P}^s_{\mathbb{L}^*}(l^j)\right) \\
&= \sum_{j=0}^{m}\left(\sum_{r+s=i,\ s<i} \mathcal{P}^r_{\mathbb{K}^*}(k_j)\mathcal{P}^s_{\mathbb{L}^*}(l^j)\right) + \sum_{j=1}^{m} k_j \mathcal{P}^i_{\mathbb{L}^*}(l^j) \\
&= \sum_{j=0}^{m}\left(\sum_{r+s=i,\ s<i} \mathcal{P}^r_{\mathbb{K}^*}(k_j)\mathcal{P}^s_{\mathbb{L}^*}(l^j)\right) \\
&\qquad + \sum_{j=1}^{m} k_j \left(\sum_{r_1+\cdots+r_j=i,\ r_1,\ldots,r_j<i} \mathcal{P}^{r_1}_{\mathbb{L}^*}(l)\cdots\mathcal{P}^{r_j}_{\mathbb{L}^*}(l)\right) \\
&\qquad + \sum_{j=1}^{m} j k_j \mathcal{P}^i_{\mathbb{L}^*}(l) l^{j-1}.
\end{aligned}$$

Since $\mathrm{minpol}_{l\in\mathbb{L}^*/\mathbb{K}^*}(X)$ is *-separable we have that

$$0 \neq \frac{\mathrm{d}}{\mathrm{d}X}\mathrm{minpol}_{l\in\mathbb{L}^*/\mathbb{K}^*}(X)\Big|_{X=l} = \sum_{j=1}^{m} j k_j l^{j-1},$$

compare [38] Corollary 1 on page 67. Hence we can solve the equation formally for $\mathcal{P}^i_{\mathbb{L}^*}(l)$ (remember, that the $\mathcal{P}^s_{\mathbb{L}^*}(l)$ are defined for $s < i$)

$$\mathcal{P}^i_{\mathbb{L}^*}(l) :=$$

$$-\frac{\sum_{j=0}^{m}\left(\sum_{r+s=i,\ s<i} \mathcal{P}^r_{\mathbb{L}^*}(k_j)\mathcal{P}^s_{\mathbb{L}^*}(l^j)\right) + \sum_{j=1}^{m} k_j\left(\sum_{r_1+\cdots+r_j=i,\ r_1,\ldots,r_j<i} \mathcal{P}^{r_1}_{\mathbb{L}^*}(l)\cdots\mathcal{P}^{r_j}_{\mathbb{L}^*}(l)\right)}{\sum_{j=1}^{m} j k_j l^{j-1}}$$

and use this to define $\mathcal{P}^i_{\mathbb{L}^*}$, so we are done by induction.

Secondly we show the uniqueness of this extension. For $i = 0$ there is nothing to show. Let $i > 0$, and assume there were another extension $\overline{\mathcal{P}^i_{\mathbb{L}^*}}$.

Since $\mathbb{L}^*/\mathbb{K}^*$ is *-separable it follows that also $\mathbb{K}^*(l)/\mathbb{K}^*$ is *-separable for any $l \in \mathbb{L}^*$. But then we have

$$\mathbb{K}^*(l) = \mathbb{K}^*(l^p) = \mathbb{K}^*(l^q) \quad \forall\ l \in \mathbb{L}^*,$$

by [38] the Corollary on page 70. Therefore $l \in \mathbb{K}^*(l^q)$, i.e., there exist $k_0, \dots, k_m \in \mathbb{K}^*$]such that

$$l = \sum_{j=1}^{m} k_j (l^q)^j + k_0.$$

Then, for any $l \in \mathbb{L}^*$, we get by induction

$$\begin{aligned}
\overline{\mathcal{P}^i_{\mathbb{L}^*}}(l) - \mathcal{P}^i_{\mathbb{L}^*}(l) &= \sum_{j=1}^{m} \left(\sum_{r+s=i} \mathcal{P}^r_{\mathbb{K}^*}(k_j) \left(\overline{\mathcal{P}^s_{\mathbb{L}^*}}(l^{qj}) - \mathcal{P}^s_{\mathbb{L}^*}(l^{qj}) \right) \right) \\
&= \sum_{j=1}^{m} k_j \left(\overline{\mathcal{P}^i_{\mathbb{L}^*}}(l^{qj}) - \mathcal{P}^i_{\mathbb{L}^*}(l^{qj}) \right) \\
&= \begin{cases} \sum_{j=1}^{m} k_j \left(\overline{\mathcal{P}^j_{\mathbb{L}^*}}(l) - \mathcal{P}^j_{\mathbb{L}^*}(l) \right)^q & \text{for } rq^j = i \\ 0 & \text{otherwise} \end{cases} \\
&= 0,
\end{aligned}$$

where we have used the induction hypothesis twice. Since 0 obviously extends to 0 the Adem-Wu relations and the Cartan formulae for the extended $\mathcal{P}^*$-action on $\mathbb{L}^*$ follow from uniqueness •

This result tells us that the notion of *-separability and $\boldsymbol{\mathcal{P}}^*$-separability are equivalent as long as we have an action of the Steenrod algebra on the smaller field. We come to the much more delicate case of inseparable field extensions.

2.3 Inseparable Extensions

In this section we ask ourselves the obvious question whether there is a “purely inseparable extension lemma”. Let us first determine exactly what “purely inseparable” means in our category.

Let $\mathbb{K}^* \subseteq \mathbb{L}^*$ be a field extension of graded fields. As we mentioned in Section 2.1 an element $l \in \mathbb{L}^*$ is said to be ***-inseparable** over $\mathbb{K}^*$ if its *-minimal polynomial is

$$\operatorname{minpol}_{l \in \mathbb{L}^*/\mathbb{K}^*}(X) = X^{p^e} - k,$$

for some $k \in \mathbb{K}^*$. The first observation we have to make is that, of course, then

$$\deg(l) p^e = \deg(k).$$

This is not enough! Have a look at the field $\mathbb{F}_2(x, y)$ generated by two transcendental elements of degree 1 over the field of two elements. Consider the following *-inseparable polynomial

$$p(X) := X^2 - xy \in \mathbb{F}_2(x, y)[X].$$

Its splitting field $\mathbb{L}$ would contain an element l of degree one such that

$$l^2 = xy \in \mathbb{F}_2(x, y).$$

This splitting field is *not* an object in our category, for otherwise, there would exist an extension of the (naturally given) Steenrod algebra action on $\mathbb{F}_2(x, y)$ to $\mathbb{L}$. However,

$$x^2y + xy^2 = \mathcal{P}^1(xy) = \mathcal{P}^1(l^2) = 2\mathcal{P}^1(l)l = 0$$

shows that this can't be done consistently. So, what we need is that the p-th powers in the little field "behave as such" under the action of the Steenrod algebra. We achieve this by taking into account the additional derivations which are given to us from the Steenrod algebra, namely $\mathcal{P}^{\Delta_i}$ for $i \geq 0$ (recall Section 1.1):

DEFINITION : *Let $\mathbb{K}^* \subseteq \mathbb{L}^*$ be an extension of graded fields over the Steenrod algebra. A polynomial $p(X) \in \mathbb{K}^*[X]$ is called* **inseparable over $\mathcal{P}^*$***, or* **$\mathcal{P}^*$-inseparable** *if* [5]

[1] *its standard derivation vanishes $\frac{d}{dX}p(X) = 0$, and*

[2] *all Steenrod derivations $\mathcal{P}^{\Delta_i}(p(X))$ also vanish, where we let $\mathcal{P}^*$ act on the variable X as if it had degree 1, i.e.,*

$$\mathcal{P}^i(X) = \begin{cases} X & \text{if } i = 0 \\ X^q & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

An element $l \in \mathbb{L}^*$ is called **$\mathcal{P}^*$-inseparable** *if its *-minimal polynomial is $\mathcal{P}^*$-inseparable. Hence its *-minimal polynomial has the form for some $e \in \mathbb{N}$*

$$X^{p^e} - k,$$

where $k \in \mathbb{K}^*$ has degree $\deg(l)p^e$ and in addition

$$\mathcal{P}^{\Delta_i}(k) = 0 \quad \forall\ i \geq 1.$$

Call a field $\mathbb{K}^*$ **$\mathcal{P}^*$-inseparably closed** *if it contains all roots of any $\mathcal{P}^*$-inseparable polynomial in $\mathbb{K}^*[X]$.*

The **$\mathcal{P}^*$-inseparable closure** of an arbitrary $\mathbb{K}^*$ can be obtained in the following way. First we take the algebraic closure of $\mathbb{K}^*$ as a graded

[5] One might wonder whether condition [1] follows from condition [2].

field, $\overline{\mathbb{K}^*}$. This exists and the classical non graded proof works more or less word-for-word, see e.g. [38] Volume 1, Chapter II, Section 14. Inside $\overline{\mathbb{K}^*}$ we can take the *-inseparable closure of $\mathbb{K}^*$ as a graded field, $\mathbb{L}^*$. Hence we get a chain of graded fields

$$\mathbb{K}^* \hookrightarrow \mathbb{L}^* \hookrightarrow \overline{\mathbb{K}^*},$$

where the first extension, $\mathbb{K}^* \hookrightarrow \mathbb{L}^*$, is *-purely inseparable, while the second is *-separable.[6] We get the $\mathcal{P}^*$-inseparable closure, $\mathbb{K}^*_{\mathcal{P}^*-\text{insep}}$, of the field $\mathbb{K}^*$ in the *category of graded fields over the Steenrod algebra* by taking the graded subfield of $\mathbb{L}^*$ generated by the roots of $\mathcal{P}^*$-inseparable polynomials:

$$\mathbb{K}^* \hookrightarrow \mathbb{K}^*_{\mathcal{P}^*-\text{insep}} \hookrightarrow \mathbb{L}^* \hookrightarrow \overline{\mathbb{K}^*}.$$

Specifically, we construct the $\mathcal{P}^*$-inseparable closure $\mathbb{K}^*_{\mathcal{P}^*-\text{insep}}$ by the following inductive procedure. Let $\mathbb{K}^* = \mathbb{K}^*_0$ and define successively

$$\mathbb{K}^*_i = \mathbb{K}^*_{i-1}(S)$$

where the set S consists of the roots of $\mathcal{P}^*$-inseparable polynomials of degree p in X, i.e., $s^p \in \mathbb{K}^*_{i-1}$. In order to extend the Steenrod action to $\mathbb{K}^*_i$ in consistency with the Cartan formulae we set

$$\mathcal{P}^j_{\mathbb{K}^*_{i-1}}(s^p) = \mathcal{P}^j_{\mathbb{K}^*_i}(s^p) = \begin{cases} \left(\mathcal{P}^k_{\mathbb{K}^*_i}(s)\right)^p & \text{for } kp = j \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathbb{K}^*_i$ contains *all* roots of $\mathcal{P}^*$-inseparable polynomials of degree p it contains also the root of

$$X^p - \left(\mathcal{P}^j_{\mathbb{K}^*_i}(s)\right)^p \in \mathbb{K}^*_{i-1}[X],$$

i.e., $\mathbb{K}^*_i$ is again a graded field over the Steenrod algebra. This action is also unique as we next show. Let $k \in \mathbb{K}^*_i$ and $k^p \in \mathbb{K}^*_{i-1}$. Assume there were two extensions $\mathcal{P}^j_{\mathbb{K}^*_i}$ and $\overline{\mathcal{P}^j_{\mathbb{K}^*_i}}$. If $j = 0$, then

$$\mathcal{P}^0_{\mathbb{K}^*_i} = \text{identity}_{\mathbb{K}^*_i} = \overline{\mathcal{P}^0_{\mathbb{K}^*_i}}.$$

Let $j > 0$. Then we had

$$\begin{aligned}\left(\mathcal{P}^j_{\mathbb{K}^*_i}(k) - \overline{\mathcal{P}^j_{\mathbb{K}^*_i}}(k)\right)^p &= \left(\mathcal{P}^j_{\mathbb{K}^*_i}(k)\right)^p - \left(\overline{\mathcal{P}^j_{\mathbb{K}^*_i}}(k)\right)^p \\ &= \mathcal{P}^{jp}_{\mathbb{K}^*_i}(k^p) - \overline{\mathcal{P}^{jp}_{\mathbb{K}^*_i}}(k^p) \\ &= \mathcal{P}^{jp}_{\mathbb{K}^*_{i-1}}(k^p) - \overline{\mathcal{P}^{jp}_{\mathbb{K}^*_{i-1}}}(k^p) \\ &= 0,\end{aligned}$$

[6] In [38] $\mathbb{L}^*$ is called the **perfect closure** of $\mathbb{K}^*$, denoted by $\mathbb{K}^{p^{-\infty}}$, see page 108 there.

where the last equation holds since by induction the two actions coincide on $\mathbb{K}^*_{i-1}$. Hence

$$\mathcal{P}^j_{\mathbb{K}^*_i}(k) - \overline{\mathcal{P}^j_{\mathbb{K}^*_i}}(k) = 0,$$

what we claimed. The Adem-Wu relations as well as the Cartan formulae follow from uniqueness. We have therefore constructed an ascending chain of $\mathcal{P}^*$-inseparable field extensions

$$\mathbb{K}^* = \mathbb{K}^*_0 \subseteq \mathbb{K}^*_1 \subseteq \cdots \subseteq \mathbb{K}^*_i \subseteq \cdots.$$

PROPOSITION 2.3.1 : *The colimit of the above chain of fields,* $\operatorname{colim}(\mathbb{K}^*_i)$*, is the* $\mathcal{P}^*$*-inseparable closure.*

PROOF : Since the colimit of graded fields over $\mathcal{P}^*$ is a graded field over $\mathcal{P}^*$ and the extensions are all $\mathcal{P}^*$-inseparable we certainly get

$$\operatorname{colim}(\mathbb{K}^*_i) \subseteq \mathbb{K}^*_{\mathcal{P}^*-\text{insep}}.$$

On the other hand any element $k \in \mathbb{K}^*_{\mathcal{P}^*-\text{insep}}$ is the root of an $\mathcal{P}^*$-inseparable polynomial, i.e.,

$$k^{p^l} \in \mathbb{K}^* \quad \text{for large enough } l \in \mathbb{N}_0.$$

Hence $k \in \mathbb{K}^*_l \subseteq \operatorname{colim}(\mathbb{K}^*_i)$ and we are done •

EXAMPLE 1 : Take the field of two elements and adjoint two transcendental generators of degree 1, $\mathbb{F}_2(x, y)$, i.e., we are looking at the field of fraction of the ring of polynomials $\mathbb{F}_2[x, y]$ in two variables over the field of two elements. If we forget the grading and take the field of fractions of the totalization, then this is certainly not an inseparably closed field since, e.g.,

$$p_1(X) := X^2 + x$$

has no root in it. However, no root of p_1, say r, leads to a *homogeneous* equation

$$r^2 + x = 0,$$

because r would had to have degree $\frac{1}{2}$, i.e, this polynomial is not $\deg(r)$-graded in the terminology of Section 2.1. Nevertheless our field is also not *-inseparably closed, because e.g.

$$p_2(X) := X^2 + xy$$

has no root. In fact in *our category*, where we take the $\mathcal{P}^*$-action into account, $\mathbb{F}_2(x, y)$ *is* $\mathcal{P}^*$-inseparably closed. Assume to the contrary that $\mathbb{F}_2(x, y)$ were not $\mathcal{P}^*$-inseparably closed. Then we construct its

$\mathcal{P}^*$-inseparable closure as above by taking the colimit of the ascending chain of $\mathcal{P}^*$-inseparable field extensions

$$\mathbb{F}_2(x,\ y) = \mathbb{K}_0^* \subseteq \mathbb{K}_1^* \subseteq \cdots \subseteq \mathbb{K}_i^* \subseteq \cdots.$$

In particular, we had

$$\mathbb{F}_2(x,\ y) = \mathbb{K}_0^* \subsetneqq \mathbb{K}_1^*,$$

hence there would be an $\mathcal{P}^*$-inseparable polynomial of degree 2

$$p_3(X) = X^2 + \lambda_1 x^2 + \lambda_2 x y + \lambda_3 y^2$$

with $\lambda_1,\ \lambda_2,\ \lambda_3 \in \mathbb{F}_2$ which does not split over $\mathbb{F}_2(x,\ y)$. Note however, $\mathcal{P}^*$-inseparability implies

$$\lambda_2 = 0,$$

and for any value of $\lambda_1,\ \lambda_3 \in \mathbb{F}_2$ the polynomial

$$X^2 + \lambda_1 x^2 + \lambda_3 y^2 = (X + \lambda_1 x + \lambda_3 y)^2$$

is not irreducible.

This example is not an accident as the next two results show.

THEOREM 2.3.2 : *Let $\mathbb{K}^*$ be $\mathcal{P}^*$-inseparably closed, let t be $\mathcal{P}^*$-transcendental*[7] *over $\mathbb{K}^*$ of degree 1. If the action of the Steenrod derivations on t is given by*[8]

$$\mathcal{P}^{\Delta_i}(t) = t^{q^i} \qquad \forall\ i \geq 0$$

then $\mathbb{K}^*(t)$ is $\mathcal{P}^*$-inseparably closed.

PROOF : Consider a $\mathcal{P}^*$-purely inseparable field extension

$$\mathbb{K}^*(t) \hookrightarrow \mathbb{L}^*.$$

We have to show that $\mathbb{K}^*(t) = \mathbb{L}^*$. So, take an $l \in \mathbb{L}^*$. Its minimal polynomial is then $\mathcal{P}^*$-inseparable of, say, degree p^m:

$$\mathrm{minpol}_{l\in\mathbb{L}^*/\mathbb{K}^*(t)}(X) = X^{p^m} - \sum_{i=0}^{d} k_i t^i,$$

where $k_i \in \mathbb{K}^*\ \forall\ i = 0, \ldots, d$, for some $d \in \mathbb{N}_0$. Hence

$$\bigstar \qquad l^{p^m} - \sum_{i=0}^{d} k_i t^i = 0.$$

[7] This terminology means that t is transcendental over $\mathbb{K}^*$ and $\mathbb{K}^*(t)$ is again a graded field over $\mathcal{P}^*$.

[8] This condition means that under the operation of the $\mathcal{P}^{\Delta}$'s the element t "behaves" as a standard linear form.

We need to show that $l \in \mathbb{K}^*(t)$, or what is the same thing $m = 0$. First, observe that

$$\deg(k_i) + i = p^m \deg(l) \ \forall \ i = 0, \dots, d,$$

by homogeneity of the equation (★). We apply a Steenrod derivation $\mathcal{P}^{\Delta_j}$ to this equation and get

$$\begin{aligned}
0 = \mathcal{P}^{\Delta_j} \left(l^{p^m} - \sum_{i=0}^{d} k_i t^i \right) &= \sum_{i=0}^{d} \left(\mathcal{P}^{\Delta_j}(k_i) t^i + k_i \mathcal{P}^{\Delta_j}(t^i) \right) \\
&= \sum_{i=0}^{d} \left(\mathcal{P}^{\Delta_j}(k_i) t^i + i k_i \mathcal{P}^{\Delta_j}(t) t^{i-1} \right) \\
&= \sum_{i=0}^{d} \left(\mathcal{P}^{\Delta_j}(k_i) t^i + i k_i t^{q^j} t^{i-1} \right) \\
&= \sum_{i=0}^{d} \left(\mathcal{P}^{\Delta_j}(k_i) t^i + i k_i t^{q^j+i-1} \right).
\end{aligned}$$

Let's suppose that $m > 0$. We are going to show that the constant term occurring in (★) is itself a p-th power, i.e., that

$$\sum_{i=0}^{d} k_i t^i = k^p$$

for some $k \in \mathbb{K}^*(t)$, which would contradict the minimality of m, because then (★) becomes

$$0 = l^{p^m} - \sum_{i=0}^{d} k_i t^i = \left(l^{p^{m-1}} - k \right)^p,$$

hence

$$0 = l^{p^{m-1}} - k.$$

We proceed by considering possible values of $j \geq 0$.

First take a j such that $q^j - 1 > d$. We get

$$\begin{aligned}
0 &= \sum_{i=0}^{d} \left(\mathcal{P}^{\Delta_j}(k_i) t^i + i k_i t^{q^j+i-1} \right) \\
&= \left(\sum_{i=0}^{d} \mathcal{P}^{\Delta_j}(k_i) t^i \right) + \left(\sum_{i=q^j-1}^{q^j+d-1} (i - q^j + 1) k_{i-q^j+1} t^i \right),
\end{aligned}$$

where we have ordered the sums according to the powers of t. Since t is transcendental over $\mathbb{K}^*$ we must have that all coefficients vanish; so the first sum gives

$$\mathcal{P}^{\Delta_j}(k_i) = 0 \ \forall \ i = 0, \ldots, d, q^j - 1 > d$$

while the second gives that

$$i k_i = 0 \qquad \forall \ i = 0, \ldots, d,$$

i.e.,

$$k_i = 0 \ \forall \ i = 0, \ldots, d \mid i \not\equiv 0 \ \mathbf{MOD} \ p.$$

So the higher Steenrod derivations, $\mathcal{P}^{\Delta_j}$ with j such that $q^j - 1 > d$, vanish on all coefficients k_i and moreover, the coefficients k_i, where i is not divisible by p, are all zero.

We next show that the lower Steenrod derivations $\mathcal{P}^{\Delta_0}, \ldots, \mathcal{P}^{\Delta_j}$ for j with $q^j - 1 \le d$ vanish on the coefficients k_i for which i is divisible by p. To this end, note that we have for our nonzero coefficients

$$\deg(k_i) = \deg(l) p^m - i \equiv 0 \ \mathbf{MOD} \ p,$$

hence $\mathcal{P}^{\Delta_0}$ evaluates to zero on them. Next let $j > 0$, but still satisfy $q^j - 1 \le d$, and apply $\mathcal{P}^{\Delta_j}$ to the equation (★). We obtain again

$$\begin{aligned} 0 &= \mathcal{P}^{\Delta_j}\left(l^{p^m} - \sum_{i=0}^{d} k_i t^i \right) \\ &= \sum_{i=0}^{d} \left(\mathcal{P}^{\Delta_j}(k_i) t^i + i k_i t^{q^j + i - 1} \right), \end{aligned}$$

whence all coefficients of the t-powers must vanish, because t is transcendental over $\mathbb{K}^*$. We compute the coefficient of t^i and find it to be

$$\begin{cases} \mathcal{P}^{\Delta_j}(k_i) & \text{if } i < q^j - 1, \\ \mathcal{P}^{\Delta_j}(k_i) + (i - q^j + 1) k_{i - q^j + 1} & \text{if } i \ge q^j - 1 \text{ and } i \le d, \\ (i - q^j + 1) k_{i - q^j + 1} & \text{if } i > d. \end{cases}$$

So all these coefficients must vanish. The last case does not give new information, but the first two give

$$\mathcal{P}^{\Delta_j}(k_i) = \begin{cases} 0 & \text{if } i < q^j - 1, \\ -(i - q^j + 1) k_{i - q^j + 1} & \text{if } i \ge q^j - 1 \text{ and } i \le d. \end{cases}$$

Since we have already shown that $k_i = 0$ whenever i is not divisible by p we get altogether

$$\mathcal{P}^{\Delta_j}(k_i) = \begin{cases} 0 & \text{for any } j \geq 0, \text{ if } i \not\equiv 0 \textbf{ MOD } p, \\ 0 & \text{for } j = 0 \text{ and } i \equiv 0 \textbf{ MOD } p, \\ 0 & \text{for } j \text{ such that } q^j - 1 > d \\ & \quad \text{and } i \equiv 0 \textbf{ MOD } p, \\ 0 & \text{for } j \text{ such that } q^j - 1 \leq d,\ i \equiv 0 \textbf{ MOD } p \\ & \quad \text{and } i < q^j - 1, \\ -(i - q^j + 1)k_{i-q^j+1} & \text{for } j \text{ such that } q^j - 1 \leq d,\ i \equiv 0 \textbf{ MOD } p \\ & \quad \text{and } q^j - 1 \leq i \leq d. \end{cases}$$

Last note that $i - q^j + 1 \not\equiv 0$ **MOD** p, unless $i \equiv 1$ **MOD** p, so we get

$$\mathcal{P}^{\Delta_j}(k_i) = 0 \ \forall\ i,\ j.$$

This in turn means that all the coefficients are p-th powers in $\mathbb{K}^*$ because $\mathbb{K}^*$ is $\mathcal{P}^*$-inseparably closed. Since taking to p-th powers is additive in characteristic p we have shown that our sum in (★)

$$l^{p^m} - \sum_{i=0}^{d} k_i t^i$$

is a p-th power and hence m was not minimal. This is a contradiction •

The above theorem has no direct generalization to the case where the transcendental elements t has higher degree, as the following example shows.

EXAMPLE 2: Let $\mathbb{F}(x,\ y)$ be the field generated by two linear transcendetal elements over a Galois field $\mathbb{F} = \mathbb{F}_q$ with standard Steenrod algebra action. Consider the subfields

$$\mathbb{F}(x^{q-1}) \hookrightarrow \mathbb{F}(x^{q-1},\ xy^q) \hookrightarrow \mathbb{F}(x,\ y).$$

Is is an easy and straightforward calculation that shows that both subfields are again closed under the action of the Steenrod algebra. As one might convince oneself equally easily the smallest field $\mathbb{F}(x^{q-1})$ is $\mathcal{P}^*$-inseparably closed.[9] Obviously the field extension

$$\mathbb{F}(x^{q-1}) \hookrightarrow \mathbb{F}(x^{q-1},\ xy^q)$$

is purely transcendental and the additional element xy^q has degree prime to the characteristic, but the extended field $\mathbb{F}(x^{q-1},\ xy^q)$ is *not*

[9] This extends easily to the general case: if $\mathbb{F}$ is a Galois field of characteristic p, if t is transcendental over $\mathbb{F}$ of degree prime to p and if $\mathbb{F}(t)$ is a field over the Steenrod algebra, then it is $\mathcal{P}^*$-inseparably closed.

$\mathcal{P}^*$-inseparably closed, because all Steenrod derivations vanish on the element

$$x^q y^q = x^{q-1} x y^q \in \mathbb{F}(x^{q-1},\ x y^q),$$

but $x y \notin \mathbb{F}(x^{q-1},\ x y^q)$.

COROLLARY 2.3.3 : *The field of fractions of the polynomial functions in n variables over $\mathbb{F}$, $\mathbb{F}(x_1, \ldots, x_n)$, is $\mathcal{P}^*$-inseparably closed.*

PROOF : A Galois field $\mathbb{F}$ is perfect. Therefore we can apply the above Theorem 2.3.2 inductively starting with $\mathbb{F} = \mathbb{K}^*$ •

We turn to the question we asked at the beginning of this section, namely, whether there is a "purely inseparable extension lemma". Let's take a purely inseparable finite field extension in the category of graded fields

$$\mathbb{L}^* = \mathbb{K}^*(l_1, \ldots, l_m)/\mathbb{K}^*,$$

where $\mathbb{K}^*$ enjoys a $\mathcal{P}^*$-action. Let's assume that all elements $l_i \in \mathbb{L}^*$ are roots of $\mathcal{P}^*$-inseparable polynomials. So take an $l \in \mathbb{L}^*$ and its minimal polynomial of degree p^e over $\mathbb{K}^*$. Then we can define the Steenrod algebra action (uniquely!) on the additional element $l \in \mathbb{L}^*$ just as we did at the beginning of this section where we extended the Steenrod algebra action successively to the fields $\mathbb{K}^*_i$. However, this construction does not necessarily lead to values in $\mathbb{L}^*$, i.e., if $l \in \mathbb{L}^*$ then we just don't know whether

$$\mathcal{P}^j_{\mathbb{L}^*}(l)$$

is in $\mathbb{L}^*$ or not, because we don't know whether $\mathbb{L}^*$ also contains the splitting field of

$$X^{p^e(\deg(l)+j(q-1))} - \left(\mathcal{P}^j_{\mathbb{L}^*}(l)\right)^{p^e} \in \mathbb{K}^*[X].$$

Finally, we want to note that, if we have an algebraic field extension

$$\mathbb{K}^* \hookrightarrow \mathbb{L}^*$$

of fields over $\mathcal{P}^*$, then, analogous to the classical situation, we can form the $\mathcal{P}^*$-inseparable closure of $\mathbb{K}^*$ inside $\mathbb{L}^*$, $\mathbb{K}^*_{\mathcal{P}^*-\text{insep},\ \mathbb{L}^*/\mathbb{K}^*}$ such that

$$\mathbb{K}^* \underset{\mathcal{P}^*-\text{purely insep.}}{\hookrightarrow} \mathbb{K}^*_{\mathcal{P}^*-\text{insep},\ \mathbb{L}^*/\mathbb{K}^*} \underset{\mathcal{P}^*-\text{sep.}}{\hookrightarrow} \mathbb{L}^*.$$

This is indeed again a field closed under the action of the Steenrod algebra: Let $l \in \mathbb{L}^*$ be $\mathcal{P}^*$-inseparable over $\mathbb{K}^*$. Then there exists an e such that

$$l^{p^e} \in \mathbb{K}^*.$$

Hence for any $i \in \mathbb{N}_0$

$$\begin{aligned}\mathcal{P}^i(l)^{p^e} &= \mathcal{P}^{ip^e}(l^{p^e}) \\ &= \mathcal{P}^{ip^e}\left(l^{p^e}\right) \\ &\in \mathbb{K}^*,\end{aligned}$$

i.e., any Steenrod power of our l is also $\boldsymbol{\mathcal{P}}^*$-inseparable over $\mathbb{K}^*$

$$\mathcal{P}^i(l) \in \mathbb{K}^*_{\boldsymbol{\mathcal{P}}^*-\text{insep},\ \mathbb{L}^*/\mathbb{K}^*}.$$

On the other hand, again, as in the classical case, we can take the $\mathcal{P}^*$-separable closure of $\mathbb{K}^*$ inside $\mathbb{L}^*$, $\mathbb{K}^*_{\boldsymbol{\mathcal{P}}^*-\text{sep},\ \mathbb{L}^*/\mathbb{K}^*}$ such that

$$\mathbb{K}^* \underset{\boldsymbol{\mathcal{P}}^*-\text{sep.}}{\hookrightarrow} \mathbb{K}^*_{\boldsymbol{\mathcal{P}}^*-\text{sep},\ \mathbb{L}^*/\mathbb{K}^*} \underset{\boldsymbol{\mathcal{P}}^*-\text{purely insep.}}{\hookrightarrow} \mathbb{L}^*.$$

As in the first case it is easily seen that the intermediate field is again closed under the action of the Steenrod algebra. If $l \in \mathbb{L}^*$ is $\mathcal{P}^*$-separable over $\mathbb{K}^*$, then Proposition 2.2.1 provides us with a recursion formula for its Steenrod powers, which in turn leads to

$$\mathcal{P}^i(l) \in \mathbb{K}^*\left(l,\ \mathcal{P}^1(l), \ldots, \mathcal{P}^{i-1}(l)\right)$$

and hence inductively

$$\mathcal{P}^i(l) \in \mathbb{K}^*(l) \in \mathbb{K}^*_{\boldsymbol{\mathcal{P}}^*-\text{sep},\ \mathbb{L}^*/\mathbb{K}^*}$$

for all $i \in \mathbb{N}_0$.

CHAPTER 3

The Integral Closure Theorem and the Unstable Part

In this chapter we are going to generalize the Integral Closure Theorem [1] of Clarence W. Wilkerson, [35], to arbitrary Galois fields.

3.1 Rings of Fractions and Their Unstable Part

Let $S \subseteq \mathrm{H}^*$ be a multiplicatively closed subset, and form the ring of fractions

$$S^{-1}\mathrm{H}^* := \left\{ \frac{h}{s} \mid h \in \mathrm{H}^*,\ s \in S \right\}.$$

There is an extension of the action of the Steenrod algebra to $S^{-1}\mathrm{H}^*$ given by requiring

$$\mathcal{P}(\xi)(\frac{h}{s})\mathcal{P}(\xi)(s) = \mathcal{P}(\xi)(h)$$

for any $\frac{h}{s} \in S^{-1}\mathrm{H}^*$, compare [34] Proposition 2.1. By construction, this action satisfies the Cartan formulae, but not the unstability condition.

It will be useful to recall from [10], Definition 2.2, the notion of the unstable part:

DEFINITION: *Let* M *be a graded module over the Steenrod algebra. For a multi index* $I = (i_1, \dots, i_s)$ *denote by* $\mathcal{P}^I$ *the product* $\mathcal{P}^{i_1} \cdots \mathcal{P}^{i_s}$*. Then the* **unstable part of** M *in degree* k *is the* $\mathbb{F}$*-vector subspace defined by*

$$\big(Un(\mathrm{M})\big)_{(k)} := \Big\{ m \in \mathrm{M}_{(k)} \mid \mathcal{P}^r\mathcal{P}^I(m) = 0,\ r > k + \deg(\mathcal{P}^I),\ \forall \text{ multi indices } I \Big\}.$$

[1] Wilkerson calls this theorem the "Generalized Serre Lemma", because it extends a result from J.-P. Serre, [26], in some way.

The unstable part is a graded submodule over $\mathcal{P}^*$, which satisfies the unstability condition (compare again [10]).

We need the following result, compare Proposition 2.2 in [35].

LEMMA 3.1.1: *Let $\mathbb{L}^*/\mathbb{K}^*$ be a $\mathcal{P}^*$-algebraic extension of graded fields over the Steenrod algebra. Then the ring extension*

$$Un(\mathbb{K}^*) \hookrightarrow Un(\mathbb{L}^*)$$

is integral.

PROOF: Take an element

$$l \in Un(\mathbb{L}^*) \subset \mathbb{L}^*.$$

We have to show that it is integral over $Un(\mathbb{K}^*)$. If l is $\mathcal{P}^*$-inseparable over $\mathbb{K}^*$, then there exists an $s \in \mathbb{N}$ such that

$$(l)^{q^s} \in \mathbb{K}^*.$$

However, this element is again unstable, since it is just a q-th power of an unstable element, i.e.,

$$(l)^{q^s} \in Un(\mathbb{K}^*)$$

and l is integral over $Un(\mathbb{K}^*)$. On the other hand, if l is separable over $\mathbb{K}^*$, then its minimal polynomial

$$\mathrm{minpol}_{l \in \mathbb{L}^*/\mathbb{K}^*}(X) = k_0 + k_1 X + \cdots + X^n \in \mathbb{K}^*[X]$$

is separable. Lets take its splitting field $\Delta^*/\mathbb{K}^*$, i.e.,

$$k_0 + k_1 X + \cdots + X^n = \prod_{i=1}^{n}(X - r_i) \in \Delta^*[X],$$

where without loss of generality $r_1 = l$. Any automorphism of Δ^* fixing $\mathbb{K}^*$ commutes with the action of the Steenrod algebra, because the extension of the Steenrod algebra action from $\mathbb{K}^*$ to Δ^* is unique by Proposition 2.2.1, compare Proposition 1.1 (c) in [35]. Hence if one root is an unstable element then all roots are unstable. Therefore the coefficients $k_0, \ldots, k_{n-1}$ are unstable elements, because they are just elementary symmetric functions in the roots, i.e., the minimal polynomial has unstable coefficients

$$\mathrm{minpol}_{l \in \mathbb{L}^*/\mathbb{K}^*}(X) = k_0 + k_1 X + \cdots + X^n \in Un(\mathbb{K}^*)[X],$$

which in turn means that l is integral over $Un(\mathbb{K}^*)$ •

For rings of fractions the unstable part can be explicitly calculated:

LEMMA 3.1.2 : *Let $S^{-1}\mathrm{H}^*$ the ring of fractions for a multiplicatively closed subset $S \subseteq \mathrm{H}^*$. Then the unstable part of $S^{-1}\mathrm{H}^*$ is*

$$Un(S^{-1}\mathrm{H}^*) = \left\{ \frac{h}{s} \mid \mathcal{P}(\xi)(s) \mid \mathcal{P}(\xi)(h) \right\}.$$

PROOF : Let $\frac{h}{s} \in S^{-1}\mathrm{H}^*$. Then the condition

$$\mathcal{P}(\xi)(s) \mid \mathcal{P}(\xi)(h)$$

is equivalent to the statement that

$$\mathcal{P}(\xi)\left(\frac{h}{s}\right) = \frac{\mathcal{P}(\xi)(h)}{\mathcal{P}(\xi)(s)}$$

is a *polynomial* in ξ of degree $\deg(h) - \deg(s)$ with highest coefficient $\left(\frac{h}{s}\right)^q$. This is just another way of saying that $\frac{h}{s}$ is unstable •

If we consider integrally closed unique factorization domains then it is equal to the unstable part of its field of fractions (compare Proposition 3.3 in [35]).

LEMMA 3.1.3 : *If H^* is an integrally closed unique factorization domain then*

$$\mathrm{H}^* = Un(FF(\mathrm{H}^*)).$$

PROOF : Let

$$\frac{h}{t} \in Un(FF(\mathrm{H}^*)).$$

By definition of the unstable part this means that

$$\mathcal{P}(\xi)(t) \mid \mathcal{P}(\xi)(h) \in FF(\mathrm{H}^*)[[\xi]],$$

and we have to show that $t \mid h$ in H^*. Since we work in a unique factorization domain there is no loss of generality to assume that $t \in \mathrm{H}^*$ is a prime element. Define the t-content of a polynomial $f(\xi) = h_0 + h_1\xi + \cdots + h_n\xi^n \in \mathrm{H}^*[\xi]$ by

$$c_t(f) := t^i,$$

where

$$i := \max\{j \in \mathbb{N}_0 \mid t^j \mid h_0, \ldots, t^j \mid h_n\}.$$

We find that the t-content of

$$\mathcal{P}(\xi)(t) = t + \mathcal{P}^1(t)\xi + \cdots + t^q\xi^{\deg(t)}$$

is t^0 or t^1. Assume that t does not divide h. Then the t-content of

$$\mathcal{P}(\xi)\left(\frac{h}{t}\right) = \frac{h}{t} + \cdots + \left(\frac{h}{t}\right)^q \xi^{\deg(h)-\deg(t)}$$

is t^{-l} for some large $l \geq q$. Since

$$\mathcal{P}(\xi)(t)\mathcal{P}(\xi)\left(\frac{h}{t}\right) = \mathcal{P}(\xi)(h)$$

we get that the t-content of $\mathcal{P}(\xi)(h)$ is

$$t^{-l+0} \quad \text{or} \quad t^{-l+1},$$

but this is a contradiction, because $-l+0 < -l+1 < 0$. So t divides h •

3.2 The Integral Closure Theorem

We come next to the generalization of Wilkerson's Integral Closure Theorem. The following lemma is the main ingredient.

LEMMA 3.2.1 : *Consider the polynomial algebra $\mathbb{F}[x_1, \ldots, x_n]$ over $\mathbb{F}$, where the generators x_i all have degree one. Let H^* be an integrally closed integral domain and let*

$$\varphi : \mathbb{F}[x_1, \ldots, x_n] \hookrightarrow \mathrm{H}^* = \overline{\mathrm{H}^*}$$

be an integral extension of unstable algebras over the Steenrod algebra. Then any linear form $l \in \mathbb{F}[x_1, \ldots, x_n]$ generates a prime ideal in H^*.

PROOF : Without loss of generality we can assume that $l = x_1$. Let $h_1,\ h_2 \in \mathrm{H}^*$ such that

$$h_1 h_2 = t x_1 \in (x_1)^e \subset \mathrm{H}^*.$$

We apply the giant Steenrod operation and get

$$\mathcal{P}(\xi)(h_1)\mathcal{P}(\xi)(h_2) = \mathcal{P}(\xi)(t)\mathcal{P}(\xi)(x_1) \in \mathrm{H}^*[\xi] \subset FF(\mathrm{H}^*)[\xi].$$

As an element in $FF(\mathrm{H}^*)[\xi]$

$$\mathcal{P}(\xi)(x_1) = x_1 + x_1^q \xi$$

is prime, because it is linear in ξ. Hence in $FF(\mathrm{H}^*)[\xi]$ it divides one of the factors on the left hand side, say

$$\mathcal{P}(\xi)(x_1) = x_1 + x_1^q \xi \;\big|\; \mathcal{P}(\xi)(h_1).$$

This means by definition that

$$\frac{h_1}{x_1} \in Un(FF(\mathrm{H}^*)).$$

By Lemma 3.1.1 the unstable part $Un(FF(\mathrm{H}^*))$ is integral over $Un\left(\mathbb{F}(x_1, \ldots, x_n)\right)$. Lemma 3.1.3 tells us that

$$Un(\mathbb{F}(x_1, \ldots, x_n)) = \mathbb{F}[x_1, \ldots, x_n].$$

Combining the two statements gives us that $Un(FF(\mathrm{H}^*))$ is integral over $\mathbb{F}[x_1, \ldots, x_n]$, i.e., $\frac{h_1}{x_1}$ is integral over $\mathbb{F}[x_1, \ldots, x_n]$ and a fortiori over H^*. Since we assumed H^* to be integrally closed this means that $\frac{h_1}{x_1} \in \mathrm{H}^*$ or

$$h_1 \in (x_1)^e \subset \mathrm{H}^*,$$

and therefore $(x_1)^e \subset \mathrm{H}^*$ is a prime ideal •

The following proof is due to Larry Smith, [30].

THEOREM 3.2.2 (Integral Closure Theorem, General Version): *Consider the polynomial algebra $\mathbb{F}[x_1, \ldots, x_n]$ over $\mathbb{F}$, where the generators x_i all have degree one. Let H^* be an integral domain and let*

$$\varphi : \mathbb{F}[x_1, \ldots, x_n] \hookrightarrow \mathrm{H}^*$$

be an integral extension of unstable algebras over the Steenrod algebra. Then φ is an isomorphism of unstable algebras over the Steenrod algebra.

PROOF : Since

$$\mathbb{F}[x_1, \ldots, x_n] \underset{\varphi}{\hookrightarrow} \mathrm{H}^* \hookrightarrow \overline{\mathrm{H}^*}$$

is again an integral extension of unstable integral domains over $\mathscr{P}^*$ we assume without loss of generality that H^* is integrally closed.

We proceed by induction on the Krull dimension n. If $n = 0$ then $\mathbb{F} = \mathrm{H}^*$ since H^* is connected and φ integral. Let $n > 0$. We know from Lemma 3.2.1 that

$$(x_1)^e \subset \mathrm{H}^*$$

is a prime ideal. Since

$$(x_1)^e \cap \mathbb{F}[x_1, \ldots, x_n] = (x_1)$$

it follows that $(x_1)^e$ lies over (x_1). We therefore obtain a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{F}[x_1, \ldots, x_n] & \xrightarrow{\mu} & \mathbb{F}[x_1, \ldots, x_n] & \longrightarrow & \mathbb{F}[x_2, \ldots, x_n] & \longrightarrow & 0 \\ & & \downarrow \varphi & & \downarrow \varphi & & \downarrow \psi & & \\ 0 & \longrightarrow & \mathrm{H}^* & \xrightarrow{\mu} & \mathrm{H}^* & \longrightarrow & \mathrm{H}^*/(x_1)^e & \longrightarrow & 0 \end{array}$$

where the maps marked μ denote multiplication by x_1 and the map ψ is induced by the inclusion φ. By Lemma 3.2.1 $\mathrm{H}^*/(x_1)^e$ is an integral domain. So ψ is an isomorphism by the induction hypothesis. Hence by the serpent lemma

$$\mu : \mathrm{coker}(\varphi) \longrightarrow \mathrm{coker}(\varphi)$$

is an isomorphism. Since μ has degree $+1$ and $\mathrm{coker}(\varphi)$ is non negatively graded it follows that $\mathrm{coker}(\varphi) = 0$ and we are done •

REMARK : Note that this theorem does not remain valid if we drop the assumption on H^* to be an integral domain. Consider e.g.

$$\mathbb{F}[x] \hookrightarrow \mathbb{F}[x, y]/(y^2)$$

where $x,\ y$ are linear generators.

Note carefully that the preceding theorem depends heavily on unstability. This is illustrated by the next example.

EXAMPLE 1 : Let $\mathbb{F} = \mathbb{F}_3$ be the field with three elements and let $\mathbb{F}[x,\ y]$ be a polynomial ring with two linear generators. Embed $\mathbb{F}[x,\ y]$ in its field of fractions $\mathbb{F}(x,\ y)$ and extend this field to

$$\mathbb{K}^* := \mathbb{F}(x,\ y,\ t),$$

where $t^2 = xy$. We receive a $\boldsymbol{\mathcal{P}}^*$-separable finite field extension

$$\begin{array}{ccl} \mathbb{F}[x,\ y] & & \\ \big\downarrow & & \\ \mathbb{F}(x,\ y) & \underset{\boldsymbol{\mathcal{P}}^*-\mathrm{sep}}{\hookrightarrow} & \mathbb{K}^* := \mathbb{F}(x,\ y,\ t), \quad t^2 = xy. \end{array}$$

Proposition 2.2.1 tells us how to extend the $\boldsymbol{\mathcal{P}}^*$-action naturally given on $\mathbb{F}(x,\ y)$ uniquely to $\mathbb{K}^*$, namely by solving the equations

$$2\mathcal{P}^i(t)t + \sum_{r+s=i;\ r,\ s<i} \mathcal{P}^r(t)\mathcal{P}^s(t) = \mathcal{P}^i(t^2) = \mathcal{P}^i(xy)$$

inductively for i, e.g.,

$$\begin{aligned} \mathcal{P}^1(t) &= -\frac{1}{t}\mathcal{P}^1(xy) \\ &= -\frac{1}{t}(x^3y + xy^3) \\ &= -\frac{1}{t}t^2(x^2 + y^2) \\ &= -t(x^2 + y^2) \end{aligned}$$

and

$$\begin{aligned} \mathcal{P}^2(t) &= -\frac{1}{t}\left(\mathcal{P}^2(xy) - \mathcal{P}^1(t)^2\right) \\ &= -\frac{1}{t}\left(x^3y^3 - t^2(x^2 + y^2)^2\right) \\ &= \frac{1}{t}\left(t^6 + t^2x^4 + t^2y^4\right) \\ &= t\left(t^4 + x^4 + y^4\right) \\ &= t\left(x^2 - y^2\right)^2 \\ &= t(x - y)^2(x + y)^2. \end{aligned}$$

So, the $\mathscr{P}^*$-action on t is no longer unstable, for otherwise we would have

$$txy = t^3 = \mathcal{P}^1(t) = -t(x^2 + y^2),$$

which contradicts the fact that $\mathbb{F}(x,\ y)$ is a purely transcendental field extension. Also, we obviously have

$$0 \neq \mathcal{P}^2(t) = t(x-y)^2(x+y)^2.$$

So, let's take the unstable part of $\mathbb{K}^*$. Then we have a diagram in the category of fields, resp. unstable algebras over $\mathscr{P}^*$ of the following form

$$\begin{array}{ccc} \mathbb{F}[x,\ y] & \overset{\varphi}{\hookrightarrow} & Un(\mathbb{K}^*) \\ \downarrow & & \downarrow \\ \mathbb{F}(x,\ y) & \underset{\mathscr{P}^*\text{-sep, finite}}{\hookrightarrow} & \mathbb{K}^*. \end{array}$$

Since $Un(\mathbb{F}(x,\ y)) = \mathbb{F}[x,\ y]$ by Lemma 3.1.3, we can apply Lemma 3.1.1 to conclude that φ is an integral ring extension. So by Theorem 3.2.2 the map φ is an isomorphism and

$$Un(\mathbb{K}^*) = \mathbb{F}[x,\ y].$$

However, let's ignore the Steenrod algebra for a moment and take, instead of the unstable part of $\mathbb{K}^*$, the integral closure of $\mathbb{F}[x,\ y]$ in $\mathbb{K}^*$: this gives

$$\begin{array}{ccc} \mathbb{F}[x,\ y] & \overset{\psi}{\hookrightarrow} & \overline{\mathbb{F}[x,\ y]}_{\mathbb{K}^*} \\ \downarrow & & \downarrow \\ \mathbb{F}(x,\ y) & \underset{\mathscr{P}^*\text{-sep, finite}}{\hookrightarrow} & \mathbb{K}^*. \end{array}$$

Then, by construction, the map ψ is an integral ring extension, but it is far from being an isomorphism, because

$$\overline{\mathbb{F}[x,\ y]}_{\mathbb{K}^*} = \mathbb{F}[x,\ y,\ t]/(t^2 - xy)$$

as one should easily see. The reason is, that this new ring is *not unstable* over the Steenrod algebra: recall from above that we found

$$\mathcal{P}^1(t) \neq t^3$$

and, e.g.,

$$\mathcal{P}^2(t) \neq 0.$$

CHAPTER 4

The Inseparable Closure

After having settled the awful technical stuff we are about to arrive at our first success: the discovery of a new exotic animal, the $\mathcal{P}^*$-inseparable closure of an unstable algebra H^*. Until now this has been known only for the prime field $\mathbb{F}_p$, and even there, only for Noetherian integral domains. However, as we show, the $\mathcal{P}^*$-inseparable closure can be constructed more generally. We just have to be more careful and patient.

4.1 The Introduction of our Exotic Animal

Let's start with a definition.

DEFINITION : *Call an unstable algebra* H^* ***$\mathcal{P}^*$-inseparably closed****, if whenever* $h \in \mathrm{H}^*$ *and*

$$\mathcal{P}^{\Delta_i}(h) = 0 \quad \forall\ i \geq 0,$$

then there exists an element $h' \in \mathrm{H}^*$ *such that*

$$(h')^p = h.$$

In other words, if all our derivations $\mathcal{P}^{\Delta_i}$ vanish on an element in H^* that element must be a p-th power[1]. Before elaborating a bit more let's append the obvious definition of a $\mathcal{P}^*$-inseparable closure[2].

---

[1] Note that we need to detect p-th powers, not only q-th powers.

[2] I agree this is just some abstract nonsense, but wait a minute and we will see that this animal really exists.

DEFINITION : *The* $\mathcal{P}^*$**-inseparable closure** *of* H^* *is a* $\mathcal{P}^*$*-inseparably closed algebra* $\sqrt{\mathrm{H}^*}$ *containing* H^* *such that the following universal property holds: Whenever we have a* $\mathcal{P}^*$*-inseparably closed algebra* H'^* *containing* H^* *there exists an embedding* $\varphi : \sqrt{\mathrm{H}^*} \hookrightarrow \mathrm{H}'^*$.

S.-P. Lam defined the $\mathcal{P}^*$-inseparable closure in the category of unstable Noetherian integral domains over $\mathcal{P}^*$, [14] §3. Moreover, he showed that his definition is equivalent to condition 1.2.2 occurring in J. F. Adam's and C. W. Wilkerson's proof that certain unstable integral domains over $\mathcal{P}^*$ are rings of invariants, see again [14] Proposition 3.2 and compare [1]. However, the above definitions are not quite direct translations, because we are taking into account the action of $\mathcal{P}^{\Delta_0}$.

It is clear that $\sqrt{\mathrm{H}^*}$ is unique. However, the hard part begins now: we have to show that $\sqrt{\mathrm{H}^*}$ exists. We do that by giving a *construction method*, which mimics the construction method of the $\mathcal{P}^*$-inseparable closure of a field given in §1.2.

First we assume that our algebra H^* is Noetherian.

Denote by $C^* \subseteq \mathrm{H}^*$ the sub algebra consisting of the $\mathcal{P}^{\Delta_i}$ constants for all $i \geq 0$, i.e.

$$C^* = C(\mathrm{H}^*) = \{h \in \mathrm{H}^* \mid \mathcal{P}^{\Delta_i}(h) = 0 \ \forall\ i \geq 0\}.$$

LEMMA 4.1.1 : *C^* is a finitely generated algebra over* $\mathbb{F}$.

PROOF : We have that the extension

$$C^* \hookrightarrow \mathrm{H}^*$$

is finite, since for any $h \in \mathrm{H}^*$ its p-th power is in C^*, $h^p \in C^*$ and H^* is Noetherian. Therefore we have that C^* is a Noetherian ring. Hence C^* is finitely generated by standard tic-tac-toe over graded connected commutative $\mathbb{F}$-algebras, see e.g. [28] §2.2 •

We have that C^* is a graded connected commutative Noetherian algebra over $\mathbb{F}$. Moreover the following lemma ensures that C^* inherits an action of the Steenrod algebra from H^*.

LEMMA 4.1.2 : *C^* is closed under the action of the Steenrod algebra $\mathcal{P}^*$.*

PROOF : Let $h \in C^*$. We have to show that

$$\mathcal{P}^j(h) \in C^* \quad \forall\ j \geq 0,$$

i.e.,

$$\mathcal{P}^{\Delta_i}\mathcal{P}^j(h) = 0 \quad \forall\ i,\ j \geq 0.$$

We have for any $i \geq 0$ and for any $j \geq 0$

$$\begin{aligned}
\mathcal{P}^{\Delta_i}\left(\mathcal{P}^j(h)\right) &= \mathcal{P}^j\left(\mathcal{P}^{\Delta_i}(h)\right) - \mathcal{P}^{\Delta_{i+1}}\left(\mathcal{P}^{j-q^i}(h)\right)\\
&= -\mathcal{P}^{\Delta_{i+1}}\left(\mathcal{P}^{j-q^i}(h)\right)\\
&= -\mathcal{P}^{j-q^i}\left(\mathcal{P}^{\Delta_{i+1}}(h)\right) + \mathcal{P}^{\Delta_{i+2}}\left(\mathcal{P}^{j-q^i-q^{i+1}}(h)\right)\\
&= \mathcal{P}^{\Delta_{i+2}}\left(\mathcal{P}^{j-q^i-q^{i+1}}(h)\right)\\
&= \cdots\\
&= 0,
\end{aligned}$$

where we made use of the commutation rules Lemma 1.1.2 and the unstability condition •

Hence we have shown that for a Noetherian H^*

$$C^* = C(\mathrm{H}^*) = \mathbb{F}\langle c_1, \ldots, c_r\rangle$$

is again an unstable Noetherian algebra[3] over $\mathcal{P}^*$. Define an algebra

$$\mathrm{H}_1^* := \left(\mathrm{H}^* \otimes_{\mathbb{F}} \mathbb{F}[\gamma_1, \ldots, \gamma_r]\right) / (\gamma_1^p - c_1, \ldots, \gamma_r^p - c_r),$$

i.e., we have adjoined all p-th roots of the algebra generators of $C(\mathrm{H}^*)$.

LEMMA 4.1.3 : H_1^* *is again a Noetherian unstable algebra over* $\mathcal{P}^*$.

PROOF : Give the new generators γ_i the obvious degree, namely

$$\deg(\gamma_i) := \frac{1}{p}\deg(c_i) \quad \forall\ i.$$

The condition

$$\mathcal{P}^{\Delta_0}(c_i) = \deg(c_i)c_i = 0 \quad \forall\ i$$

ensures that c_i is indeed of degree divisible by p, so our new object is again graded over the non negative integers. Connectivity and commutativity is free of charge. Also Noetherian is obvious. The remaining point is to define a suitable Steenrod algebra action on our new elements γ_i. We do this in the following way (recall from Section 2.3 the analogous construction on the field level):

$$\left(\mathcal{P}^j(\gamma_i)\right)^p := \mathcal{P}^{jp}(c_i) \quad \forall\ i,\ j.$$

Since this equation takes place inside $C^* = \mathbb{F}\langle c_1, \ldots, c_r\rangle$ it follows that

$$\mathcal{P}^j(\gamma_i) \in \mathbb{F}\langle \gamma_1, \ldots, \gamma_r\rangle \subseteq \mathrm{H}_1^*,$$

[3] The notation $C^* = \mathbb{F}\langle c_1, \ldots, c_r\rangle$ means that C^* is an $\mathbb{F}$-algebra generated by the elements $c_1, \ldots, c_r$; while $\mathbb{F}[\gamma_1, \ldots, \gamma_r]$ denotes a *polynomial* algebra over $\mathbb{F}$ generated by $\gamma_1, \ldots, \gamma_r$.

and H_1^* is closed under the action of the Steenrod algebra. By construction this action satisfies the Cartan formulae, the unstability condition and the Adem-Wu relations, because raising to p-th powers is an additive operation in characteristic p •

If $H_1^* = H^*$ then H^* was already $\mathcal{P}^*$-inseparably closed. If $C(H_1^*) = H_1^{*p}$ then H_1^* is $\mathcal{P}^*$-inseparably closed. Otherwise we repeat this construction: take the sub algebra $C_1^* := C(H_1^*)$ of $\mathcal{P}^{\Delta_i}$-constants in H_1^*, find a set of generators

$$C_1^* = \mathbb{F} < c_{11}, \dots, c_{1r_1} >$$

and form the algebra

$$H_2^* := H_1^* \otimes_{\mathbb{F}} \mathbb{F}[\gamma_{11}, \dots, \gamma_{1r_1}]/(\gamma_{11}^p - c_{11}, \dots, \gamma_{1r_1}^p - c_{1r_1}).$$

Again, if $H_2^{*p} = C_2^*$ stop, otherwise proceed. This way we get an ascending chain of unstable Noetherian algebras over the Steenrod algebra

$$H^* = H_0{}^* \subseteq H_1^* \subseteq \cdots \subseteq H_k^* \subseteq \cdots,$$

each extension being finite and integral.

PROPOSITION 4.1.4 : *The colimit of this chain of algebras is the $\mathcal{P}^*$-inseparable closure of H^*.*

PROOF : By construction the colimit $\mathrm{colim}(H_i^*)$ is a $\mathcal{P}^*$-inseparably closed algebra. Therefore

$$\sqrt{H^*} \hookrightarrow \mathrm{colim}(H_i^*).$$

On the other hand for any $h \in \mathrm{colim}(H_i)$ there exists an index $i_0 \in \mathbb{N}_0$ such that

$$h \in H_{i_0}^*,$$

hence

$$h^{p^{i_0}} \in H^*$$

and therefore

$$h \in \sqrt{H^*}$$

which was to be shown •

If we drop the condition on H^* being Noetherian then this construction method for the inseparable closure works in the same way. The only difference is that the whole chain of algebras H_i^* and their respective sub algebras of constants $C(H_i^*)$ are possibly no longer Noetherian [4] and the successive extensions $H_i^* \hookrightarrow H_{i+1}^*$ might not be finite.

[4] So in the non Noetherian case the whole notation would become a lot more clumsy.

4.2 The Animal and its Properties

In this section we are going to establish the main properties of $\sqrt{\mathrm{H}^*}$, and add some examples. We do not assume that H^* is Noetherian. Let's start with the basics:

PROPOSITION 4.2.1 : *The $\mathcal{P}^*$-inseparable closure of H^* has the following properties:*

[1] $\sqrt{\mathrm{H}^*}$ *is a graded, connected, commutative algebra over* $\mathbb{F}$.

[2] $Nil\,(\mathrm{H}^*) = (0)$ *if and only if* $Nil\,(\sqrt{\mathrm{H}^*}) = (0)$. *Moreover we have* $Nil\,(\mathrm{H}^*_{i+1}) = \sqrt{(Nil\,(\mathrm{H}^*_i))^e}$ *for all* $i \geq 0$, *where* $Nil\,(\mathrm{H}^*_i)^e$ *denotes the extension of the ideal* $Nil\,(\mathrm{H}^*_i)$ *in* H^*_{i+1}.

[3] H^* *is an integral domain if and only if* $\sqrt{\mathrm{H}^*}$ *is, and* H^*_{i+1} *is an integral domain if and only if* H^*_i *is.*

[4] $\mathrm{H}^* \hookrightarrow \sqrt{\mathrm{H}^*}$ *is an integral extension and* $\dim(\sqrt{\mathrm{H}^*}) = \dim(\mathrm{H}^*)$.

[5] *If* H^* *is an integrally closed domain then so is* $\sqrt{\mathrm{H}^*}$.

[6] $\sqrt{\mathrm{H}^*}$ *is an unstable algebra over the Steenrod algebra.*

PROOF :

AD [1] : The first statement is true by construction, because colimits exist in the category of graded connected commutative algebras over a field $\mathbb{F}$, see e.g. [11] Appendix A 6.3.

AD [2] : If $Nil\,(\sqrt{\mathrm{H}^*}) = (0)$ then certainly $Nil\,(\mathrm{H}^*) = (0)$, because $\mathrm{H}^* \subset \sqrt{\mathrm{H}^*}$. So, let $Nil\,(\mathrm{H}^*) = (0)$ and take an element $h \in Nil\,(\sqrt{\mathrm{H}^*})$. Then $h \in \mathrm{H}^*_{i_0}$ for some $i_0 \in \mathbb{N}_0$. Let i_0 be minimal with this property. Hence

$$h^{p^{i_0}} \in \mathrm{H}^*$$

is nilpotent. Since H^* has no nilpotent elements it follows that

$$h^{p^{i_0}} = 0 \in \mathrm{H}^*$$

and

$$h \in Nil\,(\mathrm{H}^*_{i_0}).$$

We proceed by induction on i to show that

$$Nil\,(\mathrm{H}^*_i) = (0) \quad \forall\ i \geq 0.$$

If $i = 0$ this is the hypothesis. Let $i > 0$ and let $h \in Nil\,(\mathrm{H}^*_i) \subset \mathrm{H}^*_i$. Then

$$h^p \in \mathrm{H}^*_{i-1}$$

is nilpotent and, therefore by induction, $h^p = 0$. Hence h^p is in the kernel[5] of the remembering map ϱ

$$h^p \in \ker\{\varrho : \mathbb{F}[c_{i-1,\,1}, \ldots, c_{i-1,\,r_{i-1}}] \longrightarrow C^*_{i-1}\}.$$

Therefore its p-th root h is in the kernel of the remembering map of the corresponding algebras of p-th roots[6]

$$h \in \ker\{\varrho : \mathbb{F}[\gamma_{i-1,\,1}, \ldots, \gamma_{i-1,\,r_{i-1}}] \longrightarrow \mathbb{F}\langle\gamma_{i-1,\,1}, \ldots, \gamma_{i-1,\,r_{i-1}}\rangle\}.$$

Hence

$$h = 0 \in \mathbb{F}\langle\gamma_{i-1,\,1}, \ldots, \gamma_{i-1,\,r_{i-1}}\rangle \subseteq C^*_i \subseteq \mathrm{H}^*_i$$

as claimed.

For the second statement note firstly that

$$Nil\,(\mathrm{H}^*_i) = Nil\,(\mathrm{H}^*_{i+1}) \cap \mathrm{H}^*_i,$$

because $\mathrm{H}^*_i \hookrightarrow \mathrm{H}^*_{i+1}$. Therefore

$$Nil\,(\mathrm{H}^*_i)^e \subseteq Nil\,(\mathrm{H}^*_{i+1}) \subset \mathrm{H}^*_{i+1}$$

and hence

$$\sqrt{Nil\,(\mathrm{H}^*_i)^e} \subseteq \sqrt{Nil\,(\mathrm{H}^*_{i+1})} = Nil\,(\mathrm{H}^*_{i+1}).$$

For the converse inclusion take an element $h \in Nil\,(\mathrm{H}^*_{i+1})$. If $h \in \mathrm{H}^*_i$ then nothing is to show. So assume that $h \notin \mathrm{H}^*_i$. Then by construction

$$h^p \in \mathrm{H}^*_i$$

is nilpotent, i.e.,

$$h^p \in Nil\,(\mathrm{H}^*_i) \subseteq Nil\,(\mathrm{H}^*_i)^e$$

and therefore

$$h \in \sqrt{Nil\,(\mathrm{H}^*_i)^e},$$

what we claimed.

---

[5] The following argument remains unchanged for non Noetherian algebras H^*, again only C^*_{i-1} and hence $\mathbb{F} < \gamma_{i-1,\,1},\ \gamma_{i-1,\,2} \cdots >$ might not be finitely generated.

[6] Note that the Frobenius map describes an isomorphism (of degree p) between the algebras $\mathbb{F}[c_{i-1,\,1}, \ldots, c_{i-1,\,r_{i-1}}]$ and $\mathbb{F}[\gamma_{i-1,\,1}, \ldots, \gamma_{i-1,\,r_{i-1}}]$, resp. C^*_{i-1} and $\mathbb{F}\langle\gamma_{i-1,\,1}, \ldots, \gamma_{i-1,\,r_{i-1}}\rangle$.

AD [3] : If $\sqrt{\mathrm{H}^*}$, resp. H^*_{i+1}, is an integral domain, then so is its sub algebra H^*, resp. H^*_i.

So assume that H^* is an integral domain. Let $h,\ h' \in \sqrt{\mathrm{H}^*}$ with $hh' = 0$. Then for a suitably large $l \in \mathbb{N}_0$ we have

$$h^{p^l},\ h'^{p^l} \in \mathrm{H}^* \text{ and } h^{p^l} h'^{p^l} = (hh')^{p^l} = 0.$$

Since H^* has no zero divisors h or h' is zero.

The second statement is equally easily verified. For that assume that H^*_i is an integral domain, and take two elements $h,\ h' \in \mathrm{H}^*_{i+1}$ such that their product is zero, $hh' = 0$. If both elements are in H^*_i then nothing is to show. So assume that without loss of generality $h \notin \mathrm{H}^*_i$ but $h' \in \mathrm{H}^*_i$. Then we have that $h^p \in \mathrm{H}^*_i \setminus \{0\}$ and so

$$h^p h' = 0 \in \mathrm{H}^*_i \Rightarrow h' = 0.$$

In the third case, where both elements are not in H^*_i, we have that $h^p,\ h'^p \in \mathrm{H}^*_i \setminus \{0\}$. Hence

$$hh' = 0 \Rightarrow h^p h'^p = 0$$

would contradict the assumption that H^*_i has no zerodivisors.

AD [4] : In the preceding section we have found that the $\mathscr{P}^*$-inseparable closure $\sqrt{\mathrm{H}^*}$ of H^* is just a filtered colimit of a certain chain of algebras

$$\mathrm{H}^* = \mathrm{H}^*_0 \subseteq \mathrm{H}^*_1 \subseteq \cdots \subseteq \mathrm{H}^*_k \subseteq \cdots,$$

where each extension is integral. Hence the extension

$$\mathrm{H}^* \hookrightarrow \sqrt{\mathrm{H}^*}$$

is integral, and therefore

$$\dim(\sqrt{\mathrm{H}^*}) = \dim(\mathrm{H}^*).$$

AD [5] : By [3] $\sqrt{\mathrm{H}^*}$ is an integral domain because by assumption H^* is one. Consider the diagram

$$\begin{array}{ccccc}
\mathrm{H}^* = \overline{\mathrm{H}^*} & \underset{\text{integral}}{\hookrightarrow} & \sqrt{\mathrm{H}^*} & \underset{\text{integral}}{\hookrightarrow} & \overline{\sqrt{\mathrm{H}^*}} \\
\downarrow & & \downarrow & & \downarrow \\
FF(\mathrm{H}^*) & \hookrightarrow & FF(\sqrt{\mathrm{H}^*}) & = & FF\left(\overline{\sqrt{\mathrm{H}^*}}\right).
\end{array}$$

Let $\frac{h_1}{h_2} \in FF(\sqrt{\mathrm{H}^*})$ be integral over $\sqrt{\mathrm{H}^*}$. Since $h_1,\ h_2 \in \sqrt{\mathrm{H}^*}$ there is an $l \in \mathbb{N}_0$ such that

$$h_1^{p^l},\ h_2^{p^l} \in \mathrm{H}^*$$

and hence

$$\frac{h_1^{p^l}}{h_2^{p^l}} \in FF(\mathrm{H}^*).$$

Moreover, by transitivity, $\frac{h_1}{h_2}$ is integral over H^*, hence $\left(\frac{h_1}{h_2}\right)^{p^l}$ is integral over H^*, and so

$$\left(\frac{h_1}{h_2}\right)^{p^l} \in \mathrm{H}^*.$$

Therefore

$$\frac{h_1}{h_2} \in \sqrt{\mathrm{H}^*}$$

as we claimed.

AD [6] : Since an element $h \in \sqrt{\mathrm{H}^*}$ is contained in some H_i^*, 4.1.3 defines an unstable action of the Steenrod algebra on $\sqrt{\mathrm{H}^*}$ •

A number of comments are in order.

Remark : Since the inseparable closure of H^* is just some filtered colimit we have

$$\operatorname{colim}\left(Nil\,(\mathrm{H}_i^*)\right) = Nil\,(\sqrt{\mathrm{H}^*}) = Nil\,\left(\operatorname{colim}(\mathrm{H}_i^*)\right),$$

see Exercise 22 on page 34, Chapter 2 of [3]. In the same exercise the diligent student is asked to prove the following statement: If H_i^* are integral domains for all i then their colimit is also an integral domain. This, together with the second statement of [4] in Proposition 4.2.1, implies the first of [4].

Remark : Statement [5] in the above Proposition 4.2.1 can be generalized to algebras with zero divisors, in the sense that one would have to consider the total *ring* of fractions [7] instead of the *field* of fractions and define integrally closed as being integrally closed in its ring of fractions.

Statement [4] of the preceding proposition does *not* imply that $\sqrt{\mathrm{H}^*}$ is Noetherian whenever H^* is. This is indeed a difficult question, which we will answer in Sections 6.1 and 6.3. The reverse problem, namely

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[7] For a ring H^* the **total ring of fractions** $RF(\mathrm{H}^*)$ is defined to be an over ring of H^* such that

(1) $RF(\mathrm{H}^*)$ has an identity,

(2) every element of $RF(\mathrm{H}^*)$ has the form $\frac{h_1}{h_2}$, where $h_1,\ h_2 \in \mathrm{H}^*$ and in addition h_2 is not a zero divisor, and

(3) every non zero divisor of H^* has an inverse in $RF(\mathrm{H}^*)$,

see e.g. §19 of [38]. Again, in our work we consider, of course, only homogeneous elements $h_1,\ h_2, \cdots$, compare the remarks at the beginning of Section 2.1.

assuming that $\sqrt{H^*}$ is Noetherian and asking what can we say about H^* is a lot easier, as the following discussion shows.

The following lemma was suggested by Larry Smith, [30].

LEMMA 4.2.2 : *If*

$$H_0^* \subseteq H_1^* \subseteq H_2^* \subseteq \cdots$$

is an increasing chain of graded connected $\mathbb{F}$*-algebras and if* $\operatorname{colim}(H_i^*)$ *is Noetherian, then there exists a* $r \in \mathbb{N}_0$ *such that*

$$H_r^* = H_{r+1}^* = \cdots = \operatorname{colim}(H_i^*).$$

PROOF : Choose generators $h_1, \ldots, h_m$ for $\operatorname{colim}(H_i^*)$ as an $\mathbb{F}$-algebra. For each $i = 1, \ldots, m$ we can find a $r_i \in \mathbb{N}_0$ such that $h_i \in H_{r_i}^*$. Set

$$r := \max\{r_1, \ldots, r_m\}.$$

Then

$$\operatorname{colim}(H_i^*) \subseteq H_r^* \subseteq H_{r+1}^* \subseteq \cdots \subseteq \operatorname{colim}(H_i^*)$$

and we are done •

PROPOSITION 4.2.3: *Let* H^* *be an unstable algebra* [8] *and let* $\sqrt{H^*}$ *be Noetherian. Then so is* H^*.

PROOF : Choose a homogeneous system of parameters $h_1, \ldots, h_n$ for $\sqrt{H^*}$. Then there exist indices i_j such that

$$h_j \in H_{i_j}^* \quad \text{for } j = 1, \ldots, n.$$

Set

$$i := \max\{i_1, \ldots, i_n\}$$

then we have

$$h_1, \ldots, h_n \in H_i^*$$

and by construction

$$h_1^{p^i}, \ldots, h_n^{p^i} \in H^*$$

which is still a system of parameters for $\sqrt{H^*}$. Hence $\sqrt{H^*}$ is a finite $\mathbb{F}[h_1^{p^i}, \ldots, h_n^{p^i}]$-module, and therefore so is H^*. Therefore H^* is Noetherian •

---

[8] This is not a typo. We don't need to assume anything more about H^*.

Alternative Proof : If $\sqrt{\mathrm{H}^*}$ is Noetherian, then the chain

$$\mathrm{H}^* = \mathrm{H}_0^* \subseteq \mathrm{H}_1^* \subseteq \cdots \subseteq \mathrm{H}_k^* = \sqrt{\mathrm{H}^*}$$

becomes stationary after, say k steps, by Lemma 4.2.2. Choose a set $h_1, \ldots, h_m$ of algebra generators for $\sqrt{\mathrm{H}^*}$. Then, by definition of the inseparable closure we have

$$\mathbb{F} < h_1, \ldots, h_m >= \sqrt{\mathrm{H}^*}.$$

So, $\sqrt{\mathrm{H}^*}$ is a finitely generated $\mathbb{F} < h_1^{p^k}, \ldots, h_m^{p^k}$ >-module, hence so is H^*. Therefore H^* is Noetherian •

The following characterization of the case when $\sqrt{\mathrm{H}^*}$ is Noetherian will be useful in later chapters.

Proposition 4.2.4 : *Let* H^* *be a Noetherian unstable algebra. Then the following two statements are equivalent:*

[1] $\sqrt{\mathrm{H}^*}$ *is Noetherian.*

[2] *The chain of algebras*

$$\mathrm{H}^* \stackrel{\varphi_0}{\hookrightarrow} \mathrm{H}_1^* \stackrel{\varphi_1}{\hookrightarrow} \mathrm{H}_2^* \stackrel{\varphi_2}{\hookrightarrow} \cdots$$

becomes stationary after a finite number, say r *steps, so that their colimit is indeed a finite union*

$$\operatorname{colim}\left(\mathrm{H}_i^*\right) = \bigcup_{i=0}^{r} \mathrm{H}_i^*.$$

If in addition H^* *is an integral domain then the preceding statements are equivalent to*

[3] *The chain of fields of fractions*

$$FF(\mathrm{H}^*) \hookrightarrow FF(\mathrm{H}_1^*) \hookrightarrow FF(\mathrm{H}_2^*) \hookrightarrow \cdots$$

becomes stationary after a finite number, say s *steps, so that their colimit is also a finite union*

$$\operatorname{colim}\left(FF(\mathrm{H}_i^*)\right) = \bigcup_{i=0}^{s} FF(\mathrm{H}_i^*).$$

Moreover $r \geq s$ *and* $\sqrt{\mathrm{H}^*} = \overline{\mathrm{H}_s}$.

Proof : We start with proving the equivalence of [1] and [2].

ad [1] $\Rightarrow$ [2] : This is easy. Apply Lemma 4.2.2.

AD [2] ⇒ [1] :

This is equally easy. Just observe that

$$\sqrt{\mathrm{H}^*} := \operatorname{colim}(\mathrm{H}^*_i) = \bigcup_{i=0}^{r} \mathrm{H}^*_i = \mathrm{H}^*_r,$$

hence $\sqrt{\mathrm{H}^*} = \mathrm{H}^*_r$ is Noetherian because H^*_r is a finitely generated H^*-module.

Now assume that H^* is an integral domain. We prove the implications [2] ⇒ [3] ⇒ [1].

AD [2] ⇒ [3] :

This is trivial.

AD [3] ⇒ [1] :

We have the following diagram

$$\begin{array}{ccccc} \mathrm{H}^* & \hookrightarrow & \sqrt{\mathrm{H}^*} & \hookrightarrow & \overline{\mathrm{H}^*{}_{FF(\sqrt{\mathrm{H}})}} \\ \downarrow & & \downarrow & & \\ FF(\mathrm{H}^*) & \hookrightarrow & FF(\sqrt{\mathrm{H}^*}) & & \end{array}$$

where by assumption the field extension $FF(\mathrm{H}^*) \hookrightarrow FF(\sqrt{\mathrm{H}^*})$ is finite and $\mathcal{P}^*$-purely inseparable. Hence by an oldie from Emmy, see e.g. [11] Corollary 13.13, the integral closure of H^* in the bigger field, denoted by $\overline{\mathrm{H}^*{}_{FF(\sqrt{\mathrm{H}})}}$, is again Noetherian, and, a finitely generated H^*-module. Hence so is $\sqrt{\mathrm{H}^*}$, i.e, $\sqrt{\mathrm{H}^*}$ is Noetherian.

Finally, note that if

$$\mathrm{H}^*_r = \mathrm{H}^*_{r+1} = \cdots$$

then,

$$FF(\mathrm{H}^*_r) = FF(\mathrm{H}^*_{r+1}) = \cdots$$

hence $r \geq s$, and by Proposition 4.2.6 $\overline{\mathrm{H}^*_s} = \sqrt{\mathrm{H}^*}$ •

We proceed to investigate the relation between the $\mathcal{P}^*$-inseparable closure on the algebra and on the field level. This will be of use in later chapters.

LEMMA 4.2.5: *Let H^* be an integral domain. Then the field extension*

$$FF(\mathrm{H}^*) \hookrightarrow FF(\sqrt{\mathrm{H}^*})$$

is $\mathcal{P}^*$-purely inseparable.

PROOF : By 4.2.1 [3] the $\mathcal{P}^*$-inseparable closure $\sqrt{\mathrm{H}^*}$ is also an integral domain, so we can apply the field of fraction functor. The field extension is algebraic, since the ring extension is integral. However, for any

$$\frac{h_1}{h_2} \in FF(\sqrt{\mathrm{H}^*}) \text{ with } h_1,\ h_2 \in \sqrt{\mathrm{H}^*}$$

there is an l such that

$$h_1^{p^l},\ h_2^{p^l} \in \mathrm{H}^*$$

hence

$$\frac{h_1^{p^l}}{h_2^{p^l}} \in FF(\mathrm{H}^*),$$

i.e., the field extension is *-purely inseparable. Since both fields inherit an action of the Steenrod algebra from H^*, resp. $\sqrt{\mathrm{H}^*}$, and the various actions are compatible, we remain in our category and the extension is $\mathcal{P}^*$-purely inseparable •

The following proposition shows that in the case of integral domains our notion of the $\mathcal{P}^*$-inseparable closure of an algebra fits naturally with the notion of the $\mathcal{P}^*$-inseparable closure of its field of fractions.

PROPOSITION 4.2.6: *Let* H^* *be an integral domain. The field of fractions of its* $\mathcal{P}^*$*-inseparable closure is the* $\mathcal{P}^*$*-inseparable closure of its field of fractions*

$$FF(\sqrt{\mathrm{H}^*}) = \big(FF(\mathrm{H}^*)\big)_{\mathcal{P}^*-\mathrm{insep}}.$$

PROOF : Recall our construction method for $\sqrt{\mathrm{H}^*}$. It leads to the following diagram

$$\begin{array}{ccccccccc} \mathrm{H}^* = \mathrm{H}^*_0 & \hookrightarrow & \mathrm{H}^*_1 & \hookrightarrow & \cdots & \hookrightarrow & \mathrm{colim}(\mathrm{H}^*_i) & \hookrightarrow \\ \cap & & \cap & & \cdots & & \cap & \\ FF(\mathrm{H}^*) & \hookrightarrow & FF(\mathrm{H}^*_1) & \hookrightarrow & \cdots & \hookrightarrow & FF(\mathrm{colim}(\mathrm{H}^*_i)) & \hookrightarrow \end{array}$$

where we have purely $\mathcal{P}^*$-inseparable field extensions. So certainly

$$FF(\mathrm{colim}(\mathrm{H}^*_i)) \subseteq (FF(\mathrm{H}^*))_{\mathcal{P}^*-\mathrm{insep}}.$$

Moreover we could consider the colimit, $\mathrm{colim}(FF(\mathrm{H}^*_i))$, of the sequence [9]

$$FF(\mathrm{H}^*) \hookrightarrow FF(\mathrm{H}^*_1) \hookrightarrow FF(\mathrm{H}^*_2) \hookrightarrow \cdots.$$

We have

$$\mathrm{colim}(FF(\mathrm{H}^*_i)) \subseteq FF(\mathrm{colim}(\mathrm{H}^*_i)),$$

[9] The filtered colimit of graded fields is a graded field, compare [11] appendix A 6.3.

because any $h \in \operatorname{colim}(FF(\mathrm{H}_i^*))$ is by definition contained in some $FF(\mathrm{H}_{i_0}^*)$ which in turn is contained in $FF(\operatorname{colim}(\mathrm{H}_i^*))$, i.e., we have field extensions

$$\operatorname{colim}(FF(\mathrm{H}_i^*)) \subseteq FF(\operatorname{colim}(\mathrm{H}_i^*)) \subseteq FF(\mathrm{H}^*)_{\mathcal{P}^*-\mathrm{insep}}.$$

Let now $h \in FF(\mathrm{H}^*)_{\mathcal{P}^*-\mathrm{insep}}$ then there is an l such that

$$h^{p^l} \in FF(\mathrm{H}^*)$$

and if we choose l to be minimal with this property we get

$$h \in FF(\mathrm{H}^*)_l = FF(\mathrm{H}_l^*) \subseteq \operatorname{colim}(FF(\mathrm{H}_i^*)).$$

Therefore the three fields are the same

$$\operatorname{colim}(FF(\mathrm{H}_i^*)) = FF(\operatorname{colim}(\mathrm{H}_i^*)) = FF(\mathrm{H}^*)_{\mathcal{P}^*\mathrm{insep}}$$

and we are done •

There is an obvious corollary to this.

COROLLARY 4.2.7 : *If H^* is a $\mathcal{P}^*$-inseparably closed integral domain then its field of fractions is also $\mathcal{P}^*$-inseparably closed. If H^* is in addition integrally closed then these two statements are equivalent.*

PROOF : If H^* is $\mathcal{P}^*$-inseparably closed then by Proposition 4.2.6

$$FF(\mathrm{H}^*) = FF\left(\operatorname{colim}(\mathrm{H}_i^*)\right) = \operatorname{colim}(FF(\mathrm{H}_i^*)) = FF(\mathrm{H}^*)_{\mathcal{P}^*-\mathrm{insep}}.$$

In order to prove the second statement assume that H^* is integrally closed and that the field of fractions of H^* is $\mathcal{P}^*$-inseparably closed. Then the above proposition gives again that

$$FF\left(\operatorname{colim}(\mathrm{H}_i^*)\right) = \operatorname{colim}(FF(\mathrm{H}_i^*)) = FF(\mathrm{H}^*)_{\mathcal{P}^*-\mathrm{insep}} = FF(\mathrm{H}^*).$$

So, by Proposition 4.2.1 [3], [4] and [5]

$$\mathrm{H}^* \hookrightarrow \operatorname{colim}(\mathrm{H}_i^*)$$

is an integral extension of integrally closed integral domains with the same field of fractions. Therefore they are equal •

REMARK : On the other hand if we start with a $\mathcal{P}^*$-inseparably closed field $\mathbb{K}^*$ then its unstable part, $Un(\mathbb{K}^*)$, is also $\mathcal{P}^*$-inseparably closed: If $k \in \sqrt{Un(\mathbb{K}^*)}$ then $k \in \mathbb{K}^*$, because $\mathbb{K}^*$ is $\mathcal{P}^*$-inseparably closed, and also $k^{p^e} \in Un(\mathbb{K}^*)$ for some suitably large $e \in \mathbb{N}_0$. However, the Cartan formulae give us

$$\mathcal{P}^r(k^{p^e}) = \left(\mathcal{P}^{\frac{r}{p^e}}(k)\right)^{p^e},$$

and so for any $r > \deg(k)p^e$ we have

$$0 = \mathcal{P}^{\frac{r}{p^e}}(k),$$

and therefore k is also an unstable element.

COROLLARY 4.2.8 : *The algebra of polynomial functions in n variables* $\mathbb{F}[x_1, \dots, x_n]$ *is* $\mathcal{P}^*$*-inseparably closed.*

PROOF : This follows from the above Corollary 4.2.7, because the field of fractions $\mathbb{F}(x_1, \dots, x_n)$ is $\mathcal{P}^*$-inseparably closed by Corollary 2.3.3 •

PROPOSITION 4.2.9: *The Dickson algebra* $\mathcal{D}^*(n)$ *is* $\mathcal{P}^*$*-inseparably closed.*

PROOF : Let $\sqrt{\mathcal{D}^*(n)}$ be the $\mathcal{P}^*$-inseparable closure of the Dickson algebra $\mathcal{D}^*(n)$. Then we have

$$\mathcal{D}^*(n) \hookrightarrow \sqrt{\mathcal{D}^*(n)} \hookrightarrow \mathbb{F}[V],$$

because $\mathbb{F}[V]$ is $\mathcal{P}^*$-inseparably closed by Corollary 4.2.8. Take an element $h \in \sqrt{\mathcal{D}^*(n)}$. Then there exists an $l \in \mathbb{N}_0$ such that

$$h^{p^l} \in \mathcal{D}^*(n).$$

Hence h^{p^l} is invariant under any $g \in \mathrm{GL}(n, \mathbb{F})$, i.e.,

$$h^{p^l} = g\left(h^{p^l}\right) = (gh)^{p^l}.$$

Therefore, for some $\lambda_g \in \mathbb{F}$ in the ground field, we have

$$g(h) = \lambda_g h$$

where $\lambda_g^{p^l} = 1$. Since

$$\lambda_g = \lambda_g^{p^l}$$

we get that the general linear group acts trivially on h, i.e.,

$$h \in \mathcal{D}^*(n),$$

as claimed •

4.3 Further Properties

In this section we study the relation between the ideals of an algebra H^* and its $\mathcal{P}^*$-inseparable closure and establish some technical properties which will be useful later on.

Let's denote by $\mathit{Proj}(\mathrm{H}^*)$ the spectrum of homogeneous prime ideals of H^*. Recall from the preceding section the chain of algebras that led to the inseparable closure

$$\mathrm{H}^* = \mathrm{H}_0^* \overset{\varphi_0}{\hookrightarrow} \mathrm{H}_1^* \overset{\varphi_1}{\hookrightarrow} \mathrm{H}_2^* \overset{\varphi_2}{\hookrightarrow} \cdots \subset \sqrt{\mathrm{H}^*} = \mathrm{colim}(\mathrm{H}_i^*).$$

Recall also that every ring extension φ_i, for all $i \in \mathbb{N}_0$, is integral. Therefore, by lying over, the induced maps

$$\varphi_i^* : \mathit{Proj}(\mathrm{H}_{i+1}^*) \longrightarrow \mathit{Proj}(\mathrm{H}_i^*)$$

are surjective for any $i \in \mathbb{N}_0$. The following theorem shows that these maps are also injective.

Theorem 4.3.1: *With the preceding notation we have that*

$$\varphi_i^* : Proj(\mathrm{H}_{i+1}^*) \longrightarrow Proj(\mathrm{H}_i^*)$$

is a bijection for any $i \in \mathbb{N}_0$. *Moreover, this is also true for the map*

$$\varphi^* : Proj(\sqrt{\mathrm{H}^*}) \longrightarrow Proj(\mathrm{H}^*)$$

induced by the inclusion $\varphi : \mathrm{H}^* \hookrightarrow \sqrt{\mathrm{H}^*}$.

Proof: Consider our map

$$\begin{array}{rccc} \varphi_i^* : & Proj(\mathrm{H}_{i+1}^*) & \longrightarrow & Proj(\mathrm{H}_i^*) \\ & \mathfrak{q} & \longmapsto & \mathfrak{p} := \mathfrak{q} \cap \mathrm{H}_i^* \end{array}$$

We have already seen that φ_i^* is surjective, so what's left to show that it is also injective.

Let $\mathfrak{q}_1, \ \mathfrak{q}_2 \in Proj(\mathrm{H}_{i+1}^*)$ such that

$$\varphi_i^*(\mathfrak{q}_1) = \varphi_i^*(\mathfrak{q}_2) = \mathfrak{p}$$

for some prime ideal $\mathfrak{p} \in Proj(\mathrm{H}_i^*)$. Denote by $\mathfrak{p}^e$ the extended ideal $\big(\varphi_i(\mathfrak{p})\big) \subset \mathrm{H}_{i+1}^*$. Then

$$\mathfrak{p}^e = \left(\mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \mathrm{H}_i^*\right)^e \subseteq \mathfrak{q}_1 \cap \mathfrak{q}_2 \subseteq \mathfrak{q}_j \quad j = 1, \ 2.$$

Therefore

$$\mathfrak{p}^e \subseteq \sqrt{\mathfrak{p}^e} \subseteq \mathfrak{q}_1 \cap \mathfrak{q}_2 \subseteq \mathfrak{q}_j \quad j = 1, \ 2,$$

and hence

$$\mathfrak{p} \subseteq \mathfrak{p}^e \cap \mathrm{H}_i^* \subseteq \sqrt{\mathfrak{p}^e} \cap \mathrm{H}_i^* \subseteq \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \mathrm{H}_i^* \subseteq \mathfrak{q}_1 \cap \mathrm{H}_i^* = \mathfrak{p},$$

i.e., any of the ideals of this chain contracts to $\mathfrak{p}$. If we could show that the $\sqrt{\mathfrak{p}^e} \subset \mathrm{H}_{i+1}^*$ is a prime ideal, then we would have that

$$\sqrt{\mathfrak{p}^e} = \mathfrak{q}_j \quad j = 1, \ 2,$$

since there are no proper inclusions between the prime ideals lying over a fixed prime ideal. So, what's left to show is that

$$\sqrt{\mathfrak{p}^e} \subset \mathrm{H}_{i+1}^*$$

is a prime ideal. This we do by direct calculation. Let $h, \ h' \in \mathrm{H}_{i+1}^*$ such that their product $h h' \in \sqrt{\mathfrak{p}^e}$.

Case 1 : $h, \ h' \in \mathrm{H}_i^*$.
Then

$$h h' \in \sqrt{\mathfrak{p}^e} \cap \mathrm{H}_i^* = \mathfrak{p}$$

and hence without loss of generality $h \in \mathfrak{p} \subseteq \sqrt{\mathfrak{p}^e}$.

Case 2 : $h \in H_i^*$ and $h' \notin H_i^*$.
Then $h'^p \in H_i^* \setminus \{0\}$ and therefore

$$h\,h'^p \in \sqrt{\mathfrak{p}^e} \cap H_i^* = \mathfrak{p}$$

and therefore $h \in \mathfrak{p} \subseteq \sqrt{\mathfrak{p}^e}$ or $h'^p \in \mathfrak{p} \subset \mathfrak{p}^e$, which implies $h' \in \sqrt{\mathfrak{p}^e}$.

Case 3 : Neither h nor h' is in H_i^*.
Then $h^p,\ h'^p \in H_{i+1}^* \setminus \{0\}$ and, as above, we conclude from

$$h^p\,h'^p \in \sqrt{\mathfrak{p}^e} \cap H_i^* = \mathfrak{p}$$

that, without loss of generality,

$$h^p \in \mathfrak{p} \subseteq \mathfrak{p}^e$$

and hence

$$h \in \sqrt{\mathfrak{p}^e}.$$

This proves the first part of the theorem. For the second part note that the same series of arguments as above apply to the integral extension

$$\varphi : H^* \hookrightarrow \sqrt{H^*},$$

i.e., the induced map

$$\varphi^* : Proj(\sqrt{H^*}) \longrightarrow Proj(H^*)$$

is surjective and for any $\mathfrak{q}_1,\ \mathfrak{q}_2 \in Proj(\sqrt{H^*})$ such that

$$\varphi^*(\mathfrak{q}_1) = \varphi^*(\mathfrak{q}_2) = \mathfrak{p} \in Proj(H^*),$$

we have that

$$\mathfrak{p} \subseteq \mathfrak{p}^e \cap H^* \subseteq \sqrt{\mathfrak{p}^e} \cap H^* \subseteq \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap H^* \subseteq \mathfrak{q}_1 \cap H^* = \mathfrak{p}.$$

So what is left to show is that for any prime ideal $\mathfrak{p} \subset H^*$ the radical of its extension in $\sqrt{H^*}$, $\sqrt{\mathfrak{p}^e}$ is again prime.

To this end take $h,\ h' \in \sqrt{H^*}$ such that $h\,h' \in \sqrt{\mathfrak{p}^e}$. Then there exists an index $i_0 \in \mathbb{N}_0$ such that $h,\ h' \in H_{i_0}$ and hence

$$h\,h' \in \sqrt{\mathfrak{p}^e} \cap H_{i_0}.$$

Therefore $h^{p^{i_0}},\ h'^{p^{i_0}} \in H^*$ and hence

$$h^{p^{i_0}}\,h'^{p^{i_0}} \in \sqrt{\mathfrak{p}^e} \cap H^* = \mathfrak{p}.$$

Without loss of generality assume that

$$h^{p^{i_0}} \in \mathfrak{p}.$$

Then, it follows that

$$h \in \sqrt{(\varphi_{i_0}(\mathfrak{p}))} \subset \mathrm{H}^*_{i_0}$$

and hence

$$h \in \left(\sqrt{(\varphi_{i_0}(\mathfrak{p}))}\right)^e \subseteq \sqrt{(\varphi_{i_0}(\mathfrak{p}))^e} = \sqrt{\mathfrak{p}^e} \subset \sqrt{\mathrm{H}^*},$$

which was to be shown •

LEMMA 4.3.2 : *Let* H^* *be an unstable algebra and consider the integral extension* $\mathrm{H}^* \overset{\varphi_0}{\hookrightarrow} \mathrm{H}^*_1$*. If* $\mathfrak{p} \subset \mathrm{H}^*$ *is a prime ideal and* $\mathfrak{q} \subset \mathrm{H}_1$ *lies over it, then*[10]

$$\mathrm{H}^*_1/\mathfrak{q} = \left(\mathrm{H}^*/\mathfrak{p}\right)_1.$$

PROOF : The proof is by straightforward calculation. Take an element

$$h \in \mathrm{H}^*_1/\mathfrak{q},$$

then $h + \mathfrak{q} \subset \mathrm{H}^*_1$. Hence by construction $h^p \in \mathrm{H}^*$ and therefore

$$h^p \in \mathrm{H}^*/\mathfrak{p},$$

i.e., $h \in \left(\mathrm{H}^*/\mathfrak{p}\right)_1$, what proves the one inclusion. To prove the other inclusion take an element

$$h \in \left(\mathrm{H}^*/\mathfrak{p}\right)_1.$$

By definition it follows that h^p is in $\mathrm{H}^*/\mathfrak{p}$, hence $h^p + \mathfrak{p} \subset \mathrm{H}^*$, so $h \in \mathrm{H}_1$ and therefore

$$h \in \mathrm{H}_1/\mathfrak{q},$$

as we claimed •

LEMMA 4.3.3: *Taking the* $\mathcal{P}^*$*-inseparable closure commutes with intersections, i.e., for two unstable sub algebras* H'^**,* H''^* *of an unstable algebra* H^* *it follows that*

$$\sqrt{\mathrm{H}'^*} \cap \sqrt{\mathrm{H}''^*} = \sqrt{\mathrm{H}'^* \cap \mathrm{H}''^*} \subset \mathrm{H}^*.$$

---

[10] Keep in mind that $(-)_1$ is just a functor that applied to an unstable algebra H^* produces the algebra $\mathrm{H}^*_1 = \mathrm{H}^* < \gamma_1,\ \gamma_2, \cdots >$, where we adkoined all p-th roots, recall Section 4.1.

PROOF : Let $h \in \sqrt{\mathrm{H}'^*} \cap \sqrt{\mathrm{H}''^*}$. Then there exists an $l \in \mathbb{N}_0$ such that

$$h^{p^l} \in \mathrm{H}'^* \cap \mathrm{H}''^* \subseteq \sqrt{\mathrm{H}'^* \cap \mathrm{H}''^*},$$

and hence

$$h \in \sqrt{\mathrm{H}'^* \cap \mathrm{H}''^*}.$$

On the other hand let $h \in \sqrt{\mathrm{H}'^* \cap \mathrm{H}''^*}$. Then there exists an $l \in \mathbb{N}_0$ such that

$$h^{p^l} \in \mathrm{H}'^* \cap \mathrm{H}''^*,$$

and hence

$$h \in \sqrt{\mathrm{H}'^*} \cap \sqrt{\mathrm{H}''^*},$$

as we wanted •

There is an obvious corollary.

COROLLARY 4.3.4: *The intersection of two $\mathcal{P}^*$-inseparably closed algebras is $\mathcal{P}^*$-inseparably closed.*

PROOF : •

Recall from the introduction that an element $\mathbf{t} \in \mathrm{H}^*$ is called a **Thom class** if the principal ideal it generates is a $\mathcal{P}^*$-invariant ideal[11] of height one. We then have:

LEMMA 4.3.5: *Let $\mathbf{t} \in \mathrm{H}^*$ be a Thom class. Let H^* be $\mathcal{P}^*$-inseparably closed. Then so is $\mathrm{H}^*/(\mathbf{t})$.*

PROOF : Certainly we have

$$\mathrm{H}^*/(\mathbf{t}) \subseteq \sqrt{\mathrm{H}^*/(\mathbf{t})}.$$

On the other hand let $h \in \sqrt{\mathrm{H}^*/(\mathbf{t})}$. Then

$$h^{p^l} \in \mathrm{H}^*/(\mathbf{t}) \quad \text{for some } l \in \mathbb{N}_0,$$

i.e.,

$$h^{p^l} + (\mathbf{t}) \subseteq \mathrm{H}^*.$$

In particular

$$(h+\mathbf{t})^{p^l} = h^{p^l} + \mathbf{t}^{p^l} \in \mathrm{H}^*$$

and since H^* is $\mathcal{P}^*$-inseparably closed

$$h + \mathbf{t} \in \mathrm{H}^*.$$

[11] An ideal in H^* is called $\mathcal{P}^*$**-invariant** if it is closed under the action of the Steenrod algebra; see [28] Chapter 11 for an introduction to this terminology and [23], [21], [27] and [26] for further results.

Therefore

$$h \in H^*/(\mathbf{t}),$$

as was to be shown •

More generally we have:

PROPOSITION 4.3.6 : *Let $I \subset H^*$ be a $\mathcal{P}^*$-invariant ideal. Let H^* be $\mathcal{P}^*$-inseparably closed. Then so is H^*/I.*

PROOF : Certainly we have

$$H^*/I \subseteq \sqrt{H^*/I}.$$

On the other hand let $h \in \sqrt{H^*/I}$. Then

$$h^{p^l} \in H^*/I \quad \text{for some } l \in \mathbb{N}_0,$$

i.e.,

$$h^{p^l} + I \subseteq H^*.$$

In particular, for a suitably large $l' = l'(k) \in \mathbb{N}_0$

$$(h+k)^{p^{l'}} = h^{p^{l'}} + k^{p^{l'}} \in H^*$$

for $k \in I$, and since H^* is $\mathcal{P}^*$-inseparably closed

$$h + k \in H^*.$$

Therefore

$$h \in H^*/I,$$

as was to be shown •

The next result deals with the problem what happens to quotients when the algebra H^* is not $\mathcal{P}^*$-inseparably closed.

PROPOSITION 4.3.7: *Consider the natural inclusion*

$$\varphi : H^* \hookrightarrow \sqrt{H^*}.$$

Let $J \subset \sqrt{H^*}$ be a $\mathcal{P}^*$-invariant ideal and denote by J^c its contraction via φ in H^*. Then

$$\sqrt{H^*/Rad(J)^c} = \sqrt{H^*}/Rad(J),$$

where $Rad(J)$ denotes the radical of the ideal J.

PROOF : Consider the following map induced by φ via taking appropriate quotients

$$\mathrm{H}^*/Rad\,(J)^c \hookrightarrow \sqrt{\mathrm{H}^*}/Rad\,(J).$$

Since $\sqrt{\mathrm{H}^*}/Rad\,(J)$ is $\mathcal{P}^*$-inseparably closed by Proposition 4.3.6 we have

$$\mathrm{H}^*/Rad\,(J)^c \hookrightarrow \sqrt{\mathrm{H}^*/Rad\,(J)^c} \hookrightarrow \sqrt{\mathrm{H}^*}/Rad\,(J).$$

To show the reverse inclusion, take a nonzero element $h \in \sqrt{\mathrm{H}^*}/Rad\,(J)$. Then $h + Rad\,(J) \subset \sqrt{\mathrm{H}^*}$. Therefore for an element $k \in Rad\,(J)$ we have

$$h^{p^l} + k^{p^l} \in \mathrm{H}^* \text{ and } k^{p^l} \in Rad\,(J) \cap \mathrm{H}^* = Rad\,(J)^c$$

for a suitably large $l = l(k) \in \mathbb{N}_0$. Hence

$$h^{p^l} \in \mathrm{H}^*/Rad\,(J)^c \quad \text{and} \quad h \in \sqrt{\mathrm{H}^*/Rad\,(J)^c},$$

as we claimed •

COROLLARY 4.3.8 : *For an unstable algebra* H^* *its* $\mathcal{P}^*$*-inseparable closure modulo its nil radical, i.e.,*

$$\sqrt{\mathrm{H}^*}/Nil\,(\sqrt{\mathrm{H}^*}),$$

$\mathcal{P}^*$*-inseparably closed, and, in particular, it is the* $\mathcal{P}^*$*-inseparable closure of* $\mathrm{H}^*/Nil\,(\mathrm{H}^*)$.

PROOF : Consider the natural inclusion

$$\varphi : \mathrm{H}^* \hookrightarrow \sqrt{\mathrm{H}^*}.$$

By construction this is an integral extension. The preimage of $Nil\,(\sqrt{\mathrm{H}^*}) \subset \sqrt{\mathrm{H}^*}$ is the nil radical in H^*,

$$\varphi^{-1}\left(Nil\,(\sqrt{\mathrm{H}^*})\right) = Nil\,(\sqrt{\mathrm{H}^*}) \cap \mathrm{H}^* = Nil\,(\mathrm{H}^*) \subset \mathrm{H}^*.$$

Hence by Proposition 4.3.7 we get an integral extension

$$\overline{\varphi} : \mathrm{H}^*/Nil\,(\mathrm{H}^*) \hookrightarrow \sqrt{\mathrm{H}^*/Nil\,(\mathrm{H}^*)} = \sqrt{\mathrm{H}^*}/Nil\,(\sqrt{\mathrm{H}^*}),$$

where the bigger algebra is the $\mathcal{P}^*$-inseparable closure of the little one •

We have another amazing result, which illustrates again the rigidity of the structure the Steenrod algebra forces an H^* to have.

PROPOSITION 4.3.9 : *If* H^* *is* $\mathcal{P}^*$*-inseparably closed and for any* $h \in \mathrm{H}^*$

$$\mathcal{P}^{\Delta_i}(h) = 0 \;\forall\; i \geq 0,$$

then $\mathrm{H}^* = \mathbb{F}$.

PROOF : Let $h \in \mathrm{H}^*$ be of minimal positive degree d. By assumption

$$\mathcal{P}^{\Delta_i}(h) = 0 \ \forall \ i \geq 0.$$

Since H^* is $\mathcal{P}^*$-inseparably closed h must have a p-th root in H^*, which has degree $0 < \frac{d}{p} < d$. This is a contradiction. Hence there is no element of positive degree and $\mathrm{H}^* = \mathbb{F}$ •

REMARK : The same statement is true if we consider fields $\mathbb{K}^*$ over $\mathcal{P}^*$, i.e., a $\mathcal{P}^*$-inseparably closed field $\mathbb{K}^*$ for which

$$\mathcal{P}^{\Delta_i}(k) = 0 \quad \forall \ i \geq 0 \quad \forall \ k \in \mathbb{K}^*$$

is just the ground field $\mathbb{F}$ itself. Compare that with the classical situation, where for a perfect field $\mathbb{K} = \mathbb{K}^p$!

CHAPTER 5
The Embedding Theorem I

In this chapter we are going to prove the Embedding Theorem over arbitrary Galois fields for $\boldsymbol{\mathcal{P}}^*$-inseparably closed algebras H^*.

5.1 A Dickson Algebra in $FF(\mathrm{H}^*)$

We will start this chapter with close scrutiny of the coefficients which appear in the Δ-relation, i.e., we assume that our algebra H^* is Δ-finite.

Since we are going to look at the field of fractions of H^* we assume that H^* is an integral domain.

Recall from Section 1.2 that there are nonzero elements $h_0, \ldots, h_m \in \mathrm{H}^*$ such that

$$h_0\mathcal{P}^{\Delta_0} + \cdots + h_m\mathcal{P}^{\Delta_m} = 0$$

is zero in the H^*-module $\Delta(\mathrm{H}^*)$. If we allow coefficients to be taken from the field of fractions $FF(\mathrm{H}^*)$ of H^*, we can normalize this relation to get

$$\mathbf{t}_0\mathcal{P}^{\Delta_0} + \mathbf{t}_1\mathcal{P}^{\Delta_1} + \cdots + \mathbf{t}_{m-1}\mathcal{P}^{\Delta_{m-1}} + \mathcal{P}^{\Delta_m} = 0$$

where $\mathbf{t}_i = \frac{h_i}{h_m}$ for $i = 0, \ldots, m-1$. For convenience we define

$$\mathbf{t}_m = 1.$$

OBSERVATION 5.1.1: *Note that by minimality of* $m = m(\mathrm{H}^*)$ *the coefficients* $\mathbf{t}_0, \ldots, \mathbf{t}_{m-1}, \mathbf{t}_m = 1$ *are uniquely determined in* $FF(\mathrm{H}^*)$. *Note also that* $\deg(\mathbf{t}_i) = q^m - q^i$ *by homogeneity.*

This means, if there is another $m+1$-tuple $(a_0, \ldots, a_m) \in FF(\mathrm{H}^*)$ such that

$$a_0\mathcal{P}^{\Delta_0} + \cdots + a_m\mathcal{P}^{\Delta_m} = 0$$

on H^*, then

$$(a_0, \ldots, a_m) = a_m(\mathbf{t}_1, \ldots, \mathbf{t}_m).$$

We are aiming to prove that these coefficients $\mathbf{t}_0, \ldots, \mathbf{t}_{m-1}$ generate a Dickson algebra inside $FF(\mathrm{H}^*)$ of Krull dimension m. First we prove that they are algebraically independent. We introduce the following convention:

CONVENTION : For $m = 0$ call

$$\mathscr{D}^*(0) := \mathbb{F}$$

the **trivial Dickson algebra**.

LEMMA 5.1.2 : *The elements $\mathbf{t}_0, \ldots, \mathbf{t}_{m-1} \in FF(\mathrm{H}^*)$, given uniquely by the Δ-Theorem, are algebraically independent.*

PROOF : It is enough to show that the determinant of the generalized Jacobian matrix is not zero, see Theorem A.4.1,

$$\det\left(\mathscr{P}^{\Delta_i}(\mathbf{t}_j)\right)_{i,\ j=0,\ldots,\ m-1} \neq 0.$$

By Proposition 1.2.4

$$\begin{aligned}
0 &= \mathscr{P}^{\Delta_i}\Delta_0\\
&= \mathscr{P}^{\Delta_i}(\mathbf{t}_0)\mathscr{P}^{\Delta_0} + \cdots + \mathscr{P}^{\Delta_i}(\mathbf{t}_m)\mathscr{P}^{\Delta_m} + \mathbf{t}_0\mathscr{P}^{\Delta_i}\mathscr{P}^{\Delta_0} + \cdots + \mathbf{t}_m\mathscr{P}^{\Delta_i}\mathscr{P}^{\Delta_m}\\
&= \mathscr{P}^{\Delta_i}(\mathbf{t}_0)\mathscr{P}^{\Delta_0} + \cdots + \mathscr{P}^{\Delta_i}(\mathbf{t}_m)\mathscr{P}^{\Delta_m} + \mathbf{t}_0\mathscr{P}^{\Delta_0}\mathscr{P}^{\Delta_i} + \cdots + \mathbf{t}_m\mathscr{P}^{\Delta_m}\mathscr{P}^{\Delta_i} + \mathbf{t}_0\mathscr{P}^{\Delta_i}\\
&= \mathscr{P}^{\Delta_i}(\mathbf{t}_0)\mathscr{P}^{\Delta_0} + \cdots + \left(\mathscr{P}^{\Delta_i}(\mathbf{t}_i) + \mathbf{t}_0\right)\mathscr{P}^{\Delta_i} + \cdots + \mathscr{P}^{\Delta_i}(\mathbf{t}_{m-1})\mathscr{P}^{\Delta_{m-1}},
\end{aligned}$$

where the last two equations follow, because $\mathbf{t}_m = 1$ implies $\mathscr{P}^{\Delta_i}(\mathbf{t}_m) = 0$, and $\left(\mathbf{t}_0\mathscr{P}^{\Delta_0} + \cdots + \mathbf{t}_m\mathscr{P}^{\Delta_m}\right)\mathscr{P}^{\Delta_i} = 0$ for all i by assumption. Since m was minimal we get

$$\mathscr{P}^{\Delta_i}(\mathbf{t}_j) = \begin{cases} 0 & \text{for } i = 1, \ldots, m-1,\ i \neq j \\ -\mathbf{t}_0 & \text{for } i = j. \end{cases}$$

Therefore our matrix looks like

$$\begin{aligned}
\det\left(\mathscr{P}^{\Delta_i}(\mathbf{t}_j)\right)_{i,\ j=0}^{m-1} &= \det\begin{bmatrix} \mathscr{P}^{\Delta_0}(\mathbf{t}_0) & \cdots & \mathscr{P}^{\Delta_0}(\mathbf{t}_{m-1}) \\ \cdots & \cdots & \cdots \\ \mathscr{P}^{\Delta_{m-1}}(\mathbf{t}_0) & \cdots & \mathscr{P}^{\Delta_{m-1}}(\mathbf{t}_{m-1}) \end{bmatrix}\\
&= \det\begin{bmatrix} \deg(\mathbf{t}_0)\mathbf{t}_0 & \cdots & \cdots & \cdots & \deg(\mathbf{t}_{m-1})\mathbf{t}_{m-1} \\ 0 & -\mathbf{t}_0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & -\mathbf{t}_0 & 0 \\ 0 & \cdots & \cdots & 0 & -\mathbf{t}_0 \end{bmatrix}\\
&= (-1)^{m-1}\deg(\mathbf{t}_0)\mathbf{t}_0^m\\
&= (-1)^{m-1}(q^m - 1)\mathbf{t}_0^m.
\end{aligned}$$

This determinant is nonzero if and only if $m > 0$. However, for $m = 0$ we have $\mathbf{t}_0 = \mathbf{t}_m = 1$, and, by convention, $\mathbb{F} \subseteq FF(\mathrm{H}^*)$, and we are done •

COROLLARY 5.1.3: *The elements* $\mathbf{t}_0, \ldots, \mathbf{t}_{m-1} \in FF(\mathrm{H}^*)$*, given uniquely by the* Δ*-Theorem, generate a polynomial* $\mathbb{F}$*-sub algebra* [1]

$$\mathbb{F}[\mathbf{t}_0, \ldots, \mathbf{t}_{m-1}] \hookrightarrow FF(\mathrm{H}^*)$$

of $FF(\mathrm{H}^*)$ *.*

PROOF: •

Of course, the Dickson algebra $\mathcal{D}^*(m)$ being a polynomial algebra of the appropriate Krull dimension is *as algebra over* $\mathbb{F}$ isomorphic to our $\mathbb{F}[\mathbf{t}_0, \ldots, \mathbf{t}_{m-1}]$. Next we need to show that our algebra is closed under the action of the Steenrod algebra induced from $FF(\mathrm{H}^*)$. For this we need some preparations.

Consider the following polynomial in $\mathrm{H}^*[X]$

$$\delta(X) := h_0 X + h_1 X^q + \cdots + h_m X^{q^m}.$$

As a polynomial over the field of fraction of H^*, $FF(\mathrm{H}^*)$, we normalize it, and get what we will call **Δ-polynomial** (recall $\mathbf{t}_m = 1$)

$$\Delta(X) := \mathbf{t}_0 X + \mathbf{t}_1 X^q + \cdots + \mathbf{t}_m X^{q^m}.$$

In the terminology of Chapter 2 $\Delta(X)$ is 1-graded. The following proposition is the optimal generalization of Lemma 5.6 in [1].

PROPOSITION 5.1.4: *Let* H^* *be a* Δ*-finite integral domain with* Δ*-polynomial* $\Delta(X)$*. Let* $\mathbb{K}^*$ *be a graded field containing* H^**. Then the following are equivalent*

[1] $\sum_{r=0}^{k} a_r \mathcal{P}^{\Delta_r} \equiv 0$ *on* H^**, where* $a_r \in \mathbb{K}^*$ *for all* r*.*

[2] $\Delta(X) \mid \sum_{r=0}^{k} a_r X^{q^r}$ *in* $\mathbb{K}^*[X]$*.*

PROOF: If $k < m$ then

$$\sum_{r=0}^{k} a_r \mathcal{P}^{\Delta_r} = 0 \iff a_0 = \cdots = a_k = 0$$

by minimality of m and nothing needs to be shown. If $k = m$, then by Observation 5.1.1,

$$(a_0, \ldots, a_m) = a_m(\mathbf{t}_0, \ldots, \mathbf{t}_{m-1}, 1)$$

which is equivalent to

$$\Delta(X) \mid a_m \Delta(X) = \sum_{r=0}^{m} a_r X^{q^r}.$$

[1] The term "$\mathbb{F}$-sub algebra" emphasizes that we don't know yet that this embedding is a morphism in our category.

So we may proceed by induction on k and assume that $k > m$. Then using Lemma 1.1.8

$$\begin{aligned} 0 &= \sum_{r=0}^{k} a_r \mathcal{P}^{\Delta_r} \\ &= \sum_{r=0}^{k} a_r \mathcal{P}^{\Delta_r} - a_k \sum_{i=0}^{m} \mathbf{t}_i^{q^{k-m}} \mathcal{P}^{\Delta_{k-m+i}} \\ &= \sum_{r=0}^{k-1} \tilde{a}_r \mathcal{P}^{\Delta_r} \end{aligned}$$

for certain $\tilde{a}_r \in FF(\mathrm{H}^*)$. The last equation, is, by the induction hypothesis, equivalent to

$$\Delta(X) \mid \sum_{r=0}^{k-1} \tilde{a}_r X^{q^r} = \sum_{r=0}^{k} a_r X^{q^r} - a_k \sum_{i=0}^{m} \mathbf{t}_i^{q^{k-m}} X^{q^{k-m+i}}.$$

Since,

$$\sum_{i=0}^{m} \mathbf{t}_i^{q^{k-m}} X^{q^{k-m+i}} = \left(\sum_{i=0}^{m} \mathbf{t}_i X^{q^i} \right)^{q^{k-m}} = \left(\Delta(X)\right)^{q^{k-m}},$$

we are done •

The following lemma corresponds to Lemma 5.7 [1].

LEMMA 5.1.5: *If* H^* *is a* Δ*-finite integral domain then for any* $\alpha \geq 0$

$$\Delta(X) \mid \mathcal{P}^{\alpha} \Delta(X)$$

as polynomials[2] *over the fraction field of* H^**, i.e., in* $FF(\mathrm{H}^*)[X]$.

PROOF: We want to apply the above criteria for divisibility. So, lets calculate the Steenrod powers of $\Delta(X)$:

$$\begin{aligned} \mathcal{P}^{\alpha} \Delta(X) &= \mathcal{P}^{\alpha} \left(\sum_{i=0}^{m} \mathbf{t}_i X^{q^i} \right) \\ &= \sum_{i=0}^{m} \left(\sum_{r+s=\alpha} \mathcal{P}^{r}(\mathbf{t}_i) \mathcal{P}^{s}(X^{q^i}) \right) \end{aligned}$$

[2] Note, that the Steenrod algebra acts on the additional variable X as though it had degree 1:

$$\mathcal{P}^i(X) = \begin{cases} X & \text{if } i = 0 \\ X^q & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} .$$

$$= \sum_{i=0}^{m} \left(\sum_{s=0}^{\alpha} \mathcal{P}^{\alpha-s}(\mathbf{t}_i)\mathcal{P}^{s}(X^{q^i}) \right)$$

$$= \sum_{i=0}^{m} \left(\sum_{s'=0}^{\alpha} \mathcal{P}^{\alpha-s'q^i}(\mathbf{t}_i)\mathcal{P}^{s'q^i}(X^{q^i}) \right)$$

$$= \sum_{i=0}^{m} \left(\sum_{s'=0}^{\alpha} \mathcal{P}^{\alpha-s'q^i}(\mathbf{t}_i)(\mathcal{P}^{s'}(X))^{q^i} \right)$$

$$= \sum_{i=0}^{m} \left(\mathcal{P}^{\alpha}(\mathbf{t}_i)X^{q^i} + \mathcal{P}^{\alpha-q^i}(\mathbf{t}_i)(\mathcal{P}^{1}(X))^{q^i} \right)$$

$$= \sum_{i=0}^{m} \left(\mathcal{P}^{\alpha}(\mathbf{t}_i)X^{q^i} + \mathcal{P}^{\alpha-q^i}(\mathbf{t}_i)X^{q^{i+1}} \right)$$

$$= \sum_{i=0}^{m} \left(\mathcal{P}^{\alpha}(\mathbf{t}_i)X^{q^i} \right) + \sum_{i=0}^{m} \left(\mathcal{P}^{\alpha-q^i}(\mathbf{t}_i)X^{q^{i+1}} \right).$$

The preceding proposition tells us that $\Delta(X) \mid \mathcal{P}^{\alpha}\Delta(X)$ if and only if

$$\sum_{i=0}^{m} \mathcal{P}^{\alpha}(\mathbf{t}_i)\mathcal{P}^{\Delta_i} + \sum_{i=0}^{m} \mathcal{P}^{\alpha-q^i}(\mathbf{t}_i)\mathcal{P}^{\Delta_{i+1}} = 0$$

on H^*. But notice that the left hand side in this equation is just good old $\Delta(\alpha, \mathbf{t}_0, \ldots, \mathbf{t}_{m-1}, 1)$, which is zero by Proposition 1.2.4 •

We need the following generalization of Lemma 5.8 in [1].

LEMMA 5.1.6 : *Let $\mathbb{K}^*$ be a graded field over the Steenrod algebra, and $f(X) \in \mathbb{K}^*[X]$ a 1-graded polynomial of degree m with roots $\lambda_1, \ldots, \lambda_m$ of degree 1. If, for any $\alpha \geq 0$,*

$$f(X) \mid \mathcal{P}^{\alpha} f(X),$$

then the roots are unstable elements, i.e., fulfill the unstability condition.

PROOF : Write the polynomial $f(X)$ in factored form

$$f(X) = k(X - \lambda_1) \cdots (X - \lambda_m),$$

where $k \in \mathbb{K}^*$ is suitable. Since $f(X) \mid \mathcal{P}^{\alpha} f(X)$ for any α we have

$$f(X) \mid \mathcal{P}(\xi) f(X) = \mathcal{P}(\xi)(k)\mathcal{P}(\xi)(X - \lambda_1) \cdots \mathcal{P}(\xi)(X - \lambda_m).$$

Hence, for any $i = 1, \ldots, m$, there exists an $j \in \{1, \ldots, m\}$, such that

$$\mathcal{P}(\xi)(X - \lambda_j)\big|_{X=\lambda_i} = 0.$$

From

$$\begin{aligned}\mathcal{P}^{\alpha}(X-\lambda_j) &= \mathcal{P}^{\alpha}(X)-\mathcal{P}^{\alpha}(\lambda_j)\\ &= \begin{cases} X-\lambda_j & \text{for } \alpha = 0\\ X^q - \mathcal{P}^1(\lambda_j) & \text{for } \alpha = 1\\ -\mathcal{P}^{\alpha}(\lambda_j) & \text{for } \alpha > 1,\end{cases}\end{aligned}$$

we obtain by setting $X = \lambda_i$

$$\begin{aligned}\lambda_i &= \lambda_j\\ \lambda_i^q &= \mathcal{P}^1(\lambda_j) = \mathcal{P}^1(\lambda_i)\\ \mathcal{P}^{\alpha}(\lambda_j) &= 0 \qquad \text{for } \alpha > 1,\end{aligned}$$

which in turn shows that the roots are unstable elements •

We come back to our problem of trying to show that the polynomial algebra

$$\mathbb{F}[\mathbf{t}_0, \dots, \mathbf{t}_{n-1}]$$

is isomorphic to the Dickson algebra $\mathcal{D}^*(n)$ *as algebra over the Steenrod algebra*. To continue our investigation of the Steenrod powers $\mathcal{P}^i(\mathbf{t}_j)$, recall the polynomial

$$\Delta(X) = \mathbf{t}_0 X + \mathbf{t}_1 X^q + \cdots + \mathbf{t}_m X^{q^m}.$$

By Lemma 5.1.5 we have for any $\alpha \in \mathbb{N}_0$

$$\Delta(X) \mid \mathcal{P}^{\alpha}\Delta(X),$$

hence, by Lemma 5.1.6, the polynomial $\Delta(X)$ has unstable roots. Since $\Delta(X)$ is monic, the coefficients

$$\mathbf{t}_0, \dots, \mathbf{t}_{m-1}, \ \mathbf{t}_m = 1$$

are polynomials in these roots, so are also unstable elements in $FF(\mathrm{H}^*)$. We apply Proposition 1.2.4 and get

$$\begin{aligned}&\Delta(\alpha, \ \mathbf{t}_0, \dots, \mathbf{t}_{m-1}, \ \mathbf{t}_m = 1)\\ &\quad = \mathcal{P}^{\alpha}(\mathbf{t}_0)\mathcal{P}^{\Delta_0} + \cdots + \mathcal{P}^{\alpha}(\mathbf{t}_m)\mathcal{P}^{\Delta_m} + \sum_{i=0}^{m} \mathcal{P}^{\alpha-q^i}(\mathbf{t}_i)\mathcal{P}^{\Delta_{i+1}}\\ &\quad = 0.\end{aligned}$$

Hence, we have for the $(m+1)$-tuple of coefficients of $\Delta_\alpha = \Delta(\alpha, \ \mathbf{t}_0, \dots, \mathbf{t}_m)$, by minimality of m, and Observation 5.1.1,

$$\Big(\mathcal{P}^{\alpha}(\mathbf{t}_0), \ \mathcal{P}^{\alpha}(\mathbf{t}_1) + \mathcal{P}^{\alpha-1}(\mathbf{t}_0), \ \mathcal{P}^{\alpha}(\mathbf{t}_2) + \mathcal{P}^{\alpha-q}(\mathbf{t}_1), \cdots$$
$$\cdots, \ \mathcal{P}^{\alpha}(\mathbf{t}_{j+1}) + \mathcal{P}^{\alpha-q^j}(\mathbf{t}_j), \dots, \mathcal{P}^{\alpha}(\mathbf{t}_m) + \mathcal{P}^{\alpha-q^{m-1}}(\mathbf{t}_{m-1})\Big)$$

$$= \begin{cases} \mathcal{P}^{\alpha}(\mathbf{t}_m)\cdot(\mathbf{t}_0,\ldots,\mathbf{t}_{m-1},\ 1) & \text{for } \alpha < q^{m-1} \\ \left(\mathcal{P}^{\alpha}(\mathbf{t}_m)+\mathcal{P}^{\alpha-q^{m-1}}(\mathbf{t}_{m-1})\right)\cdot(\mathbf{t}_0,\ldots,\mathbf{t}_{m-1},\ 1) & \text{for } q^{m-1}\le\alpha<q^m \\ 0\cdot(\mathbf{t}_0,\ldots,\mathbf{t}_{m-1},\ 1) & \text{otherwise,} \end{cases}$$

where the last statement is true for degree reasons.

Let's collect this in a proposition:

PROPOSITION 5.1.7: *With the preceding notations we have*

$$\Big(\mathcal{P}^{\alpha}(\mathbf{t}_0),\ \mathcal{P}^{\alpha}(\mathbf{t}_1)+\mathcal{P}^{\alpha-1}(\mathbf{t}_0),\ \mathcal{P}^{\alpha}(\mathbf{t}_2)+\mathcal{P}^{\alpha-q}(\mathbf{t}_1),\cdots$$
$$\cdots,\ \mathcal{P}^{\alpha}(\mathbf{t}_{j+1})+\mathcal{P}^{\alpha-q^j}(\mathbf{t}_j),\ldots,\mathcal{P}^{\alpha}(\mathbf{t}_m)+\mathcal{P}^{\alpha-q^{m-1}}(\mathbf{t}_{m-1})\Big)$$

$$= \begin{cases} \mathcal{P}^{\alpha}(\mathbf{t}_m)\cdot(\mathbf{t}_0,\ldots,\mathbf{t}_{m-1},\ 1) & \textit{for } \alpha < q^{m-1} \\ \left(\mathcal{P}^{\alpha-q^{m-1}}(\mathbf{t}_{m-1})\right)\cdot(\mathbf{t}_0,\ldots,\mathbf{t}_{m-1},\ 1) & \textit{for } q^{m-1}\le\alpha<q^m \\ 0\cdot(\mathbf{t}_0,\ldots,\mathbf{t}_{m-1},\ 1) & \textit{otherwise,} \end{cases}$$

from which we may read off recursively the Steenrod powers of our elements $\mathbf{t}_0,\ldots,\mathbf{t}_m$.

PROOF: •

We can now prove that $\mathbb{F}[\mathbf{t}_0,\ldots,\mathbf{t}_{m-1}]$ is a Dickson algebra *in the category of unstable algebras over* $\boldsymbol{\mathscr{P}}^*$, and hence establishes the first part of the main goal of this section.

THEOREM 5.1.8: *The isomorphism of* $\mathbb{F}$*-algebras given by*

$$\begin{array}{rccc} \varphi: & \mathbb{F}_q[\mathbf{t}_0,\ldots,\mathbf{t}_{m-1}] & \longrightarrow & \mathcal{D}^*(m)=\mathbb{F}_q[\mathbf{d}_{m,0},\ldots,\mathbf{d}_{m,m-1}] \\ & \mathbf{t}_i & \longmapsto & (-1)^{m-i}\mathbf{d}_{m,i} \end{array}$$

is an isomorphism of algebras over the Steenrod algebra.

PROOF: Is is clear that φ is an isomorphism of $\mathbb{F}$-algebras. What needs to be shown is that φ commutes with the action of the Steenrod algebra, i.e.,

$$\varphi\mathcal{P}^{\alpha}(\mathbf{t}_j)=\mathcal{P}^{\alpha}\varphi(\mathbf{t}_j)$$

for any choice of α or j. We prove this by induction on α. If $\alpha=0$ there is nothing to show, since $\mathcal{P}^0$ is the identity operator. So, we can assume that $q^j\le\alpha<q^{j+1}$ for some $j\in\mathbb{N}_0$. If $\alpha\ge q^m$ both sides are zero for degree reasons. So, assume that $j\le m-1$.

Again, we will make use of the Δ_α-relation from Proposition 1.2.4, resp. Proposition 5.1.7, which give a recursion formula for the Steenrod powers of the $\mathbf{t}_0,\ldots,\mathbf{t}_m$. So, if $j\le m-2$, then

$$\Big(\mathcal{P}^{\alpha}(\mathbf{t}_0),\ \mathcal{P}^{\alpha}(\mathbf{t}_1)+\mathcal{P}^{\alpha-1}(\mathbf{t}_0),\ \mathcal{P}^{\alpha}(\mathbf{t}_2)+\mathcal{P}^{\alpha-q}(\mathbf{t}_1),\cdots$$

$$\cdots,\ \mathcal{P}^{\alpha}(\mathbf{t}_{j+1}) + \mathcal{P}^{\alpha-q^j}(\mathbf{t}_j),\ \mathcal{P}^{\alpha}(\mathbf{t}_{j+2}), \dots, \mathcal{P}^{\alpha}(\mathbf{t}_{m-1}),\ 0\Big)$$
$$= 0,$$

i.e., whenever $q^j \leq \alpha < q^{j+1} \leq q^{m-1}$

$$\mathcal{P}^{\alpha}(\mathbf{t}_i) = \begin{cases} 0 & \text{for } i = 0,\ j+2, \dots, m \\ \mathcal{P}^{\alpha-q^{i-1}}(\mathbf{t}_{i-1}) & i = 1, \dots, j+1. \end{cases}$$

Hence by induction

$$\begin{aligned}
\varphi(\mathcal{P}^{\alpha}(\mathbf{t}_i)) &= \begin{cases} \varphi(0) & \text{for } i = 0,\ j+2, \dots, m \\ \varphi(\mathcal{P}^{\alpha-q^{i-1}}(\mathbf{t}_{i-1})) & i = 1, \dots, j+1 \end{cases} \\
&= \begin{cases} 0 & \text{for } i = 0,\ j+2, \dots, m \\ (-1)^{m-(i-1)}\mathcal{P}^{\alpha-q^{j-1}}(\mathbf{d}_{m,i-1}) & i = 1, \dots, j+1 \end{cases} \\
&= \begin{cases} (-1)^{m-i}\mathcal{P}^{\alpha}(\mathbf{d}_{m,i}) & \text{for } i = 0,\ j+2, \dots, m \\ (-1)^{m-i+1}\mathcal{P}^{\alpha-q^{j-1}}(\mathbf{d}_{m,i-1}) & i = 1, \dots, j+1 \end{cases} \\
&= \begin{cases} (-1)^{m-i}\mathcal{P}^{\alpha}(\mathbf{d}_{m,i}) & \text{for } i = 0,\ j+2, \dots, m \\ (-1)^{m-i}\mathcal{P}^{\alpha}(\mathbf{d}_{m,i}) & i = 1, \dots, j+1 \end{cases} \\
&= \mathcal{P}^{\alpha}(\varphi(\mathbf{t}_i)).
\end{aligned}$$

where we made use of the explicit formulae of the action of the Steenrod algebra on the Dickson classes given in Proposition A.2.1 of the appendix.

Finally, lets examine the case where $q^{m-1} \leq \alpha < q^m$. Then, by Proposition 5.1.7, the above equation reads as follows

$$\Big(\mathcal{P}^{\alpha}(\mathbf{t}_0),\ \mathcal{P}^{\alpha}(\mathbf{t}_1) + \mathcal{P}^{\alpha-1}(\mathbf{t}_0),\ \mathcal{P}^{\alpha}(\mathbf{t}_2) + \mathcal{P}^{\alpha-q}(\mathbf{t}_1), \cdots$$
$$\cdots,\ \mathcal{P}^{\alpha}(\mathbf{t}_{m-1}) + \mathcal{P}^{\alpha-q^{m-2}}(\mathbf{t}_{m-2}),\ \mathcal{P}^{\alpha-q^{m-1}}(\mathbf{t}_{m-1})\Big)$$
$$= \mathcal{P}^{\alpha-q^{m-1}}(\mathbf{t}_{m-1})(\mathbf{t}_0, \dots, \mathbf{t}_{m-1},\ 1).$$

So we get by induction

$$\begin{aligned}
&\varphi(\mathcal{P}^{\alpha}(\mathbf{t}_j)) \\
&= \begin{cases} \varphi\left(\mathcal{P}^{\alpha-q^{m-1}}(\mathbf{t}_{m-1})\mathbf{t}_0\right) & \text{for } j = 0 \\ \varphi\left(\mathcal{P}^{\alpha-q^{m-1}}(\mathbf{t}_{m-1})\mathbf{t}_j - \mathcal{P}^{\alpha-q^{j-1}}(\mathbf{t}_{j-1})\right) & \text{otherwise} \end{cases} \\
&= \begin{cases} \left(\mathcal{P}^{\alpha-q^{m-1}}(\varphi(\mathbf{t}_{m-1}))\right)\varphi(\mathbf{t}_0) & \text{for } j = 0 \\ \left(\mathcal{P}^{\alpha-q^{m-1}}(\varphi(\mathbf{t}_{m-1}))\right)\varphi(\mathbf{t}_j) - \mathcal{P}^{\alpha-q^{j-1}}(\varphi(\mathbf{t}_{j-1})) & \text{otherwise} \end{cases} \\
&= \begin{cases} (-1)^{m+1}\mathcal{P}^{\alpha-q^{m-1}}(\mathbf{d}_{m,m-1})\mathbf{d}_{m,0} & \text{for } j = 0 \\ (-1)^{m-j+1}\mathcal{P}^{\alpha-q^{m-1}}(\mathbf{d}_{m,m-1})\mathbf{d}_{m,j} + (-1)^{m-j}\mathcal{P}^{\alpha-q^{j-1}}(\mathbf{d}_{m,j-1}) & \text{otherwise} \end{cases}
\end{aligned}$$

$$= \begin{cases} (-1)^m \mathcal{P}^{\alpha}(\mathbf{d}_{m,0}) & \text{for } j = 0 \\ (-1)^{m-j} \mathcal{P}^{\alpha}(\mathbf{d}_{m,j}) & \text{otherwise} \end{cases}$$

$$= \mathcal{P}^{\alpha}(\varphi(\mathbf{t}_j)),$$

where we again made use of Proposition A.2.1 in the appendix. This proves the theorem •

REMARK: We could have proved the preceding theorem also by simply noting that

$$(-1)^m \mathbf{d}_{m,\,0}\mathcal{P}^{\Delta_0} + \cdots + (-1)\mathbf{d}_{m,\,m-1}\mathcal{P}^{\Delta_{m-1}} + \mathcal{P}^{\Delta_m} = 0$$

is the Δ-relation in $\mathcal{D}^*(m)$, see the remark after Theorem 1.2.3. Hence the analogous formulae as in Proposition 5.1.7 hold for the Dickson classes (with appropriate signs) as well.

REMARK: The proof shows also that the Dickson algebra of Krull dimension m we have just found inside $FF(\mathrm{H}^*)$ is maximal. For otherwise there would be a $k > m$ such that

$$\varphi : \mathcal{D}^*(k) \hookrightarrow FF(\mathrm{H}^*)$$

is a monomorphism. But then

$$(-1)^k \varphi(\mathbf{d}_{k,\,0})\mathcal{P}^{\Delta_0} + \cdots + (-1)\varphi(\mathbf{d}_{k,\,k-1})\mathcal{P}^{\Delta_{k-1}} + \mathcal{P}^{\Delta_k} = 0$$

would also be a linear relation of *minimal* length, which is a contradiction. This means that $m = m(\mathrm{H}^*)$ is an invariant of H^*.

REMARK: On the other hand, one could use the above theorem to again prove that the Δ-relation for the Dickson algebra $\mathcal{D}^*(m)$ reads as follows:

$$(-1)^m \mathbf{d}_{m,0}\mathcal{P}^{\Delta_0} + \cdots + (-1)\mathbf{d}_{m,m-1}\mathcal{P}^{\Delta_{m-1}} + \mathcal{P}^{\Delta_m} = 0.$$

This is because it is the image of the Δ-relation in $FF(\mathrm{H}^*)$ under the map given in the theorem.

5.2 Preparing the Embedding Theorem après Smith-Switzer

In this section we prepare for the proof of the general version of the Embedding Theorem. Following the path described by [31], we are going to establish the proposition in their Section 2 (see also [28] Theorem 10.5.5).

Of course, we have to assume that H^* is an integral domain, because we are going to deal with the field of fraction of H^*; or more generally with fields containing H^*. Moreover we assume that H^* is Δ-finite.

Consider the Δ-relation in H^*

$$h_0\mathcal{P}^{\Delta_0} + \cdots + h_m\mathcal{P}^{\Delta_m} = 0,$$

and recall from Section 5.1 the Δ-polynomial

$$\Delta(X) := \mathbf{t}_0 X + \mathbf{t}_1 X^q + \cdots + \mathbf{t}_m X^{q^m} \in FF(\mathrm{H}^*)[X],$$

where $\mathbf{t}_i = \frac{h_i}{h_m}$ for $i = 0, \ldots, m$, and $\mathbf{t}_m = 1$. It has a splitting field $\mathbb{E}^*$, and, since the formal derivative

$$\frac{\mathrm{d}}{\mathrm{d}X}\Delta(X) = \mathbf{t}_0 \neq 0$$

is not zero, the field extension $\mathbb{E}^*/FF(\mathrm{H}^*)$ is separable. Hence the Steenrod algebra action on H^* can be extended uniquely to $\mathbb{E}^*$ by the Separable Extension Lemma, Lemma 2.2.1. Consider the subset

$$V := \{v \in \mathbb{E}^* \mid \Delta(v) = 0\} \subseteq \mathbb{E}^*.$$

Since $\Delta(X)$ is a q-polynomial we have that

$$V = \ker\{\Delta(X) : \mathbb{E}^* \longrightarrow \mathbb{E}^*\}$$

is an $\mathbb{F}_q$-vector subspace of $\mathbb{E}^*$ of dimension m over $\mathbb{F}_q$. Hence, since $\mathbf{t}_m = 1$,

$$\Delta(X) = X \prod_{v \in V \setminus \{0\}} (X - v).$$

Let $z_1, \ldots, z_m$ be a basis of V as a vector space over $\mathbb{F}_q$.

We are now fit for the following theorem, compare Theorem 10.5.5 in [28]:

THEOREM 5.2.1: *Let H^* be a Δ-finite integral domain, with Δ-polynomial*

$$\Delta(X) = \mathbf{t}_0 X + \mathbf{t}_1 X^q + \cdots + \mathbf{t}_m X^{q^m}.$$

Let $\mathbb{E}^*$ be a splitting field for $\Delta(X)$ over $FF(\mathrm{H}^*)$, and let $z_1, \ldots, z_m$ be an $\mathbb{F}_q$-basis of the vector subspace V of $\mathbb{E}^*$ of roots of $\Delta(X)$. Then

[1] *$z_1, \ldots, z_m$ are unstable elements.*

[2] *$z_1, \ldots, z_m$ are algebraically independent.*

[3] *$\mathscr{D}^* = \mathbb{F}[z_1, \ldots, z_m]^{\mathrm{GL}(m,\, \mathbb{F})} \hookrightarrow \mathbb{F}[z_1, \ldots, z_m]$ is an inclusion of unstable algebras over the Steenrod algebra.*

[4] *If H^* is in addition Noetherian, then $\mathbb{F}[z_1, \ldots, z_m]$ is integral over H^*.*

PROOF: We take each point in turn.

AD [1] : By Lemma 5.1.5 for any $\alpha \geq 0$ we have

$$\Delta(X) \mid \mathscr{P}^\alpha \Delta(X),$$

which in turn implies, by Lemma 5.1.6, that $\Delta(X)$ has unstable roots. In particular it follows that

$$\mathscr{P}^\alpha(z_j) = \begin{cases} z_j & \text{for } \alpha = 0 \\ z_j^q & \text{for } \alpha = 1 \\ 0 & \text{otherwise} \end{cases}$$

for $j = 1, \ldots, m$.

AD [2] : Taken together with what we just proved, the determinant of the generalized Jacobian matrix (compare Theorem A.4.1) is

$$\det\left(\mathcal{P}^{\Delta_i}(z_j)\right)_{i=0,\ldots,m-1}^{j=1,\ldots,m} = \det\begin{bmatrix} z_1 & \cdots & z_m \\ \cdots & \cdots & \cdots \\ z_1^{q^{m-1}} & \cdots & z_m^{q^{m-1}} \end{bmatrix} \neq 0.$$

Therefore $z_1, \ldots, z_m$ are algebraically independent.[3]

AD [3] : By part [1] and [2] we have that

$$\mathbb{F}[z_1, \ldots, z_m]$$

an unstable algebra over the Steenrod algebra. Therefore

$$\mathbb{F}[z_1, \ldots, z_m]^{\mathrm{GL}(m,\, \mathbb{F})} \subseteq \mathbb{F}[z_1, \ldots, z_m]$$

is an inclusion as algebras over the Steenrod algebra for the tautological action of $\mathrm{GL}(m, \mathbb{F})$.

AD [4] : Recall, from the proof of Theorem 1.2.1, that there exists an integer g such that for any $h \in \mathrm{H}^*$

$$-\mathcal{P}^{\Delta_{g+1}}(h) = a_0\mathcal{P}^{\Delta_0}(h) + \cdots + a_g\mathcal{P}^{\Delta_g}(h)$$

for certain coefficients $a_0, \ldots, a_g \in \mathrm{H}^*$. By Proposition 2.2.1 this holds for $h \in \mathbb{E}^*$ and, in particular, for any $h \in V$. Hence we have

$$-z_j^{q^{g+1}} = a_0 z_j + \cdots + a_g z_j^{q^g}$$

for $j = 1, \ldots, m$, i.e., $\mathbb{F}[z_1, \ldots, z_m]$ is integral over H^* •

5.3 The Embedding Theorem I

We continue to assume that H^* is a Δ-finite integral domain of Krull dimension n. From what we have done so far we have the following diagram of unstable algebras over the Steenrod algebra:

$$\begin{array}{ccc} \mathcal{D}^*(m) & & \mathrm{H}^* \\ \uparrow \text{integral} & & \uparrow \\ \mathbb{F}[z_1, \ldots, z_m] & \overset{}{\underset{\psi}{\hookrightarrow}} & \mathrm{H}^*\langle z_1, \ldots, z_m\rangle =: \mathrm{A}^*. \end{array}$$

This notation will remain fixed throughout this section.

[3] This determinant was calculated by L. E. Dickson in [7] and is the pre Euler class (which we won't need here), for further information see Section 8.1 in [28].

Lemma 5.3.1 : *The following statements are equivalent.*
[1] *The extension ψ defined in the above diagram is integral.*
[2] $m(\mathrm{H}^*) = m = n = \dim(\mathrm{H}^*)$.

Proof : If $m < n$ then

$$\dim(\mathbb{F}[z_1, \ldots, z_m]) = m < n = \dim(\mathrm{H}^*\langle z_1, \ldots, z_m\rangle)$$

and therefore ψ can't be integral.

On the other hand, if $m = n$, then the field extension induced by ψ

$$\mathbb{F}(z_1, \ldots, z_n) \subseteq FF(\mathrm{H}^*)(z_1, \ldots, z_n)$$

is algebraic, because

$$\begin{aligned} n &= \operatorname{trzdeg}(\mathbb{F}(z_1, \ldots, z_n)/\mathbb{F}) \\ &\le \operatorname{trzdeg}(\mathrm{H}^* < z_1, \ldots, z_n > /\mathbb{F}) \\ &= \operatorname{trzdeg}(\mathrm{H}^*/\mathbb{F}) \le \dim(\mathrm{H}^*) = n. \end{aligned}$$

Therefore by Lemma 3.1.1 the ring extension

$$Un\left(\mathbb{F}(z_1, \ldots, z_n)\right) \hookrightarrow Un\left(FF(\mathrm{H}^*)(z_1, \ldots, z_n)\right)$$

is integral. Hence

$$\mathbb{F}[z_1, \ldots, z_n] = Un\left(\mathbb{F}(z_1, \ldots, z_n)\right) \hookrightarrow \mathrm{H}^* < z_1, \ldots, z_n > \subseteq Un\left(FF(\mathrm{H}^*)(z_1, \ldots, z_n)\right)$$

is integral •

For the key results of this section we need the following lemma.

Lemma 5.3.2 : *With the same notations as above;* H^* *is* $\mathcal{P}^*$*-inseparably closed if and only if* A^* *is.*

Proof : Let H^* be $\mathcal{P}^*$-inseparably closed, and, suppose to the contrary that A^* were not $\mathcal{P}^*$-inseparably closed. Then looking at the fields of fractions we would get

$$\begin{array}{ccc} FF(\mathrm{H}^*) & \underset{\text{separable}}{\hookrightarrow} & FF(A^*) \\ & & \mathcal{P}^*\text{–purely} \downarrow \text{insep.} \\ & & FF(\sqrt{A^*}), \end{array}$$

where the upper map is $\mathcal{P}^*$-separable, because $FF(A^*) = \mathbb{E}^*$ is the splitting field of the separable polynomial $\Delta(X)$, and the pure $\mathcal{P}^*$-inseparability of the map downwards follows from Lemma 4.2.5. In such a case we could find an intermediate field $\mathbb{K}^*$ between $FF(\mathrm{H}^*)$ and $FF(\sqrt{A^*})$, such that

$$FF(\mathrm{H}^*) \subseteq \mathbb{K}^*$$

is purely $\mathcal{P}^*$-inseparable, while

$$\mathbb{K}^* \subseteq FF(\sqrt{A^*})$$

is $\mathcal{P}^*$-separable, see the remarks at the end of Section 2.3. However, by Corollary 4.2.7, $FF(\mathrm{H}^*)$ is $\mathcal{P}^*$-inseparably closed, which means that $\mathbb{K}^* = FF(\mathrm{H}^*)$ and

$$FF(\mathrm{H}^*) \hookrightarrow FF(\sqrt{A^*})$$

is $\mathcal{P}^*$-separable. This is a contradiction.

To prove the converse assume that H^* is not $\mathcal{P}^*$-inseparably closed. Suppose A^* was $\mathcal{P}^*$-inseparably closed. Then, taken together Lemma 4.2.5 and Lemma 4.2.6 give the following diagram

$$\begin{array}{ccc} FF(\mathrm{H}^*) & \underset{\mathcal{P}^*\text{–purely insep.}}{\hookrightarrow} & FF(\sqrt{\mathrm{H}^*}) \\ \downarrow & & \downarrow \\ FF(A^*) & = & FF(\sqrt{A^*}). \end{array}$$

Therefore the field extension $FF(A^*)/FF(\mathrm{H}^*)$ must be $\mathcal{P}^*$-inseparable. This is a contradiction, since we adjoined the roots of a $\mathcal{P}^*$-separable polynomial, namely of $\Delta(X)$ •

LEMMA 5.3.3: *With the same notation as above, if ψ is integral, then A^* is $\mathcal{P}^*$-inseparably closed.*

PROOF: Assume that A^* is not $\mathcal{P}^*$-inseparably closed and, suppose to the contrary, that ψ were integral. By what we know already we have by Lemma 5.3.1, that $m = n$ and, H^* is not $\mathcal{P}^*$-inseparably closed, by Lemma 5.3.2. Hence by the Integral Closure Theorem, Theorem 3.2.2,

$$\psi : \mathbb{F}[z_1, \ldots, z_m] \hookrightarrow A^*$$

is an isomorphism of unstable algebras and hence

$$\mathrm{H}^* \hookrightarrow \mathbb{F}[z_1, \ldots, z_m] = A^*.$$

But notice: this hands us an embedding

$$\sqrt{\mathrm{H}^*} \hookrightarrow \mathbb{F}[z_1, \ldots, z_m],$$

because $\mathbb{F}[z_1, \ldots, z_m]$ is $\mathcal{P}^*$-inseparably closed. Recall by Corollary 4.2.8

$$\mathrm{H}^* \hookrightarrow \sqrt{\mathrm{H}^*} \hookrightarrow \sqrt{\mathbb{F}[z_1, \ldots, z_m]} = \mathbb{F}[z_1, \ldots, z_m].$$

Altogether we get at the level of fields of fractions:

$$\begin{array}{ccc} FF(\mathrm{H}^*) & \underset{\mathcal{P}^*\text{–purely insep.}}{\hookrightarrow} & FF(\sqrt{\mathrm{H}^*}) \\ \mathcal{P}^*\text{–sep.} \downarrow & & \downarrow \\ \mathbb{F}(z_1, \ldots, z_m) & = \mathbb{E}^* = & FF(A^*), \end{array}$$

where the pure $\mathcal{P}^*$-inseparability of the upper map follows, because $\Delta(X)$ is a $\mathcal{P}^*$-separable polynomial. This is a contradiction. So, ψ is not integral •

The following theorem is the key to the Embedding Theorem and the Little Imbedding Theorem in Section 7.3. This will make it possible to prove the Embedding Theorem *without* using the Integral Closure Theorem.[4]

THEOREM 5.3.4: *With the same notation as above, if A^* is $\mathcal{P}^*$-inseparably closed, then ψ is an isomorphism.*

PROOF: Let A^* be $\mathcal{P}^*$-inseparably closed. Denote by

$$I := (z_1, \ldots, z_m) \subseteq A^*$$

the ideal generated by the little algebra in the big one. First we want to show that the quotient A^*/I is trivial, i.e., $A^*/I = \mathbb{F}$. By the Δ-Theorem we know that

$$\mathcal{P}^{\Delta_m} = -\left((-1)^m \mathbf{d}_{m,0}\mathcal{P}^{\Delta_0} + \cdots + (-1)\mathbf{d}_{m,m-1}\mathcal{P}^{\Delta_{m-1}}\right)$$

on H^*, by construction on $\mathbb{F}[z_1, \ldots, z_m]$, and hence on A^*. By Lemma 1.1.8

$$\mathcal{P}^{\Delta_{r+m}} = -\left((-1)^m \mathbf{d}_{m,0}^{q^r}\mathcal{P}^{\Delta_r} + \cdots + (-1)\mathbf{d}_{m,m-1}^{q^r}\mathcal{P}^{\Delta_{r+m-1}}\right)$$

for any $r \geq 0$, and hence for any element $a \in A^*$

$$(\bigstar) \qquad \mathcal{P}^{\Delta_{r+m}}(a) \in \left(\mathbf{d}_{m,0}, \ldots, \mathbf{d}_{m,m-1}\right) \subseteq I \subseteq A^*.$$

We next show by induction on m that

$$\mathcal{P}^{\Delta_r}(a) \in I \subseteq A^* \qquad \text{for all } r \geq 0.$$

If $m = 0$ we have the following situation

$$\begin{array}{ccc} \mathcal{D}^*(0) = \mathbb{F} & & \mathrm{H}^* \\ \uparrow\wr & & \uparrow\wr \\ \mathbb{F} & \hookrightarrow & A^*. \end{array}$$

Hence equation $(\bigstar)$ gives

$$\mathcal{P}^{\Delta_r}(a) \in (0) \quad \forall\ r \geq 0.$$

So, assume that $m > 0$. In order to apply the induction hypothesis we consider the diagram

$$\begin{array}{ccc} \mathbb{F}[z_1, \ldots, z_m] & \hookrightarrow & A^* \\ \downarrow\ \mathrm{pr} & & \downarrow \\ \mathbb{F}[z_2, \ldots, z_m] & \hookrightarrow & A^*/(z_1). \end{array}$$

[4] In other words: the Integral Closure Theorem follows from the following theorem combined with proposition 8.1.2, which we first use, and therefore prove, in a later chapter.

Since z_1 is a linear form, hence a Thom class, our quotient is again an unstable algebra over $\mathcal{P}^*$, and, moreover again $\mathcal{P}^*$-inseparably closed by Lemma 4.3.5. Denote by $\bar{a}$ the image of a under the projection.

If $\bar{a} = 0$ then

$$a \in (z_1) \Rightarrow a \in I$$

and nothing needs to be shown, because I is $\mathcal{P}^*$-invariant. So assume $\bar{a} \neq 0$. By induction

$$\mathcal{P}^{\Delta_r}(\bar{a}) \in (z_2, \dots, z_m) \quad \forall\ r \geq 0.$$

Since the projection commutes with the Steenrod action we get

$$\mathrm{pr}\left(\mathcal{P}^{\Delta_r}(a)\right) = \mathcal{P}^{\Delta_r}(\bar{a}) \in \ (z_2, \dots, z_m) \quad \forall\ r \geq 0.$$

Therefore

$$\mathcal{P}^{\Delta_r}(a) \in (z_1, \dots, z_m) \qquad \text{for all } r \geq 0.$$

Since I is $\mathcal{P}^*$-invariant, the quotient A^*/I is an unstable algebra and our equation reads

$$(\bigstar) \qquad \mathcal{P}^{\Delta_r}(\tilde{a}) = 0 \in A^*/I \quad \forall\ r \geq 0,\ \forall\ \tilde{a} \in A^*/I.$$

Since A^* is in addition $\mathcal{P}^*$-inseparably closed, the quotient is also, by Proposition 4.3.6, and therefore ($\bigstar$) says that any element in A^*/I is a p-th power. So

$$A^*/I = \mathbb{F}$$

by Proposition 4.3.9. To complete the proof, take an element $a \in A^*$ of minimal positive degree. Then $a \in I$ by what we have proven so far, i.e.,

$$a = a_1 z_1 + \cdots + a_n z_n$$

for some elements $a_1, \dots, a_n \in A^*$. Since a has minimal positive degree, the coefficients $a_1, \dots, a_n$ must have degree zero, i.e.,

$$a \in \mathrm{span}_{\mathbb{F}}(z_1, \dots, z_n) \in \mathrm{Im}(\psi),$$

as we claimed •

As a corollary from this we get the Embedding Theorem over arbitrary Galois fields:

THEOREM 5.3.5 (Embedding Theorem, Version I): *Let* H^* *be an* $\mathcal{P}^*$*-inseparably closed Noetherian integral domain of Krull dimension* n*. Then* H^* *can be embedded integrally into a polynomial ring of the same Krull dimension with linear generators.*

PROOF: The algebra A^* constructed above is $\mathscr{P}^*$-inseparably closed by Lemma 5.3.2 and the extension $H^* \hookrightarrow A^*$ is integral by Theorem 5.2.1. So, Theorem 5.3.4 shows that the map ψ is an isomorphism, i.e.,

$$H^* \hookrightarrow A^* \cong \mathbb{F}[z_1, \ldots, z_n]$$

is the embedding we wanted •

REMARK: If we replaced Noetherianess by the weaker Δ-finiteness in the preceding theorem then we would still get an embedding

$$H^* \hookrightarrow A^* \cong \mathbb{F}[z_1, \ldots, z_n],$$

which, however, might not be integral. Recall Example 2 from Section 1.2:

$$\begin{array}{ccc} \mathscr{D}^*(2) & & H^* := \mathbb{F} < x,\ xy,\ xy^2,\ xy^3, \cdots > \\ \downarrow & & \downarrow \\ \mathbb{F}[x,\ y] & \overset{\psi}{\hookrightarrow} & A^*. \end{array}$$

The algebra H^* is a $\mathscr{P}^*$-inseparably closed integral domain of Krull dimension 2 with Δ-relation

$$x\mathbf{d}_{2,\,0}\mathscr{P}^{\Delta_0} + x\mathbf{d}_{2,\,1}\mathscr{P}^{\Delta_1} + x\mathscr{P}^{\Delta_2} = 0.$$

Hence by Lemma 5.3.2 the algebra A^* is also $\mathscr{P}^*$-inseparably closed, and we even have that

$$A^* = \mathbb{F}[x,\ y]$$

as one might easily see. So, as Theorem 5.3.4 predicts, the map ψ is an isomorphism. But the embedding

$$H^* \hookrightarrow \mathbb{F}[x,\ y]$$

is not integral: e.g., the element y is not integral over H^*.

REMARK: If you drop $\mathscr{P}^*$-inseparably closed you still get an embedding

$$H^* \hookrightarrow \mathbb{F}[V],$$

since the $\mathscr{P}^*$-inseparable closure will embeed. Whoops! There is the problem that $\sqrt{H^*}$ may not be Noetherian.

CHAPTER 6

Noetherianess, the Embedding Theorem II and Turkish Delights

We closed the preceding chapter with a proof of the Embedding Theorem, Theorem 5.3.5. However, we had to assume that the algebra in question, H^*, was inseparably closed. Since we can embed any H^* in its inseparable closure $\sqrt{H^*}$ one might be tempted to say, well then embed the inseparable closed $\sqrt{H^*}$ into $\mathbb{F}[V]$ and we are done. But wait! We can apply the Embedding Theorem 5.3.5 (in this version I) only if $\sqrt{H^*}$ is Noetherian. We don't know yet when this happens.[1] So, this is one of the main topics of this chapter. This allows us to prove the Embedding Theorem in its full generality. Moreover, we will be rewarded with some gorgeous results about the $\mathscr{P}^*$-invariant prime spectrum, which was part of the original motivation for these investigations.

6.1 Noetherianess

The main goal of this section is to prove that an unstable integral domain H^* is Noetherian if and only if its inseparable closure $\sqrt{H^*}$ is Noetherian. This makes then the proof of the Embedding Theorem in its full generality easy.

Recall from Chapter 4 that we have a chain of unstable algebras

$$H^* = H_0^* \overset{\varphi_0}{\hookrightarrow} H_1^* \overset{\varphi_1}{\hookrightarrow} \cdots \overset{\varphi_{i-1}}{\hookrightarrow} H_i^* \overset{\varphi_i}{\hookrightarrow} \cdots,$$

where each extension is integral, so that

$$H^* \hookrightarrow \sqrt{H^*} := \operatorname{colim}(H_i^*)$$

[1] This is precisely the reason why we emphasized up to now when the algebras involved had to be Noetherian, or when a weaker condition would be sufficient. That made the writing sometimes a bit clumsy and the reading a bit awkward, I know. I hope you forgive me now!

is also an integral ring extension.

In Proposition 4.2.3 we have seen that if $\sqrt{\mathrm{H}^*}$ is Noetherian then so is H^*. The converse of this proposition is a lot more difficult. Let's assume that H^* is the least pretentious graded ring you can think of, i.e., a polynomial ring in one linear generator over a Galois field $\mathbb{F}$. Still, this would not imply (from classical tic-tac-toe or such) that $\sqrt{\mathrm{H}^*}$ is Noetherian (just because the extension is integral). This is illustrated by the following gorgeous example of Leslie G. Roberts, [25].

EXAMPLE 1 (Leslie G. Roberts): Let $\mathbb{F}[x]$ be a polynomial ring in one indeterminate of degree 1 over the field $\mathbb{F}$. Then we map this ring diagonally into an infinite product of polynomial rings $\mathbb{F}[x_i]$ in one generator of degree 1 over the field $\mathbb{F}$

$$\begin{array}{rccc} \Delta : & \mathbb{F}[x] & \hookrightarrow & \prod_{i\in\mathbb{N}} \mathbb{F}[x_i] \\ & x & \longmapsto & (x_1,\ x_2, \cdots) \end{array}$$

Denote by X_i the element of $\prod_{i\in\mathbb{N}} \mathbb{F}[x_i]$ that is x_i in the i-th coordinate and zero elsewhere. Then we take the $\mathbb{F}$-sub algebra B in $\prod_{i\in\mathbb{N}} \mathbb{F}[x_i]$ generated by these X_i's and the image of x, $\Delta(x)$,

$$B := \mathbb{F}\langle \Delta(x), X_1,\ X_2, \cdots\rangle.$$

Then B is certainly not Noetherian, because the ideal generated by the elements of positive degree is not finitely generated. However, B is integral over $\Delta(\mathbb{F}[x]) \cong \mathbb{F}[x]$, because

$$X_i^2 - \Delta(x)X_i = 0 \quad \forall\ i \in \mathbb{N}.$$

Note that we could even put a $\mathcal{P}^*$-action onto the algebras involved; that wouldn't make any difference.

THEOREM 6.1.1: *Let* H^* *be an integral domain. Then* H^* *is Noetherian if and only if* $\sqrt{\mathrm{H}^*}$ *is Noetherian.*

PROOF: Since the "if"-part is the contents of Proposition 4.2.3, we deal with the "only if"-part.

Let H^* be Noetherian. We start by collecting some facts we know already:

[1] H^*_i is Noetherian, for any $i \geq 0$, because the extension $\mathrm{H}^* \hookrightarrow \mathrm{H}^*_i$ is integral and finite[2].

[2] H^*_i, $\forall\ i \geq 0$, is Δ-finite, by Theorem 1.2.1.

[2] Yes, they are finite, because H^* is Noetherian by assumption, recall Section 4.1

[3] The Δ-length of H_i^*, $m_i := m(H_i^*)$, is less or equal to the Krull dimension $\dim(H_i^*) = \dim(\sqrt{H^*})$, $\forall\, i \geq 0$, by Corollary 1.2.2 and Proposition 4.2.1 [4].

Case 1 : $n := \dim(H^*) = m(H^*) =: m$

Since H^* is an integral domain we can apply the constructions of Chapter 5, in particular those of Section 5.3: Let $z_1, \ldots, z_n$ be an $\mathbb{F}_q$-basis for the vector space V consisting of the roots of the Δ-polynomial of H^*. By Lemma 5.3.1 it follows that

$$\mathbb{F}[z_1, \ldots, z_n] \underset{\psi}{\hookrightarrow} H^* < z_1, \ldots, z_n >= A^*$$

is an integral ring extension. Therefore Lemma 5.3.3 implies that A^* is $\mathcal{P}^*$-inseparably closed. Hence we have a diagram

$$\begin{array}{ccc} H^* & \underset{\text{finite, int}}{\hookrightarrow} & H_i^* \\ \downarrow & & \downarrow \\ FF(H^*) & \underset{\text{purely insep}}{\hookrightarrow} & FF(H_i^*) \\ \downarrow & & \downarrow \\ \mathbb{E}^* := FF(H^*)(z_1, \ldots, z_n) & \hookrightarrow & FF(H_i^*)(z_1, \ldots, z_n) =: \mathbb{E}_i^*. \end{array}$$

By Theorem 5.2.1 the field extensions

$$FF(H^*) \hookrightarrow \mathbb{E}^* \quad \text{and} \quad FF(H_i^*) \hookrightarrow \mathbb{E}_i^*$$

are finite and separable (recall: we just adjoint the roots of the separable Δ-polynomial). Moreover, since, by Lemma 4.2.5,

$$FF(H^*) \hookrightarrow FF(H_i^*)$$

is $\mathcal{P}^*$-purely inseparable, we conclude also that

$$\mathbb{E}^* \hookrightarrow \mathbb{E}_i^*$$

is $\mathcal{P}^*$-purely inseparable. However, $\mathbb{E}^*$ is $\mathcal{P}^*$-inseparably closed by Corollary 4.2.7, because $\mathbb{E}^* = FF(H^* < z_1, \ldots, z_n >)$. Hence

$$\mathbb{E}^* = \mathbb{E}_i^*$$

for any $i \geq 0$, so we have

$$\begin{array}{ccc} FF(H^*) & \underset{\text{purely insep}}{\hookrightarrow} & FF(H_i^*) \\ \downarrow \text{sep} & & \downarrow \text{sep} \\ \mathbb{E}^* & = & \mathbb{E}_i^*\,. \end{array}$$

Therefore

$$FF(H^*) = FF(H_i^*)$$

for all $i \geq 0$. Moreover since $\mathbb{E}^*$ is $\mathscr{P}^*$-inseparably closed, and $FF(\mathrm{H}^*) \hookrightarrow \mathbb{E}^*$ is separable, also the little field $FF(\mathrm{H}^*)$ must be $\mathscr{P}^*$-inseparably closed and therefore by Proposition 4.2.6

$$FF(\mathrm{H}^*) = FF(\mathrm{H}_i^*) = FF(\sqrt{\mathrm{H}^*}).$$

Since H^* is an affine domain[3] its integral closure $\overline{\mathrm{H}^*}$ (i.e., in its field of fractions) is again Noetherian by a classic from Emmy, see e.g. [11] Corollary 13.13. Hence we have a chain of integral extensions of algebras

$$\mathrm{H}^* \hookrightarrow \sqrt{\mathrm{H}^*} \hookrightarrow \overline{\mathrm{H}^*},$$

where the smallest and the biggest are Noetherian. So take a homogeneous system of parameters $h_1, \ldots, h_n \in \mathrm{H}^*$. The quotient ring

$$\mathrm{H}^*/(h_1, \ldots, h_n)$$

is totally finite. The quotient

$$\overline{\mathrm{H}^*}/(h_1, \ldots, h_n)$$

is Noetherian and has Krull dimension 0, therefore it is Artinian and hence also totally finite, i.e., $h_1, \ldots, h_n$ is a system of parameters for the integral closure $\overline{\mathrm{H}^*}$. So $\overline{\mathrm{H}^*}$ is a finite H^*-module and hence so is $\sqrt{\mathrm{H}^*}$. In other words, $\sqrt{\mathrm{H}^*}$ is Noetherian.

CASE 2 : $n := \dim(\mathrm{H}^*) > m(\mathrm{H}^*) =: m$

We claim that there exists an index $i_0 \in \mathbb{N}_0$ such that

$$m_{i_0} := m(\mathrm{H}_{i_0}^*) > m(\mathrm{H}^*) =: m.$$

Assuming that we had proven this we would have that for some index i_k

$$m(\mathrm{H}_{i_k}^*) = \dim(\mathrm{H}_{i_k}^*) \left(= \dim(\mathrm{H}^*) = n\right).$$

Since $\mathrm{H}_{i_k}^*$ is again Noetherian we could apply Case 1 and we would be done.

So whats left to show is that the Δ-length already goes up after finitely many steps in our chain of sucessive ring extensions that leads to $\sqrt{\mathrm{H}^*}$. We assume to the contrary that for any $i \geq 0$

$$m(\mathrm{H}_i^*) = m(\mathrm{H}^*) = m < n = \dim(\mathrm{H}^*).$$

Look at the Δ-relations involved

$$h_0(i)\mathcal{P}^{\Delta_0} + \cdots + h_m(i)\mathcal{P}^{\Delta_m} = 0 \quad \text{on } \mathrm{H}_i^*,$$

[3] Remember we are in the graded connected case!

where the (i) indicates that we take coefficients from the i-th algebra H^*_i. By Observation 5.1.1, allowing coefficients in the respective field of fractions, the Δ-relations coincide

$$\frac{1}{h_m(i)} h_k(i) = \frac{1}{h_m(j)} h_k(j)$$

for all $i \geq 0$ and $k = 0, \ldots, m$. Moreover, for any element $h \in \sqrt{H^*}$ there exists an index i_0 such that $h \in H^*_{i_0}$ and hence also

$$\left(\frac{h_0(i_0)}{h_m(i_0)} \mathcal{P}^{\Delta_0} + \cdots + \frac{h_m(i_0)}{h_m(i_0)} \mathcal{P}^{\Delta_m} \right)(h) = 0,$$

so the same (normalized) Δ-relation holds on $\sqrt{H^*}$ and $\sqrt{H^*}$ is Δ-finite with Δ-length m. Recall again the constructions from Chapter 5: If $z_1, \ldots, z_m$ is a basis of the vector space over $\mathbb{F}$ of the roots of the Δ-polynomial of $\sqrt{H^*}$ then by Lemma 5.3.2

$$\sqrt{H^*} < z_1, \ldots, z_m >$$

is $\mathcal{P}^*$-inseparably closed, and so Theorem 5.3.4 shows that

$$\psi : \mathbb{F}[z_1, \ldots, z_m] \hookrightarrow \sqrt{H^*} < z_1, \ldots, z_m >$$

is an isomorphism. Therefore we have

$$\begin{array}{ccccc} H^* & \hookrightarrow & \sqrt{H^*} & \hookrightarrow & \mathbb{F}[z_1, \ldots, z_m] \\ \downarrow & & \downarrow & & \downarrow \\ FF(H^*) & \hookrightarrow & FF(\sqrt{H^*}) & \hookrightarrow & \mathbb{F}(z_1, \ldots, z_m) \end{array}$$

from which we get

$$\begin{aligned} m &= \dim(\mathbb{F}[z_1, \ldots, z_m]) \\ &= \operatorname{trzdeg}\left(\mathbb{F}(z_1, \ldots, z_m)/\mathbb{F}\right) \\ &\geq \operatorname{trzdeg}\left(FF(H^*)/\mathbb{F}\right) \\ &= \dim(H^*) \\ &= n \\ &> m \end{aligned}$$

which is a contradiction •

We can prove now the complete Embedding Theorem.

COROLLARY 6.1.2 (Embedding Theorem): *Let* H^* *be a Noetherian integral domain of Krull dimension* n. *Then* H^* *can be embedded integrally into a polynomial ring of the same Krull dimension with linear generators.*

PROOF : Since H^* is Noetherian, Theorem 6.1.1 tells us that also $\sqrt{H^*}$ is Noetherian. So, $\sqrt{H^*}$ is a Noetherian $\mathcal{P}^*$-inseparably closed unstable integral domain and hence can be embedded integrally into a polynomial ring of the desired form. Since the extension $H^* \hookrightarrow \sqrt{H^*}$ is integral we are done •

6.2 Turkish Delights

After all that hard work we are going to be rewarded with some nice results about the spectrum of $\mathcal{P}^*$-invariant prime ideals of H^*.

Throughout this section we consider only Noetherian algebras.

We start by recalling some terminology.

Consider the spectrum of homogeneous prime ideals in H^*, $Proj(H^*)$. We denote by

$$Proj_{\mathcal{P}^*}(H^*)$$

the subset of homogeneous prime ideals which are closed under the action of the Steenrod algebra, which we call **the spectrum of $\mathcal{P}^*$-invariant prime ideals of** H^*, see [21], [23], [26], [27] and [28] Chapter 11 for results concerning $\mathcal{P}^*$-invariant ideals.

PROPOSITION 6.2.1 (Turkish Delight 1): *Let H^* be a Noetherian unstable algebra over the Steenrod algebra of Krull dimension n. Then H^* has only finitely many $\mathcal{P}^*$-invariant prime ideals.*

PROOF : Since H^* is Noetherian it has only finitely many prime ideals $\mathfrak{p}_0$ of height 0, which are all $\mathcal{P}^*$-invariant by Theorem 1 in [15], see also Proposition 11.2.3 in [28] or the more general result in Theorem 3.5 [23]. Therefore $H^*/\mathfrak{p}_0$ is again an unstable algebra over the Steenrod algebra and moreover an integral domain. Hence by Corollary 6.1.2 there exists an integral extension of unstable algebras over $\mathcal{P}^*$

$$H^*/\mathfrak{p}_0 \hookrightarrow \mathbb{F}[z_1, \ldots, z_n]$$

where the polynomial ring has linear generators. Since the extension is integral, for any prime ideal $\mathfrak{p} \subset H^*/\mathfrak{p}_0$, there exists a prime ideal in $\mathbb{F}[z_1, \ldots, z_n]$ lying over it. Since we have an extension of unstable algebras over the Steenrod algebra all prime ideals in $\mathbb{F}[z_1, \ldots, z_n]$ lying over a $\mathcal{P}^*$-invariant prime ideal $\mathfrak{p} \subset H^*/\mathfrak{p}_0$ are themselves $\mathcal{P}^*$-invariant, see Theorem 2.3 in [21]. By [26] any $\mathcal{P}^*$-invariant prime ideal in $\mathbb{F}[z_1, \ldots, z_n]$ is generated by linear forms. Since there are only finitely many linear forms, there are only finitely many $\mathcal{P}^*$-invariant prime ideals in

$\mathbb{F}[z_1, \ldots, z_n]$. Therefore, there are only finitely many $\mathcal{P}^*$-invariant prime ideals in $H^*/\mathfrak{p}_0$, and hence also in H^* •

PROPOSITION 6.2.2 (Turkish Delight 2): *Let* H^* *be a Noetherian unstable algebra over the Steenrod algebra of Krull dimension* n. *Then, for any* $i = 0, \ldots, n$, *there exists a* $\mathcal{P}^*$*-invariant prime ideal in* H^* *of height* i.

PROOF: By [15], Proposition 11.2.3 in [28] or Theorem 3.5 in [23] all prime ideals $\mathfrak{p}_0 \subset H^*$ of height 0 are $\mathcal{P}^*$-invariant. Hence our statement is proven for $i = 0$. Proceeding as above, we may divide out by a height zero prime ideal $\mathfrak{p}_0$ and get

$$H^* \xrightarrow{\text{pr}} H^*/\mathfrak{p}_0,$$

which is again an unstable algebra over $\mathcal{P}^*$ of the same Krull dimension n. Moreover, it embeds as an algebra over $\mathcal{P}^*$ into a polynomial algebra with linear generators

$$H^*/\mathfrak{p}_0 \hookrightarrow \mathbb{F}[z_1, \ldots, z_n].$$

In the polynomial algebra on the right we can find, for any given height, a $\mathcal{P}^*$-invariant prime ideal, e.g., take the following saturated maximal chain

$$(0) \subsetneqq (z_1) \subsetneqq (z_1, z_2) \subsetneqq \cdots \subsetneqq (z_1, \ldots, z_n) \subsetneqq \mathbb{F}[z_1, \ldots, z_n].$$

Then

$$(0) \subsetneqq (z_1) \cap \left(H^*/\mathfrak{p}_0\right) \subsetneqq \cdots \subsetneqq (z_1, \ldots, z_n) \cap \left(H^*/\mathfrak{p}_0\right) \subsetneqq H^*/\mathfrak{p}_0$$

is a saturated maximal chain of $\mathcal{P}^*$-invariant prime ideals in $H^*/\mathfrak{p}_0$, because our extension of algebras is integral, Lemma 2.1 in [21], and in particular satisfies lying over, compare §3.1 in [4]. This implies that

$$\mathfrak{p}_0 \subsetneqq \text{pr}^{-1}\left((z_1) \cap H^*/\mathfrak{p}_0\right) \subsetneqq \cdots \subsetneqq \text{pr}^{-1}\left((z_1, \ldots, z_n) \cap H^*/\mathfrak{p}_0\right) \subsetneqq H^*$$

is a saturated maximal chain of $\mathcal{P}^*$-invariant prime ideals in H^* •

The following proposition justifies the remark in the introduction that $\mathit{Proj}_{\mathcal{P}^*}(H^*)$ forms a chain saturated poset (with respect to inclusion).

PROPOSITION 6.2.3 (Turkish Delights 3 and 4): *Let* H^* *be a Noetherian unstable algebra over the Steenrod algebra of Krull dimension* n. *Let* $\mathfrak{p} \subset H^*$ *be a* $\mathcal{P}^*$*-invariant prime ideal of height* i, $i = 0, \ldots, n$. *Then*

[1] *there exists an ascending saturated maximal chain of* $\mathcal{P}^*$*-invariant prime ideals in* H^* *starting with* $\mathfrak{p}$ *and ending with the maximal ideal* $\mathfrak{m}$.

[2] *there exists a descending saturated maximal chain of* $\mathcal{P}^*$*-invariant prime ideals in* H^* *starting at* $\mathfrak{p}$ *and ending with a* $\mathcal{P}^*$*-invariant prime ideal* $\mathfrak{p}_0$ *of height 0.*

PROOF : Let $\mathfrak{p}$ have height i. Note that if $i = 0$ the statement of the proposition is Turkish Delight 2. Otherwise, take a prime ideal of height 0 sitting inside our given $\mathfrak{p}$,

$$\mathfrak{p}_0 \subsetneq \mathfrak{p} \subset \mathrm{H}^*$$

and divide it out, to obtain

$$\mathrm{pr} : \mathrm{H}^* \longrightarrow \mathrm{H}^*/\mathfrak{p}_0.$$

Our choice of $\mathfrak{p}_0$ guarantees that $\mathfrak{p}/\mathfrak{p}_0 \subset \mathrm{H}^*/\mathfrak{p}_0$ is still a $\mathscr{P}^*$-invariant prime ideal of height i. Again, we embed the quotient algebra in a polynomials ring with linear generators

$$\mathrm{H}^*/\mathfrak{p}_0 \hookrightarrow \mathbb{F}[z_1, \dots, z_n].$$

Since this embedding is integral, the lying over and going up theorems hold. So, take a prime ideal

$$\mathfrak{q} \subset \mathbb{F}[z_1, \dots, z_n]$$

lying over our given

$$\mathfrak{q} \cap \left(\mathrm{H}^*/\mathfrak{p}_0\right) = \mathfrak{p}/\mathfrak{p}_0.$$

Then $\mathfrak{q}$ is also $\mathscr{P}^*$-invariant by Theorem 2.3 in [21]. By [26] we can find saturated ascending and descending maximal chains of $\mathscr{P}^*$-invariant prime ideals in the polynomial ring staring with $\mathfrak{q}$ and ending with the maximal, resp. staring with $\mathfrak{q}$ and ending at (0):

$$\mathfrak{q} \subsetneq \cdots \subsetneq \mathfrak{m} \subset \mathbb{F}[z_1, \dots, z_n]$$

and

$$(0) \subsetneq \cdots \subsetneq \mathfrak{q} \subset \mathbb{F}[z_1, \dots, z_n].$$

Intersect these chains with $\mathrm{H}^*/\mathfrak{p}_0$ and pull them back to H^* via pr^{-1}, and we are done •

6.3 Noetherianess II

In this section we generalize Theorem 6.1.1 to reduced algebras H^*.

Let's start with another more technical characterization of Noetherianess of $\sqrt{\mathrm{H}^*}$. Recall form Section 4.1 that the $\mathscr{P}^*$-inseparable closure of an unstable algebra H^* was defined to be a filtered colimit

$$\sqrt{\mathrm{H}^*} = \mathrm{colim}(\mathrm{H}^*_i),$$

where the field of fractions behave naturally, i.e,

$$FF(\sqrt{\mathrm{H}^*}) = \mathrm{colim}\left(FF(\mathrm{H}^*_i)\right),$$

by Proposition 4.2.6. Moreover, we have seen in Proposition 4.2.4 that $\sqrt{\mathrm{H}^*}$ is Noetherian if and only if the involved colimit is indeed a finite union. We are going to exploit this fact in the following proof, which generalizes Theorem 6.1.1 to reduced algebras.

THEOREM 6.3.1: *Let* H^* *be a reduced (i.e.,* $Nil\,(\mathrm{H}^*) = (0)$*), unstable algebra. Then* H^* *is Noetherian if and only if* $\sqrt{\mathrm{H}^*}$ *is Noetherian.*

PROOF: The "if"-part is the contents of Proposition 4.2.3. So we have to prove the "only if"-part, and we assume that H^* is Noetherian.

For a prime ideal $\mathfrak{p} \subset \mathrm{H}^*$ denote by

$$\mathrm{pr}_{\mathfrak{p}} : \mathrm{H}^* \longrightarrow \mathrm{H}^*/\mathfrak{p}$$

the canonical projection onto the quotient. Since $Nil\,(\mathrm{H}^*) = (0)$ by assumption we have the Lam-Rector Embedding, see [14] and [24],

$$L := \overset{k}{\underset{i=1}{\oplus}} \mathrm{pr}_i : \mathrm{H}^* \hookrightarrow \bigoplus_{i=1}^{k} \mathrm{H}^*/\mathfrak{p}_i$$

where the direct sum runs over all height zero prime ideals $\mathfrak{p}_i$ of H^* and the i-th component of L is projection onto $\mathrm{H}^*/\mathfrak{p}_i$. This is a finite direct sum, because H^* is Noetherian.

We receive a diagram as follows

$$\begin{array}{ccc}
\mathrm{H}^* & \overset{L}{\hookrightarrow} & \overset{k}{\underset{i=1}{\oplus}} \mathrm{H}^*/\mathfrak{p}_i \\
\downarrow & & \downarrow \\
\mathrm{H}^*_1 & & \left(\overset{k}{\underset{i=1}{\oplus}} \mathrm{H}^*/\mathfrak{p}_i \right)_1 \\
 & & || \\
 & & \overset{k}{\underset{i=1}{\oplus}} \left(\mathrm{H}^*/\mathfrak{p}_i\right)_1 \\
\downarrow & & \downarrow \\
\cdots & & \cdots
\end{array}$$

By Proposition 4.2.1 the algebra H_1 is again reduced and Noetherian and therefore admits also a Lam-Rector Embedding

$$L_1 := \overset{k_1}{\underset{i=1}{\oplus}} \mathrm{pr}_i : \mathrm{H}^*_1 \hookrightarrow \bigoplus_{i=1}^{k_1} \mathrm{H}^*_1/\mathfrak{q}_i .$$

By Theorem 4.3.1 $k_1 = k$ and by Lemma 4.3.2

$$L_1 := \oplus_{i=1}^{k} \mathrm{pr}_i : \mathrm{H}^*_1 \hookrightarrow \bigoplus_{i=1}^{k} \mathrm{H}^*_1/\mathfrak{q}_i = \bigoplus_{i=1}^{k} \left(\mathrm{H}^*/\mathfrak{p}_i\right)_1 .$$

Hence we can complete our diagram above as follows:

$$\begin{array}{ccc} H^* & \overset{L}{\hookrightarrow} & \bigoplus_{i=1}^k H^*/\mathfrak{p}_i \\ \downarrow & & \downarrow \\ H^*_1 & \overset{L_1}{\hookrightarrow} & \bigoplus_{i=1}^k \left(H^*/\mathfrak{p}_i\right)_1 . \end{array}$$

By Theorem 6.1.1 the $\mathcal{P}^*$-inseparable closure $\sqrt{H^*/\mathfrak{p}_i}$ is Noetherian, for any i. Therefore, by Proposition 4.2.4 the colimits involved are finite unions

$$\begin{aligned} \sqrt{H^*/\mathfrak{p}_i} := \operatorname{colim}_j \left((H^*/\mathfrak{p}_i)_j\right) \\ = \bigcup_{j=0}^{r_i} \left((H^*/\mathfrak{p}_i)_j\right) \\ = \left(H^*/\mathfrak{p}_i\right)_{r_i} \end{aligned}$$

for some $r_i \in \mathbb{N}_0$. Set $r := \max\{r_1, \ldots, r_k\}$ then we have

$$\begin{array}{ccc} H^* & \overset{L}{\hookrightarrow} & \bigoplus_{i=1}^k H^*/\mathfrak{p}_i \\ \downarrow & & \downarrow \\ H^*_1 & \overset{L_1}{\hookrightarrow} & \bigoplus_{i=1}^k \left(H^*/\mathfrak{p}_i\right)_1 \\ \downarrow & & \downarrow \\ H^*_2 & \overset{L_2}{\hookrightarrow} & \bigoplus_{i=1}^k \left(H^*/\mathfrak{p}_i\right)_2 \\ \downarrow & & \downarrow \\ \cdots & & \cdots \\ \downarrow & & \downarrow \\ H^*_r & \overset{L_r}{\hookrightarrow} & \bigoplus_{i=1}^k \left(H^*/\mathfrak{p}_i\right)_r \\ \downarrow & & || \\ H^*_{r+1} & \overset{L_{r+1}}{\hookrightarrow} & \bigoplus_{i=1}^k \left(H^*/\mathfrak{p}_i\right)_{r+1} \\ & & || \\ & & \sqrt{\bigoplus_{i=1}^k H^*/\mathfrak{p}_i} \end{array}$$

We claim that $H^*_r = H^*_{r+1}$, and hence by Theorem 4.2.4 $\sqrt{H^*}$ is Noetherian. To this end take an element $h \in H^*_{r+1}$. Denote by

$$\mathfrak{q}_{r,1}, \ldots, \mathfrak{q}_{r,k} \subset H^*_r$$

the minimal prime ideals of H^*_r and by

$$\mathfrak{q}_{r+1,1}, \ldots, \mathfrak{q}_{r+1,k} \subset H^*_{r+1}$$

those of H^*_{r+1}. Without loss of generality we can assume that

$$\mathfrak{q}_{r,j} = \mathfrak{q}_{r+1,j} \cap \mathrm{H}^*_r \quad \forall\ j = 1, \dots, k.$$

Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^*_r & \xrightarrow{\mathrm{pr}_{r,j}} & \mathrm{H}^*_r/\mathfrak{q}_{r,j} \\ \downarrow & & \| \\ \mathrm{H}^*_{r+1} & \xrightarrow{\mathrm{pr}_{r+1,j}} & \mathrm{H}^*_{r+1}/\mathfrak{q}_{r+1,j} \end{array}$$

for all $j = 1, \dots, k$. Then for any $h \in \mathrm{H}^*_{r+1}$ we have

$$\mathrm{pr}^{-1}_{r,j}\left(\mathrm{pr}_{r+1,j}(h)\right) - h \in \mathfrak{q}_{r+1,j}$$

for any $j = 1, \dots, k$. Therefore

$$\mathrm{pr}^{-1}_{r,j}\left(\mathrm{pr}_{r+1,j}(h)\right) - h \in \mathfrak{q}_{r+1,1} \cap \cdots \cap \mathfrak{q}_{r+1,k} = \mathit{Nil}\,(\mathrm{H}^*_{r+1}) = (0)$$

and hence

$$\mathrm{pr}^{-1}_{r,j}\left(\mathrm{pr}_{r+1,j}(h)\right) = h$$

as was to be shown •

We next extend Corollary 4.3.8 in the following way.

COROLLARY 6.3.2: *For a Noetherian unstable algebra* H^**, its* $\mathcal{P}^*$*-inseparable closure modulo its nil radical, i.e.,*

$$\sqrt{\mathrm{H}^*}/\mathit{Nil}\,(\sqrt{\mathrm{H}^*}),$$

is Noetherian and $\mathcal{P}^*$*-inseparably closed. In particular it is the* $\mathcal{P}^*$*-inseparable closure of* $\mathrm{H}^*/\mathit{Nil}\,(\mathrm{H}^*)$*.*

PROOF: The only thing that is left to show is that $\sqrt{\mathrm{H}^*}/\mathit{Nil}\,(\sqrt{\mathrm{H}^*})$ is again Noetherian. This follows, because H^* is Noetherian, hence also $\mathrm{H}^*/\mathit{Nil}\,(\mathrm{H}^*)$ and therefore so is

$$\sqrt{\mathrm{H}^*/\mathit{Nil}\,(\mathrm{H}^*)} = \sqrt{\mathrm{H}^*}/\mathit{Nil}\,(\sqrt{\mathrm{H}^*})$$

by Theorem 6.3.1 •

REMARK: This result certainly does not imply that $\sqrt{\mathrm{H}^*}$ is Noetherian in general, e.g., take a polynomial ring in infinitely many indeterminants of degree 1 and mod out the ideal which is generated by the p-th powers of the generators

$$\mathrm{H}^* = \mathbb{F}[X_1,\ X_2,\ \cdots]/(X_1^p,\ X_2^p,\ \cdots).$$

Then H^* is not Noetherian since the ideal of all elements of positive degree[4] is not finitely generated, but if we mod out the nil radical we get a Noetherian ring, namely

$$H^*/Nil\,(H^*) = \mathbb{F},$$

because every element of positive degree is nilpotent.

[4] This ideal is called the **irrelevant ideal** or the **augmentation ideal**, depending on which side of the river you live.

CHAPTER 7
The Galois Embedding Theorem, the Little Imbedding Theorem, and A Bit More

After all the work we have done so far we are prepared to take the remaining steps towards the Galois Embedding Theorem. Hence, we will assume in the first section of this chapter that our H^* is an $\mathcal{P}^*$-inseparably closed, integrally closed, integral domain. After that, we will have a look at what we can do when H^* is not $\mathcal{P}^*$-inseparably closed. In the last section we will prove the Little Imbedding Theorem, which says that we can always find a fractal of the Dickson algebra in an unstable integral domain H^*.

Throughout the whole chapter H^* is Noetherian.

7.1 The Galois Embedding Theorem

Assume H^* to be an integrally closed, $\boldsymbol{\mathcal{P}}^*$-inseparably closed, Noetherian integral domain over $\mathbb{F}$.

Then, firstly, we are in the lucky position that we can embed H^* integrally into a polynomial algebra with linear generators by Theorem 6.1.2:

$$\mathrm{H}^* \hookrightarrow \mathbb{F}[V].$$

Hence we have an algebraic extension of the field of fractions

$$FF(\mathrm{H}^*) \hookrightarrow \mathbb{F}(V) = \mathbb{E}^*,$$

which is separable, because $\mathbb{E}^*$ is the splitting field of the separable polynomial $\Delta(X)$. Therefore this field extension is also finite and normal, see [38] Theorem 13 on page 76. So, our extension is Galois with some Galois group $G \leq \mathrm{Aut}(\mathbb{F}(V))$, i.e.,

$$FF(\mathrm{H}^*) = \mathbb{F}(V)^G,$$

and we have integral extensions

$$\mathrm{H}^* \hookrightarrow \mathbb{F}[V]^G \hookrightarrow \mathbb{F}[V]$$

of integrally closed domains, where the first two have the same field of fractions. Hence they must be equal and we have proven:

THEOREM 7.1.1 (Galois Embedding Theorem): *Let* H^* *be an unstable algebra over* $\mathbb{F}$. *Then* H^* *is a ring of invariants* $\mathbb{F}[V]^G$ *if and only if* H^* *is an integrally closed,* $\mathcal{P}^*$*-inseparably closed, Noetherian integral domain.*

PROOF : •

7.2 A Bit More

An essential assumption in the preceding section was that H^* is $\mathcal{P}^*$-inseparably closed. If this is not the case then at the level of fields of fractions the extension

$$FF(\mathrm{H}^*) \hookrightarrow \mathbb{F}(V) = \mathbb{E}^*$$

is not separable. As in classical field theory we have two possibilities to bring order into this mess: the first is to find an intermediate field $\mathbb{K}^*$

$$FF(\mathrm{H}^*) \underset{\mathcal{P}^*-\text{purely insep.}}{\hookrightarrow} \mathbb{K}^* \underset{\text{sep.}}{\hookrightarrow} \mathbb{F}(V)$$

such that the first extension is $\mathcal{P}^*$-purely inseparable and the second separable. We then get:

THEOREM 7.2.1: *The field* $\mathbb{K}^*$ *is the field of fractions of the* $\mathcal{P}^*$*-inseparable closure* $\sqrt{\mathrm{H}^*}$ *of* H^*. *Moreover, if* H^* *is integrally closed, then* $\sqrt{\mathrm{H}^*}$ *is the integral closure of* H^* *in* $\mathbb{K}^*$. *Denote this by* $\overline{\mathrm{H}^*}_{\mathbb{K}^*}$. $\sqrt{\mathrm{H}^*}$ *fullfills the assumption of the Galois Embedding Theorem, so is a ring of invariants.*

PROOF : From Corollary 2.3.3 we get a $\mathcal{P}^*$-inseparable field extension

$$\mathbb{K}^* \subseteq FF(\mathrm{H}^*)_{\mathcal{P}^*-\text{insep}} \subseteq \mathbb{F}(V)_{\mathcal{P}^*-\text{insep}} = \mathbb{F}(V).$$

Therefore

$$\mathbb{K}^* = FF(\mathrm{H}^*)_{\mathcal{P}^*-\text{insep}}.$$

Then the first statement follows from Proposition 4.2.6

$$\mathbb{K}^* = FF(\sqrt{\mathrm{H}^*}).$$

Therefore we have the following situation:

$$\begin{array}{ccccccc}
\mathrm{H}^* & \hookrightarrow & \sqrt{\mathrm{H}^*} & & & \hookrightarrow & \mathbb{F}[V] \\
\downarrow & & \downarrow & & & & \downarrow \\
FF(\mathrm{H}^*) & \underset{\mathcal{P}^*-\text{purely insep.}}{\hookrightarrow} & FF(\sqrt{\mathrm{H}^*}) & = & \mathbb{K}^* & \underset{\text{sep.}}{\hookrightarrow} & \mathbb{F}(V)
\end{array}$$

where the first field extension is $\mathscr{P}^*$-purely inseparable, while the last one is separable by Proposition 4.2.6. By Proposition 4.2.1 $\sqrt{\mathrm{H}^*}$ is integrally closed whenever H^* is, hence the integral closure of H^* in $FF(\sqrt{\mathrm{H}^*})$ is

$$\overline{\mathrm{H}^*}_{FF(\sqrt{H})} \subseteq \sqrt{\mathrm{H}^*}.$$

But, notice that we have an integral extension of integrally closed domain with the same field of fractions, so they are equal.

Finally $\sqrt{\mathrm{H}^*}$ is an integrally closed, $\mathscr{P}^*$-inseparably closed integral domain. Hence $\sqrt{\mathrm{H}^*}$ is a ring of invariants. That's what we wanted •

On the other hand, we can find another intermediate field $\mathbb{L}^*$

$$FF(\mathrm{H}^*) \underset{\text{sep.}}{\hookrightarrow} \mathbb{L}^* \underset{\mathscr{P}^*\text{–purely insep.}}{\hookrightarrow} \mathbb{F}(V)$$

such that, this time the first extension is separable, and the second $\mathscr{P}^*$-purely inseparable. Recall the notation

$$\mathbb{F}[V] := \mathbb{F}[z_1, \ldots, z_n].$$

We have (compare also [37]):

THEOREM 7.2.2 : *The integral closure of* H^* *in* $\mathbb{L}^*$*, denoted by* $\overline{\mathrm{H}^*}_{\mathbb{L}^*}$*, is the polynomial algebra*

$$\mathbb{F}[z_1^{p^{e_1}}, \ldots, z_n^{p^{e_n}}]$$

for some p*-th powers* $p^{e_1}, \ldots, p^{e_n}$*. Moreover, if* H^* *is integrally closed*

$$\mathrm{H}^* = \left(\overline{\mathrm{H}^*}_{\mathbb{L}^*}\right)^G = \mathbb{F}[z_1^{p^{e_1}}, \ldots, z_n^{p^{e_n}}]^G$$

for a subgroup $G \leq \mathrm{GL}(n, \mathbb{F})$*, where the action of the general linear group on* $\overline{\mathrm{H}^*}_{\mathbb{L}^*}$ *is induced by the natural one on* $\mathbb{F}[V]$*.*

PROOF : First note that $\overline{\mathrm{H}^*}_{\mathbb{L}^*}$ is an integrally closed domain with field of fractions $\mathbb{L}^*$. Therefore we have

$$\begin{array}{ccccc}
\mathrm{H}^* & \hookrightarrow & \overline{\mathrm{H}^*}_{\mathbb{L}^*} & \hookrightarrow & \mathbb{F}[V] = \mathbb{F}[z_1, \ldots, z_n] \\
\downarrow & & \downarrow & & \downarrow \\
FF(\mathrm{H}^*) & \underset{\text{sep.}}{\hookrightarrow} & FF\left(\overline{\mathrm{H}^*}_{\mathbb{L}^*}\right) = \mathbb{L}^* & \underset{\mathscr{P}^*\text{–purely insep.}}{\hookrightarrow} & \mathbb{F}(V) = \mathbb{F}(z_1, \ldots, z_n)
\end{array}$$

where at the algebra level we have integral extensions, and at the field level, the first one is separable, and the second one is $\mathscr{P}^*$-purely inseparable. Hence for any z_i, $i = 1, \ldots, n$ there exists a p-th power p^{e_i} such that

$$z_i^{p^{e_i}} \in \mathbb{L}^*.$$

Since, in addition, $z_i^{p^{e_i}} \in \mathbb{L}^*$ is integral over $\overline{\mathrm{H}^*}_{\mathbb{L}^*}$, we have

$$\begin{array}{ccccc} \mathbb{F}[z_1^{p^{e_1}}, \ldots, z_n^{p^{e_n}}] & \hookrightarrow & \overline{\mathrm{H}^*}_{\mathbb{L}^*} & \hookrightarrow & \mathbb{F}[V] \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{F}(z_1^{p^{e_1}}, \ldots, z_n^{p^{e_n}}) & \hookrightarrow & FF\left(\overline{\mathrm{H}^*}_{\mathbb{L}^*}\right) = \mathbb{L}^* & \hookrightarrow & \mathbb{F}(V) \end{array}$$

where at algebra level we have again integral extensions of integrally closed domains, and at field level, $\mathcal{P}^*$-purely inseparable extensions. If we choose the exponents p^{e_i} to be minimal with this property we get a separable extension

$$FF(\mathrm{H}^*) \subseteq \mathbb{F}(z_1^{p^{e_1}}, \ldots, z_n^{p^{e_n}}).$$

Since we started with a separable extension

$$FF(\mathrm{H}^*) \subseteq \mathbb{L}^*$$

the field extension

$$\mathbb{F}(z_1^{p^{e_1}}, \ldots, z_n^{p^{e_n}}) \subseteq \mathbb{L}^*$$

is also separable. Hence these two fields must be equal, and, therefore

$$\mathbb{F}[z_1^{p^{e_1}}, \ldots, z_n^{p^{e_n}}] = \overline{\mathrm{H}^*}_{\mathbb{L}^*}.$$

Hence H^* is $\mathcal{P}^*$-inseparably closed in $\overline{\mathrm{H}^*}_{\mathbb{L}^*}$. The field extension

$$FF(\mathrm{H}^*) \subset \mathbb{F}(z_1^{p^{e_1}}, \ldots, z_n^{p_{e_n}})$$

is also normal, because its the splitting field of $\Delta(X)$. If we assume in addition that H^* is integrally closed the rest of the statement follows from the Galois Embedding Theorem •

7.3 The Little Imbedding Theorem

An obvious consequence of the Galois Embedding Theorem is that the Dickson algebra $\mathcal{D}^*(n) \hookrightarrow \mathrm{H}^*$ embeds integrally into an H^* which fulfills all the requirements of that theorem. However, this is also almost true if we just assume that H^* is an integral domain: H^* always contains a fractal of the Dickson algebra. Recall from the introduction that a fractal of $\mathcal{D}^*(n)$ is

$$\begin{aligned} \mathcal{D}^*(n)^{q^s} &= \mathbb{F}[\mathbf{d}_{n,0}^{q^s}, \ldots, \mathbf{d}_{n,n-1}^{q^s}] \\ &= \mathbb{F}[z_1^{q^s}, \ldots, z_n^{q^s}]^{\mathrm{GL}(n,\ \mathbb{F})}. \end{aligned}$$

We first show that we can reduce the situation to $\mathcal{P}^*$-inseparably closed algebras[1].

[1] In the following lemma H^* does not need to be Noetherian.

LEMMA 7.3.1 : *Let* $\sqrt{\mathrm{H}^*}$ *contain integrally a fractal of the Dickson algebra* $\mathcal{D}^*(n)$*, say* $\mathcal{D}^*(n)^{q^s}$ *for some* $s \geq 0$*. Then* H^* *contains a fractal of the Dickson algebra, namely* $\mathcal{D}^*(n)^{q^l}$ *for some* $l \geq s \geq 0$*.*

PROOF : By assumption we have the following diagram

$$\begin{array}{ccc} \mathrm{H}^* & \hookrightarrow & \sqrt{\mathrm{H}^*} \\ & & \uparrow \\ & & \mathcal{D}^*(n)^{q^s} \end{array}$$

Since a certain p-th power of any element in $\sqrt{\mathrm{H}^*}$ is inside H^* we can find natural numbers $l_0, \ldots, l_{n-1}$ such that

$$\left(\mathbf{d}_{n,i}^{q^s}\right)^{q^{l_i}} \in \mathrm{H}^* \ \forall\ i = 0, \ldots, n-1.$$

If we choose $l \in \mathbb{N}$ to be the maximum of $s + l_0, \ldots, s + l_{n-1}$ then

$$\mathcal{D}^*(n)^{q^l} \subseteq \mathrm{H}^*$$

•

The set up of the Galois Embedding Theorem hands us a copy of the Dickson algebra in any $\mathcal{P}^*$-inseparably closed, integrally closed, integral domain. The next result shows that it is enough to assume that H^* is a $\mathcal{P}^*$-inseparably closed integral domain, to conclude that H^* contains a copy of the top Dickson class, which implies the Little Imbedding Theorem for the case of algebras of Krull dimension 1.

PROPOSITION 7.3.2 : *Let* H^* *be a Noetherian integral domain of Krull dimension* n*. Consider the embedding*

$$\mathrm{H}^* \hookrightarrow \mathbb{F}[V]$$

given by the Embedding Theorem 6.1.2.*Then* H^* *contains a* q*-th power of the top Dickson class* $\mathbf{d}_{n,\,0} \in \mathbb{F}[V]$*. If in addition* H^* *is* $\mathcal{P}^*$*-inseparably closed then*

$$\mathbf{d}_{n,0} \in \mathrm{H}^*.$$

PROOF : By Lemma 7.3.1 we need only consider $\mathcal{P}^*$-inseparably closed integral domains H^*. We proceed by induction on the Krull dimension n.

If $n = 0$, then $\mathrm{H}^* = \mathbb{F} = \mathcal{D}^*(0)$ and nothing needs to be shown.

If $n = 1$, then the Galois Embedding Theorem 7.1.1 hands us a copy of the Dickson algebra in the integral closure of H^*, $\overline{\mathrm{H}^*}$

$$\begin{array}{ccccc} & & \mathrm{H}^* & & \\ & & \downarrow & & \\ \mathbb{F}[x^{q-1}] & \hookrightarrow & \overline{\mathrm{H}^*} & \hookrightarrow & \mathbb{F}[x]. \end{array}$$

Let $x^{\alpha} \in \mathrm{H}^*$ for a minimal $\alpha \in \mathbb{N}$. Assume $\alpha > 1$. If $\alpha \mid q-1$ then there is nothing to show, because

$$\mathbf{d}_{1,0} = x^{q-1} = \left(x^{\alpha}\right)^{\frac{q-1}{\alpha}} \in \mathrm{H}^*.$$

If $p \mid \alpha$, then $x^{\frac{\alpha}{p}} \in \mathrm{H}^*$, as H^* is $\mathcal{P}^*$-inseparably closed, i.e., contains all p-th roots. This contradicts the minimality of α. So, assume $\alpha = kp + r$ for some $k,\ r \in \mathbb{N}_0$ and $0 < r < p$, $p \nmid r$. Then certainly

$$\mathbf{d}_{1,0}^{\alpha} = x^{(q-1)\alpha} = \left(x^{\alpha}\right)^{q-1} \in \mathrm{H}^*.$$

Choose an α minimal with respect to this property. We get, by Proposition A.2.1,

$$\mathcal{P}^{\Delta_1}\left(\mathbf{d}_{1,0}^{\alpha}\right) = \alpha \mathbf{d}_{1,0}^{\alpha+1} \in \mathrm{H}^*.$$

So applying $\mathcal{P}^{\Delta_1}$ $(p-r)$-times yields

$$\begin{aligned}\mathcal{P}^{\Delta_1}\cdots\mathcal{P}^{\Delta_1}\left(\mathbf{d}_{1,0}\alpha\right) &= \alpha(\alpha+1)\cdots(\alpha+p-r-1)\mathbf{d}_{1,0}^{(\alpha+p-r)}\\ &= \alpha(\alpha+1)\cdots(\alpha+p-r-1)\mathbf{d}_{1,0}^{(k+1)p}.\end{aligned}$$

By the choice of our α the coefficient does not vanish. Since H^* is $\mathcal{P}^*$-inseparably closed it follows that

$$\mathbf{d}_{1,0}^{k+1} = \mathbf{d}_{1,0}^{\frac{(k+1)p}{p}} \in \mathrm{H}^*.$$

However, this contradicts the minimality of α, because

$$k + 1 < r + kp = \alpha,$$

unless $\alpha = 1$. Hence $\mathbf{d}_{1,0} = x^{q-1} \in \mathrm{H}^*$.

For the general case, assume that $n > 1$. Take the embedding into a polynomial ring $\mathbb{F}[V]$ given by Theorem 6.1.2. Choose a linear form $l \in \mathbb{F}[V]$, and consider the projections

$$\begin{array}{ccc}\mathrm{H}^* & \hookrightarrow & \mathbb{F}[V]\\ \downarrow & & \downarrow\\ \mathrm{H}^*/((l)\cap\mathrm{H}^*) & \hookrightarrow & \mathbb{F}[W]\end{array}$$

where $\mathrm{span}_{\mathbb{F}}(W,\ l) = V$. The ideal $(l) \subset \mathbb{F}[V]$ is a $\mathcal{P}^*$-invariant prime of height one. Hence so is the contraction $(l) \cap \mathrm{H}^* \subset \mathrm{H}^*$, by Lemma 2.1 in [21]. Therefore $\mathbb{F}[W]$ and $\mathrm{H}^*/((l)\cap\mathrm{H}^*)$ are unstable integral domains of Krull dimension $n-1$, and by Proposition 4.3.6 both are again $\mathcal{P}^*$-inseparably closed. We apply the induction hypothesis and get that the top Dickson class in $\mathbb{F}[W]$ is also in $\mathrm{H}^*/((l)\cap\mathrm{H}^*)$

$$\mathbf{d}_{n-1,0}(l) \in \mathrm{H}^*/((l)\cap\mathrm{H}^*).$$

The pre image of $\mathbf{d}_{n-1,0}(l)$ under the projection is

$$\begin{aligned}
\mathrm{pr}^{-1}\left(\mathbf{d}_{n-1,0}(l)\right) \\
&= \left(\mathbf{d}_{n-1,0}(l)\right) + \left((l)\cap \mathrm{H}^*\right) \\
&\ni \left(\mathbf{d}_{n-1,0}(l)\right) + 0 \\
&= \prod_{v\in W^*\setminus\{0\}} v.
\end{aligned}$$

We play the same game for all linear forms $l \in \mathbb{F}[V]$ and multiply all the inverse images together we get:

$$\begin{aligned}
\prod_{l\in V^*\setminus 0} \mathbf{d}_{n-1,0}(l) &= \prod_{l\in V^*\setminus 0}\left(\prod_{v\in W^*,\ \mathrm{Span}_{\mathbb{F}}(W,\ l)=V} v\right) \\
&= \prod_{W^*<V^*,\ \dim_{\mathbb{F}}(W^*)=n-1}\left(\prod_{v\in W^*\setminus 0} v\right)^{q^n-q^{n-1}} \\
&= \left(\prod_{v\in V^*\setminus 0} v\right)^{\alpha} \quad \text{for some } \alpha \in \mathbb{N}_0,
\end{aligned}$$

where the last equation follows by symmetry. Hence we have that

$$\mathbf{d}_{n,0}^{\alpha} = \left(\prod_{v\in V^*\setminus 0} v\right)^{\alpha} \in \mathrm{H}^*,$$

for some $\alpha \in \mathbb{N}_0$. Without loss of generality assume α to be minimal with this property, and assume that $\alpha > 1$. Then p does not divide α, because otherwise

$$\mathbf{d}_{n,0}^{\frac{\alpha}{p}} \in \mathrm{H}^*$$

since H^* is $\mathcal{P}^*$-inseparably closed. We express α as a multiple of p with some remainder, i.e., let

$$\alpha = kp + r,$$

for some $k,\ r \in \mathbb{N}_0$, $0 < r < p$. We get

$$\mathcal{P}^{\Delta_n}(\mathbf{d}_{n,0}^{\alpha}) = \alpha\mathbf{d}_{n,0}^{\alpha+1},$$

where we used Proposition A.2.1. So, applying $\mathcal{P}^{\Delta_n}$ $(p-r)$-times to $\mathbf{d}_{n,0}^{\alpha}$ gives

$$\begin{aligned}
\mathcal{P}^{\Delta_n}\cdots\mathcal{P}^{\Delta_n}(\mathbf{d}_{n,0}^{\alpha}) &= \alpha(\alpha+1)\cdots(\alpha+r-1)\mathbf{d}_{n,0}^{\alpha+p-r} \\
&= \alpha(\alpha+1)\cdots(\alpha+r-1)\mathbf{d}_{n,0}^{(k+1)p}.
\end{aligned}$$

The coefficient is non zero modulo p by construction, so, since H^* is $\mathcal{P}^*$-inseparably closed, it follows that

$$\mathbf{d}_{n,0}^{k+1} \in H^*.$$

As $k+1 < \alpha$, unless $\alpha = 1$, this contradicts the minimality of α. Hence

$$\mathbf{d}_{n,0} \in H^*,$$

as claimed •

THEOREM 7.3.3 (Little Imbedding Theorem): *Let* H^* *be a Noetherian integral domain. Then there exists a natural number* $l \in \mathbb{N}$*, and an inclusion*

$$\mathcal{D}^*(n)^{q^l} \hookrightarrow H^*$$

which is an integral extension of unstable algebras over $\mathcal{P}^*$*. Moreover, if* H^* *is in addition* $\mathcal{P}^*$*-inseparably closed, then* $l = 0$*.*

PROOF : By Lemma 7.3.1 it suffices to consider the case where H^* is $\mathcal{P}^*$-inseparably closed.

Consider first a $\mathcal{P}^*$-inseparably closed integral domain H^* which is also integrally closed. Then we have, by the Galois Embedding Theorem 7.1.1, that H^* is a ring of invariants, i.e.,

$$\mathcal{D}^* \hookrightarrow H^* = \mathbb{F}[V]^G \hookrightarrow \mathbb{F}[V]$$

for some group G and nothing needs to be shown.

Consider the case of a $\mathcal{P}^*$-inseparably closed integral domain H^* which is not integrally closed. Denote by $\overline{H^*}$, the integral closure of H^*. From the above we can assume that there is a Dickson algebra $\mathcal{D}^*(n)$ inside $\overline{H^*}$. We have

$$\begin{array}{ccccc} \mathcal{D}^*(n) & \hookrightarrow & \overline{H^*} & \subseteq & \mathbb{F}[V] \\ \uparrow & & \uparrow & & \\ \mathcal{D}^*(n)\cap H^* & \hookrightarrow & H^* & & \end{array}$$

where every extension is finite and integral.

We want to show by induction on n that $\mathcal{D}^*(n) \cap H^* = \mathcal{D}^*(n)$. If $n = 0$ then our diagram looks like

$$\begin{array}{ccccc} \mathbb{F} & \hookrightarrow & \mathbb{F} & \subseteq & \mathbb{F} \\ \uparrow & & \uparrow & & \\ \mathbb{F}\cap H^* & \hookrightarrow & \mathbb{F} & & \end{array}$$

and this implies

$$\mathcal{D}^*(0) \cap H^* = \mathbb{F} \cap H^* = \mathbb{F} = \mathcal{D}^*(0)$$

because H^* is connected. For $n = 1$ the result is the contents of Proposition 7.3.2 So let's suppose that $n > 1$. Let $l \in \mathbb{F}[V]$ be a linear form.

Then by taking the quotient mod l in the above diagram leads to, by Proposition A.3.2,

$$\begin{array}{ccc} \mathcal{D}^*(n-1)^q & \hookrightarrow & \overline{\mathrm{H}^*} \big/ \left((l) \cap \overline{\mathrm{H}^*}\right) \quad \subseteq \mathbb{F}[W] \\ \uparrow & & \uparrow \\ \left(\mathcal{D}^*(n) \cap \mathrm{H}^*\right) / \left((l) \cap \mathcal{D}^*(n) \cap \mathrm{H}^*\right) & \hookrightarrow & \mathrm{H}^* / \left((l) \cap \mathrm{H}^*)\right) \end{array}$$

where $\mathrm{Span}_{\mathbb{F}}(W,\ l) = V$. By induction hypothesis we have

$$\mathcal{D}^*(n-1) = \mathcal{D}^*(n-1) \cap \left(\mathrm{H}^*/((l) \cap \mathrm{H}^*)\right),$$

since $\mathrm{H}^*/((l) \cap \mathrm{H}^*)$ is again an unstable $\mathcal{P}^*$-inseparably closed integral domain by Proposition 4.3.6. Therefore we get a diagram:

$$\begin{array}{ccccc} \mathcal{D}^*(n) & \xrightarrow{\mathrm{pr}} & \mathcal{D}^*(n-1)^q & \hookrightarrow & \overline{\mathrm{H}^*} \big/ \left((l) \cap \overline{\mathrm{H}^*}\right) \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{D}^*(n) \cap \mathrm{H}^* & \xrightarrow{\mathrm{pr}} & \left(\mathcal{D}^*(n) \cap \mathrm{H}^*\right) / \left((l) \cap \mathcal{D}^*(n) \cap \mathrm{H}^*\right) & \hookrightarrow & \mathrm{H}^* / \left((l) \cap \mathrm{H}^*\right). \end{array}$$

By induction

$$\begin{aligned} \mathcal{D}^*(n-1)^q &= \left(\mathcal{D}^*(n-1) \cap \mathrm{H}^*/((l) \cap \mathrm{H}^*)\right)^q \\ &\subseteq \mathcal{D}^*(n-1)^q \cap \left(\mathrm{H}^*/((l) \cap \mathrm{H}^*)\right), \end{aligned}$$

Therefore $\mathcal{D}^*(n-1)^q = \mathcal{D}^*(n-1)^q \cap \left(\mathrm{H}^*/((l) \cap \mathrm{H}^*)\right)$ and hence we have

$$\begin{aligned} &\left(\mathcal{D}^*(n) \cap \mathrm{H}^*\right) / \left((l) \cap \mathcal{D}^*(n) \cap \mathrm{H}^*\right) \\ &\qquad \subseteq \mathcal{D}^*(n-1)^q \\ &\qquad = \mathcal{D}^*(n-1)^q \cap \left(\mathrm{H}^*/((l) \cap \mathrm{H}^*)\right) \\ &\qquad = \left(\mathcal{D}^*(n)/\left((l) \cap \mathcal{D}^*(n)\right)\right) \cap \left(\mathrm{H}^*/((l) \cap \mathrm{H}^*)\right) \\ &\qquad \subseteq \left(\mathcal{D}^*(n) \cap \mathrm{H}^*\right) / \left((l) \cap \mathcal{D}^*(n) \cap \mathrm{H}^*\right). \end{aligned}$$

Therefore the middle map in the above diagram is surjective, i.e.,

$$\begin{array}{ccccc} \mathcal{D}^*(n) & \xrightarrow{\mathrm{pr}} & \mathcal{D}^*(n-1)^q & \hookrightarrow & \overline{\mathrm{H}^*} \big/ \left((l) \cap \overline{\mathrm{H}^*}\right) \\ \uparrow & & \uparrow\wr & & \uparrow \\ \mathcal{D}^*(n) \cap \mathrm{H}^* & \xrightarrow{\mathrm{pr}} & \left(\mathcal{D}^*(n) \cap \mathrm{H}^*\right) / \left((l) \cap \mathcal{D}^*(n) \cap \mathrm{H}^*\right) & \hookrightarrow & \mathrm{H}^* / \left((l) \cap \mathrm{H}^*\right). \end{array}$$

Choose $x \in \mathcal{D}^*(n)$ not in H^*, i.e., not in $\mathcal{D}^*(n) \cap \mathrm{H}^*$, of minimal positive degree. Then this element x must lie in the kernel of the projection map and therefore

$$x \in (\mathbf{d}_{n,0}) \in \mathcal{D}^*(n)$$

$$x = x' \mathbf{d}_{n,0} \in \mathcal{D}^*(n).$$

Since x is of minimal positive degree we get that $x' \in \mathrm{H}^*$. Hence

$$x = x' \mathbf{d}_{n,0} \in (\mathbf{d}_{n,0}) \cap \mathcal{D}^*(n) \cap \mathrm{H}^*,$$

because Proposition 7.3.2 tells us that the top Dickson class is in H^*. That's a contradiction •

COROLLARY 7.3.4: *An integral domain* H^* *contains a Thom class* $\mathbf{t}$.

PROOF: Obvious, since a large enough power of the top Dickson class is contained in the fractal of the Dickson algebra which we just found inside H^*, $\mathbf{d}_{n,0}^{q^l} \in \mathrm{H}^*$ •

We finish this chapter with an example of an integral domain H^* which is not $\mathscr{P}^*$-inseparably closed, but still contains a copy of the Dickson algebra, i.e., the converse of the second statement of the Little Imbedding Theorem, Theorem 7.3.3, is not true.

EXAMPLE 1: Consider the field $\mathbb{F} = \mathbb{F}_2$ with two elements, and take a polynomial algebra in two linear generators $x,\ y$, $\mathbb{F}[x,\ y]$. The Dickson algebra in this case is

$$\mathscr{D}^*(2) = \mathbb{F}[x^2y + xy^2,\ x^2 + y^2 + xy] \hookrightarrow \mathbb{F}[x,\ y].$$

Define the algebra H^* by

[redacted] $) + ($ [redacted]

[redacted] tegrally $\mathscr{D}^*$ [redacted]

[redacted] $0 \quad \forall\ i \geq 0,$

[redacted] $x + y \notin \mathrm{H}^*.$

[redacted] e level of fields of fractions separabl [redacted] rably closed fields:

[redacted] $(2))$ $\hookrightarrow$ $FF(\mathrm{H}^*)$ $\hookrightarrow$ $FF(\mathbb{F}[x,\ y])$

[redacted] $||$ $||$ $||$

[redacted] $xy^2,\ xy(x+y))$ $\hookrightarrow$ $\mathbb{F}(x+y,\ xy)$ $\hookrightarrow$ $\mathbb{F}(x,\ y).$

[redacted] n though the algebra H^* is not $\mathscr{P}^*$-inseparably closed, its field [redacted] is, and in this particular case even a copy of the Dickson [redacted]) sits integrally inside H^*. Finally note that the integral [redacted] s

$$\mathbb{F}[x + y,\ xy],$$

[redacted] bly closed, integrally closed, Noetherian integral do- [redacted] is Embedding Theorem, 7.1.1, predicts a ring of in- [redacted] the regular representation of /2 afforded by the

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathrm{GL}(2,\ \mathbb{F}).$$

[redacted]

CHAPTER 8

The Big Imbedding Theorem, Thom Classes, Turkish Delights II and the Reverse Landweber-Stong Conjecture

In this chapter we will drop the assumption that our algebra H^* is an integral domain, and we will show that one of the most important results of the preceding chapters remains true, namely, we can always find a fractal of the Dickson algebra in H^*.

From this we can conclude that any H^* contains a Thom class, which in turn leads to further gorgeous results about the $\mathscr{P}^*$-invariant prime spectrum of H^* (Turkish Delights II), and, to a tool for proofs by induction in this category. Finally, we will give a counter example to the Reverse Landweber-Stong Conjecture.

Throughout the whole chapter H^* remains Noetherian.

8.1 The Big Imbedding Theorem

In this section we prove that we can find a fractal of the Dickson algebra (integrally, of course,) in any unstable Noetherian algebra H^* even if we drop the condition on H^* to be an integral domain.

Recall that the Dickson algebra has the following fractal property (see Propositions A.3.1 and A.3.2 in the appendix)

$$\begin{array}{ccc} \mathscr{D}^*(n)^{q^l} & \hookrightarrow & \mathbb{F}[z_1, \dots, z_n] \\ \downarrow & & \downarrow \mathrm{pr} \\ \mathscr{D}^*(n-1)^{q^{l+1}} & \hookrightarrow & \mathbb{F}[z_2, \dots, z_n] \end{array}$$

where

$$\mathscr{D}^*(n-1)^{q^{l+1}} = \mathscr{D}^*(n)^{q^l} / \left((z_1) \cap \mathscr{D}^*(n)^{q^l}\right) = \mathscr{D}^*(n)^{q^l} / (\mathbf{d}_{n,0}^{q^l}).$$

First we are going to show that, we can restrict our attention to reduced algebras H^*.

The following lemma[1] is due to Larry Smith, [30].

LEMMA 8.1.1 : *Let* H^* *be Noetherian. If* $\mathrm{H}^*/\mathit{Nil}\,(\mathrm{H}^*)$ *contains integrally a fractal of the Dickson algebra* $\mathcal{D}^{*q^l}(n)$ *then so does* H^*.

PROOF : Consider the Frobenius map

$$\Phi : \mathrm{H}^* \longrightarrow \mathrm{H}^*,\ \ h \longmapsto h^q.$$

(Obviously this map does not preserve degrees, but this is no bother to us at the moment.) The image $\Phi(\mathrm{H}^*) = \mathrm{H}^{*q} \subseteq \mathrm{H}^*$ is a sub algebra. Moreover the Cartan formulae imply

$$\mathcal{P}^k(\Phi(h)) = \begin{cases} \Phi\left(\mathcal{P}^{\frac{k}{q}}(h)\right) & \text{if } q \mid k \\ 0 & \text{otherwise.} \end{cases}$$

So, in fact $\Phi(\mathrm{H}^*) \subseteq \mathrm{H}^*$ is an unstable sub algebra. Since H^* is Noetherian the nil radical $\mathit{Nil}\,(\mathrm{H}^*)$ is a nilpotent ideal, i.e., there is an integer $s \in \mathbb{N}_0$ such that

$$\left(\mathit{Nil}\,(\mathrm{H}^*)\right)^s = (0).$$

Choose $s = q^t$, so that Φ^t annihilates $\mathit{Nil}\,(\mathrm{H}^*)$ and therefore $\Phi^t(\mathrm{H}^*)$ has no nilpotent elements.

Returning to our problem, take the fractal of the Dickson algebra inside $\mathrm{H}^*/\mathit{Nil}\,(\mathrm{H}^*)$

$$\mathcal{D}^{*q^l}(n) \subseteq \mathrm{H}^*/\mathit{Nil}\,(\mathrm{H}^*)$$

pull it back to H^* via the canonical projection map

$$\mathcal{D}^{*q^l}(n) + \mathit{Nil}\,(\mathrm{H}^*) = \mathrm{pr}^{-1}\left(\mathcal{D}^{*q^l}(n)\right) \subseteq \mathrm{H}^*$$

and push it forward[2] to H^{*q^t} via the iterated Frobenius Φ^t

$$\Phi^t\left(\mathcal{D}^{*q^l}(n) + \mathit{Nil}\,(\mathrm{H}^*)\right) \subseteq \mathrm{H}^{*q^t}.$$

Since Φ^t is an additive map we get

$$\mathcal{D}^{*q^{l+t}}(n) = \Phi^t\left(\mathcal{D}^{*q^l}(n) + \mathit{Nil}\,(\mathrm{H}^*)\right) \subseteq \mathrm{H}^{*q^t} \subseteq \mathrm{H}^*$$

and that's what we wanted •

---

[1] The following lemma remains true if we replace Noetherian by Δ-finite with finitely generated nil radical.

[2] Surely, the Dickson algebra doesn't enjoy being pulled in the one direction and pushed in the other. However, that's how life is.

This lemma allows us to restrict our attention to unstable algebras H^* with trivial nil radical $Nil(H^*) = (0)$.

By Lemma 7.3.1 we can also assume without loss of generality that in addition H^* is $\mathcal{P}^*$-inseparably closed.

Next we show, for a fixed q-th power, that an integrally embedded fractal of the Dickson algebra inside H^* is unique. For the next two results we need that H^* is reduced.[3]

PROPOSITION 8.1.2 : *Let H^* be reduced and Noetherian . If H^* contains integrally the Dickson algebra, $\mathcal{D}^*(n)$, and $n = \dim(H^*)$, then the Δ-relation for H^* is*

$$(-1)^n \mathbf{d}_{n,\,0}\mathcal{P}^{\Delta_0} + \cdots + (-1)\mathbf{d}_{n,\,n-1}\mathcal{P}^{\Delta_{n-1}} + \mathcal{P}^{\Delta_n} = 0.$$

PROOF : By Corollary 1.2.2 we have

$$n = m(\mathcal{D}^*(n)) \leq m(H^*) \leq \dim(H^*) = n,$$

hence $m(H^*) = n$, i.e., by Corollary 1.1.7 there are elements $h_0, \ldots, h_n$ such that

$$h_0\mathcal{P}^{\Delta_0} + \cdots + h_n\mathcal{P}^{\Delta_n} = 0.$$

By minimality we must have

$$h_i = (-1)^{n-i} h_n \mathbf{d}_{n,\,i} \ \forall\ i = 0, \ldots, n-1,$$

i.e.,

$$h_n\left((-1)^n \mathbf{d}_{n,\,i}\mathcal{P}^{\Delta_0} + \cdots + (-1)\mathbf{d}_{n,\,n-1}\mathcal{P}^{\Delta_{n-1}} + \mathcal{P}^{\Delta_n}\right) = 0.$$

Since by assumption $\mathcal{D}^*(n) \hookrightarrow H^*$ is an integral extension, it follows that for any prime ideal $\mathfrak{p} \subset H^*$ of height zero

$$(-1)^n \mathbf{d}_{n,\,i}\mathcal{P}^{\Delta_0} + \cdots + (-1)\mathbf{d}_{n,\,n-1}\mathcal{P}^{\Delta_{n-1}} + \mathcal{P}^{\Delta_n} = 0$$

on $H^*/\mathfrak{p}$, i.e., for any $\mathfrak{p} \in Proj(H^*)$ of height zero

$$(-1)^n \mathbf{d}_{n,\,i}\mathcal{P}^{\Delta_0} + \cdots + (-1)\mathbf{d}_{n,\,n-1}\mathcal{P}^{\Delta_{n-1}} + \mathcal{P}^{\Delta_n} \in \mathfrak{p}.$$

Therefore

$$(-1)^n \mathbf{d}_{n,\,i}\mathcal{P}^{\Delta_0} + \cdots + (-1)\mathbf{d}_{n,\,n-1}\mathcal{P}^{\Delta_{n-1}} + \mathcal{P}^{\Delta_n} \in \bigcap_{\mathfrak{p}\in Proj(H^*),\ ht(\mathfrak{p})=0} \mathfrak{p} = Nil(H^*) = (0),$$

which was to be shown •

[3] But note that we could replace the assumtion that H^* is Noetherian with H^* is Δ-finite with finitely generated nil radical.

REMARK: Let H^* be reduced. Note that we now know that there is an integral extension $\mathcal{D}^*(n) \hookrightarrow \mathrm{H}^*$ if and only if on H^* the following Δ-relation holds

$$(-1)^n \mathbf{d}_{n,0}\mathcal{P}^{\Delta_0} + \cdots + (-1)\mathbf{d}_{n,n-1}\mathcal{P}^{\Delta_{n-1}} + \mathcal{P}^{\Delta_n} = 0.$$

The "if"-part follows from Theorem 5.1.8. The "only if"-part is given by the above proposition.

PROPOSITION 8.1.3: *Let H^* be reduced and Noetherian of Krull dimension n. If H^* contains integrally a copy of the Dickson algebra $\mathcal{D}^*(n)$ then the H^*-module $\Delta(\mathrm{H}^*)$ of derivations defined in Section*1.2 *is given by*

$$\Delta(\mathrm{H}^*) = \bigoplus_{i=0}^{n-1} \mathrm{H}^*\mathcal{P}^{\Delta_i}.$$

In particular it is a free module on n generators.

PROOF: By Proposition 8.1.2 the Δ-relation for H^* is

$$(-1)^n \mathbf{d}_{n,\,0}\mathcal{P}^{\Delta_0} + \cdots + (-1)\mathbf{d}_{n,\,n-1}\mathcal{P}^{\Delta_{n-1}} + \mathcal{P}^{\Delta_n} = 0.$$

Therefore

$$\mathcal{P}^{\Delta_n} \in \bigoplus_{i=0}^{n-1} \mathrm{H}^*\mathcal{P}^{\Delta_i}.$$

Applying Lemma **???** yields

$$\mathcal{P}^{\Delta_n+r} \in \bigoplus_{i=r}^{n+r-1} \mathrm{H}^*\mathcal{P}^{\Delta_i}$$

for any $r \in \mathbb{N}_0$. Hence inductively we get

$$\mathcal{P}^{\Delta_n+r} \in \bigoplus_{i=0}^{n-1} \mathrm{H}^*\mathcal{P}^{\Delta_i}$$

for all $r \in \mathbb{N}_0$ •

The preceding proposition applies, e.g., to $\mathcal{P}^*$-inseparably closed, reduced, Noetherian algebras H^*, as the Big Imbedding Theorem will show, 8.1.8.

The uniqueness of an integrally embedded Dickson algebra (or a fractal thereof) is now easy.

THEOREM 8.1.4: *Let H^* be reduced and Noetherian. For $i = 1, 2$ denote by $\mathcal{D}^*{}_i(n)^{q^s}$ a fractal of the Dickson algebra of Krull dimension n and let*

$$\varphi_i : \mathcal{D}^*{}_i(n)^{q^s} \hookrightarrow \mathrm{H}^*$$

be integral extensions. Then

$$\mathcal{D}^*{}_1(n)^{q^s} = \mathcal{D}^*{}_2(n)^{q^s}.$$

PROOF : We consider first the case where $s = 0$, i.e., we have Dickson algebras inside H^*. Denote by $\mathbf{d}_{n,\,j}(i)$ the Dickson classes of $\mathcal{D}^*{}_i(n)$ for $i = 1,\ 2$ and $j = 0, \ldots, n-1$. Proposition 8.1.2 leads to two Δ-relations on H^*, namely

$$(-1)^n \mathbf{d}_{n,\,0}(i)\mathcal{P}^{\Delta_0} + \cdots + (-1)\mathbf{d}_{n,\,n-1}(i)\mathcal{P}^{\Delta_{n-1}} + \mathcal{P}^{\Delta_n} = 0$$

for $i = 1,\ 2$. By minimality of $n = m(H^*)$ the coefficients must coincide.

Next we consider the general case, where $s > 0$. We embed H^* into its $\mathcal{P}^*$-inseparable closure $\sqrt{H^*}$. We compose this embedding with the maps φ_i, $i = 1,\ 2$

$$\mathcal{D}^*{}_i(n)^{q^s} \overset{\varphi_i}{\hookrightarrow} H^* \hookrightarrow \sqrt{H^*}, \quad i = 1,\ 2.$$

Since $\sqrt{H^*}$ is $\mathcal{P}^*$-inseparably closed, the $\mathcal{P}^*$-inseparable closure, i.e., the Dickson algebra itself, of $\mathcal{D}^*{}_1(n)^{q^s}$, resp. of $\mathcal{D}^*{}_2(n)^{q^s}$, is contained in $\sqrt{H^*}$, so we get integral extensions

$$\Phi_i : \mathcal{D}^*{}_i(n) \hookrightarrow \sqrt{H^*}, \quad i = 1,\ 2.$$

By the first part of this proof, these two copies of the Dickson algebra must coincide

$$\mathcal{D}^*{}_1(n) = \mathcal{D}^*{}_2(n) \subset \sqrt{H^*}.$$

Hence corresponding fractals of them are equal, and in particular

$$\mathcal{D}^*{}_1(n)^{q^s} = \mathcal{D}^*{}_2(n)^{q^s} \subset H^*.$$

That's what we wanted •

REMARK : If we drop the condition on H^* that $Nil\,(H^*) = (0)$, we get that for any pair of fractals of the Dickson algebra $\mathcal{D}^{*q^{s_i}}_{i}(n)$, $i = 1,\ 2$, large enough q-th powers coincide

$$\mathcal{D}^{*q^l}_{1}(n) = \mathcal{D}^{*q^l}_{2}(n)$$

for an $l \geq s_1,\ s_2$. To be precise: suppose without loss of generality $s = s_1 \geq s_2$. Then the preceding theorem tells us that

$$\mathcal{D}^{*q^s}_{1}(n) = \mathcal{D}^{*q^s}_{2}(n) \subset H^*/Nil\,(H^*).$$

Let $l \in \mathbb{N}_0$, $l \geq s$, such that[4]

$$Nil\,(H^*)^{q^l} = (0).$$

Then pulling back the q^l-th fractals of the two Dickson algebras to H^* leads to

$$\mathcal{D}^{*q^l}_{1}(n) = \mathcal{D}^{*q^l}_{2}(n) + Nil\,(H^*)^{q^l} = \mathcal{D}^{*q^l}_{2}(n) \subset H^*$$

[4] Again, if H^* is not Noetherian, we would need here that its nil radical is finitely generated.

as claimed.

So, we assume that H^* is reduced $\mathscr{P}^*$-inseparably closed[5]. Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$ be the set of all height zero prime ideals of H^*, then

$$(0) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k.$$

Recall from the proof of Theorem 6.3.1 the Lam-Rector Embedding

$$L := \oplus_{i=1}^k \mathrm{pr}_i : \mathrm{H}^* \hookrightarrow \bigoplus_{i=1}^k \mathrm{H}^*/\mathfrak{p}_i.$$

The k factors on the right hand side are unstable integral domains of the same Krull dimension n as H^* and, by Proposition 4.3.6, are $\mathscr{P}^*$-inseparably closed. Therefore the Little Imbedding Theorem, Theorem 7.3.3, tells us that there are Dickson algebras in these factors

$$\mathscr{D}^*{}_i(n) \subseteq \mathrm{H}^*/\mathfrak{p}_i, \ \forall\ i = 1, \dots, k.$$

Hence we have

$$\begin{array}{cccc} L: & \mathrm{H}^* & \hookrightarrow & \bigoplus_{i=1}^k \mathrm{H}^*/\mathfrak{p}_i \\ & & & \big\uparrow \\ & & & \bigoplus_{i=1}^k \mathscr{D}^*{}_i(n), \end{array}$$

and what we are looking for is a common lift of all these Dickson algebras to a Dickson algebra in H^*. Denote, in analogy to the notation used above, $\mathbf{d}_{n,0}(i) \in \mathscr{D}^*{}_i(n)$ the top Dickson class in the Dickson algebra $\mathscr{D}^*{}_i(n) \subset \mathrm{H}^*/\mathfrak{p}_i$.

LEMMA 8.1.5: *With the above notation we have that*

$$\mathbf{d}_{n,0}^{q^t}(j) \in \bigcap_{i=1,\ i\neq j}^{k} \mathfrak{p}_i$$

for a suitably large $t \in \mathbb{N}_0$.

PROOF: Since we have integral extensions

$$\mathscr{D}^*{}_j(n) \hookrightarrow \mathrm{H}^*/\mathfrak{p}_j$$

any $\mathscr{P}^*$-invariant prime ideal $\mathfrak{p} \subset \mathrm{H}^*/\mathfrak{p}_j$ contracts back to a $\mathscr{P}^*$-invariant prime ideal, $\mathfrak{p} \cap \mathscr{D}^*{}_j(n)$, in the Dickson algebra, by Lemma 2.1 in [21]. Therefore for a suitable $l \in \{1, \dots, n-1\}$

$$\mathfrak{p} \cap \mathscr{D}^*{}_j(n) = \big(\mathbf{d}_{n,0}(j), \dots, \mathbf{d}_{n,l}(j)\big) \subset \mathscr{D}^*{}_j(n)$$

5 If H^* is neither reduced nor $\mathscr{P}^*$-inseparably closed, we can first mod out the nil radical and then take the inseparable closure, to make sure that our algebra is still Noetherian, compare Proposition 6.3.1, or we take the $\mathscr{P}^*$-inseparable closure and divide out its nil radical, compare Corollary 6.3.2.

by a result of Peter S. Landweber, [28] Theorem 11.4.6. In particular our top Dickson class $\mathbf{d}_{n,0}(j)$ is an element of any nontrivial $\mathcal{P}^*$-invariant prime ideal

$$\mathbf{d}_{n,0}(j) \in \mathfrak{p} \cap \mathcal{D}^*{}_j(n) \subseteq \mathfrak{p}.$$

This means that for any nontrivial $\mathcal{P}^*$-invariant radical ideal $I = \sqrt{I} \subset \mathrm{H}^*/\mathfrak{p}_j$

$$\mathbf{d}_{n,0}(j) \in \left(\sqrt{I}\right) \cap \mathcal{D}^*{}_j(n).$$

This in turn implies that for any nontrivial $\mathcal{P}^*$-invariant ideal $I \subset \mathrm{H}^*/\mathfrak{p}_j$ there is an integer $t = t(I) \in \mathbb{N}_0$ such that

$$\left(\mathbf{d}_{n,0}(j)\right)^{q^t} \in I \cap \mathcal{D}^*{}_j(n) \subset \mathcal{D}^*{}_j(n) \hookrightarrow \mathrm{H}^*/\mathfrak{p}_j.$$

In particular, for $t_j = \max\{t(\mathfrak{p}_0), \ldots, \widehat{t(\mathfrak{p}_j)}, \ldots, t(\mathfrak{p}_k)\}$,

$$\left(\mathbf{d}_{n,0}(j)\right)^{q^{t_j}} \in \mathfrak{p}_i/\mathfrak{p}_j \subset \mathrm{H}^*/\mathfrak{p}_j$$

for any $i = 1, \ldots, k,\ i \neq j$, because when $\mathfrak{p}_i$ is $\mathcal{P}^*$-invariant so is $\mathfrak{p}_i/\mathfrak{p}_j$. And therefore

$$\left(\mathbf{d}_{n,0}(j)\right)^{q^{t_j}} \in \bigcap_{i=1,\ i\neq j}^{k} \mathfrak{p}_i/\mathfrak{p}_j \subset \mathrm{H}^*/\mathfrak{p}_j,$$

i.e.,

$$\left(\mathbf{d}_{n,0}(j)\right)^{q^{t_j}} \in \bigcap_{i=1,\ i\neq j}^{k} \mathfrak{p}_i \subset \mathrm{H}^*$$

as we claimed •

REMARK: Note that $\mathbf{d}_{n,0}^{q^{t_j}}(j)$ generates a height zero $\mathcal{P}^*$-invariant principal ideal in H^*, because it is an element in all but one height zero prime ideal of H^*. In particular it follows that $\mathbf{d}_{n,0}^{q^{t_j}}(j)$ is a "universal" zero divisor in H^* for any $j = 0, \ldots, k$. [6]

Let's have an example of this.

[6] Of course, all this looks a bit odd if H^* is an integral domain. So to make sense out of this, you should keep in mind that we are looking at algebras with zero divisors, so there are really nontrivial prime ideals of height zero.

EXAMPLE 1: Let

$$\mathrm{H}^* = \mathbb{F}[x,\ y]/(x\,y)$$

be the $\mathbb{F}$-algebra generated by two linear forms x and y with the single relation $x\,y = 0$. The Lam-Rector Embedding gives

$$\begin{array}{cccc} L: & \mathbb{F}[x,\ y]/(x\,y) & \hookrightarrow & \mathbb{F}[x] \oplus \mathbb{F}[y] \\ & & & \cup \\ & & & \mathbb{F}[x^{q-1}] \oplus \mathbb{F}[y^{q-1}] \end{array}$$

where $\mathbb{F}[x^{q-1}] = \mathscr{D}^*{}_1(1)$ and $\mathbb{F}[y^{q-1}] = \mathscr{D}^*{}_2(1)$. Then certainly

$$x^{q-1},\ \text{resp. } y^{q-1} \in \mathbb{F}[x,\ y]/(x\,y)$$

are zero divisors, and generate $\mathcal{P}^*$-invariant principal ideals. In this explicit case it is easily verified that

$$\begin{array}{cccc} L: & \mathbb{F}[x,\ y]/(x\,y) & \hookrightarrow & \mathbb{F}[x] \oplus \mathbb{F}[y] \\ & \cup & & \cup \\ & \mathbb{F}[x^{q-1} + y^{q-1}] & \hookrightarrow & \mathbb{F}[x^{q-1}] \oplus \mathbb{F}[y^{q-1}] \end{array}$$

completes our diagram in the desired way, i.e., $\mathbb{F}[x^{q-1} + y^{q-1}]$ is a Dickson algebra sitting integrally inside $\mathbb{F}[x,\ y]/(x\,y)$.

$$x^{2(q-1)} - (x^{q-1} + y^{q-1})x^{q-1} = 0$$

and

$$y^{2(q-1)} - (x^{q-1} + y^{q-1})\,y^{q-1} = 0$$

are the respective integral relations for x and y over $\mathbb{F}[x^{q-1} + y^{q-1}]$. Note carefully that the common lift of the top (and here only) Dickson class is exactly the coefficient of the Δ-relation, i.e.

$$(x^{q-1} + y^{q-1})\mathscr{P}^{\Delta_0} + \mathscr{P}^{\Delta_1} = 0$$

on H^*, as it should be.

This example extends to the general case of Krull dimension 1. Continuing to use the above notation we have:

PROPOSITION 8.1.6: *Let* H^* *be Noetherian of Krull dimension 1. Then* H^* *contains a fractal* $\mathscr{D}^{*q^s}(1)$ *of the Dickson algebra of Krull dimension 1. If*

$$L : \mathrm{H}^*/Nil\,(\mathrm{H}^*) \hookrightarrow \bigoplus_{\mathfrak{p},\ ht(\mathfrak{p})=0} \mathrm{H}^*/\mathfrak{p}$$

is the Lam-Rector Embedding, then

$$\mathscr{D}^{*q^s}(1) \cong \mathbb{F}[\mathbf{d}_{1,0}^{q^s}(1) + \cdots + \mathbf{d}_{1,0}^{q^s}(k)]$$

for a suitably large $s \in \mathbb{N}_0$*. If* H^* *is* $\mathscr{P}^*$*-inseparably closed then* $s = 0$*.*

PROOF: By Lemma 8.1.1 and Lemma 7.3.1 it is no restriction to assume that H^* is reduced and $\mathcal{P}^*$-inseparably closed. Let's consider, as above, the Lam-Rector Embedding

$$\begin{array}{lccc} L: & \mathrm{H}^* & \hookrightarrow & \bigoplus_{i=1}^{k} \mathrm{H}^*/\mathfrak{p}_i \\ & \uparrow & & \uparrow \\ & \mathbb{F}[\mathbf{d}_{1,0}^{q^s}(1)+\cdots+\mathbf{d}_{1,0}^{q^s}(k)] & \hookrightarrow & \bigoplus_{i=1}^{k} \mathbb{F}[\mathbf{d}_{1,0}^{q^s}(i)] \end{array}$$

where, with the preceding notations, we choose the integer s such that

$$s = \max\{t_j \mid j = 1, \ldots, k\}.$$

We have to show that the obviously given isomorphism (*in the category of* $\mathbb{F}$-*algebras*)

$$\begin{array}{llcl} \varphi: & \mathbb{F}[\mathbf{d}_{1,0}^{q^s}(1)+\cdots+\mathbf{d}_{1,0}^{q^s}(k)] & \xrightarrow{\sim} & \mathbb{F}[\mathbf{d}_{1,0}^{q^s}] = \mathcal{D}^*(1)^{q^s} \\ & \mathbf{d}_{1,0}^{q^s}(1)+\cdots+\mathbf{d}_{1,0}^{q^s}(k) & \longmapsto & \mathbf{d}_{1,0}^{q^s} \end{array}$$

is an isomorphism in the category of unstable algebras, i.e., we have to show that

$$\mathcal{P}^r\left(\varphi\left(\mathbf{d}_{1,0}^{q^s}(1)+\cdots+\mathbf{d}_{1,0}^{q^s}(k)\right)\right) = \varphi\left(\mathcal{P}^r\left(\mathbf{d}_{1,0}^{q^s}(1)+\cdots+\mathbf{d}_{1,0}^{q^s}(k)\right)\right)$$

for any $r \geq 0$. We do so by induction on r. For $r = 0$ there is nothing to show. So let $r > 0$. Then we have[7]

$$\begin{aligned} &\mathcal{P}^r\left(\mathbf{d}_{1,0}^{q^s}(1)+\cdots+\mathbf{d}_{1,0}^{q^s}(k)\right) \\ &\quad= \mathcal{P}^r\left(\mathbf{d}_{1,0}^{q^s}(1)\right)+\cdots+\mathcal{P}^r\left(\mathbf{d}_{1,0}^{q^s}(k)\right) \\ &\quad= \left(\mathcal{P}^{\frac{r}{q^s}}(\mathbf{d}_{1,0}(1))\right)^{q^s}+\cdots+\left(\mathcal{P}^{\frac{r}{q^s}}(\mathbf{d}_{1,0}(k))\right)^{q^s} \\ &\quad= -\left(\mathcal{P}^{\frac{r}{q^s}-1}\left(\mathbf{d}_{1,0}(1)\right)\mathbf{d}_{1,\,0}(1)+\cdots+\mathcal{P}^{\frac{r}{q^s}-1}\left(\mathbf{d}_{1,0}(k)\right)\mathbf{d}_{1,\,0}(k)\right)^{q^s} \\ &\quad= -\left(\mathcal{P}^{r-q^s}\left(\mathbf{d}_{1,0}^{q^s}(1)\right)\mathbf{d}_{1,\,0}^{q^s}(1)+\cdots+\mathcal{P}^{r-q^s}\left(\mathbf{d}_{1,0}^{q^s}(k)\right)\mathbf{d}_{1,\,0}^{q^s}(k)\right) \end{aligned}$$

where we made use of Proposition A.2.1 in the appendix. So what's left to show is that for all $r > 0$

$$\begin{aligned} &\mathcal{P}^r(\mathbf{d}_{1,0}^{q^s}(1))\mathbf{d}_{1,\,0}^{q^s}(1)+\cdots+\mathcal{P}^r(\mathbf{d}_{1,0}^{q^s}(k))\mathbf{d}_{1,\,0}^{q^s}(k) = \\ &\qquad\qquad \mathcal{P}^r\left(\mathbf{d}_{1,0}^{q^s}(1)+\cdots+\mathbf{d}_{1,0}^{q^s}(k)\right)\left(\mathbf{d}_{1,0}^{q^s}(1)+\cdots+\mathbf{d}_{1,0}^{q^s}(k)\right). \end{aligned}$$

[7] Recall the convention that $\mathcal{P}^i \equiv 0$ if $i \notin \mathbb{N}_0$.

Expand the right hand side to get

$$
\begin{aligned}
&\mathscr{P}^r\left(\mathbf{d}_{1,0}^{q^s}(1)+\cdots+\mathbf{d}_{1,0}^{q^s}(k)\right)\cdot\left(\mathbf{d}_{1,0}^{q^s}(1)+\cdots+\mathbf{d}_{1,0}^{q^s}(k)\right)\\
&\quad=\left(\sum_{i=1}^{k}\mathscr{P}^r\left(\mathbf{d}_{1,0}^{q^s}(i)\right)\mathbf{d}_{1,0}^{q^s}(i)\right)\\
&\qquad\quad+\left(\sum_{i=1}^{k}\mathscr{P}^r\left(\mathbf{d}_{1,0}^{q^s}(i)\right)\left(\mathbf{d}_{1,0}^{q^s}(1)+\cdots+\widehat{\mathbf{d}_{1,0}^{q^s}(i)}+\cdots+\mathbf{d}_{1,0}^{q^s}(k)\right)\right)\\
&\quad=\left(\sum_{i=1}^{k}\mathscr{P}^r\left(\mathbf{d}_{1,0}^{q^s}(i)\right)\mathbf{d}_{1,0}^{q^s}(i)\right)+\left(\sum_{i=1}^{k}\sum_{j=1,j\neq i}^{k}\mathscr{P}^r\left(\mathbf{d}_{1,0}^{q^s}(i)\right)\left(\mathbf{d}_{1,0}^{q^s}(j))\right)\right)\\
&\quad=\left(\sum_{i=1}^{k}\mathscr{P}^r\left(\mathbf{d}_{1,0}^{q^s}(i)\right)\mathbf{d}_{1,0}^{q^s}(i)\right)+0
\end{aligned}
$$

where the last equation follows from Lemma 8.1.5, and

$$
\begin{aligned}
&\mathscr{P}^r\left(\mathbf{d}_{1,0}^{q^s}(i)\right)\left(\mathbf{d}_{1,0}^{q^s}(j)\right)\\
&\quad\in\left(\mathfrak{p}_1\cap\cdots\cap\widehat{\mathfrak{p}_i}\cap\cdots\cap\mathfrak{p}_k\right)\cdot\left(\mathfrak{p}_1\cap\cdots\cap\widehat{\mathfrak{p}_j}\cap\cdots\cap\mathfrak{p}_k\right)\\
&\quad\subseteq\left(\mathfrak{p}_1\cap\cdots\cap\widehat{\mathfrak{p}_i}\cap\cdots\cap\mathfrak{p}_k\right)\cap\left(\mathfrak{p}_1\cap\cdots\cap\widehat{\mathfrak{p}_j}\cap\cdots\cap\mathfrak{p}_k\right)\\
&\quad=\left(\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_k\right),\quad\text{for } i\neq j\\
&\quad=0
\end{aligned}
$$

by assumption. Hence putting this together gives

$$
\begin{aligned}
&\varphi\left(\mathscr{P}^r\left(\mathbf{d}_{1,0}^{q^s}(1)+\cdots+\mathbf{d}_{1,0}^{q^s}(k)\right)\right)\\
&\quad=\varphi\left(-\left(\mathscr{P}^{r-q^s}\left(\mathbf{d}_{1,0}^{q^s}(1)\right)\mathbf{d}_{1,\ 0}^{q^s}(1)+\cdots+\mathscr{P}^{r-q^s}\left(\mathbf{d}_{1,0}^{q^s}(k)\right)\mathbf{d}_{1,\ 0}^{q^s}(k)\right)\right)\\
&\quad=\varphi\left(-\mathscr{P}^{r-q^s}\left(\mathbf{d}_{1,0}^{q^s}(1)+\cdots+\mathbf{d}_{1,0}^{q^s}(k)\right)\left(\mathbf{d}_{1,0}^{q^s}(1)+\cdots+\mathbf{d}_{1,0}^{q^s}(k)\right)\right)\\
&\quad=\varphi\left(-\mathscr{P}^{r-q^s}\left(\mathbf{d}_{1,0}^{q^s}(1)+\cdots+\mathbf{d}_{1,0}^{q^s}(k)\right)\right)\varphi\left(\mathbf{d}_{1,0}^{q^s}(1)+\cdots+\mathbf{d}_{1,0}^{q^s}(k)\right)\\
&\quad=-\mathscr{P}^{r-q^s}\left(\varphi\left(\mathbf{d}_{1,0}^{q^s}(1)+\cdots+\mathbf{d}_{1,0}^{q^s}(k)\right)\right)\varphi\left(\mathbf{d}_{1,0}^{q^s}(1)+\cdots+\mathbf{d}_{1,0}^{q^s}(k)\right)\\
&\quad=-\mathscr{P}^{r-q^s}\left(\mathbf{d}_{1,0}^{q^s}\right)\mathbf{d}_{1,0}^{q^s}\\
&\quad=\mathscr{P}^r\left(\mathbf{d}_{1,0}^{q^s}\right)\\
&\quad=\mathscr{P}^r\left(\varphi\left(\mathbf{d}_{1,0}^{q^s}(1)+\cdots+\mathbf{d}_{1,0}^{q^s}(k)\right)\right).
\end{aligned}
$$

Hence we have an isomorphism in the category of unstable algebras

$$\mathbb{F}[\mathbf{d}_{1,0}^{q^s}(1)+\cdots+\mathbf{d}_{1,0}^{q^s}(k)]\cong\mathscr{D}^*(1)^{q^s}\subset\mathrm{H}^*.$$

Since H^* is $\mathcal{P}^*$-inseparably closed, it contains also the $\mathcal{P}^*$-inseparable closure of $\mathcal{D}^*(1)^{q^s}$, i.e.,

$$\mathcal{D}^*(1) = \sqrt{\mathcal{D}^*(1)^{q^s}} \subset \sqrt{H^*} = H^*.$$

By construction the extension is also integral. That's what we claimed •

One might be tempted to try to generalize Proposition 8.1.6 by taking just the sum of the *bottom* Dickson classes $\mathbf{d}^{q^s}_{n,n-1}(i)$, for $i = 1, \ldots, k$, in H^* and its appropriate Steenrod powers to produce a Dickson algebra inside H^*. This won't work as the following example shows.

EXAMPLE 2 : Let x_1, x_2, x_3 be forms of degree one, $\mathbb{F} = \mathbb{F}_2$ the field with two elements and let $H^* = \mathbb{F}[x_1, x_2, x_3]/(x_1x_2x_3)$. Then we apply the Lam-Rector Embedding and we get

$$\begin{array}{rccc} L: & \mathbb{F}[x_1, x_2, x_3]/(x_1x_2x_3) & \hookrightarrow & \mathbb{F}[x_2, x_3] \oplus \mathbb{F}[x_1, x_3] \oplus \mathbb{F}[x_1, x_2] \\ & & & \uparrow \\ & & & \mathcal{D}^*{}_1(2) \oplus \mathcal{D}^*{}_2(2) \oplus \mathcal{D}^*{}_3(2) \end{array}$$

where for pairwise distinct i, j, k we have

$$\mathbf{d}_{2,0}(i) = x_j^2 x_k + x_j x_k^2$$
$$\mathbf{d}_{2,1}(i) = x_j^2 + x_k^2 + x_j x_k.$$

The sum of the bottom Dickson classes

$$\begin{aligned} D_1 := {} & \mathbf{d}_{2,1}(1) + \mathbf{d}_{2,1}(2) + \mathbf{d}_{2,1}(3) \\ = {} & x_1x_2 + x_1x_3 + x_2x_3 \\ \in {} & H^* \end{aligned}$$

maps under $\mathcal{P}^1$ to the sum of the top Dickson classes, as it should in a proper Dickson algebra of Krull dimension 2 over the field with two elements:

$$\begin{aligned} \mathcal{P}^1(D_1) = {} & x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 \\ = {} & \mathbf{d}_{2,0}(1) + \mathbf{d}_{2,0}(2) + \mathbf{d}_{2,0}(3) \\ = {} & D_0. \end{aligned}$$

The two elements D_0, D_1, so defined, even form a polynomial sub algebra in H^*, because they are algebraically independent as the short calculation of the determinant of the generalized Jacobian determinant shows:

$$\det \begin{bmatrix} \mathcal{P}^{\Delta_0}(D_0) & \mathcal{P}^{\Delta_0}(D_1) \\ \mathcal{P}^{\Delta_1}(D_0) & \mathcal{P}^{\Delta_1}(D_1) \end{bmatrix} = \det \begin{bmatrix} D_0 & 2D_1 \\ 0 & D_0 \end{bmatrix} = D_0^2,$$

which is not a zero divisor in H^*. However $\mathbb{F}[D_0, D_1] \subseteq \mathrm{H}^*$ is *not* closed under the action of the Steenrod algebra. Consider e.g.

$$\mathcal{P}^2(D_0) = x_1^4 x_2 + x_1 x_2^4 + x_1^4 x_3 + x_1 x_3^4 + x_2^4 x_3 + x_2 x_3^4$$

which is not expressible[8] as a polynomial in D_0 and D_1.

We come to one of the main results of this chapter. Consider a $\mathscr{P}^*$-inseparably closed reduced algebra H^*. As above we have the Lam-Rector Embedding

$$\begin{array}{cccc} L: & \mathrm{H}^* & \hookrightarrow & \bigoplus_{i=1}^{k} \mathrm{H}^*/\mathfrak{p}_i \\ & & & \uparrow \\ & & & \bigoplus_{i=1}^{k} \mathcal{D}^{*q^s}_{i}(n), \end{array}$$

where we again choose

$$s := \max\{t_j \mid j = 1, \ldots, k\}.$$

Denote by $\mathbf{d}^{q^s}_{n,0}(i) \in \mathcal{D}^{*q^s}_{i}(n)$ the q^s-th fractal of the top Dickson class in the i-th summand. Define

$$\mathbf{t} := \mathbf{d}^{q^s}_{n,0}(1) + \cdots + \mathbf{d}^{q^s}_{n,0}(k) \in \mathrm{H}^*.$$

We need a technical lemma.[9]

LEMMA 8.1.7: *With the above notation and hypotheses for* $\mathrm{H}^* \neq \mathbb{F}$*, we have*

[1] $\mathbf{t}$ *is a non zero divisor in* H^*.

[2] *The ideal* $I := L^{-1}\left((\mathbf{d}^{q^s}_{n,0}(1), \ldots, \mathbf{d}^{q^s}_{n,0}(k))\right) \subset \mathrm{H}^*$ *is a* $\mathscr{P}^*$*-invariant ideal of height 1.*

PROOF: We take the statements in order.

AD [1] : Take an element $h \in \mathrm{H}^*$ such that

$$h\mathbf{t} = 0.$$

Projecting onto $\mathrm{H}^*/\mathfrak{p}_i$ gives

$$\begin{aligned} 0 &= \mathrm{pr}_i(h\mathbf{t}) \\ &= \mathrm{pr}_i(h)\mathrm{pr}_i(\mathbf{t}) \\ &= \mathrm{pr}_i(h)\mathbf{d}^{q^s}_{n,0}(i) \end{aligned}$$

[8] Note that the only polynomial of degree 5 in $\mathbb{F}[D_0, D_1]$ is $D_0 D_1$.

[9] Note that the following statement only makes sense if there is a non zero divisor in H^*. However, assume H^* consists of zero divisors and is by our assumption reduced, then (0) being the intersection of its associated prime ideals and not having embedded component, must be the maximal ideal itself, i.e., H^* was $\mathbb{F}$.

for any $i = 1, \ldots, k$. Since the Dickson class is not zero in $\mathrm{H}^*/\mathfrak{p}_i$ and $\mathrm{H}^*/\mathfrak{p}_i$ is an integral domain we deduce that

$$\mathrm{pr}_i(h) = 0,$$

i.e., $h \in \mathfrak{p}_i$ for any i. Hence

$$h \in \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k = (0).$$

So, h is zero.

AD [2] : The ideal I is the pre image of a $\mathcal{P}^*$-invariant ideal, and therefore itself $\mathcal{P}^*$-invariant by Lemma 2.1 in [21]. Since $\mathbf{t} \in I$ and $\mathbf{t}$ is not a zero divisor by part [1], the ideal I has height[10] at least one. Moreover, as ideals in H^* we have[11]

$$\begin{aligned} I := {} & L^{-1}\left(\left((\mathbf{d}_{n,0}^{q^s}(1), \ldots, \mathbf{d}_{n,0}^{q^s}(k))\right)\right) \\ \subseteq {} & L^{-1}\left(\left(\mathfrak{p}_k/\mathfrak{p}_1, \ldots, \mathfrak{p}_k/\mathfrak{p}_{k-1}, \mathbf{d}_{n,0}^{q^s}(k)\right)\right) \\ \subseteq {} & \left(\mathfrak{p}_k, \mathbf{d}_{n,0}^{q^s}(k)\right) \end{aligned}$$

where the last statement follows from Lemma 8.1.5. By construction, $\mathfrak{p}_k \subset \mathrm{H}^*$ is a prime ideal of height 0 not containing $\mathbf{d}_{n,0}^{q^s}(k)$. Therefore the ideal

$$\left(\mathfrak{p}_k, \mathbf{d}_{n,0}^{q^s}(k)\right)$$

has height 1, and hence I as height at most 1. Putting the two things together gives that I has exactly height 1 •

We come to the second important result of this chapter.

THEOREM 8.1.8 (Big Imbedding Theorem): *Let H^* be a Noetherian unstable algebra over the Steenrod algebra. Then there is a fractal of the Dickson algebra, $\mathcal{D}^*(n)^{q^r}$, integrally in H^*. If H^* is $\mathcal{P}^*$-inseparably closed then $r = 1$,*

PROOF : By Lemma 8.1.1 and Lemma 7.3.1 we can, without loss of generality, assume that H^* has no nilpotent elements and is $\mathcal{P}^*$-inseparably closed. We proceed by induction on n. The case $n = 1$ is the contents of Proposition 8.1.6, so assume $n > 1$. We have

$$\begin{array}{cccc} L: & \mathrm{H}^* & \hookrightarrow & \bigoplus_{i=1}^{k} \mathrm{H}^*/\mathfrak{p}_i \\ & & & \uparrow \\ & & & \bigoplus_{i=1}^{k} \mathcal{D}^{*q^s}_{i}(n) \end{array}$$

[10] Recall that the height of an arbitrary ideal I is defined to be the minimal height of a prime ideal containing I.

[11] I agree, this looks awfull: $L^{-1}\left(\left((\mathbf{d}_{n,0}^{q^s}(1), \ldots, \mathbf{d}_{n,0}^{q^s}(k))\right)\right)$ is the inverse image of the principal ideal $\left((\mathbf{d}_{n,0}^{q^s}(1), \ldots, \mathbf{d}_{n,0}^{q^s}(k))\right)$ generated by the k-tupel $\mathbf{d} := (\mathbf{d}_{n,0}^{q^s}(1), \ldots, \mathbf{d}_{n,0}^{q^s}(k)) \in \oplus \mathrm{H}^*/\mathfrak{p}_i$.

and

$$\mathbf{d}_{n,0}^{q^s}(j) \in \mathfrak{p}_i \ \forall\ i \neq j.$$

Consider the element

$$\mathbf{d} := \left(\mathbf{d}_{n,0}^{q^s}(1), \dots, \mathbf{d}_{n,0}^{q^s}(k)\right) \in \bigoplus_{i=1}^{k} \mathrm{H}^*/\mathfrak{p}_i.$$

By construction this element is a Thom class, i.e., it generates a height one $\mathscr{P}^*$-invariant principal ideal. The pre image under L of the ideal generated by $\mathbf{d}$ is the ideal I in Lemma 8.1.7. Hence we are in the lucky position that we have found a $\mathscr{P}^*$-invariant ideal $I \subset \mathrm{H}^*$ of height 1, mapping via L to the principal $(\mathbf{d}) \subset \bigoplus_{i=1}^{k} \mathrm{H}^*/\mathfrak{p}_i$, so we get a commutative diagram

$$\begin{array}{ccccc} L: & \mathrm{H}^* & \hookrightarrow & \bigoplus_{i=1}^{k} \mathrm{H}^*/\mathfrak{p}_i \\ & \downarrow \mathrm{pr} & \copyright & \downarrow \mathrm{pr} \\ \bar{L}: & \mathrm{H}^*/I & \hookrightarrow & \bigoplus_{i=1}^{k} \mathrm{H}^*/(\mathfrak{p}_i, \mathbf{d}_{n,0}^{q^s}(i)) \end{array}$$

By the fractal property (compare Proposition A.3.2) of the Dickson algebra the projection maps

$$\bigoplus_{i=1}^{k} \mathscr{D}^{*q^s}_{i}(n) \subseteq \bigoplus_{i=1}^{k} \mathrm{H}^*/\mathfrak{p}_i$$

onto

$$\bigoplus_{i=1}^{k} \mathscr{D}^{*q^{s+1}}_{i}(n-1) \subseteq \bigoplus_{i=1}^{k} \mathrm{H}^*/(\mathfrak{p}_i, \mathbf{d}_{n,0}^{q^s}(i)).$$

Since H^*/I has Krull dimension one less, we may apply the induction hypothesis, and conclude that there is a fractal of the Dickson algebra inside H^*/I

$$\mathscr{D}^{*q^t}(n-1) = \mathbb{F}[\mathbf{d}_{n-1,0}^{q^t}, \dots, \mathbf{d}_{n-1,n-2}^{q^t}] \subseteq \mathrm{H}^*/I.$$

Take $r := \max\{s+1,\ t\}$. We have the following situation

$$\begin{array}{ccccc} \mathrm{H}^* & \underset{L}{\hookrightarrow} & \bigoplus_{i=1}^{k} \mathrm{H}^*/\mathfrak{p}_i & \xrightarrow{\mathrm{pr}_i} & \mathrm{H}^*/\mathfrak{p}_i \\ \downarrow \mathrm{pr} & \copyright & \downarrow \mathrm{pr} & \copyright & \downarrow \mathrm{pr} \\ \mathrm{H}^*/(I^{q^{r-s}}) & \underset{\bar{L}}{\hookrightarrow} & \bigoplus_{i=1}^{k} \mathrm{H}^*/(\mathfrak{p}_i, \mathbf{d}_{n,0}^{q^r}(i)) & \xrightarrow{\mathrm{pr}_i} & \mathrm{H}^*/(\mathfrak{p}_i, \mathbf{d}_{n,0}^{q^r}(i)) \\ \cup\!\uparrow & & \cup\!\uparrow & & \cup\!\uparrow \\ \mathscr{D}^{*q^r}(n-1) & \hookrightarrow & \bar{L}(\mathscr{D}^{*q^r}(n-1)) & \longrightarrow & \mathrm{pr}_i \bar{L}(\mathscr{D}^{*q^r}(n-1)), \end{array}$$

and we want to show that the image $\mathrm{pr}_i \bar{L}(\mathcal{D}^{*q^r}(n-1))$ is isomorphic to a q^r-th-fractal of the Dickson algebra. First note that all maps involved commute with the action of the Steenrod algebra. Hence our image

$$\mathrm{pr}_i \bar{L}(\mathcal{D}^{*q^r}(n-1))$$

is again an unstable algebra. By construction it is generated by

$$\mathrm{pr}_i \bar{L}(\mathbf{d}^{q^r}_{n-1,0}), \ldots, \mathrm{pr}_i \bar{L}(\mathbf{d}^{q^r}_{n-1,n-2}).$$

These elements are algebraically independent in $\mathrm{H}^*/\left(\mathfrak{p}_i, \mathbf{d}^{q^r}_{n,0}(i)\right)$ for any $i = 1, \ldots, k$. To see this, assume to the contrary they were algebraically dependent. Then the generalized Jacobian would vanish

$$\det\left(\mathcal{P}^{\Delta_a}\left(\mathrm{pr}_i \bar{L}(\mathbf{d}^{q^r}_{n-1,b})\right)\right)_{a,b=0,\ldots,n-1} = 0 \in \mathrm{H}^*/(\mathfrak{p}_i, \mathbf{d}^{q^r}_{n,0}(i)).$$

Hence

$$h := \det\left(\mathcal{P}^{\Delta_a}\left(\mathbf{d}^{q^r}_{n-1,b}\right)\right)_{a,b=0,\ldots,n-1} \in \mathfrak{p}_i/(I^{q^{r-s}}) \subseteq \mathrm{H}^*/(I^{q^{r-s}}).$$

Since $\mathfrak{p}_i/I^{q^{r-s}}$ has height zero, we have that this determinant is a zero divisor $h \in \mathrm{H}^*/(I^{q^{r-s}})$. So, if we divide out a height zero prime ideal $\mathfrak{p} \subset \mathrm{H}^*/(I^{q^{r-s}})$ containing h, we would get

$$\det\left(\mathcal{P}^{\Delta_a}\left(\mathbf{d}^{q^r}_{n-1,b}\right)_{a,b=0,\ldots,n-1}\right) = 0 \in \mathrm{H}^*/\mathfrak{p}.$$

However, our Dickson fractal $\mathcal{D}^{*q^r}(n-1) \subseteq \mathrm{H}^*/(I^{q^{r-s}})$ sits integrally inside the quotient algebra. So, dividing out a height zero prime ideal must again give an integral extension

$$\mathcal{D}^{*q^r}(n-1) \subseteq \mathrm{H}^*/\mathfrak{p},$$

i.e., the preceding determinant is not zero in $\mathrm{H}^*/\mathfrak{p}$. This is a contradiction, so we have that the image

$$\mathrm{pr}_i \bar{L}(\mathcal{D}^{*q^r}(n-1)) \subseteq \mathrm{H}^*/(\mathfrak{p}_i,\ \mathbf{d}^{q^r}_{n,0}(i))$$

generates an unstable polynomial algebra of Krull dimension n. Since the maps involved commute with the $\boldsymbol{\mathcal{P}}^*$-action we have that our image

$$\mathrm{pr}_i \bar{L}(\mathcal{D}^{*q^r}(n-1)) \cong \mathcal{D}^{*q^r}(n-1) \subseteq \mathrm{H}^*/(\mathfrak{p}_i,\ I) = \mathrm{H}^*/\left(\mathfrak{p}_i,\ \mathbf{d}^{q^r}_{n,\ 0}(i)\right)$$

is a q^r-th fractal of the Dickson algebra sitting integrally inside $\mathrm{H}^*/(\mathfrak{p}_i, I)$. Hence[12] by Theorem 8.1.4 this is equal to the fractal of the Dickson

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[12] If $\mathrm{H}^*/(\mathfrak{p}_i, I)$ is not reduced, we can't apply Theorem 8.1.4 directly; in this case we need to observe that our fractals of the Dickson algebra are equal in the quotient obtained by dividing out the nil radical. Then, since in a Noetherian ring the nil radical is annihilated by some large q- power, we get that the two fractals of the Dickson algebra become equal after taking suitably high q-th powers, i.e., take just r to be large enough in the following discussion (compare also the remark after the theorem in question).

algebra, $\mathscr{D}^*{}_i^{q^r}(n-1)$ coming from $\mathrm{H}^*/\mathfrak{p}_i$ by projecting onto $\mathrm{H}^*/(\mathfrak{p}_i, I)$. So we can complete our diagram as follows

$$\begin{array}{ccccc}
 & & \bigoplus\limits_{i=1}^{k} \mathscr{D}^*{}_i^{q^{r-1}}(n) & \longrightarrow & \mathscr{D}^*{}_i^{q^{r-1}}(n) \\
 & & \downarrow & & \downarrow \\
\mathrm{H}^* & \underset{L}{\hookrightarrow} & \bigoplus\limits_{i=1}^{k} \mathrm{H}^*/\mathfrak{p}_i & \xrightarrow{\mathrm{pr}_i} & \mathrm{H}^*/\mathfrak{p}_i \\
\downarrow \mathrm{pr} & © & \downarrow \mathrm{pr} & © & \downarrow \mathrm{pr} \\
\mathrm{H}^*/(I^{q^{r-s}}) & \underset{\bar{L}}{\hookrightarrow} & \bigoplus\limits_{i=1}^{k} \mathrm{H}^*/(\mathfrak{p}_i, \mathbf{d}^{q^r}_{n,0}(i)) & \xrightarrow{\mathrm{pr}_i} & \mathrm{H}^*/(\mathfrak{p}_i, \mathbf{d}^{q^r}_{n,0}(i)) \\
\uparrow & & \uparrow & & \uparrow \\
\mathscr{D}^{*q^r}(n-1) & \underset{\text{diagonally}}{\hookrightarrow} & \bigoplus\limits_{i=1}^{k} \mathscr{D}^*{}_i^{q^r}(n-1) & \longrightarrow & \mathscr{D}^*{}_i^{q^r}(n-1).
\end{array}$$

We look at the appropriate fractals of the bottom Dickson classes involved:

$$\begin{array}{ccccc}
\mathbf{d}^{q^r}_{n-1,n-2} + (I^{q^{r-s}}) & & \left(\mathbf{d}^{q^{r-1}}_{n,n-1}(1), \dots, \mathbf{d}^{q^{r-1}}_{n,n-1}(k)\right) & \overset{\mathrm{pr}_i}{\longmapsto} & \mathbf{d}^{q^{r-1}}_{n,n-1}(i) \\
\downarrow \ \mathrm{pr} & & \downarrow & & \downarrow \ \mathrm{pr} \\
\mathbf{d}^{q^r}_{n-1,n-2} & \overset{\bar{L}}{\longmapsto} & \left(\mathbf{d}^{q^r}_{n-1,n-2}(1), \dots, \mathbf{d}^{q^r}_{n-1,n-2}(k)\right) & \overset{\mathrm{pr}_i}{\longmapsto} & \mathbf{d}^{q^r}_{n-1,n-2}(i).
\end{array}$$

The composition maps $\mathrm{pr}_i \circ L$ and $\mathrm{pr}_i \circ \bar{L}$ are surjective. Therefore the bottom Dickson class in the upper right corner, $\mathbf{d}^{q^{r-1}}_{n,n-1}(i) \in \mathrm{H}^*/\mathfrak{p}_i$ must have a preimage in H^* and, by commutativity of this diagram, in the pre image of the lower left bottom Dickson class

$$(\mathrm{pr}_i \circ L)^{-1}(\mathbf{d}^{q^{r-1}}_{n,n-1}(i)) \in \mathbf{d}^{q^r}_{n-1,n-2} + (I^{q^{r-s}}) = \mathrm{pr}^{-1}(\mathbf{d}^{q^r}_{n-1,n-2}) \subseteq \mathrm{H}^*.$$

Comparing degrees leads to the only possible value, unless $n = 1$ which is alreday covered by Proposition 8.1.6,

$$\mathrm{pr}_i^{-1}(\mathbf{d}^{q^{r-1}}_{n,n-1}(i)) = \mathbf{d}^{q^r}_{n-1,n-2} \in \mathrm{H}^*$$

for all possible $i = 1, \dots, k$. Or, in other words, there is an element

$$\mathbf{t}^{q^{r-1}}_{n,n-1} := \mathbf{d}^{q^r}_{n-1,n-2} = (\mathrm{pr}_i \circ L)^{-1}(\mathbf{d}^{q^{r-1}}_{n,n-1}(i)) \in \mathrm{H}^* \quad \forall\ i = 1, \dots, k,$$

mapping via pr to $\mathbf{d}^{q^r}_{n-1,\ n-2}$ and via $\mathrm{pr}_{\mathrm{i}} \circ L$ to $\mathbf{d}^{q^{r-1}}_{n,n-1}(i)$ for all $i = 1, \dots, k$. Set

$$\mathbf{t}^{q^{r-1}}_{n,i} := \mathscr{P}^{q^i q^{r-1}} \cdots \mathscr{P}^{q^{n-1} q^{r-1}}(\mathbf{t}^{q^{r-1}}_{n,n-1})$$

for $i = 0, \dots, n-2$. This definition is motivated by Corollary A.2.2, that tells us how to jump from one Dickson class to the next with an appropriate Steenrod power. We claim

$$\begin{array}{cccc}
\varphi : & \mathbb{F}[\mathbf{t}^{q^{r-1}}_{n,0}, \dots, \mathbf{t}^{q^{r-1}}_{n,n-1}] & \longrightarrow & \mathscr{D}^*(n)^{q^{r-1}} \\
 & \mathbf{t}^{q^{r-1}}_{n,i} & \longmapsto & \mathbf{d}^{q^{r-1}}_{n,i}
\end{array}$$

is an isomorphism of unstable algebras over the Steenrod algebra. To prove this we must show that

$$\varphi\left(\mathcal{P}^k\left(\mathbf{t}_{n,n-1}^{q^{r-1}}\right)\right) = \mathcal{P}^k\left(\varphi\left(\mathbf{t}_{n,n-1}^{q^{r-1}}\right)\right)$$

for any $k \geq 0$. We proceed inductively. For $k = 0$ there is nothing to show. So, assume that $k > 0$. Then by construction, for $i = 0, \dots, n-1$,

$$\begin{aligned}
&\mathcal{P}^k\left(\varphi\left(\mathbf{t}_{n,i}^{q^{r-1}}\right)\right) \\
&\quad = \mathcal{P}^k\left(\mathbf{d}_{n,i}^{q^{r-1}}\right) \\
&\quad = \left(\mathcal{P}^{\frac{k}{q^{r-1}}}\left(\mathbf{d}_{n,i}\right)\right)^{q^{r-1}} \\
&\quad = \left(\mathcal{P}^{\frac{k}{q^{r-1}}-q^{i-1}}\left(\mathbf{d}_{n,i-1}\right) - \left(\mathcal{P}^{\frac{k}{q^{r-1}}-q^{n-1}}\left(\mathbf{d}_{n,n-1}\right)\right)\mathbf{d}_{n,i}\right)^{q^{r-1}} \\
&\quad = \left(\mathcal{P}^{\frac{k}{q^{r-1}}-q^{i-1}}\left(\varphi(\mathbf{t}_{n,i-1})\right)\right. \\
&\qquad\qquad \left. - \left(\mathcal{P}^{\frac{k}{q^{r-1}}-q^{n-1}}\left(\varphi(\mathbf{t}_{n,n-1})\right)\right)\varphi(\mathbf{t}_{n,i})\right)^{q^{r-1}} \\
&\quad = \left(\varphi\left(\mathcal{P}^{\frac{k}{q^{r-1}}-q^{i-1}}(\mathbf{t}_{n,i-1})\right)\right. \\
&\qquad\qquad \left. - \left(\varphi\left(\mathcal{P}^{\frac{k}{q^{r-1}}-q^{n-1}}(\mathbf{t}_{n,n-1})\right)\right)\varphi(\mathbf{t}_{n,i})\right)^{q^{r-1}} \\
&\quad = \varphi\left(\mathcal{P}^{\frac{k}{q^{r-1}}-q^{i-1}}(\mathbf{t}_{n,i-1})\right. \\
&\qquad\qquad \left. - \left(\mathcal{P}^{\frac{k}{q^{r-1}}-q^{n-1}}(\mathbf{t}_{n,n-1})\right)(\mathbf{t}_{n,i})\right)^{q^{r-1}} \\
&\quad = \varphi\left(\mathcal{P}^{\frac{k}{q^{r-1}}}(\mathbf{t}_{n,i})\right)^{q^{r-1}} \\
&\quad = \varphi\left(\mathcal{P}^k(\mathbf{t}_{n,i}^{q^{r-1}})\right),
\end{aligned}$$

where we made use of the fact that

$$\begin{aligned}
&\mathcal{P}^k(\mathbf{t}_{n,i}^{q^{r-1}}) \\
&\quad = \left(\mathcal{P}^{\frac{k}{q^{r-1}}}(\mathbf{t}_{n,i})\right)^{q^{r-1}} \\
&\quad = \left(\mathcal{P}^{\frac{k}{q^{r-1}}-q^{i-1}}(\mathbf{t}_{n,i-1}) - \left(\mathcal{P}^{\frac{k}{q^{r-1}}-q^{n-1}}(\mathbf{t}_{n,i-1})\right)(\mathbf{t}_{n,i})\right)^{q^{r-1}} + h_j
\end{aligned}$$

for some $h_j \in \mathfrak{p}_j$ by construction. Since this is true for any $j = 1, \ldots, k$ we have that

$$\mathcal{P}^k(\mathbf{t}_{n,i}^{q^{r-1}}) = \left(\mathcal{P}^{\frac{k}{q^{r-1}} - q^{i-1}}(\mathbf{t}_{n,i-1}) - \left(\mathcal{P}^{\frac{k}{q^{r-1}} - q^{n-1}}(\mathbf{t}_{n,i})\right)(\mathbf{t}_{n,n-1})\right)^{q^{r-1}} + h$$

for some $h \in \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k = (0)$. (We could have also just observed that the action of the Steenrod algebra commutes with the Lam-Rector Embedding L, and, for the image under L, these recursion formulae are true, because $L(\mathbf{t}_{n,n-1}^{q^{r-1}})$ is a fractal of the bottom Dickson class.) So we have $\mathcal{D}^{*q^{r-1}}(n) \cong \mathbb{F}[\mathbf{t}_{n,0}^{q^{r-1}}, \ldots, \mathbf{t}_{n,n-1}^{q^{r-1}}] \subset \mathrm{H}^*$. Since we assumed that H^* is $\mathcal{P}^*$-inseparably closed we get that all p-th roots are in H^*, hence the Dickson algebra itself is in H^*

$$\mathcal{D}^*(n) \cong \mathbb{F}[\mathbf{t}_{n,0}, \ldots, \mathbf{t}_{n,n-1}] \subset \mathrm{H}^*$$

because it is the $\mathcal{P}^*$-inseparable closure of its fractals •

REMARK: Note that by Theorem 8.1.4 the Dickson algebra we just found in a $\mathcal{P}^*$-inseparably closed H^* is unique.

EXAMPLE 3: In Lemma 8.1.1 we showed that one can pull back a Dickson algebra inside $\mathrm{H}^*/Nil\,(\mathrm{H}^*)$ to H^* with a possible necessary correction: one has to perhaps take a *fractal* of the Dickson algebra. However, here are two examples were we have the Dickson algebra itself in H^* even though the nil radical is not trivial. Let $x,\ y$ have degree one. Then

[1] $\mathcal{D}^*(1) = \mathbb{F}[x^{q-1}] \hookrightarrow \mathbb{F}[x,\ y]/(y^q)$ and

[2] $\mathcal{D}^*(1) = \mathbb{F}[x^{q-1}] \hookrightarrow \mathbb{F}[x,\ y^p]/(y^{q^2})$.

Note that the second example is not even $\mathcal{P}^*$-inseparably closed.

8.2 Turkish Delights II

Obviously the Big Imbedding Theorem gives a Thom class explicitly. We are going to derive some results exemplifying why Thom classes are wonderful: they allow proofs by induction *within our category*. Note again, that we are still dealing only with Noetherian algebras.

PROPOSITION 8.2.1 (Turkish Delight 5): *In any Noetherian unstable algebra* H^* *there is a Thom class* $\mathbf{t}$.

PROOF: Since we have found a fractal of the Dickson algebra in H^*, the top Dickson class $\mathbf{d}_{n,0}^{q^s}$ serves as the desired Thom class, compare Theorem 8.1.8 •

PROPOSITION 8.2.2 (Turkish Delight 6): *The Thom class* $\mathbf{t} = \mathbf{d}_{n,0}^{q^s}$ *which we just found in* H^**, is an element of any* $\mathcal{P}^*$*-invariant prime ideal of positive height.*

Proof: Since

$$\mathscr{D}^*(n)^{q^s} \hookrightarrow \mathrm{H}^*$$

is integral, for any $\mathcal{P}^*$-invariant prime ideal

$$\mathfrak{q} \subset \mathrm{H}^*$$

of positive height, also the contracted ideal

$$\mathfrak{q} \cap \mathscr{D}^*(n)^{q^s}$$

is $\mathcal{P}^*$-invariant prime of positive height. By [16], see Theorem 11.4.6 [28] for a complete proof, the only $\mathcal{P}^*$-invariant prime ideals in $\mathscr{D}^*(n)^{q^s}$ are

$$(0) \subset (\mathbf{d}^{q^s}_{n,\,0}) \subset (\mathbf{d}^{q^s}_{n,\,0},\ \mathbf{d}^{q^s}_{n,\,1}) \subset \cdots \subset (\mathbf{d}^{q^s}_{n,\,0}, \ldots, \mathbf{d}^{q^s}_{n,\,n-1}).$$

Hence

$$\mathbf{d}^{q^s}_{n,\,0} \in \mathfrak{q} \cap \mathscr{D}^*(n)^{q^s}$$

and therefore also

$$\mathbf{d}^{q^s}_{n,\,0} \in \mathfrak{q}$$

as claimed •

So, we have a Thom class with nice properties. Since Thom classes generate (by definition) $\mathcal{P}^*$-invariant height one ideals they allow us to construct proofs by induction over the Krull dimension as we will illustrate with the next two results.

The first one reproves Turkish Delight 4 (Proposition 6.2.3).

Proposition 8.2.3 (Turkish Delight 7): *Let $\mathfrak{p} \in \mathrm{H}^*$ a $\mathcal{P}^*$-invariant prime ideal of height $i > 0$, then there exist a $\mathcal{P}^*$-invariant prime ideal $\mathfrak{q} \subset \mathfrak{p}$ of height $i-1$.*

Proof: By double induction on the Krull dimension of H^* and the height i of $\mathfrak{p}$.

Let $\dim(\mathrm{H}^*) = 1$. Then there is nothing to show since prime ideals of height zero are $\mathcal{P}^*$-invariant by a result of Peter S. Landweber, [15], see Proposition 11.2.3 in [28] for a complete proof or Theorem 3.5 in [23] for a more general statement. So, let $\dim(\mathrm{H}^*) > 1$. We induct on the height of $\mathfrak{p}$. If $i = 1$ there is nothing to show since any prime ideal of height zero is $\mathcal{P}^*$-invariant. Let $i > 1$. By Turkish Delight 5 (Proposition 8.2.1) there exists a Thom class $\mathbf{t} \in \mathfrak{p}$. Dividing out the principal ideal generated by this Thom class we get an unstable algebra $\mathrm{H}^*/(\mathbf{t})$ over the Steenrod algebra of Krull dimension one less. Hence by induction there exists a prime ideal

$$\overline{\mathfrak{q}} \subsetneq \mathfrak{p}/(\mathbf{t}) \subsetneq \mathrm{H}^*/(\mathbf{t})$$

with

$$\begin{aligned}\mathrm{ht}(\overline{\mathfrak{q}}) &= \mathrm{ht}(\mathfrak{p}/(\mathbf{t})) - 1 \\ &= i - 2.\end{aligned}$$

Hence the pre image in H^*

$$\mathfrak{q} = (\overline{\mathfrak{q}}, \mathbf{t}) \subsetneqq \mathfrak{p} \subset \mathrm{H}^*$$

is a $\mathscr{P}^*$-invariant prime ideal of height $i - 1$ •

We are now ready to prove the main result of this section; it establishes a $\mathscr{P}^*$-invariant version of Krull's principal ideal theorem and its generalization, see Theorem 1.2.16 in [5].

THEOREM 8.2.4 (Turkish Delight 8): *Let $\mathfrak{p}_i \subset \mathrm{H}^*$ be a $\mathscr{P}^*$-invariant prime ideal of height i, $i = 1, \ldots, n$. Then there exist elements $h_1, \ldots, h_i \in \mathrm{H}^*$ such that*

[1] $h_1, \ldots, h_i \in \mathfrak{p}_i$,

[2] *$\mathfrak{p}_i$ is an isolated prime ideal of $(h_1, \ldots, h_i)$, and*

[3] *$(h_1, \ldots, h_j) \subset \mathrm{H}^*$ are $\mathscr{P}^*$-invariant ideals of height j, $j = 1, \ldots, i$.*

PROOF: We use induction on the height i. The statement for $i = 1$ is the contents of Turkish Delight 6 (Proposition 8.2.2). So, suppose $i > 1$ and assume by induction that for any $\mathscr{P}^*$-invariant prime ideal of height $< i$ the result is established, and let $\mathfrak{p}$ be a $\mathscr{P}^*$-invariant prime ideal of height i. Then by Turkish Delight 7, Proposition 8.2.3, there exists a $\mathscr{P}^*$-invariant prime ideal $\mathfrak{q} \subsetneqq \mathfrak{p}$ of height $i - 1$. By the induction hypothesis there exist elements $h_1, \ldots, h_{i-1}$ such that

[1] $h_1, \ldots, h_{i-1} \in \mathfrak{q}$,

[2] $\mathfrak{q}$ is an isolated prime ideal of $(h_1, \ldots, h_{i-1})$, and

[3] The ideals $(h_1, \ldots, h_j) \subset \mathrm{H}^*$ are $\mathscr{P}^*$-invariant ideals of height j, $j = 1, \ldots, i - 1$.

The quotient $\mathrm{H}^*/(h_1, \ldots, h_{i-1})$ is again an unstable algebra over the Steenrod algebra, and hence contains a Thom class $\mathbf{t}$ by Turkish Delight 5 (Proposition 8.2.1), which moreover is contained in $\overline{\mathfrak{p}} := \mathfrak{p}/(h_1, \ldots, h_{i-1})$ by Turkish Delight 6 (Proposition 8.2.2) (note that $\overline{\mathfrak{p}}$ has height one). If we choose $h_i \in \mathrm{pr}^{-1}(\mathbf{t})$ a pre image under the projection $\mathrm{pr} : \mathrm{H}^* \longrightarrow \mathrm{H}^*/(h_1, \ldots, h_{i-1})$ then

$$(h_1, \ldots, h_i) \subseteq \mathfrak{p} \subset \mathrm{H}^*$$

is a $\mathscr{P}^*$-invariant ideal of height i and $\mathfrak{p}$ is an isolated prime ideal of it •

8.3 The Reverse Landweber-Stong Conjecture

In the preceding section we have found a Thom class **t** in any unstable algebra over $\mathcal{P}^*$. In this section we prove that if the algebra in question does not consist of zero divisors and is reduced then our naturally given Thom class is a non zero divisor.

Again H^* is always Noetherian.

PROPOSITION 8.3.1 (Turkish Delight 9): *Let Nil* $(\mathrm{H}^*) = (0)$. *If* H^* *contains a non zero divisor then it contains a Thom class* **t** *which is a non zero divisor.*

PROOF: Let's go back a step and look how we constructed a Thom class in H^*. First we took the $\mathcal{P}^*$-inseparable closure $\sqrt{\mathrm{H}^*}$. If H^* contains a non zero divisor then the Krull dimension n of H^* is positive. So we find by the Big Imbedding Theorem (Theorem 8.1.8) a nontrivial Dickson algebra integrally inside the $\mathcal{P}^*$-inseparably closure

$$\mathcal{D}^*(n) \hookrightarrow \sqrt{\mathrm{H}^*},$$

which also has the same Krull dimension n. The top Dickson class is a Thom class by Turkish Delight 5 (Proposition 8.2.1), and, by Lemma 8.1.7, gives a non zero divisor

$$\mathbf{d}_{n,0} \in \mathcal{D}^*(n) \hookrightarrow \sqrt{\mathrm{H}^*}.$$

Hence some q-th power of it is still a Thom class and a non zero divisor in H^* •

This result does not remain true if we drop the condition on H^* that *Nil* $(\mathrm{H}^*) = (0)$ as the following example shows.

EXAMPLE 1: Let $\mathbb{F} = \mathbb{F}_2$ be the field with two elements, and let x, y, z be linear form. Consider the algebra

$$\mathrm{H}^* := \mathbb{F}[x, y, z]/(yz, z^2),$$

and note that its nil radical is

$$Nil\,(\mathrm{H}^*) = (z) \neq (0).$$

Then H^* contains integrally a copy of the Dickson algebra $\mathcal{D}^*(2)$

$$\mathcal{D}^*(2) = \mathbb{F}[xy(x+y),\ x^2+y^2+xy] \hookrightarrow \mathrm{H}^*,$$

where $\mathbf{d}_{2,0} = xy(x+y)$ and $\mathbf{d}_{2,1}$. Integrality follows from

$$0 = z^2,$$
$$0 = \mathbf{d}_{2,0}x + \mathbf{d}_{2,1}x^2 + x^4,$$
$$0 = \mathbf{d}_{2,0}y + \mathbf{d}_{2,1}y^2 + y^4.$$

The top Dickson class is a Thom class as Turkish Delight 5 (Proposition 8.2.1) predicts, but it is a zero divisor

$$\mathbf{d}_{2,0}z = xy(x+y)z = 0.$$

If Turkish Delight 9 (Proposition 8.3.1) would have been true without the additional assumption that the algebra be reduced, i.e., that the nil radical be zero, then it would be a routine chore to deduce by induction on the Krull dimension the reverse version of the Landweber-Stong conjecture. Recall some notions from the introduction:

A sequence $h_1, \ldots, h_k \in \mathrm{H}^*$ of elements of positive degree in H^* is called a **regular sequence** if

[1] $h_1 \in \mathrm{H}^*$ is not a zero divisor,
[2] $h_i \in \mathrm{H}^*/(h_1, \ldots, h_{i-1})$ is not a zero divisor $\forall\ i = 2, \ldots, k$.

Then define the **depth** (or **homological codimension**) of H^*, denoted $dp(\mathrm{H}^*)$, to be the length of the longest possible regular sequence in H^*. See [28] Chapter 6 for an introduction to the homological properties of graded connected commutative Noetherian algebras.

The original **Landweber-Stong Conjecture** asserts that a ring of invariants $\mathbb{F}[V]^G$ has depth at least k, $dp(\mathrm{H}^*) \geq k$, if and only if the k *bottom* Dickson classes $\mathbf{d}_{n,n-1}, \ldots, \mathbf{d}_{n,n-k}$ form a regular sequence[13], see [17]. This conjecture was proven in 1996 by Dorra Bourguiba and Saïd Zarati, [6], using the classification of injective $\mathbb{F}[V]$-modules over $\mathscr{P}^*$ by J. Lannes and S. Zarati, see [18].

The **Reverse Landweber-Stong Conjecture**, [30], says that a ring of invariants $\mathbb{F}[V]^G$ has depth at least k if and only if the *top* k Dickson classes, $\mathbf{d}_{n,0}, \ldots, \mathbf{d}_{n,k-1}$ form a regular sequence.[14] Since the Big Imbedding Theorem hands us a fractal of the Dickson algebra in any unstable algebra over the Steenrod algebra, we can reformulate the reverse Landweber-Stong conjecture in the following way:

THE REVERSE LANDWEBER-STONG CONJECTURE: Let H^* be a Noetherian unstable algebra over the Steenrod algebra. Then H^* has depth at least k if and only if high enough q-th powers of the k top Dickson classes, $\mathbf{d}_{n,}^{q^s}, \ldots, \mathbf{d}_{n,k-1}^{q^s} \in \mathrm{H}^*$ form a regular sequence.

This conjecture was one of the original motivations to study Thom classes, their existence and their properties. However, the above example shows not only that $Nil(\mathrm{H}^*) = (0)$ is a necessary condition in Turkish

[13] Note that since we have a ring of invariants, of course, the Dickson algebra $\mathscr{D}^*(n) \hookrightarrow \mathbb{F}[V]^G$ is integrally contained in the ring of invariants.

[14] Note again that in a ring of invariants a copy of the Dickson algebra is always present.

Delight 9 (Proposition 8.3.1), it is also a counter example to the Reverse Landweber Stong Conjecture.

EXAMPLE 2: Recall that we have an integral extension of unstable Noetherian algebras

$$\mathscr{D}^*(2) = \mathbb{F}[xy(x+y),\ x^2+y^2+xy] \hookrightarrow \mathrm{H}^* := \mathbb{F}[x,\ y,\ z]/(yz,\ z^2).$$

The big algebra H^* has depth 1, but the top Dickson class

$$\mathbf{d}_{2,0} = xy(x+y) \in \mathrm{H}^*$$

is a zero divisor, as we have seen already. However, the bottom Dickson class

$$\mathbf{d}_{2,1} = x^2+y^2+xy \in \mathrm{H}^*$$

is not a zero divisor as one might prove in the following way: let $h \in \mathrm{H}^*$ such that

$$\left(x^2+y^2+xy\right) h = 0 \in (z) = Nil\,(\mathrm{H}^*) \subset \mathrm{H}^*.$$

Hence $h = z\bar{h} \in (z)$, because (z) is a prime ideal and $\mathbf{d}_{2,1} \notin (z)$. Then

$$\begin{aligned} 0 &= \left(x^2+y^2+xy\right) h \\ &= \left(x^2+y^2+xy\right) z\bar{h} \\ &= x^2 z\bar{h} + (x+y)yz\bar{h} \\ &= x^2 z\bar{h} + 0 \\ &= x^2 z\bar{h} \end{aligned}$$

Since x^2 is not a zero divisor in H^* we conclude that

$$h = z\bar{h} = 0,$$

and that's all we wanted •

We are almost at the end of our journey. What follows is an appendix with some technical proofs in all their detail. I want to say good-bye. It has been a long way and I hope you enjoyed it. And above all, I want to thank you for the company.

APPENDIX A
Technical Stuff

This appendix is full of unpleasant technical stuff.

A.1 Ad Chapter 1

In this first section of the appendix we will give the two technical proofs which were omitted in Chapter 1. Recall from Chapter 1 that we made the convention that $\mathcal{P}^i = 0$ for $i \notin \mathbb{N}_0$.

We first prove Lemma 1.1.2, i.e.,

LEMMA A.1.1 : *For all natural numbers $i \geq 1$ the following commutation rule holds*

$$\mathcal{P}^k \mathcal{P}^{\Delta_i} - \mathcal{P}^{\Delta_i} \mathcal{P}^k = \mathcal{P}^{\Delta_{i+1}} \mathcal{P}^{k-q^i}.$$

If $i = 0$ we have

$$\mathcal{P}^k \mathcal{P}^{\Delta_0} - \mathcal{P}^{\Delta_0} \mathcal{P}^k = k\mathcal{P}^k.$$

PROOF : As we already pointed out in Chapter 1 the second statement is clear, because $\mathcal{P}^{\Delta_0}$ is just multiplication with the degree of the operand.

For the first statement we translate the formula into Milnor's notation

$$\mathcal{P}^k \triangleq P^{(k,0,\cdots)}$$

and

$$\mathcal{P}^{\Delta_i} \triangleq P^{(0,\ldots,0,1_i,0,\cdots)},$$

where the index i at the 1 indicates that the 1 comes in the i-th position. Then the statement reads as follows:

$$P^{(k,0,\cdots)} P^{(0,\ldots,0,1_i,0,\cdots)} - P^{(0,\ldots,0,1_i,0,\cdots)} P^{(k,0,\cdots)} = P^{(0,\ldots,0,1_{i+1},0,\cdots)} P^{(k-q^i,0,\cdots)}.$$

Following Milnor, we introduce the notations

$$R := (r_1,\ r_2, \cdots)$$

and

$$S := (s_1,\ s_2,\ \cdots),$$

and consider infinite matrices

$$\mathbf{X} = \begin{bmatrix} * & x_{01} & x_{02} & \cdots \\ x_{10} & x_{11} & x_{12} & \cdots \\ x_{20} & x_{21} & x_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

with entries $x_{ab} \in \mathbb{N}_0$ (and nothing in the 00-position) which satisfy the following system of equations

$$r_a = \sum_b q^b x_{ab}$$
$$s_b = \sum_a x_{ab}.$$

Then the product $P^R P^S$ is given by

$$P^R P^S = \sum_{\mathbf{X}} b(\mathbf{X}) P^{T(\mathbf{X})},$$

where the index sequence $T(\mathbf{X}) = (t_1,\ t_2, \cdots)$ is the sequence of diagonal sums

$$t_c = \sum_{a+b=c} x_{ab},$$

and the coefficient $b(\mathbf{X})$ is given by

$$b(\mathbf{X}) = \frac{\prod_c t_c!}{\prod_{a,\ b} x_{ab}!}.$$

Hence, for our product $P^{(k,0,\cdots)} P^{(0,\ldots,0,1_i,0,\cdots)}$, we have to consider matrices $\mathbf{X}$ with

$$\sum_b q^b x_{ab} = \begin{cases} k & \text{if } a = 1 \\ 0 & \text{if } a > 1 \end{cases}$$
$$\sum_a x_{ab} = \begin{cases} 1 & \text{if } b = i \\ 0 & \text{if } b \le 1,\ \ne i. \end{cases}$$

Therefore our matrices look like

$$\mathbf{X} = \begin{bmatrix} * & 0 & \cdots & 0 & x_{0i} & 0 & \cdots \\ x_{10} & 0 & \cdots & 0 & x_{1i} & 0 & \cdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix},$$

with

$$x_{0i} + x_{1i} = 1,$$
$$x_{10} + q^i x_{1i} = k.$$

This system has the following solutions

$$\mathbf{X}_1 = \begin{bmatrix} * & 0 & \cdots & 0 & 1_{0i} & 0 & \cdots \\ k & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix},$$

and, if in addition $k \geq q^i$, also

$$\mathbf{X}_2 = \begin{bmatrix} * & 0 & \cdots & 0 & 0 & 0 & \cdots \\ k-q^i & 0 & \cdots & 0 & 1_{1i} & 0 & \cdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

Hence we have

$$\begin{aligned}
&\mathcal{P}^k \mathcal{P}^{\Delta_i} \\
&\quad \widehat{=}\; P^{(k,0,\cdots)} P^{(0,\dots,0,1_i,0,\cdots)} \\
&\qquad = \begin{cases} P^{(k,0,\dots,0,1_i,0\cdots)} & \text{if } k < q^i \\ P^{(k,0,\dots,0,1_i,0\cdots)} + P^{k-q^i,0,\dots,0,1_{i+1},0\cdots)} & \text{otherwise} \end{cases} \\
&\qquad = \begin{cases} P^{(0,\dots,0,1_i,0,\cdots)} P^{(k,0,\cdots)} & \text{if } k < q^i \\ P^{(0,\dots,0,1_i,0,\cdots)} P^{(k,0,\cdots)} + P^{(0,\dots,0,1_{i+1},0,\cdots)} P^{(k-q^i,0,\cdots)} & \text{otherwise} \end{cases} \\
&\quad \widehat{=}\; \mathcal{P}^{\Delta_i} \mathcal{P}^k + \mathcal{P}^{\Delta_{i+1}} \mathcal{P}^{k-q^i},
\end{aligned}$$

as we claimed •

Next comes the proof of Lemma 1.1.3.

Lemma A.1.2 : *Let $h \in \mathrm{H}^*$ be an arbitrary element of degree d. Then for any $r \geq 0$ and for any $s \geq 0$ there exists an element $\mathcal{M}_{r,d} \in \mathcal{P}^*$ such that*

$$\mathcal{P}^{\Delta_{r+s}}\left(\mathcal{M}_{r,d}(h)\right) = \left(\mathcal{P}^{\Delta_s}(h)\right)^{q^r}$$

where the notation $\mathcal{M}_{r,d}$ emphasizes the dependence on r and d.

Proof : Define inductively

$$\begin{aligned}
\mathcal{M}_{0,d} &:= \mathcal{P}^0 \\
\mathcal{M}_{1,d} &:= \mathcal{P}^{d_0-1}, && \text{where } d_0 := d = \deg(h) \\
\mathcal{M}_{r+1,d} &:= \mathcal{P}^{d_r-1}\mathcal{M}_{r,d}, && \text{where } d_r := \deg\left(\mathcal{M}_{r,d}(h)\right).
\end{aligned}$$

We will proceed by induction on r. For $r = 0$ there is nothing to prove. So, let $r > 0$. Then we have

$$\mathcal{P}^{\Delta_{r+s}}\mathcal{M}_{r,d}(h) := \mathcal{P}^{\Delta_{r+s}}\mathcal{P}^{d_{r-1}-1}\mathcal{M}_{r-1,d}(h)$$

$$\text{(1)} \quad = \left(\mathcal{P}^{d_{r-1}-1+q^{r+s-1}}\mathcal{P}^{\Delta_{r+s-1}} - \mathcal{P}^{\Delta_{r+s-1}}\mathcal{P}^{d_{r-1}-1+q^{r+s-1}}\right)\mathcal{M}_{r-1,d}(h)$$

$$\text{(2)} \quad = \mathcal{P}^{d_{r-1}-1+q^{r+s-1}}\mathcal{P}^{\Delta_{r+s-1}}\mathcal{M}_{r-1,d}(h) - 0$$

$$\text{(3)} \quad = \mathcal{P}^{d_{r-1}-1+q^{r+s-1}}\left(\left(\mathcal{P}^{\Delta_s}(h)\right)^{q^{r-1}}\right)$$

$$\text{(4)} \quad = \left(\left(\mathcal{P}^{\Delta_s(h)}\right)^{q^{r-1}}\right)^{q}$$

$$= \left(\mathcal{P}^{\Delta_s(h)}\right)^{q^r}$$

where (1) follows from Lemma A.1.1 with $i = r+s-1$ and $k = d_{r-1} - 1 + q^{r+s-1}$, (2) follows from the fact that

$$d_{r-1} - 1 + q^{r+s-1} \geq d_{r-1} = \deg\left(\mathcal{M}_{r-1,d}\right) + d,$$

(3) follows by induction and (4) by

$$\begin{aligned} d_{r-1} - 1 + q^{r+s-1} &= \deg\left(\mathcal{M}_{r-1,d}\right) + d + q^{r+s-1} - s \\ &= \left(\deg\left(\mathcal{P}^{\Delta_s}\right) + d\right) q^r. \end{aligned}$$

That's what we wanted •

As we pointed out in Chapter 1 the recursion formula for the new elements $\mathcal{M}_{r,d}$ leads to the following explicit description

$$\mathcal{M}_{r,d} = \mathcal{P}^{d_{r-1}-1} \cdots \mathcal{P}^{d_0-1}\mathcal{P}^0.$$

Finally:

LEMMA A.1.3: *The degree of the new elements $\mathcal{M}_{r,d}$ is given by*

$$\deg\left(\mathcal{M}_{r,d}\right) = q^r d_0 - q^r - d_0 + 1.$$

Moreover, there is the relation

$$d_r = q^r d_0 - q^r + 1.$$

PROOF : Again we induct on r. If $r = 0$ there is nothing to show. Let $r > 0$. Then

$$\begin{aligned}\deg\left(\mathcal{M}_{r,d}\right) &= \deg\left(\mathcal{M}_{r-1,d}\right) + \deg\left(\mathcal{P}^{d_{r-1}-1}\right)\\ &= q^{r-1}d_0 - q^{r-1} - d_0 + 1 + (d_{r-1}-1)(q-1)\\ &= q^{r-1}d_0 - q^{r-1} - d_0 + 1 + (q^{r-1}d_0 - q^{r-1})(q-1)\\ &= q^{r-1}d_0 - q^{r-1} - d_0 + 1 + q^r d_0 - q^r - q^{r-1}d_0 + q^{r-1}\\ &= q^r d_0 - q^r - d_0 + 1,\end{aligned}$$

where we made use of the induction hypothesis for both formulae. Finally,

$$d_r = \deg\left(\mathcal{M}_{r,d}\right) + d_0$$

leads to the desired second formula •

A.2 The Action of the Steenrod Algebra on the Dickson Algebra

In this section we want to show that the formulae for the action of the mod p Steenrod algebra on the Dickson algebra given in Section 3 of [31], resp. Section II of [36][1], have direct analogues over arbitrary finite fields.

Let

$$\mathcal{D}^*(n) = \mathbb{F}[\mathbf{d}_{n,0}, \dots, \mathbf{d}_{n,n-1}] \subseteq \mathbb{F}[V]^{\mathrm{GL}(n,\ \mathbb{F})}$$

be the Dickson algebra of Krull dimension n.

Recall from [28] Theorem 8.1.6 that the Dickson classes are given by the Stong-Tamagawa-formulae

$$\mathbf{d}_{n,i} = \sum_{W^* \le V^*,\ \dim(W^*)=i} \left(\prod_{v \notin W^*} v\right).$$

Equally we could have said that the Dickson classes occur as coefficients in the following polynomial

$$\begin{aligned}f(X) &= \prod_{v \in V^*}(X+v)\\ &= X^{q^n} + \sum_{i=0}^{n-1} \mathbf{d}_{n,i} X^{q^i}.\end{aligned}$$

To simplify the notation in the following formulae write

$$\mathbf{d}_{n,n} := 1 \quad \text{and} \quad \mathbf{d}_{n,i} := 0\ \forall\ i < 0.$$

[1] Note carefully that in [36] some formulae are not quite correct.

Then using the product form of $f(X)$ we get by the Cartan formulae

$$\begin{aligned}
\mathcal{P}^k(f(X)) &= \mathcal{P}^k\left(\prod_{v\in V^*}(X+v)\right)\\
&= \sum_{(i_1,\ldots,i_k)}\left(\mathcal{P}^1(X+v_{i_1})\cdots\mathcal{P}^1(X+v_{i_k})\right)\left(\prod_{V^*\setminus\{v_{i_1},\ldots,v_{i_k}\}}(X+v)\right)\\
&= \sum_{(i_1,\ldots,i_k)}\left((X^q+v_{i_1}^q)\cdots(X^q+v_{i_k}^q)\right)\left(\prod_{V^*\setminus\{v_{i_1},\ldots,v_{i_k}\}}(X+v)\right)\\
&= f(X)\left(\sum_{(i_1,\ldots,i_k)}(X+v_{i_1})^{q-1}\cdots(X+v_{i_k})^{q-1}\right),
\end{aligned}$$

i.e., $f(X)$ divides every Steenrod power of itself. On the other hand we can use the sum form of $f(X)$. Then we get

$$\begin{aligned}
\mathcal{P}^k\left(f(X)\right) &= \mathcal{P}^k\left(X^{q^n}+\sum_{i=0}^{n-1}\mathbf{d}_{n,i}X^{q^i}\right)\\
&= \begin{cases}
0 & \text{if } k>q^n\\
X^{q^{n+1}} & \text{if } k=q^n\\
\sum_{i=0}^{n-1}\left(\mathcal{P}^k(\mathbf{d}_{n,i})X^{q^i}+\mathcal{P}^{k-q^i}(\mathbf{d}_{n,i})\mathcal{P}^{q^i}(X^{q^i})\right) & \text{if } 1\le k<q^n\\
f(X) & \text{if } k=0
\end{cases}\\
&= \begin{cases}
0 & \text{if } k>q^n\\
X^{q^{n+1}} & \text{if } k=q^n\\
\sum_{i=0}^{n-1}\left(\mathcal{P}^k(\mathbf{d}_{n,i})X^{q^i}+\mathcal{P}^{k-q^i}(\mathbf{d}_{n,i})X^{q^{i+1}}\right) & \text{if } 1\le k<q^n\\
f(X) & \text{if } k=0
\end{cases}\\
&= \begin{cases}
0 & \text{if } k>q^n\\
X^{q^{n+1}} & \text{if } k=q^n\\
\sum_{i=0}^{n}\left(\mathcal{P}^k(\mathbf{d}_{n,i})-\mathcal{P}^{k-q^{i-1}}(\mathbf{d}_{n,i-1})\right)X^{q^i} & \\
 & \text{if } 1\le k<q^n\\
f(X) & \text{if } k=0.
\end{cases}
\end{aligned}$$

Combining the two formulae and comparing the highest coefficients gives

$$\mathcal{P}^k\left(f(X)\right) = -f(X)\mathcal{P}^{k-q^{n-1}}(\mathbf{d}_{n,n-1})$$

for $1 \le k < q^n$. Hence we have

$$\mathcal{P}^k(\mathbf{d}_{n,i}) = \begin{cases} 0 & \text{if } k \ge q^n \\ \mathcal{P}^{k-q^{i-1}}\mathbf{d}_{n,i-1} - \mathcal{P}^{k-q^{n-1}}(\mathbf{d}_{n,n-1})\mathbf{d}_{n,i} & \text{if } 1 \le k < q^n \\ \mathbf{d}_{n,i} & \text{if } k = 0. \end{cases}$$

This corrects Corollary 2.3 a) in [36] and extends it from $\mathbb{F}_p$ to $\mathbb{F}_q$.

Analogously, one proves the formulae for $\mathcal{P}^{\Delta_k}(\mathbf{d}_{n,i})$. First we use the product form for $f(X)$ and get

$$\begin{aligned} \mathcal{P}^{\Delta_k}\left(f(X)\right) &= \sum_{v_1 \in V^*} \mathcal{P}^{\Delta_k}(X + v_1) \prod_{v \in V^* \setminus \{v_1\}} (X + v) \\ &= \sum_{v_1 \in V^*} (X^{q^k} + v_1^{q^k}) \prod_{v \in V^* \setminus \{v_1\}} (X + v) \\ &= f(X) \sum_{v_1 \in V^*} (X + v_1)^{q^k - 1}. \end{aligned}$$

Here we made use of the derivation property of the $\mathcal{P}^{\Delta_k}$'s and how they act on linear forms. Again

$$f(X) \mid \mathcal{P}^{\Delta_k}\left(f(X)\right).$$

Using the other formula for $f(X)$ we get

$$\mathcal{P}^{\Delta_k}\left(f(X)\right) = \mathbf{d}_{n,0} X^{q^k} + \sum_{i=0}^{n-1} \left(\mathcal{P}^{\Delta_k}\left(\mathbf{d}_{n,i}\right) X^{q^i}\right).$$

Once again we compare the coefficients in the two formulae and use the defining recursion formulae for the $\mathcal{P}^{\Delta}$'s, to get

$$\mathcal{P}^{\Delta_k}(f(X)) = \begin{cases} 0 & \text{for } k < n \\ \mathbf{d}_{n,0} f(X) & \text{for } k = n \\ \mathcal{P}^{q^{k-1}} \mathcal{P}^{\Delta_{k-1}}\left(f(X)\right) & \text{for } k > n. \end{cases}$$

Recursively we recover the formulae of [31] Section 3, see also [28] §10.6, or [36] Corollary 2.3 b, (in the latter again the signs are not correct) extended to $\mathbb{F}_q$

$$\mathcal{P}^{\Delta_k}(\mathbf{d}_{n,i}) = \begin{cases} 0 & \text{if } 0 \le k < n \text{ and } k \ne i \\ -\mathbf{d}_{n,0} & \text{if } 0 \le k < n \text{ and } k = i \\ \mathbf{d}_{n,0}\mathbf{d}_{n,i} & \text{if } k = n \\ \mathcal{P}^{q^{k-1}} \mathcal{P}^{\Delta_{k-1}}\left(\mathbf{d}_{n,i}\right) & \text{if } k > n \end{cases}.$$

We collect these results in a proposition.

PROPOSITION A.2.1: *With the above notation we have*

[1]

$$\mathcal{P}^k(\mathbf{d}_{n,i}) = \begin{cases} 0 & \text{if } k \geq q^n \\ \mathcal{P}^{k-q^{i-1}}(\mathbf{d}_{n,i-1}) - \mathcal{P}^{k-q^{n-1}}(\mathbf{d}_{n,n-1})\mathbf{d}_{n,i} & \text{if } 1 \leq k < q^n \\ \mathbf{d}_{n,i} & \text{if } k = 0. \end{cases}$$

[2]

$$\mathcal{P}^{\Delta_k}(\mathbf{d}_{n,i}) = \begin{cases} 0 & \text{if } 0 \leq k < n \text{ and } k \neq i \\ -\mathbf{d}_{n,0} & \text{if } 0 \leq k < n \text{ and } k = i \\ \mathbf{d}_{n,0}\mathbf{d}_{n,i} & \text{if } k = n \\ \mathcal{P}^{q^{k-1}}\mathcal{P}^{\Delta_{k-1}}\left(\mathbf{d}_{n,i}\right) & \text{if } k > n. \end{cases}$$

PROOF: •

The following extends the analogoue formulae of §3 in [31], see also Corollary 2.4 b) of [36].

COROLLARY A.2.2: *With the above notation we have*

$$\mathcal{P}^{q^{i-1}}(\mathbf{d}_{n,\ i}) = \mathbf{d}_{n,\ i-1} \quad \forall\ i = 1, \ldots, n-1.$$

PROOF: This formula can be read off directly from part [1] of the preceding proposition •

A.3 The Fractal Property of the Dickson Algebra

For the sake of completeness we will prove the fractal property of the Dickson algebra in two versions, compare also [28] proof of Theorem 8.1.6.

PROPOSITION A.3.1 (Fractal Property, Version 1): *Consider the projection*

$$\mathrm{pr} : V^* := \mathrm{Span}_{\mathbb{F}}\{z_1, \ldots, z_n\} \longrightarrow W^* := \mathrm{Span}_{\mathbb{F}}\{z_2, \ldots, z_n\}.$$

Then the induced map on the polynomial algebras $\mathbb{F}[V] \xrightarrow{\mathrm{pr}} \mathbb{F}[W]$ *maps the Dickson algebra* $\mathcal{D}^*(n)$ *onto* $\mathcal{D}^*(n-1)^q$.

PROOF: We use the Stong-Tamagawa formula given in the preceding section:

$$\begin{aligned} \mathrm{pr}(\mathbf{d}_{n,i}) &= \mathrm{pr}\left(\sum_{V'^* \leq V^*,\ \dim(V'^*)=i}\left(\prod_{v \notin V'^*} v\right)\right) \\ &= \mathrm{pr}\left(\sum_{V'^* \leq V^*,\ \dim(V'^*)=i}\left(\prod_{v=\bar{v}+\lambda z_1 \notin V'^*,\ \lambda \in \mathbb{F}} (\bar{v} + \lambda z_1)\right)\right) \end{aligned}$$

$$= \sum_{V'^* \le V^*,\ \dim(V'^*)=i, z_1 \in V'^*} \left(\prod_{v=\bar{v}+\lambda z_1 \notin V'^*,\ \lambda \in \mathbb{F}} (\bar{v}) \right)$$
$$= \sum_{V'^* \le V^*,\ \dim(V'^*)=i, z_1 \in V'^*} \left(\prod_{v=\bar{v} \notin V'^*} \bar{v}^q \right)$$
$$= \sum_{W'^* \le W^*,\ \dim(W'^*)=i-1} \left(\prod_{v=\bar{v} \notin W'^*} \bar{v}^q \right)$$
$$= \mathbf{d}^q_{n-1,i-1}$$

giving what we wanted •

PROPOSITION A.3.2 (Fractal Property, Version 2): *We have the following isomorphism in the category of unstable algebras*

$$\mathcal{D}^*(n)/(\mathbf{d}_{n,0}) \cong \mathcal{D}^*(n-1)^q.$$

PROOF : Let $\mathrm{pr} : \mathcal{D}^*(n) \longrightarrow \mathcal{D}^*(n)/(\mathbf{d}_{n,0})$ be the canonical projection and denote by

$$\overline{\mathbf{d}_{n,i}} = \mathrm{pr}(\mathbf{d}_{n,i}) \ \forall\ i = 0, \ldots, n-1$$

the images of the Dickson classes. Define a map

$$\mathcal{D}^*(n)/(\mathbf{d}_{n,0}) = \mathbb{F}[\overline{\mathbf{d}_{n,1}}, \ldots, \overline{\mathbf{d}_{n,n-1}}] \xrightarrow{\varphi} \mathcal{D}^*(n-1)^q$$

by

$$\varphi\left(\overline{\mathbf{d}_{n,i}}\right) := \mathbf{d}^q_{n-1,i-1}.$$

We have to show that this map commutes with Steenrod powers. Since for $\mathcal{P}^k$ with $k = 0$ or $k \ge q^n$ nothing needs to be shown, we induct on k, and assume in the following, that $1 \le k < q^n$. Recalling our convention, from Section 1.1, that $\mathcal{P}^i \equiv 0$ if $i \notin \mathbb{N}_0$ we have

$$\begin{aligned}
&\varphi\left(\mathcal{P}^k\left(\overline{\mathbf{d}_{n,i}}\right)\right) \\
&\stackrel{(1)}{=} \varphi\left(\mathcal{P}^{k-q^{i-1}}\left(\overline{\mathbf{d}_{n,i-1}}\right)\right) - \varphi\left(\mathcal{P}^{k-q^{n-1}}\left(\overline{\mathbf{d}_{n,n-1}}\right)\overline{\mathbf{d}_{n,i}}\right) \\
&\stackrel{(2)}{=} \mathcal{P}^{k-q^{i-1}}\left(\varphi\left(\overline{\mathbf{d}_{n,i-1}}\right)\right) - \left(\mathcal{P}^{k-q^{n-1}}\left(\varphi\left(\overline{\mathbf{d}_{n,n-1}}\right)\right)\right)\varphi\left(\overline{\mathbf{d}_{n,i}}\right) \\
&\stackrel{(3)}{=} \mathcal{P}^{k-q^{i-1}}\left(\mathbf{d}^q_{n-1,i-2}\right) - \left(\mathcal{P}^{k-q^{n-1}}\left(\mathbf{d}^q_{n-1,n-2}\right)\right)\mathbf{d}^q_{n-1,i-1} \\
&\stackrel{(4)}{=} \left(\mathcal{P}^{\frac{k-q^{i-1}}{q}}\left(\mathbf{d}_{n-1,i-2}\right)\right)^q - \left(\mathcal{P}^{\frac{k-q^{n-1}}{q}}\left(\mathbf{d}_{n-1,n-2}\right)\right)^q \mathbf{d}^q_{n-1,i-1} \\
&\stackrel{(5)}{=} \left(\mathcal{P}^{\frac{k-q^{i-1}}{q}}\left(\mathbf{d}_{n-1,i-2}\right) - \mathcal{P}^{\frac{k-q^{n-1}}{q}}\left(\mathbf{d}_{n-1,n-2}\right)\mathbf{d}_{n-1,i-1}\right)^q
\end{aligned}$$

$$\overset{(6)}{=} \left(\mathcal{P}^{\frac{k}{q}}\left(\mathbf{d}_{n-1,i-1}\right)\right)^q$$
$$\overset{(7)}{=} \mathcal{P}^k\left(\mathbf{d}^q_{n-1,i-1}\right)$$
$$\overset{(8)}{=} \mathcal{P}^k\left(\varphi\left(\overline{\mathbf{d}_{n,i}}\right)\right),$$

where (1) follows from the fact that the projection commutes with the Steenrod action and the explicit formulae given in the preceding section for the Steenrod powers of the Dickson classes, (2) follows from induction, (3) and (8) by definition of φ, (4) and (7) by the Cartan formulae, (5) from the additivity of the Frobenius map in characteristic p and (6) again from the formulae of the preceding section •

A.4 The Generalized Jacobian

A reference for the following result is hard to find in the literature. Therefore it is included here. However, this is just a careful extension of the proof given in Lemma 5.6.1 in [28] for the classical situation.

THEOREM A.4.1: *Let* H^* *be an unstable algebra over* $\mathcal{P}^*$ *and let* $h_0, \ldots, h_m \in \mathrm{H}^*$. *If the determinant of the generalized Jacobian matrix*

$$\begin{bmatrix} \mathcal{P}^{\Delta_0}(h_0) & \cdots & \mathcal{P}^{\Delta_0}(h_m) \\ \cdots & \cdots & \cdots \\ \mathcal{P}^{\Delta_m}(h_0) & \cdots & \mathcal{P}^{\Delta_m}(h_m) \end{bmatrix}$$

is neither zero nor a zerodivisor, then the elements $h_0, \ldots, h_m$ *are algebraically independent.*

PROOF : Without loss of generality we can assume that H^* is reduced. Suppose to the contrary that $h_0, \ldots, h_m$ were algebraically dependent. Then choose a polynomial $p(x_0, \ldots, x_m) \in \mathbb{F}[x_0, \ldots, x_m]$ of minimal degree such that

$$p(h_0, \ldots, h_m) = 0.$$

Then certainly $\mathcal{P}^{\Delta_j}\left(p(h_0, \ldots, h_m)\right) = 0$ for any $j \geq 0$. Write

$$p(h_0, \ldots, h_m) = \sum_{j_0, \ldots, j_m} h_0^{j_0} \cdots h_m^{j_m}.$$

Then we have that

$$\mathcal{P}^{\Delta_i}\left(p(h_0, \ldots, h_m)\right) = \sum_{j_0, \ldots, j_m} \sum_{k=0}^{m} j_k h_0^{j_0} \cdots \mathcal{P}^{\Delta_i}(h_k) h_k^{j_k - 1} \cdots h_m^{j_m}.$$

Hence

$$\begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix} = \begin{bmatrix} \mathcal{P}^{\Delta_0}\left(p(h_0, \ldots, h_m)\right) \\ \cdots \\ \mathcal{P}^{\Delta_m}\left(p(h_0, \ldots, h_m)\right) \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{P}^{\Delta_0}(h_0) & \cdots & \mathcal{P}^{\Delta_0}(h_m) \\ \cdots & \cdots & \cdots \\ \mathcal{P}^{\Delta_m}(h_0) & \cdots & \mathcal{P}^{\Delta_m}(h_m) \end{bmatrix} \begin{bmatrix} \sum_{j_0>0,\ j_1,\dots,j_m} j_0 h_0^{j_0-1} h_1^{j_1} \cdots h_m^{j_m} \\ \cdots \\ \sum_{j_m>0,\ j_0,\dots,j_{m-1}} h_0^{j_0} \cdots h_{m-1}^{j_{m-1}} h_m^{j_m-1} \end{bmatrix}.$$

By assumption the determinant of the generalized Jacobian matrix is neither zero nor a zero divisor. Therefore the vector

$$\begin{bmatrix} \sum_{j_0>0,\ j_1,\dots,j_m} j_0 h_0^{j_0-1} h_1^{j_1} \cdots h_m^{j_m} \\ \cdots \\ \sum_{j_m>0,\ j_0,\dots,j_{m-1}} h_0^{j_0} \cdots h_{m-1}^{j_{m-1}} h_m^{j_m-1} \end{bmatrix}$$

must be the zero vector. This hands us polynomials

$$p_i(x_0,\dots,x_m) := \sum_{j_i>0, j_0,\dots,j_m} j_i x_0^{j_0} \cdots x_i^{j_i-1} \cdots x_m^{j_m}$$

for $i=0,\dots,m$, of degree less than $p(x_0,\dots,x_m)$ which evaluate to zero on the elements $h_0,\dots,h_m$. Therefore these must be the zero polynomials, i.e, the coefficients

$$j_0,\dots,j_m$$

are divisible by the characteristic of the ground field $\mathbb{F}$. This in turn implies that

$$\tilde{p}(h_0,\dots,h_m) = \sum_{j_0,\dots,j_m} h_0^{\frac{j_0}{p}} \cdots h_m^{\frac{j_m}{p}}$$

is a non zero polynomial of degree less than $p(x_0,\dots,x_m)$ evaluating to zero on the elements $h_0,\dots,h_m$. This is a contradiction •

REMARK: Note that this proof would also work for an arbitrary set of derivations, i.e., we have not used the particular form of the $\mathcal{P}^{\Delta_i}$'s.

Puh!

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