p–STUBBORN SUBGROUPS OF CLASSICAL COMPACT LIE GROUPS

by Bob Oliver

For any compact Lie group G, let $\mathcal{O}(G)$ denote the "orbit category" of G: the category whose objects are the orbits G/H for closed subgroups $H \subseteq G$, and whose morphisms are all G-maps between orbits. For any prime p, let $\mathcal{R}_p(G)$ denote the category of those orbits G/P for *p-stubborn* subgroups $P \subseteq G$: namely, those subgroups which satisfy the conditions

(a) P is p-toral (i.e., an extension of a torus by a finite p-group)

(b) $N(P)/P$ is finite

(c) $N(P)/P$ is *p-reduced*: there is no nontrivial normal p-subgroup $1 \neq Q \lhd N(P)/P$.

One of the main results in [JMO] is a decomposition for BG indexed over $\mathcal{R}_p(G)$. More precisely, for any G and p, the natural projection map

$$\underset{G/P \in \mathcal{R}_p(G)}{\underrightarrow{\operatorname{hocolim}}} (EG/P) \longrightarrow BG$$

induces an equivalence of $\mathbb{F}_p$-homology [JMO, §§1-2]. This decomposition of BG, and the category $\mathcal{R}_p(G)$, play a central role in [JMO] and [JMO2] as a tool for describing sets of homotopy classes of maps from BG to BH for any (other) compact connected Lie group H. But in most cases, explicit descriptions of the $\mathcal{R}_p(G)$, or explicit lists of the p-stubborn subgroups of G, were not necessary to obtain the results in those papers. However, recent results of Notbohm [N], proving in many cases the uniqueness of the completed classifying spaces $BG^\wedge_p$ (uniqueness among spaces with the same mod p cohomology), do require a more precise description of the p-stubborn subgroups of the classical compact Lie groups. And this provides the motivation for the present paper.

In Theorems 3 below, the p-stubborn subgroups of the matrix groups $\mathrm{U}(n)$, $\mathrm{O}(n)$, and $\mathrm{Sp}(n)$ are described explicitly for each n and p. In Theorems 5 and 7, these results are extended to describe the p-stubborn subgroups of $\mathrm{SU}(n)$ and $\mathrm{SO}(n)$. In all cases, the p-stubborn subgroups show a surprisingly simple pattern: being generated from a small collection of "basic" p-stubborn subgroups by products and wreath products.

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For a finite group G, it turns out that the "p-stubborn" subgroups are precisely the same as the "radical" p-subgroups which are used by group theorists when classifying finite simple groups. The description given here of p-stubborn subgroups of the classical matrix groups is very similar in nature to the description of radical subgroups of symmetric groups found by Alperin & Fong in [AF, Theorem 2A].

The following fundamental properties of p-stubborn subgroups will be used frequently.

Lemma 1. *Let P be a p-stubborn subgroup of a compact Lie group G. Then the following two properties hold.*

(i) Any p-toral subgroup $H \subseteq G$ which is normalized by $N(P)$ (i. e., $N(P) \subseteq N(H)$) is contained in P.

(ii) $C_{G_0}(P) \subseteq Z(P)$, and $C_G(P) = Z(P)$ if G/G_0 is a p-group.

Proof. Part (ii) is shown in [JMO, Lemma 1.5(ii)]. Part (i) is essentially shown in the same lemma, but in a slightly different formulation. For that reason, we repeat the proof here.

Assume that $H \not\subseteq P$, and that $N(P)$ normalizes H. Set $H' = \langle H, P \rangle \supsetneqq P$. Since P normalizes H, H is normal in H', and H' is p-toral since H'/H is a quotient group of P. Also, $N(P) \subseteq N(H')$; and

$$\operatorname{Ker}[N(P)/P \to N(H')/H'] = (H' \cap N(P))/P = N_{H'}(P)/P$$

is a nontrivial normal p-subgroup of $N(P)/P$ (cf. [JMO, Lemmas A.2 & A.3]). Which contradicts the assumption that P is p-stubborn. □

We now define certain p-stubborn subgroups of Σ_n, $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$: subgroups which will be seen to generate all other p-stubborn subgroups of such groups.

In the following definitions, $\sigma_0, \ldots, \sigma_{k-1} \in \Sigma_{p^k}$ will denote the permutations

$$\sigma_r(i) = \begin{cases} i + p^r & \text{if } i \equiv 1, \ldots, (p-1)p^r \pmod{p^{r+1}} \\ i - (p-1)p^r & \text{if } i \equiv (p-1)p^r + 1, \ldots, p^{r+1} \pmod{p^{r+1}}. \end{cases}$$

Note that these generate an elementary abelian p-subgroup of rank k. Also, $\zeta = e^{2\pi i/p}$ denotes a primitive p-th root of unity. Define matrices $A_1, \ldots, A_{k-1}, B_1, \ldots, B_{k-1} \in \mathrm{U}(p^k)$ by setting

$$(A_r)_{ij} = \begin{cases} \zeta^{[(i-1)/p^r]} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad \text{and} \qquad (B_r)_{ij} = \begin{cases} 1 & \text{if } \sigma_r(i) = j \\ 0 & \text{if } \sigma_r(i) \neq j \end{cases}$$

(where $[-]$ denotes greatest integer). The A_r are thus all diagonal matrices, and the B_r are the permutation matrices for the σ_r. Note that they satisfy the commutator relations

$$[A_r, A_s] = I = [B_r, B_s] = [B_r, A_s] \quad (r \neq s), \qquad \text{and} \qquad [B_r, A_r] = \zeta \cdot I.$$

Finally, let $Q(8)\subseteq S^1(j) \subseteq \mathbb{H}^*$ denote the subgroups

$$Q(8) = \{\pm 1, \pm i, \pm j, \pm k\} \quad \text{and} \quad S^1(j) = \{a + bi, aj + bk : a^2 + b^2 = 1\}.$$

Definition 2. *For each prime p and each $k \geq 0$, subgroups*

$$E_{p^k} \subseteq \Sigma_{p^k} \quad \text{and} \quad \Gamma^{\mathrm{U}}_{p^k} \subseteq \mathrm{U}(p^k) \subseteq \mathrm{O}(2p^k),\ \mathrm{Sp}(p^k)$$

are defined by setting

$$E_{p^k} = \langle \sigma_0, \ldots, \sigma_{k-1} \rangle \cong (C_p)^k$$

and

$$\Gamma^{\mathrm{U}}_{p^k} = \langle u \cdot I, A_r, B_r \mid u \in S^1, 0 \leq r \leq k-1 \rangle \subseteq \mathrm{U}(p^k).$$

If $p = 2$, then $A_r, B_r \in \mathrm{O}(2^k)$, and we define

$$\Gamma^{\mathrm{O}}_{2^k} \subseteq \bar{\Gamma}^{\mathrm{O}}_{2^k} \subseteq \mathrm{O}(2^k) \quad \text{and} \quad \Gamma^{\mathrm{Sp}}_{2^k} \subseteq \bar{\Gamma}^{\mathrm{Sp}}_{2^k} \subseteq \mathrm{Sp}(2^k)$$

by setting

$$\Gamma^{\mathrm{O}}_{2^k} = \langle -I, A_r, B_r \mid 0 \leq r \leq k-1 \rangle$$

$$\bar{\Gamma}^{\mathrm{O}}_{2^k} = \langle \alpha^{\oplus 2^{k-1}}, A_r, B_r \mid \alpha \in \mathrm{SO}(2),\ 0 \leq r \leq k-1 \rangle,$$

$$\Gamma^{\mathrm{Sp}}_{2^k} = \langle u \cdot I, A_r, B_r \mid u \in Q(8),\ 0 \leq r \leq k-1 \rangle,$$

and

$$\bar{\Gamma}^{\mathrm{Sp}}_{2^k} = \langle u \cdot I, A_r, B_r \mid u \in S^1(j),\ 0 \leq r \leq k-1 \rangle.$$

Note that the subgroup $E_{p^k} \subseteq \Sigma_{p^k}$ can be identified with the action of $(C_p)^k$ on itself. The other groups sit in central extensions

$$1 \to S^1 \to \Gamma^{\mathrm{U}}_{p^k} \to (C_p)^{2k} \to 1$$

$$1 \to \{\pm 1\} \to \Gamma^{\mathrm{O}}_{2^k} \to (C_2)^{2k} \to 1, \qquad 1 \to \mathrm{O}(2) \to \bar{\Gamma}^{\mathrm{O}}_{2^k} \to (C_2)^{2k-2} \to 1$$

$$1 \to Q(8) \to \Gamma^{\mathrm{Sp}}_{2^k} \to (C_2)^{2k} \to 1, \qquad 1 \to S^1(j) \to \bar{\Gamma}^{\mathrm{Sp}}_{2^k} \to (C_2)^{2k} \to 1$$

The groups $\Gamma^{\mathrm{O}}_{2^k}$ and $\Gamma^{\mathrm{Sp}}_{2^k}$ are both central products of copies of $D(8)$ and $Q(8)$.

When describing p-stubborn subgroups of the classical Lie groups, it will be convenient to let $\mathbb{G}$ denote one of the classes O, U, or Sp. We make here the usual identifications $\mathbb{G}(k) \times \mathbb{G}(m) \subseteq \mathbb{G}(k+m)$ and $\mathbb{G}(k) \wr \Sigma_m \subseteq \mathbb{G}(km)$ for $k, m \geq 1$. A subgroup $P \subseteq \mathbb{G}(n)$ will be called irreducible if the induced P-representation on $\mathbb{R}^n$, $\mathbb{C}^n$, or $\mathbb{H}^n$ is irreducible.

Theorem 3. *Fix a prime p, set $\mathbb{G} = \mathrm{O}$, U, or Sp, and let $G = \mathbb{G}(n)$ for some $n \geq 1$.*

(i) For any $n \geq 1$, an irreducible subgroup $1 \neq P \subseteq \mathbb{G}(n)$ is p-stubborn if and only if it is conjugate to an iterated wreath product of the form

$$P = \Gamma \wr E_{q_1} \wr \cdots \wr E_{q_r} \quad (\Gamma \subseteq \mathbb{G}(m),\ q_i = p^{t_i} > 1,\ n = m \cdot p^{t_1 + \cdots + t_r}) \tag{1}$$

where $\Gamma \subseteq \mathbb{G}(m)$ is one of the groups in the following list:

Γ	$\mathbb{G}(m)$	$N(\Gamma)/\Gamma$
$\Gamma^{\mathrm{U}}_{p^k}$	$\mathrm{U}(p^k)$	$\mathrm{Sp}_{2k}(\mathbb{F}_p)$
$\Gamma^{\mathrm{U}}_{p^k}$ (p odd)	$\mathrm{O}(2p^k)$, $\mathrm{Sp}(p^k)$	$C_2 \times \mathrm{Sp}_{2k}(\mathbb{F}_p)$
$\Gamma^{\mathrm{O}}_{2^k}$ ($k \neq 1$)	$\mathrm{O}(2^k)$	$\mathrm{O}^+_{2k}(\mathbb{F}_2)$
$\bar{\Gamma}^{\mathrm{O}}_{2^{k+1}}$	$\mathrm{O}(2^{k+1})$	$\mathrm{Sp}_{2k}(\mathbb{F}_2)$
$\Gamma^{\mathrm{Sp}}_{2^k}$	$\mathrm{Sp}(2^k)$	$\mathrm{O}^-_{2k+2}(\mathbb{F}_2)$
$\bar{\Gamma}^{\mathrm{Sp}}_{2^k}$	$\mathrm{Sp}(2^k)$	$\mathrm{Sp}_{2k}(\mathbb{F}_2)$

and where $q_1 \geq 4$ ($t_1 \geq 2$) if $\Gamma = \Gamma^{\mathrm{O}}_1 = \mathrm{O}(1)$. Here, unless otherwise specified, k can be any nonnegative integer. Furthermore, for P as in (1),

$$N_{\mathbb{G}(n)}(P)/P \cong N_{\mathbb{G}(m)}(\Gamma)/(\Gamma) \times GL_{t_1}(\mathbb{F}_p) \times \cdots \times GL_{t_r}(\mathbb{F}_p).$$

(ii) If $P \subseteq \mathbb{G}(n)$ is an arbitrary p-stubborn subgroup, then it is conjugate to a subgroup of the form $P_1 \times \ldots \times P_s$, where each P_i is an irreducible p-stubborn subgroup of some $\mathbb{G}(n_i)$, and where $n_1 + \cdots + n_s = n$. A subgroup of this form is p-stubborn if and only if the trivial factor $1 \subseteq \mathrm{O}(1)$ occurs at most once (if p is odd and $\mathbb{G} = \mathrm{O}$); and there is no factor P_i with $N_{\mathbb{G}(n_i)}(P_i)/P_i = 1$ which occurs with multiplicity exactly 2 or 4 (if $p = 2$) or 3 (if $p = 3$).

Proof. This will be shown in four steps. Fix $P \subseteq \mathbb{G}(n)$, and let V be the corresponding n-dimensional P-representation. We write $\chi_V(g)$ for the character (i.e., trace) of an element $g \in G$. (When $\mathbb{G} = \mathrm{Sp}$, $\chi_V(g)$ means the real part of the sum of the diagonal elements in the matrix g; this clearly depends only on the conjugacy class of g in G.)

Throughout the first two steps, we assume that $P \neq 1$ is irreducible and p-stubborn. Consider the subgroup

$$A = \langle g \in P : \chi_V(g) \neq 0 \rangle \subseteq P.$$

Note that $A \lhd N(P)$, and that $[P : A] < \infty$.

Step 1 Assume here that for any subgroup $K \subseteq P$ such that $K \supseteq A$ and $K \lhd N(P)$, either $V|K$ is irreducible, or it splits as a sum of isomorphic irreducible K-representations. Set

$$P' = \{g \in C_P(A) : g^p \in A, [g, P] \subseteq A\} \quad \text{and} \quad A' = Z(P').$$

By assumption, $V|A \cong W^s$ for some irreducible A-representation W. Also, A is generated by elements g for which $\chi_W(g) \neq 0$, and this is only possible if A is cyclic or S^1. In particular, A is abelian, and so $A \subseteq A' \subseteq P'$. Since $A' \lhd N(P)$ by construction, $V|A'$ also splits as a sum of isomorphic irreducible representations. In particular, $A' = Z(P')$ is abelian and has an effective irreducible representation, and this means A' must be cyclic or S^1.

If $A' \cong C_{p^m}$ and $p^m > 2$, then $V|A'$ extends in a unique way to a representation of the circle S^1. This uniqueness insures that S^1 also is normalized by $N(P)$; and hence that $S^1 \subseteq P$ by Lemma But $[P : A] < \infty$, so this is impossible; and we see that

$$A = A' = Z(P') \cong \{\pm 1\} \text{ or } S^{(3)}$$

We next claim that

$$[P, P'] \subseteq A \quad \text{and} \quad C_P(P') = A. \tag{4}$$

The first statement follows from the definition of P'. To see the second, assume that $C_P(P') \supsetneqq Z(P') = A$. Then there exists $gA \in (C_P(P')/A) \cap Z(P/A)$ of order p. But then $g \in P'$ by construction, and so $g \in P' \cap C_P(P') = Z(P') = A$.

By construction, P'/A is elementary abelian. So by (3), there is a central extension

$$1 \to A \to P' \to (C_p)^{2k} \to 1 \tag{5}$$

for some k; where $P'/A = \langle a_1, \ldots, a_k, b_1, \ldots, b_k \rangle$, and where the a_i and b_i satisfy the commutator relations

$$[a_i, a_j] = [b_i, b_j] = [b_i, a_j] = 1 \ (i \neq j) \quad \text{and} \quad [b_i, a_i] = \zeta = e^{2\pi i/p}. \tag{6}$$

Case A Assume first that $A \cong S^1$. Since $\mathrm{Ext}((C_p)^{2k}, S^1) = 0$ (S^1 is infinitely divisible), there is only one central extension of the form (5) which satisfies the commutator relations (6). Hence $P' \cong \Gamma^{\mathrm{U}}_{p^k}$. By (5) and (6), any irreducible $\mathbb{C}$-representation of P' upon which S^1 acts by the identity is a module over the appropriate twisted group ring $R = \mathbb{C}[(C_p)^{2k}]^t$; and $R \cong M_{p^k}(\mathbb{C})$ since P' does have an irreducible p^k-dimensional module. In other words, there is a unique irreducible $\mathbb{C}$-representation of P' upon which S^1 acts by the identity, and $G = \mathbb{G}(n) = \mathrm{U}(p^k)$, $\mathrm{Sp}(p^k)$, or $\mathrm{O}(2p^k)$. So P' is conjugate in G to $\Gamma^{\mathrm{U}}_{p^k}$.

Since P' has a unique representation which extends the given representation on its center, any automorphism of P' which fixes the center must be realized as conjugation in G. The image of $\mathrm{Aut}(P')$ in $\mathrm{Aut}(P'/A) \cong GL_{2k}(\mathbb{F}_p)$ is the symplectic group $\mathrm{Sp}_{2k}(\mathbb{F}_p)$. Also, any automorphism of P' which induces the identity on A and on P'/A is inner (note that there are at most $p^{2k} = |P'/A|$ such automorphisms). It follows that

$$N(P')/P' \cong \begin{cases} \mathrm{Out}(\Gamma^{\mathrm{U}}_{p^k} \operatorname{rel} S^1) \cong \mathrm{Sp}_{2k}(\mathbb{F}_p) & \text{if } \mathbb{G} = \mathrm{U} \\ \mathrm{Out}(\Gamma^{\mathrm{U}}_{p^k}) \cong C_2 \times \mathrm{Sp}_{2k}(\mathbb{F}_p) & \text{if } \mathbb{G} = \mathrm{O}, \mathrm{Sp}\,. \end{cases} \tag{8}$$

Here, when $\mathbb{G} = \mathrm{O}$ or Sp, the factor C_2 is generated by complex conjugation, regarded as an automorphism of $\Gamma^{\mathrm{U}}_{p^k} \subseteq \mathrm{U}(p^k)$. Note that $\mathrm{Sp}_{2k}(\mathbb{F}_p)$ is p-reduced for any k and p: since the upper and lower triangular matrices, for an appropriate ordering of the basis, make up two Sylow p-subgroups with trivial intersection.

Set $P'' = C_P(A)$. Then $[P : P''] \le |\mathrm{Aut}(S^1)| = 2$. By (3), $[P, P'] \subseteq A \cong S^1$. So for any $x \in P''$, conjugation by x is an automorphism of P' which induces the identity on P'/A and on A. Any such automorphism is inner (see (8)); and so $P'' \subseteq \langle P', C_P(P')\rangle$. Thus $P'' = P'$ by (4), and P'' is conjugate to $\Gamma^{\mathrm{U}}_{p^k}$.

If $\mathbb{G} = \mathrm{U}$ or if p is odd, then $P = P''$ and we are done. If $p = 2$ and $\mathbb{G} = \mathrm{O}$ or Sp, then $N(P'')/P'' \cong C_2 \times \mathrm{Sp}_{2k}(\mathbb{F}_2)$ by (8). Furthermore, $N_{\mathbb{G}(n)}(P) \subseteq N_{\mathbb{G}(n)}(P'')$ by construction of P''; so P/P'' is 2-stubborn in $N(P'')/P''$, and hence must be precisely the C_2 factor ($|P/P''| \le 2$). It follows that P is conjugate to $\bar{\Gamma}^{\mathrm{O}}_{2^{k+1}} \subseteq \mathrm{O}(2^{k+1})$ or $\bar{\Gamma}^{\mathrm{Sp}}_{2^k} \subseteq \mathrm{Sp}(2^k)$; and that $N(P)/P \cong \mathrm{Sp}_{2k}(\mathbb{F}_2)$.

Case B Now assume that $A \cong \{\pm 1\}$. In particular, $p = 2$ and $\mathbb{G} = \mathrm{O}$ or Sp. Then any central extension of the form given in (5) and (6) is isomorphic, either to a central product of copies of $D(8)$, or a central product of one copy of $Q(8)$ and copies of $D(8)$. In either case P' has a unique effective irreducible representation: of real type in the first case, and of symplectic type in the second case. In other words, $P' \subseteq \mathbb{G}(n)$ is conjugate to one of the pairs $\Gamma^{\mathrm{O}}_{2^k} \subseteq \mathrm{O}(2^k)$ or $\Gamma^{\mathrm{Sp}}_{2^k} \subseteq \mathrm{Sp}(2^k)$. As in Case A, any automorphism of P' which induces the identity on P'/A is inner; and so $P = P'$ by (4).

Finally, since P has a unique effective irreducible representation,

$$N_{\mathbb{G}(2^k)}(\Gamma^{\mathbb{G}}_{2^k})/(\Gamma^{\mathbb{G}}_{2^k}) \cong \mathrm{Out}(\Gamma^{\mathbb{G}}_{2^k}) \cong \begin{cases} \mathrm{O}^{+}_{2k}(\mathbb{F}_2) & \text{if } \mathbb{G} = \mathrm{O} \\ \mathrm{O}^{-}_{2k+2}(\mathbb{F}_2) & \text{if } \mathbb{G} = \mathrm{Sp} \end{cases}$$

(see [12, §4] for the definitions of these groups). By a theorem of Dieudonné [12, pp. 47–51], for $k \ge 3$, any nontrivial normal subgroup of $\mathrm{O}^{\pm}_{2k}(\mathbb{F}_2)$ contains its commutator subgroup, which is simple. So $\mathrm{O}^{\pm}_{2k}(\mathbb{F}_2)$ is 2-reduced when $k \ge 3$; and the following list shows that $\mathrm{O}^{\pm}_{2k}(\mathbb{F}_2)$ is 2-reduced in all other cases except for $\mathrm{O}^{+}_{2}(\mathbb{F}_2)$:

$$\mathrm{O}^{+}_{2}(\mathbb{F}_2) \cong C_2,\ \mathrm{O}^{-}_{2}(\mathbb{F}_2) \cong \Sigma_3,\ \mathrm{O}^{+}_{4}(\mathbb{F}_2) \cong \Sigma_3 \wr \Sigma_2, \text{ and } \mathrm{O}^{-}_{4}(\mathbb{F}_2) \cong \Sigma_5.$$

Thus $\Gamma^{\mathrm{O}}_{2^k}$ and $\Gamma^{\mathrm{Sp}}_{2^k}$ are 2-stubborn for all $k \ge 0$, except $\Gamma^{\mathrm{O}}_{2} \cong D(8)$.

Step 2 We now assume that there exists a subgroup $A\subseteq K\subseteq P$ such that $K\lhd N(P)$, and such that $V|K$ splits as a sum of nonisomorphic irreducible K-representations. Write $V|K\cong V_1\times\cdots\times V_r$ $(r>1)$, where each V_i is a sum of isomorphic K-representations, and where for $i\neq j$ the irreducible summands of V_i and V_j are nonisomorphic.

Since V is irreducible an an P-representation, P/K permutes the V_i transitively. In particular, they all have the same dimension: set $m=n/r=\dim(V_i)$. So (after conjugating) we may assume that $K\subseteq\mathbb{G}(m)^r$.

Let $K'/K\subseteq P/K$ be the subgroup of elements which leave each V_i invariant. Let $K_i\subseteq\mathbb{G}(m)$ be the image of K' in the i-th factor, so that

$$K'\subseteq\bar{K}=K_1\times\cdots\times K_r\subseteq\mathbb{G}(n).$$

Since $K\lhd N(P)$, the conjugation action of $N(P)$ permutes the V_i, and so $\bar{K}$ is normalized by $N(P)$. Lemma 1 now applies to show that $\bar{K}\subseteq P$.

Now set $L=P/\bar{K}$. Then L permutes the V_i effectively and transitively. Also, since $\chi_V(g)=0$ for all $g\in P\smallsetminus\bar{K}$, the action of any $1\neq x\in L$ must permute the V_i freely. So we can regard L as a free transitive subgroup of Σ_r. Then P is conjugate to $K_1\wr L$, and

$$N_{\mathbb{G}(n)}(P)/P\cong N_{\mathbb{G}(m)}(K_1)/K_1\times N_{\Sigma_r}(L)/L\cong N_{\mathbb{G}(m)}(K_1)/K_1\times\mathrm{Aut}(L). \qquad (9)$$

In particular, $\mathrm{Aut}(L)$ is p-reduced, so L must be elementary abelian, and is conjugate to $E_{p^k}\subseteq\Sigma_{p^k}$ $(r=p^k)$. Also, K_1 is p-stubborn in $\mathbb{G}(m)$. So by induction, we see that P is conjugate to an iterated wreath product $\Gamma\wr E_{q_1}\wr\cdots\wr E_{q_r}$, where $q_i=2^{t_i}$ and Γ is one of the groups listed in (2). Also, each of the $\Gamma\wr E_{q_1}\wr\cdots\wr E_{q_i}$ is p-stubborn in the appropriate $\mathbb{G}(m_i)$ and normal in $N(P)$; and so

$$N_G(P)/P\cong N_{\mathbb{G}(m)}(\Gamma)/\Gamma\times GL_{t_1}(\mathbb{F}_2)\times\cdots\times GL_{t_r}(\mathbb{F}_2).$$

Note in particular that $q_1\geq 4$ if $\Gamma=\mathrm{O}(1)$ (and $p=2$) — since $\mathrm{O}(1)\wr\Sigma_2$ is not 2-stubborn in $\mathrm{O}(2)$.

Step 3 In order to finish the proof of point (i), it remains to show that all of the groups listed there are p-stubborn (and with $N(P)/P$ as given). Fix a subgroup of the form $P=\Gamma\wr E_{q_1}\wr\cdots\wr E_{q_r}\subseteq\mathbb{G}(n)$: where $q_i=2^{t_i}$, $\Gamma\subseteq\mathbb{G}(m)$ is one of the groups listed in (2), and $q_1\geq 4$ if $\Gamma=\Gamma_1^{\mathrm{O}}=\mathrm{O}(1)$. When $r=0$, Γ was shown to be p-stubborn in Step Otherwise,

$$P'=\left(\Gamma\wr E_{q_1}\wr\cdots\wr E_{q_{r-1}}\right)^{q_r}\lhd P$$

is the subgroup generated by elements of nonzero character on V. In particular, $P'\lhd N(P)$, and so

$$N(P)/P\cong N(P')/P'\times GL_{t_r}(\mathbb{F}_p).$$

It now follows inductively that

$$N(P)/P \cong N_{\mathbb{G}(m)}(\Gamma)/\Gamma \times GL_{t_1}(\mathbb{F}_p) \times \cdots \times GL_{t_r}(\mathbb{F}_p).$$

This is p-reduced since $N(\Gamma)/\Gamma$ is (Step 1); and so P is p-stubborn.

Step 4 Now let $P \subseteq \mathbb{G}(n)$ be an arbitrary p-stubborn subgroup. Assume that the corresponding P-representation factors as a product $V_1 \times \ldots \times V_s$ of irreducible representations. In other words, after conjugating, we may assume that $P \subseteq \mathbb{G}(n_1) \times \cdots \times \mathbb{G}(n_s)$, where $n_i = \dim(V_i)$, and where the image $P_i \subseteq \mathbb{G}(n_i)$ of P in the i-th factor is irreducible for each i.

By Lemma 1, $C_{\mathbb{G}(n)}(P) = Z(P)$ — unless possibly p is odd and $\mathbb{G} = \mathrm{O}$ in which case $[C_{\mathbb{G}(n)}(P) : Z(P)] \leq 2$. In this latter case, $C_{\mathbb{G}(n)}(P)$ is a product of unitary groups and one copy of $\mathrm{O}(m)$, where $m = \dim(V^P)$. Also, $m \leq 1$, since $C_{\mathbb{G}(n)}(P)/Z(P) \supseteq \mathrm{O}(m)$ is the number of trivial summands in V.

In either case, $C_{\mathbb{G}(n)}(P)$ is abelian, so the V_i are distinct as P-representations, and are permuted by $N(P)$. In particular, $N(P) \subseteq N(P_1 \times \cdots \times P_s)$, and so Proposition 2.2 applies to show that $P = P_1 \times \cdots \times P_s$.

After reindexing, we can write $P = (P_1)^{m_1} \times \cdots \times (P_r)^{m_r}$, where the $P_i \subseteq \mathbb{G}(n_i)$ are irreducible and pairwise distinct as representations. Then

$$N(P)/P \cong \big(N_{\mathbb{G}(n_1)}(P_1)/P_1\big) \wr \Sigma_{m_1} \times \cdots \times \big(N_{\mathbb{G}(n_r)}(P_r)/P_r\big) \wr \Sigma_{m_r}.$$

In particular, since $N(P)/P$ is p-reduced by assumption, each of the $N(P_i)/P_i$ must be p-reduced, and so each of the P_i must be p-stubborn in $\mathbb{G}(q_i)$. If $N(P_i)/P_i$ is p-reduced, then the wreath product $(N(P_i)/P_i) \wr \Sigma_{m_i}$ is p-reduced if and only if either $N(P_i)/P_i \neq 1$, or Σ_{m_i} is p-reduced. Finally, Σ_m fails to be p-reduced only when (p, m) is one of the pairs $(2,2)$, $(2,4)$, or $(3,3)$; and this finishes the proof of point (ii). $\square$

For the sake of completeness, we also note the following conditions for when for two p-stubborn subgroups of $\mathbb{G}(n)$, of the form described in Theorem 3, one is contained (up to conjugacy) in the other.

Proposition 4. *Let p, and $G = \mathbb{G}(n)$ be as in Theorem 3, and let $P' \subseteq P$ be a pair of p-stubborn subgroups of G.*

(i) If P is reducible — if $P = P_1 \times \cdots \times P_s$ where $P_i \subseteq \mathbb{G}(n_i)$ is 2-stubborn and irreducible — then $P' = P'_1 \times \cdots \times P'_s$ for some subgroups $P'_i \subseteq P_i$ p-stubborn in $\mathbb{G}(n_i)$.

(ii) If P is irreducible and P' is reducible, then P is a wreath product

$$P = \Gamma \wr E_{q_1} \wr \cdots \wr E_{q_r}$$

for some $\Gamma \subseteq \mathbb{G}(q_0)$ as in Theorem 3 and some $r \geq 1$. And $P' = P'_1 \times \cdots \times P'_t$, where each P'_i is an irreducible subgroup of $\Gamma \wr E_{q_1} \wr \cdots \wr E_{q_s} \subseteq P$ (one of the standard subgroups of this form) for some $0 \geq s < r$.

(iii) Now assume that P and P' are both irreducible, and that

$$P = \Gamma \wr E_{q_1} \wr \cdots \wr E_{q_r} \quad \text{and} \quad P' = \Gamma' \wr E_{q'_1} \wr \cdots \wr E_{q'_s}$$

(up to conjugacy), where $\Gamma \subseteq \mathbb{G}(q_0)$ and $\Gamma' \subseteq \mathbb{G}(q'_0)$. Then either $P' = P$, or $s < r$ and $q_0, \ldots, q_r$ is a refinement of $q'_0, \ldots, q'_s$ (i.e., $q'_0 = q_0 \cdots q_{i_0}$, etc.).

Proof. Part (i) is clear. To see the other parts, define

$$A = \langle g \in P : Tr(g) \neq 0 \rangle \qquad \text{and} \qquad A' = \langle g \in P' : Tr(g) \neq 0 \rangle.$$

These subgroups can take the following forms (listed only for P):

(a) P is reducible. Then $A = P$.

(b) $P = \Gamma \wr E_{q_1} \wr \cdots \wr E_{q_r}$, where $(r \geq 1)$. Then $A = \big(\Gamma \wr E_{q_1} \wr \cdots \wr E_{q_{r-1}}\big)^{q_r}$.

(c) $P = \Gamma^X_{p^k}$ or $\bar{\Gamma}^X_{2^k}$. Then

$$A \cong \begin{cases} S^1 & \text{if } P = \Gamma^{\mathrm{U}}_{p^k} \\ \{\pm 1\} & \text{if } P = \Gamma^{\mathrm{O}}_{2^k} \text{ or } \Gamma^{\mathrm{Sp}}_{2^k} \\ S^1 & \text{if } P = \bar{\Gamma}^{\mathrm{O}}_{2^k} \textit{ or } \bar{\Gamma}^{\mathrm{Sp}}_{2^k}. \end{cases}$$

If P' is reducible, then $P' \subseteq A$ by (a). So P must be a wreath product as in (b), and P' factors as a product of q_r p-stubborn subgroups of $\mathbb{G}(n/q_r)$, each of which is contained in $\Gamma \wr E_{q_1} \wr \cdots \wr E_{q_{r-1}}$. Part (ii) now follows by induction.

Now assume that P and P' are both irreducible, and have the form given in (iii). If $r \geq 1$ (i.e., P is a nontrivial wreath product), then since $A' \subseteq A$, we see that $q_r | q'_s$, and that

$$\Big(\Gamma' \wr E_{q'_1} \wr \cdots \wr E_{q'_{s-1}}\Big)^{q'_s/q_r} \subseteq \Gamma \wr E_{q_1} \wr \cdots \wr E_{q_{r-1}}.$$

Point (iii) now follows by induction. □

Theorem 3 describes the p-stubborn subgroups of the matrix groups $\mathrm{O}(n)$, $\mathrm{U}(n)$, and $\mathrm{Sp}(n)$. However, what we are mostly interested in are the connected simple groups. The next two theorems describe the connection between the p-stubborn subgroups of $\mathrm{SU}(n)$ and of $\mathrm{U}(n)$; and between the 2-stubborn subgroups of $\mathrm{SO}(n)$ and $\mathrm{O}(n)$. In the first case, the categories $\mathcal{R}_p(\mathrm{SU}(n))$ and $\mathcal{R}_p(\mathrm{U}(n))$ are in fact isomorphic.

Theorem 5. *A subgroup $P \subseteq \mathrm{SU}(n)$ is p-stubborn if and only if $\langle P, Z(\mathrm{U}(n))\rangle$ is a p-stubborn subgroup of* $\mathrm{U}(n)$*; if and only if $P = \bar{P} \cap \mathrm{SU}(n)$ for some p-stubborn subgroup $\bar{P}$ of* $\mathrm{U}(n)$.

Proof. By Lemma 1(ii), for any connected G and any p-stubborn subgroup $P \subseteq G$, $P \supseteq C_G(P) \supseteq Z(G)$. And for any $P \subseteq \mathrm{SU}(n)$ such that $P \supseteq Z(\mathrm{SU}(n))$, if we set $\bar{P} = \langle P, Z(\mathrm{U}(n))\rangle$, then P is p-toral if and only if $\bar{P}$ is, $N_{\mathrm{SU}(n)}(P)/P \cong N_{\mathrm{U}(n)}(\bar{P})/\bar{P}$; and so P is p-stubborn if and only if $\bar{P}$ is. $\square$

Note the following corollary to Theorem 5. For convenience, for any prime power p^k, we write $\Gamma^{\mathrm{SU}}_{p^k} = \mathrm{SU}(p^k) \cap \Gamma^{\mathrm{U}}_{p^k}$. Note that there is a central extension $1 \to C_{p^k} \to \Gamma^{\mathrm{SU}}_{p^k} \to (C_p)^{2k} \to 1$.

Corollary 6. *If $P \subseteq \mathrm{SU}(n)$ is a finite p-stubborn subgroup, then $n = p^k$ for some k, and P is conjugate to $\Gamma^{\mathrm{SU}}_{p^k}$. Any irreducible p-stubborn subgroup of $\mathrm{SU}(p^k)$ contains a subgroup conjugate to $\Gamma^{\mathrm{SU}}_{p^k}$.*

Proof. Assume that $P \subseteq \mathrm{U}(n)$ is p-stubborn: then $P = P' \cap \mathrm{SU}(n)$ for some p-stubborn $P' \subseteq \mathrm{U}(n)$ by Theorem 5. In particular, if P is finite, then $\dim(P') = 1$ and P' is irreducible. So $n = p^k$ for some $k \geq 0$, P' is conjugate to $\Gamma^{\mathrm{U}}_{p^k}$ by Theorem 3(i), and P is conjugate to $\Gamma^{\mathrm{SU}}_{p^k} = \Gamma^{\mathrm{U}}_{p^k} \cap \mathrm{SU}(p^k)$.

The last statement follows easily from Theorems 3(i) and 5. $\square$

The relation between 2-stubborn subgroups of $\mathrm{SO}(n)$ and $\mathrm{O}(n)$ is more complicated.

Proposition 7. *For any 2-stubborn subgroup $P \subseteq \mathrm{SO}(n)$, there is a unique 2-stubborn subgroup $\bar{P} \subseteq \mathrm{O}(n)$ such that $P = \bar{P} \cap \mathrm{SO}(n)$ and $N_{\mathrm{O}(n)}(\bar{P}) = N_{\mathrm{O}(n)}(P)$. If $P_1 \subseteq P_2$ is a pair of 2-stubborn subgroups of $\mathrm{SO}(n)$, then $\bar{P}_1 \subseteq \bar{P}_2$.*

Proof. If $P \subseteq \mathrm{SO}(n)$ is 2-stubborn, then $N_{\mathrm{SO}(n)}(P)/P$ is 2-reduced, and so the intersection $\bar{P}/P$ of the 2-Sylow subgroups in $N_{\mathrm{O}(n)}(P)/P$ has order at most 2. If $\bar{P}/P = 1$, then $\bar{P} = P$ is 2-stubborn in $\mathrm{O}(n)$. Otherwise, $\bar{P} \not\subseteq \mathrm{SO}(n)$ and $N_{\mathrm{O}(n)}(\bar{P})/\bar{P} = N_{\mathrm{O}(n)}(P)/\bar{P}$ is 2-reduced; and so $\bar{P}$ is 2-stubborn and $P = \bar{P} \cap \mathrm{SO}(n)$. The uniqueness of $\bar{P}$ is clear.

Now assume that $P_1 \subseteq P_2$ are 2-stubborn in $\mathrm{SO}(n)$, and that $\bar{P}_i \supseteq P_i$ are as above. We must show that $\bar{P}_1 \subseteq \bar{P}_2$.

An inspection of the list in Theorem 3 shows that for any irreducible 2-stubborn subgroup $\bar{P} \subseteq \mathrm{O}(m)$, either $m = 2$ (and $\bar{P} = \mathrm{O}(2)$), or $\bar{P} \subseteq \mathrm{SO}(m)$, or there exist elements in $\bar{P} \smallsetminus \mathrm{SO}(m)$ of nonzero trace. So in all cases, $P = \bar{P} \cap \mathrm{SO}(m)$ is irreducible. Upon extending this to arbitrary 2-stubborn subgroups, we see that the P_i (for $i = 1, 2$) have the same decompositions into irreducible representations as the $\bar{P}_i$. In particular, we may assume that P_2 is irreducible.

Now define subgroups

$$A_i = \langle g \in P_i : Tr(g) \neq 0 \rangle \qquad \text{and} \qquad \bar{A}_i = \langle g \in \bar{P}_i : Tr(g) \neq 0 \rangle.$$

Clearly, $A_1 \subseteq A_2$. We may assume that $n \geq 3$. Then there are the following possibilities:

(a) $\bar{P}_i$ is not irreducible (and $i = 1$). Then $A_i = P_i$ and $\bar{A}_i = \bar{P}_i$.

(b) $\bar{P}_i = \Gamma \wr E_{q_1} \wr \cdots \wr E_{q_r}$, where $(r \geq 1)$. Then $\bar{A}_i = \big(\Gamma \wr E_{q_1} \wr \cdots \wr E_{q_{r-1}}\big)^{q_r}$ and $A_i = \bar{A}_i \cap P_i = \bar{A}_i \cap \mathrm{SO}(n)$.

(c) $\bar{P}_i = \Gamma^{\mathrm{O}}_{2^k}$ (some $k \geq 2$). Then $A_i = \bar{A}_i \cong \{\pm 1\}$.

(d) $\bar{P}_i = \bar{\Gamma}^{\mathrm{O}}_{2^k}$ (some $k \geq 2$). Then $A_i = \bar{A}_i \cong \mathrm{SO}(2)$.

Note that in each case, $\bar{P}_i = \langle \bar{A}_i, P_i \rangle$. So it will suffice to show that $\bar{A}_1 \subseteq \bar{A}_2$.

Assume first that $\bar{P}_2$ is of type (b). Then $\bar{A}_2$ is a product of 2-stubborn subgroups of $\bar{B} \subseteq \mathrm{O}(n/q_r)$. If $\bar{P}_1$ is reducible, then $P_1 = A_1 \subseteq A_2$, so $\bar{P}_1$ also splits as a product of 2-stubborn subgroups of $\mathrm{O}(n/q_r)$, and we get that $\bar{P}_1 \subseteq \bar{A}_2 \subseteq \bar{P}_2$ by induction on n. A similar argument applies to show that $\bar{A}_1 \subseteq \bar{A}_2$ if $\bar{P}_1$ is a wreath product. And if $\bar{P}_1 = \Gamma^{\mathrm{O}}_{2^k}$ or $\bar{\Gamma}^{\mathrm{O}}_{2^k}$ (where $2^k = n \geq 4$), then $\bar{P}_1 \subseteq \mathrm{SO}(n)$, so $P_1 = \bar{P}_1$, and there is nothing to prove.

Finally, assume that $P_2 = \bar{P}_2 = \Gamma^{\mathrm{O}}_{2^k}$ or $\bar{\Gamma}^{\mathrm{O}}_{2^k}$ (where $2^k = n \geq 4$). Then $A_2 \cong \{\pm 1\}$ or $\mathrm{O}(2)$, respectively, and $A_1 \subseteq A_2$. And an inspection of cases (a)–(d) above shows that either $\bar{P}_1 = \bar{P}_2$, or $\bar{P}_1 = \Gamma^{\mathrm{O}}_{2^k}$ and $\bar{P}_2 = \bar{\Gamma}^{\mathrm{O}}_{2^k}$. □

By Proposition 7, for each n, there is a well defined functor

$$\epsilon_n : \mathcal{R}_2(\mathrm{SO}(n)) \to \mathcal{R}_2(\mathrm{O}(n)),$$

defined by setting $\epsilon_n(\mathrm{SO}(n)/P) = \mathrm{O}(n)/\bar{P}$ whenever $P \subseteq \mathrm{SO}(n)$ and $\bar{P} \subseteq \mathrm{O}(n)$ are 2-stubborn subgroups such that $P = \bar{P} \cap \mathrm{SO}(n)$ and $N_{\mathrm{O}(n)}(\bar{P}) = N_{\mathrm{O}(n)}(P)$.

Theorem 8. *If $4 \nmid n$, then ϵ_n is an isomorphism of categories. Otherwise, its failure to be an isomorphism is described by the following points:*

(i) For any 2-stubborn subgroup $\bar{P} \subseteq \mathrm{O}(n)$, $P = \bar{P} \cap \mathrm{SO}(n)$ is 2-stubborn, and $N_{\mathrm{O}(n)}(P) = N_{\mathrm{O}(n)}(\bar{P})$ unless $\bar{P}$ has the form $\bar{P} = (\mathrm{O}(1) \wr E_4) \times P'$ for $P' \subseteq \mathrm{SO}(n-4)$ (in which case $N_{\mathrm{O}(n)}(P) \neq N_{\mathrm{O}(n)}(\bar{P})$).

(ii) Let $\bar{P} \subseteq \mathrm{O}(n)$ be any 2-stubborn subgroup. Then

$$\{g\bar{P}g^{-1} \cap \mathrm{SO}(n) : g \in \mathrm{O}(n)\}$$

consists of 1 $\mathrm{SO}(n)$*-conjugacy class if* $N(\bar{P}) \nsubseteq \mathrm{SO}(n)$*, and consists of 2 conjugacy classes if* $N(\bar{P}) \subseteq \mathrm{SO}(n)$*.*

(iii) Fix an irreducible 2-stubborn subgroup $\bar{P} \subseteq \mathrm{O}(2^k)$ *(any* $k \geq 0$*). Then* $\bar{P} \subseteq \mathrm{SO}(2^k)$ *if and only if* $\bar{P}$ *is conjugate to one of the groups*

$$\Gamma^{\mathrm{O}}_{q_0} \wr E_{q_1} \wr \cdots \wr E_{q_r} \; (q_0 \geq 4) \quad \text{or} \quad \bar{\Gamma}^{\mathrm{O}}_{q_0} \wr E_{q_1} \wr \cdots \wr E_{q_r} \; (q_0 \geq 4). \tag{1}$$

And for such $\bar{P}$*,* $N(\bar{P}) \subseteq \mathrm{SO}(2^k)$ *unless* $\bar{P}$ *is conjugate to* Γ^{O}_4*.*

Proof. If n is odd, then $\mathrm{O}(n) \cong \mathrm{SO}(n) \times \{\pm I\}$, and so ϵ_n is clearly an isomorphism of categories. If $n \equiv 2 \pmod 4$, then any 2-stubborn subgroup $\bar{P} \subseteq \mathrm{O}(n)$ has the form $\bar{P}' \times \mathrm{O}(2)$ for some $\bar{P}' \subseteq \mathrm{O}(n-2)$. In particular, if we set $P = \bar{P} \cap \mathrm{SO}(n)$, then $\mathrm{O}(n)/\bar{P} = \epsilon_n(\mathrm{SO}(n)/P)$ by (i); and $\mathrm{O}(n)/\bar{P}$ and $\mathrm{SO}(n)/P$ have the same morphisms since $\bar{P} \nsubseteq \mathrm{SO}(n)$.

(i) Let $\bar{P} \subseteq \mathrm{O}(n)$ be any 2-stubborn subgroup, and set $P = \bar{P} \cap \mathrm{SO}(n)$. By Theorem 3(ii), $\bar{P}$ splits as a product of subgroups $P_i \subseteq \mathrm{O}(n_i)$ (where $n = \sum n_i$). Let r be the number of factors for which $P_i \nsubseteq \mathrm{SO}(n_i)$. If $r = 0$, then $P = \bar{P}$ and is clearly 2-stubborn in $\mathrm{SO}(n)$. If $r \geq 2$, then $N_{\mathrm{O}(n)}(P) = N_{\mathrm{O}(n)}(\bar{P})$ ($\bar{P}$ is the product of the projections of P into the irreducible factors); and so P is again 2-stubborn. We are left with the case $r = 1$; and we can just as easily assume here that $\bar{P} \nsubseteq \mathrm{SO}(n)$ is irreducible. And a quick survey of the list in Theorem 3(i) shows that $\bar{P} = \mathrm{O}(1) \wr E_4 \subseteq \mathrm{O}(4)$ is the only case where $N(\bar{P}) \neq N(P)$.

(ii) Clear.

(iii) If $\bar{P} \subseteq \mathrm{O}(2^k)$ is irreducible and 2-stubborn, then by Theorem 3(i), $\bar{P} \subseteq \mathrm{SO}(n)$ if and only if it is one of the groups listed in (1) above. The only claim which is not easily checked is that $N_{\mathrm{O}(2^k)}(\bar{P}) \subseteq \mathrm{SO}(2^k)$ when $\bar{P} = \Gamma^{\mathrm{O}}_{2^k}$ ($k \geq 3$) or $\bar{\Gamma}^{\mathrm{O}}_{2^k}$ ($k \geq 2$). When $\bar{P} = \bar{\Gamma}^{\mathrm{O}}_{2^k}$, this follows since $N(\bar{P})/\bar{P} \cong \mathrm{Sp}_{2k-2}(\mathbb{F}_2)$ is generated by elements in $\mathrm{U}(2^{k-1}) \subseteq \mathrm{SO}(2^k)$.

Now assume $\bar{P} = \Gamma^{\mathrm{O}}_{2^k} = \langle A_0, B_0, \ldots, A_{k-1}, B_{k-1} \rangle$ (see Definition 2); and $k \geq 3$. Then for $1 \leq i \leq k-1$, A_i and B_i lie in the simply connected subgroup $\mathrm{SU}(2^{k-1}) \subseteq \mathrm{SO}(2^k)$. So the commutators $[A_i, B_i]$ ($1 \leq i \leq k-1$) all lift to the same commutator in $\tilde{P} \subseteq \mathrm{Spin}(2^k)$. By symmetry, $[A_0, B_0]$ lifts to the same commutator ($k \geq 3$); and so only one lifting of $-I$ lies in $[\tilde{P}, \tilde{P}]$. But for any $x \in \mathrm{O}(2^k) \smallsetminus \mathrm{SO}(2^k)$, $\mathrm{conj}(x)$ lifts to a unique automorphism of $\mathrm{Spin}(2^k)$ which switches the two liftings of $-I$; and so $x \notin N(P)$. Thus $N(P) = N(\bar{P}) \subseteq \mathrm{SO}(2^k)$. □

The importance of identifying those p-stubborn $P' \subseteq \mathrm{O}(n)$ for which $N(P') \subseteq \mathrm{SO}(n)$ is that these are precisely the subgroups whose conjugacy class in $\mathrm{O}(n)$ splits up into two conjugacy classes in $\mathrm{SO}(n)$.

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