Stable homotopy over the Steenrod algebra

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ABSTRACT. We apply the tools of stable homotopy theory to the study of modules over the Steenrod algebra A^*; in particular, we study the (triangulated) category $\mathsf{Stable}(A)$ of unbounded cochain complexes of injective comodules over A, the dual of A^*, in which the morphisms are cochain homotopy classes of maps. This category satisfies the axioms of a stable homotopy category (as given in [**HPS97**]); so we can use Brown representability, Bousfield localization, Brown-Comenetz duality, and other homotopy-theoretic tools to study $\mathrm{Ext}_A^{**}(\mathbf{F}_p, \mathbf{F}_p)$, which plays the role of the stable homotopy groups of spheres. We also have nilpotence theorems, periodicity theorems, a convergent chromatic tower, and a number of other results.

Contents

List of Figures

Preface

The object of study for this book is the mod p Steenrod algebra A and its cohomology Ext_A. Various people (including the author) have approached this subject by taking results in stable homotopy theory and then trying to prove analogous results for A-modules. This has proven to be successful, but the analogies were just that—there was no formal setting in which to do anything more precise than to make analogies.

In [**HPS97**], Hovey, Strickland, and the author developed "axiomatic stable homotopy theory." In particular, we gave axioms for a *stable homotopy category*; in any such category, one has available many of the tools of classical and modern stable homotopy theory—tools like Brown representability and Bousfield localization. It turns out that a category $\mathsf{Stable}(A)$ (defined in the next paragraph) related to the category of A-modules is such a category; as one might expect, the trivial module $\mathbf{F}_p$ plays the role of the sphere spectrum S^0, and $\operatorname{Ext}_A^{**}(-,-)$ plays the role of homotopy classes of maps. Since so many of the tools of stable homotopy theory are focused on the study of the homotopy groups of S^0 (and of other spectra), one should expect the corresponding tools in $\mathsf{Stable}(A)$ to help in the study of $\operatorname{Ext}_A^{**}(\mathbf{F}_p, \mathbf{F}_p)$ (and related groups). In this book we apply some of these tools (nilpotence theorems, periodicity theorems, chromatic towers, etc.) to the study of Ext over the Steenrod algebra. It is our hope that this book will serve two purposes: first, to provide a reference source for a number of results about the cohomology of the Steenrod algebra, and second, to provide an example of an in-depth use of the language and tools of axiomatic stable homotopy theory in an algebraic setting.

First we describe the category in which we work. We fix a prime p, let A^* be the mod p Steenrod algebra, and let $A = \operatorname{Hom}_{\mathbf{F}_p}(A^*, \mathbf{F}_p)$ be the (graded) dual of the Steenrod algebra. We let $\mathsf{Stable}(A)$ be the category whose objects are cochain complexes of injective left A-comodules, and whose morphisms are cochain homotopy classes of maps. This is a stable homotopy category (of a particularly nice sort—it is a monogenic Brown category—see [**HPS97**, 9.5]). We prove a number of results in $\mathsf{Stable}(A)$; some of these are analogues of results in the ordinary stable homotopy category, and some are not. Some of these are new, and some already known, at least in the setting of A^*-modules; the old results often need new proofs to apply in the more general setting we discuss here.

Note. This work arose from the study of the abelian category of (left) A^*-modules; to apply stable homotopy theoretic techniques, though, it is most convenient to work in a triangulated category. One's first guess for an appropriate category might have objects which are chain complexes of projective A^*-modules; it turns out that this category has some technical difficulties (see Remark 1.2.1). It is much more convenient to work with A-comodules instead of A^*-modules, and fortunately, one does not lose much by doing this. Most A^*-modules of interest can be viewed as A-comodules; the main effects of using comodules are things of the following sort: various arrows go the "wrong" way, $\mathrm{Ext}_A^{**}(k,k)$ is covariant in A, and one studies A by means of its quotient Hopf algebras (because those are dual to the sub-Hopf algebras of A^*).

Each chapter is divided into a number of sections; at the beginning of each chapter, we give a brief description of its contents, section by section. In this introduction, we give a brief overview of each chapter. We note that each chapter has at least one "Further discussion" section, in which we discuss issues auxiliary to the general discussion.

In Chapter 1, we set up notation and discuss results that hold in the category $\mathsf{Stable}(\Gamma)$ for any graded commutative Hopf algebra Γ over a field k, e.g., the dual of a group algebra, the dual of an enveloping algebra, or the dual of the Steenrod algebra. Aside from setting up notation for use throughout this book, the main topics of this chapter include: construction of cellular and Postnikov towers, an examination of the Adams spectral sequence associated to particular homology theories on $\mathsf{Stable}(\Gamma)$, and some remarks on Bousfield classes and Brown-Comenetz duality.

NOTE. While some of Chapter 1 may be well-known, we recommend that the reader look over Section 1.2 and the first part of Section 1.3 (at least the definition of the functor H) before reading later parts of the book. These sections introduce notation that gets used throughout the book.

In Chapter 2 we specialize to the case in which p is a prime, $k = \mathbf{F}_p$ is the field with p elements, and A is the dual of the mod p Steenrod algebra. Recall from [**Mil58**] that as algebras, we have

$$A \cong \begin{cases} \mathbf{F}_2[\xi_1, \xi_2, \xi_3, \dots], & \text{if } p = 2, \\ \mathbf{F}_p[\xi_1, \xi_2, \xi_3, \dots] \otimes E[\tau_0, \tau_1, \tau_2, \dots], & \text{if } p \text{ is odd.} \end{cases}$$

The coproduct on A is determined by

$$\Delta\colon\ \xi_n \longmapsto \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \xi_i,$$

$$\Delta\colon\ \tau_n \longmapsto \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \tau_i + \tau_n \otimes 1.$$

In this chapter, we discuss two tools with which to study $\mathsf{Stable}(A)$: quotient Hopf algebras of A and P_t^s-homology. We use these tools to prove two theorems: the first is a vanishing line theorem (given conditions on the P_t^s-homology groups of X, then $\mathrm{Ext}_A^{s,t}(\mathbf{F}_p, X) = 0$ when $s > mt - c$, for some numbers m and c). The second is a "self-map" theorem: given a finite A-comodule M, we construct a non-nilpotent element of $\mathrm{Ext}_A^{**}(M,M)$ satisfying certain properties.

Let $p = 2$. In Chapter 3 we develop analogues in the category $\mathsf{Stable}(A)$ of the nilpotence theorem of Devinatz, Hopkins, and Smith, as well as the stratification theorem of Quillen. In fact we give two nilpotence theorems: in one we describe a single ring object (like BP) that detects nilpotence; more precisely, there is a quotient Hopf algebra D of A so that, if M is a finite-dimensional A-comodule, an element $z \in \mathrm{Ext}_A^{**}(M,M)$ is nilpotent under Yoneda composition if and only if its restriction to $\mathrm{Ext}_D^{**}(M,M)$ is nilpotent. The second nilpotence theorem is similar, but uses a family of ring objects (somewhat like the Morava K-theories) to detect nilpotence. These are versions in $\mathsf{Stable}(A)$ of the nilpotence theorems of [**DHS88**] and [**HSb**]. We strengthen these results when studying $\mathrm{Ext}_A^{**}(\mathbf{F}_2, \mathbf{F}_2)$, by "identifying" the image of $\mathrm{Ext}_A^{**}(\mathbf{F}_2, \mathbf{F}_2) \to \mathrm{Ext}_D^{**}(\mathbf{F}_2, \mathbf{F}_2)$ (and similarly for the other nilpotence theorem). One can view this as an analogue of Quillen's theorem [**Qui71**, 6.2], in which he identifies the cohomology of a compact Lie group up to F-isomorphism.

Again, let $p = 2$. In Chapter 4, we discuss applications of the theorems from the previous chapter. In ordinary stable homotopy theory, the nilpotence theorems lead to the periodicity theorem and the thick subcategory theorem (see [**Hop87**]); in our setting, things are a bit harder, so we get a weak version of a periodicity theorem, and only a conjecture as to a classification of the thick subcategories of finite objects in $\mathsf{Stable}(A)$. More precisely, if M is a finite-dimensional A-comodule, then we produce a number of central non-nilpotent elements in $\mathrm{Ext}_A^{**}(M, M)$ by using the "variety of M": the kernel of $\mathrm{Ext}_D^{**}(\mathbf{F}_2, \mathbf{F}_2) \to \mathrm{Ext}_D^{**}(M, M)$.

One of our analogues of Quillen's theorem says that the elements of the kernel of $\mathrm{Ext}_A^{**}(\mathbf{F}_2, \mathbf{F}_2) \to \mathrm{Ext}_D^{**}(\mathbf{F}_2, \mathbf{F}_2)$ are nilpotent, and it identifies the image. This identification is not explicit, so we discuss a small list of examples. We also imitate [**Rav84**] to show that the objects that detect nilpotence have strictly smaller Bousfield class than the sphere.

Let p be any prime. In Chapter 5 we consider Steenrod algebra analogues of chromatic theory and the functors L_n and L_n^f. The latter turns out be more tractable; in fact, it is a generalization (from the module setting) of Margolis' killing construction [**Mar83**, Chapter 21]. We show that $L_n \neq L_n^f$ if $n > 1$, at least at the prime 2. We compute L_n^f on some particular ring spectra, and show that, at least for these rings, it turns "group cohomology" into "Tate cohomology." We use this result to show that the chromatic tower constructed using the functors L_n^f converges for any finite object. (This is an extension of a theorem of Margolis [**Mar83**, Theorem 22.1].)

We also have several appendices: In Appendix A.1, we describe a model category whose associated homotopy category is $\mathsf{Stable}(A)$; the results in this section are due to Hovey. In Appendix A.2 we prove a theorem due to Hopkins and Smith [**HSa**], that the property of having a vanishing line with given slope, at some term of the Adams spectral sequence and with some intercept, is generic. (We prove this in the context of Adams spectral sequences in $\mathsf{Stable}(A)$, which are trigraded; hence we actually discuss vanishing *planes*.) In Appendix B, we discuss the nilpotence of certain classes in $\mathrm{Ext}_A^{**}(\mathbf{F}_p, \mathbf{F}_p)$ when A is the Steenrod algebra; we use these results in Chapter 3 to prove our nilpotence theorems.

In this book we have a mix of results: some are extensions of older results to the cochain complex setting, and some are new. For each older result, if the proof in the literature extends easily to our setting, then we do not include a proof; otherwise, we at least give a sketch. It appears that when one uses the language of stable homotopy theory, one tends to change arguments with spectral sequences into simpler arguments with cofibration sequences (see Lemma 1.3.10, for example), so even though the setting is potentially more complicated, some of the proofs simplify. In such cases, we often give in to temptation and include the new proof in its entirety (as, for example, with the vanishing line theorem 2.3.1). Obviously, we include full proofs of all of the new results, and we give complete references for all of the old results.

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CHAPTER 1

Stable homotopy over a Hopf algebra

In this chapter we discuss stable homotopy theory over any graded commutative Hopf algebra Γ over a field k; a major focus of study is $\mathrm{Ext}_{\Gamma}^{**}(k,k)$, where Ext denotes *comodule* Ext—derived functors of Hom in the category of left A-comodules. This material applies when Γ is the dual of a group algebra, the dual of an enveloping algebra, or the dual of the Steenrod algebra; in these cases, $\mathrm{Ext}_{\Gamma}^{**}(k,k)$ is the ordinary cohomology of Γ^* with coefficients in k.

Our goal in this chapter is to establish some notation, make some basic definitions, and prove some general facts about the category $\mathsf{Stable}(\Gamma)$ of cochain complexes of injective Γ-comodules.

In more detail: we start in Section 1.1 with brief recollections about Hopf algebras, comodules, and homological algebra. In Section 1.2 we define our setting for the rest of the book, the category $\mathsf{Stable}(\Gamma)$. We also set up the some important notation; for instance, we explain the grading conventions on morphisms and (co)homology functors in $\mathsf{Stable}(\Gamma)$—if X is an injective resolution of a left comodule M, then the (s,t)-homotopy group $\pi_{s,t}X$ is equal to $\mathrm{Ext}_{\Gamma}^{s,t}(k,M)$. In Section 1.3 we construct some particular ring objects in our category, one object HB for every quotient Hopf algebra B of Γ. To be precise, HB is an injective resolution of $\Gamma \,\square_B k$; this is a ring spectrum in $\mathsf{Stable}(\Gamma)$. So for instance, if we were working with $\Gamma = (kG)^*$, we would have one such object HB for every subgroup B of G, and the object HB would have homotopy groups $\pi_* HB = H^*(B,k)$. In Subsection 1.3.1 we establish some notation for Hopf algebra extensions, and we prove one or two useful results about extensions with small kernel. For example, given an extension of Hopf algebras of the form

$$E[x] \to B \to C,$$

the associated extension spectral sequence has only one possible differential; if B is a quotient Hopf algebra of Γ, then in the category $\mathsf{Stable}(\Gamma)$, this manifests itself as a cofibration sequence $HB \to HB \to HC$. In Section 1.4 we set up cellular towers and Postnikov towers in $\mathsf{Stable}(\Gamma)$, and we prove a Hurewicz theorem and a few useful lemmas. In Section 1.5 we discuss the Adams spectral sequence based on the homology theory associated to the ring spectrum HB, for B a “conormal” quotient of Γ. This turns out to be the same, up to a regrading, as the Lyndon-Hochschild-Serre spectral sequence associated to the Hopf algebra extension

$$\Gamma \,\square_B k \to \Gamma \to B.$$

In Section 1.6 we define Bousfield classes and Brown-Comenetz duality, and we recall some results of Ravenel's relating the two.

In Section 1.7 we apply some of this work to the study of stable homotopy over a group algebra. We point, for example, out that a corollary of work of Benson, Carlson, and Rickard is a classification of the Bousfield lattice in $\mathsf{Stable}(kG^*)$, for G a p-group and k a field of characteristic p.

1.1. Recollections

In this section we review material on Hopf algebras, comodules, and homological algebra. This material is standard; many readers are probably familiar with it.

1.1.1. Hopf algebras. We start with the definition of a Hopf algebra. This is standard; one reference is [**MM65**].

Definition 1.1.1. Fix a field k. A *Hopf algebra* over k is a graded k-vector space $\Gamma = \bigoplus_{i\in\mathbf{Z}} \Gamma_i$ together with the following structure maps:

- a *unit* map $\eta\colon\ k \to \Gamma$,
- a *multiplication* map $\mu\colon\ \Gamma \otimes_k \Gamma \to \Gamma$,
- a *counit* map $\varepsilon\colon\ \Gamma \to k$,
- a *comultiplication* map $\Delta\colon\ \Gamma \to \Gamma \otimes_k \Gamma$,
- and a *conjugation* map $\chi\colon\ \Gamma \to \Gamma$.

The maps η and μ give Γ the structure of an associative unital k-algebra—i.e., the following two diagrams commute (all tensor products are over k):

$$\begin{tikzcd}
\Gamma\otimes\Gamma\otimes\Gamma \arrow[r, "1\otimes\mu"] \arrow[d, "\mu\otimes 1"'] & \Gamma\otimes\Gamma \arrow[d, "\mu"] \\
\Gamma\otimes\Gamma \arrow[r, "\mu"] & \Gamma,
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Gamma\otimes k \arrow[d, "1\otimes\eta"'] \arrow[dr, "="] & \\
\Gamma\otimes\Gamma \arrow[r, "\mu"] & \Gamma. \\
k\otimes\Gamma \arrow[u, "\eta\otimes 1"] \arrow[ur, "="'] &
\end{tikzcd}$$

Dually, the maps ε and Δ make Γ into a coassociative coalgebra—the dual diagrams to those above (i.e., diagrams as above, but with all the arrows reversed) commute. We give $\Gamma\otimes\Gamma$ an algebra structure via the composite

$$\Gamma\otimes\Gamma\otimes\Gamma\otimes\Gamma \xrightarrow{1\otimes T\otimes 1} \Gamma\otimes\Gamma\otimes\Gamma\otimes\Gamma \xrightarrow{\mu\otimes\mu} \Gamma\otimes\Gamma,$$

where $T\colon\ \Gamma\otimes\Gamma \to \Gamma\otimes\Gamma$ is the "twist" map: $T(a\otimes b) = (-1)^{|a||b|} b\otimes a$. (Dually, we can give $\Gamma\otimes\Gamma$ the structure of a coalgebra.) We insist that the maps Δ and ε be algebra maps; equivalently, we insist that μ and η be coalgebra maps. Lastly, the conjugation map χ makes the following diagram commute:

$$\begin{tikzcd}
\Gamma \arrow[r, "\Delta"] \arrow[d, "\varepsilon"'] & \Gamma \otimes \Gamma \arrow[r, "1\otimes\chi"] & \Gamma \otimes \Gamma \arrow[d, "\mu"] \\
k \arrow[rr, "\eta"] & & \Gamma.
\end{tikzcd}$$

Also, the same diagram, except with $\chi \otimes 1$ replacing $1 \otimes \chi$, also commutes.

CONVENTION 1.1.2. We work throughout with graded vector spaces; every map between graded spaces is a graded map, and every element from such a vector space is assumed to be homogeneous, unless otherwise indicated. Given $v \in V$, we write $|v|$ for the *degree* of v. Also, all unmarked tensor products are over the ground field.

Since the structure maps in the previous definition are assumed to be graded maps, then the image of η lies in Γ_0, and the kernel of ε contains $\bigoplus_{i\neq 0} \Gamma_i$.

DEFINITION 1.1.3. Let Γ be a Hopf algebra over a field k. We say that Γ is *commutative* if it is commutative as an algebra—i.e., the following diagram commutes:

$$\begin{tikzcd}
\Gamma \otimes \Gamma \arrow[d, "T"'] \arrow[dr, "\mu"] & \\
\Gamma \otimes \Gamma \arrow[r, "\mu"] & \Gamma.
\end{tikzcd}$$

(As in Definition 1.1.1, T is the twist map.) Γ is *cocommutative* if the dual diagram commutes; Γ is *bicommutative* if it is both commutative and cocommutative. We say that $\Gamma = \bigoplus_i \Gamma_i$ is *connected* if $\Gamma_i = 0$ when $i < 0$, and $\eta\colon k \to \Gamma_0$ is an isomorphism.

Note that if each homogeneous piece Γ_i is finite-dimensional, then the graded dual Γ^* of Γ has the structure of a Hopf algebra. Then Γ is commutative if and only if Γ^* is cocommutative, and so forth.

EXAMPLE 1.1.4. (a) The homology of a topological group G with coefficients in a field k is a cocommutative Hopf algebra; it is connected if and only if G is connected.

(b) For any group G, the group algebra kG is a cocommutative Hopf algebra. It is commutative if and only if G is abelian. It is graded trivially: every element is homogeneous of degree zero. If G is finite, then the vector space dual of kG is a commutative Hopf algebra.

(c) For any Lie algebra L, its universal enveloping algebra $U(L)$ is a cocommutative Hopf algebra. It is commutative if and only if L is abelian. As with kG, it is graded trivially (unless L is graded itself).

(d) Similarly, if k has characteristic p, for any restricted Lie algebra L, its restricted universal enveloping algebra $V(L)$ is a cocommutative Hopf algebra.

(e) Fix a prime p. The mod p Steenrod algebra A^* is a cocommutative Hopf algebra; its dual A_* is a commutative Hopf algebra. Both A^* and A_* are connected. Starting in the next chapter, we will focus almost exclusively on this example.

DEFINITION 1.1.5. An element γ of a Hopf algebra Γ is *primitive* if $\Delta(\gamma) = \gamma \otimes 1 + 1 \otimes \gamma$. We write $P\Gamma$ for the vector space of all primitives of Γ. We say that γ is *grouplike* if $\Delta(\gamma) = \gamma \otimes \gamma$.

1.1.2. Comodules. We move on to a brief discussion of modules and comodules. Again, [**MM65**] is one of the standard references; [**Boa**] is also quite useful.

DEFINITION 1.1.6. Let Γ be a coalgebra over a field k. A k-space M is a (left) Γ-*comodule* if there is a structure map $\psi\colon M \to \Gamma \otimes M$, called the *coaction* map, making the following diagram commute:

$$\begin{array}{ccc} M & \xrightarrow{\psi} & \Gamma \otimes M. \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle 1\otimes\psi} \\ \Gamma \otimes M & \xrightarrow{\psi\otimes 1} & \Gamma \otimes \Gamma \otimes M \end{array}$$

(In other words, ψ defines a "coassociative" coaction.) A (left) module over a k-algebra is defined dually, of course. We say that $M = \oplus_i M_i$ is *bounded below* if $M_i = 0$ for all sufficiently small i.

Throughout, we use left comodules and left modules; from here on, we will omit the word "left."

Given two Γ-comodules M and N over a Hopf algebra Γ, then $M \otimes N$ is naturally a Γ-comodule, via the structure map

$$M \otimes N \xrightarrow{\psi_M \otimes \psi_N} \Gamma \otimes M \otimes \Gamma \otimes N \xrightarrow{1\otimes T \otimes 1} \Gamma \otimes \Gamma \otimes M \otimes N \xrightarrow{\mu \otimes 1 \otimes 1} \Gamma \otimes M \otimes N.$$

This is called the "diagonal" coaction. We can also put the "left" coaction on $M \otimes N$, in which the structure map is

$$M \otimes N \xrightarrow{\psi_M \otimes 1} \Gamma \otimes M \otimes N.$$

We rarely use this comodule structure; when we do, we denote the comodule by $M \overset{L}{\otimes} N$. If we want to explicitly distinguish the diagonal coaction from the left coaction, we write $M \overset{\Delta}{\otimes} N$ for the tensor product with the diagonal coaction.

We will use the following lemma once or twice. It is fairly standard; see [**Boa**, 5.7], for example.

LEMMA 1.1.7. *Let M be a Γ-comodule. Then $\Gamma \overset{\Delta}{\otimes} M$ with the diagonal coaction is naturally isomorphic, as a Γ-comodule, to $\Gamma \overset{L}{\otimes} M$ with the left coaction. In particular, $\Gamma \overset{\Delta}{\otimes} M$ is a direct sum of copies of Γ.*

PROOF. One can check that the following two composites are mutually inverse Γ-comodule maps:

$$\Gamma \overset{L}{\otimes} N \xrightarrow{1\otimes\psi_N} \Gamma \otimes \Gamma \otimes N \xrightarrow{1\otimes\chi\otimes 1} \Gamma \otimes \Gamma \otimes N \xrightarrow{\mu} \Gamma \otimes N,$$

$$\Gamma \otimes N \xrightarrow{1\otimes\psi_N} \Gamma \otimes \Gamma \otimes N \xrightarrow{\mu\otimes 1} \Gamma \overset{L}{\otimes} N.$$

$\square$

LEMMA 1.1.8. *Let Γ be a Hopf algebra over k, and assume that each Γ_i is finite-dimensional. Let Γ^* denote the (graded) dual of Γ. Then every Γ-comodule* *has a natural* Γ^**-module structure. If* Γ *is finite-dimensional, then the categories* Γ-Comod *and* Γ^*-Mod *are equivalent.*

PROOF. Let M be a Γ-comodule. Then we make it a Γ^*-module via the structure map

$$\Gamma^* \otimes M \xrightarrow{1\otimes\psi} \Gamma^* \otimes \Gamma \otimes M \xrightarrow{\mathrm{ev}\otimes 1} k \otimes M = M.$$

If Γ is finite-dimensional, then one chooses dual bases (γ_i) and (g_i) for Γ and Γ^*. Given a Γ^*-module N with structure map $\varphi\colon \Gamma^* \otimes M \to M$, we make N into a Γ-comodule via the map

$$\begin{aligned} N &\to \Gamma \otimes N, \\ n &\longmapsto \sum_j \gamma_j \otimes n_j, \end{aligned}$$

where we sum over all j so that n is "hit" by g_j: we have $\varphi(g_j \otimes n_j) = n$. Since Γ is finite-dimensional, this is a finite sum, and hence an element of the tensor product.

We leave the rest of the proof to the reader. □

1.1.3. Homological algebra. Now we discuss a little homological "coalgebra." [**Boa**] is a good reference for this material, as well. One can also dualize discussions of homological algebra for modules, as found in any number of places (such as [**CE56, Wei94, Ben91a**]).

Since we are working with comodules rather than modules, we work with the notions of cofree and injective comodules, which are dual to the notions of free and projective, respectively.

DEFINITION 1.1.9. Let Γ be a k-coalgebra. A comodule M is *injective* if the functor $\mathrm{Hom}_\Gamma(-, M)$ is exact. A comodule M is *projective* if $\mathrm{Hom}_\Gamma(M, -)$ is exact. The forgetful functor $U\colon$ Γ-$\mathsf{Comod} \to k$-Mod has a right adjoint, C:

$$\mathrm{Hom}_k(UM, N) \cong \mathrm{Hom}_\Gamma(M, CN).$$

CN is called the *cofree* comodule on N.

See [**Boa**] for the following two results.

LEMMA 1.1.10. *A comodule is injective if and only if it is a summand of a cofree comodule.*

LEMMA 1.1.11. Γ *is cofree, and hence injective, as a* Γ*-comodule.*

In other words, every comodule map $M \to \Gamma$ is adjoint to a vector space map $M \to k$, and hence is determined by the preimage of $1 \in \Gamma$.

Injective comodules are much more important to us than projective comodules, because of the following.

EXAMPLE 1.1.12.
(a) On one hand, if Γ is finite-dimensional, then injective comodules are the same as projective comodules.
(b) On the other hand, if $\Gamma = A_*$ is the dual of the mod p Steenrod algebra, then it seems likely that there are no nonzero projective comodules. Essentially, an element in a projective comodule over A_* should need to have infinitely many elements in its diagonal, so it would have to be viewed as a completed comodule, not a comodule proper.

By Lemma 1.1.7, $\Gamma \otimes M$ is injective for any comodule M; indeed, $\Gamma \otimes M$ is isomorphic to $C(UM)$. The map $M \to \Gamma \otimes M$ which is adjoint to the identity on UM gives us the start of an injective resolution of M.

DEFINITION 1.1.13. Given a comodule M, a sequence of injective comodules $I_\bullet = (I_0 \to I_1 \to I_2 \to \dots)$ is an *injective resolution* of M if the sequence

$$0 \to M \to I_0 \to I_1 \to I_2 \to \dots$$

is exact.

As is well-known, injective resolutions are unique up to chain homotopy equivalence. We are interested in injective comodules because they are useful in computing various derived functors, particularly Ext.

DEFINITION 1.1.14. Let Γ be a Hopf algebra over k, and let M and N be Γ-comodules. Then $\mathrm{Ext}_\Gamma^s(M,N)$ is the sth derived functor of $\mathrm{Hom}_\Gamma(M,N)$. To compute it, one takes an injective resolution $I_\bullet$ of N, and defines $\mathrm{Ext}_\Gamma^s(M,N)$ to be the sth cohomology group of the cochain complex $\mathrm{Hom}_\Gamma(M, I_\bullet)$.

Throughout this book, when we say Ext, we mean Ext in this setting: the category of comodules over a Hopf algebra.

LEMMA 1.1.15. *Let Γ be a connected Hopf algebra over a field k. Then the vector space $\mathrm{Ext}_\Gamma^1(k,k)$ is isomorphic to $P\Gamma$, the space of primitives of Γ.*

PROOF. This is dual to the statement that in the category of modules over an algebra, Ext^1 is in bijection with the generators of the algebra. One proves this by constructing the first few terms of a minimal resolution of k as a Γ-comodule. We leave the details to the reader. $\square$

We point out that for any comodule, there is a canonical injective resolution, known as the *cobar complex*. We will do a few computations with it in Appendix B.3; we refer the reader to [**Ada56, HMS74**] for details.

1.2. The category $\mathsf{Stable}(\Gamma)$

In this section, we define the category in which we work, and we introduce notation which we will use for the rest of this book.

Let Γ be a graded commutative Hopf algebra over a field k; we work in the category $\mathsf{Stable}(\Gamma)$ whose objects are cochain complexes of injective graded left Γ-comodules, and whose morphisms are cochain homotopy classes of graded maps. We call objects of this category *spectra*, and we write the morphisms from X to Y as $[X,Y]$. We will omit the word "graded" from this point on; all comodules and maps are understood to be graded.

The category $\mathsf{Stable}(\Gamma)$ is a *stable homotopy category* in the sense of [**HPS97**]; hence one can perform many standard stable homotopy theoretic constructions in $\mathsf{Stable}(\Gamma)$. For instance, rather than having exact sequences, one has "exact triangles," also known as "cofibrations" or "cofiber sequences." We freely use other language and results from [**HPS97**], often without explicit citation. $\mathsf{Stable}(\Gamma)$ is *generated* by the injective resolutions of the simple comodules—that is, if X is an object so that $[S,X]_{**} = 0$ whenever S is an injective resolution of a simple comodule, then X is a contractible cochain complex. If the trivial comodule k is the only simple (say, if Γ is connected, or if Γ is the dual of the mod p group algebra of a p-group), then we say that $\mathsf{Stable}(\Gamma)$ is *monogenic*, at least in the graded sense. In this case, the stable homotopy constructions are even more familiar.

If $\mathsf{Stable}(\Gamma)$ is monogenic and the homotopy of the sphere object (defined below) is countable, then $\mathsf{Stable}(\Gamma)$ is a *Brown category*, so that homology functors are representable. This is the case when $\Gamma = A$, the dual of the Steenrod algebra. ($\mathsf{Stable}(\Gamma)$ can be a Brown category even if it is not monogenic; since we are focusing on the Steenrod algebra in this book, though, we often assume monogenicity.)

REMARK 1.2.1. (a) Mahowald and Sadofsky studied the category $\mathsf{Stable}(\Gamma)$ in their paper [**MS95**], with $\Gamma = A$.

(b) If Γ is finite-dimensional, then one could just as well work with the category of cochain complexes of injective Γ^*-modules, because in the finite-dimensional case, the categories of Γ-comodules and Γ^*-modules are equivalent, with injective comodules corresponding to injective modules. When Γ^* is not finite-dimensional, in particular when $\Gamma^* = A^*$ is the Steenrod algebra, there are technical problems with the category of cochain complexes of injective A^*-modules. For example, there are no maps from $\mathbf{F}_2$ to A^*, so the "homotopy" of the (injective) module A^* would be zero; therefore in the module setting, we would not have the implication $\pi_* X = 0 \Rightarrow X = 0$.

(c) We also note that, regardless of the dimension of Γ, the category $\mathsf{Stable}(\Gamma)$ is rather different from the derived category of Γ-comodules, because homology isomorphisms are not necessarily invertible in $\mathsf{Stable}(\Gamma)$. For instance, if $\Gamma = \mathbf{F}_2[x]/(x^2)$ with x primitive, then the (periodic) cochain complex

$$\begin{gathered} \dots \longrightarrow \Gamma \longrightarrow \Gamma \longrightarrow \Gamma \longrightarrow \Gamma \longrightarrow \dots, \\ x \mapsto 1, \; x \mapsto 1, \end{gathered}$$

has no homology, and hence is zero in the derived category. On the other hand, it is non-contractible in $\mathsf{Stable}(\Gamma)$; if we write $\mathrm{Ext}_\Gamma^{**}(\mathbf{F}_2, \mathbf{F}_2) = \mathbf{F}_2[v]$, then this complex is a ring spectrum with homotopy groups (as defined below) equal to $\mathbf{F}_2[v, v^{-1}]$.

The category $\mathsf{Stable}(\Gamma)$ has arbitrary *coproducts*; we use the symbol $\vee$ to denote the coproduct.

For objects X and Y of $\mathsf{Stable}(\Gamma)$, we write $X \wedge Y$ for $X \otimes_k Y$, and we call this the *smash product* of X and Y. This operation is commutative, associative, and unital: if S is an injective resolution of the trivial comodule k, then S is the unit of the smash product. We call S the *sphere* spectrum. We grade morphisms as follows: first of all, we let $[X, Y]_{0,0} = [X, Y]$. We write

$$S = (I_0 \to I_1 \to I_2 \to \dots),$$

and we let $S^{i,j}$ be the cochain complex which is $s^{-j} I_{n+i}$ in homological degree n (here s denotes the "internal" suspension functor $s \colon \Gamma\text{-}\mathsf{Comod} \to \Gamma\text{-}\mathsf{Comod}$). For integers i and j we define the *(i,j)-suspension functor* $\Sigma^{i,j}$ by

$$\begin{gathered} \Sigma^{i,j} \colon \mathsf{Stable}(\Gamma) \to \mathsf{Stable}(\Gamma), \\ X \longmapsto S^{i,j} \wedge X. \end{gathered}$$

Then we let $[X, Y]_{i,j} = [\Sigma^{i,j} X, Y]_{0,0}$, and we write $[X, Y]_{**}$ for $\bigoplus_{i,j} [X, Y]_{i,j}$.

REMARK 1.2.2. The grading is the usual Ext grading: if X and Y are injective resolutions of comodules M and N, respectively, then $[X, Y]_{i,j} = \mathrm{Ext}_\Gamma^{i,j}(M, N)$.

Hence if Γ is concentrated in degree 0 (e.g., if $\Gamma = (kG)^*$), then one may as well work with Γ-comodules concentrated in degree 0, in which case $[-,-]_{ij} = 0$ if $j \neq 0$, and $[-,-]_{i0} = \mathrm{Ext}^i_\Gamma(-,-)$. We will follow the (somewhat odd) tradition in homotopy theory of drawing pictures of Ext (and hence of $[-,-]$) using the Adams spectral sequence grading: $\mathrm{Ext}^{s,t}$ is drawn with s on the vertical axis and $t-s$ on the horizontal axis.

Unfortunately, because of the form of long exact sequences in Ext, cofiber sequences look like this:

$$\ldots \to \Sigma^{1,0} Z \to X \to Y \to Z \to \Sigma^{-1,0} X \to \ldots.$$

So one needs to take a little care when translating proofs from ordinary homotopy theory to this setting. Given a spectrum X, we define the *homology functor* associated to X, X_{ij}, by

$$\begin{aligned} X_{ij} \colon \mathsf{Stable}(\Gamma) &\to \mathsf{Ab}, \\ Y &\longmapsto [S, X \wedge Y]_{i,j}, \end{aligned}$$

and we define the *cohomology functor* associated to X, X^{ij}, by

$$\begin{aligned} X^{ij} \colon \mathsf{Stable}(\Gamma)^{\mathrm{op}} &\to \mathsf{Ab}, \\ Y &\longmapsto [Y, X]_{-i,-j}. \end{aligned}$$

When $X = S$, we have a special notation for X_{ij}: we define the (i,j)-*homotopy group* of Y to be $\pi_{ij} Y = S_{ij} Y = [S, Y]_{ij}$. Note that if $X \to Y \to Z$ is a cofiber sequence, then we have

$$\ldots \to \pi_{i-1,j} Z \to \pi_{i,j} X \to \pi_{i,j} Y \to \pi_{i,j} Z \to \pi_{i+1,j} X \to \ldots$$

(and simliarly for other homology functors). As with morphisms, we write $\pi_{**}(-)$ for $\bigoplus_{i,j} \pi_{i,j}(-)$ (and similarly for other homology and cohomology functors). Also, given a spectrum X, we write $X_{i,j}$ for $\pi_{i,j} X = X_{i,j} S$, and X_{**} for $\bigoplus_{i,j} X_{i,j}$.

We say that an object R in $\mathsf{Stable}(\Gamma)$ is a *ring spectrum* if there is a multiplication map $\mu \colon R \wedge R \to R$ and a unit map $S \to R$, making the appropriate diagrams commute.

We often abuse notation and let $S^0 = S^{0,0} = S$. One of our main goals is to get as much information as possible about $\pi_{**} S^0 = \mathrm{Ext}^{**}_\Gamma(k,k)$.

1.3. The functor H

We assume that Γ is a graded commutative Hopf algebra over a field k, and we work in the category $\mathsf{Stable}(\Gamma)$. The quotient coalgebras and Hopf algebras of Γ carry useful information; in this section we construct a spectrum HB in $\mathsf{Stable}(\Gamma)$ for each quotient coalgebra B of Γ, and we study the properties of the functor H.

Recall that if B is a quotient coalgebra of Γ and if M is a B-comodule, then the *cotensor product* $\Gamma \,\square_B M$ is defined to be the equalizer of the two maps

$$\Gamma \otimes M \xrightarrow{1_\Gamma \otimes \psi_M} \Gamma \otimes B \otimes M,$$

$$\Gamma \otimes M \xrightarrow{\psi_\Gamma \otimes 1_M} \Gamma \otimes B \otimes M.$$

Here ψ_Γ is the right B-comodule structure map on Γ, and ψ_M is the left B-comodule structure map on M. (The tensor products are over k.)

DEFINITION 1.3.1. We define a covariant functor H from quotient coalgebras of Γ to spectra by defining HB to be an injective resolution of $\Gamma \,\square_B k$.

For example, $Hk = \Gamma$, so that $Hk_{**}(X)$ is the homology of the cochain complex X; also, $H\Gamma = S^0$. H provides a useful source of (co)homology functors on $\mathsf{Stable}(\Gamma)$. The general philosophy is that if one has a quotient B of Γ, rather than studying B by working in $\mathsf{Stable}(B)$, one studies B by looking at HB in the category $\mathsf{Stable}(\Gamma)$. This is borne out by Corollary 1.3.5, as well as the other results in this section.

Fix a quotient coalgebra B of Γ. Given an Γ-comodule M, we let $M{\downarrow_B}$ denote its *restriction* to B; this is the B-comodule with structure map $M \to \Gamma \otimes M \to B \otimes M$. Recall from [**MM65**] and [**Rad77**] that if B happens to be a quotient Hopf algebra of Γ, then $\Gamma{\downarrow_B}$ is injective as a right (and a left) B-comodule.

PROPOSITION 1.3.2. *For any quotient coalgebra B of Γ over which $\Gamma{\downarrow_B}$ is injective as a right B-comodule, we have $HB_{ij} \cong \operatorname{Ext}_B^{ij}(k,k)$. Furthermore, we have the following.*

(a) *If B is a quotient Hopf algebra of Γ, then HB is a commutative ring spectrum.*
(b) *If B is a quotient coalgebra of B over which $\Gamma{\downarrow_B}$ is injective as a right comodule, then HB has many of the properties of a ring spectrum*:
    (i) *HB_{**} is a k-algebra, and for any spectrum Y, $HB_{**}Y$ is a right module over HB_{**}.*
    (ii) *There is a "unit map" $S^0 \to HB$ which induces an algebra map $\pi_{**}S^0 \to HB_{**}$.*
    (iii) *More generally, if $B \twoheadrightarrow C$ are quotient coalgebras of Γ over which Γ is injective, then the induced map $HB_{**} \to HC_{**}$ is an algebra map.*

PROOF. That $HB_{**} \cong \operatorname{Ext}_B^{**}(k,k)$ follows from Lemma 1.3.4 below.

Part (a) is clear—since Γ is commutative as an algebra, then so is B, and the commutative product on B induces one on HB. The quotient map $\Gamma \twoheadrightarrow B$ induces the unit map $S^0 = H\Gamma \to HB$, and hence a Hurewicz map $\pi_{**}(X) \to HB_{**}(X)$. This Hurewicz map is the same as the *restriction map* $\operatorname{res}_{\Gamma,B}\colon \operatorname{Ext}_\Gamma^{**}(k,k) \to \operatorname{Ext}_B^{**}(k,k)$.

For part (b), the unit map is induced from the quotient $\Gamma \twoheadrightarrow B$, as in (a). The induced map in (iii) is just the restriction map. The rest of (b) follows from Lemma 1.3.4 and Corollary 1.3.5. ∎

EXAMPLE 1.3.3. Note that if B is a quotient coalgebra of Γ, then HB need not be a ring spectrum. For example, suppose that k has characteristic p, and let $\Gamma = k[x]$ with x primitive. Then $B = k[x^p]$ is a quotient coalgebra of Γ, and HB is an injective resolution of $M = k[x]/(x^p)$. A multiplication on HB would induce one on M (by taking homology), and it is easy to see that M is not a Γ-comodule algebra: the comodule structure on M is given by $x \mapsto 1 \otimes x + x \otimes 1 \in \Gamma \otimes M$, so if this coaction were multiplicative, we would have

$$0 = x^p \longmapsto x^p \otimes 1 \neq 0.$$

LEMMA 1.3.4. *Suppose that B is a quotient coalgebra of Γ so that $\Gamma{\downarrow_B}$ is injective as a right B-comodule.*

(a) *Given a Γ-comodule M and a B-comodule N, we have an isomorphism*

$$\operatorname{Ext}_B^{**}(M{\downarrow_B}, N) \cong \operatorname{Ext}_\Gamma^{**}(M, \Gamma \,\square_B N).$$

(b) *If N is a Γ-comodule, then we have an isomorphism of Γ-comodules*

$$\Gamma \,\Box_B N \cong (\Gamma \,\Box_B k) \otimes N.$$

(c) *Hence if M and N are both Γ-comodules, then we have*

$$\mathrm{Ext}_B^{**}(M{\downarrow_B}, N{\downarrow_B}) \cong \mathrm{Ext}_\Gamma^{**}(M, (\Gamma \,\Box_B k) \otimes N).$$

PROOF. Part (a) is (the coalgebra version of) Shapiro's lemma. See [**Wei94**, Lemma 6.3.2], [**Ben91a**, Corollary 2.8.4], or any other good reference on homological algebra, for a statement in terms of modules and algebras. We leave it to the reader to dualize to our version; see [**Rav86**, A1.3.13] for a related result.

Part (b), the "shearing isomorphism," seems to be well-known, but it may be difficult to locate a proof in the published literature (see [**HSb**] and [**Boa**], for instance). Let N_{tr} denote N with the trivial coaction of Γ, so that $\Gamma \otimes N_{\mathrm{tr}}$ is the tensor product $\Gamma \overset{L}{\otimes} N$ with the left coaction (as compared with $\Gamma \otimes N = \Gamma \overset{\Delta}{\otimes} N$, the tensor product with the diagonal coaction). The desired isomorphism is induced by one between $\Gamma \otimes N_{\mathrm{tr}}$ and $\Gamma \otimes N$, because $\Gamma \,\Box_B N$ is a subcomodule of the former, and $(\Gamma \,\Box_B k) \otimes N$ a subcomodule of the latter. Hence Lemma 1.1.7 completes the proof of (b).

Part (c) follows from parts (a) and (b). □

Given a quotient coalgebra B of Γ and given objects X and Y of $\mathsf{Stable}(\Gamma)$, we let $[X, Y]^B$ denote the set of cochain homotopy classes of maps from X to Y, viewed as cochain complexes of left B-comodules.

COROLLARY 1.3.5. *Suppose that B is a quotient coalgebra of Γ so that $\Gamma{\downarrow_B}$ is injective as a right B-comodule. Given objects X and Y of $\mathsf{Stable}(\Gamma)$, we have $[X, HB \wedge Y]_{**} \cong [X, Y]^B_{**}$. In particular, HB_{**} is an algebra and $HB_{**}Y$ is a right module over it.*

COROLLARY 1.3.6. *The spectrum Hk is a field spectrum: it is a ring spectrum, and any Hk-module spectrum is a wedge of suspensions of Hk. In particular, for any X, $Hk \wedge X$ is a wedge of suspensions of Hk.*

PROOF. By Proposition 1.3.2(b), $Hk = \Gamma$ is a ring spectrum. By Lemma 1.1.7, for any Γ-comodule M, $\Gamma \otimes M$ is a direct sum of copies of Γ; one can check (by [**Boa**, 5.7], for instance) that given a comodule map $M \to N$, the induced map $\Gamma \otimes M \to \Gamma \otimes N$ sends each summand of Γ either isomorphically to a summand, or to zero. So if X is any cochain complex of Γ-comodules, then $Hk \wedge X = \Gamma \otimes X$ splits into a direct sum of cochain complexes of the forms

$$0 \to \Gamma \xrightarrow{=} \Gamma \to 0$$

and

$$0 \to \Gamma \to 0.$$

□

1.3.1. Remarks on Hopf algebra extensions. Many results about group cohomology (and indeed about Hopf algebra cohomology in general) are proved using the spectral sequence associated to an extension. If one has a group extension in which the quotient is cyclic of prime order, the associated spectral sequence is particularly tractable, and hence quite useful. In this subsection we remind the reader of standard notation related to Hopf algebra extensions, and then we focus on extensions with small kernel. In Section 1.5 we discuss the spectral sequence associated to a general Hopf algebra extension.

DEFINITION 1.3.7. (a) Suppose that $C \hookrightarrow \Gamma$ is an inclusion of augmented algebras over a field k, and let IC be the augmentation ideal of C—the kernel of $C \to k$. We say that C is *normal* in Γ if the left ideal of Γ generated by C (i.e., $\Gamma \cdot IC$) is equal to the right ideal generated by C (i.e., $IC \cdot \Gamma$). If C is normal in Γ, then we let $\Gamma//C = \Gamma \otimes_C k = k \otimes_C \Gamma$. In this case,

$$C \to \Gamma \to \Gamma//C$$

is an extension of augmented k-algebras.

(b) Dually, suppose that $\Gamma \to B$ is a surjective map of coaugmented coalgebras over k, and let JB denote the coaugmentation coideal of B—the cokernel of $k \to B$. We say that B is a *conormal* quotient of Γ if $\Gamma \,\square_B k = k \,\square_B \Gamma$. If B is a conormal quotient of Γ, then

$$\Gamma \,\square_B k \to \Gamma \to B$$

is an extension of coaugmented k-coalgebras.

As far as this definition goes, for us Γ will usually be a Hopf algebra with commutative multiplication, so that every subalgebra of Γ will be normal. We will be more interested in quotients of Γ; if $\Gamma \to B$ is a surjective map of commutative Hopf algebras over k, then $\Gamma \,\square_B k$ is the algebra kernel. So B is conormal if the algebra kernel is also a subcoalgebra of Γ.

REMARK 1.3.8. If Γ is a commutative Hopf algebra and B is a quotient coalgebra of Γ over which $\Gamma{\downarrow}_B$ is injective as a right comodule, then Lemma 1.3.4 tells us that $\mathrm{Ext}_B^{**}(k,k) \cong \mathrm{Ext}_\Gamma^{**}(k, \Gamma \,\square_B k)$. If, in addition, B is conormal, then there is a right coaction of Γ on $\Gamma \,\square_B k$, which induces a left coaction of Γ on $\mathrm{Ext}_B^{**}(k,k)$. The coaction of B is clearly trivial, so we get a left coaction of $\Gamma \,\square_B k$ on $\mathrm{Ext}_B^{**}(k,k)$; indeed, we get a left coaction of $\Gamma \,\square_B k$ on $\mathrm{Ext}_B^{**}(k,M)$ for any left B-comodule M.

For the remainder of this section, we assume that the ground field k has characteristic $p > 0$.

NOTATION 1.3.9. Given a homogeneous element x in a graded vector space, let $|x|$ denote its *degree*. Let $E[x]$ denote the Hopf algebra $k[x]/(x^2)$ with x primitive, where $|x|$ is odd if p is odd. Let $D[x] = k[x]/(x^p)$ with x primitive, where $|x|$ is even if p is odd. Recall from Lemma 1.1.15 that if B is a Hopf algebra, then $\mathrm{Ext}_B^{1,*}(k,k) = HB_{1,*}$ is isomorphic to the vector space of primitives of B. If $y \in B$ is primitive, we let $[y]$ denote the associated element of $HB_{1,*}$. Recall further that if p is odd, then there is a Steenrod operation

$$\beta\widetilde{\mathscr{P}}^0\colon\ \mathrm{Ext}_B^{s,t}(k,k) \to \mathrm{Ext}_B^{s+1,pt}(k,k).$$

(See [**May70, Wil81**], as well as Appendix B.1.)

Here is our analysis of Hopf algebra extensions with small kernel.

LEMMA 1.3.10. *Fix a graded commutative Hopf algebra Γ over a field k of characteristic $p > 0$.*

(a) *Suppose that there is a Hopf algebra extension of the form*

$$E[x] \to B \to C,$$

where B is a quotient Hopf algebra of Γ. Then x is primitive in B, so there is a nonzero element $h = [x]$ in $\mathrm{Ext}_B^{1,|x|}(k,k) = HB_{1,|x|}$. We also let $h\colon \Sigma^{1,|x|}HB \to HB$ denote the corresponding self-map of HB—i.e., the composite

$$\Sigma^{1,|x|}HB = S^{1,|x|} \wedge HB \xrightarrow{h\wedge 1} HB \wedge HB \xrightarrow{\mu} HB,$$

where μ is the multiplication map. Then there is a cofiber sequence

$$\Sigma^{1,|x|}HB \xrightarrow{h} HB \to HC \to \Sigma^{0,|x|}HB.$$

(b) *Suppose that p is odd and there is a Hopf algebra extension of the form*

$$D[x] \to B \to C,$$

where B is a quotient Hopf algebra of Γ. Then x is primitive in B; we let $b = \beta\widetilde{\mathscr{P}}^0[x]$ in $\mathrm{Ext}_B^{2,p|x|}(k,k) = HB_{2,p|x|}$. We also let $b\colon \Sigma^{2,p|x|}HB \to HB$ denote the corresponding self-map of HB. Then there is a cofiber sequence

$$\Sigma^{2,p|x|}HB \xrightarrow{b} HB \to \widetilde{HC} \to \Sigma^{1,p|x|}HB,$$

where $\widetilde{HC}$ is defined by a cofibration

$$\Sigma^{1,|x|}HC \to \widetilde{HC} \to HC \to \Sigma^{0,|x|}HC.$$

The element $b = \beta\widetilde{\mathscr{P}}^0[x]$ in (b) is also the p-fold Massey product of $[x]$ with itself, as mentioned in [**May70**, 11.11].

PROOF. Part (a): It is clear that x is primitive, so we only need to discuss the putative cofibration. There is a short exact sequence of $E[x]$-comodules

$$0 \to k \to E[x] \to \Sigma^{|x|}k \to 0.$$

We apply the (exact) functor $\Gamma \,\square_B -$ to this, noting that $E[x] = B \,\square_C k$:

$$0 \to \Gamma \,\square_B k \to \Gamma \,\square_C k \to \Sigma^{|x|}\Gamma \,\square_B k \to 0.$$

Taking injective resolutions then gives the desired cofibration.

Part (b): Note that there are two short exact sequences of $D[x]$-comodules:

$$0 \to k \to D[x] \to \Sigma^{|x|}k[x]/(x^{p-1}) \to 0,$$
$$0 \to k[x]/(x^{p-1}) \to D[x] \to \Sigma^{(p-1)|x|}k \to 0.$$

We apply the functor $\Gamma \,\square_B -$ to these and take injective resolutions; writing $\widetilde{HB}$ for the injective resolution of $\Gamma \,\square_B (k[x]/(x^{p-1}))$, we have the following cofiber sequences:

$$\Sigma^{1,|x|}\widetilde{HB} \xrightarrow{\eta} HB \to HC \to \Sigma^{0,|x|}\widetilde{HB},$$

$$\Sigma^{1,(p-1)|x|}HB \xrightarrow{\eta'} \widetilde{HB} \to HC \to \Sigma^{0,(p-1)|x|}HB.$$

It is standard (e.g., [**Ben91b**, pp. 137–8]) that the map b is the composite $\eta \circ \eta'$, and the 3×3 lemma (or the octahedral axiom—see [**HPS97**, A.1.1–A.1.2]) allows us to identify the cofiber of $\eta \circ \eta'$ in terms of the cofibers of η and η'. $\square$

1.4. Some classical homotopy theory

We assume that Γ is a graded commutative Hopf algebra over a field k, and we work in the category $\mathsf{Stable}(\Gamma)$. For this section, we assume that $\mathsf{Stable}(\Gamma)$ is monogenic (i.e., the trivial comodule k and its suspensions are the only simple comodules). We also assume that Γ is non-negatively graded: $\Gamma_n = 0$ if $n < 0$.

Because $\pi_{ij} S^0 = \operatorname{Ext}_{\Gamma}^{i,j}(k,k)$, then $\pi_{**} S^0$ is concentrated in the first quadrant. More precisely, $\pi_{ij} S^0 = 0$ if $j < 0$ and (unless $i = j = 0$) if $i \leq 0$; furthermore, $\pi_{00} S^0 = k$. Since $\pi_{**} S^0$ is "connected" in this sense, we can construct cellular towers and hence Postnikov towers, as in the usual stable homotopy category (and indeed in any *connective* stable homotopy category—see [**HPS97**, Section 7]). We also have a Hurewicz theorem. Since we are working in a bigraded setting rather than the singly graded setting of [**HPS97**, Section 7], we state the relevant results; we leave the proofs as an exercise for the reader. (See also the appropriate parts of [**Mar83**].)

Definition 1.4.1. Given a spectrum X, we say that

$$\dots \to X^{n-1} \to X^n \to X^{n+1} \to \dots$$

is a *cellular tower* for X if

(i) $\operatorname{wcolim} X^n = X$,
(ii) The fiber of $X^n \to X^{n+1}$ is a coproduct of spheres $S^{i,j}$ with $i + j = n$,
(iii) $\varprojlim Hk_{**} X^n = 0$.

Because of the connectivity properties of $\pi_{ij} S^0$, we have the following theorem, guaranteeing existence of cellular and Postnikov towers.

Theorem 1.4.2. (a) *Any spectrum X has a cellular tower.*
(b) *For any spectrum X and integer n, there is a cofiber sequence*

$$X[n,\infty] \to X \to X[-\infty, n-1],$$

so that
(i) $\pi_{ij} X[n,\infty] = 0$ *if* $i + j < n$,
(ii) $\pi_{ij} X[-\infty, n-1] = 0$ *if* $i + j \geq n$,
(c) *If for some integer n we have spectra X and Y with $X = X[n,\infty]$ and $Y = Y[-\infty, n-1]$, then $[X,Y]_{**} = 0$.*
(d) *Hence for any spectrum X, we can construct a* Postnikov tower:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & X[-\infty,r] & \longrightarrow & X[-\infty,r-1] & \longrightarrow & X[-\infty,r-2] & \longrightarrow & \dots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & X[r] & & X[r-1] & & X[r-2] & & \end{array}$$

(Here $X[r]$ is a coproduct of factors of the form $\Sigma^{i,j} Hk$, where $i + j = r$.) The sequential colimit of $X[-\infty,r]$ is 0, and the sequential limit of $X[-\infty,r]$ is X.
(e) *Dually, we can construct a diagram*

$$\begin{array}{ccccccccc} \dots & \longrightarrow & X[r,\infty] & \longrightarrow & X[r-1,\infty] & \longrightarrow & X[r-2,\infty] & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & X[r] & & X[r-1] & & X[r-2] & & \end{array}$$

The sequential colimit of the "connective covers" $X[r,\infty]$ is X, and the sequential limit is 0.

Parts (d) and (e) are easy in our context: if k is the only simple comodule, then every injective is a direct sum of copies of $\Gamma = Hk$. So to construct $X[r,\infty]$, for example, one just truncates the cochain complex X at bidegrees (s,t) with $s+t<r$.

Note that if Γ is a connected Hopf algebra (Definition 1.1.3), then $\pi_{ij}S^0=0$ if $j<i$; since the Steenrod algebra is connected, we use this pattern for our definition of connectivity. At the other extreme, if Γ is concentrated in degree zero, then $\pi_{st}S^0=0$ if $t\neq 0$; this leads to a weaker notion of connectivity.

DEFINITION 1.4.3. Given a spectrum X, if there exist numbers i_0 and j_0 so that $\pi_{ij}X=0$ when $i<i_0$ or $j-i<j_0$, then we say that X is (i_0,j_0)-*connective*. We say that X is *connective* if X is (i_0,j_0)-connective for some unspecified i_0 and j_0. If for some i_0 and j_0, we have $\pi_{ij}X=0$ when $i<i_0$ or $j<j_0$, we say that X is *weakly* (i_0,j_0)-*connective*.

Here is the second main theorem of this section, a bigraded version of the Hurewicz theorem.

THEOREM 1.4.4. *If X is (i_0,j_0)-connective, then the Hurewicz map $\pi_{ij}X\to Hk_{ij}X$ is an isomorphism when $i<i_0$ or $j-i\leq j_0$. Similarly, if X is weakly (i_0,j_0)-connective, then $\pi_{ij}X\to Hk_{ij}X$ is an isomorphism when $i<i_0$ or $j\leq j_0$.*

REMARK 1.4.5. We point out that one can use the bigrading to generalize these results somewhat; for example, one can discuss cellular towers with "slope $\frac{u}{v}$"—towers as above, but with the fiber of $X^n\to X^{n+1}$ equal to a coproduct of spheres $S^{i,j}$ with $vi+uj=n$. Using the form of connectivity of the sphere spectrum, one can construct cellular towers of any given nonpositive slope. Similar remarks hold for Postnikov towers.

Similarly, one can define "connectivity with slope $\frac{u}{v}$" for nonpositive $\frac{u}{v}$—i.e., $\pi_{**}X=0$ below a line of slope $\frac{u}{v}$—and then show that the Hurewicz map is an isomorphism at and below the line of connectivity.

Furthermore, if Γ happens to be connected, then $\pi_{ij}S^0=0$ when $j<i$ and when $i<0$. In this case, one can construct cellular towers and prove Hurewicz theorems for any slope $m\leq 0$ or $m>1$.

One could also work with connectivity determined by arbitrary non-increasing curves rather than lines of nonpositive slope and get the same sorts of results. We have no need to work with anything approaching this level of generality.

If Γ is connected and X is an injective resolution of a bounded below comodule M, then X is $(0,0)$-connective, but it also satisfies a stronger property. This property occurs several times in this work, so we make it into a definition.

DEFINITION 1.4.6. We say that a spectrum X is *comodule-like*, or *CL*, if X satisfies the following conditions:

(i) There exists an integer i_0 such that $\pi_{i*}X=0$ if $i<i_0$,
(ii) There exists an integer j_0 such that $\pi_{ij}X=0$ if $j-i<j_0$,
(iii) There exists an integer i_1 so that $(Hk)_{i*}X=0$ for $i>i_1$.

(When X is an injective resolution of M, we may take $i_0=0=i_1$, and j_0 to be the degree of the bottom class of M.)

Note that X is a CL-spectrum if and only if X has a cellular tower built of spheres $S^{i,j}$ with $i_0 \leq i \leq i_1$ and $j - i \geq j_0$.

We will need the following lemmas later. First we need to recall a few definitions from [**HPS97**, 1.4.3, 2.1.1].

Definition 1.4.7. (a) A full subcategory D of $\mathsf{Stable}(\Gamma)$ is *localizing* if it is "closed under cofibrations and coproducts": if $X \to Y \to Z$ is a cofibration and two of X, Y, and Z are in D, then so is the third; if $\{X_\alpha\}$ is a set of objects in D, then $\bigoplus X_\alpha$ is in D. Given an object Y, we let $\mathrm{loc}(Y)$ denote the localizing subcategory generated by Y, i.e., the intersection of all of the localizing subcategories containing Y.

(b) Similarly, a full subcategory D of $\mathsf{Stable}(\Gamma)$ is *thick* if it is closed under cofibrations and retracts (if Y is in D and there are maps $X \to Y \to X$ so that the composite is an isomorphism, then X is in D); and $\mathrm{thick}(Y)$ denotes the thick subcategory generated by Y.

(c) A property P of spectra is *generic* if the full subcategory of spectra satisfying P is thick.

(d) An object X of $\mathsf{Stable}(\Gamma)$ is *finite* if and only if it is small (in the categorical sense), if and only if it is in $\mathrm{thick}(S^0)$.

If Γ is connected, then an object X is finite if and only if X is connective and has $\dim_k Hk_{**}X < \infty$. So if Γ is connected and B is a quotient Hopf algebra of Γ, then an object X of $\mathsf{Stable}(\Gamma)$ is finite if and only if its restriction $X{\downarrow_B}$ is finite in $\mathsf{Stable}(B)$.

Lemma 1.4.8. *Suppose that X is a spectrum and that there is a line of nonpositive slope above which the homotopy of X is zero (i.e., for some u, v with $\frac{u}{v} \leq 0$, there is an n so that if $vi + ij \geq n$, then $\pi_{ij}X = 0$). Then X is in the localizing subcategory generated by $\Gamma = Hk$.*

Proof. The Postnikov tower of slope $\frac{u}{v}$ for such an X displays X as being a colimit of objects of $\mathrm{loc}(\Gamma)$; hence X is itself an object of $\mathrm{loc}(\Gamma)$. □

Lemma 1.4.9. *If D is a localizing subcategory of $\mathsf{Stable}(\Gamma)$ which contains a nonzero finite spectrum, then $\mathrm{loc}(\Gamma) \subseteq \mathsf{D}$.*

Proof. It suffices to show that $\Gamma \in \mathrm{ob}\,\mathsf{D}$ if D is as given. Let Y be a nonzero finite object of D; then $Y \wedge \Gamma$ is nonzero (by the Hurewicz theorem 1.4.4—remember that $\Gamma = Hk$) and is contained in $\mathrm{ob}\,\mathsf{D}$. On the other hand, Corollary 1.3.6 tells us that $Y \wedge \Gamma$ is a direct sum of suspensions of Γ, so by the Eilenberg swindle [**HPS97**, 1.4.9], $\Gamma \in \mathrm{ob}\,\mathsf{D}$. □

We will see in Corollary 4.5.7 that as long as Γ is infinite-dimensional, the containment $\mathrm{loc}(\Gamma) \subset \mathsf{D}$ is strict (i.e., $\mathrm{loc}(\Gamma)$ contains no nonzero finite spectrum).

1.5. The Adams spectral sequence

As in the rest of this chapter, we assume that Γ is a graded commutative Hopf algebra over a field k. We discuss the generalized Adams spectral sequence associated to the homology theory HB_{**} in this section, for B a conormal quotient of Γ (see Definitions 1.3.1 and 1.3.7). In particular, we note that it is the same as the spectral sequence associated to a Hopf algebra extensions, and we derive a few consequences.

For the general approach to the Adams spectral sequence, see [**Ada74**]. We also recall a construction of the spectral sequence in Appendix A.2.

THEOREM 1.5.1. *Suppose that B is a conormal quotient Hopf algebra of Γ and fix a spectrum X. Then the Adams spectral sequence based on the homology theory HB_{**} has E_2-term*

$$E_2^{s,t,u} = \mathrm{Ext}^{s,t,u}_{HB_{**}HB}(HB_{**}, HB_{**}X),$$

with differentials

$$d_r \colon E_r^{s,t,u} \to E_r^{s+r,t+r-1,u-r+1},$$

and abuts to $\pi_{s+u,t+u}X$.

Given a Γ-comodule M, the change-of-rings spectral sequence associated to the extension

$$\Gamma \,\square_B k \to \Gamma \to B$$

has E_2-term

$$'E_2^{p,q,v} = \mathrm{Ext}^{p,v}_{\Gamma \,\square_B k}(k, \mathrm{Ext}^{q,*}_B(k, M)) \tag{1.5.2}$$

and converges to

$$\mathrm{Ext}^{p+q,v}_{\Gamma}(k, M).$$

(The action of $\Gamma \,\square_B k$ on $\mathrm{Ext}^{**}_B(k, M)$ was discussed in Remark 1.3.8.) The differentials are indexed as follows:

$$d_r \colon {}'E_r^{p,q,v} \to {}'E_r^{p+r,q-r+1,v}.$$

(See [**Sin73**, II, §5], for example; alternatively, one can dualize the construction of the Lyndon-Hochschild-Serre spectral sequence for the computation of group cohomology. See [**Ben91b**, 3.5] for a construction of this as the spectral sequence associated to a double complex, for instance.)

PROPOSITION 1.5.3. *Suppose that B is a conormal quotient of Γ, and suppose that X is an injective resolution of a Γ-comodule M. Then the HB-based Adams spectral sequence abutting to $\pi_{**}X$ is isomorphic, up to a regrading, to the change-of-rings spectral sequence associated to the extension*

$$\Gamma \,\square_B k \to \Gamma \to B,$$

abutting to $\mathrm{Ext}^{**}_{\Gamma}(k, M)$. The regrading is as follows: for all $r \geq 2$, the $E_r^{s,t,u}$-term of the Adams spectral sequence is isomorphic to the ${}'E_r^{s,u,t+u}$-term of the change-of-rings spectral sequence.

An example of the regrading is the isomorphism of E_2-terms

$$\mathrm{Ext}^{p,v}_{\Gamma \,\square_B k}(k, \mathrm{Ext}^{q,*}_B(k, M)) \cong \mathrm{Ext}^{p,v-q,q}_{HB_{**}HB}(HB_{**}, HB_{**}X).$$

PROOF. This is an exercise in homological algebra. The key observation is that since B is conormal, then $\Gamma \,\square_B k$ is a trivial B-comodule. Hence

$$HB_{**}HB = \mathrm{Ext}^{**}_B(k, \Gamma \,\square_B k) \cong (\Gamma \,\square_B k) \otimes \mathrm{Ext}^{**}_B(k, k) = \Gamma \,\square_B k \otimes HB_{**}.$$

If X is an injective resolution of a Γ-comodule M, then combining X with an Adams tower for X yields a double complex; using the above isomorphism, one can easily show that the resulting spectral sequence is the change-of-rings spectral sequence.

The shift in gradings comes from a more precise statement of the above isomorphism:

$$\operatorname{Ext}_B^{\ell,m}(k, \Gamma \,\square_B k) \cong \bigoplus_{i+j=m} (\Gamma \,\square_B k)_i \otimes \operatorname{Ext}_B^{\ell,j}(k,k).$$

$\square$

Hence the *HB*-based Adams spectral sequence has some nice properties. For example, we have a convergence result: if X and M are as in the proposition, then the Adams spectral sequence converges to $\pi_{**}X = \operatorname{Ext}_\Gamma^{**}(k, M)$. (We also show in Proposition A.2.5 that the spectral sequence converges to $\pi_{**}X$ whenever Γ is a connected Hopf algebra and X is connective—i.e., every connective spectrum is "*HB*-complete".)

Also, if $X = S^0$ (or more generally if X is an injective resolution of a commutative Γ-comodule algebra), then one has Steenrod operations acting on this spectral sequence, as described in [**Sin73**] (see also [**Saw82**]). We need the following result in Chapter 3.

PROPOSITION 1.5.4. *Suppose that Γ is a Hopf algebra over the field $\mathbf{F}_p$, and suppose that B is a conormal quotient of Γ. Consider the HB-based Adams spectral sequence converging to $\pi_{**}S^0$. Given $y \in E_2^{0,t,u}$, with t and u even if p is odd, then for each n, y^{p^n} survives to $E_{p^n+1}^{0,p^n t,p^n u}$.*

(Note that the result is the same whether one is using the Adams grading (Theorem 1.5.1) or the change-of-rings grading (equation (1.5.2)): the elements in $E_2^{0,q,v}$ in the change-of-rings grading correspond to elements in $E_2^{0,v-q,q} = E_2^{0,t,u}$ in the Adams grading; those with q and v even correspond to those with t and u even.)

PROOF. This follows from properties of Steenrod operations on this spectral sequence, as discussed in [**Sin73**] and [**Saw82**]. Suppose we have a spectral sequence $E_r^{s,t}$ which is a spectral sequence of algebras over the Steenrod algebra. Fix $z \in E_r^{s,t}$, and fix an integer k. If $p = 2$, then [**Sin73**, 1.4] tells us to which term of the spectral sequence $\operatorname{Sq}^k z$ survives (the result depends on r, s, t, and k). For instance, if $z \in E_r^{0,t}$, then $\operatorname{Sq}^t(z) = z^2$ survives to $E_{2r-1}^{0,2t}$. [**Saw82**, 2.5] is the corresponding result at odd primes. $\square$

By the way, Singer's results [**Sin73**] are stated in the case of an extension of commutative Hopf algebras

$$B \to \Gamma \to C$$

where C is also cocommutative (actually, he works in the dual situation). This cocommutativity condition is not, in fact, necessary, as forthcoming work of Singer shows [**Sin**].

We also need the theorem of Hopkins and Smith [**HSa**] that the presence of a vanishing plane in the Adams spectral sequence is a generic property. Suppose that E is a spectrum satisfying the following conditions (cf. [**Ada74**, III.15] and [**Rav86**, 2.2.5]):

(a) E is a commutative associative ring spectrum.
(b) $E_{**}E$ is flat over E_{**}.

(c) E is $(0,0)$-connective (Definition 1.4.3), and the unit map $\eta\colon S^0 \to E$ induces an isomorphism on $\pi_{0,0}$.
(d) $Hk_{ij}E$ is a finite-dimensional k-vector space for each i and j.

THEOREM 1.5.5 (Hopkins-Smith). *Suppose that Γ is a connected Hopf algebra. Consider the Adams spectral sequence $E_r^{s,t,u}$ based on a ring spectrum E satisfying the above conditions. Fix numbers $m \leq 0$ and n. The full subcategory of $\mathsf{Stable}(\Gamma)$ consisting of all spectra X that satisfy the property "there exist numbers r and b so that $E_r^{s,t,u}(X) = 0$ when $s \geq m(s+u) + n(t+u) + b$" is thick.*

Since we are not aware of any published proof of this, we give a proof in Appendix A.2.

1.6. Bousfield classes and Brown-Comenetz duality

Again, we suppose that Γ is a graded commutative Hopf algebra over a field k. We assume that $\mathsf{Stable}(\Gamma)$ is monogenic. In this section we collect some useful results about Bousfield classes, Brown-Comenetz duality, and their interaction.

We remind the reader that the *Bousfield class* $\langle X \rangle$ of an object X in an arbitrary stable homotopy category is the collection of X*-acyclic* objects—all objects Z with $X \wedge Z = 0$. We order Bousfield classes by reverse inclusion, so $\langle X \rangle \geq \langle Y \rangle$ means that $X \wedge Z = 0 \Rightarrow Y \wedge Z = 0$. X and Y are *Bousfield equivalent* if $\langle X \rangle = \langle Y \rangle$. We define the operation $\vee$ on Bousfield classes by $\langle X \rangle \vee \langle Y \rangle = \langle X \vee Y \rangle$; one can show that this is the least-upper bound of $\langle X \rangle$ and $\langle Y \rangle$. A theorem of Ohkawa [**Ohk89**] says that there is a set of Bousfield classes; hence there is also a greatest-lower bound operation (take the least-upper bound of all the classes less than or equal to both $\langle X \rangle$ and $\langle Y \rangle$). For certain well-behaved classes of spectra, the greatest-lower bound is given by $\wedge$, but that is not true in general.

These concepts were introduced (in the ordinary stable homotopy category) by Bousfield in [**Bou79a**] and [**Bou79b**]; Ravenel proved a number of fundamental results about them in [**Rav84**]. See [**HPS97**, Section 3.6] for a discussion of Bousfield classes in a general stable homotopy category.

For any spectrum X, we define its *Brown-Comenetz dual* IX to be the spectrum that represents the cohomology functor

$$Y \longmapsto \mathrm{Hom}_k^{**}(\pi_{**}(X \wedge Y), k).$$

Hence $\pi_{**}IX = \mathrm{Hom}_k^{**}(\pi_{**}X, k)$. (In a general stable homotopy category, one should take Hom over $R = \pi_0(S^0)$ into the injective hull of R, essentially. See [**BC76**] for the analogue in the ordinary stable homotopy category.)

We say that the homotopy of a spectrum X is of *finite type* if $\pi_{ij}X$ is finite-dimensional over k for each bidegree (i,j).

PROPOSITION 1.6.1. [**Rav84**] *Let X and Y be spectra.*
(a) *Given a map $f\colon X \to X$ with cofiber X/f and telescope $f^{-1}X$, we have $\langle X/f \rangle \vee \langle f^{-1}X \rangle = \langle X \rangle$ and $\langle X/f \rangle \wedge \langle f^{-1}X \rangle = 0$.*
(b) *If X is weakly connective* (*Definition* 1.4.3)*, then IX is in the localizing subcategory generated by $Hk = \Gamma$.*
(c) *Suppose that X has finite type homotopy. If $[Y,X]_{**} = 0$, then $\pi_{**}(Y \wedge IX) = 0$.*
(d) *If X is a ring spectrum and Y is an X-module spectrum, then we have $\langle Y \rangle = \langle X \wedge Y \rangle \leq \langle X \rangle$.*

(e) *If X is a ring spectrum, then IX is an X-module spectrum*; *hence, $\langle IX \rangle \leq \langle X \rangle$.*
(f) *Suppose that X is a noncontractible ring spectrum with finite type homotopy, and Y is an X-module spectrum. If $[Y, X]_{**} = 0$, then $\langle Y \rangle < \langle X \rangle$.*

See [**Rav84**] for the proofs. (See also Lemma 1.4.8 for (b).)
Combining part (a) of this with Lemma 1.3.10 gives us a corollary.

Corollary 1.6.2. (a) *Suppose that there is a Hopf algebra extension of the form*

$$E[x] \to B \to C,$$

giving a cofiber sequence

$$\Sigma^{1,|x|} HB \xrightarrow{h} HB \to HC \to \Sigma^{0,|x|} HB.$$

Then $\langle HB \rangle = \langle HC \rangle \vee \langle h^{-1} HB \rangle$, and $HC \wedge h^{-1} HB = 0$.

(b) *Suppose that there is a Hopf algebra extension of the form*

$$D[x] \to B \to C,$$

with $|x| > 0$, giving cofiber sequences

$$\Sigma^{2,p|x|} HB \xrightarrow{b} HB \to \widetilde{HC} \to \Sigma^{1,p|x|} HB,$$

$$\Sigma^{1,0} HC \to \Sigma^{1,|x|} HC \to \widetilde{HC} \to HC.$$

Then $\langle HB \rangle = \langle HC \rangle \vee \langle b^{-1} HB \rangle$, and $HC \wedge b^{-1} HB = 0$.

Proof. Part (a) is immediate; part (b) follows once we show that $\langle HC \rangle = \langle \widetilde{HC} \rangle$. By the second cofibration above, it suffices to show that $f^{-1} HC$ is contractible. For degree reasons, though, $\pi_{**} f^{-1} HC = 0$. $\square$

1.7. Further discussion

In this section, we note that we can apply the technology of this chapter to group algebras, and we give one or two examples.

Let k be a field of characteristic $p > 0$, and let G be a finite group. Then kG is a finite-dimensional cocommutative Hopf algebra, so we can use stable homotopy theory to study $\mathsf{Stable}(kG^*)$. By Remark 1.2.1, we could just as well work with the category of cochain complexes of injective kG-modules; in either case, morphisms in the category relate to group cohomology, as mentioned in Remark 1.2.2. Since kG is concentrated in degree 0, then morphisms are only singly graded.

As a result, most of the results in this chapter apply to $\mathsf{Stable}(kG^*)$. If G is p-group, then the trivial module k is the only simple; so all of the results of the chapter apply to $\mathsf{Stable}(kG^*)$ for G a p-group. To illustrate, Lemma 1.3.10 translates to the following, when $p = 2$.

Corollary 1.7.1. *Let k be a field of characteristic 2, and let G be a finite group. Any group extension*

$$1 \to B \to G \to \mathbf{Z}/2 \to 1$$

gives rise to a cofibration in $\mathsf{Stable}(kG^*)$:

$$\Sigma^1 HG \xrightarrow{z} HG \to HB,$$

where $z \in H^1(G; k)$ is the inflation of the polynomial generator in $H^1(\mathbf{Z}/2; k)$.

As mentioned in Subsection 1.3.1, when one has a group extension like

$$1 \to B \to G \to \mathbf{Z}/2 \to 1,$$

one can often replace arguments involving the associated spectral sequence with simpler arguments involving the cofibration in the corollary. It is an amusing exercise to prove Chouinard's theorem [**Cho76**] this way, for instance.

We should also discuss the Bousfield lattice in $\mathsf{Stable}(kG^*)$. In any stable homotopy category, the partially ordered set of Bousfield classes, together with the meet and join operations, contains important structural information about the category. It turns out that when G is a finite p-group, there is a complete description of the Bousfield lattice, essentially due to Benson, Carlson, and Rickard [**BCR96, BCR**].

Given a p-group G and an algebraically closed field k of characteristic p, they define $V_G(k)$ to be the maximal ideal spectrum of the (graded) commutative Noetherian ring $H^*(G;k)$. Then for each closed homogeneous irreducible subvariety V of $V_G(k)$, they define a kG-module $\kappa(V)$ satisfying various properties, as outlined below. (Throughout, they work in the category $\mathsf{StMod}(kG)$, so that $\kappa(V)$ is only well-defined up to projective summands. One could just as easily work in $\mathsf{Stable}(kG^*)$, in which case $\kappa(V)$ is a cochain complex, well-defined up to chain-homotopy equivalence. Naturally, we choose the latter course.)

THEOREM 1.7.2 ([**BCR96**]). *The objects $\kappa(V)$ satisfy the following:*

(a) *There is a Bousfield class decomposition of $\langle S^0 \rangle$:*

$$\langle S^0 \rangle = \bigvee_{V \subseteq V_G(k)} \langle \kappa(V) \rangle$$

(where the wedge is taken over all closed homogeneous irreducible V).

(b) *If $V \neq W$, then $\kappa(V) \wedge \kappa(W) = 0$.*

(c) *For any $V \subseteq V_G(k)$ and objects X and Y, if $\kappa(V) \wedge X \wedge Y = 0$, then either $\kappa(V) \wedge X = 0$ or $\kappa(V) \wedge Y = 0$.*

By [**HPS97**, 5.2.3], this implies the following.

COROLLARY 1.7.3 ([**BCR**]). *When G is a p-group, the thick subcategories of finitely generated kG-modules are in one-to-one correspondence with collections of closed homogeneous subvarieties of $V_G(k)$ which are closed under specialization.*

Their result also has the following corollary.

COROLLARY 1.7.4. *Each $\langle \kappa(V) \rangle$ is a minimal nonzero Bousfield class. Hence the Bousfield lattice is a Boolean algebra on the classes $\langle \kappa(V) \rangle$. Indeed, for every X,*

$$\langle X \rangle = \bigvee_{V \,:\, \kappa(V) \wedge X \neq 0} \langle \kappa(V) \rangle.$$

PROOF. To show that $\langle \kappa(V) \rangle$ is minimal, we let $E = \bigvee_{W \neq V} \kappa(W)$. Then $\langle S^0 \rangle = \langle \kappa(V) \rangle \vee \langle E \rangle$, and $\kappa(V) \wedge E = 0$—in other words, $\kappa(V)$ is *complemented.* If X is any object with $\langle X \rangle < \langle \kappa(V) \rangle$, then there is a spectrum Z with $X \wedge Z = 0$ but $\kappa(V) \wedge Z \neq 0$. Hence $\kappa(V) \wedge X \wedge Z = 0$; by Theorem 1.7.2(c), this implies that $\kappa(V) \wedge X = 0$. Since $\langle S^0 \rangle = \langle \kappa(V) \rangle \vee \langle E \rangle$, then we have $\langle X \rangle = \langle X \wedge E \rangle$. On the other hand, we have $\kappa(V) \wedge E = 0$ and $\langle \kappa(V) \rangle > \langle X \rangle$, so $X \wedge E = 0$. Hence $\langle X \rangle = \langle X \wedge E \rangle = \langle 0 \rangle$, so $X = 0$.

Hence for any spectrum Y, if $\kappa(V) \wedge Y \neq 0$, then $\langle \kappa(V) \wedge Y \rangle = \langle \kappa(V) \rangle$. Combined with Theorem 1.7.2(a), this gives the desired description of the Bousfield lattice. $\square$

These results describe much of the global structure of the category $\mathsf{Stable}(kG^*)$ (and also of $\mathsf{StMod}(kG)$), but they leave open the question of classifying all localizing subcategories (Definition 1.4.7). We conjecture that they are in one-to-one correspondence with arbitrary collections of closed homogeneous subvarieties of $V_G(k)$. This sort of general structure is discussed in [**HPS97**, Chapter 6].

One can construct the objects $\kappa(V)$ in $\mathsf{Stable}(\Gamma)$ for any Γ whose cohomology ring is Noetherian—see [**HPS97**, 6.0.8, 6.1.4] (this includes all finite-dimensional commutative Hopf algebras, by work of Friedlander and Suslin [**FS97**]). The analogues of parts (a) and (b) of Theorem 1.7.2 hold, but it is not clear whether part (c) does. It is natural to conjecture that a similar description of the Bousfield lattice is valid, as long as the Hopf algebra Γ is suitably well-behaved. For instance, if every quasi-elementary quotient of Γ is elementary (these terms are defined in Section 2.1), then one might expect this. In such a situation, one could try to imitate the work in [**BCR96**]; if the quasi-elementaries do not coincide with the elementaries, one still might be able to imitate Benson et al., if one had a good enough understanding of the quasi-elementary quotients of Γ.

(In particular, if Γ is a finite-dimensional quotient of the dual of the mod 2 Steenrod algebra, its quasi-elementary quotients are all elementary; if Γ is a finite-dimensional quotient of the dual of the odd primary Steenrod algebra, the quasi-elementary quotients are more complicated. In both cases, though, we would conjecture that the analogues of 1.7.2–1.7.4 hold.)

CHAPTER 2

Basic properties of the Steenrod algebra

The results and constructions in Chapter 1 hold as long as Γ is a graded commutative Hopf algebra over a field (occasionally with the assumption that $\mathsf{Stable}(\Gamma)$ is monogenic or that Γ is connected). Now we start to make use of particular properties of the Steenrod algebra.

We define the dual A of the Steenrod algebra, and we give a classification of the quotient Hopf algebras of A in Section 2.1. We also define several important families of quotient Hopf algebras of A: the $A(n)$'s, the elementary quotients, and the quasi-elementary quotients. We classify the latter two families, at least at the prime 2, and for each (quasi-)elementary E, we compute the homotopy groups of the ring spectrum HE. In Section 2.2 we introduce P^s_t-homology, a well-known tool for studying Ext over the Steenrod algebra. In our setting, P^s_t-homology is a homology theory, and hence is represented by an object P^s_t in $\mathsf{Stable}(A)$; we compute $\pi_{**}P^s_t$, and we perform a few other computations.

Starting in Section 2.3, we begin to get to the main results. We discuss a vanishing line theorem in Section 2.3: given conditions on the P^s_t-homology groups of an object X, then $\pi_{ij}X = 0$ when $mi > j + c$ for some numbers m and c. (This is an extension to the cochain complex setting of theorems of Anderson-Davis [**AD73**] and Miller-Wilkerson [**MW81**].) In Section 2.4 we use the vanishing line theorem to construct "self-maps of finite objects" in $\mathsf{Stable}(A)$. For example, if M is an A-comodule and $\dim_{\mathbf{F}_p} M$ is finite, and if X is an injective resolution of M, then we construct a cochain map $\Sigma^n X \to X$ which is non-nilpotent under composition. We also establish that these self-maps have certain nice properties. (This is an extension to the cochain complex setting of a result of the author [**Pal92**].)

In Section 2.5 we mention a few topological applications of the vanishing line and self-map results, and one or two other issues.

2.1. Quotient Hopf algebras of A

In this section we define the dual A of the mod p Steenrod algebra, we give a classification of quotient Hopf algebras of A, and we discuss two important families of these quotient Hopf algebras: namely, the $A(n)$'s and the elementary quotients. Margolis' book [**Mar83**] is a good reference for all of these topics. In a subsection, we also discuss the quasi-elementary quotients of A.

Fix a prime number p and let A be the dual of the mod p *Steenrod algebra*. Recall from [**Mil58**] Milnor's description of A: as an algebra, we have

$$A = \begin{cases} \mathbf{F}_2[\xi_1, \xi_2, \xi_3, \dots], & \text{if } p = 2, \\ \mathbf{F}_p[\xi_1, \xi_2, \xi_3, \dots] \otimes E[\tau_0, \tau_1, \tau_2, \dots], & \text{if } p \text{ is odd,} \end{cases}$$

where $|\xi_n| = 2^n - 1$ if $p = 2$, $|\xi_n| = 2(p^n - 1)$ and $|\tau_n| = 2p^n - 1$ if p odd. The coalgebra map $\Delta\colon A \to A \otimes A$ is determined by

$$\Delta\colon \xi_n \longmapsto \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \xi_i,$$

$$\Delta\colon \tau_n \longmapsto \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \tau_i + \tau_n \otimes 1.$$

(In both of these formulas, we take ξ_0 to be 1.)

One can check that A is a connected commutative Hopf algebra over the field $\mathbf{F}_p$ (or see [**Mil58**]); as a result, the work in the previous chapter applies to the study of $\mathsf{Stable}(A)$.

The quotient Hopf algebras of A have been classified by Anderson-Davis ($p = 2$) and Adams-Margolis (p odd). We state the classification theorem here; see [**Mar83**] or the original papers for the proof. (Recall that we defined conormal quotient Hopf algebras in Definition 1.3.7.)

THEOREM 2.1.1. [**AD73, AM74**]

(a) *Every quotient Hopf algebra B of A is of the form*

$$B = \begin{cases} A/(\xi_1^{2^{n_1}}, \xi_2^{2^{n_2}}, \dots), & p = 2, \\ A/(\xi_1^{p^{n_1}}, \xi_2^{p^{n_2}}, \dots; \tau_0^{e_0}, \tau_1^{e_1}, \dots), & p \text{ odd}, \end{cases}$$

for some exponents $n_1, n_2, \dots \in \{0, 1, 2, \dots\} \cup \{\infty\}$ and $e_0, e_1, \dots \in \{1, 2\}$. These exponents satisfy the following conditions:

  (i) *For all primes p: for each i and r with $0 < i < r$, either $n_r \geq n_{r-i} - i$ or $n_r \geq n_i$.*

  (ii) *For p odd: if $e_r = 1$, then for each i and j with $1 \leq i \leq r$ and $i + j = r$, either $n_i < j$ or $e_j = 1$.*

(b) *Conversely, any set of exponents $\{n_i\}$ and $\{e_i\}$ satisfying conditions* (i)-(ii) *above determines a quotient Hopf algebra of A.*

(c) *Let B be a quotient Hopf algebra of A as in* (a). *Then B is a conormal quotient Hopf algebra of A if and only if, for $p = 2$,*

  (i) $n_1 \leq n_2 \leq n_3 \leq \dots,$

and for p odd,

  (i) $n_1 \leq n_2 \leq n_3 \leq \dots,$

  (ii) $e_0 \leq e_1 \leq e_2 \leq \dots,$

  (iii) $e_k = 1 \Rightarrow n_k = 0.$

For part (a), if some $n_i = \infty$, that means that one does not divide out by any power of ξ_i. Similarly, if some $e_i = 2$, then one does not mod out by τ_i.

Part (a) says that there is a (monomorphic) function from the set of quotient Hopf algebras of A to the set of sequences either of the form $(n_1, n_2, \dots)$, or of the form $(n_1, n_2, \dots; e_0, e_1, \dots)$; part (b) gives the image of this function. For $p = 2$, given a Hopf algebra B, one can view the sequence of exponents $n_1, n_2, \dots$ as a function

$$\begin{aligned} \{1, 2, \dots\} &\to \{0, 1, 2, \dots\} \cup \{\infty\}, \\ i &\longmapsto n_i. \end{aligned}$$

We refer to this as the *profile function* of B. There is, of course, a similar function when p is odd. We will occasionally give graphical representations of quotient Hopf

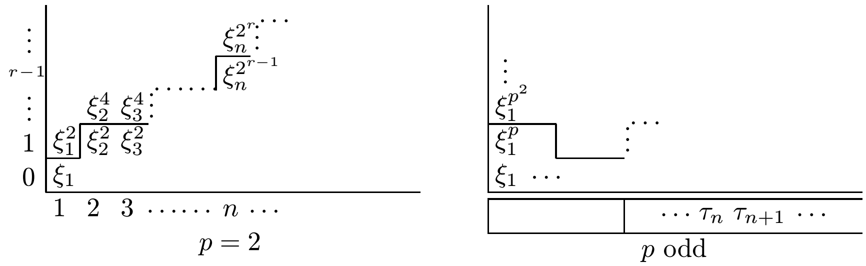


FIGURE 2.1.A. Graphical representation of a quotient Hopf algebra of A. For $p = 2$, this is a bar chart; the nth column is height $r - 1$ if one is dividing out by $\xi_n^{2^r}$. For p odd, this is a similar bar chart, together with an extra row at the bottom; in this row, one marks which τ_n's are nonzero in the quotient.

algebras via their profile functions, as in [**Mar83**, p. 234–5]. See Figure 2.1.A, for example.

Here is a simple, but useful, result. See Notation 1.3.9 for the definition of $D[x]$ and $E[x]$.

LEMMA 2.1.2. (a) *Suppose that B is a quotient Hopf algebra of A, and that for some s and t, we have*

- $\xi_t^{p^s} \neq 0$ *in B,*
- $\xi_t^{p^{s+1}} = 0$ *in B,*
- $\xi_j^{p^s} = 0$ *in B for all $j < t$.*

Then there is a Hopf algebra extension

$$D[\xi_t^{p^s}] \to B \to C.$$

(b) *Fix p odd. Suppose that B is a quotient Hopf algebra of A, and that for some n we have*

- $\tau_n \neq 0$ *in B,*
- $\tau_j = 0$ *in B for all $j < n$,*
- $\xi_j = 0$ *in B for all $j \leq n$.*

Then there is a Hopf algebra extension

$$E[\tau_n] \to B \to C.$$

PROOF. For part (a), one only has to check that given the conditions on B, then $\xi_t^{p^s}$ is primitive in B (and similarly for part (b)). This check is straightforward. $\square$

REMARK 2.1.3. (a) Hence the results of Lemma 1.3.10 apply. The usual notation is:

$$h_{ts} = [\xi_t^{p^s}] \in HB_{1,|\xi_t^{p^s}|} = \mathrm{Ext}_B^{1,|\xi_t^{p^s}|}(\mathbf{F}_p, \mathbf{F}_p),$$

$$b_{ts} = \beta\widetilde{\mathscr{P}}^0(h_{ts}) \in HB_{2,|p\xi_t^{p^s}|} = \mathrm{Ext}_B^{2,p|\xi_t^{p^s}|}(\mathbf{F}_p, \mathbf{F}_p),$$

$$v_n = [\tau_n] \in HB_{1,|\tau_n|} = \mathrm{Ext}_B^{1,|\tau_n|}(\mathbf{F}_p, \mathbf{F}_p).$$

(b) Also, note that if B is a finite-dimensional quotient of A, then one can always find an integer n or a pair (s,t) so that the hypotheses of Lemma 2.1.2 hold.

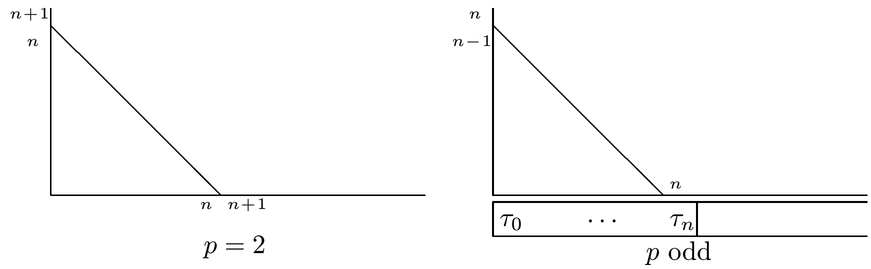


FIGURE 2.1.B. Profile function for $A(n)$. The actual profile functions have a staircase shape, which we abbreviate as lines with slope -1.

At the prime 2, for instance, one can take s and t to be as follows:

$$t = \min\{n \mid \xi_n \neq 0 \text{ in } B\},$$
$$s = \max\{i \mid \xi_t^{2^i} \neq 0 \text{ in } B\}.$$

This provides an inductive procedure for studying finite quotients of A.

We need to use several different families of quotient Hopf algebras of A; here is the first family. These quotients are quite well-known; see [**Mar83**, p. 235], for example.

EXAMPLE 2.1.4. We define $A(n)$ as follows; see also Figure 2.1.B:

$$A(n) = \begin{cases} A/(\xi_1^{2^{n+1}}, \xi_2^{2^n}, \dots, \xi_{n+1}^2, \xi_{n+2}, \xi_{n+3}, \dots), & p = 2, \\ A/(\xi_1^{p^n}, \xi_2^{p^{n-1}}, \dots, \xi_n^p, \xi_{n+1}, \xi_{n+2}, \dots; \tau_{n+1}, \tau_{n+2}, \dots), & p \text{ odd}. \end{cases}$$

Then $A(n)$ is a quotient Hopf algebra of A, and the map $A \to A(n)$ is an isomorphism below degree

$$\begin{aligned} |\xi_1^{2^{n+1}}| &= 2^{n+1}, & p &= 2, \\ |\xi_1^{p^n}| &= 2(p-1)p^n, & p &\text{ odd}. \end{aligned}$$

One important property of the $A(n)$'s is that the dual A^* of A is the union of the duals of the $A(n)$'s, and this gives A^* the structure of a "P-algebra" (see [**Mar83**, Chapter 13] for the precise definition). In our setting, this translates into the following (cf. [**Mar83**, Proposition 13.4]).

PROPOSITION 2.1.5. *Suppose that Y is a spectrum so that for each i, $\pi_{ij}Y = 0$ when $j \ll 0$. Then Y is the sequential limit of*

$$\dots \to HA(3) \wedge Y \to HA(2) \wedge Y \to HA(1) \wedge Y \to HA(0) \wedge Y.$$

Hence for any spectrum X, there is a Milnor exact sequence

$$0 \to \varprojlim{}^1 [X, HA(n) \wedge Y]_{i-1,j} \to [X, Y]_{i,j} \to \varprojlim [X, HA(n) \wedge Y]_{i,j} \to 0.$$

PROOF. We write the cochain complex Y as $\dots \to Y_j \to Y_{j+1} \to \dots$. We may assume that for each j, the injective comodule Y_j is bounded below. Because $A \twoheadrightarrow A(n)$ is an isomorphism in a range of dimensions increasing with n, the inverse system of comodules

$$\dots \to (A \,\square_{A(n)} \mathbf{F}_p) \otimes Y_j \to (A \,\square_{A(n-1)} \mathbf{F}_p) \otimes Y_j \to (A \,\square_{A(n-2)} \mathbf{F}_p) \otimes Y_j \to \dots$$

stabilizes in any given degree, and the inverse limit is Y_j. We let $(A \,\square_{A(n)} \mathbf{F}_p) \otimes Y$ denote the cochain complex which is $(A \,\square_{A(n)} \mathbf{F}_p) \otimes Y_j$ in degree j; then the inverse system of cochain complexes stabilizes in each bidegree, and the inverse limit is Y. This finishes the proof. (Note that $(A \,\square_{A(n)} \mathbf{F}_p) \otimes Y$ is isomorphic to $HA(n) \wedge Y$ in $\mathsf{Stable}(A)$.) $\square$

We consider one other interesting family of quotient Hopf algebras. See [**Wil81**] for some results related to these sorts of Hopf algebras.

DEFINITION 2.1.6. We say that a connected commutative Hopf algebra B over a field k of characteristic p is *elementary* if it is isomorphic (as a Hopf algebra) to a tensor product of Hopf algebras of the forms $k[x]/(x^{p^n})$ with x primitive (and $|x|$ even, if p is odd), and $E[y]$ with y primitive (and $|y|$ odd, if p is odd). (In other words, B is bicommutative and its dual B^* has $z^p = 0$ for all z in the augmentation ideal.)

See [**Mar83**], [**Lin**] and [**Wil81**] for the following.

PROPOSITION 2.1.7. (a) *Suppose that $p = 2$. A quotient Hopf algebra B of A is elementary if and only if it has the form*

$$B = A/(\xi_1^{2^{n_1}}, \xi_2^{2^{n_2}}, \dots),$$

where for some r, we have

(i) *if $i < r$, then $n_i = 0$,*

(ii) *if $i \geq r$, then $n_i \leq r$.*

Conversely, any quotient algebra B given by exponents n_i satisfying (i)–(ii) *is an elementary quotient Hopf algebra of A.*

(b) *Suppose that p is odd. A quotient Hopf algebra B of A is elementary if and only if it has the form*

$$B = A/(\xi_1, \xi_2, \dots; \tau_0^{e_0}, \tau_1^{e_1}, \dots) \cong E[\tau_0^{1-e_0}, \tau_1^{1-e_1}, \dots]$$

for any exponents e_0, e_1, $\dots$, or

$$B = A/(\xi_1^{p^{n_1}}, \xi_2^{p^{n_2}}, \dots; \tau_0^{e_0}, \tau_1^{e_1}, \dots),$$

where for some r, we have

(i) *if $i < r$, then $n_i = 0$,*

(ii) *if $i \geq r$, then $n_i \leq r$,*

(iii) *if $i < r$, then $e_i = 1$.*

Conversely, any quotient algebra B given by these exponents n_i and e_i is an elementary quotient Hopf algebra of A.

We also describe the maximal elementary quotient Hopf algebras of A; see Figure 2.1.C. Every elementary quotient Hopf algebra of A is a quotient of one of these.

COROLLARY 2.1.8. (a) *Suppose that $p = 2$. The maximal elementary quotient Hopf algebras of A are*

$$E(m) = A/(\xi_1, \dots, \xi_m, \xi_{m+1}^{2^{m+1}}, \xi_{m+2}^{2^{m+1}}, \xi_{m+3}^{2^{m+1}}, \dots), \ m \geq 0.$$

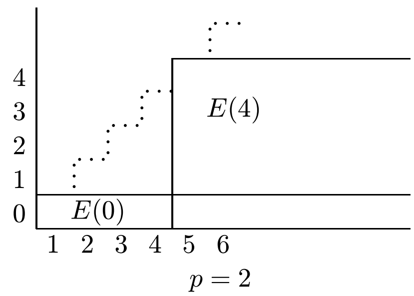


FIGURE 2.1.C. Profile functions for maximal elementary quotients of A at the prime 2. For reference, we have included a staircase above which are the elements $\xi_t^{2^s}$ with $s \geq t$.

(b) *Suppose that p is odd. The maximal elementary quotient Hopf algebras of A are*

$$E(-1)' = A/(\xi_1, \xi_2, \dots) \cong E[\tau_0, \tau_1, \dots],$$

$$E(m)' = A/(\xi_1, \dots, \xi_m, \xi_{m+1}^{p^{m+1}}, \xi_{m+2}^{p^{m+1}}, \xi_{m+3}^{p^{m+1}}, \dots; \tau_0, \dots, \tau_m), \ m \geq 0.$$

Note that the quotient Hopf algebras $E(m)$ and $E(m)'$ are conormal for all m.

We use the elementary quotient Hopf algebras of A to prove the vanishing line theorem of Section 2.3; we need to know their coefficient rings. The following is standard.

PROPOSITION 2.1.9. (a) *Let $p = 2$, and assume that E is an elementary quotient Hopf algebra of A. Then*

$$HE_{**} = \mathbf{F}_2[h_{ts} \mid \xi_t^{2^s} \neq 0 \text{ in } E].$$

Here, $|h_{ts}| = (1, |\xi_t^{2^s}|)$.

(b) *Let p be odd, and assume that E is an elementary quotient Hopf algebra of A. Then*

$$\begin{aligned} HE_{**} = E[h_{ts} \mid \xi_t^{p^s} \neq 0 \text{ in } E] &\otimes \mathbf{F}_p[b_{ts} \mid \xi_t^{p^s} \neq 0 \text{ in } E] \\ &\otimes \mathbf{F}_p[v_n \mid \tau_n \neq 0 \text{ in } E]. \end{aligned}$$

Here, $|h_{ts}| = (1, |\xi_t^{p^s}|)$, $|b_{ts}| = (2, p|\xi_t^{p^s}|)$, and $|v_n| = (1, |\tau_n|)$.

Indeed, if E is an elementary quotient of A with $\xi_t^{p^s} \neq 0$, then $\xi_t^{p^s}$ is primitive in E. Following Notation 1.3.9 and Remark 2.1.3(a), we have $h_{ts} = [\xi_t^{p^s}]$; similarly, if τ_n is non-zero in E, then it is primitive and we have $v_n = [t_n]$.

2.1.1. Quasi-elementary quotients of A. We also need to consider one other family of quotient Hopf algebras, the "quasi-elementary" Hopf algebras. It turns out that when $p = 2$, these coincide with elementary Hopf algebras; when p is odd, there are quasi-elementary Hopf algebras which are not elementary, but we do not have a complete classification.

The quasi-elementary quotients of A arise in the nilpotence theorem 3.1.6 of Section 3.1, and indeed in many of the results of Chapters 3 and 4; the lack of a classification at odd primes is one obstacle to proving those theorems when p is odd. So when p is 2, we have already studied these; when p is odd, we have no immediate use for them. Hence the contents of this subsection may be safely ignored, except for the term "quasi-elementary."

DEFINITION 2.1.10. Recall from Notation 1.3.9 that we have a Steenrod operation $\beta\widetilde{\mathscr{P}}^0$ acting on Ext, when p is odd. We say that a connected commutative Hopf algebra B over a field k is *quasi-elementary* if no product of the form

$$\begin{array}{ll} \prod_{w\in S} w, & p=2, \\ (\prod_{u\in S_{\text{odd}}} u)(\prod_{v\in S_{\text{even}}} \beta\widetilde{\mathscr{P}}^0 v), & p \text{ odd}, \end{array}$$

is nilpotent, for any finite sets $S \subset \operatorname{Ext}_B^{1,*}(k,k) - \{0\}$, $S_{\text{odd}} \subset \operatorname{Ext}_B^{1,\text{odd}}(k,k) - \{0\}$, and $S_{\text{even}} \subset \operatorname{Ext}_B^{1,\text{even}}(k,k) - \{0\}$.

Every elementary Hopf algebra is quasi-elementary; Wilkerson [**Wil81**, Section 6] gives several examples of quasi-elementary Hopf algebras which are not elementary. In particular, one of his examples is a quotient of A, when p is odd. See [**Wil81**, Counterexample 6.3] for the following.

EXAMPLE 2.1.11. Suppose that p is odd, and let B be this quotient Hopf algebra of A:

$$B = \mathbf{F}_p[\xi_1, \xi_2, \xi_3]/(\xi_1^p, \xi_2^{p^2}, \xi_3^p).$$

Then B is quasi-elementary, but not elementary.

When p is 2, things are a bit nicer. See [**Wil81**, Theorem 6.4] for the following.

PROPOSITION 2.1.12. *Suppose that $p = 2$. A quotient Hopf algebra B of A is elementary if and only if it is quasi-elementary.*

Note that we have classified the elementary quotients of A in Proposition 2.1.7 and Corollary 2.1.8. We have also computed the coefficient rings of these Hopf algebras in Proposition 2.1.9.

At odd primes, we do not have a classification of the quasi-elementary Hopf algebras, so we content ourselves with the following. This would follow from Conjecture 3.4.1.

CONJECTURE 2.1.13. *Suppose that p is odd. Then every quasi-elementary quotient Hopf algebra of A is a quotient of*

$$D = A/(\xi_1^p, \xi_2^{p^2}, \xi_3^{p^3}, \ldots, \xi_n^{p^n}, \ldots).$$

See [**Pal97**] for some results relating to quasi-elementary Hopf algebras.

2.2. P_t^s-homology

In this section we discuss P_t^s-homology; this tool has been used by many authors to study Ext_A, so it is reasonable to expect it to be useful in the present setting. [**Mar83**, Section 19.1] is a good reference for basic results on P_t^s-homology of modules; in this section, we prove analogues of some of those results.

Let A^* be the dual of A. Note that an A-comodule M with structure map ψ is naturally an A^*-module, via the map

$$A^* \otimes M \xrightarrow{1\otimes\psi} A^* \otimes A \otimes M \xrightarrow{\text{ev}\otimes 1} M.$$

We remind the reader that if one dualizes with respect to the monomial basis for A, then P_t^s is the element of A^* dual to $\xi_t^{p^s}$, and (when p is odd) Q_n is the element dual to τ_n. If $s < t$, then $(P^s_t)^p = 0$, so in this case one may define the P^s_t-*homology* of an A-comodule M by

$$H(M, P^s_t) = \frac{\ker(M \xrightarrow{(P^s_t)^{p-1}} M)}{\operatorname{im}(M \xrightarrow{P^s_t} M)}.$$

For all n we have $Q_n^2 = 0$, so one may define $H(M, Q_n)$ similarly. One *may* do these things, but that does not mean that one should do them when working in the context of cochain complexes of injective A-comodules.

Definition 2.2.1. Given integers s and t with $0 \leq s < t$, we define the P^s_t-*homology spectrum*, P^s_t, to be the cochain complex with jth term

$$(P^s_t)_j = \begin{cases} \Sigma^{kp|\xi_t^{p^s}|}A, & j = 2k, \\ \Sigma^{(kp+1)|\xi_t^{p^s}|}A, & j = 2k+1, \end{cases}$$

with differentials given as in the following diagram:

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & A & \longrightarrow & A & \longrightarrow & A & \longrightarrow & A & \longrightarrow & \dots \\ & & \xi_t^{p^s} & \longmapsto & 1,\ \xi_t^{(p-1)p^s} & \longmapsto & 1,\ \xi_t^{p^s} & \longmapsto & 1 & & \end{array}$$

(A is a rank 1 cofree comodule, so a comodule map into A is determined by what hits 1—see Lemma 1.1.11.) We define the *connective P^s_t-homology spectrum*, p^s_t, to be the cochain complex obtained by truncating the complex for P^s_t at homological dimension zero; in other words, $p^s_t = H\mathbf{F}_p[\xi_t^{p^s}]/(\xi_t^{p^{s+1}})$. Similarly, when p is odd, we define the Q_n-*homology spectrum*, Q_n, to be the cochain complex with jth term $(Q_n)_j = \Sigma^{j|\tau_n|}A$, and with differentials given by

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & A & \longrightarrow & A & \longrightarrow & A & \longrightarrow & A & \longrightarrow & \dots \\ & & \tau_n & \longmapsto & 1,\ \tau_n & \longmapsto & 1,\ \tau_n & \longmapsto & 1 & & \end{array}$$

We define the *connective Q_n-homology spectrum*, q_n, to be the truncation of the Q_n complex below homological degree zero; it is equal to $HE[\tau_n]$. For any spectrum X, we define its P^s_t-*homology* to be $(P^s_t)_{**}X$, and we define its Q_n-*homology* to be $(Q_n)_{**}X$.

In the remainder of this section, we compute the coefficient rings of these spectra and we prove a few results for later use.

Proposition 2.2.2. (a) *If $s < t$, then $\mathbf{F}_p[\xi_t^{p^s}]/(\xi_t^{p^{s+1}})$ is a quotient coalgebra of A over which A is injective as a right comodule. For any $n \geq 0$, $E[\tau_n]$ is a quotient Hopf algebra of A (and hence A is injective as a right comodule over it).*

(b) *Hence,*

$$(p^s_t)_{**} \cong \begin{cases} \mathbf{F}_2[h_{ts}], & p = 2, \\ \mathbf{F}_p[b_{ts}] \otimes E[h_{ts}], & p \text{ odd}, \end{cases}$$

where $|h_{ts}| = (1, |\xi_t^{p^s}|)$ and $|b_{ts}| = (2, p|\xi_t^{p^s}|)$. Also,

$$(q_n)_{**} \cong \mathbf{F}_p[v_n],$$

where $|v_n| = (1, |\tau_n|)$.

(c) *Hence*

$$(P_t^s)_{**} = \begin{cases} \mathbf{F}_2[h_{ts}^{\pm 1}], & p = 2, \\ \mathbf{F}_p[b_{ts}^{\pm 1}] \otimes E[h_{ts}], & p \text{ odd}, \end{cases}$$
$$(Q_n)_{**} = \mathbf{F}_p[v_n^{\pm 1}],$$

and for any connective X,

$$(P_t^s)_{**}X = \begin{cases} h_{ts}^{-1}(p_t^s)_{**}X, & p = 2, \\ b_{ts}^{-1}(p_t^s)_{**}X, & p \text{ odd}, \end{cases}$$
$$(Q_n)_{**}X = v_n^{-1}(q_n)_{**}X.$$

PROOF. Part (a) is well-known. It is dual to the statement that A^*, the dual of A, is free over the subalgebra $\mathbf{F}_p[P_t^s]/(P_t^s)^p$; i.e., $H(A^*, P_t^s) = 0$. (And similarly, A^* is free over $E[Q_n]$; i.e., $H(A^*, Q_n) = 0$.) See [**AM71**], [**MP72**], or [**Mar83**, Proposition 19.1].

Parts (b) and (c) are standard, trivial, or both (using Lemma 1.3.4, for instance). See [**Mar83**, Propositions 19.2–3] for an alternate formulation of part (c). □

By the way, we note that for all primes, the spectrum $p_t^0 = H\mathbf{F}_p[\xi_t]/(\xi_t^p)$ is a ring spectrum, while P_t^0 is a field spectrum for $p = 2$ (perhaps P_t^0 is an example of an "Artinian ring spectrum" when p is odd). Similarly, when p is odd, $q_n = HE[\tau_n]$ is a ring spectrum and Q_n is a field spectrum. The spectra p_t^s for $s > 0$ are not ring spectra, but are examples of the sort discussed in Proposition 1.3.2(b).

REMARK 2.2.3. As one might expect, there is a relationship between the module definition of P_t^s-homology (given just before Definition 2.2.1) and the homology functor represented by the P_t^s-homology spectrum: if X is an injective resolution of a comodule M, then

$$(P_t^s)_{**}X = H_*(M, P_t^s) \otimes (P_t^s)_{**}.$$

(Here we are viewing the singly-graded vector space $H_*(M, P_t^s)$ as doubly-graded by putting $H_i(M, P_t^s)$ in bidegree $(0, i)$.) In particular, $(P_t^s)_{**}X = 0$ if and only if $H_*(M, P_t^s) = 0$. Similarly, we have

$$(Q_n)_{**}X = H_*(M, Q_n) \otimes (Q_n)_{**}.$$

It is convenient to have alternate notation for the spectra P_t^s and Q_n, based on the "slopes" of the polynomial generators in their coefficient rings.

NOTATION 2.2.4. We define the *slope* of P_t^s to be $\frac{p|\xi_t^{p^s}|}{2}$, and the slope of Q_n to be $|\tau_n|$. Note that these spectra all have distinct slopes. The set

$$\begin{array}{ll} \{|\xi_t^{2^s}| : s < t\}, & p = 2, \\ \{\frac{p|\xi_t^{p^s}|}{2} : s < t\} \cup \{|\tau_n| : n \geq 0\}, & p \text{ odd}, \end{array}$$

is called the set of *slopes* of A; the phrase "fix a slope n" means "fix an element n of this set." Given a slope n, we let $Z(n)$ denote the corresponding P_t^s or Q_n spectrum. For example, when $p = 2$, we have

$$Z(1) = P_1^0,\ Z(3) = P_2^0,\ Z(6) = P_2^1,\ Z(7) = P_3^0,\ \ldots,\ Z(|\xi_t^{2^s}|) = P_t^s,\ \ldots.$$

When p is odd, we have

$$Z(1) = Q_0,\ Z(p-1) = Q_1,\ Z(p^2-p) = P_1^0,\ Z(p^2-1) = Q_2,\ \ldots,$$
$$Z(\frac{p|\xi_t^{p^s}|}{2}) = P_t^s,\ \ldots,\ Z(|\tau_m|) = Q_m,\ \ldots.$$

We let $z(n)$ be the connective cover of $Z(n)$. We let y_n denote the element of A, either $\xi_t^{p^s}$ or τ_m, with slope n.

See [**Mar83**, 19.21] for the following at the prime 2.

Proposition 2.2.5. *Let B be a quotient Hopf algebra of A. Then $(P_t^s)_{**}HB \neq 0$ if and only if $\xi_t^{p^s} \neq 0$ in B. For p odd, $(Q_n)_{**}HB \neq 0$ if and only if $\tau_n \neq 0$ in B.*

We will use this result in Section 5.3.

Proof. We will give the proof for P_t^s and leave the Q_n proof for the reader.

By Remark 2.2.3, we are interested in $H(A \,\square_B \mathbf{F}_p, P_t^s)$. If $\xi_t^{p^s} \neq 0$ in B, then $1 \in A \,\square_B \mathbf{F}_p$ generates a nonzero homology class.

To prove the converse, first we reduce to the case when B is finite. We define $B(n)$ to be the quotient Hopf algebra of A defined by the following pushout diagram of Hopf algebras:

$$\begin{array}{ccc} A & \longrightarrow & B. \\ \downarrow & & \downarrow \\ A(n) & \longrightarrow & B(n) \end{array}$$

($A(n)$ is defined in Example 2.1.4.) In other words, $B(n)$ is the quotient of B induced by the map $A \twoheadrightarrow A(n)$. Each $B(n)$ is finite, and one can apply Proposition 2.1.5 to see that $(P_t^s)_{**}HB \xrightarrow{\cong} \varprojlim_n (P_t^s)_{**}HB(n)$.

So it suffices to show that if B is a finite quotient of A in which $\xi_t^{p^s} = 0$, then $(P_t^s)_{**}HB = 0$. We do this by induction on $\dim_{\mathbf{F}_p} B$. The induction starts with $B = \mathbf{F}_p$, in which case $H\mathbf{F}_p = A$. Then

$$(P_t^s)_{**}H\mathbf{F}_p = (H\mathbf{F}_p)_{**}P_t^s,$$

the homology of the cochain complex P_t^s. This complex is acyclic, so its homology is zero. (Alternatively, see the remark following the proof.) This starts the induction.

Now fix a finite-dimensional B with $\xi_t^{p^s} = 0$ in B, and assume that for every proper quotient C of B, we have $(P_t^s)_{**}HC = 0$. As noted in Remark 2.1.3, we have a Hopf algebra quotient C of B with Hopf algebra kernel either $\mathbf{F}_p[\xi_r^{p^q}]/(\xi_r^{p^{q+1}})$ or $E[\tau_m]$. So by Lemma 1.3.10 this leads to a cofiber sequence (in which we neglect suspensions)

$$HB \xrightarrow{f} HB \to Z,$$

where either $Z = HC$ or Z is the cofiber of a self-map of HC. In either case, $(P_t^s)_{**}HC = 0 \Rightarrow (P_t^s)_{**}Z = 0$, so $(P_t^s)_{**}f$ is an isomorphism.

Now we argue essentially as in the proof of Lemma 2.3.11 to see that boundedness of the comodule $A \,\square_B \mathbf{F}_p$ implies that $(P_t^s)_{**}HB = 0$. To be precise, we first have to specify the degree of the map f. There are three cases:

(1) $p = 2$: then $f\colon \Sigma^{1,|\xi_r^{2^q}|}HB \to HB$.
(2) p odd, Hopf algebra kernel $E[\tau_m]$: then $f\colon \Sigma^{1,|\tau_m|}HB \to HB$.

(3) p odd, Hopf algebra kernel $\mathbf{F}_p[\xi_r^{p^q}]/(\xi_r^{p^{q+1}})$; then $f\colon \Sigma^{2,p|\xi_r^{p^q}|}HB \to HB$.

We deal with cases (1) and (3); (2) is handled the same way. By our computations in Proposition 2.2.2, we know that for all i and j, we have an isomorphism

$$(P_t^s)_{ij}HB \cong (P_t^s)_{i+2,j+p|\xi_t^{p^s}|}HB.$$

In cases (1) and (3), f induces an isomorphism

$$(P_t^s)_{ij}HB \cong (P_t^s)_{i+2,j+p|\xi_r^{p^q}|}HB.$$

Combining these, for each integer k we get an isomorphism

$$(P_t^s)_{i,j}HB \cong (P_t^s)_{i,j+pk(|\xi_t^{p^s}|-|\xi_r^{p^q}|)}HB.$$

Now, $\xi_r^{p^q} \neq 0$ in B and $\xi_t^{p^s} = 0$, so $\xi_r^{p^q} \neq \xi_t^{p^s}$. In particular, these two elements have different degrees. By Remark 2.2.3, for fixed i, $(P_t^s)_{i,*}HB$ is equal (up to suspension) to $H_*(A \,\square_B \mathbf{F}_p, P_t^s)$; since $A \,\square_B \mathbf{F}_p$ is a bounded below comodule, then this homology must be zero in small enough degrees. By the above isomorphism, we may then conclude that it is zero in all degrees. $\square$

We remark that it is easy to show that $(P_t^s)_{**}A = 0$—this follows by a result of Milnor-Moore [**MM65**], as in [**Mar83**, Proposition 19.1]. It seems as though there should be a similar proof of Proposition 2.2.5, but we have not been able to find one.

Recall from [**HPS97**, A.2.4] that if X is any object in $\mathsf{Stable}(A)$, then DX denotes the *Spanier-Whitehead dual* of X. DX has the property that for any Y,

$$[X \wedge Y, S^0] = [Y, DX].$$

We mention the following easy fact.

Proposition 2.2.6. *Let E and X be spectra with X finite. Then $E_{**}X = 0$ if and only if $E_{**}DX = 0$.*

Proof. If X is E-acyclic, then so is $X \wedge Y$ for any Y. In particular, $DX \wedge X \wedge DX$ is E-acyclic. Since DX is a retract of this when X is finite, we are done. $\square$

Corollary 2.2.7. *Let X be a finite spectrum. Then $(P_t^s)_{**}X = 0$ if and only if $(P_t^s)_{**}DX = 0$; and for p odd, $(Q_n)_{**}X = 0$ if and only if $(Q_n)_{**}DX = 0$.*

One should be able to get more precise information about the relationship between $(P_t^s)_{**}X$ and $(P_t^s)_{**}DX$, as in [**Mar83**, 19.12], but we do not need it.

We end this section with a note on operations on P_t^s-homology. The next result follows from Remark 2.2.3.

Proposition 2.2.8. *We have*

$$(P_t^s)_{**}P_t^s = H_*(A \,\square_{D[\xi_t^{p^s}]} \mathbf{F}_p, P_t^s) \otimes (P_t^s)_{**},$$
$$(Q_n)_{**}Q_n = H_*(A \,\square_{E[\tau_n]} \mathbf{F}_p, Q_n) \otimes (Q_n)_{**}.$$

Margolis has calculated $H_*(A \,\square_{D[\xi_t^{p^s}]} \mathbf{F}_p, P_t^s)$ for $s = 0$ at the prime 2 [**Mar83**, 19.26]; similar calculations work at odd primes for $H_*(A \,\square_{E[\tau_n]} \mathbf{F}_p, Q_n)$.

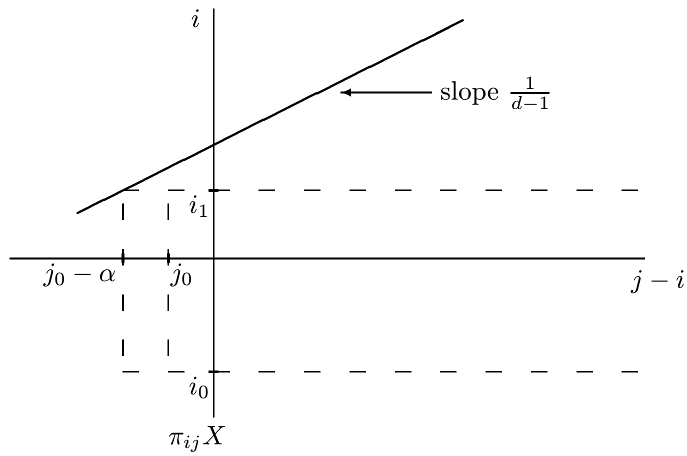


FIGURE 2.3.A. Vanishing line at the prime 2. In this picture, $\pi_{ij}X = 0$ above the slanted line (the vanishing line), to the left of the vertical line $j - i = j_0$, and below the horizontal line $i = i_0$. α is a number depending only on the slope d.

2.3. Vanishing lines for homotopy groups

In this section we prove several theorems relating the vanishing of P^s_t- and Q_n-homology groups of an object X to the homotopy groups of X; these are versions in the cochain complex category of well-known theorems about modules over the Steenrod algebra. These results are at the heart of many of the other results of the book.

Recall that if X is an injective resolution of a comodule M, then $\pi_{**}(X) = \mathrm{Ext}^{**}_A(\mathbf{F}_p, M)$.

We make heavy use of Notation 2.2.4 in this section. *CL*-spectra are defined in Definition 1.4.6.

Here are our main theorems. In the setting of modules over the Steenrod algebra, the first is due to Anderson-Davis [**AD73**] ($p = 2$) and Miller-Wilkerson [**MW81**] (p odd), and the second to Adams-Margolis [**AM71**] ($p = 2$) and Moore-Peterson [**MP72**] (p odd). Both of the results are proved in [**MW81**]; we follow those proofs.

THEOREM 2.3.1 (Vanishing line theorem). *Let X be a CL-spectrum. Suppose that there is a number d so that $Z(n)_{**}X = 0$ for all slopes n with $n < d$. Then $\pi_{**}X$ has a vanishing line of slope d: for some c, we have $\pi_{ij}X = 0$ when $j < di - c$.*

Of course, this vanishing line has slope $\frac{1}{d-1}$ in $(j - i, i)$-coordinates. See Figure 2.3.A for a picture.

THEOREM 2.3.2. *Let X be a CL-spectrum, and assume that $Z(n)_{**}X = 0$ for all slopes n; then $\pi_{**}X$ has a "horizontal" vanishing line: for some c, we have $\pi_{ij}X = 0$ when $i > c$.*

COROLLARY 2.3.3. *Under the hypotheses of Theorem 2.3.2, X is in the localizing subcategory generated by A.*

PROOF. This follows from Lemma 1.4.8. ☐

REMARK 2.3.4. (a) When we speak of phenomena "above" a vanishing line, we mean "above" in the grading of Figure 2.3.A—i.e., in the region in which π_{ij} might be zero.

(b) We will see in the proof that the "intercept" c of the vanishing line in Theorem 2.3.1 may be given by $c = (d-1)i_1 - j_0 + \alpha$, where $\alpha = \alpha(d)$ is a number depending only on d. In the grading of Figure 2.3.A, the homotopy π_{ij} is zero above the line of slope $\frac{1}{d-1}$ through the point $(j_0 - \alpha, i_1)$. To compute α, we find the smallest integer n so that the map $A \to A(n)$ is an isomorphism through degree d (i.e., so that $2^{n+1} > d$ when $p = 2$, or $2(p-1)p^n > d$ when p is odd); then

$$\alpha = \begin{cases} \displaystyle\sum_{\substack{s+t \leq n+1 \\ |\xi_t^{p^s}| \leq d}} |\xi_t^{p^s}|, & \text{if } p = 2, \\ \displaystyle\sum_{\substack{s+t \leq n \\ |\xi_t^{p^s}| \leq d}} (d + (p-1)|\xi_t^{p^s}|) + \sum_{\substack{i \leq n \\ |\tau_i| \leq d}} |\tau_i|, & \text{if } p \text{ is odd.} \end{cases}$$

For Theorem 2.3.2, the intercept is i_1—the homotopy π_{ij} is zero if $i > i_1$.

(c) For both of these theorems (in fact, for all of the results of this section), we can weaken condition (ii) of Definition 1.4.6 slightly—it suffices to assume that for all i, there is a $j_0 = j_0(i)$ so that $\pi_{ij}X = 0$ if $j - i < j_0$. With this assumption, one replaces j_0 in the above formulas for c with $\min\{j_0(i) \mid i_0 \leq i \leq i_1\}$.

Let M be a bounded below A-comodule with injective resolution X; then it is easy to see that X is *CL*. Furthermore, we know from [**Mar83**, Theorem 13.12] that $\pi_{st}X = \operatorname{Ext}_A^{st}(\mathbf{F}_p, M)$ has a "horizontal" vanishing line if and only if M is injective. Hence our theorems do indeed provide generalizations of the previously cited ones.

One can generalize these results somewhat. The same proofs carry over essentially unchanged, so to keep the notation simple we prove Theorems 2.3.1 and 2.3.2, rather than the following.

THEOREM 2.3.5. *Let X be a CL-spectrum, and fix a quotient Hopf algebra B of A.*

(a) *Suppose that there is a number d so that $Z(n)_{**}X = 0$ for all slopes n with $n < d$ and $y_n \neq 0$ in B. Then $HB_{**}X$ has a vanishing line of slope d: for some c, we have $HB_{ij}X = 0$ when $j < di - c$.*

(b) *If $Z(n)_{**}X = 0$ for all slopes n with $y_n \neq 0$ in B, then $HB_{**}X$ has a "horizontal" vanishing line; hence $HB \wedge X$ is in the localizing subcategory generated by A.*

Of course, HB_{**} relates to Ext_B the same way π_{**} relates to Ext_A, so one should view this result as a vanishing line theorem over B. (One could also work in $\mathsf{Stable}(B)$ and prove a result about π_{**} in that category, but it seems better to work in $\mathsf{Stable}(A)$ whenever possible.)

Remark 2.3.4 also applies here.

2.3.1. Proof of Theorems 2.3.1 and 2.3.2 for $p = 2$. We give the proofs of Theorems 2.3.1 and 2.3.2 when $p = 2$. In the next subsection we indicate the changes necessary when working at odd primes.

Recall from Lemmas 1.3.10 and 2.1.2, as well as Remark 2.1.3, that if B is a finite-dimensional quotient Hopf algebra of A, then for some s and t we have the following conditions:

- $\xi_t^{2^s} \neq 0$ in B,
- $\xi_t^{2^{s+1}} = 0$ in B,
- $\xi_j^{2^s} = 0$ in B for all $j < t$.

Hence there is a Hopf algebra extension

$$E[\xi_t^{2^s}] \to B \to C,$$

hence an element $h_{ts} \in HB_{1,|\xi_t^{2^s}|}$ and a cofibration

$$\Sigma^{1,|\xi_t^{2^s}|} HB \xrightarrow{h_{ts}} HB \to HC.$$

Given this situation, we have the following result. (Given a spectrum Y and a self-map $v\colon \Sigma^{ij} Y \to Y$, we say that v *acts nilpotently* on $\pi_{**}Y$ if for all $y \in \pi_{**}Y$, some power of $\pi_{**}(v)$ annihilates y.)

LEMMA 2.3.6. *Assume that B is finite-dimensional, and fix s and t as above.*

(a) *If $s \geq t$, then $h_{ts} \in HB_{**}$ is nilpotent; hence the self-map $h_{ts}\colon HB \to HB$ is nilpotent.*

(b) *Given an object X, consider the cofibration*

$$\Sigma^{1,|\xi_t^{2^s}|} HB \wedge X \xrightarrow{h_{ts} \wedge 1_X} HB \wedge X \to HC \wedge X.$$

If $h_{ts} \wedge 1_X$ acts nilpotently on $HB_{**}X$, and if $HC_{**}X$ has a vanishing line of slope d, then $HB \wedge X$ has a vanishing line of slope d. The difference in intercepts depends only on d and $|\xi_t^{2^s}|$—it is independent of X.

REMARK 2.3.7. In fact, the nilpotence of h_{ts} does not depend on B being finite-dimensional. See Theorem B.2.1 for a generalization, due to Lin.

PROOF. Lin proved part (a) in [**Lin**, Corollary 3.2]; see also Anderson-Davis [**AD73**] and Miller-Wilkerson [**MW81**, Proposition 4.1].

For part (b), we look at the long exact sequence in homotopy coming from the given cofibration (we write h for $\pi_{**}(h_{ts} \wedge 1_X)$):

$$\dots \to HC_{i-1,j}X \to HB_{i-1,j-|\xi_t^{2^s}|}X \xrightarrow{h} HB_{ij}X \to HC_{ij}X \to \dots.$$

Fix a bidegree (i,j) above the vanishing line for $HC_{**}X$. There are two cases: suppose first that $|\xi_t^{2^s}| > d$. The exact sequence tells us that

$$HB_{i-1,j-|\xi_t^{2^s}|}X \xrightarrow{h} HB_{ij}X$$

is an epimorphism. Since $|\xi_t^{2^s}| > d$, then $(i-1, j-|\xi_t^{2^s}|)$ is above the vanishing line for $HC_{**}X$. Hence

$$HB_{i-k,j-k|\xi_t^{2^s}|}X \xrightarrow{h} HB_{i-(k-1),j-(k-1)|\xi_t^{2^s}|}X$$

is an epimorphism for all $k > 0$. Also since $|\xi_t^{2^s}| > d$, then one can see that for $k \gg 0$, $(i-k, j-(k-1)|\xi_t^{2^s}|)$ is a bidegree above the vanishing line for $HC_{**}X$, so

$$HB_{i-k,j-k|\xi_t^{2^s}|}X \xrightarrow{h} HB_{i-(k-1),j-(k-1)|\xi_t^{2}|}X$$

is an isomorphism. Since h acts nilpotently on $HB_{**}X$, though, these groups must be zero for k large; since they surject onto $HB_{ij}X$, then $HB_{ij}X = 0$. Note that in this case, the vanishing line for $HC_{**}X$ is the same as that for $HB_{**}X$.

Suppose, on the other hand, that $|\xi_t^{2^s}| \leq d$. If (i,j) is a bidegree above the vanishing line, then so is $(i+1,j)$, so the map

$$HB_{i,j-|\xi_t^{2^s}|}X \xrightarrow{h} HB_{i+1,j}X$$

is an isomorphism. Arguing as above, we see that the group $HB_{i,j-|\xi_t^{2^s}|}X$ must be zero. In this case, the intercept of the HB vanishing line changes by $|\xi_t^{2^s}|$: $HB_{ij}X = 0$ when $j < di - c - |\xi_t^{2^s}|$. □

LEMMA 2.3.8. *Fix an object X satisfying condition* (i) *of Definition* 1.4.6*.*

(a) *Given an extension*

$$E[\xi_t^{2^s}] \to B \to C$$

where B and C are quotients of A, if $|\xi_t^{2^s}| \geq d$ and if $HC_{**}X$ has a vanishing line of slope d, then $HB_{**}X$ has a vanishing line of slope d. In fact, $HB_{**}X$ has the same vanishing line as $HC_{**}X$.

(b) *Given finite-dimensional quotients $B \twoheadrightarrow C$ of A, if $HC_{**}X$ has a vanishing line of slope d and if the map $B \twoheadrightarrow C$ is an isomorphism in dimensions less than d, then $HB_{**}X$ has a vanishing line of slope d. In fact, $HB_{**}X$ has the same vanishing line as $HC_{**}X$.*

PROOF. Part (a): (This proof is based on that of [**MW81**, Proposition 3.2].) We assume that $HC_{i,j}X = 0$ when $j < di-c$, and we want to show that $HB_{i,j}X = 0$ when $j < di - c$. We prove this by induction on i. By Lemma 1.3.10, we have a cofibration

$$\Sigma^{1,|\xi_t^{2^s}|}HB \wedge X \xrightarrow{h_{ts}} HB \wedge X \to HC \wedge X.$$

This gives us a long exact sequence in homotopy:

$$\ldots \to HB_{i-1,j-|\xi_t^{2^s}|}X \to HB_{i,j}X \to HC_{i,j}X \to \ldots. \tag{2.3.9}$$

Since X satisfies condition (i) of Definition 1.4.6, then for i sufficiently small and for all j, we have

$$\pi_{i,j}(X) = 0 = HB_{i,j}X = HC_{i,j}X.$$

This starts the induction: if i_0 is the smallest value of i for which $HC_{i,j}X$ is nonzero, then we have an inclusion $HB_{i_0,j}X \hookrightarrow HC_{i_0,j}X$ (for all j).

The inductive step is also easy: if we have i and j with $j < di - c$, then $j - |\xi_t^{2^s}| < d(i-1) - c$; hence both $HC_{i,j}X$ and $HB_{i-i,j-|\xi_t^{2^s}|}X$ are zero. So we apply exactness in the long exact sequence (2.3.9).

Part (b): Given $B \twoheadrightarrow C$ with B finite-dimensional (or more generally, with C of finite index in B), then there is a sequence of extensions

$$\begin{gathered}
E[\xi_{t_1}^{2^{s_1}}] \to B \to B_1, \\
E[\xi_{t_2}^{2^{s_2}}] \to B_1 \to B_2, \\
\vdots \\
E[\xi_{t_n}^{2^{s_n}}] \to B_{n-1} \to C.
\end{gathered}$$

(See [**HSb**, Lemma A.11] or [**MW81**, Lemmas 3.4–3.5], for example.) If, furthermore, $B \twoheadrightarrow C$ is an isomorphism through degree $d-1$, then each $\xi_{t_i}^{2^{s_i}}$ has degree at least d. So apply induction and part (a). □

Lemma 2.3.8 is a special case of the following; we have stated and proved them separately because at odd primes we need a variant of the former, while the latter holds as stated.

LEMMA 2.3.10. *Fix an object X satisfying condition* (i) *of Definition* 1.4.6. *Given a surjection of coalgebras $f\colon A \twoheadrightarrow B$, if f is an isomorphism below degree d, and if $HB_{**}X$ has a vanishing line of slope d, then $HA_{**}X$ has a vanishing line of slope d. In fact, they have the same vanishing line.*

PROOF. We have an injection of comodules $A \square_A \mathbf{F}_2 \to A \square_B \mathbf{F}_2$, and the cokernel is zero below dimension d. Taking injective resolutions gives a cofibration

$$\Sigma^{1,0} Z \to HA \to HB \to Z,$$

where Z is $(0,d)$-connective (Definition 1.4.3). So to show that $HA_{ij}X = \pi_{ij}X = 0$ if $j < di - c$, one argues by induction on i just as in Lemma 2.3.8(a). $\square$

LEMMA 2.3.11. *Let X be an object satisfying conditions* (ii) *and* (iii) *of Definition* 1.4.6. *Suppose that E is a finite-dimensional elementary quotient Hopf algebra of A. If $(P^s_t)_{**}X = 0$ whenever $\xi_t^{2^s} \neq 0$ in E, then $HE_{**}X$ has a horizontal vanishing line. In fact, given i_1 as in condition* (iii), *then $HE_{i*}X = 0$ if $i > i_1$.*

PROOF. Since the dual of E is an exterior algebra, we abuse notation and write $E = E[x_1, x_2, \ldots, x_n]$ where each x_k is primitive; in other words, each x_k is equal to $\xi_t^{2^s}$ for some s and t. We point out that the x_k's have distinct degrees.

Now HE_{**} is a polynomial algebra on n generators; we write $v_k \in HE_{1,|x_k|}$ for the generator corresponding to x_k. We also use v_k to denote the corresponding self-map of HE, and we let $HE/(v_k)$ be its cofiber. Note that the self-map v_k induces multiplication by v_k on homotopy, so that

$$\pi_{**}(HE/(v_k)) = (\pi_{**}HE)/(v_k).$$

We define $HE/(v_{k_1}, \ldots, v_{k_m})$ similarly (assuming that the numbers $k_1, \ldots, k_m$ are distinct). Hence if ℓ is an integer with $1 \leq \ell \leq n$ and $\ell \notin \{k_1, \ldots, k_m\}$, then $v_\ell\colon HE \to HE$ induces a self-map of $HE/(v_{k_1}, \ldots, v_{k_m})$.

Also note that, if $x_k = \xi_t^{2^s}$, then by a change-of-coalgebras, we have

$$(P^s_t)_{**}X = v_k^{-1}(HE/(v_1, \ldots, \hat{v}_k, \ldots, v_n))_{**}X.$$

We refer to this as the *x_k-homology* of X.

We claim for any set $\{k_1, \ldots, k_m\} \subseteq \{1, \ldots, n\}$,

$$(HE/(v_{k_1}, \ldots, v_{k_m}))_{**}(X)$$

has a horizontal vanishing line. We prove this by induction on $n - m$. (We will have proved the lemma when $n - m = n$.)

We have to deal with the first few cases before we can apply the inductive step. If $n - m = 0$, then $\{k_1, \ldots, k_m\} = \{1, \ldots, n\}$, so $HE/(v_{k_1}, \ldots, v_{k_m}) = H\mathbf{F}_2$, and by assumption, $H\mathbf{F}_{2**}X$ has a horizontal vanishing line with intercept i_1. If $n - m = 1$, then we let k be the integer so that

$$\{k\} \cup \{k_1, \ldots, k_m\} = \{1, 2, \ldots, n\}.$$

To simplify the notation, we let $HE_m = HE/(v_{k_1}, \ldots, v_{k_m})$. Then we have a cofibration

$$HE_m \wedge X \xrightarrow{v_k} HE_m \wedge X \to H\mathbf{F}_2 \wedge X.$$

By hypothesis, $H\mathbf{F}_{2**}X$ has a horizontal vanishing line, so the map v_k induces an isomorphism in π_{ij} for $i > i_1$. On the other hand, the x_k-homology of X is zero, and the x_k-homology of X is equal to $v_k^{-1}(HE_m)_{**}X$. Since v_k induces an isomorphism for $i > i_1$, and since this localization is zero, then v_k must be zero when $i > i_1$. So $(HE_m)_{**}X$ has a horizontal vanishing line with intercept i_1.

Now fix m with $n - m \geq 2$, and assume that

$$(HE/(v_{\ell_1}, \ldots, v_{\ell_t}))_{**}X$$

has a horizontal vanishing line whenever $n - t < n - m$. As above, let $HE_m = HE/(v_{k_1}, \ldots, v_{k_m})$. Pick distinct integers $k, \ell \leq n$ which are not in $\{k_1, \ldots, k_m\}$, and consider the following diagram (in which each row and column is a cofibration):

$$\begin{array}{ccccc}
\Sigma^{2,|x_k|+|x_\ell|}HE_m & \xrightarrow{v_k} & \Sigma^{1,|x_\ell|}HE_m & \longrightarrow & \Sigma^{1,|x_\ell|}HE_m/(v_k) \\
{\scriptstyle v_\ell}\downarrow & & {\scriptstyle v_\ell}\downarrow & & \downarrow \\
\Sigma^{1,|x_k|}HE_m & \xrightarrow{v_k} & HE_m & \longrightarrow & HE_m/(v_k) \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^{1,|x_k|}HE_m/(v_\ell) & \longrightarrow & HE_m/(v_\ell) & \longrightarrow & HE_m/(v_k, v_\ell)
\end{array}$$

Now smash this diagram with X. By induction, all of the spectra $HE_m/(v_k, v_\ell)\wedge X$, $HE_m/(v_k)\wedge X$, and $HE_m/(v_\ell)\wedge X$ have horizontal vanishing lines with intercept i_1. Now we apply $\pi_{ij}(-)$; for $i > i_1$, the maps labeled v_k and v_ℓ induce isomorphisms on π_{ij}, so we have

$$\pi_{i,j}HE_m \wedge X \xrightarrow[\cong]{v_\ell^{-1}\circ v_k} \pi_{i,j+|x_k|-|x_\ell|}HE_m \wedge X.$$

This isomorphism, combined with the facts that $|x_k| \neq |x_\ell|$ and that $\pi_{ij}(HE_m \wedge X)$ is zero when $j \ll 0$, implies that $\pi_{ij}(HE_m \wedge X) = 0$ for all j; i.e., $\pi_{**}(HE_m \wedge X)$ has the predicted horizontal vanishing line. This completes the inductive step, and hence the proof. $\square$

Proof of Theorem 2.3.1. By Lemma 2.3.10, if we know that $HB_{**}X$ has a vanishing line of slope d for all finite-dimensional quotient Hopf algebras B of A, then $\pi_{**}(X)$ will, also. (For example, the map $A \to A(n)$ is an isomorphism through degree $2^{n+1} - 1$, so apply the lemma to the case $B = A(n)$ where $2^{n+1} - 1 \geq d$.)

Now we show that $HB_{**}X$ has a vanishing line of slope d for all quotients B of A with $\dim_{\mathbf{F}_2} B < \infty$, by induction on $\dim_{\mathbf{F}_2} B$: The case where $\dim_{\mathbf{F}_2} B = 1$ (i.e., $B = \mathbf{F}_2$) is taken care of by condition (iii), so we move on to the inductive step. By Remark 2.1.3, there is a Hopf algebra extension

$$E[\xi_t^{2^s}] \to B \to C.$$

By hypothesis, $HC_{**}X$ has a vanishing line of slope d, and we want to produce a vanishing line for $HB_{**}X$. There are several cases:

(1) If $s \geq t$, then we are done by Lemma 2.3.6.

(2) If $s < t$ and $|\xi_t^{2^s}| > d$, then we are done by Lemma 2.3.8(a).

(3) If $s < t$ and $|\xi_t^{2^s}| \leq d$, then we have to prove something. We may assume that $\xi_j = 0$ in B for $j < t$. Let

$$B = \mathbf{F}_2[\xi_t, \xi_{t+1}, \xi_{t+2}, \dots]/(\xi_t^{2^{s+1}}, \xi_{t+1}^{2^{n_1}}, \xi_{t+2}^{2^{n_2}}, \xi_{t+3}^{2^{n_3}}, \dots),$$
$$E = \mathbf{F}_2[\xi_t, \xi_{t+1}, \xi_{t+2}, \dots]/(\xi_t^{2^{s+1}}, \xi_{t+1}^{2^{m_1}}, \xi_{t+2}^{2^{m_2}}, \xi_{t+3}^{2^{m_3}}, \dots),$$

where $m_i = \max(\min(n_i, s - i), 0)$. By Proposition 2.1.7, E is an elementary quotient Hopf algebra of A (and of B). Furthermore, if $\xi_v^{2^u} \neq 0$ in E, then $|\xi_v^{2^u}| \leq |\xi_t^{2^s}|$. By hypothesis on X, then, $(P_v^u)_{**}X = 0$ when $\xi_v^{2^u} \neq 0$ in E. So we apply Lemma 2.3.11 to conclude that $HE_{**}X$ has a horizontal vanishing line with intercept i_1. Since X satisfies condition (ii) of Definition 1.4.6, then $HE_{**}X$ also have a vanishing line of slope $|\xi_t^{2^s}| + 1$ (the line with this slope through the point (i_1, j_0)). We then apply Lemma 2.3.8(b) to the case $B \twoheadrightarrow E$ to conclude that $HB_{**}X$ has a vanishing line of slope $|\xi_t^{2^s}| + 1$ (and in fact the same vanishing line).

Now, the map h_{ts} acts at slope $|\xi_t^{2^s}|$, and hence acts nilpotently on $HB_{**}X$. By Lemma 2.3.6(b), then, the vanishing line for $HC_{**}X$, which has slope d, gives one for $HB_{**}X$. □

Proof of Theorem 2.3.2. By Proposition 2.1.5, it suffices to show that there is a uniform horizontal vanishing line for all of the groups $HA(n)_{**}X$. Lemma 2.3.11 and Lemma 2.3.8 together show that each $HA(n)_{**}X$ has a horizontal vanishing line, and in fact these results identify the vanishing line: $HA(n)_{i*}X = 0$ if $i > i_1$, with i_1 as in condition (iii) of Definition 1.4.6. □

2.3.2. Changes necessary when p is odd. The above proofs of Theorems 2.3.1 and 2.3.2 go through with a few changes when the prime is odd; we indicate those changes in this subsection.

We start with the same set-up as for the prime 2: we have an extension of Hopf algebras in one of the following forms:

$$\text{(E)} \quad E[\tau_n] \to B \to C,$$
$$\text{(D)} \quad D[\xi_t^{p^s}] \to B \to C.$$

If case (E) arises, then Lemma 2.3.6(b) carries over as stated (writing v_n for the homotopy element associated to the element τ_n, rather than h_{ts}).

Otherwise extension (D) arises, and we may assume that we have

- $\xi_t^{p^s} \neq 0$ in B,
- $\xi_t^{p^{s+1}} = 0$ in B,
- $\xi_j^{p^s} = 0$ in B for all $j < t$.

Lemma 1.3.10 then gives an element $b_{ts} \in HB_{2,p|\xi_t^{p^s}|}$, and hence a cofibration

$$\Sigma^{2,p|\xi_t^{p^s}|} HB \xrightarrow{b_{ts}} HB \to \widetilde{HC} \to \Sigma^{1,p|\xi_t^{p^s}|} HB, \tag{2.3.12}$$

where $\widetilde{HC}$ is the cofiber of a self-map of HC, as in Lemma 1.3.10. Hence if $HC_{**}X$ has a vanishing line of slope d, then so does $\widetilde{HC}_{**}X$. More precisely, if $HC_{ij}X = 0$ when $j < di - c$, then $\widetilde{HC}_{ij}X = 0$ when $j < di - c + \min(0, |\xi_t^{p^s}| - d)$.

Here is the odd prime analogue of Lemma 2.3.6.

Lemma 2.3.13. (a) *Given s and t as above so that we have extension* (D), *if $s \geq t$, then $b_{ts} \in HB_{**}$ is nilpotent*; *hence the self-map $b_{ts}\colon HB \to HB$ is nilpotent.*

(b) *Given extension* (D) *and an object* X*, consider the cofibration*

$$\Sigma^{2,p|\xi_t^{p^s}|} HB \wedge X \xrightarrow{b_{ts} \wedge 1_X} HB \wedge X \to \widetilde{HC} \wedge X.$$

If $b_{ts} \wedge 1_X$ *acts nilpotently on* $HB_{**}X$*, and if* $\widetilde{HC}_{**}X$ *has a vanishing line of slope* d*, then* $HB \wedge X$ *has a vanishing line of slope* d*. The difference in intercepts depends only on* d *and* $|\xi_t^{p^s}|$*—it is independent of* X*.*

Proof. For part (a), see [**MW81**, Proposition 4.1].
Part (b) is proved just as in the $p = 2$ case. $\square$

The odd prime version of Lemma 2.3.8 is as follows.

Lemma 2.3.14. *Fix an object* X *satisfying condition* (i) *of Definition* 1.4.6*.*

(a) *Given an extension*

$$D[\xi_t^{p^s}] \to B \to C,$$

if $\frac{p|\xi_t^{2^s}|}{2} \geq d$ *and if* $HC_{**}X$ *has a vanishing line of slope* d*, then* $HB_{**}X$ *has a vanishing line of slope* d*. In fact,* $HB_{**}X$ *has the same vanishing line as* $\widetilde{HC}_{**}X$*.*

(b) *Given an extension*

$$E[\tau_n] \to B \to C,$$

if $|\tau_n| \geq d$ *and if* $HC_{**}X$ *has a vanishing line of slope* d*, then* $HB_{**}X$ *has a vanishing line of slope* d*. In fact,* $HB_{**}X$ *has the same vanishing line as* $HC_{**}X$*.*

(c) *Given finite-dimensional quotients* $B \twoheadrightarrow C$ *of* A*, if* $HC_{**}X$ *has a vanishing line of slope* d *and if the map* $B \twoheadrightarrow C$ *is an isomorphism in odd dimensions less than* d *and in even dimensions less than* $\frac{2d}{p}$*, then* $HB_{**}X$ *has a vanishing line of slope* d*. Furthermore, the difference in intercept between the two vanishing lines is independent of* X*.*

Proof. Part (a) is proved just as is Lemma 2.3.8(a), but based on the cofibration (2.3.12). Part (b) is the same as Lemma 2.3.8(a) (except for the characteristic of the ground field, which is not relevant). Part (c) is proved, as in Lemma 2.3.8(b), by induction and parts (a) and (b). Since in (a), the vanishing line for $\widetilde{HC}_{**}X$ may have a different intercept than that for $HC_{**}X$, the intercept for $HB_{**}X$ will change as the induction proceeds, but it will change by amounts dependent only on d. $\square$

Lemma 2.3.10 holds as stated, regardless of the prime involved.
Here is the analogue of Lemma 2.3.11.

Lemma 2.3.15. *Let* X *be an object satisfying conditions* (ii) *and* (iii) *of Definition* 1.4.6*. Suppose that* E *is a finite-dimensional elementary quotient Hopf algebra of* A*. If* $(P_t^s)_{**}X = 0$ *whenever* $\xi_t^{2^s} \neq 0$ *in* E*, and* $(Q_n)_{**}X = 0$ *whenever* $\tau_n \neq 0$ *in* E*, then* $HE_{**}X$ *has a horizontal vanishing line. In fact, given* i_1 *as in condition* (iii)*, then* $HE_{i*}X = 0$ *if* $i > i_1$*.*

The proof is the same as that for Lemma 2.3.11, using cofibrations of the form (2.3.12) repeatedly. We also need to point out that the slopes $\frac{p|\xi_t^{p^s}|}{2}$ and $|\tau_n|$ are all distinct. (Note also that in this case, we may choose d as large as we like, so that a horizontal vanishing line for $HC_{**}X$ induces the same one for $\widetilde{HC}_{**}X$.)

Finally, given these modified tools, the proofs of the main theorems go through essentially unchanged.

2.4. Self-maps via vanishing lines

In this section we use vanishing lines to construct self-maps of finite spectra. For each finite X, we construct one self-map; we give many others in Theorem 4.1.3. We make heavy use of Notation 2.2.4 in this section.

Our approach for this section is based on some recent work in ordinary stable homotopy theory, as described in work of Ravenel [**Rav86**] and Hopkins and Smith [**HSb**]. For now, we view the objects P^s_t and Q_n (i.e., the $Z(n)$'s) as the analogues of Morava K-theories. This is not a perfect analogy, because the Morava K-theories detect nilpotence, while the $Z(n)$'s do not.

The following definition is based on the definition of "v_n-map" in ordinary stable homotopy theory; see [**HSb**] and [**Rav92**]. We will investigate and generalize this definition in Section 4.2.

NOTATION 2.4.1. Fix a slope n. The ring $z(n)_{**}$ has a polynomial subalgebra; we call the polynomial generator u_n. (In the notation of Remark 2.1.3 and Proposition 2.2.2, u_n is one of h_{ts}, b_{ts}, or v_n.)

DEFINITION 2.4.2. Fix a spectrum X and a slope n.

(a) A self-map $f \in [X, X]_{**}$ is a *u_n-map* if for some j,
$$Z(n)_{**} f = u_n^j \wedge 1_X.$$
(b) We say that X is of *type n* if $Z(d)_{**}X = 0$ for $d < n$, and $Z(n)_{**}X \neq 0$.

Notice that if $Z(n)_{**}X = 0$, then the zero map of X is a u_n-map.

We point out that in the ordinary stable homotopy category, Ravenel showed in [**Rav84**, 2.11] that for any finite spectrum X, $K(n)_* X \neq 0 \Rightarrow K(n+1)_* X \neq 0$. The analogous statement here, with $Z(n)_{**}$ rather than $K(n)_*$, does not hold. See Proposition 4.8.1 (and also [**Pal96b**, Prop. 3.10 and Thm. A.1]) for the correct statement when $p = 2$ (and for a guess in the odd prime case). We do know that if X is a nonzero finite spectrum, then by Theorem 2.3.2 and Corollary 4.5.7, for some n we have $Z(n)_{**}X \neq 0$. (One could also use the Atiyah-Hirzebruch spectral sequence to show this.)

The following first appeared (for modules) in [**Pal92**]. It is a slight generalization of a result of Hopkins and Smith [**HSb**]. See Theorems 3.1.2 and 4.1.3 for stronger results when $p = 2$.

THEOREM 2.4.3. *Fix a finite spectrum X of type n. Then for some k, there is a non-nilpotent u_n-map*
$$v\colon \Sigma^{k,kn} X \to X.$$

To prove this, we need a "relative vanishing line" result; this is a generalization of a standard result—see [**Rav86**, 3.4.9], for instance.

LEMMA 2.4.4. *Fix a slope n. Suppose that X is a CL spectrum, and suppose that X is of type at least n (hence $\pi_{**}X$ has a vanishing line of slope n). Given $m \geq 0$, let M be the number below which degree $h\colon A \to A(m)$ is an isomorphism. Then the Hurewicz map $h\colon \pi_{**}X \to HA(m)_{**}X$ is an isomorphism above a line of slope n: for some c independent of m, h is an isomorphism on π_{ij} when $j < ni + M - c$.*

PROOF. This is a consequence of the form of the intercept in the vanishing line theorem—see Remark 2.3.4. Let W denote the fiber of the map $S^0 \to HA(m)$. Since the kernel of the map $A \to A(m)$ is zero below degree M, if we choose numbers i_0, i_1, and j_0 so that X satisfies Definition 1.4.6, then $W \wedge X$ satisfies the conditions with the numbers i_0, i_1, and $j_0 + M$. $\square$

We have the following result based on work of Lin [**Lin**] and Wilkerson [**Wil81**]; this is a corollary of Theorem 3.3.5.

PROPOSITION 2.4.5. *Fix m and consider the quotient Hopf algebra $A(m)$ of A.*

(a) *Let $p = 2$. Fix integers $s < t$ with $\xi_t^{2^s}$ nonzero in $A(m)$ (i.e., with $s + t \leq m + 1$). Then the restriction map*

$$\mathrm{Ext}^{**}_{A(m)}(\mathbf{F}_2, \mathbf{F}_2) \to \mathrm{Ext}^{**}_{\mathbf{F}_2[\xi_t]/(\xi_t^{2^{s+1}})}(\mathbf{F}_2, \mathbf{F}_2) \cong \mathbf{F}_2[h_{t0}, h_{t1}, \ldots, h_{ts}]$$

is surjective modulo nilpotence (i.e., the algebra cokernel consists entirely of nilpotent elements). Hence for some $i = i(m)$, there is a non-nilpotent element

$$w \in \mathrm{Ext}^{i, i|\xi_t^{2^s}|}_{A(m)}(\mathbf{F}_2, \mathbf{F}_2) = HA(m)_{i, i|\xi_t^{2^s}|}$$

which restricts to h_{ts}^i.

(b) *Let p be odd. Fix integers $s < t$ with $\xi_t^{p^s}$ nonzero in $A(m)$ (i.e., with $s + t \leq m$). Then the restriction map*

$$\begin{array}{ccc} \mathrm{Ext}^{**}_{A(m)}(\mathbf{F}_p, \mathbf{F}_p) & \longrightarrow & \mathrm{Ext}^{**}_{\mathbf{F}_p[\xi_t]/(\xi_t^{p^{s+1}})}(\mathbf{F}_p, \mathbf{F}_p) \\ & & \| \\ & & \mathbf{F}_p[b_{t0}, \ldots, b_{ts}] \otimes E[h_{t0}, \ldots, h_{ts}] \end{array}$$

is surjective modulo nilpotence. Hence for some $j = j(m)$, there is a non-nilpotent element

$$w \in \mathrm{Ext}^{2j, jp|\xi_t^{p^s}|}_{A(m)}(\mathbf{F}_p, \mathbf{F}_p) = HA(m)_{2j, jp|\xi_t^{p^s}|}$$

which restricts to b_{ts}^j.

(c) *Let p be odd. Fix an integer t with τ_t nonzero in $A(m)$ (i.e., with $t \leq m$). Then the restriction map*

$$\mathrm{Ext}^{**}_{A(m)}(\mathbf{F}_p, \mathbf{F}_p) \to \mathrm{Ext}^{**}_{E[\tau_t]}(\mathbf{F}_p, \mathbf{F}_p) \cong \mathbf{F}_p[v_t]$$

is surjective mod nilpotence. Hence for some $k = k(m)$, there is a non-nilpotent element

$$w \in \mathrm{Ext}^{k, k|\tau_t|}_{A(m)}(\mathbf{F}_p, \mathbf{F}_p) = HA(m)_{k, k|\tau_t|}$$

which restricts to v_t^k.

We need the following lemma. Recall from Notation 1.3.9 that $D[y]$ is the Hopf algebra $\mathbf{F}_p[y]/(y^p)$ with y primitive.

LEMMA 2.4.6. *Fix a slope n, and choose an integer m large enough that y_n is nonzero in $A(m)$ (and hence so that $D[y_n]$ or $E[y_n]$ is a quotient coalgebra of $A(m)$ over which $A(m)$ is injective).*

(a) *Then the map $HA(m)_{**} \to z(n)_{**}$ is an algebra map, and some power of the polynomial generator u_n of $z(n)_{**}$ is in the image.*

(b) *For any object X, the following diagram commutes.*

$$\begin{array}{ccc} [X, HA(m) \wedge X]_{**} & \longrightarrow & [X, z(n) \wedge X]_{**} \\ {\scriptstyle -\wedge X}\uparrow & & \uparrow{\scriptstyle -\wedge X} \\ HA(m)_{**} & \longrightarrow & z(n)_{**} \end{array}$$

(c) *For any object X with $Z(n)_{**}X \neq 0$, the map $z(n)_{**} \to [X, z(n) \wedge X]_{**}$ is an injection.*

PROOF. Part (a) follows from Proposition 1.3.2(b) and Proposition 2.4.5.

Part (b) follows from the fact that $[X, z(n) \wedge X]_{**}$ is isomorphic to cochain homotopy classes of $D[y_n]$-linear (resp., $E[y_n]$-linear) self-maps of X, and the horizontal maps are just restriction.

For part (c), we merely note that if $Z(n)_{**}X \neq 0$, then the identity map on X is not cochain homotopic to zero over $D[y_n]$ (resp., over $E[y_n]$). $\square$

PROOF OF THEOREM 2.4.3. By Lemma 2.4.4, if we choose m large enough, then the map $[X, X]_{**} \to [X, HA(m) \wedge X]_{**}$ is an isomorphism in the bidegrees (k, kn), for all integers k. Now we apply Lemma 2.4.6 to find a non-nilpotent element $w \in HA(m)_{k,kn}$ (for some k) which maps nontrivially to $z(n)_{**}$ and to $[X, z(n) \wedge X]_{**}$. The lift of $w \wedge 1_X$ to $[X, X]_{k,kn}$ clearly has the desired properties. $\square$

We will need the following lemma in Section 4.7.

LEMMA 2.4.7. *Fix a finite type n spectrum X. Then for some k, the u_n-map $v \in [X, X]_{**}$ constructed in Theorem 2.4.3 is central in a band parallel to the vanishing line. This band includes the origin.*

PROOF. Consider the element $w \in HA(m)_{k,kn}$, as given by Proposition 2.4.5. Since $HA(m)_{**}$ is a commutative ring, w maps to a central element in $[X, HA(m) \wedge X]_{**}$. By Lemma 2.4.4, $[X, X]_{**} \to [X, HA(m) \wedge X]_{**}$ is an isomorphism in a band parallel to the vanishing line; by choice of m, that band includes the origin. Hence the lift of w to $[X, X]_{**}$ is central in that band. $\square$

2.5. Further discussion

As mentioned in Subsection 2.1.1, one of the main gaps in this theory, when p is odd, is the lack of a classification of the quasi-elementary quotient Hopf algebras of A. See Appendix B.3 for a discussion of conjectures and results related to this issue.

We note that the vanishing line theorem 2.3.1 has been used many times in many papers; it provides a convenient way to get valuable information about the Adams E_2-term. Combined with newer results, such as Theorem A.2.6, it is even more powerful.

Theorem 2.4.3, the result that ensures the existence of a non-nilpotent self-map of any finite object, also has been used in topological applications. For example, Hopkins and Smith used it to prove the periodicity theorem [**HSb**]: they had constructed a particular spectrum X, and they used the theorem to find a v_n-map of X at the E_2-term of the Adams spectral sequence. Later, Theorem 2.4.3 was used by Sadofsky and the author in [**PS94**] to give a new proof of the periodicity theorem: we used it not only to produce a v_n-map, but also to construct the spectrum X in question. This was done by taking iterated cofibers: when $p = 2$, the cofiber of $u_1\colon S^0 \to S^0$ has a u_3-map (3 is the next slope after 1); the cofiber of this map has a u_6-map, etc. This was done with modules over the Steenrod algebra, and then realized at the spectrum level in the ordinary stable homotopy category.

See the results of Section 4.7 for a similar application of Theorem 2.4.3, but in $\mathsf{Stable}(A)$.

CHAPTER 3

Quillen stratification and nilpotence

The vanishing line theorem of Section 2.3 is a nilpotence theorem of a sort: if X is a finite spectrum, then any self-map of X with slope smaller than that of the vanishing line of X must be nilpotent (because some power of it will lie above the vanishing line). We used the vanishing line theorem in Section 2.4 to construct a non-nilpotent self-map of any finite spectrum.

In this chapter we give some related, but stronger, results; we work mostly at the prime 2. We let Q denote the category of quasi-elementary quotients of A, with morphisms given by quotient maps. We can assemble the individual Hurewicz (restriction) maps $S^0 \to HE$ into

$$S^0 \to \varprojlim{}_{\mathsf{Q}} HE.$$

Since the maximal quasi-elementary quotients are conormal, there is an action of A on $\varprojlim{}_{\mathsf{Q}} HE_{**}$; we prove that

$$\pi_{**}S^0 \to (\varprojlim{}_{\mathsf{Q}} HE_{**})^A$$

is an F-isomorphism—its kernel consists of nilpotent elements, and some pth power of every element in the target is actually in the image. One can view this as an analogue of the Quillen stratification theorem [**Qui71**, 6.2], which identifies $\pi_{**}S^0$ up to F-isomorphism in the category $\mathsf{Stable}(kG^*)$, for G a finite group and k an algebraically closed field of characteristic p.

Similarly, suppose that we write D for the following quotient of A:

$$D = A/(\xi_1^p, \xi_2^{p^2}, \xi_3^{p^3}, \dots, \xi_n^{p^n}, \dots).$$

Then D is conormal, and the Hurewicz map induces an F-isomorphism

$$\pi_{**}S^0 \to HD_{**}^A.$$

These F-isomorphism theorems first appeared in [**Pal**].

We have weaker results with nontrivial coefficients: if R is a ring spectrum, then an element $\alpha \in \pi_{**}R$ is nilpotent if its image under the Hurewicz map in $HD_{**}R$ is zero, or if its image in $HE_{**}R$ is zero for all quasi-elementary quotients E of A. (These nilpotence theorems are improvements on results of the author in [**Pal96a**].) See also Theorem 4.2.2 for a statement about lifting invariants from $HD_{**}R$ to $\pi_{**}R$.

We state these results more precisely in Section 3.1. In Section 3.2, we prove the two theorems—nilpotence and F-isomorphism—involving D; we then use these these to prove the analogous results for quasi-elementary Hopf algebras in Section 3.3. We end the chapter with Sections 3.4 and 3.5; in the first of these, we indicate why our proofs only work at the prime 2; this leads us to a conjecture about the nilpotence of particular classes in Ext over quotients of the dual of the

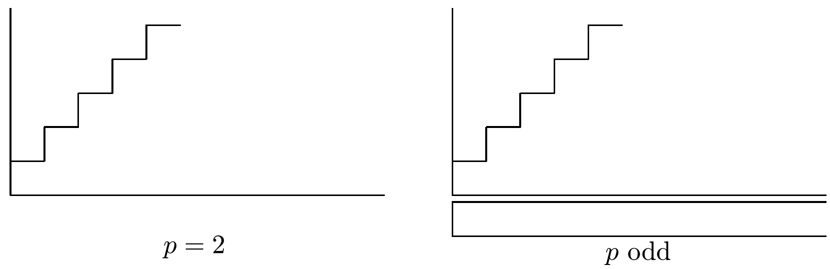


FIGURE 3.1.A. Profile function for D.

odd primary Steenrod algebra. In the second of these, we discuss a few possible generalizations of the main results of this chapter.

Except for Sections 3.4 and 3.5, we work at the prime 2 in this chapter.

3.1. Statements of theorems

Let $p = 2$.

In this section we present two F-isomorphism results and two nilpotence results.

Let D be the following quotient Hopf algebra of A:

$$D = A/(\xi_1^p, \xi_2^{p^2}, \xi_3^{p^3}, \dots, \xi_n^{p^n}, \dots).$$

See Figure 3.1.A. Note that D and A have the same quasi-elementary quotients—i.e., every quasi-elementary quotient map $A \to E$ factors as $A \to D \to E$. As a result, it turns out that HD plays a similar role in $\mathsf{Stable}(A)$ to that of BP in the ordinary stable homotopy category, at least as far as detection of nilpotence goes. We write $\eta_D\colon S^0 = HA \to HD$ for the unit map of the ring spectrum HD, and similarly for $\eta_E\colon S^0 \to HE$ for E quasi-elementary.

3.1.1. Quillen stratification. We state our results describing $\pi_{**}S^0$ up to F-isomorphism. The results in this subsection first appeared in [**Pal**].

DEFINITION 3.1.1. (a) Given a Hopf algebra B and a B-comodule M, we define the B-*invariants* of M to be

$$M^B = \operatorname{Hom}_B(\mathbf{F}_p, M) \subseteq M.$$

(This is the same as the *primitives* PM of M: if $\psi\colon M \to B \otimes M$ is the coaction map, then we let $PM = \{m \in M \mid \psi(m) = 1 \otimes m\}$.)

(b) Following Quillen [**Qui71**], if $\varphi\colon R \to S$ is a map of graded commutative $\mathbf{F}_p$-algebras, we say that φ is an *F-isomorphism* if it satisfies the following properties.

(i) Every $x \in \ker\varphi$ is nilpotent. (Hence $\sqrt{\ker\varphi}$ is the nilradical of R.)

(ii) For any element $y \in S$, there is an integer n so that $y^{p^n} \in \operatorname{im}\varphi$.

In (ii), if one can choose the same n for every y, then we say that φ is a *uniform* F-isomorphism.

Here is our first result. Theorem 2.1.1 tells us that D is conormal, so by Remark 1.3.8, there is a coaction of $A \,\square_D \mathbf{F}_2$ on HD_{**}.

THEOREM 3.1.2 (Quillen stratification, I). *The Hurewicz map $\pi_{**}S^0 \to HD_{**}$ factors through*

$$\varphi\colon \pi_{**}S^0 \to HD_{**}^{A \,\square_D \mathbf{F}_2},$$

and φ is an F-isomorphism.

Here is our second result, an analogue of Quillen's theorem [**Qui71**, 6.2], which identifies group cohomology up to F-isomorphism. We have defined in Definition 2.1.10 (see also Proposition 2.1.12) the notion of a quasi-elementary quotient Hopf algebra of A; we let Q denote the category of quasi-elementary quotients of A, with morphisms given by quotient maps. In Section 3.3, we construct a coaction of A on $\varprojlim_{\mathsf{Q}} HE_{**}$.

THEOREM 3.1.3 (Quillen stratification, II). *The map $\pi_{**}S^0 \to \varprojlim_{\mathsf{Q}} HE_{**}$ factors through*

$$\gamma\colon \pi_{**}S^0 \to \left(\varprojlim{}_{\mathsf{Q}} HE_{**}\right)^A,$$

and γ is an F-isomorphism.

We prove Theorem 3.1.2 in Section 3.2; we show that Theorem 3.1.3 follows from Theorem 3.1.2 in Section 3.3.

REMARK 3.1.4. (a) One can view Theorem 3.1.3, and hence Theorem 3.1.2, as giving an analogue of the Quillen stratification theorem [**Qui71**, 6.2]: given a finite group G and an algebraically closed field k of characteristic p, we let A be the category whose objects are the elementary abelian p-subgroups of G, and whose morphisms are generated by inclusions and compositions. Then the natural map

$$H^*(G,k) \to \varprojlim{}_{E\in\mathsf{A}} H^*(E,k)$$

is an F-isomorphism. The role of the conjugation maps in the category A is played, in our results, by the process of taking invariants.

(b) We have an explicit formula for the algebra $\varprojlim HE_{**}$ (Proposition 4.4.1), as well as for the coaction of A on $\varprojlim HE_{**}$ (Proposition 4.4.4). One can use this to predict the presence of large families of non-nilpotent elements in $\pi_{**}S^0$. See Section 4.4 for details.

(c) We do not expect the F-isomorphisms in these results to be uniform, although we do not have much evidence either way.

(d) One can also view Theorem 3.1.2 as an analogue of Nishida's theorem [**Nis73**]—in the ordinary stable homotopy category, π_*S^0 is isomorphic to $\mathbf{Z}$, mod nilpotence; however, while Nishida's theorem does not immediately lead to guesses as to further structure in the ordinary stable homotopy category, our analogue does. For instance, see Section 4.6 for a suggested classification of thick subcategories of finite spectra in $\mathsf{Stable}(A)$.

(e) Of course, if A' is any quotient Hopf algebra of A, then similar results hold: for instance, the map $HA'_{**} \to (HD'_{**})^{A'}$ is an F-isomorphism. The proofs for Theorems 3.1.3 and 3.1.2 carry over easily.

3.1.2. Nilpotence. We move on to our nilpotence theorems. The first of these is based on the nilpotence theorem of Devinatz, Hopkins, and Smith [**DHS88**]: the ring spectrum BP detects nilpotence.

THEOREM 3.1.5 (Nilpotence theorem, I). *The ring spectrum HD detects nilpotence*:

(a) *Fix a ring spectrum R and $\alpha \in \pi_{**}R$. Then α is nilpotent if and only if $\eta_D \wedge \alpha \in HD_{**}R$ is nilpotent.*
(b) *Fix a finite spectrum Y and a self-map $f\colon Y \to Y$. Then $f \in [Y,Y]_{**}$ is nilpotent if and only if $\eta_D \wedge f \in [Y, HD \wedge Y]_{**}$ is nilpotent.*
(c) *Fix a finite spectrum F, an arbitrary spectrum X, and a map $f\colon F \to X$. Then $f \in [F,X]_{**}$ is smash-nilpotent if $\eta_D \wedge f \in [F, HD \wedge X]_{**}$ is zero.*

The corresponding result for modules is [**Pal96a**, Theorems 3.1 and 4.2]. In [**Pal96a**] we prove this in enough generality so that the proof goes through here without difficulty. We also give a (slightly different) proof in Section 3.2.

The second nilpotence theorem is, more or less, an analogue of the $K(n)$ nilpotence theorem in [**HSb**]. The following appeared for bounded below modules in [**Pal96a**, Theorems 1.1 and 4.3].

THEOREM 3.1.6 (Nilpotence theorem, II). *The collection of ring spectra*

$$\{HE \mid E \text{ quasi-elementary}\}$$

detects nilpotence:

(a) *Fix a ring spectrum R and $\alpha \in \pi_{**}R$. Then α is nilpotent if and only if $\eta_E \wedge \alpha \in HE_{**}R$ is nilpotent for all quasi-elementary quotients E of A.*
(b) *Fix a finite spectrum Y and a self-map $f\colon Y \to Y$. Then f is nilpotent if and only if $\eta_E \wedge f$ is nilpotent for all quasi-elementary quotients E of A.*
(c) *Fix a finite spectrum F, an arbitrary spectrum X, and a map $f\colon F \to X$. Then f is smash-nilpotent if $\eta_E \wedge f\colon F \to HE \wedge X$ is zero, for all quasi-elementary quotients E of A.*

The proof of the analogous result in [**Pal96a**] does not apply to nonconnective situations, so we give a new proof below.

REMARK 3.1.7. (a) Note that $\mathbf{F}_2$ is a quasi-elementary quotient Hopf algebra of A, so $H\mathbf{F}_2$ is included as one of the detecting spectra in Theorem 3.1.6 (as compared to [**HSb**], when $H\mathbf{F}_2$ is included for parts (a) and (c), but not for (b)).
(b) Fix a quotient Hopf algebra A' of A, and let D' be the quotient of A' induced by $A \twoheadrightarrow D$. One can generalize Theorem 3.1.5 in an obvious sort of way: if Y is a finite spectrum and $f\colon Y \to HA' \wedge Y$ is a "self-map," then

$$\begin{aligned} & f \in [Y, HA' \wedge Y]_{**} \text{ is nilpotent} \\ & \quad \Leftrightarrow \eta_{D'} \wedge f \in [Y, HD' \wedge Y]_{**} \text{ is nilpotent} \\ & \quad \Leftrightarrow \eta_{E'} \wedge f \in [Y, HE' \wedge Y]_{**} \text{ is nilpotent for all } E'. \end{aligned}$$

(Here E' ranges over all quasi-elementary quotients of A'.) There are similar versions of the ring spectrum and smash-nilpotence results. The proofs are straightforward generalizations of the ones below, so we omit them.

(c) The quotient Hopf algebra D is "best possible," in the sense that if B is quotient of A which does not map onto D, then there are non-nilpotent elements in $\pi_{**}S^0$ which are in the kernel of $\pi_{**}S^0 \to HB_{**}$. One can see this from Theorem 3.1.2.

(d) Theorem 3.1.5 identifies a single ring spectrum HD which detects nilpotence, but we do not know its coefficient ring completely. See Propositions 3.3.4 and 4.4.1 for partial information. On the other hand, we have computed the coefficient rings of HE for E quasi-elementary in Proposition 2.1.9—they are polynomial rings.

(e) We also mention that one does not need to use all of the quasi-elementary quotient Hopf algebras of A to detect nilpotence; for instance, one can use only the maximal quasi-elementary quotients, or only the finite-dimensional ones.

(f) In fact, one can see from the proof of Theorem 3.1.6 that one only needs the following spectra to detect nilpotence:

$$\{h_{m+1,j}^{-1} H(E(m)/(\xi_{m+1}^{2^{j+1}})) \mid m \geq 0,\, 0 \leq j \leq m\}.$$

3.2. Nilpotence and F-isomorphism via the Hopf algebra D

In this section we show that the spectrum HD detects a lot of information: we prove that it detects $\pi_{**}S^0$ modulo nilpotent elements (Theorem 3.1.2), and that it detects non-nilpotent elements of $\pi_{**}R$ for any ring spectrum R (Theorem 3.1.5).

The Hopf algebra D is a conormal quotient of A, by Theorem 2.1.1. So there is a Hopf algebra extension

$$A \,\square_D \mathbf{F}_p \to A \to D,$$

and a spectral sequence (as in Section 1.5) with

$$E_2^{s,t} = \operatorname{Ext}^s_{A\,\square_D \mathbf{F}_p}(\mathbf{F}_p, \operatorname{Ext}^t_D(\mathbf{F}_p, \mathbf{F}_p)) \Rightarrow \operatorname{Ext}^{s+t}_A(\mathbf{F}_p, \mathbf{F}_p).$$

The restriction (Hurewicz) map factors through the edge homomorphism

$$\operatorname{Ext}^t_A(\mathbf{F}_p, \mathbf{F}_p) \to E_2^{0,t}.$$

This gives the factorization of the Hurewicz map $h\colon\ \pi_{**}S^0 \to HD_{**}$ as advertised in Theorem 3.1.2:

$$\varphi\colon\ \pi_{**}S^0 \to HD_{**}^{A\,\square_D \mathbf{F}_2}.$$

It remains to prove Theorem 3.1.5—the spectrum HD detects nilpotence—and to verify conditions (i) and (ii) of Definition 3.1.1(b)—every element of $\ker \varphi$ is nilpotent, and for every element $y \in HD_{**}^{A\,\square_D \mathbf{F}_p}$, there is an integer m so that $y^{2^m} \in \operatorname{im} \varphi$.

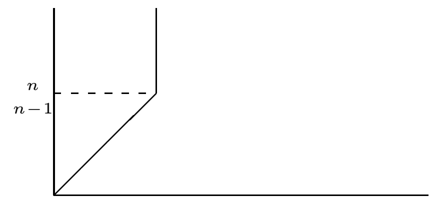
Note that condition (i) follows from Theorem 3.1.5(a) with $R = S^0$. Also, the proof of Theorem 3.1.5 and the verification of (ii) are quite similar, so first we lay the groundwork for both.

For each integer $n \geq 1$, we let

$$D(n) = A/(\xi_1^2, \xi_2^4, \ldots, \xi_n^{2^n}).$$

We let $D(0) = A$. See Figure 3.2.B. Each $D(n)$ is a conormal quotient Hopf algebra of A, and we have a diagram of Hopf algebra surjections:

$$A = D(0) \twoheadrightarrow D(1) \twoheadrightarrow D(2) \twoheadrightarrow \ldots.$$



$p = 2$

FIGURE 3.2.B. Profile function for $D(n)$. As in Figure 2.1.B, the diagonal line is an abbreviation for a staircase shape.

D is the colimit of this diagram.

First we discuss how to lift information from HD_{**} to $HD(n)_{**}$ for some n. We have the following lemma.

LEMMA 3.2.1. (a) *We have* $HD_{**} = \varinjlim HD(n)_{**}$.
(b) *We have* $HD_{**}^{A \,\square_D \mathbf{F}_p} = \varinjlim (HD(n)_{**}^{A \,\square_{D(n)} \mathbf{F}_p})$.

PROOF. Since homotopy commutes with direct limits, part (a) is clear.

Part (b): The coaction of A on $HD(n)_{**}$ is defined in Remark 1.3.8. Note that each restriction map $HD(n)_{**} \to HD(n+1)_{**}$ is an A-comodule map (and in fact, a map of comodule algebras.) We take injective resolutions of these comodules, and apply $\pi_{0,*}(-)$ (i.e., $\mathrm{Hom}_A^*(\mathbf{F}_p, -)$); since homotopy commutes with colimits, we have

$$\mathrm{Hom}_A(\mathbf{F}_p, HD_{**}) = \mathrm{Hom}_A(\mathbf{F}_p, \varinjlim HD(n)_{**}) = \varinjlim \mathrm{Hom}_A(\mathbf{F}_p, HD(n)_{**}).$$

Now, $\mathrm{Hom}_A(\mathbf{F}_p, HD(n)_{**}) = HD(n)_{**}^A = HD(n)_{**}^{A \,\square_{D(n)} \mathbf{F}_p}$, by conormality, so we have the desired result. $\square$

Now we discuss how to take information about $HD(n)_{**}$ and get information about $HD(n-1)_{**}$. (We want to know about $\pi_{**}S^0 = HD(0)_{**}$, so we will eventually want to use downward induction on n.)

Not only is each $D(n)$ a conormal quotient of A, it is also a conormal quotient of $D(n-1)$. The Hopf algebra kernel of the quotient map is easy to identify: we have an extension of Hopf algebras

$$\mathbf{F}_2[\xi_n^{2^n}] \to D(n-1) \to D(n),$$

where $\xi_n^{2^n}$ is primitive in the kernel. So given any $D(n-1)$-comodule M, there is a change-of-rings spectral sequence (see (1.5.2)) with

$$E_2^{s,t,u}(M) = \mathrm{Ext}^{s,u}_{\mathbf{F}_2[\xi_n^{2^n}]}(\mathbf{F}_2, \mathrm{Ext}^{t,*}_{D(n)}(\mathbf{F}_2, M)) \Rightarrow \mathrm{Ext}^{s+t,u}_{D(n-1)}(\mathbf{F}_2, M).$$

By Proposition 1.5.3, in the category $\mathsf{Stable}(D(n-1))$ of cochain complexes of injective left $D(n-1)$-comodules, this spectral sequence is the same (up to regrading) as the $HD(n)$-based Adams spectral sequence. Because of this, for parts of the proof we will work in the category $\mathsf{Stable}(D(n-1))$. We also use the grading on the spectral sequence as given here; we do *not* use the Adams spectral sequence grading from Section 1.5. Throughout, we abuse notation somewhat, writing $\mathrm{Ext}^{**}_{D(n)}(\mathbf{F}_2, X)$ for $\pi_{**}(HD \wedge X)$.

We need to establish an important property of this spectral sequence, that it has a nice vanishing plane at some E_r-term. We write $E_r^{s,t,u}(X)$ for the spectral sequence converging to $\mathrm{Ext}_{D(n-1)}^{s+t,u}(\mathbf{F}_2, X)$.

PROPOSITION 3.2.2. *Fix a connective spectrum X and an integer m. For some r and some c, we have $E_r^{s,t,u}(X) = 0$ when $ms + t - u > c$.*

We prove this using Theorem 1.5.5, which says that vanishing planes in Adams spectral sequences are generic (Definition 1.4.7). (And again, this is an Adams spectral sequence, as long as we work in the category $\mathsf{Stable}(D(n-1))$.)

We fix a connective spectrum X.

LEMMA 3.2.3. *For each integer m, there is a finite $D(n-1)$-comodule W so that*

$$E_2^{s,t,u}(W \wedge X) = 0$$

when $ms + t - u > c$, where c depends only on the connectivity of X.

(In Adams spectral sequence grading, this vanishing plane is of the form

$$E_2^{p,q,v}(W \wedge X) = 0$$

when $mp - q > 0$, i.e., when

$$p \geq \frac{-1}{m-1}(p+v) + \frac{1}{m-1}(q+v).$$

In particular, the coefficients $\frac{-1}{m-1}$ and $\frac{1}{m-1}$) satisfies the hypotheses of Theorem 1.5.5.)

PROOF. Choose an integer $j \geq n$ so that $|\xi_n^{2^j}| = 2^j(2^n - 1) > m$, and let $W = W_j = \mathbf{F}_2[\xi_n^{2^n}]/(\xi_n^{2^j})$, with the apparent $D(n-1) \,\square_{D(n)} \mathbf{F}_p$-comodule structure. Then W_j is a trivial $D(n)$-comodule (by definition), and as coalgebras we have

$$\mathbf{F}_2[\xi_n^{2^n}] \cong W_j \otimes \mathbf{F}_2[\xi_n^{2^j}].$$

Hence the E_2-term of the spectral sequence for $W_j \wedge X$ looks like

$$\begin{aligned} E_2^{s,t,u} &\cong \mathrm{Ext}^{s,u}_{\mathbf{F}_2[\xi_n^{2^n}]}(\mathbf{F}_2, \mathrm{Ext}^{t,*}_{D(n)}(\mathbf{F}_2, W_j \wedge X)) \\ &\cong \mathrm{Ext}^{s,u}_{\mathbf{F}_2[\xi_n^{2^n}]}(\mathbf{F}_2, W_j \otimes \mathrm{Ext}^{t,*}_{D(n)}(\mathbf{F}_2, X)) \\ &\cong \mathrm{Ext}^{s,u}_{\mathbf{F}_2[\xi_n^{2^j}]}(\mathbf{F}_2, \mathrm{Ext}^{t,*}_{D(n)}(\mathbf{F}_2, X)). \end{aligned}$$

The Hopf algebra $\mathbf{F}_2[\xi_n^{2^j}]$ is $|\xi_n^{2^j}|$-connected, so if L is a comodule which is zero below degree t, then

$$\mathrm{Ext}^{s,u}_{\mathbf{F}_2[\xi_n^{2^j}]}(\mathbf{F}_2, L) = 0$$

when $u < |\xi_n^{2^j}|s + t$. Now note that $\mathrm{Ext}^{t,*}_{D(n)}(\mathbf{F}_2, X)$ is zero below degree $t + c$ for some c dependent only on the connectivity of X. $\square$

Next we need to show that X is in $\mathrm{thick}(W_j \wedge X)$. We start with the following lemma, which describes how to build W_j out of $\mathbf{F}_2$ in a nice way.

LEMMA 3.2.4. *For $j \geq n$, let $W_j = \mathbf{F}_2[\xi_n^{2^n}]/(\xi_n^{2^j})$. Then there is a short exact sequence of $D(n-1) \,\square_{D(n)} \mathbf{F}_2$-comodules*

$$0 \to W_j \to W_{j+1} \to \Sigma^{|\xi_n^{2^j}|} W_j \to 0.$$

The connecting homomorphism in $\mathrm{Ext}^{**}_{D(n-1)}(\mathbf{F}_2, -)$ is multiplication by

$$h_{nj} = [\xi_n^{2^j}] \in \mathrm{Ext}^{**}_{D(n-1)}(\mathbf{F}_2, \mathbf{F}_2).$$

Replacing W_j by its injective resolution gives a cofibration sequence

$$\Sigma^{1,|\xi_n^{2^j}|} W_j \xrightarrow{h_{nj}} W_j \to W_{j+1} \to W_j.$$

(We are abusing notation a bit here, by writing W_j for both the module and its injective resolution. We will continute this practice for the remainder of this section.)

PROOF. This follows from Lemma 1.3.10. □

Suppose that $j \geq n$; then Theorem B.2.1(a) tells us that the element $h_{nj} \in HD(n-1)_{**}$ is nilpotent. Hence we have the following:

LEMMA 3.2.5. *If $j \geq n$, then $X \in \mathrm{thick}(W_j \wedge X)$.*

PROOF. We show by downward induction on i that for $j \geq i \geq n$, $W_i \wedge X$ is in $\mathrm{thick}(W_j \wedge X)$; the lemma is proved when $i = n$, since $W_n = S^0$.

The induction starts (trivially) with $i = j$. Suppose that $i < j$. By the cofibration sequence in Lemma 3.2.4, together with the nilpotence of h_{ni}, we see that $W_i \wedge X \in \mathrm{thick}(W_{i+1} \wedge X)$, and hence in $\mathrm{thick}(W_j \wedge X)$ by induction. □

PROOF OF PROPOSITION 3.2.2. This follows immediately from Lemma 3.2.3, Lemma 3.2.5, and Theorem 1.5.5. □

3.2.1. Nilpotence: Proof of Theorem 3.1.5. Now we prove Theorem 3.1.5, and hence verify condition (i) of Definition 3.1.1(b).

PROOF OF THEOREM 3.1.5. The basic idea of the proof is, of course, based on that of the nilpotence theorem in [**DHS88**]. As in that proof, one can reduce to the ring spectrum case—part (a)—in which the ring is connective. So we let R be a connective ring spectrum, we fix $\alpha \in \pi_{**}R$ and assume that $HD_{**}\alpha$ is nilpotent. By raising α to a power, we may assume that $HD_{**}\alpha = 0$. We want to show that α is nilpotent in $\pi_{**}R = HD(0)_{**}R$.

By Lemma 3.2.1, since $HD_{**}y = 0$, then we must have $HD(n)_{**}y = 0$ for some n. We want to show that if $HD(n)_{**}y = 0$, then $HD(n-1)_{**}y^j = 0$ for some j; the result will follow by downward induction on n.

Consider the change-of-rings spectral sequence

$$E_2^{s,t,u}(R) = \mathrm{Ext}^{s,u}_{\mathbf{F}_2[\xi_n^{2^n}]}(\mathbf{F}_2, \mathrm{Ext}^{t,*}_{D(n)}(\mathbf{F}_2, R)) \Rightarrow \mathrm{Ext}^{s+t,u}_{D(n-1)}(\mathbf{F}_2, R).$$

Write z for $HD(n-1)_{**}y$. Since $HD(n)_{**}y = 0$, then z must be represented by a class $\tilde{z} \in E_2^{p,q,v}$ with $p > 0$.

Choose an integer m so that $mp + q - v > 0$. Then for any c, we can find a j so that

$$mpj + qj - vj > c.$$

By Proposition 3.2.2 (with $X = R$), for some r and c we have $E_r^{s,t,u} = 0$ when $ms+t-u > c$. As noted above, we can choose j so that $\tilde{z}^j$ lies above this vanishing plane, so at the E_r-term for which we have the vanishing plane, $\tilde{z}^j$ must be zero. Modulo terms of higher filtration, $\tilde{z}^j$ is zero at E_∞, but the higher filtration pieces are also above the vanishing plane, and therefore zero. So $z^j = 0$ in the abutment, $HD(n-1)_{**}$, which is what we wanted to show. □

3.2.2. F-isomorphism: Proof of Theorem 3.1.2. We need to show that the map

$$\varphi\colon \pi_{**}S^0 \to HD_{**}^{A\,\square_D \mathbf{F}_2}$$

is an F-isomorphism. By Theorem 3.1.5, it is a monomorphism mod nilpotents, so we have to show that it is an epimorphism mod nilpotents; in other words, we have to verify condition (ii) of Definition 3.1.1(b).

Verification of condition (ii). Fix $y \in HD_{i,j}^{A\,\square_D \mathbf{F}_2}$. We show that there is an integer m so that $y^{2^m} \in \operatorname{im}\varphi$. By Lemma 3.2.1(b), there is an n so that y lifts to $HD(n)_{**}^{A\,\square_{D(n)}\mathbf{F}_2}$. (Alternatively, one can use Lemma 3.2.1(a) to lift y to $HD(n)_{**}$ for some n, and then use Lemma 3.2.6 to show that some power of that lift is invariant.) Now we show that some power of y lifts to $HD(n-1)_{**}^{A\,\square_{D(n-1)}\mathbf{F}_2}$; since $D(0) = A$, then downward induction on n will finish the proof.

Since y is invariant under the $A\,\square_{D(n)}\mathbf{F}_2$-coaction, then it is also invariant under the coaction of $D(n-1)\,\square_{D(n)}\mathbf{F}_2$ (since the latter is a quotient Hopf algebra of the former). So y represents a class at the E_2-term of the change-of-rings spectral sequence

$$E_2^{s,t,u}(\mathbf{F}_2) = \operatorname{Ext}^{s,u}_{\mathbf{F}_2[\xi_n^{2^n}]}(\mathbf{F}_2, \operatorname{Ext}^{t,*}_{D(n)}(\mathbf{F}_2,\mathbf{F}_2)) \Rightarrow \operatorname{Ext}^{s+t,u}_{D(n-1)}(\mathbf{F}_2,\mathbf{F}_2).$$

Also, by assumption, y lies in the (t,u)-plane, say $y \in E_2^{0,q,v}$. Choose m large enough so that $m+q-v-1$ is positive. By Proposition 3.2.2 (with $X = S^0$), we know that for some c and r, we have $E_r^{s,t,u} = 0$ for $c < ms+t-u$ (hence the same as true for $E_{r'}^{s,t,u}$, for all $r' \geq r$).

For each $i \geq 0$, Proposition 1.5.4 tells us that the possible differentials on y^{2^i} are

$$d_{j+1}(y^{2^i}) \in E_{j+1}^{j+1,2^i q-j,2^i v},$$

for $j \geq 2^i$. Choose i so that

$$2^i > \max(r-1, \frac{c-m}{m+q-v-1}),$$

and fix $j \geq 2^i$. Then we have a vanishing plane at the E_{2^i+1}-term (and hence at the E_{j+1}-term); we claim that the element $d_{j+1}(y^{2^i})$ lies above the vanishing plane, and so is zero. We just have to verify the inequality specified by the vanishing plane:

$$\begin{aligned}
m(j+1) + 2^i q - j - 2^i v &= (m-1)j + m + 2^i(q-v) \\
&\geq (m-1)2^i + m + 2^i(q-v) \\
&= 2^i(m+q-v-1) + m \\
&> \frac{c-m}{m+q-v-1}(m+q-v-1) + m \\
&= c.
\end{aligned}$$

Hence y^{2^i} is a permanent cycle. For degree reasons, it cannot be a boundary; hence it gives a nonzero element of E_∞, and hence a nonzero element of $HD(n-1)_{**}$.

It only remains to show that y^{2^i}, or at least some power $y^{2^{i+j}}$, is invariant under the A-coaction. Let $\rho\colon\ HD(n-1)_{**} \to HD(n)_{**}$ denote the Hurewicz (restriction) map. This map detects nilpotence: if $x \in \ker\rho$, then x is nilpotent. (This follows from Remark 3.1.7(b), for instance; alternatively, this is the main inductive step in proving Theorem 3.1.5.) Hence by Lemma 3.2.6 below, $y^{2^{i+j}}$ is invariant for some j.

This completes the verification of condition (ii) of Definition 3.1.1(b), and hence the proof of Theorem 3.1.2. □

We have used the following.

LEMMA 3.2.6. *Suppose that R and S are commutative A-comodule algebras, with an A-linear map $\rho\colon\ R \to S$ that detects nilpotence: every $x \in \ker\rho$ is nilpotent. Given $z \in R$ so that $\rho(z) \in S$ is invariant under the A-coaction, then z^{2^n} is also invariant, for some n.*

PROOF. Since $\rho(z)$ is invariant, then the coaction on z is of the form

$$z \longmapsto 1 \otimes z + \sum_i a_i \otimes x_i,$$

where each x_i is in $\ker\rho$, and hence is nilpotent. Since R is an A-comodule algebra, then we see that

$$z^{2^n} \longmapsto 1 \otimes z^{2^n} + \sum_i a_i^{2^n} \otimes x_i^{2^n}.$$

This is a finite sum (since these are comodules), so for n sufficiently large, z^{2^n} is invariant. □

3.3. Nilpotence and F-isomorphism via quasi-elementary quotients

In this section we use our F-isomorphism and nilpotence theorems for HD—Theorems 3.1.2 and 3.1.5—to prove analogous theorems for the quasi-elementary quotients of A—Theorems 3.1.3 and 3.1.6.

3.3.1. Nilpotence: Proof of Theorem 3.1.6. We start with Theorem 3.1.6, because we use it in the proof of Theorem 3.1.3. We need a few preliminary results. Suppose we have a map $f\colon\ S^0 \to X$. We define $X^{(\infty)}$ to be the sequential colimit of the following diagram:

$$S^0 \xrightarrow{f} X \xrightarrow{f\wedge 1} X \wedge X \xrightarrow{f\wedge 1\wedge 1} X \wedge X \wedge X \to \dots,$$

and we let $f^{(\infty)}\colon\ S^0 \to X^{(\infty)}$ be the obvious map. We recall the following from [**HSb**, Lemma 2.3].

LEMMA 3.3.1. *Given a map $f\colon\ S^0 \to X$ and a ring spectrum E with unit map $\eta\colon\ S^0 \to E$, the following are equivalent.*

(a) $E \wedge X^{(\infty)} = 0$.
(b) $\eta \wedge f^{(\infty)}\colon\ S^0 \to E \wedge X^{(\infty)}$ *is zero.*
(c) $\eta \wedge f^{(n)}\colon\ S^0 \to E \wedge X^{(n)}$ *is zero for* $n \gg 0$.
(d) $1_E \wedge f^{(n)}\colon\ E = E \wedge S^0 \to E \wedge X^{(n)}$ *is zero for* $n \gg 0$.

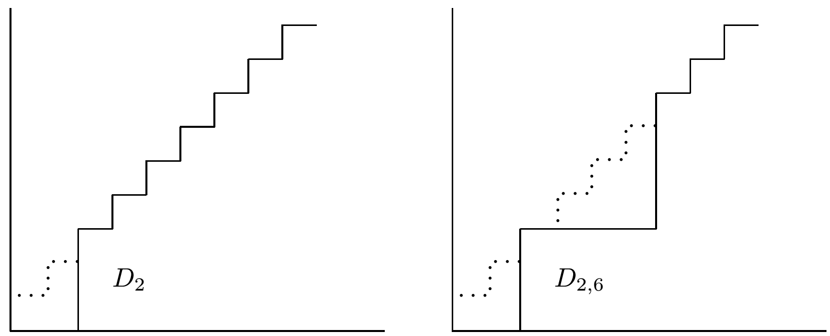


FIGURE 3.3.C. Profile functions for D_r and $D_{r,q}$.

For integers $q > r \geq 0$, we define the following quotient Hopf algebras of D:

$$D_r = D/(\xi_1, \ldots, \xi_r),$$
$$D_{r,q} = D_r/(\xi_{r+2}^{2^{r+1}}, \ldots, \xi_q^{2^{r+1}}).$$

See Figure 3.3.C. Recall from Corollary 2.1.8 that the maximal quasi-elementary quotients of A are called $E(m)$, $m \geq 0$.

LEMMA 3.3.2. *We have* $\varinjlim_r HD_r = H\mathbf{F}_2$ *and* $\varinjlim_q HD_{r,q} = HE(r)$.

PROOF. We leave this as an exercise. $\square$

LEMMA 3.3.3. *Let i, j, q, r, and R be integers.*

(a) *Suppose that $q > r \geq i \geq 0$ and $q - r > j \geq 0$, and consider the Hopf algebra $B = D_{r,q}/(\xi_{r+1}^{2^{i+1}}, \xi_{q+1}^{2^{r+2+j}})$. By Lemma 2.1.2, there are Hopf algebra extensions*

$$E[\xi_{r+1}^{2^i}] \to B \to C_1,$$
$$E[\xi_{q+1}^{2^{r+1+j}}] \to B \to C_2,$$

leading to elements $h_{r+1,i}$ and $h_{q+1,r+1+j}$ in HB_{**}. Then $h_{r+1,i}h_{q+1,r+1+j}$ is nilpotent.

(b) *Whenever $q > r$ and $r \geq i \geq 0$, there is a Bousfield equivalence*

$$\langle h_{r+1,i}^{-1} H(D_r/(\xi_{r+1}^{2^{i+1}}))\rangle = \langle h_{r+1,i}^{-1} H(D_{r,q}/(\xi_{r+1}^{2^{i+1}}))\rangle.$$

(c) *Whenever $q > R$, we have a Bousfield class decomposition*

$$\begin{aligned}\langle HD\rangle &= \langle HD_{R+1}\rangle \vee \bigvee_{r=0}^{R} \bigvee_{i=0}^{r} \langle h_{r+1,i}^{-1} H(D_r/(\xi_{r+1}^{2^{i+1}}))\rangle \\ &= \langle HD_{R+1}\rangle \vee \bigvee_{r=0}^{R} \bigvee_{i=0}^{r} \langle h_{r+1,i}^{-1} H(D_{r,q}/(\xi_{r+1}^{2^{i+1}}))\rangle.\end{aligned}$$

PROOF. For (a), the nilpotence of $h_{r+1,i}h_{q+1,r+j}$ is due to Lin [**Lin**]; we recall the statement of this result as Theorem B.2.1(b). (The nilpotence of this product is also the content of [**Wil81**, Theorem 6.4], as well as being essentially equivalent to the classification of quasi-elementary quotients of A in Propositions 2.1.7 and 2.1.12.)

Part (b) follows from part (a) and Corollary 1.6.2, by induction: one can get from $D_r/(\xi_{r+1}^{2^{i+1}})$ to $D_{r,q}/(\xi_{r+1}^{2^{i+1}})$ by dividing out by one $\xi_t^{2^s}$ at a time, where $s \geq r+1$. In other words, one has a sequence of extensions of the form

$$E[\xi_t^{2^s}] \to B \to C,$$

and hence cofibrations of the form

$$HB \xrightarrow{h_{ts}} HB \to HC.$$

One inverts $h_{r+1,i}$ in each term; then by the nilpotence of $h_{r+1,i}h_{ts}$, one gets an equivalence of Bousfield classes

$$\langle h_{r+1,i}^{-1} HB \rangle = \langle h_{r+1,i}^{-1} HC \rangle.$$

Part (c) is similar to part (b). □

Proof of Theorem 3.1.6. We imitate the proof of [**HSb**, Theorem 3]. As in that proof, one can reduce to the smash-nilpotence case, and using Spanier-Whitehead duality, one can reduce to the case where $F = S^0$. Suppose we have a map $f\colon S^0 \to X$ so that $\eta_E \wedge f = 0$ for all quasi-elementary E. We want to show that $\eta \wedge f^{(n)}\colon S^0 \to HD \wedge X^{(n)}$ is zero for some n; then Theorem 3.1.5(c) will tell us that f is smash-nilpotent. By Lemma 3.3.1, this map is zero if and only if $HD \wedge X^{(\infty)} = 0$. We use a Bousfield class argument to show this.

By assumption, $HE \wedge X^{(\infty)} = 0$ for all quasi-elementary E. By Lemma 3.3.3(c), we must show that

$$HD_{R+1} \wedge X^{(\infty)} = 0,$$

for some R, and then that

$$h_{r+1,i}^{-1} H(D_{r,q}/(\xi_{r+1}^{2^{i+1}})) \wedge X^{(\infty)} = 0,$$

for all $r \leq R$, $i \leq r$, and $q \gg 0$.

First we show that $HD_{R+1} \wedge X^{(\infty)} = 0$. By Lemma 3.3.1, it is equivalent to show that $\eta \wedge f^{(\infty)}\colon S^0 \to HD_{R+1} \wedge X^{(\infty)}$ is zero. Let R go to infinity; then by Lemma 3.3.2, the map

$$S^0 \to \varinjlim_R HD_{R+1} \wedge X^{(\infty)} = H\mathbf{F}_2 \wedge X^{(\infty)}$$

is null. Since homotopy commutes with direct limits, then for some R, the map $S^0 \to HD_{R+1} \wedge X^{(\infty)}$ is null.

One uses the same argument to show that

$$h_{r+1,i}^{-1} H(D_{r,q}/(\xi_{r+1}^{2^{i+1}})) \wedge X^{(\infty)} = 0,$$

but using the second equality of Lemma 3.3.2, rather than the first. □

3.3.2. F-isomorphism: Proof of Theorem 3.1.3. Now we work on Theorem 3.1.3: the map

$$\pi_{**} S^0 \to \left(\varprojlim_{\mathsf{Q}} HE_{**}\right)^A$$

is an F-isomorphism. First we construct the coaction of A on $\varprojlim_{\mathsf{Q}} HE_{**}$. We let $\overline{\mathsf{Q}}$ denote the full subcategory of Q consisting of the conormal quasi-elementary quotient Hopf algebras of A. Since the maximal quasi-elementary quotients are conormal, we see that $\overline{\mathsf{Q}}$ is final in Q; hence we have

$$\varprojlim_{\mathsf{Q}} HE_{**} = \varprojlim_{\overline{\mathsf{Q}}} HE_{**}.$$

On the right we have an inverse limit of comodules over A; we give this the induced A-comodule structure. So we have (since taking invariants is an inverse limit)

$$\begin{aligned}\left(\varprojlim{}_{\mathsf{Q}} HE_{**}\right)^A &= \left(\varprojlim{}_{\overline{\mathsf{Q}}} HE_{**}\right)^A \\ &= \varprojlim{}_{\overline{\mathsf{Q}}} \left(HE_{**}^A\right) \\ &= \varprojlim{}_{\overline{\mathsf{Q}}} \left(HE_{**}^{A//E}\right).\end{aligned}$$

To finish the proof of Theorem 3.1.3, we show that we can compute HD_{**} up to F-isomorphism in terms of the coefficient rings HE_{**} for E quasi-elementary.

Proposition 3.3.4. *The natural map*

$$HD_{**} \to \varprojlim{}_{\mathsf{Q}} HE_{**},$$

is an F-isomorphism, as is the induced map

$$HD_{**}^A \to \left(\varprojlim{}_{\mathsf{Q}} HE_{**}\right)^A.$$

See Proposition 4.4.1 for an explicit computation of the ring $\varprojlim{}_{\mathsf{Q}} HE_{**}$.

Proof of Theorem 3.1.3. This follows immediately from Theorem 3.1.2 and Proposition 3.3.4. $\square$

Proof of Proposition 3.3.4. We need to show that the restriction map

$$\rho\colon\ HD_{**} \to \varprojlim_{E\in\mathsf{Q}} HE_{**}$$

is an F-isomorphism, so we need to show two things: every element in the kernel of f is nilpotent, and we can lift some 2^nth power of any element in the range of f.

By the nilpotence theorem 3.1.6 (or more precisely, by the generalization in Remark 3.1.7(b)), we know that an element $z \in HD_{**}$ is nilpotent if and only if $z^n \in \ker\rho$ for some n.

The second statement follows, almost directly, from a result of Hopkins and Smith [**HSb**, Theorem 4.12], stated as Theorem 3.3.5 below, combined with the fact that the Hopf algebra D is a direct limit of finite-dimensional Hopf algebras. We explain.

For each integer $r \geq 1$, we let $B(r)$ be the following quotient Hopf algebra of D:

$$B(r) = D/(\xi_1, \xi_2, \dots, \xi_r).$$

Then each $B(r)$ is conormal in D, the kernel $K(r) = D\,\square_{B(r)}\mathbf{F}_2$ is finite-dimensional, and we have $D = \varinjlim{}_r K(r)$. Note that this colimit stabilizes in any given degree.

Given a quasi-elementary quotient E of D, we define the quotient $F(r)$ of E to be the pushout of $B(r) \leftarrow D \to E$, and we let $E(r)$ be the Hopf algebra kernel of $E \to F(r)$. In other words, we have the following diagram of Hopf algebra extensions:

$$\begin{tikzcd} K(r) \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & B(r) \arrow[d] \\ E(r) \arrow[r] & E \arrow[r] & F(r). \end{tikzcd}$$

Since $E = \varinjlim_r E(r)$, then given an element $y \in HE_{**}$, for r sufficiently large, y is in the image of the inflation map $\operatorname{Ext}^{**}_{E(r)}(\mathbf{F}_2, \mathbf{F}_2) \to \operatorname{Ext}^{**}_{E}(\mathbf{F}_2, \mathbf{F}_2)$. Now, $K(r) \to E(r)$ is a quotient map of finite-dimensional graded connected commutative Hopf algebras; let $\operatorname{res}_{K(r),E(r)}$ denote the induced map on Ext. Then given $y \in \operatorname{Ext}^{**}_{E(r)}(\mathbf{F}_2, \mathbf{F}_2)$, for some n we have $y^{p^n} \in \operatorname{im}(\operatorname{res}_{K(r),E(r)})$, by Theorem 3.3.5. The following commutative diagram finishes the proof of the first statement:

$$\begin{array}{ccc} \operatorname{Ext}^{**}_{K(r)}(\mathbf{F}_2, \mathbf{F}_2) & \longrightarrow & \operatorname{Ext}^{**}_{D}(\mathbf{F}_2, \mathbf{F}_2) \\ \downarrow & & \downarrow \\ \operatorname{Ext}^{**}_{E(r)}(\mathbf{F}_2, \mathbf{F}_2) & \longrightarrow & \operatorname{Ext}^{**}_{E}(\mathbf{F}_2, \mathbf{F}_2). \end{array}$$

Now we want to show that the natural map

$$HD^{A \,\square_D \mathbf{F}_2}_{**} \to \varprojlim_{\mathsf{Q}} HE^{A \,\square_E \mathbf{F}_2}_{**}$$

is an F-isomorphism. We treat the $A \,\square_D \mathbf{F}_2$-coaction on HD_{**} as an A-coaction with trivial D-coaction (and similarly for $A \,\square_E \mathbf{F}_2$ coacting on HE_{**}). So we want to compare HD^A_{**} with $\varprojlim HE^A_{**}$. We write f and $\tilde{f}$ for the maps

$$f\colon\ HD_{**} \to \varprojlim_{\mathsf{Q}} HE_{**},$$
$$\tilde{f}\colon\ HD^A_{**} \to \varprojlim_{\mathsf{Q}} HE^A_{**}.$$

The maximal quasi-elementary quotients are conormal; hence the conormal quasi-elementary quotients are final in the inverse system of all quasi-elementary quotients; hence f is an F-isomorphism. So it is clear that if $x \in HD^A_{**}$ is in the kernel of $\tilde{f}$, then x is nilpotent. Furthermore, given $y \in \varprojlim HE^A_{**}$, we know that y is in the image of f; hence by Lemma 3.2.6, some 2^nth power of y must be in the image of $\tilde{f}$. $\square$

We have used the following theorem. This first appeared in [**HSb**], and is a generalization of results in [**Wil81**]. In this theorem, Ext is taken in the category of modules, rather than comodules; we have actually used the theorem which is dual to this one.

Theorem 3.3.5 (Theorem 4.12 in [**HSb**]). *Suppose that $\Lambda \subseteq \Gamma$ is an inclusion of finite-dimensional graded connected cocommutative Hopf algebras over a field k of characteristic $p > 0$. For any $\lambda \in \operatorname{Ext}^{**}_{\Lambda}(k, k)$, there is a number m so that λ^{p^m} is in the image of the restriction map $\operatorname{Ext}^{**}_{\Gamma}(k, k) \to \operatorname{Ext}^{**}_{\Lambda}(k, k)$.*

3.4. Further discussion: nilpotence at odd primes

There are two main obstructions to proving the above results at odd primes. The first is clear: we do not have a classification of the quasi-elementary quotients of A; hence we cannot prove the quasi-elementary versions of the theorems. (Equivalently, we do not have an odd primary analogue of Theorem B.2.1(b).) The second is perhaps more important, since it affects the HD versions of the theorems, and more surprising, because the corresponding result at the prime 2 seems rather standard. This is the property that allows one to prove Lemma 3.2.5: at the prime 2, the element $h_{nj} = [\xi_n^{2^j}] \in HD(n)_{**}$ is nilpotent, if $j \geq n$ (see Theorem B.2.1).

Recall from Notation 1.3.9 and Lemma 1.3.10 that primitives y in a Hopf algebra B give rise to classes $[y] \in \operatorname{Ext}^{1,|y|}_B(k, k)$. When p is odd, even-dimensional primitives y in B give rise to classes $\beta\widetilde{\mathscr{P}^0}[y]$ in $\operatorname{Ext}^{2,p|y|}_B(k, k)$.

CONJECTURE 3.4.1. *Fix an odd prime p. Fix a quotient Hopf algebra B of A, and suppose that $\xi_t^{p^s}$ is primitive in B. If $s \geq t$, then $b_{ts} = \beta\widetilde{\mathscr{P}}^0[\xi_t^{p^s}]$ is nilpotent in HB_{**}.*

As at the prime 2, we define D as follows:

$$D = A/(\xi_1^p, \xi_2^{p^2}, \xi_3^{p^3}, \dots).$$

One can see from the proofs when $p = 2$ that the odd prime versions of Theorems 3.1.2 and 3.1.5 would follow from this conjecture. (One has to make a slight change in Lemma 3.2.4—as in the proof of Lemma 1.3.10, there are two related short exact sequences of comodules, leading to a cofibration in which the connecting homomorphism is a map of homological degree 2. Other than that, everything goes through as written.)

We discuss the status of Conjecture 3.4.1 in Appendix B.3.

3.5. Further discussion: miscellany

The nilpotence theorem [**DHS88**] in stable homotopy theory has far-reaching implications for the global structure of the stable homotopy category—see [**Hop87**] and [**Rav92**], for instance. We develop analogues of some of the structural results in the next chapter, but there are still many gaps. We discuss those in Sections 4.6 and 4.9 below.

If Γ is a Hopf algebra, then the quasi-elementary quotients of Γ give the right homology functors to consider for detecting nilpotence, essentially by the definition of "quasi-elementary." Indeed, if Γ is commutative and finite-dimensional, it is not too hard to prove an analogue of Theorem 3.1.6, either directly (as in [**Pal97**]) or via Chouinard's theorem (as in [**HPS97**, 9.6.10–9.6.11]). If, in addition, Γ is connected, then Theorem 3.3.5 allows one to prove an analogue of Theorem 3.1.3—see [**Pal97**]. Actually, by Theorem 3.3.5, some power of every element in the inverse limit is invariant, so one gets an F-isomorphism

$$\operatorname{Ext}_\Gamma^{**}(k,k) \to \varprojlim_{\mathsf{Q}} \operatorname{Ext}_E^{**}(k,k),$$

where the inverse limit is over Q, the category of quasi-elementary quotients of Γ.

Here are some related questions: one already has Quillen stratification for group algebras; can one prove it, as we do, with vanishing lines or planes? Can one use our approach to Quillen stratification for the Steenrod algebra to study the cohomology of other Hopf algebras? Given a Hopf algebra Γ, one should study the pullback of the quasi-elementary quotients of Γ, so one might want to assume that those quotients are somewhat well-behaved. Even more generally, in an arbitrary stable homotopy category one could consider a subcategory of appropriately chosen ring spectra, and look at the inverse limit of that.

Can one refine the description of $\pi_{**}S^0$ as given by Theorem 3.1.2? In particular, can one describe the nilpotence height of elements in the kernel, or say which powers of classes in the codomain of

$$\varphi \colon \pi_{**}S^0 \to HD_{**}^{A \,\square_D \mathbf{F}_2}$$

lift to the domain?

Since one has a version of Quillen stratification for the Steenrod algebra, one might look for analogues of other group-theoretic results. Chouinard's theorem [**Cho76**] is an example: a kG-module M is projective if and only if it is projective upon restriction to kE for every elementary abelian subgroup E of G. An analogue in $\mathsf{Stable}(A)$ might be: for any spectrum X, X is in $\operatorname{loc}(A)$ if and only if $HD \wedge X$ is in $\operatorname{loc}(A)$ if and only if $HE \wedge X$ is in $\operatorname{loc}(A)$ for every quasi-elementary quotient E of A. While one can prove things like this in $\mathsf{Stable}(\Gamma)$ for finite-dimensional Hopf algebras Γ—see [**Pal97**]—the infinite-dimensionality of A might cause a problem. One should be able to prove something like this when X is a finite spectrum, but one would like a version of Chouinard's theorem without any such restrictions.

CHAPTER 4

Periodicity and other applications of the nilpotence theorems

The nilpotence theorem in ordinary stable homotopy theory [**DHS88**] has a number of important consequences: the periodicity theorem and the thick subcategory theorem of [**HSb**] are examples. In this chapter we study applications of our nilpotence and Quillen stratification theorems—Theorems 3.1.2, 3.1.3, 3.1.5, and 3.1.6.

One of our main results of this chapter is a version of the periodicity theorem: if R is a finite ring spectrum, then we produce a number of central non-nilpotent elements in $\pi_{**}R$ via the "variety of R over D," which is essentially the kernel of $HD_{**}\eta\colon HD_{**} \to HD_{**}R$. Equivalently, this gives families of central non-nilpotent self-maps of any finite spectrum X. This is our analogue of the periodicity theorem of Hopkins and Smith. We state this precisely in Section 4.1, and we prove it in Sections 4.2 and 4.3.

Theorem 3.1.2 says that $\pi_{**}S^0$ is F-isomorphic to the A-invariants in HD_{**}; in Section 4.4 we discuss some examples of invariant elements in HD_{**}.

In Section 4.5, we show that the objects that detect nilpotence—HD and $\bigvee HE$—have strictly smaller Bousfield classes than that of the sphere. The role of D (as well as the action of A on D) has led us to a conjectured thick subcategory theorem, which we give in Section 4.6. We also discuss a few properties of "varieties" of spectra over D in this section.

In Section 4.7 we construct finite spectra (analogues of generalized Toda $V(n)$'s) with vanishing lines of various slopes, using Theorem 2.4.3, and we examine some properties of these spectra. These will be used in studying chromatic phenomena in Chapter 5. We end the chapter with two sections of miscellany—one on slope supports of finite spectra, and a very brief section with a few additional questions and remarks.

As in the previous chapter, we work at the prime 2 (unless otherwise stated).

4.1. The periodicity theorem

We start this chapter by giving our version of the periodicity theorem. This is a weak analogue of Theorem 3.1.2, with nontrivial coefficients.

The following definition was motivated by work in modular group representation theory of Alperin, Benson, Carlson, and the rest of the group theoretic alphabet.

Definition 4.1.1. Given a finite spectrum X, we define the *ideal* of X, $I(X)$, to be the radical of

$$\ker\left(HD_{**} \xrightarrow{-\wedge X} [X, HD\wedge X]_{**}\right).$$

Since X is finite, this is the same as the radical of the annihilator ideal in HD_{**} of $HD_{**}X$ via the action by composition (see Proposition 4.6.2). The ideal $I(X)$ is also invariant under the A-coaction, so that the A-coaction on HD_{**} induces one on $HD_{**}/I(X)$.

DEFINITION 4.1.2. Given an element $y \in \mathrm{Ext}_D^{**}(\mathbf{F}_2, \mathbf{F}_2) = HD_{**}$ and an object X in $\mathsf{Stable}(B)$, a map $z\colon X \to X$ is a *y-map* if $HD_{**}z\colon HD_{**}X \to HD_{**}X$ is multiplication by y^n for some n.

THEOREM 4.1.3 (Periodicity theorem). *Let X be a finite spectrum. For every*

$$y \in \Big(HD_{**}/I(X)\Big)^A,$$

X has a y-map which is central in the ring $[X, X]_{**}$.

(There is an equivalent statement involving elements in the homotopy of a finite ring spectrum R, which we give as Theorem 4.2.2 below.)

We conjecture that $(HD_{**}/I(X))^A$ is F-isomorphic to the center of $[X, X]_{**}$. See Section 4.6 for this and related ideas.

As an application, we have Corollary 4.1.5 below, which first appeared as [**Pal96b**, Theorem 4.1]. We need a bit of notation to state it.

DEFINITION 4.1.4. Let Slopes denote the set of slopes of A (Notation 2.2.4). Given a spectrum X, we define its *slope support* to be the set

$$\{n \mid Z(n)_{**}X \neq 0\} \subseteq \text{Slopes}\,.$$

Since we are working at the prime 2, we have a bijection

$$\begin{aligned} \text{Slopes} &\longleftrightarrow \{(t,s) \mid t > s \geq 0\}, \\ |\xi_t^{2^s}| &\longleftrightarrow (t,s). \end{aligned}$$

Let Slopes$'$ denote the right-hand side of this bijection. We say that a subset T of Slopes$'$ is *admissible* if T satisfies the following conditions:

$$\begin{gathered} (t,s) \in T \Rightarrow (t+1,s) \in T, \\ (t,s) \in T \Rightarrow (t+1,s-1) \in T, \\ \text{card}(\text{Slopes}' \setminus T) < \infty. \end{gathered}$$

(We call such sets "admissible" because those are the possible slope supports of finite spectra—see Section 4.8 and [**Pal96b**].)

Recall from Notations 2.2.4 and 2.4.1 that for each slope n, there is a spectrum $Z(n)$ and a non-nilpotent element $u_n \in Z(n)_{**}$. There is a ring map $HD_{**} \to Z(n)_{**}$, and we show in Section 4.4 that some power of u_n lifts to HD_{**}. Hence we can consider the existence of u_n-maps, à la Definition 4.1.2.

COROLLARY 4.1.5. *Let X be a finite spectrum and let $T \subseteq$ Slopes be the slope support of X. If $n \in T$ and $T \setminus \{n\}$ is admissible (viewed as a subset of* Slopes$'$*), then there is a (non-nilpotent) u_n-map in $[X, X]_{**}$.*

PROOF. One can see that $I(X)$ contains $\{h_{t,s} \mid (t,s) \in T \subseteq \text{Slopes}'\}$, and hence u_n is invariant in $HD_{**}/I(X)$. $\square$

4.2. Properties of y-maps

In this section we lay the groundwork for proving Theorem 4.1.3. In particular, we study analogues of v_n-maps [**Hop87, HSb**]. In ordinary stable homotopy theory, v_n-maps are defined via the Morava K-theories—these are field spectra, and hence have various convenient properties, such as Künneth isomorphisms. Our analogues are defined via HD, and hence are not quite as easy to work with. Nonetheless, our versions of v_n-maps, called "y-maps," share many of the same properties as v_n-maps in ordinary stable homotopy theory; in particular, we show in this section that the property of having a y-map is generic in our setting.

Fix a quotient Hopf algebra B of A which maps onto D. We work in the category $\mathsf{Stable}(B)$. We start by expanding Definition 4.1.2 a bit:

DEFINITION 4.2.1. Fix $y \in \operatorname{Ext}_D^{**}(\mathbf{F}_p, \mathbf{F}_p) = HD_{**}$, and let X be an object in $\mathsf{Stable}(B)$. A map $z\colon X \to X$ is a *y-map* if $HD_{**}z\colon HD_{**}X \to HD_{**}X$ is multiplication by y^n for some n. Similarly, if R is a ring object in $\mathsf{Stable}(B)$, then an element $\alpha \in \pi_{**}R$ is a *y-element* if for some n, we have $HD_{**}\alpha = y^n$ as maps $HD_{**} \to HD_{**}R$.

Note that the set of y-maps from X to itself is in bijection with the set of y-elements in the ring spectrum $X \wedge DX$, by Spanier-Whitehead duality. So the following is equivalent to Theorem 4.1.3.

THEOREM 4.2.2 (Periodicity theorem for ring spectra). *Let R be a finite ring spectrum with unit map $\eta\colon S^0 \to R$. Let $I = \sqrt{\ker HD_{**}\eta}$. For every*

$$y \in \Big(HD_{**}/I\Big)^A,$$

there is a y-element which is central in $\pi_{**}R$.

The main tool for proving Theorems 4.1.3 and 4.2.2 is the following. See Definition 1.4.7 for the definition of "thick subcategory."

THEOREM 4.2.3. *Suppose that B is a quotient Hopf algebra of A with $A \twoheadrightarrow B \twoheadrightarrow D$. Fix $y \in HD_{**}$. The full subcategory C consisting of finite objects of $\mathsf{Stable}(B)$ having a y-map is thick.*

In other words, the property of having a y-map is generic.

The proof is a simple modification of the proof in [**HSb**] that having a v_n-map is a generic property. We devote most of this section to the details. We start with a variant of the notion of y-map, and a general lemma.

Suppose that we have $B \twoheadrightarrow D$. Since a change-of-coalgebras isomorphism (as in Lemma 1.3.4) gives

$$\operatorname{Ext}_B^{**}(M, (B \,\Box_D \mathbf{F}_p) \otimes M) \cong \operatorname{Ext}_D^{**}(M, M),$$

it is sometimes useful to consider $[X, HD \wedge X]_{**}$. Note that the ring $HD_{**} = [S^0, HD]_{**}$ acts on this, via the smash product. We have the following.

DEFINITION 4.2.4. Fix $y = HD_{**}$, and let X be an object in $\mathsf{Stable}(B)$. Write $\eta\colon S^0 \to HD$ for the unit map. A map $z\colon X \to X$ is a *strong y-map* if $\eta \wedge z = y^n \wedge 1 \in [X, HD \wedge X]_{**}$ for some n.

For instance, the map produced in Theorem 2.4.3 is a strong y-map (where $y = u_n$), and hence a y-map by the following lemma.

LEMMA 4.2.5. *Let X be an object in $\mathsf{Stable}(B)$. If $z\colon X \to X$ is a strong y-map, then it is a y-map.*

PROOF. Since η is the unit map of the ring spectrum HD, then the following composite equals $1 \wedge z$:

$$HD \wedge X \xrightarrow{1\wedge\eta\wedge z} HD \wedge HD \wedge X \xrightarrow{\mu\wedge 1} HD \wedge X.$$

By assumption, $1 \wedge \eta \wedge z = 1 \wedge y^n \wedge 1$, so this composite also equals $y^n \wedge 1$. We compute the induced map on π_{**} by

$$S^0 \to HD \wedge X \xrightarrow{y^n\wedge 1} HD \wedge X.$$

Hence z is a y-map. $\square$

We move on to the proof of Theorem 4.2.3. Here is an easy lemma.

LEMMA 4.2.6. *Let p be a prime. Fix elements y_1 and y_2 of an $\mathbf{F}_p$-algebra. If y_1 and y_2 commute and $y_1 - y_2$ is nilpotent, then $y_1^{p^n} = y_2^{p^n}$ for some $n \gg 0$.*

For the remainder of the section, we consider a fixed element y of HD_{**}. We work at the prime 2.

LEMMA 4.2.7. *Let R be a ring spectrum in $\mathsf{Stable}(B)$, and fix y-elements $\alpha, \beta \in \pi_{**}R$. If α and β commute, then there exist positive integers i and j so that $\alpha^i = \beta^j$.*

PROOF. We may assume that $HD_{**}(\alpha - \beta) = 0$ by raising α and β to suitable powers; hence $\alpha-\beta$ is nilpotent. Since α and β commute, then Lemma 4.2.6 finishes the proof. $\square$

LEMMA 4.2.8. *Let R be a finite ring object in $\mathsf{Stable}(B)$, and fix a y-element $\alpha \in \pi_{**}R$. For some $i > 0$, the element α^i is central in $\pi_{**}R$.*

PROOF. Let $\ell(\alpha), r(\alpha) \in \mathrm{End}(\pi_{**}R)$ denote left and right multiplication by α, respectively. More precisely, $\ell(\alpha)$ is induced by the following self-map of R:

$$R \xrightarrow{\alpha\wedge 1} R \wedge R \xrightarrow{\mu} R,$$

and similarly for $r(\alpha)$. Since $HD_{**}\alpha$ is central in $HD_{**}R$, then $\ell(\alpha) - r(\alpha)$ maps to zero in $\mathrm{End}(HD_{**}R)$, and so is nilpotent by Theorem 3.1.5. By Lemma 4.2.6, we conclude that α is central in $\pi_{**}R$. $\square$

COROLLARY 4.2.9. *Let R be a finite ring spectrum in $\mathsf{Stable}(B)$. For any y-elements $\alpha, \beta \in \pi_{**}R$, there exist positive integers i and j so that $\alpha^i = \beta^j$.*

COROLLARY 4.2.10. *Let X be a finite spectrum in $\mathsf{Stable}(B)$, and let f and g be two y-maps of X. Then $f^i = g^j$ for some positive integers i and j.*

COROLLARY 4.2.11. *Suppose that X_1 and X_2 have y-maps y_1 and y_2. Then there are positive integers i and j so that for every Z and every $f : Z \wedge X_1 \to X_2$, the following diagram commutes:*

$$\begin{tikzcd}
Z \wedge X_1 \arrow[r, "f"'] \arrow[d, "1\wedge y_1^i"'] & X_2 \arrow[d, "y_2^j"] \\
Z \wedge X_1 \arrow[r, "f"'] & X_2
\end{tikzcd}$$

Proof. $DX_1 \wedge X_2$ has two y-maps: $Dy_1 \wedge 1$ and $1 \wedge y_2$. Now we apply Corollary 4.2.10 and Spanier-Whitehead duality. $\square$

Proof of Theorem 4.2.3. If Y has a (central) y-map g_Y and if X is a retract of Y, then the induced self-map of X

$$X \to Y \xrightarrow{g_Y} Y \to X$$

is easily seen to be a y-map.

Suppose that $X_1 \to X_2 \to X_3$ is a cofibration, and that X_1 and X_2 have y-maps y_1 and y_2, respectively. We may assume that $HD_{**}y_1$ and $HD_{**}y_2$ both are multiplication by y^n. By Corollary 4.2.11, we can find a map y_3 so that this diagram commutes:

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d, "y_1"] & X_2 \arrow[r] \arrow[d, "y_2"] & X_3 \arrow[d, "y_3"] \\
X_1 \arrow[r] & X_2 \arrow[r] & X_3
\end{tikzcd}$$

We claim that some power of y_3 is a y-map. So we compare $HD_{**}(y_3)$ and y^n in $\mathrm{End}(HD_{**}X_3)$. We have the following commutative diagram:

$$\begin{tikzcd}
HD \wedge X_2 \arrow[r] \arrow[d, "1\wedge y_2 - y^n \wedge 1"'] & HD \wedge X_3 \arrow[r] \arrow[d, "1\wedge y_3 - y^n \wedge 1"] & HD \wedge \Sigma X_1. \arrow[d, "1\wedge y_1 - y^n \wedge 1"] \\
HD \wedge X_2 \arrow[r] & HD \wedge X_3 \arrow[r] & HD \wedge \Sigma X_1
\end{tikzcd}$$

Since y_1 and y_2 are y-maps, the left- and right-hand vertical maps induce zero on π_{**}, so by a simple diagram chase, one can see that $\pi_{**}(1 \wedge y_3 - y^n \wedge 1)^2 = 0$. Hence y_3^2 is a y-map. $\square$

4.3. The proof of the periodicity theorem

In this section we prove Theorem 4.1.3. Fix a finite object X in $\mathsf{Stable}(A)$. For each element $y \in (HD_*/I(X))^A$, we want to show that X has a y-map (Definition 4.1.2).

The basic pattern of the proof is the same as that of Theorem 3.1.2: we inductively work our way from D to A via the Hopf algebras $D(n)$ (defined in Section 3.2).

As in Lemma 3.2.1, since X is finite, then $[X, HD \wedge X]_{**}$ is the colimit of $[X, HD(n) \wedge X]_{**}$. So for some n sufficiently large, the element $\eta \wedge y \in [X, HD \wedge X]_{**}$ lifts to a strong y-map in $[X, HD(n) \wedge X]_{**} = [X, X]_{**}^{D(n)}$. By Lemma 4.2.5, this gives a y-map in $[X, X]_{**}^{D(n)}$, and by Lemma 4.2.8, we may assume that this y-map is central. This starts the induction.

Now, assume that X has a y-map when viewed as an object in $\mathsf{Stable}(D(n))$. We want to show that X still has a y-map, but when viewed as an object in $\mathsf{Stable}(D(n-1))$. In the latter category, for any object Y we have the Adams spectral sequence based on $HD(n)$:

$$\mathrm{Ext}^{s,u}_{D(n-1)\,\square_{D(n)}\mathbf{F}_2}(\mathbf{F}_2, \mathrm{Ext}^{t,*}_{D(n)}(\mathbf{F}_2, Y)) \Rightarrow \mathrm{Ext}^{s+t,u}_{D(n-1)}(\mathbf{F}_2, Y).$$

In particular, if we let W_j be defined as in Lemmas 3.2.3 and 3.2.4, then we have

$$\begin{aligned}(4.3.1)\qquad &\mathrm{Ext}^{s,u}_{D(n-1)\,\square_{D(n)}\mathbf{F}_2}(\mathbf{F}_2, \mathrm{Ext}^{t,*}_{D(n)}(\mathbf{F}_2, X \wedge W_j \wedge DX \wedge DW_j))\\ &\Rightarrow \mathrm{Ext}^{s+t,u}_{D(n-1)}(\mathbf{F}_2, X \wedge W_j \wedge DX \wedge DW_j)\\ &= [X \wedge W_j, X \wedge W_j]^{D(n-1)}_{**}.\end{aligned}$$

Lemma 3.2.5 tells us that $\mathrm{thick}(S^0) = \mathrm{thick}(W_j)$ for any $j \geq n$, so $\mathrm{thick}(X) = \mathrm{thick}(X \wedge W_j)$. Hence by Theorem 4.2.3, it suffices to show that $X \wedge W_j$ has a y-map for some j.

The idea is that for j sufficiently large, the endomorphisms of $X \wedge W_j$ over $D(n-1)$ should be more or less the same as the endomorphisms of X over $D(n)$. Since X has a y-map over $D(n)$, then $X \wedge W_j$ should have one over $D(n-1)$, and hence X should have one, by genericity.

The details are as follows. Since X is finite, then there is a number a so that $\mathrm{Ext}^{s,u}_{D(n)}(\mathbf{F}_2, X \wedge DX)$ is zero when $u - a < s$. The comodule W_j is nonzero between degrees 0 and $2^j|\xi_n|$, so

$$\mathrm{Ext}^{s,u}_{D(n)}(\mathbf{F}_2, X \wedge W_j \wedge DX \wedge DW_j)$$

is zero when $u - (a - 2^j|\xi_n|) < s$. By Lemmas 3.2.5 and 3.2.3, then, we see that the E_2-term for the Adams spectral sequence (4.3.1) has the following vanishing plane:

$$E_2^{s,t,u}(X \wedge W_j \wedge DX \wedge DW_j) = 0 \quad \text{when} \quad 2^j|\xi_n|s + t - u - (2^j|\xi_n| - a) > 0.$$

The inductive hypothesis tells us that we have a y-map in $\mathrm{Ext}^{**}_{D(n)}(X, X)$. We use $\tilde{y}$ to denote this element, as well as its image in $\mathrm{Ext}^{**}_{D(n)}(X \wedge W_j, X \wedge W_j)$. Now,

$$\mathrm{Ext}^{**}_D(\mathbf{F}_2, \mathbf{F}_2)/I(X) \to \mathrm{Ext}^{**}_{D(n)}(X \wedge W_j, X \wedge W_j)$$

is a map of $D(n-1)\,\square_{D(n)}\mathbf{F}_2$-comodules, so since y is assumed to be in the invariants of $\mathrm{Ext}^{**}_D(\mathbf{F}_2, \mathbf{F}_2)/I(X)$, then $\tilde{y}$ is invariant under the $D(n-1)\,\square_{D(n)}\mathbf{F}_2$-coaction. Hence $\tilde{y}$ represents an element at the E_2-term of the spectral sequence. We claim that, when j is large enough, $\tilde{y}$ is a permanent cycle.

Suppose that $\tilde{y} \in \mathrm{Ext}^{p,q}_{D(n)}(X, X)$; then it gives a class in $E_2^{0,p,q}$. The rth differential on this class would lie in $E_r^{r,p-r+1,q}$. So we only have to check that for all $r \geq 2$, this group is above the vanishing plane, and hence zero. We check our inequality:

$$2^j|\xi_n|r + (p - r + 1) - q - (2^j|\xi_n| - a) \overset{?}{>} 0.$$

Whatever p, q, and a are, we can choose j large enough so that this holds for all $r \geq 2$. Hence $\tilde{y}$ is a permanent cycle in the spectral sequence for $X \wedge W_j$; it obviously cannot support a differential, so it survives to give a nonzero class at E_∞. Since the resulting self-map of X over $D(n-1)$ restricts to the y-map $\tilde{y}$ over $D(n)$, then one can check that it is a y-map over $D(n-1)$. This completes the inductive step, and with it, the proof of Theorem 4.1.3.

Remark 4.3.2. We have used the nilpotence part of the "Quillen stratification" theorem 3.1.2 (i.e., we have used Theorem 3.1.5) in the proof of Theorem 4.2.3, and hence in the proof of the periodicity theorem 4.1.3. We have not used the other part of Theorem 3.1.2—that some power of any invariant element in HD_{**} lifts to $\pi_{**}S^0$. Indeed, this follows from the periodicity theorem, so it gives us an alternate proof of Theorem 3.1.2.

4.4. Computation of some invariants in HD_{**}

Theorem 3.1.3 gives an F-isomorphism

$$\pi_{**}S^0 \to (\varprojlim_{\mathsf{Q}} HE_{**})^A.$$

We compute $\varprojlim HE_{**}$ in Proposition 4.4.1 below; in Proposition 4.4.4 we give a formula for the coaction of A on this inverse limit, and then we give a few examples of invariant elements.

The maximal quasi-elementary quotients of A when $p = 2$ are the Hopf algebras $E(m)$, $m \geq 0$. We recall from Corollary 2.1.8 and Proposition 2.1.9 their definition and the computation of their coefficient rings:

$$E(m) = A/(\xi_1, \ldots, \xi_m, \xi_{m+1}^{2^{m+1}}, \xi_{m+2}^{2^{m+1}}, \xi_{m+3}^{2^{m+1}}, \ldots),$$
$$HE(m)_{**} = \mathbf{F}_2[h_{ts} \mid t \geq m+1,\, s \leq m].$$

The bidegrees of the polynomial generators are given by $|h_{ts}| = (1, |\xi_t^{2^s}|)$. Since it is easy to see the effects of the maps in Q on the coefficients, we immediately have the following.

PROPOSITION 4.4.1. *There is an isomorphism*

$$\varprojlim HE_{**} \xrightarrow{\cong} \mathbf{F}_2[h_{ts} \mid s < t] \,/\, (h_{ts}h_{vu} \mid u \geq t),$$

where $|h_{ts}| = (1, |\xi_t^{2^s}|)$.

Proposition 3.3.4 gives us an F-isomorphism $HD_{**} \to \varprojlim HE_{**}$, which we can compose with this isomorphism; hence, to compute HD_{**} up to F-isomorphism, one does the following:

- one takes the coefficient rings HE_{**}, as E ranges over the maximal quasi-elementary quotients of A,
- tensors them together,
- identifies polynomial generators from different E's if they have the same name, and
- divides out by the product of two polynomial generators if they do not come from the same E.

We want to describe the coaction of A on $\varprojlim HE_{**}$. The coaction of $A \,\square_{E(m)} \mathbf{F}_2$ on $HE(m)_{**}$ is determined by the coaction on the polynomial generators h_{ts}.

PROPOSITION 4.4.2. *Fix* $m \geq 0$. *Let* $\chi\colon A \to A$ *denote the conjugation map of* A, *and for* $n \geq 1$ *let* $\zeta_n = \chi(\xi_n)$. *Let* $\zeta_0 = 1$. *Under the coaction map*

$$HE(m)_{**} \to (A \,\square_{E(m)} \mathbf{F}_2) \otimes HE(m)_{**},$$

we have

$$h_{ts} \longmapsto \sum_{j=0}^{m-s} \sum_{i=m+1}^{t-j} \zeta_j^{2^s} \xi_{t-i-j}^{2^{i+j+s}} \otimes h_{i,j+s}.$$

(The ζ_j part comes, essentially, from the right coaction of A on itself, while the ξ_{t-i-j} comes from the left coaction.)

PROOF. (We assume that $t > m+1$ and that $s < m$; the special cases when $t = m+1$ or $s = m$ are even easier to deal with.) First we find all of the terms in the coaction h_{ts} of the form $a \otimes h_{ij}$ for $a \in A$ primitive, i.e., $a = \xi_1^{2^n}$ for some n. For degree reasons, the only possible such terms are $\xi_1^{2^s} \otimes h_{t-1,s+1}$ and $\xi_1^{2^{s+t-1}} \otimes h_{t-1,s}$. By computations as in the proofs of Lemmas A.3 and A.5 of [**Pal96b**], we see that both of these terms do appear; that is, we can see that

$$h_{ts} \longmapsto 1 \otimes h_{ts} + \zeta_1^{2^s} \otimes h_{t-1,s+1} + \xi_1^{2^{s+t-1}} \otimes h_{t-1,s} + \text{other terms},$$

where the "other terms" are of the form $b \otimes h_{ij}$, with $b \in A$ non-primitive. So the formula given in the proposition is "correct on the primitives"; once we have verified that the formula is co-associative, we will have finished the proof. This verification is a straight-forward (although slightly messy) computation; it could be left to the diligent reader, but due to lack of space constraints, we include it in Lemma 4.4.3 below. $\square$

LEMMA 4.4.3. *The formula*

$$h_{ts} \longmapsto \sum_{j=0}^{m-s} \sum_{i=m+1}^{t-j} \zeta_j^{2^s} \xi_{t-i-j}^{2^{i+j+s}} \otimes h_{i,j+s}.$$

defines a coassociative coaction of A on $HE(m)_{**}$.

PROOF. We write Δ for the coproduct on A, and we write ψ for the coaction map of A on $HE(m)_{**}$. We have to verify that $(\Delta \otimes 1) \circ \psi = (1 \otimes \psi) \circ \psi$, so we compute both of these on h_{ts}. First, we have $(\Delta \otimes 1) \circ \psi(h_{ts})$:

$$\begin{aligned}
(\Delta \otimes 1)(\psi(h_{ts})) &= (\Delta \otimes 1) \left(\sum_{j=0}^{m-s} \sum_{i=m+1}^{t-j} \zeta_j^{2^s} \xi_{t-i-j}^{2^{i+j+s}} \otimes h_{i,j+s} \right) \\
&= \sum_{j=0}^{m-s} \sum_{i=m+1}^{t-j} \Delta(\zeta_j^{2^s}) \Delta(\xi_{t-i-j}^{2^{i+j+s}}) \otimes h_{i,j+s} \\
&= \sum_{j=0}^{m-s} \sum_{i=m+1}^{t-j} \left(\sum_{n=0}^{j} \zeta_n^{2^s} \otimes \zeta_{j-n}^{2^{s+n}} \right) \left(\sum_{q=0}^{t-i-j} \xi_{t-i-j-q}^{2^{i+j+s+q}} \otimes \xi_q^{2^{i+j+s}} \right) \otimes h_{i,j+s} \\
&= \sum_{j=0}^{m-s} \sum_{i=m+1}^{t-j} \sum_{n=0}^{j} \sum_{q=0}^{t-i-j} \zeta_n^{2^s} \xi_{t-i-j-q}^{2^{i+j+s+q}} \otimes \zeta_{j-n}^{2^{s+n}} \xi_q^{2^{i+j+s}} \otimes h_{i,j+s}.
\end{aligned}$$

Note that we can read off the "coefficient" of $h_{a,b+s}$: it is

$$\sum_{n=0}^{b} \sum_{q=0}^{t-a-b} \zeta_n^{2^s} \xi_{t-a-b-q}^{2^{a+b+s+q}} \otimes \zeta_{b-n}^{2^{s+n}} \xi_q^{2^{a+b+s}}.$$

On the other hand, for $(1 \otimes \psi) \circ \psi(h_{ts})$ we have

$$(1 \otimes \psi)(\psi(h_{ts})) = (1 \otimes \psi)(\sum_{j=0}^{m-s} \sum_{i=m+1}^{t-j} \zeta_j^{2^s} \xi_{t-i-j}^{2^{i+j+s}} \otimes h_{i,j+s})$$

$$= \sum_{j=0}^{m-s} \sum_{i=m+1}^{t-j} \zeta_j^{2^s} \xi_{t-i-j}^{2^{i+j+s}} \otimes \psi(h_{i,j+s})$$

$$= \sum_{j=0}^{m-s} \sum_{i=m+1}^{t-j} \zeta_j^{2^s} \xi_{t-i-j}^{2^{i+j+s}} \otimes \sum_{\ell=0}^{m-j-s} \sum_{k=m+1}^{i-\ell} \zeta_\ell^{2^{j+s}} \xi_{i-k-\ell}^{2^{k+\ell+j+s}} \otimes h_{k,\ell+j+s}$$

$$= \sum_{j=0}^{m-s} \sum_{i=m+1}^{t-j} \sum_{\ell=0}^{m-j-s} \sum_{k=m+1}^{i-\ell} \zeta_j^{2^s} \xi_{t-i-j}^{2^{i+j+s}} \otimes \zeta_\ell^{2^{j+s}} \xi_{i-k-\ell}^{2^{k+\ell+j+s}} \otimes h_{k,\ell+j+s}.$$

So here the coefficient of $h_{a,b+s}$ is (upon setting $k = a$ and $\ell + j = b$, so that j ranges from 0 to b, and i ranges from $a + b - j$ to $t - j$):

$$\sum_{j=0}^{b} \sum_{i=a+b-j}^{t-j} \zeta_j^{2^s} \xi_{t-i-j}^{2^{i+j+s}} \otimes \zeta_{b-j}^{2^{j+s}} \xi_{i-a-b+j}^{2^{a+b+s}}.$$

Hence this formula is indeed coassociative. □

Since the $E(m)$'s are maximal quasi-elementary quotients, the following is an immediate corollary.

PROPOSITION 4.4.4. *Under the coaction map*

$$\varprojlim_{\mathsf{Q}} HE_{**} \to A \otimes \varprojlim_{\mathsf{Q}} HE_{**},$$

we have

$$h_{ts} \longmapsto \sum_{j=0}^{\lfloor \frac{s+t-1}{2} \rfloor} \sum_{i=j+s+1}^{t-j} \zeta_j^{2^s} \xi_{t-i-j}^{2^{i+j+s}} \otimes h_{i,j+s}.$$

Dualizing, we have this formula for the action of A^* on $\varprojlim HE_{**}$:

$$\mathrm{Sq}^{2^k}(h_{ts}) = \begin{cases} h_{t-1,s} & \text{if } s < t-1 \text{ and } k = s+t-1, \\ h_{t-1,s+1} & \text{if } s+1 < t-1 \text{ and } k = s, \\ 0 & \text{otherwise.} \end{cases}$$

We give a graphical depiction of the (co)action in Figure 4.4.A, in which we indicate the coaction by the primitives (i.e., the 2^nth powers of ξ_1).

We end the section with a few examples.

EXAMPLE 4.4.5. Let R denote the ring $\varprojlim HE_{**}$.

(a) The element $h_{t,t-1} \in R$ in bidegree $(1, |\xi_t^{2^{t-1}}|) = (1, 2^{t-1}(2^t - 1))$ is an invariant, for $t \geq 1$. Indeed, we know that h_{10} lifts to an element of the same name in $\pi_{1,1}S^0$; also h_{21}^4 lifts to an element in $\pi_{4,24}S^0$ (the element known as g or $\overline{\kappa}$—see [**Zac67**]). We do not know which power of $h_{t,t-1}$ survives for $t \geq 3$.

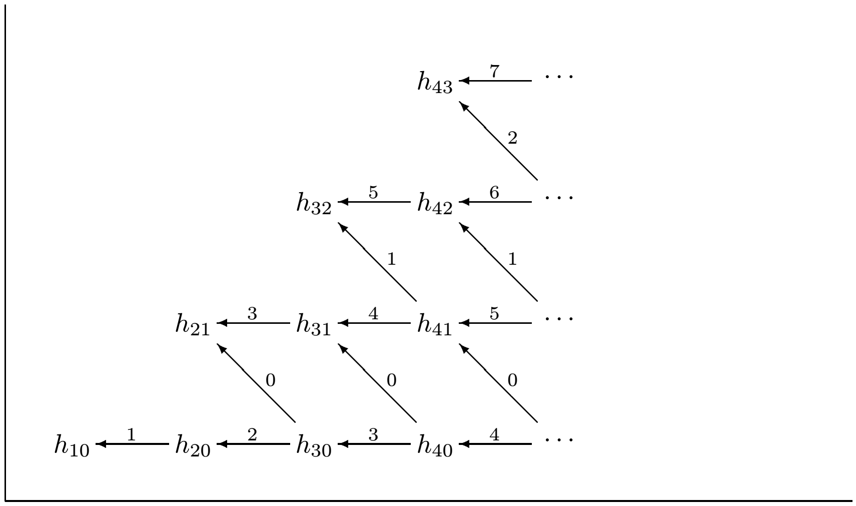


FIGURE 4.4.A. Graphical depiction of coaction of A on $\varprojlim HE_{**}$. An arrow labeled by k represents an action by $\mathrm{Sq}^{2^k} \in A^*$, or equivalently a "coaction" by $\xi_1^{2^k} \in A$—in other words, a term of the form $\xi_1^{2^k} \otimes$ (target) in the coaction on the source.

(b) We have some families of invariants. $h_{20} \in R$ is not invariant: we have

$$h_{20} \longmapsto 1 \otimes h_{20} + \xi_1^2 \otimes h_{10}.$$

But since $h_{10}h_{21} = 0$ in R, then $h_{20}^i h_{21}^j$ is invariant for all $i \geq 0$ and $j \geq 1$. It turns out that more of these elements lift to $\pi_{**}S^0$ than one might expect from Theorem 3.1.2: the elements in the "Mahowald-Tangora wedge" [**MT68**] are lifts of the elements $h_{20}^i h_{21}^j$ for all $i \geq 0$ and $j \geq 8$. (See [**MPT71**]; Zachariou [**Zac67**] first verified this for elements of the form $h_{20}^{2i} h_{21}^{2(i+j)}$ for $i, j \geq 0$.) These elements are distributed over a wedge between lines of slope $\frac{1}{2}$ and $\frac{1}{5}$ (in the Adams spectral sequence $(t-s, s)$ grading).

(c) Similarly, while h_{30} and h_{31} are not invariant, the monomials

$$\{h_{30}^i h_{31}^j h_{32}^k \mid i \geq 0,\ j \geq 0,\ k \geq 1\}$$

are invariant elements. Hence some powers of them lift to $\pi_{**}S^0$. We do not know what powers of them lift, but they will be distributed over a wedge between lines of slope $\frac{1}{6}$ and $\frac{1}{27}$. Continuing in this pattern, we find that for $n \geq 1$, we have sets of invariant elements

$$\{h_{n0}^{i_0} h_{n1}^{i_1} \ldots h_{n,n-1}^{i_{n-1}} \mid i_0, \ldots, i_{n-2} \geq 0,\ i_{n-1} \geq 1\}.$$

The lifts of these elements lie in a wedge between lines of slope $\frac{1}{2^n-2}$ and $\frac{1}{2^{n-1}(2^n-1)-1}$. Hence the family of elements in the Mahowald-Tangora wedge is not a unique phenomenon—we have infinitely many such families, and when $n \geq 3$ they give more than a lattice of points in $\pi_{**}S^0$.

(d) Margolis, Priddy, and Tangora [**MPT71**, p. 46] have found some other non-nilpotent elements, such as $x \in \pi_{10,63}S^0$ and $B_{21} \in \pi_{10,69}S^0$. These both come from the invariant

$$z = h_{40}^2 h_{21}^3 + h_{20}^2 h_{21}^2 h_{41} + h_{30}^2 h_{21} h_{31}^2 + h_{20}^2 h_{31}^3$$

in R. While we do not know which power of z lifts to $\pi_{**}S^0$, we do know that B_{21} maps to the product $h_{20}^3 h_{21}^2 z$, and x maps to $h_{20}^5 z$. In other words, we have at least found some elements in the ideal

$$(z) \subseteq (R)^A$$

which lift to $\pi_{**}S^0$.

(e) Computer calculations have lead us to a few other such "sporadic" invariant elements (i.e., invariant elements that do not belong to any family—any family that we know of, anyway):

$$h_{20}^8 h_{31}^4 + h_{30}^8 h_{21}^4 + h_{21}^{11} h_{31}$$

in bidegree $(12, 80)$, an element in bidegree $(9, 104)$ (a sum of 8 monomials in the variables $h_{i,0}$ and $h_{i,1}$, $2 \leq i \leq 5$), and an element in bidegree $(13, 104)$ (a sum of 12 monomials in the same variables). We do not know what powers of these elements lift to $\pi_{**}S^0$, nor are we aware of any elements in the ideal that they generate which are in the image of the restriction map from $\pi_{**}S^0$.

4.5. Computation of a few Bousfield classes

In [**HPS97**, Section 5.1] we show that if a spectrum E is Bousfield equivalent to the sphere, then E detects nilpotence. This is true in any stable homotopy category, and it is not very deep. While we do not vouch for the depth of the nilpotence theorems of Chapter 3, we at least point out that they are not examples of this generic nilpotence theorem.

Many of the results in this section hold at all primes; some only hold at the prime 2. Unless otherwise indicated, fix an arbitrary prime p.

THEOREM 4.5.1. *We have $\langle S^0 \rangle > \langle HD \rangle$, and when $p = 2$, $\langle HD \rangle > \bigvee_E \langle HE \rangle$, where the wedge is taken over all quasi-elementary quotients E of A.*

The proof is quite similar to that of analogous results in [**Rav84**]. We need a few lemmas, first.

LEMMA 4.5.2. *Suppose that B and C are quotient Hopf algebras of A that fit into a Hopf algebra extension*

$$B \,\square_C \mathbf{F}_p \hookrightarrow B \twoheadrightarrow C.$$

(a) *Suppose that $\dim_{\mathbf{F}_p} B \,\square_C \mathbf{F}_p = \infty$. If B and C are conormal quotient Hopf algebras of A, then $[HC, HB]_{**} = 0$. Hence $HC \wedge IHB = 0$; hence $\langle HB \rangle > \langle HC \rangle$.*

(b) *Suppose that $\dim_{\mathbf{F}_p} B \,\square_C \mathbf{F}_p < \infty$.*
   - (i) *If $B \,\square_C \mathbf{F}_p$ contains some τ_n or some $\xi_t^{p^s}$ with $s < t$, then $\langle HB \rangle > \langle HC \rangle$.*
   - (ii) *Otherwise, if $p = 2$, then $\langle HB \rangle = \langle HC \rangle$.*

PROOF. Part (a): The statement that $[HC, HB]_{**} = 0$ implies the rest of the lemma, by Proposition 1.6.1, so we only have to verify that. For that verification, we prove the corresponding statement about A^*-modules, and let the reader translate back to A-comodules and to $\mathsf{Stable}(A)$.

Let B^* be the dual of B, C^* the dual of C, and $B^*//C^*$ the dual of $B \,\square_C \mathbf{F}_p$. We show that

$$\mathrm{Ext}^{**}_{A^*}(A^*//B^*, A^*//C^*) \cong \mathrm{Ext}^{**}_{B^*}(\mathbf{F}_p, A^*//C^*) = 0.$$

By Lemma 4.5.3 below, as a B^*-module, $A^*//C^*$ is a direct sum of copies of $B^*//C^*$, so it suffices to show that $\mathrm{Ext}^{**}_{B^*}(\mathbf{F}_p, B^*//C^*) = 0$. From the Hopf algebra extension

$$C^* \to B^* \to B^*//C^*,$$

we get a spectral sequence (as in Section 1.5)

$$E_2^{p,q} = \mathrm{Ext}^p_{B^*//C^*}(\mathbf{F}_p, \mathrm{Ext}^q_{C^*}(\mathbf{F}_p, B^*//C^*)) \Rightarrow \mathrm{Ext}^{**}_{B^*}(\mathbf{F}_p, B^*//C^*).$$

We claim that $E_2^{**} = 0$. Since $B^*//C^*$ is a trivial C^*-module, then

$$\mathrm{Ext}^{**}_{C^*}(\mathbf{F}_p, B^*//C^*) \cong B^*//C^* \otimes \mathrm{Ext}^{**}_{C^*}(\mathbf{F}_p, \mathbf{F}_p)$$

as $B^*//C^*$-modules, and hence is free, and hence injective, over $B^*//C^*$. So $E_2^{p,*} = 0$ if $p > 0$, and

$$E_2^{0,*} = \mathrm{Hom}_{B^*//C^*}(\mathbf{F}_p, \bigoplus_\alpha \Sigma^{n_\alpha} B^*//C^*).$$

But $\mathrm{Hom}_{B^*//C^*}(\mathbf{F}_p, B^*//C^*) = 0$ (see Lemma 4.5.4); this finishes the calculation.

Part (b)(i) follows from Corollary 1.6.2 and induction, using the non-nilpotence of the classes h_{ts} (when $s < t$), b_{ts} (when $s < t$), and v_n. Part (b)(ii) is similar, but uses the nilpotence of h_{ts} when $s \geq t$: see Theorem B.2.1. □

We have used the following two lemmas in the proof of Lemma 4.5.2.

LEMMA 4.5.3. *Suppose B^* and C^* are normal sub-Hopf algebras of A^*, with $C^* \subseteq B^*$. Then as a B^*-module, $A^*//C^*$ is a direct sum of suspensions of $B^*//C^*$.*

PROOF. We have isomorphisms of B^*-modules (with B^* acting on the left):

$$\begin{aligned} A^* \otimes_{C^*} \mathbf{F}_p &\cong A^* \otimes_{B^*} B^* \otimes_{C^*} \mathbf{F}_p \\ &\cong A^*//B^* \otimes B^*//C^*. \end{aligned}$$

Now by normality, $A^*//B^*$ is a trivial B^*-module, so this tensor product is a direct sum of copies of $B^*//C^*$, indexed by a vector space basis of $A^*//B^*$. □

LEMMA 4.5.4. *Suppose B^* and C^* are normal sub-Hopf algebras of A^*, with $C^* \subseteq B^*$. Then*

$$\mathrm{Ext}^s_{B^*//C^*}(\mathbf{F}_p, B^*//C^*) = 0$$

for all $s > 0$. If $\dim_{\mathbf{F}_p} B^*//C^* = \infty$, then

$$\mathrm{Hom}_{B^*//C^*}(\mathbf{F}_p, B^*//C^*) = 0.$$

PROOF. $B^*//C^*$ is self-injective, so the Ext group in question is zero if $s > 0$. So we need to show that $\mathrm{Hom}_{B^*//C^*}(\mathbf{F}_p, B^*//C^*) = 0$, if $B^*//C^*$ is infinite-dimensional. An element of this Hom group corresponds to an element $x \in B^*//C^*$ which supports no operations by elements of $B^*//C^*$; we want to show that any such x must be zero. Fix such an x, and assume that $x \neq 0$.

Using the classification of (normal) sub-Hopf algebras of A^* in Theorem 2.1.1, we see that there are two possibilities: either (1) $B^*//C^*$ contains P_t^s's for arbitrarily large values of t, or (2) for some fixed t, $B^*//C^*$ contains P_t^s for all $s \gg 0$. (By "$B^*//C^*$ contains P_t^s," we mean that P_t^s is in B^* and not in C^*, so that it represents a nonzero class in the quotient.) In case (1), we choose P_t^s in $B^*//C^*$

so that $2^t > |x|$. Then it is an easy exercise in multiplication with the Milnor basis to show that $xP_t^s \neq 0$. In case (2), we argue similarly to show that $P_t^s x \neq 0$ for $s \gg 0$. We leave the details to the reader. $\square$

(Perhaps $\mathrm{Ext}_B^{**}(k, B)$ should be zero for any graded connected Hopf algebra B over a field k, if B is nonzero in infinitely many degrees. We have not been able to show this, though.)

Here are some consequences of Lemma 4.5.2(a).

Corollary 4.5.5. $[HD, S^0]_{**} = 0$.

Proof. Apply part (a) of the lemma with $B = A$ and $C = D$. $\square$

Corollary 4.5.6. *Let $p = 2$. Then $[HE, HD]_{**} = 0$ for any quasi-elementary quotient E of A.*

Proof. This follows immediately from Lemma 4.5.2(a), since the maximal quasi-elementary quotient Hopf algebras of A are conormal quotients of both A and D. $\square$

Corollary 4.5.7. $[A, HD]_{**} = 0$. *Hence if X is a finite object in* $\mathrm{loc}(A)$*, then* $X = 0$.

Proof. For the first statement, we apply the lemma with $B = D$ and $C = \mathbf{F}_p$. This then implies that $[Y, HD]_{**} = 0$ for all Y in $\mathrm{loc}(A)$. If X is finite with Spanier-Whitehead dual DX, then $[X, HD]_{**} = HD_{**}(DX)$. So it suffices to show that if X is finite and nontrivial, then $HD_{**}X \neq 0$. Well, $HD_{**}X = 0$ if and only if X is contractible when viewed as a cochain complex of comodules over D, in which case the homology of X must be zero. On the other hand, by the Hurewicz Theorem 1.4.4, if X is finite and nontrivial, then it has nonzero homology. $\square$

Proof of Theorem 4.5.1. It is clear (by Proposition 1.6.1(c), for instance) that we have

$$\langle S^0 \rangle \geq \langle HD \rangle \geq \bigvee_{\substack{E \text{ quasi-} \\ \text{elem.}}} \langle HE \rangle$$

(where the second inequality is only known to be valid when $p = 2$). By Corollary 4.5.5 and Proposition 1.6.1(f), we see that $\langle S^0 \rangle > \langle HD \rangle$; in particular, $HD \wedge IS^0 = 0$. Hence the first inequality is strict.

When $p = 2$, by Corollary 4.5.6 and Proposition 1.6.1(c)–(e), we see that $HD \wedge IHD \neq 0$, while $HE \wedge IHD = 0$ for all quasi-elementary E. Hence the second inequality is strict. $\square$

Remark 4.5.8. Let $p = 2$ and let $X = \bigvee HE$. One can in fact show a stronger result—that X has no complement in HD. Suppose otherwise: suppose that there were an object G so that $\langle HD \rangle = \langle X \rangle \vee \langle G \rangle$ and $0 = X \wedge G$. Smashing the former equality with IHD gives

$$\langle HD \wedge IHD \rangle = \langle X \wedge IHD \rangle \vee \langle G \wedge IHD \rangle,$$

But $HE \wedge IHD = 0$ for all quasi-elementary E by Corollary 4.5.6 and Proposition 1.6.1(b), so we have

$$\langle HD \wedge IHD \rangle = \langle G \wedge IHD \rangle.$$

Also, $\langle IHD\rangle \leq \langle H\mathbf{F}_p\rangle \leq \langle X\rangle$ by Proposition 1.6.1(a), so we have

$$\langle HD \wedge IHD\rangle = \langle G \wedge IHD\rangle \leq \langle G \wedge X\rangle = \langle 0\rangle.$$

But $\langle HD \wedge IHD\rangle \geq \langle IHD\rangle \neq 0$.

4.6. Ideals and thick subcategories

Let X be a finite spectrum; as in Section 4.1, we define the ideal $I(X) \leq HD_{**}$ to be the radical of the kernel of

$$HD_{**} \xrightarrow{-\wedge X} [X, HD \wedge X]_{**}.$$

The periodicity theorem 4.1.3 tells us that for any element y in $(HD_{**}/I(X))^A$, we can lift some power of y to a central element in $[X,X]_{**}$; therefore, the ideal $I(X)$ is worth studying. In this section, we establish some properties of $I(X)$—finite generation and invariance—and we examine a possible relation with a classification of thick subcategories of finite objects in $\mathsf{Stable}(A)$.

4.6.1. Ideals. Let $p = 2$.

We can define the ideal $I(X)$ for any object X in $\mathsf{Stable}(D)$; as we impose conditions on X—first finiteness, then X being defined over A rather than D—we find more properties of $I(X)$. One of those properties is invariance:

Definition 4.6.1. Suppose that R is a commutative comodule algebra over A (i.e., a commutative algebra and a left A-comodule, compatibly); we write ψ for the comodule structure map. We say that an ideal I of R is *invariant* under the A-coaction if for all $x \in I$, $\psi(x)$ is of the form $\sum a_j \otimes x_j \in A \otimes R$, where each x_j lies in I.

Proposition 4.6.2. (a) *Let X be a finite object of $\mathsf{Stable}(D)$. Then*

$$\sqrt{\ker(HD_{**} \to [X, HD \wedge X]_{**})} = \sqrt{\operatorname{ann}_{HD_{**}}(HD_{**}X)}.$$

(b) *Let X be a finite object of $\mathsf{Stable}(D)$. Then $I(X)$ is a finitely generated ideal.*
(c) *Let X be a finite object of $\mathsf{Stable}(A)$. Then $I(X)$ is invariant under the A-coaction.*

The proof of part (a) is based on similar work in [**Ben91b**, Section 5.7].

Proof. Part (a): Let

$$I = \sqrt{\ker(HD_{**} \to [X, HD \wedge X]_{**})},$$
$$J = \sqrt{\operatorname{ann}_{HD_{**}}(HD_{**}X)}.$$

The Yoneda action of HD_{**} on $HD_{**}X$ factors through the composition action (since HD_{**} is commutative). Hence $I \leq J$. On the other hand, since X is finite, then it is in thick(S^0). So any element of HD_{**} that annihilates $[S^0, HD \wedge X]_{**}$ will also annihilate $[X, HD \wedge X]_{**}$. Hence $J \leq I$.

Part (b) [Sketch of proof]: To see that $I(X)$ is finitely generated, note that since X is finite, then "most" of D acts trivially on X: we let $B(n)$ denote the following quotient Hopf algebra of D:

$$B(n) = D/(\xi_1, \xi_2, \dots, \xi_n) = \mathbf{F}_2[\xi_{n+1}, \xi_{n+2}, \dots]/(\xi_{n+1}^{2^{n+1}}, \xi_{n+2}^{2^{n+2}}, \dots).$$

Then $B(n)$ should "act trivially on X" for n large enough. More precisely, there is an Atiyah-Hirzebruch spectral sequence with

$$E_2 = H\mathbf{F}_{p**}X \otimes HB(n)_{**} \Rightarrow HB(n)_{**}X.$$

Since X is finite, then its homology $H\mathbf{F}_{p**}X$ is bounded. So if n is large enough, then for degree reasons, the elements in the image of the edge homomorphism

$$H\mathbf{F}_{p**}X \to E_2 = H\mathbf{F}_{p**}X \otimes HB(n)_{00}$$

are all permanent cycles. This is a spectral sequence of modules over $HB(n)_{**}$, so everything must be a permanent cycle, and we find that

$$HB(n)_{**}X \cong H\mathbf{F}_{p**}X \otimes HB(n)_{**}.$$

Consider the ring map $HD_{**} \to HB(n)_{**}$. The annihilator in HD_{**} of $HD_{**}X$ is contained in the annihilator in HD_{**} of $HB(n)_{**}X$. Since

$$\operatorname{ann}_{B(n)_{**}}(HB(n)_{**}X) = (0),$$

then $I(X)$ is contained in the radical of the kernel of $HD_{**} \to HB(n)_{**}$. We can calculate this kernel, up to radical, by imitating the arguments in Section 4.4; we find that we have an F-isomorphism

$$HB(n)_{**} \to \mathbf{F}_2[h_{ts} \mid s < t, n < t] \,/\, (h_{ts}h_{vu} \mid u \geq t),$$

and hence the radical of the kernel of $HD_{**} \to HB(n)_{**}$ is equal to

$$K = (h_{ts} \mid s < t \leq n).$$

If we view K as an ideal of the Noetherian ring

$$\mathbf{F}_2[h_{ts} \mid s < t \leq n]/(h_{ts}h_{vu} \mid u \geq t) \subseteq HD_{**},$$

then we see that not only is K finitely generated, but so is any subideal of it.

Part (c): Now we show that $I(X)$ is invariant under the A-coaction. By Remark 1.3.8, both HD_{**} and $HD_{**}X$ are $A \,\square_D \mathbf{F}_p$-comodules, and one can check that the action map

$$HD_{**} \otimes HD_{**}X \to HD_{**}X$$

is a map of $A \,\square_D \mathbf{F}_p$-comodules. Suppose that $y \in I(X)$, and that under the $A \,\square_D \mathbf{F}_p$-coaction, y maps to $\sum_i a_i \otimes y_i$. (We may assume that the a_i's are linearly independent elements of A.) We want to show that each y_i is in $I(X)$; i.e., that some power of y_i annihilates $HD_{**}X$.

Fix $x \in HD_{**}X$. If we assume that $y^{2^k}x = 0$, then we claim that $y_i^{2^k}x = 0$ for each i. Suppose that under the $A \,\square_D \mathbf{F}_p$-coaction, we have

$$x \longmapsto 1 \otimes x + \sum_{j=1}^{n} b_j \otimes x_j. \tag{4.6.3}$$

We prove, by induction on n, that $y_i^{2^k}x = 0$ for all i. When $n = 0$ (i.e., when x is primitive in $HD_{**}X$), then we have

$$\begin{gathered} HD_{**}X \to (A \,\square_D \mathbf{F}_p) \otimes HD_{**}X, \\ 0 = y^{2^k}x \longmapsto (\sum_i a_i \otimes y_i)^{2^k}(1 \otimes x) = \sum_i a_i^{2^k} \otimes y_i^{2^k}x. \end{gathered}$$

Since the a_i's are linearly independent, we conclude that $y_i^{2^k}x = 0$ for all i.

Suppose that the 2^kth power of each y_i annihilates every element of HD_{**} which has at most n terms in its diagonal, and fix x with diagonal as in (4.6.3). Then by coassociativity, each x_j has n terms or fewer in its diagonal, so we have

$$\begin{aligned} 0 = y^{2^k} x &\longmapsto (\sum_i a_i \otimes y_i)^{2^k} (1 \otimes x + \sum_j b_j \otimes x_j) \\ &= \sum_i (a_i^{2^k} \otimes y_i^{2^k} x + \sum_j a_i^{2^k} b_j \otimes y_i^{2^k} x_j) \\ &= \sum_i a_i^{2^k} \otimes y_i^{2^k} x. \end{aligned}$$

Hence $y_i^{2^k} x = 0$ for each i. □

4.6.2. A thick subcategory conjecture. Since this subsection consists primarily of conjectures, we may as well let p be an arbitrary prime (although we have more evidence for the conjectures when $p = 2$).

Theorems 3.1.2 and 4.1.3 provide support for several conjectures about the "global structure" of the category $\mathsf{Stable}(A)$. For instance, we have the following suggested analogue of the result of Hopkins and Smith [**HSb**, Theorem 11], in which they identify the center of $[X, X]_*$ up to F-isomorphism, for any finite p-local spectrum X. Given a ring R, we let $Z(R)$ denote the center of R.

Conjecture 4.6.4. *For any finite spectrum X, there is an F-isomorphism*

$$Z[X, X]_{**} \to (HD_{**}/I(X))^A.$$

We should point out there is not even an obvious map between these two rings.

Theorem 4.1.3 also suggests a conjectured classification of thick subcategories of finite spectra in $\mathsf{Stable}(A)$; we spend most of this section discussing this conjecture and related ideas.

Conjecture 4.6.5. *The thick subcategories of finite spectra in $\mathsf{Stable}(A)$ are in one-to-one correspondence with the finitely generated radical ideals of HD_{**} which are invariant under the coaction of $A \,\square_D \mathbf{F}_p$.*

As above, given a finite spectrum X, we let $I(X)$ denote the radical of the kernel of $HD_{**} \to [X, HD \wedge X]_{**}$. The conjectured bijection should send an invariant ideal I to the full subcategory $\mathsf{D}(I)$ with objects

$$\{X \text{ finite} \mid I(X) \supseteq I\}.$$

This is clearly a thick subcategory. The other arrow in the bijection should send a thick subcategory D of the finite objects in $\mathsf{Stable}(A)$ to the ideal $I(\mathsf{D})$, defined by

$$I(\mathsf{D}) = \bigcap_{X \in \text{ob } \mathsf{D}} I(X).$$

This is a finitely generated radical invariant ideal by Proposition 4.6.2.

Example 4.6.6. Here is a bit more evidence for Conjecture 4.6.5.

(a) By Example 4.4.5(a), we have maps

$$\begin{aligned} h_{10}\colon\ S^{1,1} &\to S^0, \\ h_{21}^4\colon\ S^{4,24} &\to S^0. \end{aligned}$$

We let S^0/h_{10} and S^0/h_{21}^4 denote the cofibers of these. One can easily compute the ideals of these cofibers:

$$I(S^0/h_{10}) = \sqrt{(h_{10})},$$
$$I(S^0/h_{21}^4) = \sqrt{(h_{21})}.$$

Since $h_{10}h_{21} = 0$ in HD_{**}, then Conjecture 4.6.5 would tell us that

$$\mathrm{thick}(S^0/h_{10}, S^0/h_{21}^4) = \mathrm{thick}(S^0).$$

Let $\mathsf{C} = \mathrm{thick}(S^0/h_{10}, S^0/h_{21}^4)$. One can show directly that S^0 is contained in C, using the octahedral axiom: the cofiber of the map $h_{10}h_{21}^4\colon\ S^{5,25} \to S^0$ fits into a cofibration with S^0/h_{10} and S^0/h_{21}^4, and hence is in C. But since $h_{10}h_{21}^4$ is zero, then this cofiber is just $S^0 \vee S^{4,25}$; hence S^0 is in C.

(b) By Example 4.4.5(b), we have non-nilpotent self-maps of the sphere spectrum called $h_{20}^i h_{21}^j$, for certain exponents i and j. If i and j are both positive, then $I(S^0/(h_{20}^i h_{21}^j)) = \sqrt{(h_{20}h_{21})}$, so Conjecture 4.6.5 would imply that $\mathrm{thick}(S^0/(h_{20}^i h_{21}^j))$ is independent of i and j. Arguing as in part (a), one can see that this is true.

(c) Lastly, we point out that $\mathrm{thick}(S^0) \neq \mathrm{thick}(S^0/(h_{20}^i h_{21}^j))$, even though the two spectra S^0 and $S^0/(h_{20}^i h_{21}^j)$ have the same slope supports (Definition 4.1.4). For instance, if we let $d_0 = h_{20}^2 h_{21}^2$, then $S^0/d_0 \wedge d_0^{-1}S^0 = 0$, and hence $X \wedge d_0^{-1}S^0 = 0$ for every X in $\mathrm{thick}(S^0/d_0)$.

Here is a sketch of part of the proof of Conjecture 4.6.5.

CONJECTURE 4.6.7. *Given any invariant finitely generated radical ideal I of HD_{**}, there is a finite spectrum X so that $I(X) = I$.*

IDEA OF PROOF. Suppose that I is generated by classes $y_1, y_2, \dots, y_n$, with $y_i \in HD_{s_i,t_i}$. We order these so that $s_1 \leq s_2 \leq \cdots \leq s_n$; then $(y_1, \dots, y_i)$ is invariant for each $i \leq n$. For each $i \geq 0$, we define spectra X_i inductively so that $I(X_i) = \sqrt{(y_1, \dots, y_i)}$. We start by letting $X_0 = S^0$; then $I(X_0) = \sqrt{(0)}$. Given X_{i-1} with $I(X_{i-1}) = \sqrt{(y_1, \dots, y_{i-1})}$, Theorem 4.1.3 tells us that then X_{i-1} has a y_i-map; we let X_i be the cofiber of this map. Clearly $I(X_i) \supseteq (y_1, \dots, y_i)$; if we could prove equality, we would be done. $\square$

This would show that the composite $I \mapsto \mathsf{D}(I) \mapsto I(\mathsf{D}(I))$ is the identity. For the composite $\mathsf{D} \mapsto I(\mathsf{D}) \mapsto \mathsf{D}(I(\mathsf{D}))$, one needs to show that every thick subcategory D is of the form $\mathsf{D} = \mathsf{D}(I)$ for some invariant ideal I.

One might also conjecture that there is a bijection between the set of localizing subcategories of $\mathsf{Stable}(A)$ and the set of all radical ideals of HD_{**}, but that seems a bit much to expect without any evidence.

We end this section with one other body of ideas, based on work of Nakano and the author [**NP**] (and this was based, in turn, on work of Friedlander and Parshall [**FP86, FP87**], among others). Let $p = 2$. We let W be the vector space

$$W = \mathrm{Span}_{\mathbf{F}_2}(P_t^s \mid s < t).$$

We view W as being an inhomogeneous sub-vector space of the Steenrod algebra A^*, and we let $V_D(\mathbf{F}_2)$ be the following subset of W:

$$V_D(\mathbf{F}_2) = \{y \in W \mid y^2 = 0\}.$$

We do *not* require that the elements y be homogeneous. One can show (as in the proof of [**NP**, 1.7]) that $V_D(\mathbf{F}_2)$ consists precisely of linear combinations of commuting P^s_t's—i.e., it is a union of affine spaces, one such space for each maximal elementary quotient of A. Therefore it is equal to the prime ideal spectrum of $\varprojlim_{\mathsf{Q}} HE_{**}$, and hence is homeomorphic to the prime ideal spectrum of HD_{**}. At odd primes, we define

$$W_{\mathrm{ev}} = \mathrm{Span}_{\mathbf{F}_p}(P^s_t \mid s < t),$$
$$W_{\mathrm{odd}} = \mathrm{Span}(Q_n \mid n \geq 0),$$

and then let $V_D(\mathbf{F}_p)$ be the set of all elements $y = (y_1, y_2)$ in $W_{\mathrm{ev}} \oplus W_{\mathrm{odd}}$ with $y_1^p = 0$ and $y_2^2 = 0$. We do not have as nice a description of $V_D(\mathbf{F}_p)$ at odd primes; because of this, and for other technical reasons, it might be best to restrict the following discussion to the case $p = 2$.

Given an element $y \in V_D(\mathbf{F}_p)$, we can construct the *y-homology spectrum* $H(y)$, just as we did the P^s_t-homology spectrum in Definition 2.2.1 . Then given a spectrum X, we define its *rank variety*, $V_D(X)$, to be the following subset of $V_D(\mathbf{F}_p)$:

$$V_D(X) = \{y \in V_D(\mathbf{F}_p) \mid H(y)_{**}X \neq 0\}.$$

We say that a subset V of $V_D(\mathbf{F}_p)$ is *realizable* if $V = V_D(X)$ for some finite spectrum X.

CONJECTURE 4.6.8. *Let X be a finite spectrum. The rank variety $V_D(X)$ determines the thick subcategory generated by X, and hence the ring of central self-maps of X, up to F-isomorphism. In other words, there is a bijection between the invariant finitely generated radical ideals of HD_{**} and the realizable subsets of $V_D(\mathbf{F}_p)$.*

This bijection should come about by the following: if X is a finite spectrum, then there should be a homeomorphism (actually, an "inseparable isogeny" [**Ben91b**, p. 172]) between the prime ideal spectrum of $HD_{**}/I(X)$ and $V_D(X)$.

4.7. Construction of spectra of specified type

Let p be a prime.

In this section we construct certain objects for later use; these are analogues of the "generalized Toda $V(n)$ spectra," as used by Mahowald-Sadofsky [**MS95**], among others. Some of the basic ideas are standard; most of the rest are due to them. A few of the details are different in our setting.

We make heavy use of Notations 2.2.4 and 2.4.1 in this section.

The results of this section follow from Theorem 2.4.3 and Lemma 2.4.7, but they have the flavor of other results in this chapter; hence we include them here. (In particular, the results holds at all primes, not just when $p = 2$.)

We start by constructing the relevant spectra. See Notation 2.2.4 and 2.4.1 for the terminology used here.

PROPOSITION 4.7.1. *Let p be a prime, and fix a slope n. Let $1 = d_1 < d_2 < \cdots < d_m$ be the slopes less than n. For any integers $k_1, \ldots, k_m$, there are integers $j_1, \ldots, j_m$ with $k_i \leq j_i$ for each i, so that there is a spectrum $F = F(u_{d_1}^{j_1}, \ldots, u_{d_m}^{j_m})$ satisfying the following.*

(a) *When $n = 1$, $F = S^0$.*
(b) *F is $Z(d)_{**}$-acyclic for $d < n$, and $Z(n)_{**}F \neq 0$.*

(c) *Hence F has a u_n-map; call it u. If $Z(n)_{**}(u)$ is multiplication by $u_n^{j_{m+1}}$, then $F(u_{d_1}^{j_1}, \dots, u_n^{j_{m+1}})$ is the fiber of u. More precisely, there is a fiber sequence*

$$F(u_{d_1}^{j_1}, \dots, u_n^{j_{m+1}}) \to F(u_{d_1}^{j_1}, \dots, u_{d_m}^{j_m}) \xrightarrow{u_n^{j_{m+1}}} \Sigma^{-j_n, -nj_n} F(u_{d_1}^{j_1}, \dots, u_{d_m}^{j_m}).$$

(d) *F is self-dual, as is its u_n-map u. That is, for some number q, we have $DF \cong \Sigma^q F$, and u maps to itself under the chain of isomorphisms*

$$[F, F] \cong [DF, DF] \cong [\Sigma^q F, \Sigma^q F] \cong [F, F].$$

Proof. The statements of (a)-(c) indicate how the spectra are constructed. Starting with S^0, one applies Theorem 2.4.3 to find a u_1-map $u_1^{j_1}$ of it, and one lets $F(u_1^{j_1})$ be the fiber. By definition, essentially, $u_1^{j_1}$ induces an isomorphism on $Z(1)_{**}$, so $F(u_1^{j_1})$ is $Z(1)_{**}$-acyclic; $u_1^{j_1}$ induces zero on $Z(d)_{**}$ for $d > 1$, so $F(u_1^{j_1})$ is not $Z(d)_{**}$-acyclic for any larger value of d. One proceeds inductively. This proves (a)-(c).

Part (d) is also straightforward; it is proved by induction on m. We leave the details to the reader. □

Up to suspension, the object $F(u_{d_1}^{j_1}, \dots, u_{d_m}^{j_m})$ is the analogue in $\mathsf{Stable}(A)$ of Mahowald and Sadofsky's spectrum (in the usual stable homotopy category) $M(p^{j_0}, v_1^{j_1}, \dots, v_{n-1}^{j_{n-1}})$. We have used the letter F rather than M since our spectra are iterated fibers rather than cofibers, as in [**MS95**]. Also, the letter M can get somewhat overworked; in particular, we want to avoid confusion with the functor M_n^f of Section 5.3.

Now we establish the main properties of the spectra $F(u_{d_1}^{j_1}, \dots, u_{d_m}^{j_m})$.

Theorem 4.7.2. *Fix a prime p, a slope n, and other notation as in Proposition 4.7.1, and let $F = F(u_{d_1}^{j_1}, \dots, u_{d_m}^{j_m})$ be as in that result. Then F satisfies the following properties.*

(a) *For any finite spectrum W which is $Z(d)_{**}$-acyclic for all $d < n$, W is in $\mathrm{thick}(F)$.*
(b) *Hence the Bousfield class of F is independent of the choice of exponents j_i.*
(c) *Let u denote the u_n-map of F. Suppose that $\ell_1, \dots, \ell_m$ are integers so that $W = F(u_{d_1}^{\ell_1}, \dots, u_{d_m}^{\ell_m})$ exists, and let v denote its u_n-map as given by Proposition 4.7.1(b). Then there are integers i and j so that $u^i \wedge 1_W = 1_F \wedge v^j$ as self-maps of $F \wedge W$.*
(d) *Suppose that $\ell_1, \dots, \ell_m$ are integers so that $F(u_{d_1}^{\ell_1}, \dots, u_{d_m}^{\ell_m})$ exists, and so that $j_i \leq \ell_i$ for each i. Then there is a map*

$$F(u_{d_1}^{j_1}, \dots, u_{d_m}^{j_m}) \to F(u_{d_1}^{\ell_1}, \dots, u_{d_m}^{\ell_m})$$

commuting with "projection to the top cell"—i.e., the map $F(\dots) \to S^0$.

Proof. (a): For each $i \leq m$, let $F(U_i) = F(u_{d_1}^{j_1}, \dots, u_{d_i}^{j_i})$. We have cofiber sequences

$$F(U_i) \to F(U_{i-1}) \xrightarrow{u_{d_i}^{j_i}} F(U_{i-1}).$$

We claim that $1_W \wedge u_{d_i}^{j_i}$ is a nilpotent self map of $W \wedge F(U_{i-1})$. Once we know this, then we see that $W \wedge F(U_{i-1})$ is in the thick subcategory generated by $W \wedge F(U_i)$ for each i. By induction, $W \wedge F(U_0) = W$ is in the thick subcategory generated by $W \wedge F(U_m)$, which is a subcategory of the thick subcategory generated by $F(U_m) = F$.

The claim that $1_W \wedge u_{d_i}^{j_i}$ is nilpotent follows by application of the vanishing line theorem 2.3.1: the group

$$[W \wedge F(U_{i-1}), W \wedge F(U_{i-1})]_{**} = \pi_{**}(W \wedge DW \wedge F(U_{i-1}) \wedge DF(U_{i-1}))$$

has a vanishing line of slope n, and $1_W \wedge u_{d_i}^{j_i}$ acts along a line of slope $d_i < n$.

(b): This follows immediately from (a).

(c): We would like to use the properties of y-maps discussed in Section 4.2, but these depend on the nilpotence theorem 3.1.5, and hence would force us to work at the prime 2. With a bit of care, we can avoid this dependence.

By Proposition 4.7.1(b), $F \wedge W$ is $Z(d)_{**}$-acyclic for $d < n$; hence $[F \wedge W, F \wedge W]_{**}$ has a vanishing line of slope at least n. If the slope is larger than n, then the powers of both $u \wedge 1_W$ and $1_F \wedge v$ would eventually lie above the vanishing line, and hence would both be zero. So we may assume that the vanishing line has slope equal to n. By our choice of u_n-map u, Lemma 2.4.7 tells us that $u \wedge 1_W$ is central in a band parallel to the vanishing line; $1_F \wedge v$ lies in that band, on the line of slope n through the origin. Therefore $u \wedge 1_W$ and $1_F \wedge v$ commute. By our choice of maps u and v, after raising them to powers, we may assume that both of our u_n-maps agree when restricted to

$$[F \wedge W, HA(m) \wedge F \wedge W]_{**},$$

for large enough m. But this ring is isomorphic to $[F \wedge W, F \wedge W]_{**}$ in the bidegree of interest; hence our self-maps agree in $[F \wedge W, F \wedge W]_{**}$.

(d): This follows from (c). We prove it by induction on m. When $m = 1$, then the following diagram commutes:

$$\begin{tikzcd}
F(u_1^{j_1}) \arrow[r] & S^0 \arrow[r, "u_1^{j_1}"] \arrow[d, equal] & S^0. \arrow[d, "u_1^{\ell_1 - j_1}"] \\
F(u_1^{\ell_1}) \arrow[r] & S^0 \arrow[r, "u_1^{\ell_1}"] & S^0
\end{tikzcd}$$

Hence there is an induced map $F(u_1^{j_1}) \to F(u_1^{\ell_1})$.

Assume that we have a map

$$\begin{tikzcd}
F(u_{d_1}^{j_1}, \ldots, u_{d_{m-1}}^{j_{m-1}}) \arrow[r, "f"] \arrow[d, equal] & F(u_{d_1}^{\ell_1}, \ldots, u_{d_{m-1}}^{\ell_{m-1}}). \arrow[d, equal] \\
F & W
\end{tikzcd}$$

We abbreviate these spectra as F and W, as indicated. Consider the following diagram, in which the rows are fiber sequences:

$$\begin{tikzcd}
F(u_{d_m}^{j_m}) \arrow[r] & F \arrow[r, "u_{d_m}^{j_m}"] \arrow[d, equal] & F. \arrow[d, "u_{d_m}^{\ell_m - j_m}"] \\
F(u_{d_m}^{\ell_m}) \arrow[r] & F \arrow[r, "u_{d_m}^{\ell_m}"] \arrow[d, "f"] & F \arrow[d, "f"] \\
W(u_{d_m}^{\ell_m}) \arrow[r] & W \arrow[r, "u_{d_m}^{\ell_m}"] & W
\end{tikzcd}$$

Clearly the top right square commutes. We claim that part (c) implies that the lower right square commutes. Given this, we get maps

$$F(u_{d_m}^{j_m}) \to F(u_{d_m}^{\ell_m}) \to W(u_{d_m}^{\ell_m}).$$

The composite is the desired map.

To see that the lower right square commutes, we imitate the proof of Corollary 4.2.11. The key is to use Spanier-Whitehead duality so that the question is whether $1 \wedge u_{d_m}^{\ell_m}$ and $Du_{d_m}^{\ell_m} \wedge 1$ agree as self-maps of the spectrum $DF \wedge W$. By part (c), it suffices to show that F is self-dual (up to suspension), as is its u_n-map u. But this is Proposition 4.7.1(d). □

We point out that since $F = F(u_{d_1}^{j_1}, \ldots, u_{d_m}^{j_m})$ is well-defined up to Bousfield class, and since its u_n-map is essentially unique, then the telescope $u_n^{-1}F$ is well-defined up to Bousfield class. But the telescope of a u_n-map of an arbitrary finite spectrum of type n could have a different Bousfield class. For example, at the prime 2, there is a non-nilpotent self-map of the sphere called d_0:

$$d_0 \colon S^{4,18} \to S^0.$$

(Under the map $\pi_{**}S^0 \to HD_{**}$, the element d_0 maps to $h_{20}^2 h_{21}^2$—see Example 4.4.5(b).) Although S^0 and S^0/d_0 are both type 0, they are not Bousfield-equivalent—S^0 sees $d_0^{-1}S^0$, while S^0/d_0 does not. Hence the telescopes $h_0^{-1}S^0$ and $h_0^{-1}(S^0/d_0)$ probably have distinct Bousfield classes.

We also point out that this theorem has the following obvious generalization, at least at the prime 2. See Conjecture 4.6.7 for related work.

Theorem 4.7.3. *Let $p = 2$, and fix a finitely generated invariant radical ideal $I \leq HD_{**}$. Write $I = (y_1, \ldots, y_m)$, and order the generators so that $(y_1, \ldots, y_i)$ is invariant for each $i \leq m$. For any integers $k_1, \ldots, k_m$, there are integers $j_1, \ldots, j_m$ with $k_i \leq j_i$ for each i, so that there is a spectrum $F = F(y_1^{j_1}, \ldots, y_m^{j_m})$ satisfying the following.*

(a) *When $I = \sqrt{(0)}$, then $F = S^0$.*
(b) *I is contained in the ideal of F.*
(c) *Hence if $y \in HD_{**}/I$ is invariant, then F has a y-map, u.*
(d) *F is self-dual, as are its y-maps.*
(e) *For any finite spectrum W with $I \leq I(W)$, then W is in* thick(F).
(f) *Hence the Bousfield class of F is independent of the choice of exponents j_i.*
(g) *Fix y and u as in (c). For any finite spectrum W and any y-map v on W, there are integers i and j so that $u^i \wedge 1_W = 1_F \wedge v^j$ as self-maps of $F \wedge W$.*

(h) *If $\ell_1, \ldots \ell_m$ are integers with $\ell_i \geq j_i$ and such that $F(y_1^{\ell_1}, \ldots, y_m^{\ell_m})$ exists, then there is a map $F(y_1^{\ell_1}, \ldots, y_m^{\ell_m}) \to F(y_1^{j_1}, \ldots, y_m^{j_m})$.*

4.8. Further discussion: slope supports

Let p be any prime. In this section, we discuss slope supports (Definition 4.1.4). This material was originally introduced in [**Pal96b**] as an approach to the periodicity theorem and thick subcategory conjecture. Since those topics now seem more closely related to $I(X)$, the ideal of X, slope supports appear to be more peripheral. On the other hand, since P_t^s-homology groups do determine vanishing lines and other useful information, studying slope supports may be worthwhile.

Our main result in this section is Proposition 4.8.1, which gives a classification, when $p = 2$, of the possible supports of finite spectra.

Recall from Definition 4.1.4 that Slopes is the set of slopes of A, whereas Slopes$'$ is the set

$$\text{Slopes}' = \begin{cases} \{(t,s) \mid t > s \geq 0\}, & p = 2, \\ \{(t,s) \mid t > s \geq 0\} \cup \{n \mid n \geq 0\}, & p \text{ odd}. \end{cases}$$

There is, of course, a (meaningful) bijection between Slopes and Slopes$'$, with the slope $\frac{p|\xi_t^{p^s}|}{2} \in$ Slopes corresponding to $(t,s) \in$ Slopes$'$, and when p is odd, $|\tau_n| \in$ Slopes corresponding to $n \in$ Slopes$'$.

As in Definition 4.1.4, we say that the slope support of a spectrum X is the set

$$\operatorname{supp}(X) = \{n \mid Z(n)_{**}X \neq 0\} \subseteq \text{Slopes}\,.$$

(See Notation 2.2.4 for the definition of $Z(n)$.) We may also view $\operatorname{supp}(X)$ as being a subset of Slopes$'$, using the bijection above. We say that a subset T of Slopes$'$ is admissible if T satisfies the following conditions:

(i) When $p = 2$:

$$(t,s) \in T \Rightarrow (t+1, s) \in T,$$
$$(t,s) \in T \Rightarrow (t+1, s-1) \in T,$$
$$\operatorname{card}(\text{Slopes}' \setminus T) < \infty.$$

(ii) When p is odd: the above conditions, as well as:

$$n \in T \Rightarrow n+1 \in T.$$

We provide a bit of justification for the term "admissible" in the following.

PROPOSITION 4.8.1. [**Pal96b**, Prop. 3.10 and Thm. A.1] *If $T \subseteq$ Slopes$'$ is admissible, then T is the slope support of some finite spectrum. If $p = 2$, then the converse holds: if X is any finite spectrum, then* $\operatorname{supp} X$ *is admissible.*

We conjecture that the converse is also true when p is odd. We can at least prove the following: for X finite, if $(Q_n)_{**}X = 0$, then $(Q_{n-1})_{**}X = 0$ (see [**Pal96b**, A.8]). We also note that there are no restrictions on the slope support of an arbitrary spectrum: the connective P_t^s-homology spectrum p_t^s has slope support equal to $\{(s,t)\}$, so any subset of Slopes$'$ may be realized as the support of a wedge of p_t^s's (and similarly at odd primes, using the p_t^s's and the q_n's).

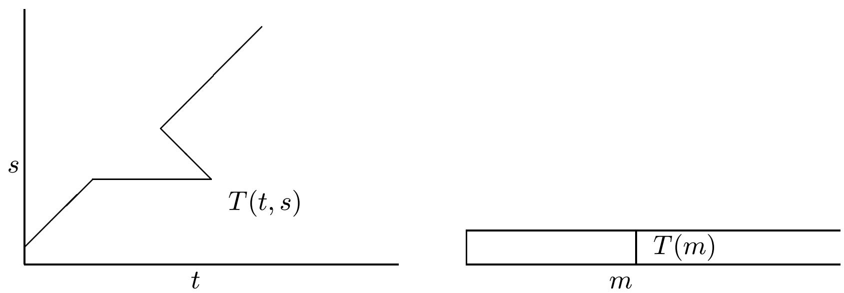


FIGURE 4.8.A. $T(t,s)$ and $T(m)$ as subsets of Slopes$'$.

PROOF. We include a sketch of the proof that every admissible T is the slope support of some finite spectrum, since the corresponding result in [**Pal96b**] assumed that $p = 2$. We focus on the case when p is odd; the reader can imitate this proof or refer to [**Pal96b**] for the $p = 2$ case.

Every admissible $T \subseteq \text{Slopes}'$ can be written as

$$T = T(t_1, s_1) \cap \cdots \cap T(t_n, s_n) \cap T(m)$$

for some numbers t_i, s_i, m, where $T(t,s)$ is the largest admissible set not containing (t,s)—i.e., it is the complement of

$$\{(t,s),\ (t-1,s),\ (t-1,s+1),\ (t-2,s),\ (t-2,s+1),\ (t-2,s+2),\ \ldots\},$$

and $T(m)$ is the complement of $\{0, \ldots, m\}$. See Figure 4.8.A. If we can find sufficiently nice finite spectra $X(t,s)$ and $X(m)$ so that $\operatorname{supp} X(t,s) = T(t,s)$ and $\operatorname{supp} X(m) = T(m)$, then we can realize any slope support T by the spectrum $X(t_1,s_1) \wedge \ldots \wedge X(t_n,s_n) \wedge X(m)$. (One might worry that although $Z(d)_{**}X(t_i,s_i) \neq 0$ and $Z(d)_{**}X(t_j,s_j) \neq 0$ for some d, one might have $Z(d)_{**}X(t_i,s_i) \wedge X(t_j,s_j) = 0$. This does not happen for injective resolutions of finite comodules—see [**NP**, 1.5]—and it certainly does not happen for the examples we construct in the next paragraph. This is what we mean by "sufficiently nice.")

We recall from [**Mit85**] that the quotient Hopf algebra

$$P(n) = A/(\xi_1^{p^{n+1}}, \xi_2^{p^n}, \ldots, \xi_{n+1}^p, \xi_{n+2}, \xi_{n+3}, \ldots; \tau_0, \tau_1, \tau_2, \ldots)$$

has the structure of an A-comodule, extending the $P(n)$-comodule structure. The same clearly holds for

$$V(m) = E[\tau_0, \ldots, \tau_m].$$

We will use $P(n)$ and $V(m)$ to refer to the Hopf algebras, the comodules, or their injective resolutions, depending on the context. It is well-known that $H(P(n), P_t^s)$ is zero if $\xi_t^{p^s} \neq 0$ in $P(n)$—see Proposition 2.2.2 or [**MP72**]. On the other hand, if $\xi_t^{p^s} = 0$ in $P(n)$, then one can see by examining the Poincaré series that $H(P(n), P_t^s) \neq 0$. So $Z(d)_{**}P(n) = 0$ if and only if $y_d \neq 0$ in $P(n)$, and similarly for $V(m)$. Hence as subsets of Slopes$'$, we have

$$\operatorname{supp} P(n) = T(n+1, 0),$$
$$\operatorname{supp} V(m) = T(m).$$

Also, there is a "pth power" functor Φ on the category of A-comodules; if M is evenly graded, then ΦM is defined to be

$$(\Phi M)_n = \begin{cases} M_r & n = pr, \\ 0 & n \text{ not divisible by } p, \end{cases}$$

with comodule structure $\psi_{\Phi M} \colon \Phi M \to A \otimes \Phi M$ given by

$$\psi_{\Phi M}(\Phi x) = (F \otimes i) \circ \psi_M(x).$$

Here, F is the Frobenius map on A defined by $F(a) = a^p$, ψ_M is the comodule structure map on M, and $i \colon M \to \Phi M$ is the map (which multiplies degrees by p) that sends y to Φy. One can readily see that if $s < t$, then

$$\operatorname{supp} \Phi^s P(t-1) = T(t,s).$$

Hence an injective resolution of the comodule

$$\Phi^{s_1} P(t_1 - 1) \otimes \cdots \otimes \Phi^{s_n} P(t_n - 1) \otimes V(m)$$

has slope support

$$T = T(t_1, s_1) \cap \cdots \cap T(t_n, s_n) \cap T(m).$$

See [**Pal96b**, A.1] for the proof of the converse when $p = 2$. □

4.9. Further discussion: miscellany

We have already mentioned several conjectures related to global structure of the category $\mathsf{Stable}(A)$ when $p = 2$—see Section 4.6. As far as working at odd primes, one has to prove analogues of Theorems 3.1.2 and 3.1.5 first; see Section 3.4 for a discussion of those issues.

We presented some preliminary computations of A-invariants in HD_{**} in Section 4.4; it would be nice to have further results. Extensive computer calculations could be useful, and obviously it would be nice to find new families of invariants. Along similar lines, we could use more information about invariant ideals of HD_{**}—basic properties, examples, and of course a classification would be helpful.

It is also natural to wonder if one can prove a version of the periodicity theorem 4.1.3 which uses quasi-elementary quotients of A instead of D.

CHAPTER 5

Chromatic structure

In this chapter we discuss "chromatic" results in $\mathsf{Stable}(A)$. We start in Section 5.1 by discussing Margolis' killing construction [**Mar83**, Chapter 21]. This is the analogue, in our setting, of the functor L_n^f in the ordinary stable homotopy category. We give several different constructions of the functor, and we prove various properties (e.g., for X with nice connectivity properties, if X has finite type homotopy, then so does $L_n^f X$). We also define an analogue of the functor L_n, and we show that $L_n \neq L_n^f$ if $n > 1$, at least at the prime 2.

We have been using the functor H heavily throughout this book; the homotopy groups of HB, for B a quotient of A, are the cohomology groups of the Hopf algebra B. In Section 5.2 we construct $\widehat{H}(-)$, a version of this functor whose homotopy groups are the Tate cohomology groups of B. The spectra $\widehat{H}A(m)$ turn out to be equal to $L_n^f HA(m)$, for n sufficiently large compared to m; we use this result in Section 5.3 to prove that the "chromatic tower," the tower

$$L_0^f X \leftarrow L_1^f X \leftarrow L_2^f X \leftarrow L_3^f X \leftarrow \dots,$$

converges to X, if X is finite. (This is an extension of a theorem of Margolis [**Mar83**, Theorem 22.1].)

In Section 5.4 we discuss some other questions related to chromatic issues, such as constructing chromatic towers in different orders, and relating the chromatic tower construction to the multiple complex construction of Benson and Carlson.

5.1. Margolis' killing construction

In this section we present Margolis' killing construction. This is a (smashing) localization functor that kills off P_t^s- and Q_n-homology for $\xi_t^{p^s}$ and τ_n of large slope.

We make use of Notation 2.2.4 and 2.4.1 in this section.

WARNING. Note that u_n is an element of either $Z(n)_{1,n}$ or $Z(n)_{2,2n}$, depending on the prime and the form of y_n—see Proposition 2.2.2. So as to avoid dividing all of our arguments into cases, we will abuse notation and write u_n^i for the power of u_n in $Z(n)_{i,*}$ (and similarly for u_n-maps: a self-map $u_n^i\colon X \to X$ has bidegree (i, in)). Hence when $y_n = P_t^s$ and p is odd, only even powers of u_n make sense.

We defined the notions of "thick" and "localizing" subcategories in Definition 1.4.7.

DEFINITION 5.1.1. Recall from [**Mil92**] and [**HPS97**, 3.3.3] that given any thick subcategory C of finite spectra, there is a functor $L_{\mathsf{C}}^f\colon \mathsf{Stable}(A) \to \mathsf{Stable}(A)$, called *finite localization away from* C, with the properties:

(i) L_{C}^f is exact; when viewed as a functor $L_{\mathsf{C}}^f\colon \mathsf{Stable}(A) \to L_{\mathsf{C}}^f\mathsf{Stable}(A)$ to the category of L_{C}^f-local spectra, it has a right adjoint.

(ii) There is a natural transformation $1 \to L^f_{\mathsf{C}}$.
(iii) L^f_{C} is idempotent—for any X, the map $L^f_{\mathsf{C}}X \to L^f_{\mathsf{C}}L^f_{\mathsf{C}}X$ induced by the natural transformation in (ii) is an equivalence.
(iv) L^f_{C} is Bousfield localization with respect to the spectrum $L^f_{\mathsf{C}}S^0$.
(v) For any X, $L^f_{\mathsf{C}}X = X \wedge L^f_{\mathsf{C}}S^0$.
(vi) For any finite X, $L^f_{\mathsf{C}}X = 0$ if and only if $X \in \mathsf{C}$; for any X, $L^f_{\mathsf{C}}X = 0$ if and only if X is in the localizing subcategory generated by C.

Properties (i)–(iii) say that L^f_{C} is a *localization* functor [**HPS97**, 3.1.1]; property (v) says that L^f_{C} is *smashing* [**HPS97**, 3.3.2]. We write C^f_{C} for the corresponding *acyclization* functor; that is, $C^f_{\mathsf{C}}X$ is the fiber of $X \to L^f_{\mathsf{C}}X$. Since L^f_{C} is smashing, then $C^f_{\mathsf{C}}X = X \wedge C^f_{\mathsf{C}}S^0$.

DEFINITION 5.1.2. For any slope n, let L^f_n denote finite localization away from the thick subcategory of finite spectra which are $\bigvee_{d \leq n} Z(d)$-acyclic, and write C^f_n for the corresponding acyclization.

In Section 4.7, we constructed a finite spectrum $F(u_1^{j_1}, \ldots, u_n^{j_n})$ for an appropriately chosen set of exponents $j_1, \ldots, j_n$, indexed by slopes. We will also write $F(U_n)$ for this spectrum. If d is the next largest slope after n, then by construction, $F(U_n)$ has a u_d-map. Let $\mathrm{Tel}(d)$ denote its telescope. Theorem 4.7.2 tells us that the Bousfield class $\langle F(U_n) \rangle$ is independent of the exponents $j_1, \ldots, j_n$; as noted after that theorem, the same is true of $\langle \mathrm{Tel}(d) \rangle$.

The following is based on Mahowald and Sadofsky's analysis in [**MS95**] of the functor L^f_n in the ordinary stable homotopy category.

PROPOSITION 5.1.3. *The functor L^f_n can be described in any of the following ways*:
(a) *as finite localization away from the thick subcategory generated by $F(U_n)$;*
(b) *as Bousfield localization with respect to $\bigvee_{d \leq n} \mathrm{Tel}(d)$;*
(c) *and as a colimit*: *if we let $F(u_1^{i_1}, \ldots, u_n^{i_n}) \to S^0$ be projection to the top cell with cofiber $\overline{F}(u_1^{i_1}, \ldots, u_n^{i_n})$, then the map $X \to L^f_n X$ is given by*

$$X \to \varinjlim_{i_1, \ldots, i_n} X \wedge \overline{F}(u_1^{i_1}, \ldots, u_n^{i_n}).$$

(*The maps in the direct system will be defined in the proof.*)

On the other hand, for n large, L^f_n is not Bousfield localization with respect to $\bigvee_{d \leq n} Z(d)$. See Proposition 5.1.10.

PROOF. By Theorem 4.7.2(a), the thick subcategory of finite $\bigvee_{d \leq n} Z(d)$-acyclic spectra is equal to the thick subcategory generated by $F(u_1^{i_1}, \ldots, u_n^{i_n})$ for any choice of exponents $i_1, \ldots, i_n$. This proves part (a).

Part (b) is essentially a Bousfield class computation. Repeated application of Proposition 1.6.1(a) gives us

$$\langle S^0 \rangle = \langle F(U_n) \rangle \vee \langle \mathrm{Tel}(1) \rangle \vee \cdots \vee \langle \mathrm{Tel}(n) \rangle, \tag{5.1.4}$$

as well as pairwise orthogonality of the Bousfield classes on the right. We need to show that a spectrum X is $\bigvee_{d \leq n} \mathrm{Tel}(d)$-acyclic if and only if it is L^f_n-acyclic. Assume that $\bigvee_{d \leq n} \mathrm{Tel}(d)_{**}X = 0$. By the decomposition of Bousfield classes (5.1.4), we see that $\langle X \rangle = \langle X \wedge F(U_n) \rangle$, so $L^f_n X = 0$ if and only if $L^f_n X \wedge F(U_n) = 0$. But certainly $L_n^f F(U_n) = 0$, so since L_n^f is smashing, then $L_n^f F(U_n) \wedge X = 0$. To show the converse, it suffices to show that $F(U_n)$ is $\bigvee_{d \leq n} \mathrm{Tel}(d)$-acyclic; this follows by the orthogonality of the above Bousfield classes.

To prove (c), we first need to construct the direct system. By "projection to the top cell," we mean the composite $F(U_n) \to \ldots \to F(U_1) \to F(U_0) = S^0$. Using properties of the spectra $F(U_k)$ and their u_j-maps (Theorem 4.7.2), we can choose the u_j-maps compatibly; i.e., if we have exponents i_j and ℓ_j so that $i_j \leq \ell_j$ and $F(u_1^{i_1}, \ldots, u_n^{i_n})$ and $F(u_1^{\ell_1}, \ldots, u_n^{\ell_n})$ are defined, then there is a map

$$F(u_1^{i_1}, \ldots, u_n^{i_n}) \to F(u_1^{\ell_1}, \ldots, u_n^{\ell_n})$$

which commutes with projection to the top cell. We let $\overline{F}(u_1^{i_1}, \ldots, u_n^{i_n})$ denote the cofiber of $F(u_1^{i_1}, \ldots, u_n^{i_n}) \to S^0$, and we define the spectrum $L'_n X$ to be the following colimit:

$$\begin{array}{ccc} X & \longrightarrow & \varinjlim_{i_1,\ldots,i_n} \overline{F}(u_1^{i_1}, \ldots, u_n^{i_n}) \wedge X. \\ \| & & \| \\ X & \xrightarrow{g'} & L'_n X \end{array}$$

For example, if $n = 1$, then we have $S^0 \xrightarrow{g'} u_1^{-1} S^0$, where $u_1^{-1} S^0$ is the mapping telescope of $u_1\colon S^0 \to S^0$, and $\mathrm{fiber}(g') = F(u_1^{\infty})$ is the analogue of the mod p^{∞} Moore spectrum.

We need to verify that L'_n agrees with L_n^f. To do this, we note that L'_n is smashing, and we show that $g'\colon S^0 \to L'_n S^0$ is an $L_n^f S^0$-equivalence and that $L'_n S^0$ is $L_n^f S^0$-local. By construction, the fiber of g' is in the localizing subcategory generated by the spectra $F(u_1^{i_1}, \ldots, u_n^{i_n})$, and hence g' is an $L_n^f S^0$-equivalence.

To show that $L'_n S^0$ is local, we have to show that $[W, L'_n S^0]_{**} = 0$ for any L_n^f-acyclic W. For any finite localization, the acyclics are the localizing subcategory generated by the finite acyclics, so it suffices to show this when W is a finite acyclic. Note that if W is a finite acyclic, then so is DW, the Spanier-Whitehead dual of W. Since

$$[W, L'_n S^0]_{**} = [S^0, DW \wedge L'_n S^0]_{**} = \pi_{**} L'_n DW,$$

it suffices to show that $L'_n W = 0$ for any finite acyclic W. This is easy: if W is an acyclic, then for each slope $j \leq n$, if k is the next largest slope after j, then by vanishing lines, every u_k-map of $F(u_1^{i_1}, \ldots, u_j^{i_j}) \wedge W$ is nilpotent. Therefore, a cofinal set of maps in the direct system defining $L'_n W$ is zero. $\square$

Recall from Definition 2.4.2 that a spectrum is of type n if it is $\bigvee_{d<n} Z(d)_{**}$-acyclic, but not $Z(n)_{**}$-acyclic. By Theorem 2.4.3, any finite type n spectrum has a u_n-map.

Proposition 5.1.5. *Suppose that X is a finite type n spectrum with u_n-map u. Then $L_n^f X = u^{-1} X$.*

Proof. As in the proof of Proposition 5.1.3(c), we have to check that the map $X \to u^{-1} X$ is an L_n^f-equivalence, and that $u^{-1} X$ is L_n^f-local. For the former, note that we have a diagram of cofibrations

$$\begin{tikzcd}
X/(u) \arrow[r] \arrow[d] & X \arrow[r, "u"] \arrow[d, equal] & X \arrow[d, "u"] \\
X/(u^2) \arrow[r] \arrow[d] & X \arrow[r, "u^2"] \arrow[d, equal] & X \arrow[d, "u"] \\
X/(u^3) \arrow[r] \arrow[d] & X \arrow[r, "u^3"] \arrow[d, equal] & X \arrow[d, "u"] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

The sequential colimit is the cofibration

$$X/(u^\infty) \to X \to u^{-1}X.$$

Note that each spectrum $X/(u^n)$ is finite of type greater than n, and so is in the thick subcategory generated by $F(U_n)$ by Theorem 4.7.2(a). Hence $X/(u^\infty)$ is in $\mathrm{loc}(F(U_n))$; hence $X \to u^{-1}X$ is an L_n^f-equivalence.

To show that $u^{-1}X$ is L_n^f-local, we have to show that $[F(U_n), u^{-1}X]_{**} = 0$. We compute:

$$\begin{aligned}
[F(U_n), u^{-1}X]_{**} &= \pi_{**}(DF(U_n) \wedge u^{-1}X) \\
&= \pi_{**}(u^{-1}DF(U_n) \wedge X) \\
&= u^{-1}\pi_{**}(DF(U_n) \wedge X).
\end{aligned}$$

Since $F(U_n)$ is of type greater than n, it has a vanishing line of slope greater than n; the same then goes for $DF(U_n) \wedge X$. Since u acts at slope n, it acts nilpotently on $\pi_{**}(DF(U_n) \wedge X)$. $\square$

Here is the main theorem of this section. It tells us that L_n^f "kills off" certain P_t^s-homology groups; we also have a connectivity result.

THEOREM 5.1.6 (Margolis' killing construction). *Fix a slope n and let X be a spectrum.*

(a) *In the cofiber sequence $C_n^f X \xrightarrow{f} X \xrightarrow{g} L_n^f X$, we have*
  (i) *$Z(d)_{**}g$ is an isomorphism if $d \leq n$, and*
  (ii) *$Z(d)_{**}f$ is an isomorphism if $d > n$.*

  In other words,
  (i)′ *$Z(d)_{**}C_n^f X = 0$ if $d \leq n$, and*
  (ii)′ *$Z(d)_{**}L_n^f X = 0$ if $d > n$.*

(b) *Suppose that X is a CL-spectrum (Definition 1.4.6). Then $L_n^f X$ is "bounded to the left": for each i, $\pi_{ij}L_n^f = 0$ for $j \ll 0$. If, in addition, X has finite type homotopy, then so does $L_n^f X$.*

PROOF. Part (a): By the theory of finite localization in [**HPS97**], the spectrum $C_n^f X$ is in the localizing subcategory generated by the finite $\bigvee_{d \leq n} Z(d)$-acyclics, and hence is $\bigvee_{d \leq n} Z(d)$-acyclic itself. This proves (i).

Since homology commutes with direct limits, we see that $Z(d)_{**}\,\mathrm{Tel}(k)$ is zero if $d \neq k$; in particular, $\bigvee_{k \leq n} \mathrm{Tel}(k)$ is $\bigvee_{d > n} Z(d)$-acyclic. Since L_n^f is Bousfield localization with respect to $\bigvee_{k\leq n} \mathrm{Tel}(k)$, then we conclude that $L_n^f S^0$ is $\bigvee_{d>n} Z(d)$-acyclic. Since L_n^f is smashing, this proves (ii).

Part (b): Given X as in the statement, we show that the maps in the direct system defining $L_n^f X$ (Proposition 5.1.3(c)) are isomorphisms on π_{**} in a range increasing with the exponents.

As above, we let $F(U_k) = F(u_1^{i_1}, \dots, u_k^{i_k})$ and we assume that d is the next largest slope after k. We write $F(U_k)(u_d^{i_d}) = F(u_1^{i_1}, \dots, u_d^{i_d})$ for the fiber of the u_d-map on $F(U_k)$. Consider the following diagram:

$$\begin{tikzcd}
\dots \arrow[r] & \Sigma^{-i_d+1,-i_d d}F(U_k) \arrow[r] \arrow[d, "u_d^{i_d}"] & F(U_k)(u_d^{i_d}) \arrow[r] \arrow[d, "f_1"] & F(U_k) \arrow[r, "u_d^{i_d}"] \arrow[d, equal] & \dots \\
\dots \arrow[r] & \Sigma^{-2i_d+1,-2i_d d}F(U_k) \arrow[r] \arrow[d, "u_d^{i_d}"] & F(U_k)(u_d^{2i_d}) \arrow[r] \arrow[d, "f_2"] & F(U_k) \arrow[r, "u_d^{2i_d}"] \arrow[d, equal] & \dots \\
\dots \arrow[r] & \Sigma^{-3i_d+1,-3i_d d}F(U_k) \arrow[r] \arrow[d, "u_d^{i_d}"] & F(U_k)(u_d^{3i_d}) \arrow[r] \arrow[d, "f_3"] & F(U_k) \arrow[r, "u_d^{3i_d}"] \arrow[d, equal] & \dots \\
& \vdots & \vdots & \vdots &
\end{tikzcd}$$

It suffices to show that the maps $f_r \wedge 1_X$ are isomorphisms in a range increasing with r. Well, $f_r \wedge 1_X$ is an isomorphism whenever the map

$$\Sigma^{-ri_d+1,-ri_d d}F(U_k) \wedge X \xrightarrow{u_d^{i_d}\wedge 1_X} \Sigma^{-(r+1)i_d+1,-(r+1)i_d d}F(U_k) \wedge X \tag{5.1.7}$$

from the left column is. The fiber is $\Sigma^{-ri_d+1,-ri_d d}F(U_k)(u_d^{i_d}) \wedge X$; this spectrum has a vanishing line with some slope $\ell > d$, and we can use Remark 2.3.4 to compute its intercept. To do this, we compute $(H\mathbf{F}_p)_{**}F(U_k)(u_d^{i_d})$—each u_j-map induces the zero map on $(H\mathbf{F}_p)_{**}$, so this computation is easy. We find that

(i) $(H\mathbf{F}_p)_{i*} = 0$ for $i \ll 0$,
(ii) $(H\mathbf{F}_p)_{i*} = 0$ for $i > 0$, and
(iii) $(H\mathbf{F}_p)_{*j} = 0$ for $j < N$ for some fixed number N.

By the Hurewicz Theorem 1.4.4, (i) and (iii) also hold for π_{**}; hence we can compute the equation of the vanishing line for $F(U_k)(u_d^{i_d})$. Smashing with X moves the intercept a bit, so we find that $\pi_{ij}F(U_k)(u_d^{i_d}) \wedge X = 0$ when $j < \ell i + N'$ for some N' which is independent of r. Therefore the map (5.1.7) is an isomorphism on π_{ij} when $j + ri_d d < \ell(i + ri_d - 1) + N'$, i.e., when $j < \ell i + ri_d(\ell - d) + (N' - \ell)$. Since $\ell > d$, then as r increases, this range increases.

So we see that for each s, then the graded groups π_{s*} of the direct system have a uniform bound to the left; hence the same is true of the direct limit. If X also has finite type homotopy, then since the homotopy of each $F(U_k)$ is of finite type, the same goes for $F(U_k) \wedge X$; since the homotopy of the direct system stabilizes at each bidegree, we are done. □

REMARK 5.1.8. (a) Much of this was already known in the module setting. For $p = 2$, this is due to Margolis [**Mar83**, Theorem 21.1]; part (a) (and the connectivity in part (b)) for arbitrary primes can be found in [**Pal92**, Theorem 3.1]. Given an A-comodule M and a slope n, then the module-version of the killing construction (dualized to the category of comodules) gives comodules $M\langle n+1,\infty\rangle$ and $M\langle 1,n\rangle$, well-defined up to injective summands, and an injective comodule J so that the following is short exact:

$$0 \to M\langle n+1,\infty\rangle \to M \oplus J \to M\langle 1,n\rangle \to 0.$$

Being well-defined up to injective summands translates in our setting to being well-defined up to an object of $\mathrm{loc}(A)$ (cf. Lemma 5.1.9). So if X is an injective resolution of M, then an injective resolution for $M\langle 1,n\rangle$ is a connective cover for $L_n^f X$.

(b) By studying the proof of Theorem 5.1.6(b), one can show that $C_n^f S^0$ has a nice vanishing curve: for $i \geq 0$, $\pi_{ij} C_n^f S^0$ has a vanishing line of slope n, and if d is the slope preceding n, then for $i < 0$, $\pi_{ij} C_n^f S^0$ has a vanishing line of slope d. For example, when $n = 1$, we have

$$\begin{array}{ccccc} C_1^f S^0 & \longrightarrow & S^0 & \longrightarrow & L_1^f S^0. \\ \| & & \| & & \| \\ F(u_1^\infty) & \longrightarrow & S^0 & \longrightarrow & u_1^{-1} S^0 \end{array}$$

So

$$\pi_{**} L_1^f S^0 = u_1^{-1} \pi_{**} S^0 = h_{10}^{-1} \pi_{**} S^0 = \mathbf{F}_2[h_{10}^{\pm 1}].$$

Hence $\pi_{ij} C_1^f S^0$ has a vanishing line of slope 2 for positive i, and a vanishing line of slope 1 (a vertical line, in $(i, j-i)$-coordinates) for i negative. See also Figure 5.1.A.

(c) One can generalize this construction. We set $p = 2$ and use the periodicity theorem 4.1.3 and its corollary Theorem 4.7.3. Let I be a finitely generated invariant ideal of HD_{**}, and let C be the thick subcategory of finite spectra X with $I(X) \supseteq I$. Then we have a finite localization functor L_I^f and a cofibration

$$C_I^f X \xrightarrow{f} X \xrightarrow{g} L_I^f X.$$

If the ideal I is generated by classes u_{d_i}, then $Z(d)_{**} f$ is an isomorphism for $d \notin \{d_i\}$, and $Z(d)_{**} g$ is an isomorphism if $d \in \{d_i\}$. The analogue of Proposition 5.1.3 holds. Our proof of Theorem 5.1.6(b), on the other hand, does not work in this situation; it may be that these finite type and connectivity results only hold when I is as in the theorem.

We need the following property of L_n^f in Section 5.2.

Lemma 5.1.9. *For any slope n, $L_n^f A = 0$. Hence any spectrum X in the localizing subcategory generated by A is L_n^f-acyclic.*

Proof. Since the category of L_n^f-acyclics contains a finite spectrum $F(U_n)$, then it contains A by Lemma 1.4.9. □

We close this section with a few remarks about the telescope conjecture (see [**Rav84**] and [**HPS97**, 3.3.8]). Let L_n denote Bousfield localization with respect to $\bigvee_{d \leq n} Z(d)$.

Proposition 5.1.10. *Let $p = 2$. We have $L_1 = L_1^f$. If $n > 1$, then $L_n \neq L_n^f$.*

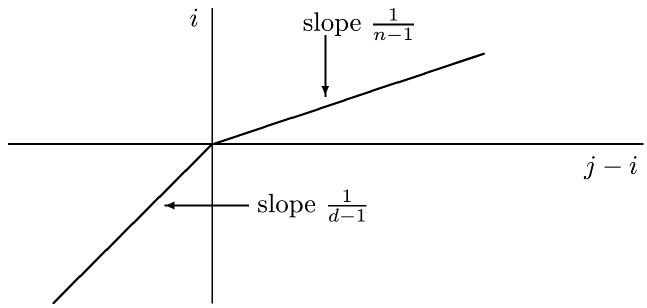


FIGURE 5.1.A. "Vanishing curve" for $\pi_{ij}(C_n^f S^0)$: $\pi_{ij}(C_n^f S^0)$ is zero above the two indicated lines. Here, d is the slope preceding n, and slopes are labeled in $(i, j-i)$-coordinates. These lines are drawn through the origin for convenience, but their intercepts may actually be nonzero.

One can interpret the statement "$L_n = L_n^f$" as an analogue of the telescope conjecture, so this result would say that the telescope conjecture fails except when $n = 1$. On the other hand, it seems more proper to refer to the statement "every smashing localization functor is a finite localization" as the telescope conjecture [**HPS97**, 3.3.8]. Since we do not know whether L_n is smashing, this result does not necessarily present a counterexample to this version of the telescope conjecture.

PROOF. We know that the sphere has a u_1-map; a simple computation shows that $u_1^{-1}S^0 = Z(1)$, so that Bousfield localization with respect to $L_1^f S^0 = u_1^{-1}S^0$ is the same as that with respect to $Z(1)$.

There is a non-nilpotent element $d_0 \in \pi_{4,18}S^0 = \operatorname{Ext}_A^{4,18}(\mathbf{F}_2, \mathbf{F}_2)$. We claim that $d_0^{-1}S^0$ is L_n-acyclic for all n, and L_n^f-local for $n \geq 3$. Since 3 is the next slope after 1, this covers all of the bases.

For degree reasons, d_0 induces the zero map on $Z(d)_{**}$ for all d. (If an element $\alpha \in \pi_{ij}S^0$ is nonzero on $Z(d)_{**}S^0 = Z(d)_{**}$, then we must have $d = j/i$.) Hence $Z(d)_{**}(d_0^{-1}S^0) = 0$ for all d, and $d_0^{-1}S^0$ is L_n-acyclic for all n. To show that $d_0^{-1}S^0$ is L_n^f-local, we show that $[F, d_0^{-1}S^0]_{**} = 0$ for any finite $\bigvee_{d \leq n} Z(d)$-acyclic F. We compute:

$$\begin{aligned} [F, d_0^{-1}S^0]_{**} &= [S^0, DF \wedge d_0^{-1}S^0]_{**} \\ &= [S^0, d_0^{-1}DF]_{**} \\ &= d_0^{-1}\pi_{**}DF. \end{aligned}$$

Since F is a finite $\bigvee_{d \leq n} Z(d)_{**}$-acyclic, so is DF (by Proposition 2.2.6). By Theorem 2.3.1, $\pi_{**}DF$ has a vanishing line of some slope $m > n$. Since $n \geq 3$, we know that $m \geq 6$, so the slope m is larger than $\frac{9}{2}$, the slope of d_0. Hence d_0 acts nilpotently on $\pi_{**}DF$; i.e., $d_0^{-1}\pi_{**}DF = 0$. □

When p is odd, one can again show that $L_1 = L_1^f$. There is every reason to expect that there are non-nilpotent elements in $\operatorname{Ext}_A^{**}(\mathbf{F}_p, \mathbf{F}_p)$ which are not detected by any single $Z(d)$; once one knew this, one could conclude that $L_n \neq L_n^f$ for n large.

5.2. A Tate version of the functor H

In this section we introduce a "Tate" version of the functor H, and we relate it to the functor L_n^f, at least when applied to the quotient Hopf algebra $A(m)$. We will use our computations here to prove chromatic convergence in Section 5.3.

Let B be a quotient Hopf algebra of A. Note that practically all of our results hold in the category $\mathsf{Stable}(B)$; the main exceptions are the strict inequalities in Theorem 4.5.1. Let $\mathrm{res}_{A,B}\colon \mathsf{Stable}(A) \to \mathsf{Stable}(B)$ denote the forgetful functor (also known as *restriction*); $\mathrm{res}_{A,B}$ has a right adjoint, *induction*, written $\mathrm{ind}_{B,A}$, and defined by $X \mapsto A \,\square_B X$ (cf. Lemma 1.3.4(a)). These functors are exact (i.e., they take cofibrations to cofibrations), and restriction preserves the smash product and the sphere object. (In the language of [**HPS97**, Section 3.4], the restriction functor is a *stable morphism*.)

When B is a finite-dimensional Hopf algebra, we may consider another stable homotopy category associated to it, called the *stable category of B-comodules*, written $\mathsf{StComod}(B)$. We provide a brief review of the relevant results here; see [**HPS97**, Section 9.6] for a few more details. The objects of $\mathsf{StComod}(B)$ are B-comodules, and the morphisms $\underline{\mathrm{Hom}}_B^*$ are defined as follows: define the morphisms of degree zero to be $\underline{\mathrm{Hom}}_B(X,Y) = \mathrm{Hom}_B(X,Y)/\simeq$, where $f \simeq g\colon X \to Y$ if $f-g$ factors through an injective comodule. Hence two comodules are equivalent in $\mathsf{StComod}(B)$ if they differ by injective summands. Define the desuspension functor Σ^{-1} by the short exact sequence

$$0 \to X \to B \otimes X \to \Sigma^{-1} X \to 0,$$

where $B \otimes X$ is the cofree comodule on X (Definition 1.1.9). This functor is invertible: ΣX is any comodule which fits into a short exact sequence

$$0 \to \Sigma X \to I \to X \to 0,$$

where I is injective. (Since B is finite, then injectives and projectives are the same, and there are enough projectives.) We let

$$\underline{\mathrm{Hom}}_B^i(X,Y) = \underline{\mathrm{Hom}}_B(\Sigma^i X, Y).$$

$\mathsf{StComod}(B)$ is a stable homotopy category. Indeed, on $\mathsf{Stable}(B)$ one has finite localization (Definition 5.1.1) away from the thick subcategory generated by the finite spectrum $B = H\mathbf{F}_p$; $\mathsf{StComod}(B)$ is equivalent to the full subcategory of $\mathsf{Stable}(B)$ of L_B^f-local objects (see [**HPS97**, 9.6.3–4]). So we have a functor $L_B^f\colon \mathsf{Stable}(B) \to \mathsf{StComod}(B)$ with a right adjoint $J\colon \mathsf{StComod}(B) \to \mathsf{Stable}(B)$. (Every localization functor has a right adjoint, namely inclusion of the local objects into the category.)

Definition 5.2.1. We note that the object HB is $\mathrm{ind}_{B,A}(S^0)$, and we define the object $\widehat{H}B$ to be $\mathrm{ind}_{B,A}(J(S^0))$.

Fix a slope n. On each of the categories $\mathsf{Stable}(A)$, $\mathsf{Stable}(B)$, and $\mathsf{StComod}(B)$, we can define the functor L_n^f to be smashing with $L_n^f S^0$, or with its image under res or $L_B^f \circ \mathrm{res}$. This gives us the commuting diagram of functors

$$\begin{tikzcd}
\mathsf{Stable}(A) \arrow[r, "\mathrm{res}"] \arrow[d, "L_n^f"] & \mathsf{Stable}(B) \arrow[r, "L_B^f"] \arrow[d, "L_n^f"] & \mathsf{StComod}(B), \arrow[d, "L_n^f"] \\
L_n^f\mathsf{Stable}(A) \arrow[r, "\mathrm{res}"] & L_n^f\mathsf{Stable}(B) \arrow[r, "L_B^f"] & L_n^f\mathsf{StComod}(B)
\end{tikzcd}$$

along with the commuting diagram of their right adjoints

$$\begin{tikzcd}
\mathsf{Stable}(A) & \mathsf{Stable}(B) \arrow[l, "\mathrm{ind}"'] & \mathsf{StComod}(B). \arrow[l, "J"'] \\
L_n^f\mathsf{Stable}(A) \arrow[u, "R_{\mathsf{Stable}(A)}"] & L_n^f\mathsf{Stable}(B) \arrow[l, "\mathrm{ind}"'] \arrow[u, "R_{\mathsf{Stable}(B)}"] & L_n^f\mathsf{StComod}(B) \arrow[l, "J"'] \arrow[u, "R_{\mathsf{St}(B)}"]
\end{tikzcd}$$

(The first diagram commutes by the definition of the vertical functors, and the second diagram commutes as a result—given a commuting square of left adjoints, their right adjoints also commute.) Recall from Notation 2.2.4 that y_d is the element of A—either $\xi_t^{p^s}$ or τ_n—"with slope d."

PROPOSITION 5.2.2. *Fix a quotient Hopf algebra B of A, and let*

$$N = \max\{d \mid y_d \neq 0 \text{ in } B\}.$$

Then if n is a slope larger than N, we have

$$L_n^f HB = \widehat{H}B.$$

PROOF. We claim that the two maps

$$\alpha\colon\ HB \to L_n^f HB,$$
$$\beta\colon\ HB \to \widehat{H}B,$$

are the same. To show this, we show that the spectrum $\widehat{H}B$ is L_n^f-local, and that the map $\beta\colon\ HB \to \widehat{H}B$ is an L_n^f-equivalence. By Lemma 5.2.4 below, the fiber of β has homotopy concentrated in the third quadrant, so by Lemma 1.4.8, it is in $\mathrm{loc}(A)$. In particular, it is L_n^f-acyclic by Lemma 5.1.9; hence β is an L_n^f-equivalence.

To show that $\widehat{H}B$ is local, we show that it is in the image of the right adjoint $R_{\mathsf{Stable}(A)}$ of $L_n^f\colon\ \mathsf{Stable}(A) \to L_n^f\mathsf{Stable}(A)$. We claim, in fact, that $R_{\mathsf{St}(B)}$ is the identity functor. Given this claim, then we use the commutativity of the diagram of right adjoints: we have

$$\begin{aligned}
\widehat{H}B &= \mathrm{ind}(JS^0) \\
&= \mathrm{ind}(J(R_{\mathsf{St}(B)}S^0)) \\
&= R_{\mathsf{Stable}(A)}(\mathrm{ind}(JS^0)).
\end{aligned}$$

Hence $\widehat{H}B$ is in the image of $R_{\mathsf{Stable}(A)}$, and so it is local.

It remains to verify the claim that $R_{\mathsf{St}(B)}$ is the identity functor, or what is the same, that

$$L_n^f\colon\ \mathsf{StComod}(B) \to \mathsf{StComod}(B)$$

is the identity functor. So it suffices to show that the cofiber of $S^0 \to L_n^f S^0$ is zero in the category $\mathsf{StComod}(B)$. This cofibration is obtained by applying the functor

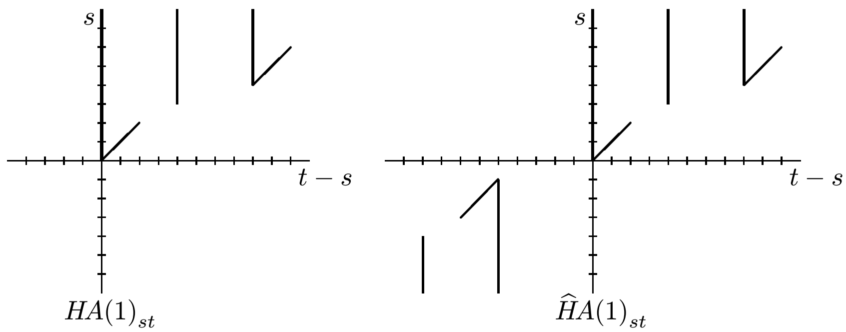


FIGURE 5.2.A. The coefficients of $HA(1)$ and $\widehat{H}A(1)$ when $p = 2$. Since the top cell of $A(1)$ is in degree 6, the third quadrant of $\widehat{H}A(1)_{**}$ is the same as the first quadrant, reflected across the origin and then translated by $(-1, -6)$.

$L^f_B \circ \operatorname{res}$ to this cofibration in $\mathsf{Stable}(A)$:

$$C^f_n S^0 \to S^0 \to L^f_n S^0.$$

So the cofiber under consideration is $L^f_B(\operatorname{res} C^f_n S^0)$. By Proposition 5.1.3, $L^f_n S^0$ is finite-localization away from the thick subcategory generated by $F(U_n)$; but $F(U_n)$ has no $Z(d)$-homology for any d with $y_d \in B$, so $\operatorname{res}(F(U_n))$ is in $\operatorname{loc}(B)$. Since $\operatorname{res}(F(U_n))$ is finite, it is in fact in $\operatorname{thick}(B)$. By fundamental properties of finite localizations (Definition 5.1.1), $\operatorname{res}(C^f_n S^0)$ is in $\operatorname{loc}(F(U_n))$, and hence in $\operatorname{loc}(B)$. Hence $L^f_B(\operatorname{res} C^f_n S^0) = 0$, as desired. □

We are particularly interested in the case $B = A(m)$ (this quotient of A is defined in Example 2.1.4).

COROLLARY 5.2.3. *Fix an integer $m \geq 0$. For n sufficiently large, we have*

$$L^f_n HA(m) = \widehat{H}A(m).$$

In particular, n should be larger than

$$\max\{d \mid y_d \neq 0 \text{ in } A(m)\} = \begin{cases} |\xi_{m+1}| = 2^{m+1} - 1 & \text{if } p = 2, \\ \frac{p}{2}|\xi_m| = p^{m+1} - p & \text{if } p \text{ is odd.} \end{cases}$$

Now we "compute" the homotopy groups of $\widehat{H}B$. See Figure 5.2.A for an example.

LEMMA 5.2.4. *Let B be a finite-dimensional quotient Hopf algebra of A, and consider $\pi_{ij}\widehat{H}B$.*

(a) *When $ij < 0$, then $\pi_{ij}\widehat{H}B = 0$. (In other words, the homotopy is concentrated in the first and third quadrants.)*
(b) *When i and j are nonnegative, the map $HB \to \widehat{H}B$ induces an isomorphism $\pi_{ij}\widehat{H}B \cong \pi_{ij}HB$.*
(c) *Let d be the maximal degree in which B is nonzero. When i and j are negative, we have an isomorphism $\pi_{ij}\widehat{H}B \cong \pi_{-1-i,-d-j}HB$.*

PROOF. We work in the category $\mathsf{Stable}(B)$. We have adjoint functors

$$L\colon \mathsf{Stable}(B) \to \mathsf{StComod}(B),$$
$$J\colon \mathsf{StComod}(B) \to \mathsf{Stable}(B),$$

and we want to compute the homotopy of JS^0 by comparing it to that of $S^0 \in \mathsf{Stable}(B)$. To do this, we need to recall the description of the functor J from [**HPS97**, 9.6.7]. Let I denote an injective resolution of the B-comodule $\mathbf{F}_p$ (i.e., $I \cong S^0$ in $\mathsf{Stable}(B)$); then $\mathrm{Hom}_{\mathbf{F}_p}(I, \mathbf{F}_p)$ is a projective resolution of $\mathbf{F}_p$. We splice these resolutions together to get the "Tate complex" $t_B(\mathbf{F}_p)$:

$$\ldots \to \mathrm{Hom}(I_1, \mathbf{F}_p) \to \mathrm{Hom}(I_0, \mathbf{F}_p) \to I_0 \to I_1 \to \ldots.$$

Then the functor J is defined by $J(M) = t_B(\mathbf{F}_p) \otimes M$. It is clear that the map $S^0 \to J(S^0)$ induces an isomorphism on π_{i*} for $i \geq 0$. If we take I to be a minimal injective resolution, then $\mathrm{Hom}(I, \mathbf{F}_p)$ is a minimal projective resolution; by minimality, $[S^0, JS^0]_{i*} = [\mathbf{F}_p, JS^0]_{i*}$ gives the primitives in JS^0 in degree i, and there is one primitive for each summand isomorphic to B. Since $\mathrm{Hom}(B, \mathbf{F}_p) \cong \Sigma^{-d} B$, we get the reflection and translation as described in (c). $\square$

5.3. Chromatic convergence

It is easy to see that given a spectrum X and slopes n_1 and n_2 with $n_1 < n_2$, then the map $X \to L^f_{n_1} X$ factors through $X \to L^f_{n_2} X$. Hence we get a tower of cofibrations:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d, equal] & \vdots \arrow[d] \\
C^f_3 X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & L^f_3 X \arrow[d] \\
C^f_2 X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & L^f_2 X \arrow[d] \\
C^f_1 X \arrow[r] & X \arrow[r] & L^f_1 X
\end{tikzcd}$$

(Strictly speaking, we have defined L^f_n only when n is a slope. For a general n, we define L^f_n, as in Definition 5.1.2, to be finite localization away from the finite spectra which are $\bigvee_{d \leq n} Z(d)$-acyclic. So if n is not a slope, then $L^f_n = L^f_{n-1}$.) We may as well just focus on the right hand column, giving the following diagram:

$$\begin{tikzcd}
0 & L^f_1 X \arrow[l] & L^f_2 X \arrow[l] & L^f_3 X \arrow[l] & \ldots. \arrow[l] \\
& M^f_1 X \arrow[u, "="] & M^f_2 X \arrow[u] & M^f_3 X \arrow[u] &
\end{tikzcd}$$

Here, $M^f_n X = X \wedge M^f_n S^0$ is defined to be the fiber of $L^f_n X \to L^f_{n-1} X$. We call this diagram the *chromatic tower* for X. (Clearly if n is not a slope, then $M^f_n X = 0$.)

The following theorem, in the module setting for $p = 2$, is due to Margolis [**Mar83**, Theorem 22.4] (see also [**Pal94**]).

THEOREM 5.3.1 (Chromatic convergence). *If X is finite, then $X = \varprojlim_n L_n^f X$.*

Indeed, the proof shows that the tower of groups $\pi_{**} L_n^f X$ is pro-constant.

In order to prove Theorem 5.3.1, we need the following proposition. This proposition may hold for general spectra, not just for finite spectra and rings, but we do not know how to prove that. $Z(n)$ is defined in Notation 2.2.4.

PROPOSITION 5.3.2. *Suppose that n is a slope. If R is either a finite spectrum or a ring spectrum, and if $Z(n)_{**}R = 0$, then $M_n^f R = 0$.*

PROOF. If X is a finite spectrum, then $X \wedge DX$ is a ring spectrum; arguing as in Proposition 2.2.6, we have $Z(n)_{**}X = 0 \Leftrightarrow Z(n)_{**}(X \wedge DX) = 0$ and $M_n^f X = 0 \Leftrightarrow M_n^f(X \wedge DX) = 0$. So the finite case follows from the ring spectrum case.

Suppose that m is the slope preceding n. Then we have a cofiber sequence

$$M_n^f R \to L_n^f R \xrightarrow{\overline{H}} L_m^f R.$$

By the octahedral axiom, we also have a cofiber sequence

$$C_n^f R \xrightarrow{H} C_m^f R \to M_n^f R.$$

We start by looking at the map H, or rather, the maps related to it in the direct limit defining $L_n^f R$. Consider the following diagram. (As in Section 5.1, $F(U_m)$ denotes $F(u_1^{i_1}, \ldots, u_m^{i_m})$, and $F(U_m)(u_n^{i_n})$ denotes the fiber of the $u_n^{i_n}$-map on $F(U_m)$.)

$$\begin{array}{ccccccc}
F(U_m) \wedge R & \longrightarrow & F(U_m)(u_n^{i_n}) \wedge R & \xrightarrow{h} & F(U_m) \wedge R & \xrightarrow{u_n^{i_n} \wedge 1_R} & F(U_m) \wedge R. \\
\Big\downarrow{\scriptstyle u_n^{i_n} \wedge 1_R} & & \Big\downarrow & & \Big\| & & {\scriptstyle u_n^{i_n} \wedge 1_R}\Big\downarrow \\
F(U_m) \wedge R & \longrightarrow & F(U_m)(u_n^{2i_n}) \wedge R & \xrightarrow{h} & F(U_m) \wedge R & \xrightarrow{u_n^{2i_n} \wedge 1_R} & F(U_m) \wedge R.
\end{array}$$

The u_n-map $u_n^{i_n}$ on $F(U_m)$ is adjoint to a u_n-element $\alpha \in \pi_{**} F(U_m) \wedge DF(U_m)$. So there are two u_n-elements in the ring spectrum $(F(U_m) \wedge DF(U_m)) \wedge R$: $\alpha \wedge 1_R$ and $1_{F(U_m) \wedge DF(U_m)} \wedge 0 = 0$. They obviously commute, so Lemma 4.2.7 tells us that some power of $\alpha \wedge 1_R$ is null. If we write η: $S^0 \to R$ for the unit map of R, then $\alpha \wedge 1_R$ is adjoint to $u_n^{i_n} \wedge \eta$: $F(U_m) \to F(U_m) \wedge R$, and the self-map $u_n^{i_n} \wedge 1_R$ factors through this map; hence some iterate of it is null as well.

So for a large enough exponent i_n, the map h is a map onto a wedge summand. Furthermore, the vertical maps $u_n^{i_n} \wedge 1_R$ in the above diagram are null.

To get information about $H\colon L_n^f R \to L_m^f R$, we map everything into R and take cofibers; we see that the cofiber of h is still a map onto a direct summand, and the vertical maps are still null. Taking direct limits gives the desired isomorphism. $\square$

COROLLARY 5.3.3. *Fix m. For n sufficiently large, $M_n^f HA(m) = 0$; hence the chromatic tower for $HA(m)$ stabilizes.*

PROOF. This follows from the previous result and Proposition 2.2.5. $\square$

In particular, at the prime 2, $M_n^f HA(m) = 0$ if $n > |\xi_{m+1}|$; at odd primes, $M_n^f HA(m) = 0$ if $n > \frac{p|\xi_m|}{2}$. Note that by Corollary 5.2.3, the chromatic tower for $HA(m)$ converges to $\widehat{H}A(m)$, not to $HA(m)$. So while we can compute the limit of the chromatic tower here, we also see that we do not have chromatic convergence in this case.

PROOF OF THEOREM 5.3.1. If X is finite, then smashing with X commutes with inverse limits. Since $L_n^f X = X \wedge L_n^f S^0$, then chromatic convergence for S^0 implies chromatic convergence for all finite spectra.

By Theorem 5.1.6(b), $L_n^f S^0$ has finite type homotopy bounded to the left, so by Proposition 2.1.5, we see that

$$L_n^f S^0 = \varprojlim{}_m HA(m) \wedge L_n^f S^0 = \varprojlim{}_m L_n^f HA(m).$$

Hence we have

$$\begin{aligned}\varprojlim{}_n L_n^f S^0 &= \varprojlim{}_n (\varprojlim{}_m L_n^f HA(m)) \\ &= \varprojlim{}_m (\varprojlim{}_n L_n^f HA(m)) \\ &= \varprojlim{}_m \widehat{H}A(m).\end{aligned}$$

Now we use Lemma 5.2.4 to compute the homotopy groups of $\varprojlim{}_m \widehat{H}A(m)$: in the first quadrant, the inverse limit stabilizes in each bidegree to the homotopy of $\varprojlim{}_m HA(m) = S^0$, by Proposition 2.1.5. In the second and fourth quadrants, there is no homotopy at any stage in the inverse system. In the third quadrant, for each bidegree (i, j) there is an m_0 so that $\pi_{ij}\widehat{H}A(m) = 0$ for all $m \geq m_0$; so the inverse limit has no homotopy in the third quadrant, either. Hence the map $S^0 \to \varprojlim{}_m \widehat{H}A(m)$ is an isomorphism in homotopy. □

One could probably get another proof of chromatic convergence by making precise, and then using, the result mentioned in Remark 5.1.8(b).

By the way, Margolis gets chromatic convergence in his setting for all bounded below modules; since we do not have convergence for $HA(n)$ (which is an injective resolution of a bounded below comodule), we cannot expect our result to generalize, precisely as stated, to other spectra. On the other hand, perhaps one could modify the proof of Theorem 5.3.1 to show that for any *CL*-spectrum X, then $X \to \varprojlim L_n^f X$ is a connective cover. For this, the alternate formulation of *CL* after Definition 1.4.6 might be useful—X is *CL* if and only if there are numbers i_0, i_1, and j_0 so that X has a cellular tower built of spheres $S^{i,j}$ with $i_0 \leq i \leq i_1$ and $j - i \geq j_0$.

5.4. Further discussion

It seems most convenient to consider the chromatic tower as above, in which we kill off the P_t^s-homology groups of X in order of slope. This ordering is what allows us to prove chromatic convergence, via the "connectivity result" of Theorem 5.1.6(b). But at the prime 2, at least, we can kill off the generators of HD_{**} in many different orders: we can start with any invariant element in HD_{**}, take its cofiber, and continue (as in Theorem 4.7.3 and Remark 5.1.8). Can one still prove convergence? Does other structure reveal itself when one works with other generators of HD_{**} or with other orderings of the same generators?

As noted after the statement of Proposition 5.1.10, the validity of the strong form of the telescope conjecture is not known in $\mathsf{Stable}(A)$. If one could show that Bousfield localization with respect to $\bigvee_{d \leq n} Z(d)$ were smashing, that would disprove it; we would be interested in any progress along these lines.

It is conceivable that the finite-type result of Theorem 5.1.6(b) could have bearing on the convergence (in the ordinary stable homotopy category) of the tower

$$\dots \to L_2^f X \to L_1^f X \to L_0^f X.$$

After all, the v_n-localized Adams spectral sequence in [**MS95**] has an E_2-term with some relation to what we call $\pi_{**} L_n^f S^0$, so whether one can prove convergence this way, computations and other information about $\pi_{**} L_n^f S^0$ could have applications to the finite localization functor L_n^f in ordinary stable homotopy theory.

Work of Mahowald and others has led to calculations similar to that of $\widehat{H}A(m)$ in Lemma 5.2.4. This may provide more connections between our functor L_n^f and topology.

Chromatic convergence for the sphere gives a convergent filtration of $\pi_{**} S^0 = \operatorname{Ext}_A^{**}(\mathbf{F}_p, \mathbf{F}_p)$. When $p = 2$, Mahowald and Shick [**MS87**] give another convergent filtration; are these filtrations the same? The slope of the element ξ_{n+1} is $2^{n+1} - 1$; this element corresponds to the periodicity operator $v_n = h_{n+1,0}$, which has slope $2^{n+1} - 1$. Mahowald and Shick also construct something they call $v_n^{-1} \operatorname{Ext}_A^{**}(\mathbf{F}_2, \mathbf{F}_2)$. How does this compare to $\pi_{**}(L^f_{2^{n+1}-1} S^0)$? (Shick [**Shi88**] has done similar work at odd primes, and one can naturally ask the same questions about that.)

In parallel with the study of chromatic phenomena in the ordinary stable homotopy category, one should try to understand the filtration pieces in the chromatic tower, and other "monochromatic" objects: objects X so that $M_n^f = X$. Other than the trivial case of $M_1^f X = h_{10}^{-1} X$, we have no information about these objects.

Lastly, it seems possible that the chromatic tower constructed here (as well as those with other orderings) are the Steenrod algebra analogues of the multiple complexes of Benson and Carlson [**BC87**]. Is this a good analogy? If so, does it give any new insight into $\mathsf{Stable}(A)$, $\mathsf{Stable}(kG)$, or $\mathsf{Stable}(\Gamma)$ for an arbitrary commutative Hopf algebra Γ? (Evens and Siegel [**ES96**] have extended the multiple complex construction to modules over finite-dimensional cocommutative Hopf algebras, or equivalently to comodules over finite-dimensional commutative Hopf algebras, so their work is relevant here.)

APPENDIX A

Two technical results

A.1. An underlying model category

Let Γ be a graded commutative Hopf algebra over a field k. In this section we (briefly) describe a model category whose associated homotopy category is equivalent to $\mathsf{Stable}(\Gamma)$. The main results here (Theorems A.1.3 and A.1.4) are due to Hovey; see [**Hov97**] for the details.

Recall that Quillen [**Qui67**] defined the notion of a *closed model category*; this is a category, like the category of topological spaces, in which one has a notion of a well-behaved homotopy relation between maps. This allows one to define a new category, the *associated homotopy category*, which has the same objects as the original one, but where the morphisms are the homotopy classes of morphisms in the original category. (Briefly, a closed model structure is determined by specifying three classes of maps—*weak equivalences*, *fibrations*, and *cofibrations*—satisfying certain properties. See [**Qui67**], as well as [**DS95**] and [**Hov97**], for details.)

These days, by the way, one often says "model category" rather than "closed model category."

Model categories are useful because, while one can do many constructions working entirely in a homotopy category, for certain more delicate operations one needs to work at the "point-set" level—i.e., in the model category. For example, while Adams' definition in [**Ada74**] of the homotopy category of spectra is extremely useful, there are constructions one cannot do unless one has a model category underlying it. Hence today one has various definitions of model categories of spectra, each with its own advantages and disadvantages.

In our work with the Steenrod algebra, we have not needed a model category underlying $\mathsf{Stable}(\Gamma)$. Nonetheless, it seems like a good idea, pedagogically and for future applications, to set up such a model category.

We assume that Γ is a graded commutative Hopf algebra over a field k. We let $\mathsf{Ch}(\Gamma)$ denote the category whose objects are cochain complexes of Γ-comodules (not necessarily injective ones), and whose morphisms are cochain maps. We put a model category structure on $\mathsf{Ch}(\Gamma)$; to do this, we need to specify the weak equivalences, the fibrations, and the cofibrations in $\mathsf{Ch}(\Gamma)$.

Notation A.1.1. Given cochain complexes X and Y, let $[X, Y]$ denote the set of cochain homotopy classes of maps from X to Y. Given a Γ-comodule M, let $S^i M$ denote the cochain complex which is M in degree i, and zero elsewhere. Let $L(k)$ denote an injective resolution of the trivial module k, and let $\mathscr{S}$ denote the set of simple comodules of Γ.

For i an integer, $M \in \mathscr{S}$, and X any object of $\mathsf{Ch}(\Gamma)$, we define $\pi_i(X; M)$ (the "ith homotopy group of X with coefficients in M") by

$$\pi_i(X; M) = [S^i M, L(k) \otimes X].$$

We say that a map $f\colon X \to Y$ in $\mathsf{Ch}(\Gamma)$ is a *weak equivalence* if and only $\pi_i(f;M)$ is an isomorphism for all integers i and all simple comodules M. We say that a map $f\colon X \to Y$ is a *fibration* if and only if each component $f_n\colon X_n \to Y_n$ of f is an epimorphism with injective kernel. We say that a map $f\colon X \to Y$ is a *cofibration* if and only if each component $f_n\colon X_n \to Y_n$ is a monomorphism.

REMARK A.1.2. One can show (see [**Hov97**]) that a map f is a weak equivalence if and only if $1_{L(k)} \otimes f$ is a cochain homotopy equivalence. Also, note that $\pi_i(X;k) = [S^i k, L(k) \otimes X]$ is the ith homology group of the cochain complex of primitives of $L(k) \otimes X$; this is a useful alternate description of $\pi_*(-;k)$.

THEOREM A.1.3. [**Hov97**] *With weak equivalences, fibrations, and cofibrations defined as above,* $\mathsf{Ch}(\Gamma)$ *is a model category. Its associated homotopy category is equivalent to* $\mathsf{Stable}(\Gamma)$.

(Note that with this model structure, every object of $\mathsf{Ch}(\Gamma)$ is cofibrant.)

One can in fact give $\mathsf{Ch}(\Gamma)$ the structure of a *cofibrantly generated* model category, as follows; we refer to [**Hov97**] for the proofs. Given a Γ-comodule M, we let $D^n M$ denote the (contractible) cochain complex which is M in degrees n and $n+1$, zero elsewhere, with differential given by the identity map:

$$\dots \to 0 \to 0 \to M \xrightarrow{1_M} M \to 0 \to 0 \to \dots .$$

Both S^n (defined in Notation A.1.1) and D^n are functors from $\mathsf{Ch}(\Gamma)$ to itself. We let J denote the following set of maps in $\mathsf{Ch}(\Gamma)$:

$$J = \{D^n f \mid n \in \mathbf{Z},\ f\colon M \hookrightarrow N \text{ an inclusion of finite-dimensional comodules}\}.$$

We define I by

$$I = J \cup \{S^{n+1}M \hookrightarrow D^n M \mid n \in \mathbf{Z},\ M \text{ simple}\}.$$

(Here the map $S^{n+1}M \hookrightarrow D^n M$ is the obvious inclusion.)

THEOREM A.1.4. [**Hov97**] *With I and J defined as above,* $\mathsf{Ch}(\Gamma)$ *has the structure of a cofibrantly generated model category, in which I is the set of generating cofibrations and J is the set of generating trivial cofibrations.*

A.2. Vanishing planes in Adams spectral sequences

In this appendix, we describe a result of Hopkins and Smith [**HSa**]: for any nice ring spectrum E and any number m, the collection of spectra X so that the E-based Adams spectral sequence converging to $\pi_* X$ has a vanishing line of slope m at some E_r-term forms a thick subcategory. We also give a convergence result for the Adams spectral sequence in $\mathsf{Stable}(A)$.

We work in a stable homotopy category like $\mathsf{Stable}(A)$—one in which homotopy groups are bigraded, and cofibrations look like this:

$$\dots \to \Sigma^{1,0} Z \to X \to Y \to Z \to \Sigma^{-1,0} X \to \dots .$$

Hence the Adams spectral sequence is trigraded, so we discuss vanishing planes rather than lines. (In a short subsection below, we also give the original statement due to Hopkins and Smith—the statement of the corresponding theorem in the ordinary stable homotopy category.)

Recall that we presented the Adams spectral sequence in this setting in Theorem 1.5.1. We defined "generic" in Definition 1.4.7. If a spectrum W is (w_1, w_2)-connective (Definition 1.4.3) but neither (w_1+1, w_2)-connective nor (w_1, w_2+1)-connective, we write $\|W\| = (w_1, w_2)$.

THEOREM A.2.1. *Suppose that E is a spectrum satisfying the conditions given before Theorem* 1.5.5*, and consider the E-based Adams spectral sequence*

$$E_*^{***}(X) \Rightarrow \pi_{***}(X).$$

Fix numbers $m \leq 0$ and n. The following properties on a spectrum X are each generic.

(i) *There exist numbers r and b so that for all s, t, and u with*
$$s \geq m(s+u) + n(t+u) + b,$$
we have $E_r^{s,t,u}(X) = 0$.

(ii) *There exist numbers r and b so that for all finite spectra W with $\|W\| = (w_1, w_2)$ and for all s, t, and u with*
$$s \geq m(s+u-w_1) + n(t+u+w_2) + b,$$
we have $E_r^{s,t,u}(X \wedge W) = 0$.

Note that the "slope" (m, n) of the vanishing plane is fixed, but the intercept b and term r of the spectral sequence may vary in these generic conditions.

REMARK A.2.2. (a) We want E to be a nice ring spectrum so we can identify the E_2-term and so we have some convergence information. For the proof of the theorem, convergence is important, but the form of the E_2-term is not. Hence, if we can guarantee convergence by some other means, then we can discard the assumptions on E. For example, in our application of this genericity result in Chapter 3, we know that the Adams spectral sequence coincides with the spectral sequence associated to a Hopf algebra extension (Proposition 1.5.3), and hence has good convergence properties. (In this application, it is also easy to verify the conditions mentioned in the theorem.)

(b) We assume that $m \leq 0$ for convenience in stating and proving Lemma A.2.4 below. It is certainly possible that the result holds regardless of the value of m.

One proves this theorem by showing that the purported generic conditions are equivalent to conditions on composites of maps in the Adams tower; then one shows that those conditions are generic.

We start by describing a construction of the Adams spectral sequence. Given a ring spectrum E, we let $\overline{E}$ denote the fiber of the unit map $S^0 \to E$. For any integer $s \geq 0$, we let

$$F_s X = \overline{E}^{\wedge s} \wedge X,$$
$$K_s X = E \wedge \overline{E}^{\wedge s} \wedge X.$$

We use these to construct the following diagram of cofibrations, which we call the *Adams tower for* X:

$$\begin{tikzcd} X \arrow[r, equal] & F_0X \arrow[d] & F_1X \arrow[l, "g"'] \arrow[d] & F_2X \arrow[l, "g"'] \arrow[d] & \cdots \arrow[l, "g"'] . \\ & K_0X & K_1X & K_2X \end{tikzcd}$$

This construction satisfies the definition of an "E_*-Adams resolution" for X, as given in [**Rav86**, 2.2.1]—see [**Rav86**, 2.2.9]. Note also that $F_sX = X \wedge F_sS^0$, and the same holds for K_sX—the Adams tower is functorial and exact.

Given the Adams tower for X, if we apply π_{**}, we get an exact couple and hence a spectral sequence. This is called the *E-based Adams spectral sequence.* More precisely, we let

$$D_1^{s,t,u} = \pi_{s+u,t+u} F_sX,$$
$$E_1^{s,t,u} = \pi_{s+u,t+u} K_sX.$$

If we let $g\colon F_{s+1}X \to F_sX$ denote the natural map, then $g_* = \pi_{s+u,t+u}(g)$ is the map $D_1^{s+1,t+1,u-1} \to D_1^{s,t,u}$. Then we have the following exact couple (the triples of numbers indicate the tridegrees of the maps):

$$\begin{tikzcd} D_1^{s,t,u} \arrow[dr, "{(0,0,0)}"'] & & D_1^{s+1,t+1,u-1} \arrow[ll, "{(-1,-1,1)}"'] \\ & E_1^{s,t,u} \arrow[ur, "{(1,0,0)}"'] & \end{tikzcd}$$

This leads to the following rth derived exact couple, where $D_r^{s,t,u}$ is the image of g_*^{r-1}, and the map $D_r^{s+1,t+1,u-1} \to D_r^{s,t,u}$ is the restriction of g_*:

$$\begin{tikzcd} D_r^{s,t,u} \arrow[dr, "{(r-1,r-1,-r+1)}"'] & & D_r^{s+1,t+1,u-1} \arrow[ll, "{(-1,-1,1)}"'] \\ & E_r^{s+r-1,t+r-1,u-r+1} \arrow[ur, "{(1,0,0)}"'] & \end{tikzcd}$$

Unfolding this exact couple leads to the following exact sequence:

(A.2.3)
$$\cdots \to E_r^{s,t+1,u-1} \to D_r^{s+1,t+1,u-1} \to D_r^{s,t,u} \to E_r^{s+r-1,t+r-1,u-r+1} \to \cdots .$$

Fix numbers $m \leq 0$ and n. With respect to the E-based Adams spectral sequence $E_*^{***}(-)$, we have the following conditions on a spectrum X:

(1) There exist numbers r and b so that for all s, t, and u with $s \geq m(s+u) + n(t+u) + b$, the map $g_*^{r-1}\colon \pi_{s+u,t+u}(F_{s+r-1}X) \to \pi_{s+u,t+u}(F_sX)$ is zero.

(2) There exist numbers r and b so that for all s, t, and u with $s \geq m(s+u) + n(t+u) + b$, we have $E_r^{s,t,u}(X) = 0$.
(3) There exist numbers r and b so that for all finite spectra W with $\|DW\| = (-w_1, -w_2)$ and for all s with $s \geq mw_1 + nw_2 + b$, then the composite $W \to F_{s+r-1}X \to F_sX$ is null. (Here, DW denotes the Spanier-Whitehead dual of W.)
(4) There exist numbers r and b so that for all finite spectra W with $\|W\| = (w_1, w_2)$ and for all s, t, and u with $s \geq m(s+u-w_1) + n(t+u-w_2) + b$, we have $E_r^{s,t,u}(X \wedge W) = 0$.

Each condition depends on a pair of numbers r and b, and we write $(1)_{r,b}$ to mean that condition (1) holds with the numbers specified, and so forth.

LEMMA A.2.4. *Fix numbers $m \leq 0$, n, $r \geq 1$, and b. We have the following implications:*
(a) $(1)_{r,b} \Rightarrow (2)_{r,b+r-1}$.
(b) $(2)_{r,b} \Rightarrow (1)_{r,b-m}$.
(c) $(3)_{r,b} \Rightarrow (4)_{r,b+r-1}$.
(d) $(4)_{r,b} \Rightarrow (3)_{r,b-m}$

(Obviously, $(3)_{r,b} \Rightarrow (1)_{r,b}$ and $(4)_{r,b} \Rightarrow (2)_{r,b}$, but we do not need these facts.)

We assume that $r \geq 1$ so that we have the inequality $r \geq 1 + m$. Since r is the term of an Adams spectral sequence, assuming that $r \geq 1$ is not much of a restriction.

PROOF. As above, we write g for the map $F_{s+1}X \to F_sX$ and g_* for the map $D_1^{s+1,t+1,u-1} \to D_1^{s,t,u}$, so that $D_r^{s,t,u}$ is the image of

$$g_*^{r-1} \colon \pi_{s+u,t+u}F_{s+r-1}X \to \pi_{s+u,t+u}F_sX.$$

(a): Assume that if $s \geq m(s+u) + n(t+u) + b$, then

$$g_*^{r-1} \colon \pi_{s+u,t+u}(F_{s+r-1}X) \to \pi_{s+u,t+u}(F_sX)$$

is zero; i.e., that $D_r^{stu} = 0$. If $s \geq m(s+u) + n(t+u) + b$, then since $r \geq m$, we have $s + r \geq m((s+r) + (u-r+1)) + n((t+r-1) + (u-r+1)) + b$. So we see that $D_r^{s+r,t+r-1,u-r+1} = 0$. By the long exact sequence (A.2.3), we conclude that $E_r^{s+r-1,t+r-1,u-r+1} = 0$ when $s \geq m(s+u) + n(t+u) + b$. Reindexing, we find that $E_r^{p,q,v} = 0$ when $p - r + 1 \geq m(p+v) + n(q+v) + b$; i.e., condition $(2)_{r,b+r-1}$ holds.

(b): If $E_r^{s,t,u}(X) = 0$ when $s \geq m(s+u) + n(t+u) + b$, then (since $r - 1 \geq m$) $E_r^{s+r-1,t+r-2,u-r+2}(X) = 0$ also. So by the exact sequence (A.2.3), we see that $D_r^{s+1,t,u} \to D_r^{s,t-1,u+1}$ is an isomorphism under the same condition. This map is induced by $g_* \colon \pi_{s+1+u,t+u}F_{s+1}X \to \pi_{s+u+1,t+u}F_sX$, so we conclude that when $s \geq m(s+u) + n(t+u) + b$, we have

$$\varprojlim{}_q \pi_{s+u+1,t+u}F_qX = D_r^{s,t-1,u+1},$$
$$\varprojlim{}^1_q \pi_{s+u+1,t+u}F_qX = 0.$$

But by convergence of the spectral sequence, we know that $\varprojlim_q \pi_{s+u+1,t+u}F_qX = 0$, so $D_r^{s,t-1,u+1} = \operatorname{im} g_*^{r-1} = 0$. Reindexing gives $D_r^{p,q,v} = 0$ when $p \geq m(p+v-1) + n(q+1+v-1) + b$; i.e., $(2)_{r,b}$ implies $(1)_{r,b-m}$.

Parts (c) and (d) are similar:

(c): Fix a finite spectrum W with $\|W\| = \|D(DW)\| = (w_1, w_2)$. By condition (3), we see that the composite

$$\Sigma^{s+u,t+u} DW \to F_{s+r-1}X \to F_s X$$

is null when $s \geq m(s+u-w_1) + n(t+u-w_2) + b$. This is adjoint to

$$S^{s+u,t+u} \to F_{s+r-1}(X \wedge W) \to F_s(X \wedge W).$$

Hence $D_r^{stu}(X \wedge W) = 0$ when $s \geq m(s+u-w_1) + n(t+u-w_2) + b$, in which case $D_r^{s+r,t+r-1,u-r+1}(X \wedge W)$ is also zero. So by the long exact sequence (A.2.3), $E_r^{s+r-1,t+r-1,u-r+1}(X \wedge W) = 0$ when $s \geq m(s+u-w_1) + n(t+u-w_2) + b$. Reindexing as in part (a) gives us condition $(4)_{r,b+r-1}$.

(d): Fix a finite spectrum W with $\|DW\| = (-w_1, -w_2)$. Condition (4) tells us that $E_r^{s,t,u}(X \wedge DW) = 0$ when $s \geq m(s+u+w_1) + n(t+u+w_2) + b$. By the exact sequence (A.2.3), we see that $D_r^{s,t,u}(X \wedge DW) = 0$ under the same condition (as in part (b)), so the composite

$$S^{s+u,t+u} \to F_{s+r-1}(X \wedge DW) \to F_s(X \wedge DW)$$

is null when $s \geq m(s+u+w_1) + n(t+u+w_2) + b - m$. Hence the composite

$$S^0 \to F_{s+r-1}(X \wedge DW) \to F_s(X \wedge DW)$$

is null when $s \geq mw_1 + nw_2 + b - m$, as is

$$W \to F_{s+r-1}X \to F_s X,$$

by Spanier-Whitehead duality. So $(4)_{r,b}$ implies $(3)_{r,b-m}$. ☐

It is easy to prove Theorem A.2.1, once we have the lemma.

Proof of Theorem A.2.1. The proofs of the genericity of the two statements are similar, so we only prove that condition (i) is generic.

We know by Lemma A.2.4 that condition (i) is equivalent, up to a reindexing, to

($*$) There exist numbers r and b so that for all s, t, and u with $s \geq m(s+u) + n(t+u) + b$, the map $g^{r-1}\colon\ F_{s+r-1}X \to F_s X$ is zero on $\pi_{s+u,t+u}$.

This is generic, by the usual sort of argument: since the Adams tower is functorial, if Y is a retract of X, then the Adams tower for Y is a retract of the Adams tower for X. So if $F_{s+r-1}X \to F_s X$ is zero on $\pi_{s+u,t+u}$, then so is $F_{s+r-1}Y \to F_s Y$. (Given $S^{s+u,t+u} \to F_{s+r-1}Y$, then consider

$$\begin{tikzcd}
S^{s+t,t+u} \arrow[r] & F_{s+r-1}Y \arrow[r] \arrow[d, "i"] & F_s Y \arrow[d, "i"] \\
& F_{s+r-1}X \arrow[r] \arrow[d, "j"] & F_s X \arrow[d, "j"] \\
& F_{s+r-1}Y \arrow[r] & F_s Y
\end{tikzcd}$$

Since $\pi_{s+u,t+u}F_{s+r-1}X \to \pi_{s+u,t+u}F_s X$ is 0, then the map $S^{s+u,t+u} \to F_s X$ is null. But $S^{s+u,t+u} \to F_s Y$ factors through this map, and hence is also null.)

Given a cofibration sequence $X \to Y \to Z$ in which X and Z satisfy conditions $(*)_{r,b}$ and $(*)_{r',b'}$, respectively, we show that Y satisfies $(*)_{r+r'-1,\max(b,b'-r+1)}$. Consider the following commutative diagram, in which the rows are cofibrations:

$$\begin{tikzcd}
F_{s+r+r'-2}X \arrow[r] \arrow[d] & F_{s+r+r'-2}Y \arrow[r] \arrow[d, "\alpha"] & F_{s+r+r'-2}Z \arrow[d, "\beta"] \\
F_{s+r-1}X \arrow[r] \arrow[d, "\gamma"] & F_{s+r-1}Y \arrow[r] \arrow[d, "\delta"] & F_{s+r-1}Z \arrow[d] \\
F_sX \arrow[r] & F_sY \arrow[r] & F_sZ
\end{tikzcd}$$

We assume that $s \geq m(s+u) + n(t+u) + \max(b, b'-r+1)$, so that we have

$$s \geq m(s+u) + n(t+u) + b,$$
$$s + r - 1 \geq m(s+u) + n(t+u) + b'.$$

If we apply $\pi_{s+u,t+u}$ to this diagram, then $\pi_{s+u,t+u}\beta = 0$; hence $\pi_{s+u,t+u}\alpha$ factors through $\pi_{s+u,t+u}F_{s+r-1}X$. Since $\pi_{s+u,t+u}\gamma = 0$, though, then $\pi_{s+u,t+u}(\delta \circ \alpha) = 0$.

This shows that condition $(*)$, and hence condition (i), is generic. □

We end this section with a convergence result. We say that a spectrum X is *E-complete* if the inverse limit of the Adams tower for X is contractible.

Proposition A.2.5. *We work in* $\mathsf{Stable}(A)$. *Suppose that E is a ring spectrum as in Theorem* A.2.1. *Then every connective spectrum X is E-complete.*

Proof. This is an easy connectivity result: our conditions on E ensure that if X is connective, then F_sX is (i_s, j_s)-connective, where i_s increases with s. In particular, for all i and j, $\pi_{ij}F_sX = 0$ for s large enough; hence the inverse limit of the Adams tower will have no homotopy, and the $\lim^1$ term will be zero. □

Note that this applies when $E = HB$, for B any quotient Hopf algebra of A.

A.2.1. Vanishing lines in ordinary stable homotopy. We give the statement of the original Hopkins-Smith result in ordinary stable homotopy theory. Given a connective spectrum W, we write $|W|$ for its connectivity.

Theorem A.2.6 ([**HSa**]). *Suppose that E is a ring spectrum as in* [**Rav86**, 2.2.5]*, and consider the E-based Adams spectral sequence $E_*^{**}(X) \Rightarrow \pi_*(X)$. Fix a number $m \geq 0$. The following properties on a spectrum X are each generic.*

(i) *There exist numbers r and b so that for all s and t with $s \geq m(t-s) + b$, we have $E_r^{s,t}(X) = 0$.*
(ii) *There exist numbers r and b so that for all finite spectra W with $|W| = w$ and for all s and t with $s \geq m(t-s-w) + b$, we have $E_r^{s,t}(X \wedge W) = 0$.*

The proof above can be easily modified to apply here.

The same proof (in the case $m = 0$) also shows the following (using the language of [**Chr97**]).

Corollary A.2.7. *If I is an ideal of maps that is part of a projective class, then the following property is generic*:

- *There exist numbers r and b so that for all $s \geq b$, the composite*

$$g^{r-1} \colon F_{s+r-1}X \to F_s X$$

is in I.

APPENDIX B

Steenrod operations and nilpotence in $\mathrm{Ext}_{\Gamma}^{**}(k,k)$

Let A be the dual of the mod p Steenrod algebra. In this appendix we recall a few results about Steenrod operations in the cohomology of any Hopf algebra Γ, and we focus in particular on the quotient Hopf algebras B of A. Then we discuss the nilpotence of certain classes in $\mathrm{Ext}_{B}^{**}(\mathbf{F}_p,\mathbf{F}_p)$, which we need to prove the nilpotence theorems of Chapter 3.

B.1. Steenrod operations in Hopf algebra cohomology

In this short section we recall a few facts about Steenrod operations in Hopf algebra cohomology. May's paper [**May70**, Section 11] is our basic reference; see also [**Sin73**], [**Wil81**], [**Saw82**], [**BMMS86**], and [**Rav86**] for related results.

Suppose that Γ is a Hopf algebra over the field $\mathbf{F}_p$. Then there are Steenrod operations acting on $\mathrm{Ext}_{\Gamma}^{**}(\mathbf{F}_p,\mathbf{F}_p)$:

(a) If $p=2$, then for each $n\geq 0$ there are the following operations:
$$\widetilde{\mathrm{Sq}}^{n}\colon\ \mathrm{Ext}_{\Gamma}^{s,t}(\mathbf{F}_2,\mathbf{F}_2)\to \mathrm{Ext}_{\Gamma}^{s+n,2t}(\mathbf{F}_2,\mathbf{F}_2).$$

(b) If p is odd, then for each $n\geq 0$ there are the following operations (where $q=2p-2$):
$$\widetilde{\mathscr{P}}^{n}\colon\ \mathrm{Ext}_{\Gamma}^{s,2t}(\mathbf{F}_p,\mathbf{F}_p)\to \mathrm{Ext}_{\Gamma}^{s+qn,2pt}(\mathbf{F}_p,\mathbf{F}_p),$$
$$\beta\widetilde{\mathscr{P}}^{n}\colon\ \mathrm{Ext}_{\Gamma}^{s,2t}(\mathbf{F}_p,\mathbf{F}_p)\to \mathrm{Ext}_{\Gamma}^{s+qn+1,2pt}(\mathbf{F}_p,\mathbf{F}_p).$$

Note that $\beta\widetilde{\mathscr{P}}^{n}$ must be treated as a single operation, not the composite of two operations.

These satisfy the usual properties of Steenrod operations: the Cartan formula, the Adem relations, and an instability condition: if $x\in \mathrm{Ext}_{\Gamma}^{s,t}(\mathbf{F}_p,\mathbf{F}_p)$ (with s and t even if p is odd), then
$$\begin{aligned}\widetilde{\mathrm{Sq}}^{s}(x)&=x^2, \quad p=2,\\ \widetilde{\mathscr{P}}^{\frac{s}{2}}(x)&=x^p, \quad p \text{ odd}.\end{aligned}$$

Note that at odd primes, the operations are zero on classes in $\mathrm{Ext}_{\Gamma}^{s,t}(\mathbf{F}_p,\mathbf{F}_p)$ when t is odd. This is an artifact of the grading conventions on the operations. To remedy this, one can either use a different grading convention (see [**May70**] and [**BMMS86**, IV.2] for two different conventions), or one can define operations indexed by half-integers, as in [**Rav86**, 1.5.2].

EXAMPLE B.1.1. (a) Fix a Hopf algebra Γ. Recall from Lemma 1.1.15 that $\mathrm{Ext}_{\Gamma}^{1}$ is isomorphic to the vector space of primitives in Γ. If x is primitive, then we write $[x]$ for the corresponding element of Ext. We have the following Steenrod operation: $\widetilde{\mathscr{P}}^{0}[x]=[x^p]$. (At the prime 2, $\widetilde{\mathrm{Sq}}^{0}[x]=[x^2]$.) One can compute $\widetilde{\mathscr{P}}^0$ and $\widetilde{\mathrm{Sq}}^0$ on general Ext classes by a similar formula—these operations are induced by the following map on the cobar complex:

$$[x_1|x_2|\dots|x_n] \mapsto [x_1^p|x_2^p|\dots|x_n^p].$$

See [**May70**] for a proof.

(b) If $\Gamma = \mathbf{F}_p[x]/(x^2)$ with x primitive, then $\mathrm{Ext}_{\Gamma}^{**}(\mathbf{F}_p, \mathbf{F}_p) \cong \mathbf{F}_p[h]$, where $h = [x]$. If $p=2$, then $\widetilde{\mathrm{Sq}}^1(h) = h^2$, and $\widetilde{\mathrm{Sq}}^n(h) = 0$ for $n \neq 1$. If p is odd (in which case $|x|$ must be odd), then all operations vanish on h (because of our grading conventions).

(c) If p is odd and $\Gamma = \mathbf{F}_p[x]/(x^p)$ with x primitive, then $\mathrm{Ext}_{\Gamma}^{**}(\mathbf{F}_p, \mathbf{F}_p) \cong E[h] \otimes \mathbf{F}_p[b]$, where $h = [x]$ and b is the p-fold Massey product of h with itself. We have $\beta\widetilde{\mathscr{P}}^0(h) = b$, and all other operations on h are zero; also, $\widetilde{\mathscr{P}}^1(b) = b^p$, and all other operations on b are zero.

B.2. Nilpotence in $HB_{**} = \mathrm{Ext}_B^{**}(\mathbf{F}_2, \mathbf{F}_2)$

In this section we recall a result of Lin which we use in the proofs of the main results of Chapter 3. Lin's result first appeared as [**Lin**, 3.1 and 3.2]. Wilkerson [**Wil81**, 6.4] has also proved a related result.

Let A be the dual of the mod 2 Steenrod algebra. Recall from Notation 1.3.9 and Remark 2.1.3 that if $\xi_t^{2^s}$ is primitive in a quotient Hopf algebra B of A, then we let $h_{ts} = [\xi_t^{2^s}]$ denote the corresponding element of $HB_{1,*}$.

THEOREM B.2.1. *Suppose that*

$$B = A/(\xi_1^{2^{n_1}}, \xi_2^{2^{n_2}}, \dots)$$

is a quotient Hopf algebra of A so that, for some integer m, we have $n_i < \infty$ for $i = 1, 2, \dots, m-1$. Fix μ so that $\xi_m^{2^\mu}$ is primitive in B.

(a) *If $\mu \geq m$, then $h_{m,\mu}$ is nilpotent in HB_{**}.*

(b) *Fix an integer $\ell \leq m$, and suppose that for some λ, $\xi_\ell^{2^\lambda}$ is primitive in B. If $\ell \leq \mu$, then $h_{\ell,\lambda} h_{m,\mu}$ is nilpotent in HB_{**}.*

(Part (a) is a corollary of part (b)—just set $\ell = m$ and $\lambda = \mu$.)

For example, the class $h_1 = h_{11} \in \mathrm{Ext}_A^{1,2}(\mathbf{F}_2, \mathbf{F}_2)$ is nilpotent. This is easy to show directly: the (reduced) diagonal $\xi_2 \mapsto \xi_1^2 \otimes \xi_1$ in A gives the relation $h_{11}h_{10} = 0$ in $\mathrm{Ext}_A^{**}(\mathbf{F}_2, \mathbf{F}_2)$. Applying the Steenrod operation $\widetilde{\mathrm{Sq}}^1$ gives

$$h_{12}h_{10}^2 + h_{11}^3 = 0,$$

so multiplying through by h_{11} and using $h_{11}h_{10} = 0$ yields $h_{11}^4 = 0$. Note that applying $\widetilde{\mathrm{Sq}}^0$ to this gives $h_{1n}^4 = 0$ for all $n \geq 1$.

For our purposes, part (a) is one of the key ingredients in the proof that restriction to the quotient Hopf algebra D detects nilpotence. Part (b) is used in the classification of quasi-elementary quotients of A, essentially, and is also used in showing that restricting to the quasi-elementary quotients detects nilpotence.

B.3. Nilpotence in $HB_{**} = \mathrm{Ext}_B^{**}(\mathbf{F}_p, \mathbf{F}_p)$ when p is odd

In this section we discuss the odd-primary analogue of Theorem B.2.1.

Fix an odd prime p, and let A be the dual of the mod p Steenrod algebra. Recall from Remark 2.1.3 that if $\xi_t^{p^s}$ is primitive in a quotient Hopf algebra B of A, then $h_{ts} = [\xi_t^{p^s}]$ is the corresponding element of $HB_{1,*} = \mathrm{Ext}_B^{1,*}(\mathbf{F}_p, \mathbf{F}_p)$, and $b_{ts} \in HB_{2,*}$ is defined to be $\beta\widetilde{\mathscr{P}}^0(h_{ts})$. (Alternatively, b_{ts} is equal to the p-fold Massey product of h_{ts} with itself.) If τ_n is primitive in B, we let $v_n = [\tau_n]$ be the corresponding element of $HB_{1,*}$.

For convenience, we restate Conjecture 3.4.1. This would be the odd-primary analogue of Theorem B.2.1(a).

CONJECTURE B.3.1. *Fix integers s and t. Suppose that*

$$B = A/(\xi_1^{p^{n_1}}, \xi_2^{p^{n_2}}, \dots; \tau_0^{e_0}, \tau_1^{e_1}, \dots)$$

is a quotient Hopf algebra of A in which $\xi_t^{p^s}$ is nonzero and primitive. If $s \geq t$, then b_{ts} is nilpotent in HB_{**}.

By the way, one could consider a partial analogue of Theorem B.2.1(b): under conditions on s, t, and n, the product $b_{ts}v_n$ is nilpotent.

PROPOSITION B.3.2. *Fix integers s, t, and n. Suppose that B is a quotient Hopf algebra of A in which $\xi_i^{p^s} = 0$ for $i < t$, and $\tau_j = 0$ for $j < n$; hence $\xi_t^{p^s}$ and τ_n are primitive in B. If $n \leq s$, then $b_{ts}v_n$ is nilpotent in $\mathrm{Ext}_B^{**}(\mathbf{F}_p, \mathbf{F}_p)$.*

PROOF. The coproduct

$$\tau_{n+t} \mapsto \sum_{i=0}^{n+t} \xi_{n+t-i}^{p^i} \otimes \tau_i + \tau_{n+t} \otimes 1,$$

together with the conditions on B, gives the following relation in the cobar complex for B:

$$\sum_{i=n}^{n+t-1} h_{n+t-i,i} v_i = 0.$$

Hence $h_{tn}v_n = -\sum_{i=n+1}^{n+t-1} h_{n+t-i,i} v_i$. Applying Steenrod operations gives the following (here one needs to use half-integer indexed operations $\widetilde{\mathscr{P}}^{\frac{n}{2}}$, or one needs to use operations $\mathscr{P}^k$ which are indexed differently):

$$b_{tn} v_n^p = \pm \sum_{i=n+1}^{n+t-1} b_{n+t-i,i} v_i^p,$$

$$b_{t,n+1} v_n^{p^2} = \pm \sum_{i=n+1}^{n+t-1} b_{n+t-i,i+1} v_i^{p^2},$$

$$\vdots$$

$$b_{t,s} v_n^{p^{s+n-1}} = \pm \sum_{i=n+1}^{n+t-1} b_{n+t-i,i+s-n} v_i^{p^{s+n-1}}.$$

Since $\xi_i^{p^s} = 0$ for $i < t$, then we see that $b_{ts} v_n^{p^{s+n-1}} = 0$. □

Another analogue of Theorem B.2.1(b) would be that under conditions on s, t, v, and u, the product $b_{ts}b_{vu}$ is nilpotent. Conjecture B.3.1 should be a special case of such a result, and since we do not know how to prove this conjecture, we will not be able to prove such an analogue of Theorem B.2.1(b).

For the remainder of this section, we provide more evidence and partial results towards Conjecture B.3.1.

We point out that if $s < t$, then $\mathbf{F}_p[\xi_t]/(\xi_t^{p^{s+1}})$ is a quotient Hopf algebra of B; the cohomology of this quotient is

$$E[h_{t0}, \dots, h_{ts}] \otimes \mathbf{F}_p[b_{t0}, \dots, b_{ts}].$$

The element b_{ts} is non-nilpotent when restricted to this quotient, and hence non-nilpotent in Ext_B. When $s \geq t$, though, surprisingly little seems to be known about the nilpotence (or lack thereof) of b_{ts}. For example, while it is easy to verify that $h_{11}^4 = 0$ in $\mathrm{Ext}_A^{**}(\mathbf{F}_2, \mathbf{F}_2)$ at the prime 2 (see Section B.2), we have not been able to locate or prove a similar result for b_{11} at an arbitrary odd prime. Working at the prime 3, Nakamura [**Nak75**] proved that $b_{11}^2 = h_{11}z$ for some z. Since h_{11} is in odd total degree, then $h_{11}^2 = 0$; hence $b_{11}^4 = 0$.

Note that if an element $\xi_t^{p^{s_0}}$ is primitive in a Hopf algebra B, then so is $\xi_t^{p^s}$ for any $s \geq s_0$.

LEMMA B.3.3. *Fix a Hopf algebra B, and fix integers $s_0 \geq t \geq 1$. Suppose that $\xi_t^{p^{s_0}}$ is primitive in B.*

(a) *If b_{t,s_0} is nilpotent, then so is b_{t,s_0+1}.*
(b) *Conversely, assume that $\xi_1^{p^{s_0-t}} = \cdots = \xi_{t-1}^{p^{s_0-t}} = 0$ in B, and that $s_0 \geq t$. If b_{t,s_0+1} is nilpotent, then so is b_{t,s_0}.*

As an application of (a), Nakamura's calculation implies that $b_{1,n}^4 = 0$ for all $n \geq 1$, when $p = 3$. One might conjecture (based on very little evidence) that at any odd prime p, $b_{1,n}^q = 0$ for all $n \geq 1$, where $q = 2(p-1)$. As an application of (b), b_{11} is nilpotent in Ext_B if and only if b_{1n} is nilpotent for $n \geq 1$; as another application of (b), if the high powers of ξ_t are zero in a quotient B of A—for instance if B is finite—then b_{ts} is nilpotent when $s \geq t$.

PROOF. Part (a) follows from the relation $\widetilde{\mathscr{P}}^0(b_{ts}^i) = b_{t,s+1}^i$.

For part (b), we have the following coproduct in B:

$$\xi_{2t} \mapsto \sum_{i=0}^{2t} \xi_{2t-i}^{p^i} \otimes \xi_i.$$

Since $\xi_i^{p^{s_0-t}} = 0$ when $i < t$, this simplifies to

$$\xi_{2t}^{p^{s_0-t}} \mapsto \xi_{2t}^{p^{s_0-t}} \otimes 1 + \xi_t^{p^{s_0}} \otimes \xi_t^{p^{s_0-t}} + 1 \otimes \xi_{2t}^{p^{s_0-t}}.$$

Also since $\xi_i^{p^{s_0-t}} = 0$ when $i < t$, then $\xi_t^{p^s}$ is primitive in B for every $s \geq s_0 - t$. So the above coproduct in B translates to the relation $h_{t,s_0-t}h_{t,s_0} = 0$ in Ext_B. We apply Steenrod operations to this relation: applying $\widetilde{\mathscr{P}}^{p^t} \dots \widetilde{\mathscr{P}}^p(\beta\widetilde{\mathscr{P}}^1)(\beta\widetilde{\mathscr{P}}^0)$ gives

$$b_{t,s_0}^{p^t+1} = \pm b_{t,s_0-t}^{p^t} b_{t,s_0+t}$$

If b_{t,s_0+1} is nilpotent, then so is b_{t,s_0+t}, by part (a). Hence so is b_{t,s_0}. $\square$

We have the following conjecture, as a special case of Conjecture B.3.1. To some extent, this would generalize Nakamura's result at the prime 3; on the other hand, he determines a much smaller nilpotence height than this would.

CONJECTURE B.3.4. *Fix an odd prime p.*

(a) *The element $b_{11} \in \mathrm{Ext}_A^{2,2p^2(p-1)}(\mathbf{F}_p, \mathbf{F}_p)$ is nilpotent.*
(b) *Fix $t \geq 1$ and let $j = \frac{p+1}{2}$. Then b_{tt} is nilpotent in $\mathrm{Ext}_B^{**}(\mathbf{F}_p, \mathbf{F}_p)$, where*

$$B = \mathbf{F}_p[\xi_t, \xi_{2t}, \ldots, \xi_{jt}, \xi_{jt+1}, \xi_{jt+2}, \xi_{jt+3}, \ldots].$$

We have a sketch of a proof (which contains a gap), but it is a bit lengthy, and so we relegate it to a subsection. We also include a few other technical results in that subsection.

B.3.1. Sketch of proof of Conjecture B.3.4, and other results.

SKETCH OF PROOF. The proof involves some Massey product manipulations. May's paper [**May69**] is the standard reference for Massey products; many of the key results are reproduced in [**Rav86**, A.1.4].

As remarked after Lemma 1.3.10, the element b_{ts} is the p-fold Massey product of $h_{ts} = [\xi_t^{p^s}]$ with itself.

Part (a): In A, we have the coproduct

$$\Delta\colon\ \xi_2 \longmapsto \xi_2 \otimes 1 + \xi_1^p \otimes \xi_1 + 1 \otimes \xi_2.$$

This produces the relation $h_{10}h_{11} = 0$ in Ext_A. Applying the Steenrod operation $\beta\widetilde{\mathscr{P}}^0$ gives the relation

$$h_{11}b_{11} - b_0 h_{12} = 0.$$

Then for any $k \geq 1$, we apply $\widetilde{\mathscr{P}}^{p^{k-1}} \ldots \widetilde{\mathscr{P}}^p \widetilde{\mathscr{P}}^1$ to get

$$h_{1,k+1} b_{11}^{p^k} - b_{10}^{p^k} h_{1,k+2} = 0.$$

Using induction gives the following formula, valid for all $k \geq 2$:

$$h_{11} b_{11}^{1+p+p^2+\cdots+p^{k-2}} = h_{1,k} b_{10}^{1+p+p^2+\cdots+p^{k-2}}. \tag{B.3.5}$$

So we let

$$\begin{aligned} N &= 1 + 1 + (1+p) + (1+p+p^2) + \cdots + (1+p+p^2+\cdots+p^{p-2}) \\ &= p\frac{p^{p-1}-1}{(p-1)^2}. \end{aligned}$$

and we look at b_{11}^N. Recall that b_{11} is the Massey product $\langle \underbrace{h_{11}, \ldots, h_{11}}_{p} \rangle$. This Massey product has no indeterminacy.

LEMMA B.3.6. *The element b_{11}^N is contained in the p-fold Massey product*

$$\langle h_{11}, h_{11}b_{11}, h_{11}b_{11}^{1+p}, \ldots, h_{11}b_{11}^{1+p+\cdots+p^{p-2}} \rangle.$$

PROOF. By definition definition of Massey products, if $y = \langle a_1, \ldots, a_n \rangle$ (with no indeterminacy), then for any elements $x_1, \ldots, x_n$, we have

$$\begin{aligned} y x_1 \ldots x_n &= \langle a_1, \ldots, a_n \rangle x_1 \ldots x_n \\ &\in \langle a_1 x_1, \ldots, a_n x_n \rangle. \end{aligned}$$

So we apply this to $b_{11}^N = \langle h_{11}, \ldots, h_{11} \rangle b_{11}^{N-1}$. $\square$

By our computations with Steenrod operations (i.e., equation (B.3.5)), we have

$$\begin{aligned}&\langle h_{11}, h_{11}b_{11}, h_{11}b_{11}^{1+p}, \ldots, h_{11}b_{11}^{1+p+\cdots+p^{p-2}}\rangle \\ &= \langle h_{11}, h_{12}b_{10}, h_{13}b_{10}^{1+p}, \ldots, h_{1p}b_{10}^{1+p+\cdots+p^{p-2}}\rangle.\end{aligned}$$

LEMMA B.3.7. *The Massey product*

$$\langle h_{11}, h_{12}b_{10}, h_{13}b_{10}^{1+p}, \ldots, h_{1p}b_{10}^{1+p+\cdots+p^{p-2}}\rangle$$

contains $\langle h_{11}, h_{12}, h_{13}, \ldots, h_{1p}\rangle b_{10}^{N-1} = 0$.

We let $\zeta_n = \chi(\xi_n)$, where $\chi\colon A \to A$ is the canonical anti-automorphism. Hence $\zeta_1 = -\xi_1$, and

$$\Delta(\zeta_n) = \sum_{i=0}^{n} \zeta_i \otimes \zeta_{n-i}^{p^i}.$$

PROOF. We have to prove two things: that the Massey product contains $\langle h_{11}, h_{12}, h_{13}, \ldots, h_{1p}\rangle b_{10}^{N-1}$, and that $\langle h_{11}, h_{12}, h_{13}, \ldots, h_{1p}\rangle = 0$. To prove the first of these, one follows the proof of Lemma B.3.6. To prove the second, we show that for all $n \geq 2$, the n-fold Massey product $\langle h_{10}, h_{11}, h_{12}, \ldots, h_{1,n-1}\rangle = 0$; this goes by induction on n. Once we know this, then applying the Steenrod operation $(\widetilde{\mathscr{P}^0})^i$ (to either the Massey product or to the proof) gives $\langle h_{1,i}, h_{1,i+1}, h_{1,i+2}, \ldots, h_{1,i+n-1}\rangle = 0$ for any i.

Indeed, we show that for each n, the cobar element $d[\zeta_n]$ is equal to

$$\langle h_{10}, h_{11}, h_{12}, \ldots, h_{1,n-1}\rangle;$$

hence this Massey product is cohomologous to zero. We also show that this Massey product has no indeterminacy. (Hence $d[\zeta_n^{p^i}]$ kills $\langle h_{1,i}, \ldots, h_{1,i+n-1}\rangle$.)

When $n = 2$, the coproduct

$$\zeta_2 \mapsto \zeta_1 \otimes \zeta_1^p = \xi_1 \otimes \xi_1^p$$

gives have the relation $h_{10}h_{11} = 0$. This starts the induction. When $n = 3$, we have

$$\zeta_3 \longmapsto \zeta_2 \otimes \zeta_1^{p^2} + \zeta_1 \otimes \zeta_2^p.$$

Since $d[\zeta_2] = h_{10}h_{11}$, then $d[\zeta_2^p] = h_{11}h_{12}$, so we have

$$\langle h_{10}, h_{11}, h_{12}\rangle = [\zeta_2]h_{12} + h_{10}[\zeta_2^p] = d[\zeta_3] = 0.$$

Since $h_{1n} \in \operatorname{Ext}_A^{1,p^n q}(\mathbf{F}_p, \mathbf{F}_p)$, then the indeterminacy of this Massey product is of the form

$$h_{10}x + yh_{12},$$

where $x \in \operatorname{Ext}_A^{1,q(p+p^2)}(\mathbf{F}_p, \mathbf{F}_p)$ and $y \in \operatorname{Ext}_A^{1,q(1+p)}(\mathbf{F}_p, \mathbf{F}_p)$. Since we know that $\operatorname{Ext}^{1,*}$ is in one-to-one correspondence with the primitives of A, then $\operatorname{Ext}^{1,i}$ is nonzero only when i is of the form $p^j q$ for some j. Hence both x and y must be zero, and there is no indeterminacy.

Now, we assume that $\langle h_{10}, h_{11}, \ldots, h_{1,i-1}\rangle = 0$ via ζ_i, for all $i < n$. Then $\zeta_i^{p^j}$ kills $\langle h_{1,j}, h_{1,j+1}, \ldots, h_{1,j+i-1}\rangle$ for all $i < n$ and all j, and so

$$\langle h_{10}, h_{11}, \ldots, h_{1,n-1}\rangle = \sum_{i=1}^{n-1} [\zeta_i | \zeta_{n-i}^{p^i}].$$

Hence the coproduct on ζ_n shows that this is zero. There is no indeterminacy for the same reason as when $n = 3$. □

So the Massey product in Lemma B.3.6 contains both b_{11}^N and 0; hence b_{11} is an element of the indeterminacy. Therefore we need some information about that indeterminacy. The indeterminacy of a p-fold Massey product is contained in the union of certain $(p-1)$-fold matric Massey products (see [**May69**, 2.3]); if every entry in the p-fold Massey product has odd total degree, the same is true of each entry in each matrix in the shorter Massey products.

Here is a general conjecture about "short" Massey products at an odd characteristic. This is the gap in our proof.

CONJECTURE B.3.8. *Fix an odd prime p, and fix $n < p$. Consider an n-fold matric Massey product $\langle V_1, \ldots, V_n\rangle$, in which each entry of each matrix V_i has odd total degree. Then any element of this matric Massey product is nilpotent.*

(As in [**Rav86**, A.1.4], whenever we discuss matric Massey products, we assume that the matrices involved have entries with compatible degrees, so that their products have homogeneous degrees, etc. See [**May69**, 1.1] for details.)

The conjecture is trivially true when $n < 3$, by graded commutativity; in particular, it is true when $p = 3$. Indeed, when $p = 3$, we see that every element of the indeterminacy is nilpotent of height p. One can also show that the conjecture is true when taking the Massey product of one-dimensional classes in the cohomology of a space [**Dwy**]; otherwise, we do not have much evidence for it. Meanwhile, it has the following consequence.

CONJECTURE B.3.9. *In particular, if $n < p$ and if a_i has odd total degree for $i = 1, \ldots, n$, then for any element x contained in $\langle a_1, \ldots, a_n\rangle$ is nilpotent.*

Since b_{11}^N is an element of the indeterminacy of a p-fold Massey product of odd-dimensional classes, we may conclude that b_{11}^N is nilpotent. This would finish the proof of Conjecture B.3.4(a).

Part (b) of the conjecture would be proved similarly. One starts with the relation $h_{t0}h_{tt} = 0$ in Ext_B and applies Steenrod operations to get the following replacement for (B.3.5):

$$h_{tt}b_{tt}^{p^{t-1}+p^{2t-1}+\cdots+p^{kt-1}} = \pm h_{t,(k+1)t}b_{t0}^{p^{t-1}+p^{2t-1}+\cdots+p^{kt-1}}.$$

If we set

$$M = 1 + p^{t-1} + (p^{t-1} + p^{2t-1}) + \cdots + (p^{t-1} + p^{2t-1} + \cdots + p^{(p-1)t-1}),$$

then we get

$$\begin{aligned}
b_{tt}^M &= \langle h_{tt}, \ldots, h_{tt}\rangle b_{tt}^{M-1} \\
&\in \langle h_{tt}, h_{tt}b_{tt}^{p^{t-1}}, h_{tt}b_{tt}^{p^{t-1}+p^{2t-1}}, \ldots, h_{tt}b_{tt}^{p^{t-1}+\cdots+p^{(p-1)t-1}}\rangle \\
&= \langle h_{tt}, h_{t,2t}b_{t0}^{p^{t-1}}, h_{t,3t}b_{t0}^{p^{t-1}+p^{2t-1}}, \ldots, h_{t,pt}b_{t0}^{p^{t-1}+\cdots+p^{(p-1)t-1}}\rangle \\
&\ni \langle h_{tt}, h_{t,2t}, h_{t,3t}, \ldots, h_{t,pt}\rangle b_{t0}^{M-1} \\
&= 0.
\end{aligned}$$

So Conjecture B.3.9 implies that b_{tt}^M is nilpotent. □

As remarked above, the gap in the proof—Conjecture B.3.8—is not a gap when $p = 3$, so Conjecture B.3.4 holds at the prime 3:

PROPOSITION B.3.10. *Fix $p=3$.*
(a) *The element $b_{11} \in \mathrm{Ext}_A^{2,36}(\mathbf{F}_3,\mathbf{F}_3)$ is nilpotent. Indeed, $b_{11}^{15}=0$.*
(b) *Fix $t \geq 1$. Then b_{tt} is nilpotent in $\mathrm{Ext}_B^{**}(\mathbf{F}_3,\mathbf{F}_3)$, where*

$$B = \mathbf{F}_3[\xi_t, \xi_{2t}, \xi_{2t+1}, \xi_{2t+2}, \xi_{2t+3}, \xi_{2t+4}, \dots].$$

Arguing similarly, we see:

PROPOSITION B.3.11. *Fix $p=3$. The element $b_{22} \in \mathrm{Ext}_B^{2,432}(\mathbf{F}_3,\mathbf{F}_3)$ is nilpotent, where $B = A/(\xi_1)$.*

PROOF. The coproduct $\xi_4 \mapsto \xi_2^{p^2} \otimes \xi_2$ gives the relation $h_{20}h_{22}=0$ in Ext_B. Hence $h_{2,i}h_{2,i+2}=0$ for all i.

As in the "proof" of Conjecture B.3.4, we find that b_{22}^{16} is contained in the Massey product $\langle h_{22}, h_{24}b_{20}^3, h_{26}b_{20}^{3+9}\rangle$; this Massey product also contains the element $\langle h_{22}, h_{24}, h_{26}\rangle b_{20}^{15}$. We claim that the three-fold Massey product $\langle h_{22}, h_{24}, h_{26}\rangle$ is nilpotent.

This Massey product is defined because the product $h_{2,i}h_{2,i+2}$ is killed by $\xi_4^{3^i}$. So we consider the diagonal on ξ_6:

$$\begin{aligned} \xi_6 &\mapsto \xi_4^{3^2} + \xi_3^{3^3} \otimes \xi_3 + \xi_2^{3^4} \otimes \xi_4 \\ &= \langle h_{20}, h_{22}, h_{24}\rangle + h_{30}h_{33}. \end{aligned}$$

We have used the fact that $\xi_3^{3^i}$ is primitive in $A/(\xi_1)$, and hence gives rise to a 1-dimensional Ext class $h_{1,i}$. Hence we have

$$h_{30}h_{33} = -\langle h_{20}, h_{22}, h_{24}\rangle.$$

By graded commutativity, $(h_{30}h_{33})^2=0$; hence the same is true of the Massey product.

We find that $b_{22}^{16} = \langle h_{22}, h_{24}, h_{26}\rangle b_{10}^{15} + x$ for some class x in the indeterminacy of a three-fold Massey product. Both x and $\langle h_{22}, h_{24}, h_{26}\rangle$ are nilpotent, and hence so is b_{22}. □

COROLLARY B.3.12. *Fix $p=3$. The element $b_{22} \in \mathrm{Ext}_C^{2,432}(\mathbf{F}_3,\mathbf{F}_3)$ is nilpotent, where $C = A/(\xi_1^3)$.*

PROOF. If we know that b_{22} is nilpotent in $\mathrm{Ext}_{A/(\xi_1)}$ and we want to know about its nilpotency in $\mathrm{Ext}_{A/(\xi_1^3)}$, then by Lemma 1.3.10 it suffices to determine whether $b_{10}b_{22}$ is nilpotent in $\mathrm{Ext}_{A/(\xi_1^3)}$: b_{22} is nilpotent over C if and only if $b_{10}b_{22}$ is nilpotent over C.

This is easy, though: the coproduct of ξ_3 reduces in C to

$$\xi_3 \mapsto \xi_2^3 \otimes \xi_1,$$

giving the relation $h_{10}h_{21}=0$ in $\mathrm{Ext}_C^{**}(\mathbf{F}_3,\mathbf{F}_3)$. Applying $\beta\widetilde{\mathscr{P}}^0$ gives

$$h_{11}b_{21} - b_{10}h_{22} = 0,$$

and so $b_{10}h_{22}=0$ ($h_{11}=0$ over C, since $\xi_1^3=0$ in C). Applying $\beta\widetilde{\mathscr{P}}^3$ then gives

$$b_{10}^3 b_{22} = 0.$$

□

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Index