COHEN-MACAULAY AND GORENSTEIN COMPLEXES FROM A TOPOLOGICAL POINT OF VIEW

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Abstract. The main invariant to study the combinatorics of a simplicial complex K is the associated face ring or Stanley-Reisner algebra. Reisner respectively Stanley explained in which sense Cohen-Macaulay and Gorenstein properties of the face ring are reflected by geometric and/or combinatoric properties of the simplicial complex. We give a new proof for these result by homotopy theoretic methods and constructions. Our approach is based on ideas used very successfully in the analysis of the homotopy theory of classifying spaces.

1. Introduction

The main tool and invariant for understanding the combinatorics of a finite simplicial complex is the associated face ring or Stanley-Reisner algebra which is a quotient of a polynomial algebra generated by the vertices. It is of interest to which extend algebraic properties of the face ring are reflected by combinatorial or geometric properties of the simplicial complex. For example, Reisner characterized all simplicial complexes whose face ring is Cohen-Macaulay [18]. And Stanley proved a similar result for Gorenstein face rings [20]. In this paper we will look at these result with the eyes of a topologist and reprove both results with methods and ideas from homotopy theory, in particular from the homotopy theory of classifying spaces. For the topological proof we introduce and discuss some new spaces associated with simplicial complexes, which, as we feel, deserve interest in their own right.

Let K be an abstract simplicial complex with m vertices given by the set $V = \{1, ..., m\}$ That is, $K = \{\sigma_1, ..., \sigma_r\}$ consists of a finite set of faces $\sigma_i \subset V$, which is closed with respect to formation of subsets. The dimension of K is denoted by $\dim K = n - 1$. That is every face σ of K has order $\sharp\sigma \leq n$ and there exists a face μ of order $\sharp\mu = n$. We consider the empty set $\emptyset$ as a face of K.

For a commutative ring R with unit we denote by $R(K)$ the associated Stanley-Reisner algebra of K over the ring R. It is the quotient $R[V]/(v_\sigma : \sigma \notin K)$, where $R[V] \overset{\text{def}}{=} R[v_1, ...v_m]$ is the polynomial algebra on m-generators and $v_\sigma \overset{\text{def}}{=} \prod_{j\in\sigma} v_j$. We can think of $R(K)$ as a graded object. Since we want to bring topology into the game we will choose the topological grading and give the generators of $R(K)$ and $R[V]$ the degree 2.

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Each abstract simplicial complex K has a geometric realization, denoted by $|K|$. Let $\Delta(V)$ denote the full simplicial complex whose faces are given by all subsets of V. One realization of $\Delta(V)$ is given by the standard $m-1$-dimensional simplex $|\Delta(V)| \overset{\text{def}}{=} \{\Sigma_{i=1}^m t_i e_i : 0 \leq t_i \leq 1, \Sigma_i t_i = 1\} \subset \mathbb{R}^m$, where $e_1, ..., e_m$ denotes the standard basis. And a topological realization of K is given by the subset $|K| \subset |\Delta(V)|$ defined by the subset relation $K \subset \Delta(V)$. We define the homology $H_*(K)$ and cohomology $H^*(K)$ of K as the homology $H^*(|K|)$ and cohomology $H^*(|K|)$ of the topological realization.

Before we can state the theorems of Reisner and Stanley, we have to recall some notions. We call a simplicial complex K Cohen-Macaulay or Gorenstein over a field $\mathbb{F}$ if $\mathbb{F}(K)$ is a Cohen-Macaulay or Gorenstein algebra over $\mathbb{F}$. We call K a Gorenstein* complex if it is Gorenstein and if V equals the union of all minimal missing simplices of K. A subset $\mu \subset V$ is minimal missing if $\mu \notin K$ and for each $\sigma \subsetneq \mu$ we have $\sigma \in K$. Moreover, for any face $\sigma \in K$, the link of σ is defined as the simplicial complex

$$\operatorname{link}_K(\sigma) \overset{\text{def}}{=} \{\tau \setminus \sigma : \sigma \subset \tau \in K\}.$$

Now the theorems of Reisner and Stanley read as follows:

Theorem 1.1. *(Reisner* [18]*) Let $\mathbb{F}$ be a field and K a simplicial complex. Then, K is Cohen-Macaulay over $\mathbb{F}$ if and only if for each face σ of K, including the empty face,*

$$\widetilde{H}^i(\operatorname{link}_K(\sigma); \mathbb{F}) = 0 \textit{ for } i < \dim \operatorname{link}_K(\sigma)$$

Theorem 1.2. *(Stanley* [20]*) Let $\mathbb{F}$ be a field and K a simplicial complex. Then, K is Gorenstein* over $\mathbb{F}$ if and only if*

$$\widetilde{H}^i(\operatorname{link}_K(\sigma); \mathbb{F}) \cong \begin{cases} \mathbb{F} & \textit{for } i = \dim \operatorname{link}_K(\sigma) \\ 0 & \textit{for } i \neq \dim \operatorname{link}_K(\sigma) \end{cases}$$

For a definition of Cohen-Macaulay and Gorenstein properties see either [2], [20] or Section 4. There also exists a version of the second statement which deals with general Gorenstein complexes. But for simplification we formulated the result for Gorenstein* complexes.

Reisner used methods of commutative algebra, in particular the machinery of local cohomology of modules, to prove his theorem. Another proof by Hochster (unpublished, see [20]) is based on similar methods and a detailed analysis of the Poincaré series of $R(K)$. Similar ideas were used by Stanley in his approach towards Gorenstein complexes. In this paper we want to give a different proof for both results. Our proof is based on topological constructions related to and based on topological interpretations of the combinatorial data of K. For example, there exists a topological space $c(K)$ such that $H^*(X; \mathbb{Z}) \cong \mathbb{Z}(K)$ as a graded algebra. These spaces can be constructed as the Borel construction of toric manifolds [4], as a (pointed) colimit of a particular diagram [3] or as the homotopy colimit of the same diagram [15]. This last construction is the most appropriate for doing homotopy theory and is the one which we will use in this paper. Using the homotopy colimit construction, if $\dim K = n - 1$, one can construct a very interesting non trivial map $f : c(K) \longrightarrow BU(n)$ [17]. The Chern classes of the associated vector bundle are given by the elementary symmetric polynomials in the generators of $\mathbb{Z}(K)$ (see Section 3). The homotopy fibre X_K is a finite CW-complex, which contains a large amount of information about the associated Stanley-Reisner algebra.

Theorem 1.3.
(i) A simplicial complex K is Cohen-Macaulay over $\mathbb{F}$, if and only if $H^*(X_K;\mathbb{F})$ is concentrated in even degrees.
(ii) K is Gorenstein over $\mathbb{F}$ if and only if $H^*(X_K;\mathbb{F})$ is a Poincaré duality algebra and concentrated in even degrees.

This theorem translates Cohen-Macaulay and Gorenstein properties of $\mathbb{F}(K)$ into conditions on X_K and is the key result necessary for our proof of the results of Reisner and Stanley.

The paper is organised as follows. In the next two sections we provide the basic topological ingredients necessary for the proof of Theorem 1.1 and Theorem 1.2. In particular, we recall the above mentioned homotopy colimit construction for the space $c(K)$ in Section 2 and discuss the map $f_K : c(K) \longrightarrow BU(n)$ in Section 3. In Section 4 we provide definitions for Cohen-Macaulay and Gorenstein algebras appropriate for our purpose, express these properties in terms of the homotopy fibre X_K and prove Theorem 1.3. In Section 5 we discuss homotopy fixed point sets and study them for particular actions of elementary abelian groups on X_K. The final three sections are devoted to a proof of Theorem 1.1 and Theorem 1.2.

We will fix the following notation throughout. K always denotes a $(n-1)$-dimensional abstract finite simplicial complex with m-vertices. The set of vertices will be denoted by $V \stackrel{\text{def}}{=} V_K \stackrel{\text{def}}{=} \{1, ..., m\}$. We denote by $\textsc{ab}$ the category of abelian groups, by $\textsc{Top}$ the category of topological spaces and by $\textsc{Top}^+$ the category of pointed topological spaces. Mostly, we are working over commutative rings with unit or fields. In particular, R will always denote such a commutative ring and $\mathbb{F}$ a field. When we deal with torsion groups, we will use the topological convention for the grading. Projective resolutions are considered as non positively graded cochain complexes and torsion groups are non positively graded objects denoted by $\operatorname{Tor}_A^j(M,N)$, where $j \leq 0$.

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2. Pointed and unpointed homotopy colimits

Given a category $\textsc{c}$ and functor $F : \textsc{c} \longrightarrow \textsc{Top}$, the colimit $\operatorname{colim}_{\mathrm{C}} F$ behaves particularly poorly in the context of homotopy theory, since object-wise equivalent diagrams may well have homotopy inequivalent colimits. The standard procedure for dealing with this situation is to introduce the left derived functor, known as the *homotopy colimit*. Following [11], for example, $\operatorname{hocolim}_{\mathrm{C}} F$ may be described by the two-sided bar construction. In a similar fashion, we can construct a pointed homotopy colimit $\operatorname{hocolim}^+_{\mathrm{C}} G$ for a functor $G : \textsc{c} \longrightarrow \textsc{Top}^+$. Composing G with the forgetful functor $\phi : \textsc{Top}^+ \longrightarrow \textsc{Top}$ induces an identity $\operatorname{colim}_{\mathrm{C}} \phi G = \operatorname{colim}^+_{\mathrm{C}} G$ and a cofibre sequence

$$B\textsc{c} \longrightarrow \operatorname{hocolim}_{\mathrm{C}} \phi G \longrightarrow \operatorname{hocolim}^+_{\mathrm{C}} G$$

[1]. Here, $B\mathrm{C}$ is the classifying space of the category C, that is the topological realization of the nerve $N(\mathrm{C})$. For details of the homotopy colimit construction see [1] [6] or [7].

The cohomology of (pointed) homotopy colimits can be calculated with the help of the Bousfield-Kan spectral sequence [1]. And this tool is more important for our purpose than the actual construction of the homotopy colimit. In both cases, this is a first quadrant spectral sequence and has in the case of the unpointed homotopy colimit the form

$$E_2^{i,j} \stackrel{\text{def}}{=} \lim{}^i_{\mathrm{C}^{op}} H^j(F) \Longrightarrow H^{i+j}(\mathrm{hocolim}_{\mathrm{C}}\, F)$$

and in the case of pointed homotopy colimit the form

$$E_2^{i,j} \stackrel{\text{def}}{=} \lim{}^i_{\mathrm{C}^{op}} \widetilde{H}^j(F) \Longrightarrow \widetilde{H}^{i+j}(\mathrm{hocolim}_{\mathrm{C}}\, F)$$

In both cases, differentials $d_r : E_r^{*,*} \longrightarrow E_r^{*,*}$ have the degree $(r, 1-r)$.

Let $\phi : \mathrm{C}^{op} \longrightarrow \mathrm{AB}$ be a (covariant) functor, e.g. $\phi \stackrel{\text{def}}{=} H^*(F)$. Higher derived limits of ϕ can be thought of as the cohomology groups of a certain cochain complex $(C^*(\mathrm{C}, F), \delta)$. The groups are defined as

$$C^r(\mathrm{C}; F) \stackrel{\text{def}}{=} \prod_{c_o \to c_1 \to \ldots \to c_r} F(c_0) \quad \text{ for } r \geq 0.$$

Here, the morphism are morphisms in C and not in C^{op}. The differential $\delta : C^n(\mathrm{C}; F) \longrightarrow C^{n+1}(\mathrm{C}; F)$ is given by the alternating sum $\sum_{k=0}^r (-1)^k \delta^k$ where δ_k is defined on $u \in C^n(\phi)$ by

$$\delta^k(u)(c_0 \to \ldots \to c_{n+1}) \stackrel{\text{def}}{=} \begin{cases} u(c_0 \to \ldots \to \widehat{c}_k \to \ldots \to c_{n+1}) & \text{for } k \neq 0 \\ \phi(c_o \to c_1) u(c_1 \to \ldots \to c_n) & \text{for } k = 0. \end{cases}$$

We may and will replace $C^*(\mathrm{C}; F)$ by it's quotient $N^*(\mathrm{C}, F)$ of normalised cochains, given by the product over chains $c_0 \to c_1 \ldots \to c_r$ with distinct objects.

In most cases we consider, the Bousfield-Kan spectral sequence has a particular simple form. All higher limits will vanish. Following Dwyer [7] we turn this property into the following definition.

Definition 2.1. Let $F : \mathrm{C} \longrightarrow \mathrm{TOP}$ be a functor and R a commutative ring with unit. We call a map $f : \mathrm{hocolim}_{\mathrm{C}}\, F \longrightarrow Y$ a sharp R-homology decomposition, if $\lim^i_{\mathrm{C}^{op}} H^*(F; R) = 0$ for $i \geq 1$ and if f induces an isomorphism $\lim_{\mathrm{C}^{op}} H^*(F; R) \cong H^*(Y; R)$. If F takes values in TOP^+, then we replace cohomology by reduced cohomology.

Using the subset relation on the faces, a simplicial complex K can be interpreted as a poset and therefore as a category which we denote by $\mathrm{CAT}(K)$. This category contains the empty set $\emptyset$ as an initial object. If we want to exclude $\emptyset$ we denote this by $K^\times$ respectively by $\mathrm{CAT}(K^\times)$. In these cases the classifying space $B\mathrm{CAT}(K)$ is equivalent to the cone of the topological realization $|K|$ and $B\mathrm{CAT}(K^\times) \simeq |K|$.

For a pointed topological space Y we can define functors

$$Y^K : \mathrm{CAT}(K) \longrightarrow \mathrm{TOP}^+, \quad Y^K : \mathrm{CAT}(K) \longrightarrow \mathrm{TOP},$$

which assigns the Cartesian product Y^σ to each face σ of K. The value of Y^K on $\sigma \subset \tau$ is the inclusion $Y^\sigma \subset Y^\tau$ where the superfluous coordinates are set to $*$. We note that

$Y^K(\emptyset) \stackrel{\text{def}}{=} *$. Moreover, this functor comes with a natural transformation $Y^K \longrightarrow 1_{Y^V}$ where 1_{Y^V} denotes the constant functor mapping each face to $Y^V = Y^m$ and each morphism to the identity map.

Since $\text{CAT}(K)$ has an initial object, the classifying space $B\text{CAT}(K)$ is contractible [1] and the above cofibre sequence degenerates to a homotopy equivalence

$$\operatorname{hocolim}_{\text{CAT}(K)} Y^K \xrightarrow{\simeq} \operatorname{hocolim}^+_{\text{CAT}(K)} Y^K.$$

We want to specialise further. Let $T \stackrel{\text{def}}{=} S^1$ denote the 1-dimensional torus and $BT = \mathbb{C}P^\infty$ the classifying space of T respectively the infinite dimensional complex projective space. For the functor $BT^K : \text{CAT}(K) \longrightarrow \text{Top}$, in fact for all functors of the form Y^K, the higher derived limits of the Bousfield-Kan spectral sequence vanish and the spectral sequence collapses at the E_2-term. Only the actual inverse limit contributes something non trivial, namely the associated Stanley-Reisner algebra [15]. Defining $c(K) \stackrel{\text{def}}{=} \operatorname{hocolim}_{\text{CAT}(K)} BT^K$ we can formulate this as follows.

Theorem 2.2. [15]
(i) $H^*(\operatorname{hocolim}_{\text{CAT}(K)} BT^K; R) \cong R(K)$.
(ii) $\operatorname{hocolim}_{\text{CAT}(K)} BT^K \longrightarrow c(K)$ *is a sharp R-homology decomposition as well as* $\operatorname{hocolim}^+_{\text{CAT}(K)} BT^K \longrightarrow c(K)$.
(iii) The natural map $c(K) \longrightarrow BT^m$ realizes the algebra map $\mathbb{Z}[V] \longrightarrow \mathbb{Z}(K)$.

Proof. Part (i) and (iii) and the first half of Part (ii)are already proven in [15]. The second claim of Part (ii) follows from the equivalence between the pointed and unpointed homotopy colimit, from the fact that reduced cohomology is a natural retract from cohomology and from part (i). □

For our purpose we will also need another homology decomposition for our space $c(K)$. For two simplicial complexes K and L we define the join product by $K * L \stackrel{\text{def}}{=} \{(\sigma, \tau) : \sigma \in K, \tau \in L\}$. Then, by construction, we have $c(K * L) = c(K) \times c(L)$. We also notice that, for the full simplex $\Delta(V) \stackrel{\text{def}}{=} \{\sigma \subset V\}$ on a vertex set V, we have $c(\Delta(V)) \simeq BT^V$. This follows from the fact that $\text{CAT}(\Delta(V))$ has V as a terminal object and that therefore $\operatorname{hocolim}_{\text{CAT}(\Delta(V))} BT^{\Delta(V)} \simeq BT^K(V) = BT^V$. For every face $\sigma \in K$ we denote by $\text{st}(\sigma) \stackrel{\text{def}}{=} \text{st}_K(\sigma) \stackrel{\text{def}}{=} \{\tau \in K : \sigma \cup \tau \in K\}$ the star of σ. This is again a simplicial complex and, since $\text{st}(\sigma) = \Delta(\sigma) * \text{link}(\sigma)$ we have $c(\text{st}(\sigma)) \simeq BT^\sigma \times c(\text{link}(\sigma))$.Moreover, for $\sigma \subset \tau \in K$, we have $\text{st}_K(\tau) \subset \text{st}_K(\sigma)$ which induces a pointed map $c(\text{st}_K(\tau)) \longrightarrow c(\text{st}_K(\sigma))$. This establishes a functor

$$\text{cst}_K : \text{CAT}(K)^{op} \longrightarrow \text{Top}^+$$

defined by $\text{cst}_K(\sigma) \stackrel{\text{def}}{=} c(\text{st}_K(\sigma))$. Since the category $\text{CAT}(K)^{op}$ has an terminal object, namely $\emptyset$, we have an obvious homotopy equivalence $\operatorname{hocolim}^+_{\text{CAT}(K)^{op}} \text{cst}_K \simeq \text{cst}_K(\emptyset) = c(K)$. But restricting cst_K to the full subcategory $\text{CAT}(K^\times)^{op}$ produces a map $\operatorname{hocolim}^+_{\text{CAT}(K^\times)^{op}} \text{cst}_K \longrightarrow c(K)$ which turns out to be an equivalence as well.

Theorem 2.3.
(i) $\mathrm{hocolim}^+_{\mathrm{CAT}(K^\times)^{op}}\ \mathrm{cst}_K \longrightarrow c(K)$ *is a homotopy equivalence and a sharp R-homology decomposition.*
(ii) There exists a cofibration

$$|K| \longrightarrow \mathrm{hocolim}_{\mathrm{CAT}(K^\times)^{op}}\ \mathrm{cst}_K \longrightarrow c(K).$$

(iii)

$$\lim^i_{\mathrm{CAT}(K^\times)^{op}} H^*(\mathrm{cst}_K; R) \cong \begin{cases} R(K) \oplus \widetilde{H}^0(K; R) & \text{for } i = 0 \\ H^i(K; R) & \text{for } i > 0. \end{cases}$$

The rest of this section is devoted to a proof of this theorem. We will compare the two homotopy colimits, $\mathrm{hocolim}^+_{\mathrm{CAT}(K^\times)^{op}}\,\mathrm{cst}_K$ and $\mathrm{hocolim}^+_{\mathrm{CAT}(K)}\,BT^K$ and do this in several steps. First we show that is sufficient to take the pointed homotopy colimit of BT^K over the category $\mathrm{CAT}(K^\times)$.

Proposition 2.4. *We have an equivalence*

$$\mathrm{hocolim}^+_{\mathrm{CAT}(K^\times)}\,BT^K \xrightarrow{\simeq} \mathrm{hocolim}^+_{\mathrm{CAT}(K)}\,BT^K.$$

Moreover, $\mathrm{hocolim}^+_{\mathrm{CAT}(K^\times)}\,BT^K \longrightarrow c(K)$ *is also a sharp R-homology decomposition.*

Proof. Since $\widetilde{H}^*(*) = 0$, we have an isomorphism

$$N(\mathrm{CAT}(K^\times)^{op}, \widetilde{H}^*(BT^K)) \cong N(\mathrm{CAT}(K)^{op}, \widetilde{H}^*(BT^K))$$

of normalised cochain complexes. This shows that the map between both pointed homotopy colimits produces an isomorphism between higher limits, an isomorphism between the Bousfield-Kan spectral sequences and therefore an isomorphism in cohomology. Moreover, both homotopy colimits are simply connected. Hence, by the Whitehead theorem, the map also induces an isomorphism between the homotopy groups and is therefore a homotopy equivalence, which proves the second part. $\square$

Now, we construct a category C which contains both, $\mathrm{CAT}(K^\times)^{op}$ and $\mathrm{CAT}(K^\times)$. This will allow to compare the pointed homotopy colimits $\mathrm{hocolim}^+_{\mathrm{CAT}(K^\times)}\,BT^K$ and $\mathrm{hocolim}^+_{\mathrm{CAT}(K^\times)^{op}}\,\mathrm{cst}_K$. To do this we will distinguish between the objects of $\mathrm{CAT}(K^\times)^{op}$, denoted by τ^{op}, and the objects of $\mathrm{CAT}(K^\times)$, denoted by τ. The objects of C are given by the union of the objects of $\mathrm{CAT}(K^\times)^{op}$ and $\mathrm{CAT}(K^\times)$. That is each face τ of K generates two objects in C, namely τ and τ^{op}. There exists at most one morphism between two objects. And there are morphism $\rho \to \tau$, $\tau^{op} \to \rho^{op}$ and $\tau \to \rho^{op}$ if and only if $\rho \subset \tau$. We have obvious inclusions $\phi : \mathrm{CAT}(K^\times)^{op} \longrightarrow \mathrm{C}$ and $\psi : \mathrm{CAT}(K^\times) \longrightarrow \mathrm{C}$.

Since for $\rho \subset \tau \in K$, the set τ is a face of $\mathrm{st}_K(\rho)$, there exists a well defined pointed map $BT^\tau \longrightarrow c(\mathrm{st}_K(\rho))$. These maps are compatible with the pointed inclusions $BT^\alpha \subset BT^\beta$ as well as with the pointed maps $c(\mathrm{st}_K(\beta)) \to c(\mathrm{st}_K(\alpha))$ for $\alpha \subset \beta$. Therefore, we can define a functor $\mathrm{F} : \mathrm{C} \longrightarrow \mathrm{TOP}^+$ such that $\mathrm{F}(\tau) \stackrel{\mathrm{def}}{=} BT^\tau$ and $\mathrm{F}(\tau^{op}) \stackrel{\mathrm{def}}{=} \mathrm{cst}_K(\tau)$. In particular, $\mathrm{F}\,\psi = BT^K$ and $\mathrm{F}\,\phi = \mathrm{cst}_K$.

Given a functor $\Phi : \mathrm{D}' \longrightarrow \mathrm{D}$, for each object $d \in D$, we can define the over category $\Phi \downarrow d$. The objects are given by morphisms $i' : \Phi(d') \to d$ in D, where d' is an object of D'. And a morphism $(i' : \Phi(d') \to d) \to (i'' : \Phi(d'') \to d)$ is given by a morphism $j : d' \to d''$ of D' such that $i''\Phi(j) = i'$. The under category $d \downarrow \rho$ is defined similarly. As usual, we say that Φ is left cofinal if all over categories $\Phi \downarrow d$ and right cofinal if all under categories $d \downarrow \rho$ are contractible; i.e. the classifying spaces are contractible.

The following series of statement shows how to compare the two homotopy colimits in question.

Lemma 2.5.
(i) For each object $\tau \in \mathrm{CAT}(K^{\times}) \subset \mathrm{C}$, the under category $\tau \downarrow \phi$ is contractible.
(ii) For each object $\tau^{op} \in \mathrm{CAT}(K^{\times})^{op} \subset \mathrm{C}$, there exists an isomorphism of categories $\mathrm{CAT}(\mathrm{link}_K(\tau)) \cong \psi \downarrow \tau^{op}$ induced by $\rho \mapsto (\rho \cup \tau \to \tau^{op})$.

Proof. The first claim follows from the fact that $\tau \to \tau^{op}$ is a terminal object of $\tau \downarrow \phi$. The second claim follows from an easy straight forward calculation. □

Proposition 2.6.
(i) $\mathrm{hocolim}^{+}_{\mathrm{CAT}(K^{\times})^{op}} \mathrm{cst}_K \simeq \mathrm{hocolim}^{+}_{\mathrm{C}} \mathrm{F}$.
(ii) $\lim{}^{i}_{\mathrm{CAT}(K^{\times})} \widetilde{H}^*(\mathrm{cst}_K) \cong \lim{}^{i}_{\mathrm{C}^{op}} \widetilde{H}^*(\mathrm{F})$

Proof. Since for every object $\tau^{op} \in \mathrm{C}$ the under category $\tau^{op} \downarrow \phi$ is obviously contractible, Lemma 2.5 implies that the inclusion functor $\mathrm{CAT}(K^{\times})^{op} \to \mathrm{C}$ is right cofinal. Since the restriction of $\mathrm{F}\,|_{\mathrm{CAT}(K^{\times})^{op}} = \mathrm{cst}_K$, the equivalence between the pointed homotopy colimits in part (i) follows from [1].

For the isomorphism in (ii) we need that $(\mathrm{CAT}(K^{\times})^{op})^{op} \to \mathrm{C}^{op}$ is left cofinal [1], i.e. all under categories $c \downarrow \phi^{op} \cong (\phi \downarrow c)^{op}$ are contractible. But this follows as above. □

Proposition 2.7.
(i) $\mathrm{hocolim}^{+}_{\mathrm{CAT}(K^{\times})} BT^K \simeq \mathrm{hocolim}^{+}_{\mathrm{C}} \mathrm{F}$.
(ii) $\lim{}^{i}_{\mathrm{CAT}(K^{\times})^{op}} \widetilde{H}^*(BT^K) \cong \lim{}^{i}_{\mathrm{C}^{op}} \widetilde{H}^*(\mathrm{F})$. *In particular, for $i \geq 1$,* $\lim{}^{i}_{\mathrm{C}^{op}} \widetilde{H}^*(\mathrm{F}) = 0$.

Proof. The left Kan extension $L \overset{\mathrm{def}}{=} L_{BT^K}$ of the functor $BT^K : \mathrm{CAT}(K^{\times}) \longrightarrow \mathrm{TOP}^+$ along the functor $\mathrm{CAT}(K^{\times}) \longrightarrow \mathrm{C}$ is defined by $L(c) \overset{\mathrm{def}}{=} \mathrm{hocolim}^{+}_{\psi \downarrow c} BT^K$. And $\mathrm{hocolim}_{\mathrm{C}} L \simeq \mathrm{hocolim}_{\mathrm{CAT}(K^{\times})} BT^K$ [1]. By Lemma 2.5 and Theorem 2.2, there exists a natural transformation $L \longrightarrow \mathrm{F}$, which induces a homotopy equivalence at each object. This proves the first part.

For the second part we apply the composition of functor spectral sequence (e.g. see [10] where the dual situation for colimits is discussed in detail). That is there exists a spectral sequence

$$E_2^{i,j} \overset{\mathrm{def}}{=} \lim{}^{i}_{\mathrm{C}^{op}} \lim{}^{j}_{c \downarrow \psi^{op}} \widetilde{H}^*(BT^K) \Longrightarrow \lim{}^{i+j}_{\mathrm{CAT}(K^{\times})^{op}} \widetilde{H}^*(BT^K).$$

By Theorem 2.2 and Lemma 2.5 $\lim{}^{j}_{c \downarrow \psi^{op}} \widetilde{H}^*(BT^K) = 0$ for $j \geq 1$ and $\lim{}_{c \downarrow \psi^{op}} \widetilde{H}^*(BT^K) = \widetilde{H}^*(\mathrm{F}(c))$. This proves part (ii). □

Proof of Theorem 2.3. The first part follows from Proposition 2.6 and Proposition 2.7, and the second part from the general relation between pointed and unpointed homotopy colimits as discussed above.

In the rest of the proof all limits are taken over $\text{CAT}(K^{\times})^{op}$. Let 1_R denote the constant functor. Since a category and it's opposite category have the same geometric realization, we have $\lim^i 1_R \cong H^i(K)$. The short exact sequence

$$0 \longrightarrow \widetilde{H}^*(\text{cst}_K) \longrightarrow H^*(\text{cst}_K) \longrightarrow 1_R \longrightarrow 0$$

of functors establishes a long exact sequence of the higher limits. By part (i), this long exact sequence splits into a short exact sequence

$$0 \longrightarrow \widetilde{H}^*(c(K)) \cong \lim{}^0 \widetilde{H}^*(\text{cst}_K) \longrightarrow \lim{}^0 H^*(\text{cst}_K) \longrightarrow \lim{}^0 1_R \cong H^0(K) \longrightarrow 0$$

and isomorphisms $\lim^i H^*(\text{cst}_K) \cong \lim^i 1_R \cong H^i(K)$ for $i \geq 1$. The short exact sequence can be rewritten as

$$0 \longrightarrow R(K) \longrightarrow \lim{}^0 R(\text{st}_K) \longrightarrow \widetilde{H}^0(K) \longrightarrow 0,$$

which proves part (iii). $\square$

Remark 2.8. For later purpose we will calculate the higher limits for particular functors. Let M be an abelian group and $\psi_M : \text{CAT}(K)^{op} \longrightarrow \text{AB}$ be the atomic functor given by $\phi_M(\emptyset) \stackrel{\text{def}}{=} M$ and $\phi_M(\sigma) \stackrel{\text{def}}{=} 0$ for $\sigma \neq \emptyset$.Let $1_M : \text{CAT}(K)^{op} \longrightarrow \text{AB}$ denote the constant functor which maps all objects to M and all morphisms to the identity. Then, we have a short exact sequence

$$0 \longrightarrow \phi_M \longrightarrow 1_M \longrightarrow \psi_M \stackrel{\text{def}}{=} 1_M/\phi_M \longrightarrow 0.$$

Since $\lim_{\text{CAT}(K)^{op}} 1_M \cong M \cong \lim_{\text{CAT}(K)^{op}} \psi_M$, the long exact sequence for the higher limits establishes isomorphisms $\lim^i_{\text{CAT}(K)^{op}} \psi_M \cong \lim^{i+1} \phi_M$. Since $N^*(\text{CAT}(K)^{op}; \psi_M) \cong N^*(\text{CAT}(K^{\times})^{op}; 1_M)$, we get a sequence of isomorphisms

$$H^i(K) \cong \lim{}^i_{\text{CAT}(K^{\times})^{op}} 1_M \cong \lim{}^i_{\text{CAT}(K)^{op}} \psi_M \cong \lim{}^{i+1}_{\text{CAT}(K)^{op}} \phi_M.$$

By construction, this composition is natural with respect to maps between simplicial complexes.

3. A vector bundle over $c(K)$

In [17], Theorem 2.2 was used to construct a particular nontrivial map $c(K) \longrightarrow BU(n)$, whose construction we recall next.

Let $T^m \longrightarrow U(m)$ denote the maximal torus of the unitary group $U(m)$ given by diagonal matrices. The cohomology $H^*(BU(m); \mathbb{Z}) \cong \mathbb{Z}[c_1, ..., c_m]$ of the classifying space $BU(m)$ is a polynomial algebra generated by the Chern classes c_i and $H^*(BT^m; \mathbb{Z}) \cong \mathbb{Z}[V]$ is a polynomial algebra as well which we identify with the polynomial algebra generated by the set V of vertices. The above map induces an isomorphism $H^*(BU(m); \mathbb{Z}) \cong \mathbb{Z}[V]^{\Sigma_m} \cong \mathbb{Z}[\sigma_1, ..., \sigma_m]$, where we identify Σ_m with the Weyl group of $U(m)$ and where Σ_m acts on

$\mathbb{Z}[V]$ by permutations . Here, σ_j denotes the j-th elementary symmetric polynomial. We can and will identify the Chern classes c_j with the elementary symmetric polynomials σ_j.

Since $\dim K = n-1$, a monomial v_τ vanishes in $\mathbb{Z}(K)$ if $\sharp\tau \geq n+1$. Hence, the composition $\mathbb{Z}[c_1,...,c_m] \longrightarrow \mathbb{Z}[V] \longrightarrow \mathbb{Z}(K)$ factors through $\mathbb{Z}[c_1,...,c_n]$ and establishes a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[c_1,...,c_m] & \longrightarrow & \mathbb{Z}[V] \cong \mathbb{Z}[v_1,....,v_m] \\ \downarrow & & \downarrow \\ \mathbb{Z}[c_1,...,c_n] & \xrightarrow{\psi} & \mathbb{Z}(K) \end{array}$$

The left vertical arrow is induced by the canonical inclusion $BU(n) \longrightarrow BU(m)$, i.e. we put $c_j = 0$ for $n+1 \leq j \leq m$. The following statement concerning a topological realization of ψ is proven in [17].

Theorem 3.1. [17] *There exists a topological realization $f_K : c(K) \longrightarrow BU(n)$ of ψ, i.e. $H^*(f;\mathbb{Z}) = \psi$, which is unique up to homotopy. Moreover, the diagram*

$$\begin{array}{ccc} c(K) & \xrightarrow{f_K} & BU(n) \\ \downarrow & & \downarrow \\ BT^m & \longrightarrow & BU(m) \end{array}$$

commutes up to homotopy.

The map f_K is constructed as follows. Let $T^n \subset U(n)$ denote the maximal torus of $U(n)$ given by diagonal matrices. Since $\sharp\tau \leq n$, for each face $\tau \in K$, we can think of τ as a subset of the set $\{1,...,n\}$ which we also denote by n. Such an inclusion establishes a monomorphism $T^\tau \longrightarrow T^n \subset U(n)$ and, passing two classifying spaces, a map $f_\tau : BT^\tau \longrightarrow BU(n)$. Moreover, for a different inclusion $\tau \subset n$, the two monomorphisms $T^\tau \longrightarrow T^n$ differ only by a permutation. Hence, they are conjugate in $U(n)$ and the induced maps between the associated classifying spaces are homotopic [19]. This establishes a map from the 1-skeleton of the homotopy colimit into $BU(n)$ unique up to homotopy. There exists an obstruction theory for extending this map to a map $\operatorname{hocolim} BT^K \longrightarrow BU(n)$ as well as for the uniqueness question of such extensions [21]. The obstruction groups are higher limits of the form $\lim^j \pi_i(\mathrm{map}(BT^\tau, BU(n))_{f_\sigma})$. If we pass to completion, i.e. we replace $BU(n)$ by it's p-adic completion $BU(n)^\wedge_p$, the mapping space can be identified with $(BT^\tau \times BU(n\backslash\tau))^\wedge_p$ [14]. For $j = i, i+1$ and target $BU(n)^\wedge_p$, these higher limits do vanish [17], which is sufficient to prove existence and homotopical uniqueness of maps $f_K : c(K) \longrightarrow BU(n)^\wedge_p$ realizing ψ for all primes [1] [21]. Rationally, the map ψ can be realized, since the rationalisation $BU(n)_0 \simeq \prod_{i=1}^n K(\mathbb{Q}, 2i)$ of $BU(n)$ is a product of rational Eilenberg-MacLane spaces in even degrees. An arithmetique square argument then establishes a map $f_K \longrightarrow BU(n)$ and also shows that the homotopy class of this map is uniquely determined (for details see [17]).

As already mentioned in the introduction, we define X_K as the homotopy fibre of $f_K : c(K) \longrightarrow BU(n)$. We are particularly interested in the top degrees of $H^*(X_K)$.

Proposition 3.2.
(i) X_K has the homotopy type of a finite CW-complex of dimension $\leq n^2+n$.
(ii)

$$H^i(X_K) \cong \begin{cases} 0 & \text{for } i > n^2+n \\ H^{n-1}(K) & \text{for } i = n^2+n \end{cases}$$

(iii) If $\widetilde{H}^i(K) = 0$ for $i < n-1$, then $H^{n^2+n-1}(X_k) = 0$ and there exists a short exact sequence

$$0 \longrightarrow H^{n^2+n-2}(X_K) \longrightarrow \prod_{i \in V} \widetilde{H}^{n-2}(\mathrm{link}_K(\{i\})) \longrightarrow H^{n-1}(K) \longrightarrow 0$$

Proof. In the proof cohomology is always taken with coefficients in R. The composition $BT^\sigma \longrightarrow c(K) \xrightarrow{f_K} BU(n)$ is natural with respect to maps in $\mathrm{CAT}(K)$. Interpreting this map as the classifying map of a $U(n)$-principal bundle establishes a diagram of $U(n)$-principal bundles $Y(\sigma) \longrightarrow BT^\sigma$ over $\mathrm{CAT}(K)$. By construction, $Y(\sigma) \simeq U(n)/T^\sigma$. Since $U(n)$ acts freely on $Y(\sigma)$, the Borel construction $Y(\sigma)_{hU(n)} \stackrel{\text{def}}{=} Y(\sigma) \times_{U(n)} EU(n)$ projects to $Y(\sigma)/U(n) = BT^\sigma$ by a homotopy equivalence. These projections are natural with respect to morphisms in $\mathrm{CAT}(K)$. Since Borel constructions commute with taking homotopy colimits [6] we get a commutative diagram of fibrations

$$\begin{tikzcd} \mathrm{hocolim}_{\mathrm{CAT}(K)} Y(-) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{hocolim}_{\mathrm{CAT}(K)} BT^K \arrow[r] \arrow[d, "\simeq"'] & BU(n) \arrow[d, equal] \\ X_K \arrow[r] & c(K) \arrow[r] & BU(n) \end{tikzcd}$$

We can calculate $H^*(X_K)$ with the help of the Bousfield-Kan spectral sequence. Since the normalised cochain complex $N^i(\mathrm{CAT}(K)^{op}, \phi)$ vanishes for $i > n$ for any functor ϕ, we have $\lim^i_{\mathrm{CAT}(K)} \phi = 0$ for $i > n$. Moreover, $H^i(U(n)/T^\sigma) = 0$ for $i > n^2 = \dim U(n)$. Since $\lim^i H^j(Y(-))$ is always a finitely generated abelian group, this shows that $H^*(X_K)$ vanishes in degrees $> n^2+n$ and is a finitely generated abelian group in each degree. By construction, X_K is simply connected, and is therefore homotopy equivalent to a finite CW-complex of dimension $\leq n^2+n$.

The above argument also shows, that the group $E_2^{n,n^2} \cong \lim^n_{\mathrm{CAT}(K)} H^{n^2}(Y(-))$ is the only possibly non vanishing term of total degree n^2+n and survives in the spectral sequence. In particular, $E_2^{n,n^2} \cong H^{n^2+n}(X_K)$. On the other hand, for any $\sigma \neq \emptyset$ we have $\dim U(n)/T^\sigma < n^2$. Hence, the functor $H^{n^2}(Y(-))$ has it's only non vanishing value for $\sigma = \emptyset$ and $H^{n^2}(Y(\emptyset)) \cong H^{n^2}(U(n)) \cong R$. Hence, by Remark 2.8, $E_2^{n,n^2} \cong \widetilde{H}^{n-1}(K)$. This proves the second part of the claim.

In fact, Remark 2.8 shows that $\widetilde{H}^{i-1}(K) \cong \lim^i_{\mathrm{CAT}(K)^{op}} H^{n^2}(Y(-)) \cong E_2^{i,n^2}$. Hence, if $\widetilde{H}^j(K)$ vanishes for $j \neq n-1$, the only term of total degree n^2+n-1 is given by $\lim^n H^{n^2-1}(Y(-))$. Let $\phi \stackrel{\text{def}}{=} H^{n^2-1}(Y(-))$. Since $\dim U(n)/T^\tau \leq n^2 - \sharp\tau$, the functor ϕ vanishes for all faces τ of order ≥ 2. Moreover, for each vertex $i \in V$ the projection

$U(n) \longrightarrow U(n)/T^{\{i\}}$ induces an isomorphism $H^{n^2-1}(U(n)/T^{\{i\}}) \xrightarrow{\cong} H^{n^2-1}(U(n))$. This follows from an analysis of the Serre spectral sequence of the fibration $T^{\{i\}} \longrightarrow U(n) \longrightarrow U(n)/T^{\{i\}}$.

For $r = 0, 1$ we define functors ϕ_r by $\phi_r(\sigma) \stackrel{\text{def}}{=} \phi(\sigma)$ if $\sharp\sigma = r$ and $\phi_r(\sigma) \stackrel{\text{def}}{=} 0$ otherwise. We get short exact sequences of functors

$$0 \longrightarrow \phi_0 \longrightarrow \phi \longrightarrow \phi_1 \longrightarrow 0$$

and

$$0 \longrightarrow \phi \longrightarrow 1_R \longrightarrow \psi \stackrel{\text{def}}{=} 1_R/\phi \longrightarrow 0$$

where 1_R denotes the constant functor. The functor ψ is non trivial only for faces of order ≥ 2.

In [16] higher limits of functors defined on $\textsc{cat}(K)$ are discussed in detail. Those results show, that $\lim^j 1_R = 0$ for $j \geq 1$, that $R = \lim{}^0 1_R \cong \lim{}^0 \phi$, that $\lim{}^j \psi \cong \lim^{j+1} \phi$ for $j \geq 1$ and that $\lim^j \psi = 0$ for $j \geq n-1$. In particular, $0 = \lim{}^{n-1}\psi = \lim{}^n \phi$. This proves that $H^{n^2+n-1}(X_K) = 0$.

The first of the above sequences establishes an exact sequence

$$0 = \lim{}^{n-1}\phi_0 \longrightarrow \lim{}^{n-1}\phi \longrightarrow \lim{}^{n-1}\phi_1 \longrightarrow \lim{}^n\phi_0 \longrightarrow \lim{}^n\phi = 0$$

By part (i) the fourth term can be identified with $\widetilde{H}^{n-1}(K)$, by [16] the third term with $\prod_{i\in V} \widetilde{H}^{n-2}(\text{link}(\{i\}))$, and the second term with $H^{n^2+n-2}(X_K)$. This finishes the proof of the third part. $\square$

Corollary 3.3. *Let $L \subset K$ be a subcomplex of the same dimension. Then, the composition*

$$H^{n-1}(K) \cong H^{n^2+n}(X_K) \longrightarrow H^{n^2+n}(X_L) \cong H^{n-1}(L)$$

is the map induced in cohomology by the inclusion.

Proof. This follows from the above proof and Remark 2.8 $\square$

We are also interested in the top degree of $H^*(X_{\text{st}(\{i\})})$.

Lemma 3.4. *$X_{\text{st}(\{i\})}$ has the homotopy type of a finite CW-complex of dimension $\leq n^2 + n - 2$ and $H^{n^2+n-2}(X_{\text{st}(\{i\})}) \cong H^{n-2}(\text{link}(\{i\}))$.*

Proof. Since $c(\text{st}(\{i\})) \simeq BT^{\{i\}} \times c(\text{link}(\{i\}))$ we have a commutative diagram of fibrations

$$\begin{tikzcd}
X_{\text{link}(\{i\})} \arrow[r] \arrow[d] & c(\text{link}(\{i\})) \times BT^{\{i\}} \arrow[r] \arrow[d, "\simeq"'] & BU(n-1) \times BT^{\{i\}} \arrow[d] \\
X_{\text{st}(\{i\})} \arrow[r] & c(\text{st}(\{i\})) \arrow[r] & BU(n)
\end{tikzcd}$$

Since the homotopy fibre of the right vertical arrow is homotopy equivalent to the $n-1$-dimensional complex projective space $\mathcal{P}(n-1)$, this establishes a fibration $X_{\text{link}(\{i\})} \longrightarrow X_{\text{st}(\{i\})} \longrightarrow \mathcal{P}(n-1)$. This shows that $X_{\text{st}(\{i\})}$ is simply connected and has the homotopy type of a finite CW-complex of dimension $\leq (n-1)^2 + (n-1) + 2(n-1) = n^2 + n - 2$

and that $H^{n^2+n-2}(X_{\mathrm{st}(\{i\})}) \cong H^{(n-1)^2+(n-1)}(\mathrm{link}(\{i\}))$. The last equation follows from an analysis of the Serre spectral sequence of the above fibration. $\square$

For later purpose we need the following lemma.

Lemma 3.5. *$R(K)$ is a finitely generated $R[c_1, ..., c_n]$-module.*

Proof. Since $R[V] \longrightarrow R(K)$ is a surjection and since $R[V]$ is a finitely generated $R[c_1, ..., c_m]$-module, the same holds for $R(K)$ as $R[c_1, ..., c_m]$-module. By Theorem 3.1 the map $R[c_1, ..., c_m] \longrightarrow R(K)$ factors through $R[c_1, ..., c_n]$, which implies the statement. $\square$

We finish this section with the following observation:

Remark 3.6. The composition $c(\mathrm{st}\,(\tau)) \longrightarrow c(K) \xrightarrow{f_K} BU(n)$ makes $R(\mathrm{st}(\tau))$ into a $R[c_1, ..., c_n] \cong H^*(BU(n); R)$-module and, with respect to this structure, all differentials of the normalised chain complex $N^*(\mathrm{CAT}(K); H^*(c(\mathrm{st}_K); R)$ become $H^*(BU(n); R)$-linear. Hence, $\lim^i H^*((\mathrm{cst}_K); R) = \lim^i R(\mathrm{st}_K)$ is an $H^*(BU(n); R)$-module. The proof of Theorem 2.3 shows that part (iii) can be refined. There exists a short exact sequence

$$0 \longrightarrow R(K) \cong H^*(c(K); R) \longrightarrow \lim{}^0 H^*(\mathrm{cst}_K); R) \cong \lim{}^0 R(\mathrm{st}_K) \longrightarrow \widetilde{H}^0(K; R) \longrightarrow 0$$

of $H^*(BU(n); R)$-modules. Here, $H^*(BU(n); R)$ acts on $\lim^i R(\mathrm{st}_K) \cong H^i(K; R)$ as well as on $\widetilde{H}^0(K; R)$ via the augmentation $H^*(BU(n); R) \longrightarrow R$.

4. Cohen-Macaulay and Gorenstein conditions

In this section we assume that $R = \mathbb{F}$ is a field and cohomology is always taken with coefficients in $\mathbb{F}$. In particular, $H^*(-) \overset{\mathrm{def}}{=} H^*(-; \mathbb{F})$.

Let A^* be a non negatively graded commutative algebra over $\mathbb{F}$. We say that A^* is connected if $A^0 \cong \mathbb{F}$ and $\mathbb{F}$-finite if $A^j = 0$ for j large and A^j is a finitely generated $\mathbb{F}$-module in each degree. We call a finite sequence of elements $a_1, .., a_r \in A$ a homogeneous system of parameters, a hsop for short, if they are homogeneous and algebraically independent and if the quotient $A^*/(a_1, ..., a_r)$ is $\mathbb{F}$-finite. We say that the sequence is a regular sequence, if, for all i, a_{i+1} is not a zero divisor in $A/(a_1, ..., a_i)$.

We call A^* Cohen-Macaulay, if there exists a sequence $a_1, ..., a_n$ which is both, a hsop and regular. If A^* is Cohen-Macaulay, then it is known, that every hsop is also a regular sequence [2].

A Noetherian local ring S is called Gorenstein if S considered as a module over itself has a finite injective resolution. If A^* is a commutative connected non negatively graded algebra, we can use the following equivalent definition [20, Theorem I.12.4]. That is, A^* is Gorenstein, if A^* is Cohen-Macaulay and if for any hsop $a_1, ...a_n$ of A^*, we have $\mathrm{soc}(A^*/a_1, ..., a_n) \cong \mathbb{F}$. The socle $\mathrm{soc}(B^*)$ of a non negatively graded algebra B^* is defined as $\mathrm{soc}(B^*) \overset{\mathrm{def}}{=} \{b \in B^* : B^+ b = 0\} \cong \mathrm{Hom}_{B^*}(\mathbb{F}, B^*)$ where B^+ denotes the elements of positive degree.

We call A^* a Poincaré duality algebra, or PD-algebra for short, if there exists a class $[A] \in \mathrm{Hom}_{\mathbb{F}}(A^*, \mathbb{F})$ such that the composition $A^* \otimes A^* \longrightarrow A^* \xrightarrow{[A]} \mathbb{F}$ is a non degenerate bilinear form. In particular, every PD-algebra is connected. As a straight forward argument shows, the above condition is equivalent to the fact that $\operatorname{soc} A^* \stackrel{\mathrm{def}}{=} \operatorname{Hom}_{A^*}(\mathbb{F}, A^*) \cong \mathbb{F}$. That is, a $\mathbb{F}$-finite non negatively graded algebra is a PD-algebra if and only if it is Gorenstein. If $A^* \cong H^*(X; \mathbb{F})$ for a topological space, then we call X a Poincaré duality space over $\mathbb{F}$, a $\mathbb{F}$-PD-space for short, if A^* is a PD-algebra. In this case $[X] \stackrel{\mathrm{def}}{=} [A] \in H_*(X)$ is the fundamental class of X.

We want to describe Cohen-Macaulay and Gorenstein properties of the face ring $\mathbb{F}(K)$ in terms of the map $f_K : c(K) \longrightarrow BU(n)$ described in Theorem 3.1 and it's homotopy fibre X_K. The next result contains part (i) of Theorem 1.3.

Theorem 4.1. *The following conditions are equivalent:*
(i) $\mathbb{F}(K)$ *is Cohen-Macaulay.*
(ii) The map $c(K) \longrightarrow BU(n)$ *makes* $\mathbb{F}(K)$ *into a finitely generated free* $H^*(BU(n))$*-module.*
(iii) The cohomology ring $H^*(X_K)$ *is concentrated in even degrees.*
(iv) $\operatorname{Tor}^j_{H^*(BU(n))}(\mathbb{F}(K), \mathbb{F}) = 0$ *for* $j \leq -1$.

For the proof we need the following lemma. Again we denote by σ_j the j-th elementar symmetric polynomial in the generators of $\mathbb{F}(K)$ and identify $H^*(BU(n))$ with $\mathbb{F}[\sigma_1, ..., \sigma_n]$, that is with image of the map $H^*(BU(n)) \longrightarrow H^*(c(K))$.

Lemma 4.2. *The sequence* $\sigma_1, .., \sigma_n \in \mathbb{F}(K)$ *is a hsop for* $\mathbb{F}(K)$.

Proof. By Lemma 3.5 $\mathbb{F}(K)/(\sigma_1, ..., \sigma_n)$ is $\mathbb{F}$-finite. We have only to show that the elements $\sigma_1, ..., \sigma_n$ are algebraic independent in $\mathbb{F}(K)$.

Let μ be a maximal face of K, that is $\sharp\mu = n$. The composition $BT^\mu \longrightarrow c(K) \longrightarrow BU(n)$ is induced by a maximal torus inclusion $T^\mu \longrightarrow U(n)$ (see Section 3). The images of $\sigma_1, ..., \sigma_n$ in $H^*(BT^\mu) \cong \mathbb{F}[\mu]$ are given by the elementary symmetric polynomials and therefore are algebraic independent as well as $\sigma_1, ...\sigma_1$ in $\mathbb{F}(K)$. □

Proof of Theorem 4.1: If $\mathbb{F}(K)$ is Cohen-Macaulay, then every hsop is given by a regular sequence. And if we have a hsop of $\mathbb{F}(K)$ given by a regular sequence, then $\mathbb{F}(K)$ is Cohen-Macaulay [2]. In the light of Lemma 4.2 this shows that the first two conditions are equivalent.

If $H^*(X_K; \mathbb{Z})$ is concentrated in even degrees (part (iii)), then, by degree reasons, the Serre spectral sequence for the fibration $X_K \longrightarrow c(K) \longrightarrow BU(n)$ collapses at the E_2 page and $H^*(c(K)) \cong H^*(X_K) \otimes_{\mathbb{F}} H^*(BU(n))$ as $H^*(BU(n))$-module. This shows that $\mathbb{F}(K)$ is a finitely generated free $H^*(BU(n))$-module and therefore Cohen-Macaulay, which is part (i).

If $\mathbb{F}(K)$ is Cohen-Macaulay, then, by (ii), it is a finitely generated free module over $H^*(BU(n))$. In particular, $\operatorname{Tor}^j_{H^*(BU(n))}(\mathbb{F}(K), \mathbb{F}) = 0$ for $j \leq -1$. This is part (iv).

If condition (iv) is satisfied, the Eilenberg-Moore spectral sequence for calculating $H^*(X_K; \mathbb{F})$ collapses at the E_2-page and shows that $H^*(X_K; \mathbb{F}) \cong \mathbb{F}(K) \otimes_{H^*(BU(n);\mathbb{F})} \mathbb{F}$ and that $H^*(X_K; \mathbb{F})$ is concentrated in even degrees, which is condition (iii). This proves the equivalence of (i), (iii) and (iv). □

Proof of Theorem 1.3 (ii): By our definition of Gorenstein, $\mathbb{F}(K)$ is Gorenstein if and only if $\mathbb{F}(K)$ is Cohen-Macaulay and $\mathbb{F}(K)\otimes_{\mathbb{F}[\sigma_1,...,\sigma_n]}\mathbb{F} \cong \mathbb{F}(K)/(\sigma_1,...,\sigma_n)$ is a PD-algebra. Hence the equivalence of the two conditions follows from the first part of the Theorem. $\square$

5. Homotopy fixed point sets

For an action of a group G on a space X we can think of the fixed point set as the mapping space $X^G = \mathrm{map}_G(*, X)$ of G-equivariant maps from a point into X. The notion is not flexible enough for doing homotopy theory, since a homotopy equivalence between G-spaces, which happens to be G-equivariant in addition, does not induce an equivalence between the fixed-point sets in general. We therefore are interested in homotopy fixed point sets which do have this property. They are defined as the equivariant mapping space $X^{hG} \overset{\mathrm{def}}{=} \mathrm{map}_G(EG, X)$ where EG is a contractible G-CW-complex with a free G-action. The projection $EG \longrightarrow *$ induces a map $X^G \longrightarrow X^{hG}$.

Applying the Borel construction establishes a fibration

$$X \longrightarrow X_{hG} \overset{\mathrm{def}}{=} X \times_G EG \xrightarrow{\pi} BG.$$

A straight forward argument shows that we can equivalently define the homotopy fixed point set as the space $\Gamma(X_{hG} \to BG)$ of sections of this fibration. The latter definition also allows to define homotopy fixed point sets in more general situations. A proxy G-action on X is a fibration $X \longrightarrow E \xrightarrow{\pi} BG$, where we think of E as the Borel construction of this action. We define $X^{hG} \overset{\mathrm{def}}{=} \Gamma(E \to BG)$. This establishes a fibration

$$X^{hG} \longrightarrow \mathrm{map}(BG, E)_{\{id\}} \longrightarrow \mathrm{map}(BG, BG)_{id}$$

Here, the middle term consist of all lifts of the identity id of BG up to homotopy, i.e. of all maps $g : BG \longrightarrow E$ such that $\pi f \simeq id$. If G is a finite abelian group the base space is homotopy equivalent to BG [12], and composition of maps yields an equivalence $BG \times X^{hG} \xrightarrow{\simeq} \mathrm{map}(BG, E)_{\{id\}}$. This equivalence fits into a commutative diagram of fibrations

$$\begin{tikzcd}
X^{hG} \arrow[r] \arrow[d] & BG \times X^{hG} \arrow[r] \arrow[d] & BG \arrow[d, equal] \\
X \arrow[r] & E \arrow[r] & BG
\end{tikzcd}$$

where the vertical arrows are induced by evaluation at a basepoint.

Typical examples of proxy actions arise from pull back constructions. Let $X \longrightarrow E \xrightarrow{\pi} B$ be a fibration and $f : BG \longrightarrow B$ be a map. The pull back construction establishes a commutative diagram

$$\begin{tikzcd}
E' \arrow[r, "\pi'"] \arrow[d] & BG \arrow[d, "f"'] \\
E \arrow[r, "\pi"] & B
\end{tikzcd}$$

and applying the mapping space functor yields a pull back diagram

$$\begin{array}{ccc} \mathrm{map}(BG, E')_{\{id\}} & \xrightarrow{\pi'_*} & \mathrm{map}(BG, BG)_{id} \\ \downarrow & & \downarrow f \\ \mathrm{map}(BG, E)_{\{f\}} & \xrightarrow{\pi_*} & \mathrm{map}(BG, B)_f. \end{array}$$

The fibre of both horizontal arrows is given by the homotopy fixed point set X^{hG}. Here, the left mapping space in the bottom row consists of all maps $BG \longrightarrow E$ which are homotopic to a lift of f. If G is finite and abelian the composition $BG \simeq \mathrm{map}(BG, BG)_{id} \longrightarrow \mathrm{map}(BG, B)_f \xrightarrow{ev} B$ equals the map $f : BG \longrightarrow B$.

The main goal of this section is to show that, in favourable cases, $H^*(X^{hG}; \mathbb{F}_p)$ is concentrated in even degrees or a PD-algebra, if $H^*(X; \mathbb{F}_p)$ satisfies these properties. For our method of proof we have to make some restrictions. According to our situation, we have the spaces X_K in mind, we will always assume that:

(1) X is $\mathbb{F}_p$-finite and p-complete.
(2) $H^*(X; \mathbb{F}_p)$ is concentrated in even degrees.

In particular, $H^*(X; \mathbb{F}_p)$ is a graded algebra, commutative in the non graded sense. Spaces satisfying both conditions will be called special.

We also restrict ourselves to particular proxy actions. We say that a proxy action $X \longrightarrow E \longrightarrow BG$ is orientable if $\pi_1(BG)$ acts trivially on $H^*(X; \mathbb{Z}_p^\wedge)$.

Remark 5.1. If G is an elementary abelian p-group, and the G-action extends to a torus action of T^r, i.e. the proxy action is induced by a pull back of a fibration $X \longrightarrow E' \longrightarrow BT^r$, the group G always acts trivially on the cohomology of X and the proxy action is orientable.

Moreover, if the proxy action $X \longrightarrow E \longrightarrow BG$ is orientable, the Serre spectral sequence for $H^*(-; \mathbb{Z}_p^\wedge)$ collapses at the E_2-page by degree reasons. And the same holds for $H^*(-; \mathbb{F}_p)$. In particular, $H^*(E; \mathbb{F}_p) \cong H^*(BG; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p)$ as $H^*(BG; \mathbb{F}_p)$-module [5, Corollary 2.5] .

The first part of the next theorem is due to Dehon and Lannes [5, Corollary 2.10].

Theorem 5.2. *Let G be an elementary abelian p-group and $X \longrightarrow E \longrightarrow BG$ an orientable proxy G-action on a space X.*
(i) If X is special, then so is X^{hG}.
(ii) If X is special and a $\mathbb{F}_p$-PD-space, then so is each component of X^{hG}.

The proof of the second part needs some preparation. For the rest of this section we make the following assumptions. $H^*(-)$ denotes $H^*(-; \mathbb{F}_p)$, G is an elementary abelian p-group and $H^G \overset{\mathrm{def}}{=} H^*(BG)$. Moreover, X is special and the proxy G-action $X \longrightarrow E \longrightarrow BG$ is orientable. In particular, $H^*(E) \cong H^G \otimes H^*(X)$ as H^G-module (see Remark 5.1).

For such actions $H^*(E)$ is not concentrated in even degrees and hence not commutative in the non graded sense. To avoid technical difficulties we will use the following construction. We denote by $J \subset H^G$ the ideal generated by all classes of degree 1. And for an graded H^G-module M we denote by $\bar{M}$ the quotient M/JM. If $G \cong (\mathbb{Z}/p)^r$, the composition

$H^*(BT^r) \longrightarrow H^G \longrightarrow \bar{H}^G$ is an isomorphism, where the first map is induced by the canonical inclusion $G \cong (\mathbb{Z}/p)^r \subset T^r$. In particular, $\bar{H}^G$ is a polynomial algebra generated by elements of degree 2. In fact, we can think of it as the polynomial part and as a subalgebra of H^G. Hence, every H^G-module is naturally an $\bar{H}^G$-module.

Lemma 5.3. *$H^*(X)$ is a PD-algebra if and only if $\bar{H}^*(E)$ is Gorenstein.*

Proof. By assumption, $H^*(E) \cong H^*(X) \otimes H^G$ as H^G-module. And hence, $\bar{H}^*(E) \cong H^*(X) \otimes \bar{H}^G$ as $\bar{H}^G$-module. Since $\bar{H}^G$ is a polynomial algebra, this implies that $\bar{H}^*(E)$ is Cohen-Macaulay. And hence, $\bar{H}^*(E)$ is Gorenstein if and only if $H^*(X) \cong \bar{H}^*(E) \otimes_{\bar{H}^G} \mathbb{F}_p$ is a PD-algebra. □

Let $S \subset H^G$ denote the multiplicative subset generated by all Bockstein images in degree 2 of nontrivial elements of degree 1. That is the subset of all images of non trivial elements of $H^*(BT^r)$ of strictly positive degree. For any H^G-module M we denote by $S^{-1}M$ the localised module over $S^{-1}H^G$. Let $X \longrightarrow E \longrightarrow BG$ be a G-proxy action on Y. Following [8] (Corollary 1.2 and the following remark), there exists a map $S^{-1}H^*(E) \longrightarrow S^{-1}(H^G \otimes H^*(Y^{hG}))$ between the localised modules. Under favourable circumstances this is an isomorphism, which, as shown in [5, Section 2], always hold if X is special and if the G-action is orientable. We collect this into the following theorem.

Theorem 5.4. *([8] [5]) Let $X \longrightarrow E \longrightarrow BG$ be an orientable G-proxy action on a special space X. Then there exist isomorphisms*

$$S^{-1}H^*(E) \cong S^{-1}(H^G \otimes H^*(X^{hG})) \cong (S^{-1}H^G) \otimes H^*(X^{hG}).$$

Since the multiplicative subset $S \subset H^G$ consist of elements of even degree, it also is a multiplicative subset of $\bar{H}^G$. We can think of the localised module $S^{-1}\bar{H}^*(E) \cong \bar{H}^*(E) \otimes_{\bar{H}^G} S^{-1}\bar{H}^G$ in a different way. Since $\bar{H}^G \longrightarrow \bar{H}^*(E)$ is a monomorphisms, the set S gives rise to a multiplicative subset $S' \subset \bar{H}^*(E)$. Then multiplication induces an isomorphism $\bar{H}^*(E) \otimes_{\bar{H}^G} S^{-1}\bar{H}^G \cong (S')^{-1}\bar{H}^*(E)$, where the latter localisation is obtained by localising the algebra $\bar{H}^*(E)$ with respect to the subset S'.

Proof of Theorem 5.2 (ii): Let us assume that X is an $\mathbb{F}_p$-PD-space. Then, $\bar{H}^*(E)$ is Gorenstein (Lemma 5.3) as well as $S^{-1}\bar{H}^*(E)$ [2, Proposition 3.1.19]. Since

$$S^{-1}\bar{H}^*(E) \cong H^*(X^{hG}) \otimes S^{-1}\bar{H}^G \cong \bigoplus_g H^*(X_g^{hG}) \otimes S^{-1}\bar{H}^G$$

as algebras, each of the summands is Gorenstein. Here, the direct sum is taken over components of X^{hG}. Let X_g^{hG} denotes the component associated to a section $g : BG \longrightarrow E$ of $\pi : E \longrightarrow BG$. Now we give the algebra $H^*(X_g^{hG}) \otimes S^{-1}\bar{H}^G$ a different grading induced by the grading of the first factor. That is elements of $S^{-1}H^G$ get degree 0. Since $\bar{H}^G \cong \mathbb{F}_p[t_1, ..., t_n]$, the sequence $\{t_1 - 1, ...t_n - 1\} \subset H^*(X_g^{hG}) \otimes S^{-1}\bar{H}^G$ is regular, homogeneous and consists of elements of degree 0. Hence $H^*(X_g^{hG}) \cong H^*(X_g^{hG}) \otimes S^{-1}\bar{H}^G/(t_1-1, ..., t_n-1)$ is a connected $\mathbb{F}_p$-finite Gorenstein algebra and therefore a PD-algebra (see Section 4). □

6. Cohen-Macaulay and Gorenstein properties for links of faces

In this section we want to prove that Cohen-Macaulay or Gorenstein properties are inherited to the Stanley Reisner algebras of links of faces. We call a simplicial complex pure, if all maximal faces have the same dimension. We first consider the case of algebras over $\mathbb{F}_p$.

Theorem 6.1. *If $\mathbb{F}_p(K)$ is Cohen-Macaulay respectively Gorenstein, then K is pure and, for each face $\tau \in K$ the algebra $\mathbb{F}_p(\mathrm{link}_K(\tau))$ is also Cohen-Macaulay respectively Gorenstein.*

For the proof we will use the space X_K given by the fibration $X_K \longrightarrow c(K) \longrightarrow BU(n)$ described in Section 3. Since $BU(n)$ is simply connected, the p-adic completion maintains the fibration [1]. Since we are working with $\mathbb{F}_p$ as coefficients and since for simply connected spaces completion induces an isomorphism in mod-p cohomology, we can and will assume that all spaces are completed. To simplify notation, we will always drop the notation for completion. The proof also relies on the interpretation of $\mathbb{F}_p(\mathrm{link}(\tau))$ as a certain mapping space given in [16], which we recall next.

For each face $\tau \in K$ we denote by $G^\tau \subset T^\tau$ the maximal elementary abelian subgroup of the torus T^τ. Let $g_\tau : BG^\tau \longrightarrow c(K)$ denote the composition $BG^\tau \longrightarrow BT^\tau \longrightarrow \mathrm{hocolim}_{\mathrm{CAT}(K)} BT^K \simeq c(K)$. The composition $BG^\tau \xrightarrow{g_\tau} c(K) \xrightarrow{f_K} BU(n)$ establishes a proxy G^τ-action $X_K \longrightarrow E \longrightarrow BG^\tau$. Applying the mapping space functor we get a fibration

$$X^{hG^\tau} \longrightarrow \mathrm{map}(BG^\tau, c(K))_{\{f_K g_\tau\}} \xrightarrow{(f_K)_*} \mathrm{map}(BG^\tau, BU(n))_{f_K g_\tau}$$

(see Section 5). The composition $f_K g_\tau$ is induced from a coordinate-wise inclusion $G^\tau \subset T^\tau \subset T^n \subset U(n)$ into the set of diagonal matrices (see Section 3). The centraliser of this image equals $T^\tau \times U(n \setminus \tau)$ where we again think of n as the set $\{1, ..., n\}$ and τ becomes a subset of the set n via the coordinate-wise inclusion. Then, by construction, $f_K g_\tau$ factors through a map $\mathrm{id} \times \mathrm{const} : BT^\tau \longrightarrow BT^\tau \times BU(n \setminus \tau)$ where the fist coordinate is the identity and the second the constant map. There also exists a map $BT^\tau \times c(\mathrm{link}(\tau)) = c(\mathrm{st}(\tau)) \longrightarrow c(K)$ as constructed in Section 2 and the map g_τ factors through $i \times \mathrm{const} : BG^\tau \longrightarrow BT^\tau \times c(\mathrm{link}(\tau))$. Moreover all these maps fit into a diagram

$$\begin{tikzcd}
X_{\mathrm{link}(\tau)} \arrow[r] \arrow[d] & BT^\tau \times c(\mathrm{link}(\tau)) \arrow[r, "id \times f_{\mathrm{link}(\tau)}"] \arrow[d] & BT^\tau \times BU(n \setminus \tau) \arrow[d] \\
X_K \arrow[r] & c(K) \arrow[r, "f_K"] & BU(n).
\end{tikzcd}$$

Applying the mapping space functor, there exit the following equivalences (of p-completed) spaces; $BT^\tau \xrightarrow{\simeq} \mathrm{map}(BG^\tau, BT^\tau)_i$, $BT^\tau \times BU(n \setminus \tau) \xrightarrow{\simeq} \mathrm{map}(BG^\tau, BU(n)_{f_K g_\tau}$ [9] and $BT^\tau \times c(\mathrm{link}(\tau)) \xrightarrow{\simeq} \mathrm{map}(BG^\tau, c(K))_{g_\tau}$ [16]. Putting all this information together we have a commutative diagram

$$\begin{array}{ccccc}
X_{\mathrm{link}(\tau))} & \longrightarrow & BT^\tau \times c(\mathrm{link}(\tau)) & \xrightarrow{f_{\mathrm{link}(\tau))}} & BU(n \setminus \tau) \\
\downarrow & & \simeq\downarrow & & \simeq\downarrow \\
(X_K)^{hG^\tau}_{g_\tau} & \longrightarrow & \mathrm{map}(BG^\tau, c(K))_{g_\tau} & \longrightarrow & \mathrm{map}(BG^\tau, BU(n))_{f_K g_\tau}
\end{array}$$

of horizontal fibrations. Since $BT^\tau \times BU(n \setminus \tau)$ is simply connected, the map $X_K^{hG^\tau} \longrightarrow \mathrm{map}(BG^\tau, c(K))_{\{f_K g_\tau\}}$ induces a bijection between the components of both spaces. Hence, the bottom left space is connected. This proves the following proposition.

Proposition 6.2. $X_{\mathrm{link}(\tau)} \simeq (X_K)^{hG^\tau}_{g_\tau}$.

Now we are in the position to prove Theorem 6.1.

Proof of Theorem 6.1 If $\mathbb{F}_p(K)$ is Cohen-Macaulay, the fibre X_K is special (Theorem 1.3(i)). Since the map $f_K g_\tau : BG^\tau \longrightarrow BU(n)$ factors through BT^τ, the proxy G^τ action extends to a torus action and is orientable (see Remark 5.1). We can apply Theorem 5.2. That is that $(X_K)^{hG^\tau}_{g_\tau} \simeq X_{\mathrm{link}(\tau)}$ is special and that $\mathbb{F}_p(\mathrm{link}(\tau))$ is Cohen-Macaulay (Theorem 1.3(i)). If $\mathbb{F}_p(K)$ is Gorenstein, we use Theorem 1.3(ii) instead of the first part.

Finally we have to show that K is pure. The above argument shows that, if $\mathbb{F}_p(K)$ is Cohen-Macaulay, then $\mathbb{F}_p(\mathrm{link}(\tau))$ is a free $H^*(BU(n \setminus \tau))$-module. If $\mu \in K$ is a maximal simplex, then $\mathrm{link}(\mu)$ is the empty complex and $\mathbb{F}_p(\mathrm{link}(\mu)) \cong \mathbb{F}_p$. This implies that μ has order n and that K is pure. □

We finally give up the restriction on the coefficients.

Corollary 6.3. *If $\mathbb{F}(K)$ is Cohen-Macaulay respectively Gorenstein, then K is pure and, for every face $\tau \in K$, the Stanley-Reisner algebra $\mathbb{F}(\mathrm{link}_K(\tau))$ is also Cohen-Macaulay respectively Gorenstein.*

Proof. We will use Theorem 6.1 and Theorem 1.3. If $\mathbb{F}$ is a field of characteristique $p > 0$, then $H^*(X_K) \cong H^*(X_K; \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}$. This shows that $\mathbb{F}(K)$ is Cohen-Macaulay if and only if $\mathbb{F}_p(K)$ is so. The same holds for links. This covers the Cohen-Macaulay case as well as the Gorenstein case and also shows that K is pure.

Let $\mathbb{Z}_{(p)}$ denote the localisation of $\mathbb{Z}$ at the prime p. Then, for $\mathbb{F} = \mathbb{Q}$ we can argue as follows. Since X_K is of the homotopy type of a finite CW-complex (Proposition 3.2), $H^*(X_K; \mathbb{Z})$ has only finitely many torsion primes. Hence, $\mathbb{Q}(K)$ is Cohen-Macaulay if and only if $H^*(X_K; \mathbb{Q})$ is concentrated in even degrees if and only if, for almost all primes, $H^*(X_K; \mathbb{Z}_{(p)})$ is concentrated in even degrees and torsion free if and only if, for almost all primes, $H^*(X_K; \mathbb{F}_p)$ is concentrated in even degrees if and only if, for almost all primes, $\mathbb{F}_p(K)$ is Cohen-Macaulay. Now, Theorem 6.1 and and the above chain of equivalent statements applied in the case of $\mathrm{link}(\tau)$ proves the claim for Cohen-Macaulay algebras over $\mathbb{F} = \mathbb{Q}$. In the Gorenstein case, we only have to notice that $H^*(X_K; \mathbb{Q})$ is a PD-algebra if and only $H^*(X_K; \mathbb{F}_p)$ is a PD-algebra for almost all primes.

For a general field of characteristique 0 we deduce the claim from the case $\mathbb{F} = \mathbb{Q}$ in the same manner as for fields of characteristique $p > 0$ from $\mathbb{F} = \mathbb{F}_p$. □

7. Proof of Theorem 1.1

Theorem 1.1 follows easily from the following statement by induction.

Theorem 7.1. *$\mathbb{F}(K)$ is Cohen-Macaulay if and only if $\mathbb{F}(\mathrm{link}_K(\tau))$ is Cohen-Macaulay for all faces $\tau \neq \emptyset$ of K and $\widetilde{H}^r(K;\mathbb{F}) = 0$ for $0 \leq r < n-1$.*

Proof of Theorem 1.1: If $\dim K = 0$ then $\mathbb{F}(K)$ is always Cohen-Macaulay. In fact, in this case $c(K)$ is the m-fold wedge product of BS^1's and X_k the $(m-1)$-fold wedge product of S^2's, whose cohomology is concentrated in even degrees. On the other hand the set of conditions on the cohomology of the links of K is an empty set. This proves the statement in this case.

The general case follows by induction over the dimension of K and the above theorem. □

In the following cohomology is always with $\mathbb{F}$-coefficients. We define $H^*(-) \stackrel{\text{def}}{=} H^*(-;\mathbb{F})$ for the rest of this section. Also, for simplification, we set $P \stackrel{\text{def}}{=} H^*(BU(n)) \cong \mathbb{F}[\sigma_1, ..., \sigma_n]$.

The proof of Theorem 7.1 is based on the homological analysis of particular double complexes. To fix notation, we will recall the general concept next. For details see [13]

Let A be a $\mathbb{F}$-algebra. A differential graded A-module (C^*, d_C) is a cochain complex of A-modules such that d_C is A-linear. A double complex or differential bigraded module $(M^{*,*}, d_h, d_v)$ over A is a bigraded A- module $M^{*,*}$ with two A-linear maps $d_h : M^{*,*} \longrightarrow M^{*,*}$ and $d_v : M^{*,*} \longrightarrow M^{*,*}$ of bidegree $(1,0)$ and $(0,1)$ such that $d_h d_h = 0 = d_v d_v$ and $d_h d_v + d_v d_h = 0$. We think of d_h as the horizontal and of d_v as the vertical differential. To each double complex $M^{*,*}$ we associate a total complex $Tot^*(M)$ which is a differential graded module over A defined by $Tot^n(M) \stackrel{\text{def}}{=} \oplus_{i+j=n} M^{i,j}$ with differential $D \stackrel{\text{def}}{=} d_h + d_v$. Examples are given by the tensor products of two differential graded modules (B^*, d_B) and (C^*, d_C). If we set $M^{i,j} \stackrel{\text{def}}{=} B^i \otimes_A C^j$, $d_h \stackrel{\text{def}}{=} d_B \otimes 1$ and $d_v \stackrel{\text{def}}{=} (-1)^j 1 \otimes d_C$, we get a double complex such that $Tot^*(M) = B^* \otimes_A C^*$.

For a double complex $(M^{*,*}, d_h, d_v)$, we can take horizontal or vertical cohomology groups denoted by $H_h^*(M^{*,*})$ and $H_v^*(M^{*,*})$. The boundary maps d_h and d_v induce again boundary maps on these cohomology groups. We can consider cohomology groups of the form $H_h^*(H_v^*(M^{*,*}))$ and $H_v^*(H_h^*(M^{*,*}))$.

If $(M^{*,*}, d_h, d_v)$ is bounded below, that is $M^{i,j} = 0$ if i or j is small enough, there exist two spectral sequences converging towards $H^*(Tot(M), D)$. In one case, we have $E_2^{i,j} = H_h^i(H_v^j(M))$ and in the other case $E_2^{i,j} \stackrel{\text{def}}{=} H_v^j(H_h^i(M))$. In the first case the differential have degree $(r, 1-r,)$ and in the second case degree $(1-r, r)$.

Let $N^* \stackrel{\text{def}}{=} N^*(\text{CAT}(K^\times)^{op}, H^*(\text{cst}_K))$ denote the normalised cochain complex for the functor $H^*(\text{cst}_K) : \text{CAT}(K^\times)^{op} \longrightarrow \text{AB}$ considered in Section 2. Actually, the complex N^* is a bigraded object. It inherits an internal degree from the grading of $H^*(\text{cst}_K)$, which we will not consider in most cases.

We collect the main properties of N^* in the next proposition.

Proposition 7.2.
(i) $N^i = 0$ *for* $i < 0$ *or* $i \geq n$.
(ii) (N^*, d_N) *is a differential graded* P*-module as well as* $H^*(N^*, d_N) \cong \lim^*_{\mathrm{CAT}(K^\times)^{op}} , \mathbb{F}(\mathrm{st}_K)$
(iii) There exist an isomorphism $H^i(N^*, d_N) \cong \widetilde{H}^i(K)$ *for* $i \geq 1$ *of* P*-modules and a short exact sequence*

$$0 \longrightarrow \mathbb{F}(K) \longrightarrow H^0(N^*, d_L) \cong \lim_{\mathrm{CAT}(K^\times)^{op}}{}^0\, \mathbb{F}(\mathrm{st}_K) \longrightarrow \widetilde{H}^0(K) \longrightarrow 0$$

of P*-modules.*

Proof. The first part follows from the fact that $N^s(\mathrm{CAT}(K^\times)^{op}, H^*(\mathrm{cst}_k)) = 0$ for $s \geq n$, the second and the third part from Remark 3.6. □

We can also look at the projective resolution of the trivial P-module $\mathbb{F}$ given by the Koszul complex $Q^* \stackrel{\text{def}}{=} \Lambda^* \otimes_{\mathbb{F}} P$. According to our convention we make this into a cochain complex and give the generators of the exterior algebra $\Lambda^* \stackrel{\text{def}}{=} \Lambda^*(x_1, ..., x_n)$ the degree -1. As usual the differential d_Q is defined by $d_Q(x_i) \stackrel{\text{def}}{=} \sigma_i$ and $d_Q(y) = 0$ for $y \in P$ and has degree 1. Again, Q^* is a differential graded P-module, bounded below and above by $Q^j = 0$ for $j > 0$ or $j < -n$. The differential bigraded P-module $N^* \otimes_P Q^*$ is then bounded. In particular, both above mentioned spectral sequences converge towards $H^*(\mathrm{Tot}(N^* \otimes_P Q^*))$.

Proof of Theorem 7.1: If K is the empty complex, there is nothing to show. If K is a 0-dimensional complex, then $\mathbb{F}(K)$ is always Cohen-Macaulay as discussed in the proof of Theorem 1.1 and Theorem 1.2.

Now we assume that $\dim K \geq 1$. i.e. $n \geq 2$. We start with the assumption that $\mathbb{F}(K)$ is Cohen-Macaulay. By Theorem 6.3, we know that K is pure and that for all $\tau \in K$, the algebra $\mathbb{F}(\mathrm{link}(\tau))$ is also Cohen-Macaulay as well as $\mathbb{F}(\mathrm{st}(\tau))$. In particular, since $\dim \mathrm{st}(\tau) = \dim K$, the algebra $R(\mathrm{st}_K(\tau))$ is a finitely generated free module over P. We have to show that $\widetilde{H}^i(K) = 0$ for $i < n - 1$.

We look at the above constructed double complex $N^* \otimes_P Q^*$. All modules N^i and Q^j are free P-modules. Hence, the functors $N^* \otimes_P -$ and $- \otimes_P Q^*$ are exact. We get

$$H_h^i(H_v^j(N^* \otimes_P Q^*)) \cong H_h^i(N^* \otimes_P H_v^j(Q^*)) \cong \begin{cases} 0 & \text{for } j \neq 0 \\ H^i(N^* \otimes_P \mathbb{F}) & \text{for } j = 0 \end{cases}$$

In particular, the E_2-term is concentrated in one horizontal line given by $j = 0$, the spectral sequence collapses and $H^i(\mathrm{Tot}(N^* \otimes_P Q^*)) \cong H^i(N^* \otimes_P \mathbb{F}) = 0$ for $i < 0$ or $i \geq n$.

Considering the second spectral sequence we get

$$\begin{aligned} H_v^j(H_h^i(N^* \otimes_P Q^*)) &\cong H_v^j(H_h^i(N^*) \otimes_P Q^*) \\ &\cong \begin{cases} \mathrm{Tor}_P^j(\widetilde{H}^i(K), \mathbb{F}) \cong \mathrm{Tor}_P^j(\mathbb{F}, \mathbb{F}) \otimes_{\mathbb{F}} \widetilde{H}^i(K) & \text{for } i > 0 \\ H^*(X_K) \oplus \mathrm{Tor}_P^0(\mathbb{F}, \mathbb{F}) \otimes \widetilde{H}^0(K) & \text{for } i = 0 \text{ and } j = 0 \\ \mathrm{Tor}_P^0(\mathbb{F}, \mathbb{F}) \otimes \widetilde{H}^0(K) & \text{for } i = 0 \text{ and } j \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

For $i = 0$ this follows from Proposition 7.2 and the fact that $\mathrm{Tor}_P^0(\mathbb{F}(K), \mathbb{F}) \cong H^*(X_K)$. By degree reasons there is no differential ending at or starting from $H_v^{-n}(H_h^0(N^* \otimes P) \cong$

$\Lambda^n(x_1, ..., x_n) \otimes \widetilde{H}^0(K) \cong \widetilde{H}^0(K)$. Since $n \geq 2$ this group has total degree $-n+1 < 0$ and must therefore vanish. Hence, the whole column given by $i = 0$ and $j \leq -1$ vanishes and $E_2^{0,0} \cong H^*(X_K)$. By induction we can apply this argument successively for $i = 1, ..., n-2$. In these cases the whole columns $E_2^{i,j}$ vanish. For $i = n-1$ there might be a non trivial differential $E_2^{n-1,n} \longrightarrow E_2^{0,0}$ and we cannot conclude anymore that $E_2^{n-1,j} = 0$. This shows that $\widetilde{H}^i(K) = 0$ for $0 \leq i \leq n-2$ and proves one direction of the claim.

Now we assume that, for $\emptyset \neq \tau \in K$, all algebras $\mathbb{F}(\mathrm{link}(\tau))$ are Cohen-Macaulay and that $\widetilde{H}^j(K) = 0$ for $j < \dim K = n-1 \geq 1$ In particular, K is connected. This implies that for any pair μ, μ' of maximal faces of K, there exists a chain of maximal faces $\mu = \mu_1, ..., \mu_s = \mu'$ such that the intersection $\mu_j \cap \mu_{j+1}$ is non empty. Since $\mathrm{link}\, i$ is pure for all vertices $\{i\} \in K$ (Theorem 6.3), this implies that all maximal simplices of K have the same order, that K is pure, that for all faces $\tau \in K$ the dimension of $\mathrm{link}(\tau)$ equals $n - 1 - \sharp\tau$ and that $H(BT^\tau \times c(\mathrm{link}(\tau))) \cong \mathbb{F}(\mathrm{st}_K(\tau))$ is a finitely generated free module over P. Now we consider again both spectral sequences. In this case, we get for $H_h^i(H_v^j(N^* \otimes_P Q^*))$ the same result as above. And, since $\widetilde{H}^0(K) = 0$, Proposition 7.2 shows that

$$H_v^j(H_h^i(N^* \otimes_P Q^*)) \cong H_v^j(H_h^i(N^*) \otimes_P Q^*) \cong \begin{cases} \mathrm{Tor}_P^j(\mathbb{F}(K), \mathbb{F}) & \text{for } i = 0 \\ \mathrm{Tor}_P^j(\widetilde{H}^{n-1}(K), \mathbb{F}) & \text{for } i = n \\ 0 & \text{otherwise} \end{cases}$$

Hence the E_2-page is concentrated in two vertical lines given by $i = 0, n$. Considering again total degrees shows that $\mathrm{Tor}_P^j(\mathbb{F}(K), \mathbb{F}) = 0$ for $j \neq 0$ and that $\mathbb{F}(K)$ is a finitely generated free P-module (Theorem 4.1). This proves the other implication of the claim. □

We can draw the following consequence from the above proof.

Corollary 7.3. *If $\mathbb{F}(K)$ is Cohen-Macaulay, then there exists a short exact sequence*

$$0 \longrightarrow \widetilde{H}^{n-1}(K) \longrightarrow H^*(X_K) \longrightarrow \lim H^*(X_{\mathrm{st}(-)}) \longrightarrow \Lambda^{n-1}(n) \otimes \widetilde{H}^{n-1}(K) \longrightarrow 0$$

8. Proof of Theorem 1.2

The proof of Theorem 1.2 is an easy consequence of the following statement.

Theorem 8.1. *$\mathbb{F}(K)$ is Gorenstein* if and only if $\mathbb{F}(\mathrm{link}_K(\tau))$ is Gorenstein* for all faces $\tau \neq \emptyset$ of K and*

$$\widetilde{H}^i(K; \mathbb{F}) \cong \begin{cases} \mathbb{F} & \textit{for } i = \dim K \\ 0 & \textit{for } i \neq \dim K \end{cases}$$

Proof of Theorem 1.2: We argue as in the proof of Theorem 1.1. If $\dim K = 0$, then K is always Cohen-Macaulay. And K is Gorenstein* if and only if $X_k \simeq S^2$ if and only if $m = 2$ if and only if $\widetilde{H}^0(K; \mathbb{F}) = \mathbb{F}$.

The general case follows by induction over the dimension of K and the above theorem. □

The proof of Theorem 8.1 needs some preparation.

Remark 8.2. We call a simplicial complex reduced, if for every vertex i, the inclusion $\mathrm{st}(\{i\}) \subset K$ is proper. If $\mathrm{st}(\{i\}) = K$ then $K = \{i\} * \mathrm{link}(\{i\})$. Hence, any complex K can be written as a joint product $\Delta * L$ of a full simplex Δ and a reduced complex L. Moreover, K is reduced if and only if the union of all minimal missing faces equals the set V of vertices.

We call a simplicial complex K $\mathbb{F}$-spherical if it satisfies the geometric condition of the Gorenstein property, i.e. if

$$\widetilde{H}^i(\mathrm{link}(\tau);\mathbb{F}) \cong \begin{cases} \mathbb{F} & \text{for } i = \dim \mathrm{link}(\tau) \\ 0 & \text{for } i \neq \dim \mathrm{link}(\tau) \end{cases}$$

for every face $\tau \in K$ in including $\emptyset$.

Proposition 8.3. *Let $\mathbb{F}(K)$ be Gorenstein. Then the following holds:*
(i) K is reduced if and only if $\widetilde{H}^{n-1}(K) \neq 0$. In fact, if this is the case, then $\widetilde{H}^{n-1}(K) \cong \mathbb{F}$.
(ii) If K is reduced, then, for every vertex $i \in V$, the link $\mathrm{link}(\{i\})$ is also reduced.

Proof. By Corollary 7.3 we have an exact sequence

$$0 \longrightarrow \widetilde{H}^{n-1}(K) \longrightarrow H^*(X_K) \longrightarrow \prod_{i\in V} H^*(X_{\mathrm{st}(\{i\})})$$

Let $d \overset{\text{def}}{=} \dim X_K$. If $\widetilde{H}^{n-1}(K) = 0$ then the second map becomes a monomorphism and there exists an $i \in V$ such that $\mathbb{F} \cong H^d(X_K) \longrightarrow H^d(X_{\mathrm{st}(\{i\})})$ is a monomorphisms. On the other hand, if $L \subset K$ is a subcomplex of the same dimension as K such that $\mathbb{F}(K)$ and $\mathbb{F}(L)$ are Cohen-Macaulay, then the map $H^*(X_K) \longrightarrow H^*(X_L)$ is an epimorphism. Hence, $H^d(X_K) \cong H^d(X_{\mathrm{st}(\{i\})}) \cong \mathbb{F}$ for some vertex $i \in K$. Since both algebras are PD-algebras (Theorem 6.3), this implies that $H^*(X_K) \cong H^*(X_{\mathrm{st}(\{i\})})$. Comparing the two fibrations defining X_K and $X_{\mathrm{st}(\{i\})}$ shows that $\mathbb{F}(\mathrm{st}(\{i\})) \cong \mathbb{F}(K)$, that $\mathrm{st}(\{i\}) = K$, and that K is not reduced.

If K is not reduced, then, as the cone of a subcomplex, $|K|$ is contractible and $\widetilde{H}^{n-1}(K) = 0$. This proves the equivalence in part (i). If one of the conditions is satisfied, then, since $H^*(X_K)$ is PD-algebra, Lemma 3.2 shows that $\mathbb{F} \cong H^{n^2+n}(X_K) \cong \widetilde{H}^{n-1}(K)$.

Since Gorenstein algebras are Cohen-Macaulay, Theorem 7.1 and Proposition 3.2 (ii) tell us that there exists a short exact sequence

$$0 \longrightarrow H^{n^2+n-2}(X_K) \longrightarrow \prod_{i\in V} \widetilde{H}^{n-2}(\mathrm{link}(\{i\}) \longrightarrow \widetilde{H}^{n-1}(K) \cong \mathbb{F} \longrightarrow 0.$$

Since X_K is a PD-space of dimension $n^2 + n$, the first term in the above sequence is isomorphic to $H_2(X_K) \cong \mathbb{F}^{m-1}$. Hence, the middle term must be isomorphic to $\mathbb{F}^m$. Since for each vertex $i \in V$, $\dim_{\mathbb{F}} \widetilde{H}^{n-2}(\mathrm{link}(\{i\})) \leq 1$, this shows that $\widetilde{H}^{n-2}(\mathrm{link}(\{i\})) \cong \mathbb{F}$ and that $\mathrm{link}(\{i\})$ is reduced. □

For $\tau \subset V$ we denote by $K_\tau \subset K$ the full subcomplex which consist of all faces $\rho \in K$ such that $\rho \subset V \setminus \tau$. If $\tau = \{i\}$ is a vertex, we denote this complex by K_i.

Lemma 8.4. *If K is $\mathbb{F}$-spherical, then, for each vertex $i \in K$, the complex K_i is Cohen-Macaulay and $\widetilde{H}^{n-1}(K_i) = 0$.*

For the proof we need some preparation. The inclusions $K_i \subset K$ and $\mathrm{st}(\{i\}) \subset K$ induce epimorphisms $\mathbb{F}(K) \longrightarrow \mathbb{F}(K_i)$ and $\beta_i : \mathbb{F}(K) \longrightarrow \mathbb{F}(\mathrm{st}(\{i\}))$ of P-modules. The kernel of the first map is the ideal $v_i\mathbb{F}(K)$ generated by $v_i \in \mathbb{F}(K)$. And the second epimorphism induces an isomorphism $v_i\mathbb{F}(K) \cong v_i\mathbb{F}(\mathrm{st}(\{i\}))$. This follows from the fact that for any face $\tau \in K$ the monomial $v_i v_\tau = 0$ in $\mathbb{F}(\mathrm{st}(\{i\}))$ if and only if $\tau \cup \{i\} \notin K$. Moreover, since $\mathrm{st}(\{i\}) = \{i\} * \mathrm{link}(\{i\})$, multiplication by v_i induces an isomorphism $\mathbb{F}(\mathrm{st}(\{i\})) \longrightarrow v_i\mathbb{F}(\mathrm{st}(\{i\}))$. In fact, all the above maps are $\mathbb{F}(K)$-linear, where $\mathbb{F}(K)$ acts on $\mathbb{F}(K_i)$ and $\mathbb{F}(\mathrm{st}(\{i\}))$ via the above projections. Moreover they fit together to a short exact sequence

$$0 \longrightarrow \mathbb{F}(\mathrm{st}(\{i\})) \longrightarrow \mathbb{F}(K) \longrightarrow \mathbb{F}(K_i) \longrightarrow 0$$

of $\mathbb{F}(K)$-modules. The first map is given by $q \mapsto v_i q'$ where $\beta_i(q') = q$. Applying the functor $-\otimes_P$ establishes an epimorphism $\alpha_i : H^*(X_K) \longrightarrow H^*(X_{\mathrm{st}(\{i\})})$ and an $H^*(X_K)$-linear map

$$\psi_i : H^*(X_{\mathrm{st}(\{i\})}) \longrightarrow v_i H^*(X_K)$$

given by $\psi_i(a) = v_i a'$ where $\alpha_i(a') = a$. In fact, we will show that this map is an isomorphism (see Corollary 8.5).

Proof of Lemma 8.4: Since for two vertices $i, j \in K$ we have $\mathrm{link}_K(\{j\})_i$ equals $\mathrm{link}_K(\{j\})$ if the simplex $\{i,j\} \notin K$ and equals $\mathrm{link}_{K_i}(\{j\})$ if $\{i,j\} \in K$, we only have to prove that $\widetilde{H}^r(K_i) = 0$ for $r \leq n-1$. And this claim we prove via an induction over the dimension of K.

For $n = 1$, $\mathbb{F}$-spherical implies that K consists only of two vertices. And for $n = 2$, $\mathbb{F}$-spherical implies that K is a triangulation of S^1. In both cases, the claim is straightforward.

Now let us assume that $n \geq 3$. By excision, $H^*(K, K_i) \cong H^*(\mathrm{st}_K(\{i\}), \mathrm{link}_K(\{i\})) \cong H^*(\Sigma\, \mathrm{link}_K(\{i\}))$. Here $\Sigma\, \mathrm{link}_K(\{i\})$ denotes the suspension of $\mathrm{link}_K(\{i\})$, actually of the geometric realization of $\mathrm{link}_K(\{i\})$. Moreover, the map $H^*(K, K_i) \longrightarrow H^*(K)$ can be identified with $H^*(\Sigma\, \mathrm{link}_K(\{i\})) \longrightarrow H^*(K)$ induced by the last arrow in the cofibration sequence $\mathrm{link}_K(\{i\}) \longrightarrow K_i \longrightarrow K \longrightarrow \Sigma\, \mathrm{link}_K(\{i\})$. Since $\Sigma\, \mathrm{link}_K(\{i\})$ is $\mathbb{F}$-spherical, it suffices to show that $H^{n-1}(K_i) = 0$.

Let $j \in V$ such that $\tau \overset{\mathrm{def}}{=} \{i,j\} \in K$. In the following, K will also denote the geometric realization of K. Because of the identities $\mathrm{link}_K(\tau) = \mathrm{link}_{\mathrm{link}_K(\{i\})}(\{j\}) = \mathrm{link}_{\mathrm{link}_K(\{j\})}(\{i\})$, $\mathrm{link}_{K_i}(\{j\}) = \mathrm{link}_K(\{j\})_i$ and $\mathrm{link}_{K_j}(\{i\}) = \mathrm{link}_K(\{i\})_j$ all rows and columns in the homotopy commutative diagram

$$\begin{array}{ccccc}
\mathrm{link}_K(\tau) & \longrightarrow & \mathrm{link}_{K_j}(\{i\}) & \longrightarrow & \mathrm{link}_K(\{i\}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{link}_{K_i}(\{j\}) & \longrightarrow & K_\tau & \longrightarrow & K_i \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{link}_K(\{j\}) & \longrightarrow & K_j & \longrightarrow & K
\end{array}$$

consist of cofibrations. Passing to suspensions we can extend the diagram to the right and to the bottom yielding a homotopy commutative 4×4-diagram, whose bottom right square looks like

$$\begin{array}{ccc} K & \longrightarrow & \Sigma\,\mathrm{link}_K(\{j\}) \\ \downarrow & & \downarrow \\ \Sigma\,\mathrm{link}_K(\{i\}) & \longrightarrow & \Sigma^2\,\mathrm{link}_K(\tau). \end{array}$$

In the induced diagram in cohomology in degree $n-1$

$$\begin{array}{ccc} H^{n-1}(\Sigma^2\,\mathrm{link}_K(\tau)) & \longrightarrow & H^{n-1}(\Sigma\,\mathrm{link}_K(\{j\})) \\ \downarrow & & \downarrow \\ H^{n-1}(\Sigma\,\mathrm{link}_K(\{i\})) & \longrightarrow & H^{n-1}(K) \end{array}$$

the left vertical and top horizontal arrows are isomorphisms by induction hypothesis. Therefore, the other two arrow are either both isomomorphisms or both trivial. And both are isomorphisms if and only if $H^{n-1}(K_i) = H^{n-1}(K_j) = 0$.

For $n \geq 2$, the complex K is connected and we can connect each pair of vertices by 1-dimensional faces. The above argument now shows that the maps $H^{n-1}(K, K_i) \longrightarrow H^{n-1}(K)$ are either isomorphisms for all vertices or trivial for all vertices. Hence it is sufficient to show this map is at least nontrivial for at least one vertex, or equivalently, that $H^{n-1}(K_i) = 0$ for at least one vertex.

Since $H^*(X_K)$ is generated by classes of degree 2, a generator of $H^{n^2+n}(X_K) \cong \mathbb{F}$ can be represented by a monomial a which can be written as $v_i a'$ for a suitable vertex $i \in K$. We fix this vertex. By the above considerations we have an exact sequence

$$H^{n^2+n-2}(X_{\mathrm{st}(\{i\})}) \longrightarrow H^{n^2+n}(X_K) \longrightarrow H^{n^2+n}(X_{K_i}).$$

The first map is given by multiplication with v_i and therefore an isomorphism. By Corollary 3.3, the latter map can be identified with the map $H^{n-1}(K) \longrightarrow H^{n-1}(K_i)$, which, since $\dim link_K(\{i\}) = n-2$, is an epimorphism. Hence, we have $0 = H^{n^2+n}(X_{K_i}) \cong H^{n-1}(K_i)$, which completes the proof. □

Corollary 8.5. *IF K is $\mathbb{F}$-spherical, then, for each vertex $i \in K$, multiplication by v_i induces an isomorphism $H^*(X_{\mathrm{st}(\{i\})}) \longrightarrow v_i H^*(X_K)$.*

Proof. All terms of the exact sequence

$$0 \longrightarrow \mathbb{F}(\mathrm{st}(\{i\})) \longrightarrow \mathbb{F}(K) \longrightarrow \mathbb{F}(K_i) \longrightarrow 0$$

are Cohen-Macaulay (Corollary 6.3, Theorem 1.1, Lemma 8.4. Hence, applying the functor $\otimes_P \mathbb{F}$ establishes a short exact sequences

$$0 \longrightarrow H^*(X_{\mathrm{st}(\{i\})}) \longrightarrow H^*(X_K) \longrightarrow H^*(X_{K_i}) \longrightarrow 0.$$

By construction, the first map is given by multiplication by v_i. □

Proof of Theorem 8.1: If K is Gorenstein*, then Proposition 8.3 shows that $H^{n-1}(K) \cong \mathbb{F}$ and, together with Corollary 6.3, that for each face $\tau \in K$ the algebra $\mathbb{F}(\mathrm{link}(\tau))$ is Gorenstein*.

For the opposite conclusion it suffices to show that $\mathrm{soc}(H^*(X_K) \cong H^{n^2+n}(X_K) \cong \mathbb{F}$ (Theorem 1.3(ii)). By induction we can assume that K is $\mathbb{F}$-spherical.

For each vertex $i \in V$ the map $H^*(X_K) \longrightarrow H^*(X_{\mathrm{st}(\{i\})})$ is an epimorphism. Hence, this map maps $\mathrm{soc}\, H^*(X_K)$ to $\mathrm{soc}\, H^*(X_{\mathrm{st}(\{i\})}) \cong H^{n^2+n-2}(X_{\mathrm{st}(\{i\})})$. Moreover, we have an exact sequence $0 \longrightarrow H^{n-1}(K) \longrightarrow H^*(X_K) \longrightarrow \prod_i H^*(X_{\mathrm{st}(\{i\})})$ (Corollary 7.3), and hence all elements $\mathrm{soc}\, H^*(X_K)$ have degree $\geq n^2+n-2$. Let $a \in H^{n^2+n-2}(X_K)$. This class maps to $0 \neq b \in H^{n^2+n-2}(X_{\mathrm{st}(\{i\})}) \cong \mathbb{F}$ for some vertex $i \in V$. And, by Lemma 8.5, $0 \neq v_i b = v_i a \in H^{n^2+n}(X_K)$. This shows that $\mathrm{soc}\, H^*(X_K) \cong H^{n^2+n}(X_K) \cong H^{n-1}(K) \cong \mathbb{F}$ and finishes the proof. $\square$

References

[1] A.K. Bousfield and D.M. Kan, *Homotopy limits, completions and localisations*, SLNM 304, Springer Verlag.

[2] W. Bruns and H.J. Herzog. *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics 39, Cambridge University Press, 1998.

[3] Victor M Buchstaber and Taras E Panov. *Torus Actions and Their Applications in Topology and Combinatorics*, volume 24 of University Lecture Series, American Mathematical Society, 2002.

[4] M.W. Davis and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Mathematical Journal 62 (1991), 417–451.

[5] F.-X. Dehon and J. Lannes, *Sur les espaces fonctionnels dont la source est le clasifiant d'un group de Lie compact commutatif*, Inst. Hautes Études Sci. Publ. Math. 89 (1999), 127-177.

[6] E. Dror-Farjoun, *Cellular spaces, null spaces and homotopy localisation*, SLNM 1622, Springer 2000.

[7] W.G. Dwyer, *Classifying spaces and homology decompositions* In: W.G. Dwyer and H.W. Henn *Homotopy theoretic methods in group cohomology*, Advanced Courses in Mathematics CRM Barcelona, pages 1–53, Birkhäuser, 2001.

[8] W.G. Dwyer and C.W. Wilkerson, *Smith theory and the functor T*, Comm. Math. Helv. 66 (1991), 1-17.

[9] W.G. Dwyer and A. Zabrodsky, *Maps between classifying spaces*, in: Algebraic Topology, proceedings of the Barcelona conference, SLNM 1298 (1987), 106-119.

[10] P. Gabriel and M. Zisman. *Calculus of Fractions and Homotopy Theory*, Ergebnisse der Mathematik, Springer Verlag, 1967.

[11] J. Hollender and R.M. Vogt. *Modules of topological spaces, applications to homotopy limits and E_∞ structures*, Archiv der Mathematik 59 (1992), 115–129.

[12] Hurewicz.

[13] J. McCleary, *A users guide to spectral sequences*, Cambridge Studies in Advanced Mathematics 58, Cambridge University Press, 2001.

[14] D. Notbohm, *Maps between classifying space*, Math. Z. 207 (1991), 153-168.

[15] D. Notbohm and N. Ray, *On Davis-Januszkiewicz homotopy types I; Formality and Rationalisation*, to appear in Alg. Geom. Topology.

[16] D. Notbohm and N. Ray, *On Davis-Januszkiewicz homotopy types II*, in preparation.

[17] D. Notbohm and N. Ray, in preparation.

[18] G. Reisner, *Cohen-Macaulay quotients of polynomial rings*, Adv. Math. 21 (1976), 30-49.

[19] G. Segal, *Classifying spaces and spectral sequences*, IHES Publ. Math. 34 (1968), 105-112.

[20] R.P. Stanley, *Combinatorics and commutative algebra* Progress in Mathematics 41, Birkhäuser, 1996.
[21] Z. Wojtkowiak, *On maps from holim F to Z*, Algebraic Topology, Barcelona 1986, SLNM 1298, Springer Verlag 1987, 227-236.

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