

## DEGREE BOUNDS AND THE REGULAR REPRESENTATION

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ABSTRACT. Let  $\rho : G \hookrightarrow \mathrm{GL}(V)$  be a permutation representation of a finite group  $G$ . We describe universal algebra generating set for the associated ring of polynomial invariants over any field as long as the invariants are Cohen-Macaulay. This has many consequences for permutation invariants as well as for invariants of arbitrary representations. In particular, it implies Killius' Conjecture for permutation invariants and Schmid's Inequality for the general nonmodular case.

### 1. INTRODUCTION

Let  $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$  be a faithful representation of a finite group  $G$  over a field  $\mathbb{F}$ . The representation  $\rho$  induces an action of  $G$  on the vector space  $V = \mathbb{F}^n$  of dimension  $n$  and hence on the ring of polynomial functions  $\mathbb{F}[V] = \mathbb{F}[x_1, \dots, x_n]$ , where we chose a basis  $x_1, \dots, x_n$  of the dual space  $V^*$ . Our interest is focused on the subring of invariants

$$\mathbb{F}[V]^G = \{f \in \mathbb{F}[V] \mid gf = f \ \forall g \in G\},$$

which is a graded connected Noetherian commutative algebra, see [11] for more information on rings of polynomial invariants of finite groups.

We denote by  $\beta(\mathbb{F}[V]^G)$  the maximal degree of an  $\mathbb{F}$ -algebra generator of  $\mathbb{F}[V]^G$  in a minimal homogeneous generating set. By Noether's Bound we have that

$$\beta(\mathbb{F}[V]^G) \leq |G|$$

in the nonmodular case (where  $|G| \in \mathbb{F}^\times$ ), see Theorem 2.3.3 in [11]. Many refinements of this statement have been proven over the past years as well as many results valid in the modular case (where  $|G| = 0 \in \mathbb{F}$ ), see [8] for a survey on these matters. This paper has been motivated by the question whether there is a "worst" representation.

In [12] Schmid showed that

$$\beta(\mathbb{F}[V]^G) \leq \beta(\mathbb{F}[\mathbb{F}G]^G)$$

whenever the ground field  $\mathbb{F}$  has characteristic zero. It is known that this inequality does not remain valid in the modular case, where  $\mathrm{char}(\mathbb{F}) = p \mid |G|$ , see Example 1 of Section 3.2 in [11] for a counterexample. However, it was a long standing conjecture

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that Schmid's inequality holds in the general nonmodular case. Indeed, it was generalized to the strong nonmodular case (i.e.,  $\text{char}(\mathbb{F}) = 0$  or  $\text{char}(\mathbb{F}) > |G|$ ) by Smith, see [15], and for abelian groups  $G$  in the general nonmodular case by Sezer, see [13]. Furthermore, Knop proved this result for nonmodular representations where the characteristic of the ground field is zero or  $p > 3/8|G| + 1$ , see Corollary 7.5 in [5].

In this paper we prove Schmid's inequality for the general nonmodular case. Indeed, the proof follows easily from Corollary 2.12. Section 2 of this paper is devoted to the proof of that result as well as of Theorem 2.11 which gives us precise information on the set of algebra generators. In Section 3 we deduce several results on degree bounds for polynomial invariants. In particular we prove Killius' Conjecture in Theorem 3.1 for permutation representations and we prove Schmid's Inequality in Theorem 3.2 for the general nonmodular case.

## 2. RINGS OF INVARIANTS OF PERMUTATION REPRESENTATIONS

Let  $\rho : G \hookrightarrow \text{GL}(n, \mathbb{K})$  be a permutation representation of a finite group  $G$ . We note that  $\rho$  factorizes through the prime field  $\mathbb{F} = \mathbb{F}_p$ , resp.  $\mathbb{F} = \mathbb{Q}$ :

$$\begin{array}{ccc} \rho : G & \hookrightarrow & \text{GL}(n, \mathbb{K}) \\ & \searrow & \uparrow \\ & & \text{GL}(n, \mathbb{F}). \end{array}$$

We obtain for the respective rings of invariants

$$\mathbb{K}[V]^G = \mathbb{K} \otimes_{\mathbb{F}} \mathbb{F}[V]^G.$$

Thus a minimal set of  $\mathbb{F}$ -algebra generators of  $\mathbb{F}[V]^G$  minimally generates  $\mathbb{K}[V]^G$  as an  $\mathbb{K}$ -algebra as well. Therefore we can assume in what follows that the fields involved are prime fields.

Furthermore, any permutation representation  $\rho : G \hookrightarrow \text{GL}(n, \mathbb{Q})$  factorizes through  $\text{GL}(n, \mathbb{Z})$

$$\begin{array}{ccc} \rho : G & \hookrightarrow & \text{GL}(n, \mathbb{Q}) \\ & \searrow & \uparrow \\ & & \text{GL}(n, \mathbb{Z}). \end{array}$$

Taking coefficients mod  $p$  we obtain a faithful representation  $G \hookrightarrow \text{GL}(n, \mathbb{F}_p)$  which we will call  $\rho$  also. Our goal is to explicitly describe a common algebra generating set for both rings,  $\mathbb{Q}[V]^G$  and  $\mathbb{F}_p[V]^G$ , in the case that both rings are Cohen-Macaulay. (Of course,  $\mathbb{Q}[V]^G$  is always Cohen-Macaulay, see Theorem 5.5.2 in [11].)

We note that any element in a ring of permutation invariants is a sum of orbit sums. Therefore, we can consider any invariant  $f \in \mathbb{F}_p[V]^G$  as an element in  $\mathbb{Z}[V]^G \subseteq \mathbb{Q}[V]^G$ . On the other hand, after clearing denominators we can consider an invariant  $f \in \mathbb{Q}[V]^G$  as a (possibly zero) element  $f \in \mathbb{F}_p[V]^G$ .

Let  $s_1, \dots, s_m \in \mathbb{Z}[V] \subseteq \mathbb{Q}[V]$ . We note that  $s_1, \dots, s_m$  are algebraically independent in  $\mathbb{Z}[V]$  if and only if they are algebraically independent in  $\mathbb{Q}[V]$ . We have the following lemma.

**Lemma 2.1.** *Let  $s_1, \dots, s_m \in \mathbb{Z}[V] \subseteq \mathbb{Q}[V]$ . If  $s_1, \dots, s_m$  are algebraically independent in  $\mathbb{F}_p[V]$ , then they are algebraically independent in  $\mathbb{Q}[V]$ .*

*Proof.* Let  $s_1, \dots, s_m \in \mathbb{F}_p[V]$  be algebraically independent. Assume there is a polynomial  $P(X_1, \dots, X_m) \in \mathbb{Q}[X_1, \dots, X_m]$  such that

$$P(s_1, \dots, s_m) = 0.$$

By clearing denominators we can assume that  $P(X_1, \dots, X_m) \in \mathbb{Z}[X_1, \dots, X_m]$ . Without loss of generality we assume that the greatest common divisor of the coefficients of  $P(X_1, \dots, X_m)$  is 1 (or  $P(X_1, \dots, X_m) = 0$ ). Since the  $s_j$ 's are algebraically independent over  $\mathbb{F}_p$  we have that  $P(X_1, \dots, X_m) \equiv 0 \pmod{p}$ . This implies that the coefficients of  $P(X_1, \dots, X_m)$  are divisible by  $p > 1$ . Hence  $P(X_1, \dots, X_m) = 0 \in \mathbb{Z}[X_1, \dots, X_m]$ .  $\square$

We note that a ring of polynomial invariants  $\mathbb{F}[V]^G$  of a permutation representation is a finite module over the ring of invariants of the full symmetric group in  $n$  letters,  $\Sigma_n$ , in its defining representation, i.e., the extension

$$\mathbb{F}[V]^{\Sigma_n} \hookrightarrow \mathbb{F}[V]^G$$

is finite, see Exercise (1) in Section 12.6 of [9]. Denote by  $e_1, \dots, e_n$  the elementary symmetric functions in  $x_1, \dots, x_n$ . Then

$$\mathbb{F}[V]^{\Sigma_n} = \mathbb{F}[e_1, \dots, e_n]$$

is a Noether normalization for  $\mathbb{F}[V]^G$ . We note that this is true for any ground field, i.e., the elementary symmetric functions form a universal system of parameters for any ring of permutation invariants over any field  $\mathbb{F}$ . Furthermore we have

$$\mathbb{F}[V]^G = \sum_{i=1}^k \mathbb{F}[e_1, \dots, e_n] f_i$$

for some homogeneous invariants  $f_1, \dots, f_k$ . We note that this sum is direct if and only if  $\mathbb{F}[V]^G$  is Cohen-Macaulay, see Theorem 5.1.3 in [11].

Next we show that we can choose also a universal set of secondary invariants across characteristics as long as the rings of invariants involved remain Cohen-Macaulay.

**Proposition 2.2.** *Assume that  $\mathbb{F}_p[V]^G$  is Cohen-Macaulay. Let  $s_1, \dots, s_n, f_1, \dots, f_k$  be homogeneous invariant polynomials of positive degree. If*

$$(\bullet) \quad \mathbb{F}_p[V]^G = \bigoplus_{i=1}^k \mathbb{F}_p[s_1, \dots, s_n] f_i,$$

then

$$(\circ) \quad \mathbb{Q}[V]^G = \bigoplus_{i=1}^k \mathbb{Q}[s_1, \dots, s_n] f_i.$$

*Proof.* Assume that  $\mathbb{F}_p[V]^G = \bigoplus_{i=1}^k \mathbb{F}_p[s_1, \dots, s_n] f_i$ . Consider the extension

$$\sum_{i=1}^k \mathbb{Q}[s_1, \dots, s_n] f_i \subseteq \mathbb{Q}[V]^G.$$

We note that  $\mathbb{Q}[s_1, \dots, s_n]$  remains a polynomial algebra by Lemma 2.1. Since Poincaré series of rings of permutation invariants are independent of the characteristic of the ground field (see Proposition 3.2.7 in [11]), it is enough to show that this sum is direct.

We proceed by induction on  $k$ . Let  $k = 1$  and assume that

$$(1) \quad \alpha_1 f_1 = 0$$

for some  $\alpha_1 = \alpha_1(s_1, \dots, s_n) \in \mathbb{Q}[s_1, \dots, s_n]$ . By clearing denominators we can assume that

$$\alpha_1(X_1, \dots, X_n) \in \mathbb{Z}[X_1, \dots, X_n].$$

Without loss of generality we assume that  $p \nmid \alpha_1(X_1, \dots, X_n)$ . From Equation (1) we have that  $\alpha_1 f_1 \equiv 0 \pmod{p}$ . By assumption it follows that  $\alpha_1(s_1, \dots, s_n) \equiv 0 \pmod{p}$ . Since  $s_1, \dots, s_n$  are algebraically independent in  $\mathbb{F}_p[V]^G$  it follows that  $\alpha_1(X_1, \dots, X_n) \in \mathbb{F}_p[X_1, \dots, X_n]$  is the zero polynomial. Since  $p$  does not divide  $\alpha_1(X_1, \dots, X_n)$  we have that  $\alpha_1(X_1, \dots, X_n) = 0$  and hence  $\alpha_1(s_1, \dots, s_n) = 0 \in \mathbb{Z}[s_1, \dots, s_n] \subseteq \mathbb{Q}[s_1, \dots, s_n]$  as claimed.

Next let  $k > 1$  and assume there is a relation

$$(2) \quad \sum_{i=1}^k \alpha_i f_i = 0$$

for some  $\alpha_i = \alpha_i(s_1, \dots, s_n) \in \mathbb{Q}[s_1, \dots, s_n]$ ,  $i = 1, \dots, k$ . By clearing denominators we assume that the polynomials  $\alpha_i(X_1, \dots, X_n)$  have integer coefficients. By Equation (2) it follows that  $\sum_{i=1}^k \alpha_i f_i \equiv 0 \pmod{p}$ . Since the sum  $(\bullet)$  is direct it follows that

$$\alpha_i(s_1, \dots, s_n) \equiv 0 \pmod{p}.$$

Since  $s_1, \dots, s_n$  are algebraically independent in  $\mathbb{F}_p[V]^G$ , it follows that the polynomials  $\alpha_i(X_1, \dots, X_n) \in \mathbb{F}_p[X_1, \dots, X_n]$  are zero. Thus

$$\alpha_i(X_1, \dots, X_n) = p^{d_i} \alpha'_i(X_1, \dots, X_n) \in \mathbb{Z}[X_1, \dots, X_n]$$

for some  $d_i \in \mathbb{N}$  with  $p \nmid \alpha'_i(X_1, \dots, X_n)$ . Without loss of generality set  $d_k = \min\{d_1, \dots, d_k\}$ . Dividing the Equation (2) by  $p^{d_k}$  leads to

$$\sum_{i=1}^k \alpha''_i f_i = 0$$

where  $p$  does not divide  $\alpha''_k = \alpha'_k(s_1, \dots, s_n)$ . Hence  $\alpha_k(s_1, \dots, s_n) = 0$  and we are done by induction.  $\square$

*Remark 2.3.* The preceding result shows that a set of invariants  $s_1, \dots, s_n$  forming a homogeneous system of parameters for  $\mathbb{F}_p[V]^G$ , forms also a system of parameters for  $\mathbb{Q}[V]^G$ . Furthermore once we have chosen a universal system of parameters we obtain a universal set of homogeneous secondary generators  $f_1, \dots, f_k$  across characteristics as long as the ring of invariants remains Cohen-Macaulay.

*Remark 2.4.* The converse of the preceding result remains true if and only if  $s_1, \dots, s_n$  remain algebraically independent as elements in  $\mathbb{F}[V]$ .

*Remark 2.5.* If the system of parameters  $s_1, \dots, s_n \in \mathbb{F}_p[V]^G$  is not homogeneous, but none of the  $s_i$ 's has a constant term, then the preceding result remains true with some slight modification: By assumption the system of parameters  $s_1, \dots, s_n$  is contained in the irrelevant ideal  $\mathfrak{m} \subset \mathbb{F}_p[V]^G$ . Thus by localizing we obtain

$$\mathbb{F}_p[V]_{\mathfrak{m}}^G = \bigoplus_{i=1}^k \mathbb{F}_p[s_1, \dots, s_n]_{\mathfrak{n}} f_i$$

where  $\mathfrak{n} = \mathfrak{m} \cap \mathbb{F}_p[s_1, \dots, s_n]$ , see Proposition 2.2.11 in [2]. The preceding proof then shows that

$$\mathbb{Q}[V]_{\mathfrak{m}'}^G = \bigoplus_{i=1}^k \mathbb{Q}[s_1, \dots, s_n]_{\mathfrak{n}'} f_i$$

for  $\mathfrak{m}' \subseteq \mathbb{Q}[V]^G$  the irrelevant ideal and  $\mathfrak{n}' = \mathfrak{m}' \cap \mathbb{Q}[s_1, \dots, s_n]$  its contraction.

Thus we have found a common set of primary and secondary invariants. Of course it follows that

$$\mathbb{F}_p[V]^G = \mathbb{F}_p[s_1, \dots, s_n, f_1, \dots, f_k]$$

and

$$\mathbb{Q}[V]^G = \mathbb{Q}[s_1, \dots, s_n, f_1, \dots, f_k].$$

However, in both cases the set of algebra generators given should not be minimal. Our next goal is to show that in both rings the same secondary invariants can be deleted from the list of algebra generators.

**Proposition 2.6.** *We continue with the same notation as above. Assume that the invariants  $s_1, \dots, s_n, f_1, \dots, f_k$  are homogeneous. Then*

$$f_k \in \mathbb{F}_p[s_1, \dots, s_n, f_1, \dots, f_{k-1}]$$

if and only if

$$f_k \in \mathbb{Q}[s_1, \dots, s_n, f_1, \dots, f_{k-1}].$$

*Proof.* Assume that

$$f_k \in \mathbb{Q}[s_1, \dots, s_n, f_1, \dots, f_{k-1}].$$

We write

$$(\star) \quad f_k = \sum_I p_I(s_1, \dots, s_n) f^I$$

where  $f^I = f_1^{i_1} \cdots f_{k-1}^{i_{k-1}}$  and  $p_I$  are polynomials in the elements of the system of parameters. For degree reasons we have for each  $I$  appearing in the sum  $(\star)$

$$(*) \quad f^I = \lambda_I f_k + \sum_{i=1}^{k-1} q_{I,i}(s_1, \dots, s_n) f_i$$

for some coefficients  $\lambda_I \in \mathbb{Q}$ . By linear independence we obtain

$$\sum_I \lambda_I = 1.$$

Thus for all  $I_0$  such that  $\lambda_{I_0} \neq 0$  we have

$$f_k = \lambda_{I_0}^{-1} f^{I_0} + \sum_{i=1}^{k-1} \lambda_{I_0}^{-1} q_{I_0,i} f_i$$

and hence

$$\mathbb{Q}[V]^G = \bigoplus_{i=1}^{k-1} \mathbb{Q}[s_1, \dots, s_n] f_i \oplus \mathbb{Q}[s_1, \dots, s_n] f^{I_0}$$

whenever  $\lambda_{I_0} \neq 0$ . We claim that there exists at least one  $I_0$  so that we can do the same over the prime field  $\mathbb{F}_p$ . For that we multiply Equation (\*) by a minimal  $\mu \in \mathbb{Z}$  such that

$$\mu f^{I_0} = (\mu \lambda_{I_0}) f_k + \sum_{i=1}^{k-1} \mu q_{I_0, i} f_i$$

is an equation with integer coefficients. We reduce modulo  $p$ . If  $\mu \equiv 0 \pmod{p}$  then

$$\mu \lambda_{I_0} \equiv 0 \pmod{p}$$

as well as

$$\mu q_{I_0, i} \equiv 0 \pmod{p}$$

because the sum ( $\bullet$ ) is direct. This contradicts that  $\mu$  was chosen to be minimal. Thus  $\mu$  is not divisible by  $p$ . We find

$$\mu f^{I_0} = (\mu \lambda_{I_0}) f_k + \sum_{i=1}^{k-1} \mu q_{I_0, i} f_i \not\equiv 0 \pmod{p}.$$

Since  $\sum \lambda_{I_0} = 1$  and no denominator of Equation (\*) is divisible by  $p$  (recall  $\mu \neq 0 \pmod{p}$ ), there exists an index  $I_0$  such that  $\mu \lambda_{I_0} \not\equiv 0 \pmod{p}$ . Therefore

$$f_k \equiv (\mu \lambda_{I_0})^{-1} \mu f^{I_0} - \sum_{i=1}^{k-1} (\mu \lambda_{I_0})^{-1} \mu q_{I_0, i} f_i \pmod{p},$$

and

$$\mathbb{F}_p[V]^G = \bigoplus_{i=1}^{k-1} \mathbb{F}_p[s_1, \dots, s_n] f_i \oplus \mathbb{F}_p[s_1, \dots, s_n] f^{I_0}$$

as desired.

We prove the converse and assume that

$$f_k \equiv \sum_I p_I(s_1, \dots, s_n) f^I \pmod{p},$$

for some polynomials  $p_I(s_1, \dots, s_n) \in \mathbb{F}_p[s_1, \dots, s_n]$  and  $I = (i_1, \dots, i_{k-1})$ . For every multi-index  $I$  appearing in this sum we have

$$f^I \equiv \lambda_I f_k + \sum_{i=1}^{k-1} q_{I, i}(s_1, \dots, s_n) f_i \pmod{p}$$

with  $\sum_I \lambda_I \equiv 1 \pmod{p}$ . Thus for all  $I_0$  such that  $\lambda_{I_0} \not\equiv 0 \pmod{p}$  we can write

$$f_k = \lambda_{I_0}^{-1} f^{I_0} - \lambda_{I_0}^{-1} \sum_{i=1}^{k-1} q_{I_0, i}(s_1, \dots, s_n) f_i \pmod{p}.$$

Lifting to  $\mathbb{Z}[V]^G$  gives the equation

$$f^{I_0} = \lambda_{I_0} f_k + \sum_{i=1}^{k-1} q_{I_0, i}(s_1, \dots, s_n) f_i + pH$$

for some invariant  $H \in \mathbb{Z}[V]^G$ . Thus as a rational invariant we can write  $H$  as

$$H = \mu f_k + \sum_{i=1}^{k-1} q_{H, i}(s_1, \dots, s_n) f_i$$

for suitable  $\mu \in \mathbb{Q}$  and  $q_{H,i}(s_1, \dots, s_n) \in \mathbb{Q}[s_1, \dots, s_n]$ . Thus we have

$$f^{I_0} = (\lambda_{I_0} + p\mu)f_k + \sum_{i=1}^{k-1} q_{I_0,i}f_i + \sum_{i=1}^{k-1} pq_{H,i}f_i.$$

If  $\lambda_{I_0} + p\mu \neq 0$  for some index  $I_0$  we are done. Otherwise we have  $\mu = -\frac{\lambda_{I_0}}{p}$  and thus

$$f^{I_0} = \sum_{i=1}^{k-1} q_{I_0,i}f_i + \sum_{i=1}^{k-1} pq_{H,i}f_i$$

for all  $I_0$ . We multiply this equation with the minimal  $\alpha \in \mathbb{N}$  such that

$$\alpha f^{I_0} = \sum_{i=1}^{k-1} \alpha q_{I_0,i}f_i + \sum_{i=1}^{k-1} \alpha pq_{H,i}f_i$$

has integer coefficients. Modulo  $p$  this relation has to become trivial (for otherwise  $f^{I_0} \in \oplus_{i=1}^{k-1} \mathbb{F}_p[s_1, \dots, s_n]f_i$ ) and thus  $\alpha \equiv 0 \pmod{p}$ . Let  $\alpha = p^s \alpha'$  such that  $p \nmid \alpha'$ . Then  $p^{s+1}$  is the maximal  $p$ -power dividing a denominator in  $q_{H,i}$ . Thus there is an  $i_0$  such that  $\alpha pq_{H,i_0}$  is not zero modulo  $p$ . This is a contradiction, because the sum  $(\bullet)$  is direct.  $\square$

*Remark 2.7.* The preceding result actually shows that a secondary invariant  $f_k$  that is redundant as an algebra generator can be replaced by some  $f^{I_0} = f_1^{i_1} \dots f_{k-1}^{i_{k-1}}$ . Thus inductively we find

$$\mathbb{F}_p[V]^G = \mathbb{F}_p[s_1, \dots, s_n, f_1, \dots, f_l]$$

if and only if

$$\mathbb{Q}[V]^G = \mathbb{Q}[s_1, \dots, s_n, f_1, \dots, f_l].$$

*Remark 2.8.* Assume that the invariants  $s_1, \dots, s_n, f_1, \dots, f_k$  are inhomogeneous, but have no constant terms. Then they are contained in the maximal ideal generated by all homogeneous invariants of positive degree. By localizing at that maximal ideal, the preceding proof extends to that case, cf. Remark 2.5.

We are now prepared to prove that a minimal algebra generating set of  $\mathbb{F}_p[V]^G$  generates also  $\mathbb{Q}[V]^G$ .

**Proposition 2.9.** *Let  $\mathbb{F}_p[V]^G = \mathbb{F}_p[h_1, \dots, h_m]$ . Then*

$$\mathbb{Q}[V]^G = \mathbb{Q}[h_1, \dots, h_m].$$

*Proof.* By assumption we have

$$\mathbb{F}_p[V]^G = \bigoplus_{i=1}^k \mathbb{F}_p[s_1, \dots, s_n]f_i = \mathbb{F}_p[h_1, \dots, h_m].$$

Without loss of generality we assume that none of the  $h_i$ 's is redundant. Since this ring of invariants is Cohen-Macaulay there are  $\mathbb{F}_p$ -linear combinations of the  $h_i$ 's

$$L_j(h_1, \dots, h_m) \quad j = 1, \dots, n$$

such that  $L_1, \dots, L_n \in \mathbb{F}_p[V]^G$  forms a system of parameters. If the  $L_i$ 's are not homogeneous we need to localize at the irrelevant ideal in order to be able to

proceed. To simplify notation we assume they are homogeneous and write

$$\mathbb{F}_p[V]^G = \mathbb{F}_p[L_1, \dots, L_n, h_{n+1}, \dots, h_m] = \bigoplus_{i=n+1}^m \mathbb{F}_p[L_1, \dots, L_n]h_i \oplus \bigoplus_{i=1}^l \mathbb{F}_p[L_1, \dots, L_n]F_i$$

for suitable invariants  $F_i$ . By Proposition 2.2 we have

$$\mathbb{Q}[V]^G = \bigoplus_{i=n+1}^m \mathbb{Q}[L_1, \dots, L_n]h_i \oplus \bigoplus_{i=1}^l \mathbb{Q}[L_1, \dots, L_n]F_i.$$

Thus by Lemma 2.6 we have

$$\mathbb{Q}[V]^G = \mathbb{Q}[L_1, \dots, L_n, h_{n+1}, \dots, h_m]$$

as desired.  $\square$

*Remark 2.10.* We could try to reverse the preceding proof, and construct linear combinations of the algebra generators  $L_1, \dots, L_n \in \mathbb{Q}[V]^G$ . The proof then goes through if and only if  $L_1, \dots, L_n$  remain algebraically independent in  $\mathbb{F}_p[V]^G$ .

Let

$$\{x_1, \dots, x_n\} = B_1 \sqcup \dots \sqcup B_r$$

be a decomposition into disjoint  $G$ -orbits. We denote by  $e_{ij}$  the  $j$ th elementary symmetric function in the elements of  $B_i$ . Then the  $e_{ij}$ 's form a homogeneous system of parameters for  $\mathbb{F}[V]^G$  over any field  $\mathbb{F}$ . Thus the preceding results can be combined to the following:

**Theorem 2.11.** *Let  $\mathbb{F}$  and  $\mathbb{K}$  be fields. Let  $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$  be a permutation representation of a finite group  $G$  and consider the “same” representation over the field  $\mathbb{K}$ . Assume both rings of invariants,  $\mathbb{F}[V]^G$  and  $\mathbb{K}[V]^G$ , are Cohen-Macaulay. Then*

$$\mathbb{F}[V]^G = \mathbb{F}[s_{11}, \dots, s_{r|B_r|}, f_1, \dots, f_l]$$

*if and only if*

$$\mathbb{K}[V]^G = \mathbb{K}[s_{11}, \dots, s_{r|B_r|}, f_1, \dots, f_l].$$

*Proof.*  $\square$

From this description we find the following degree bound for the algebra generators.

**Corollary 2.12.** *Assume that  $G$  acts by permutations and  $\mathbb{F}_p[V]^G$  is Cohen-Macaulay. Then*

$$\beta(\mathbb{F}_p[V]^G) = \max\{\beta(\mathbb{Q}[V]^G), \delta_V\},$$

*where  $\delta_V$  is the maximal degree of an elementary symmetric function in the orbit elements of  $B_1, \dots, B_r$  that is not redundant as algebra generator.*

*Proof.* This is immediate from the description given in Theorem 2.11.  $\square$



## 3. APPLICATIONS TO GENERAL INVARIANT THEORY

We apply the results of the preceding sections to several problems of degree bounds in invariant theory of arbitrary representations of finite groups.

First we note that we can prove Killius' Conjecture for permutation representations, see [4] or Problem 7.3 in [8].

**Theorem 3.1** (Killius' Conjecture for Permutation Invariants). *Let  $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$  be a permutation representation of a finite group  $G$ . Assume that  $\mathbb{F}[V]^G$  is Cohen-Macaulay. Then*

$$\beta(\mathbb{F}[V]^G) \leq |G|.$$

*Proof.* Since the length of an orbit  $B_i$  does not exceed the group order this result follows from Corollary 2.12.  $\square$

**Theorem 3.2.** *Let  $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$  be a faithful representation of a finite group. Let  $V = \mathbb{F}^n$  be a projective  $\mathbb{F}G$ -module and let  $\mathbb{F}[n\mathbb{F}G]^G$  be Cohen-Macaulay. Then*

$$\beta(\mathbb{F}[V]^G) \leq \beta(\mathbb{F}[n\mathbb{F}G]^G).$$

*Proof.* Since  $V$  is a projective  $\mathbb{F}G$ -module, the Noether map

$$\eta_G^V : \mathbb{F}[n\mathbb{F}G]^G \longrightarrow \mathbb{F}[V]^G$$

is surjective, see Proposition 3.1 in [10]. Thus  $\beta(\mathbb{F}[n\mathbb{F}G]^G) \geq \beta(\mathbb{F}[V]^G)$ , cf. proof of Proposition 4.1 in [10].

By Corollary 2.12 we have

$$\beta(\mathbb{F}[n\mathbb{F}G]^G) = \max\{\beta(\mathbb{Q}[n\mathbb{Q}G]^G), \delta_{n\mathbb{F}G}\} = \max\{\beta(\mathbb{C}[n\mathbb{C}G]^G), \delta_{n\mathbb{F}G}\}.$$

By Weyl's Theorem the invariants of the  $n$ -fold regular representation are generated by polarizations and hence

$$\beta(\mathbb{C}[n\mathbb{C}G]^G) = \beta(\mathbb{C}[\mathbb{C}G]^G).$$

Since  $\mathbb{F}G \hookrightarrow n\mathbb{F}G$  is a direct summand we have by Theorem 1.2 in [7] that

$$\mathrm{CM} - \mathrm{defect}(\mathbb{F}[n\mathbb{F}G]^G) \leq \mathrm{CM} - \mathrm{defect}(\mathbb{F}[n\mathbb{F}G]^G) = 0.$$

Thus  $\mathbb{F}[n\mathbb{F}G]^G$  is Cohen-Macaulay, and a second application of Corollary 2.12 gives

$$\beta(\mathbb{F}[n\mathbb{F}G]^G) = \max\{\beta(\mathbb{Q}[\mathbb{Q}G]^G), \delta_{\mathbb{F}G}\} = \max\{\beta(\mathbb{C}[\mathbb{C}G]^G), \delta_{\mathbb{F}G}\}.$$

Combining these inequalities gives

$$\begin{aligned} \beta(\mathbb{F}[V]^G) &\leq \beta(\mathbb{F}[n\mathbb{F}G]^G) = \max\{\beta(\mathbb{C}[n\mathbb{C}G]^G), \delta_{n\mathbb{F}G}\} \\ &= \max\{\beta(\mathbb{C}[\mathbb{C}G]^G), \delta_{n\mathbb{F}G}\} \\ &\stackrel{(1)}{\leq} \max\{\beta(\mathbb{C}[\mathbb{C}G]^G), \delta_{\mathbb{F}G}\} = \beta(\mathbb{F}[n\mathbb{F}G]^G), \end{aligned}$$

where (1) follows since an elementary symmetric function is irredundant as an algebra generator for  $\mathbb{F}[n\mathbb{F}G]^G$  if it is so for  $\mathbb{F}[n\mathbb{F}G]^G$ .  $\square$

**Theorem 3.3** (Schmid's Inequality). *Let  $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$  be a faithful representation of a finite group  $G$ . Assume that the ground field  $\mathbb{F}$  has characteristic zero or  $p \nmid |G|$ . Then  $\beta(\mathbb{F}[V]^G) \leq \beta(\mathbb{F}[n\mathbb{F}G]^G)$ .*

*Proof.* Since  $\rho$  is a nonmodular representation the  $\mathbb{F}G$ -module  $V = \mathbb{F}^n$  is projective, and  $\mathbb{F}[n\mathbb{F}G]^G$  is Cohen-Macaulay. Thus the Theorem 3.2 applies.  $\square$

*Remark 3.4.* We note that the argument used in the preceding proof shows that rings of invariants of sums of permutation representations that are Cohen-Macaulay are generated by polarizations independent of the ground field, see [3], [5], and [14] for recent progress on polarizations.

*Remark 3.5.* Finally, note that the preceding results yield also that the relative Noether bound holds for Cohen-Macaulay permutation invariants, cf. [8].

#### REFERENCES

1. Michael F. Atiyah and Ian G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing Co., Reading MA 1969.
2. Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics 39, Cambridge University Press, Cambridge 1993.
3. Frank D. Grosshans, Vector Invariants in Arbitrary Characteristic, preprint 2006.
4. Nelson Killius, talk at the Dagstuhl Workshop on Computational Invariant Theory, Dagstuhl 1996.
5. Friedrich Knop, On Noether's and Weyl's Bound in Positive Characteristic, pp. 175-188 in: *CRM Conference Proceedings* Volume 35, edited by Eddy Campbell and David Wehlau, AMS, Providence RI 2004.
6. M. D. Neusel, Invariants of some Abelian  $p$ -Groups in Characteristic  $p$ , *Proceedings of the AMS* 125 (1997) 1921-1931.
7. Mara D. Neusel, Comparing the Depths of Rings of Invariants, pp. 189-192 in: *CRM Conference Proceedings* Volume 35, edited by Eddy Campbell and David Wehlau, AMS, Providence RI 2004.
8. Mara D. Neusel, Degree Bounds - An Invitation to Postmodern Invariant Theory -, *Topology and its Applications* 154 (2007), 792-814.
9. Mara D. Neusel, *Invariant Theory*, Student Mathematical Library Volume 36, AMS, Providence RI 2007.
10. Mara D. Neusel and Müfit Sezer, The Noether Map I, *Forum Mathematicum*, to appear.
11. Mara D. Neusel and Larry Smith, *Invariant Theory of Finite Groups*, Mathematical Surveys and Monographs Volume 94, AMS, Providence RI 2002.
12. Barbara Schmid, Finite Groups and Invariant Theory, pp. 35-66 in: *Séminaire d'Algèbre P. Dubreil et M.-P. Malliavin 1989-1990*, Lecture Notes in Mathematics 1478, Springer-Verlag, Heidelberg, Berlin 1991.
13. Müfit Sezer, Sharpening the Generalized Noether Bound in the Invariant Theory of Finite Groups, *Journal of Algebra* 254 (2002), 252-263.
14. Gerald W. Schwarz, When Polarizations Generate, preprint 2006.
15. Larry Smith, On a Theorem of Barbara Schmid, *Proceedings of the AMS*, 128 (2000), 2199-2201.

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