

# PHANTOM ELEMENTS AND ITS APPLICATIONS

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ABSTRACT. In our previous work[11], a relation between Tsukiyama problem about self homotopy equivalence was found by using a generalization of phantom map. In this note, fundamental result is established for such a generalization. This is the first time one can deal with phantom maps to space not satisfying finite type condition. Application to Forgetful map is also discussed briefly.

## 1. INTRODUCTION

The main aim of this paper is to study phantom map, its generalization and applications. After the discovery of the first example of phantom map by Adams and Walker[1], theory of phantom map receives a lot of attention. The main aim of these previous studies is however to understand it, e.g., the computation and the properties of phantom maps. The first application of theory of phantom maps was given by Harper and Roitberg[5],[12] who applied it to compute  $SNT(X)$  and  $Aut(X)$ . Recently applications are also found by Roitberg[13] and Pan[10] where several conjectures of McGibbon were settled. On the other hand, a remarkable connection was established by Pan and Woo[11] between Tsukiyama problem about self homotopy equivalence and a generalization of phantom map. A byproduct of this connection is that a special case of Tsukiyama problem is almost equivalent to the famous Halperin conjecture in rational homotopy theory[4].

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A well known characterization of map between nilpotent spaces of finite type to be phantom map is the following

**Theorem 1.1.** *Let  $X, Y$  be nilpotent CW complexes of finite type with  $Y$  1-connected and  $f : X \rightarrow Y$  be any map. Then the followings are equivalent*

- $f$  is a phantom map
- $e \circ f \simeq *$  where  $e : Y \rightarrow \hat{Y}$  is the profinite completion
- $f \circ \tau \simeq *$  where  $\tau : X_\tau \rightarrow X$  is the homotopy fiber of the rationalization

On the other hand, in our previous paper[11], we generalized the concept of phantom map to that of phantom element and announced a theorem characterizing an element to be a phantom element which generalizes Theorem1.1. In this paper we will generalize further so that we can deal with space which is not of finite type.

**Theorem 1.2.** *Let  $X$  be nilpotent CW complex of finite type,  $Y$  be 1-connected such that  $\pi_n(Y)$  is reduced group for  $n \geq 2$  and  $g : X \rightarrow Y$  be any map. Then the followings are equivalent:*

- $\alpha \in \pi_j(\text{map}_*(X, Y); g)$  is a phantom element
- $(e_*)_{\#}(\alpha) = 0$  where  $(e_*)_{\#} : \pi_j(\text{map}_*(X, Y); g) \rightarrow \pi_j(\text{map}_*(X, \hat{Y}); \hat{g})$
- $(\tau^*)_{\#}(\alpha) = 0$  where  $(\tau^*)_{\#} : \pi_j(\text{map}_*(X, Y); g) \rightarrow \pi_j(\text{map}_*(X_\tau, Y); g_\tau)$

Note that the assumption that  $Y$  is 1-connected is not a real restriction by an observation of Zabrodsky[15]. We will give a complete proof of this theorem in this paper.

As an application we have(Corollary of Proposition 3.3)

**Corollary 1.3.** *Let  $P$  be 1-connected finite dimensional CW complex or such that  $H^*(P, \mathbb{Z}_p)$  is locally finite over  $A_p$  for each prime  $p$  and be of type  $F_0$ . Assume further that  $\pi_n \text{Baut}(P)$  is reduced group for  $n \geq 2$ . Assume further that  $P_{(0)}$  satisfying one of the following.*

- $P$  is rationally equivalent to Kähler manifold
- $H^*(P; \mathbb{Q})$  as an algebra has at most 3 generators
- $P$  is rationally equivalent to  $G/U$  where  $G$  is a compact Lie group and  $U$  is a closed subgroup of maximal rank

Then for all  $m \geq 1$ ,  $H$  and every principal  $K(H, 2m)$ -bundle with total space homotopy equivalent to  $P$ , Forgetful map is injective.

The organization of this paper is as follows. In section2 Theorem1.2 will be proved. The applications to Forgetful map will be discussed in section3. In this paper, We will use the following notations:

- $H$  will denote a finitely generated abelian group
- $map(X, Y)$  is the space of continuous mappings from  $X$  to  $Y$
- $map_*(X, Y)$  is the subspace of pointed mappings from  $(X, x_0)$  to  $(Y, y_0)$
- $l : X \rightarrow X_{(0)}$  is the rationalization
- Let  $\tau : X_\tau \rightarrow X$  be the homotopy fiber of  $l$ . Then  $X_\tau \xrightarrow{\tau} X \rightarrow X_{(0)}$  is a cofibration up to homotopy
- $e_p : Y \rightarrow \hat{Y}_{\mathbb{Z}_p^\infty}$  is Bousfield-Kan's  $p$ -completion. Let  $\hat{Y} = \prod_p \hat{Y}_{\mathbb{Z}_p^\infty}$  and  $e = (e_2, e_3, \dots) : Y \rightarrow \hat{Y}$ . Let  $Y_\rho$  be the homotopy fiber of  $e$

The readers should refer to [11] for all the other notations which have not been explained here.

In concluding the Introduction, we'd like to give the following

**Conjecture 1.4.** *The condition that  $\pi_n Y$  is reduced group in this paper can be removed.*

## 2. PHANTOM ELEMENTS

Let's begin with definition.

**Definition 2.1.** Let spaces  $X$  be a CW complex,  $Y$  be a space and  $g : X \rightarrow Y$  any map. Then an element  $\alpha \in \pi_j(map_*(X, Y); g)$  is called a  $g$ -phantom element if  $(i_n^*)_{\#}(\alpha) = 0$  for all  $n \geq 0$  where  $(i_n^*)_{\#} : \pi_j map_*(X, Y) \rightarrow \pi_j map_*(X^n, Y)$  is the homomorphism induced by the inclusion  $i_n : X^n \rightarrow X$ . Denoted by

$$Ph_j^g(X, Y) = \{\alpha \in \pi_j(map_*(X, Y); g) | \alpha \text{ is a } g\text{-phantom element}\}$$

Obviously if  $g = \text{constant}$  and  $j = 0$ , then  $\alpha$  is a  $g$ -phantom element iff it represents the homotopy class of a map which is a phantom map.

The main aim of this section is to prove Theorem 1.2. Before that, let's give some results necessary to the proof.

**Lemma 2.2.** *Let  $Y$  be 1-connected such that  $\pi_n(Y)$  is reduced group for  $n \geq 2$ . Then*

- $\pi_n(\hat{Y}) = \prod_p Ext(\mathbb{Z}_{p^\infty}, \pi_n(Y))$
- For  $W$  a finite CW complex,  $e_* : \pi_j map_*(W, Y)_f \rightarrow \pi_j map_*(W, \hat{Y})_{\hat{f}}$  is injective

*Proof.* The first statement follows from the fact that  $Hom(\mathbb{Z}_{p^\infty}, B) = 0$  for a reduced group since otherwise there will be nontrivial divisible subgroup in  $B$ .

To prove the second statement, note that the induced map  $\pi_n(Y_\rho) \rightarrow \pi_n(Y)$  is trivial since  $\pi_n(Y)$  is reduced group and  $\pi_n(Y_\rho)$  is rational thus divisible by the arithmetic square Theorem [3]. It follows that  $e_* : \pi_n(Y) \rightarrow \pi_n(\hat{Y})$  is injective and an easy induction argument shows what we want for  $j \geq 1$ . For  $j = 0$ , it can be proved by an argument similar to that of Theorem 2.5.3 of [6].  $\square$

**Proposition 2.3.** *Let  $X$  be nilpotent space and  $Y$  be 1-connected such that  $\pi_n(Y)$  is reduced group for  $n \geq 2$ . Then the followings hold:*

- $[\Sigma^n X_{(0)}, \hat{Y}] = *$ ,  $\tilde{H}^n(X_{(0)}, \pi_i(\hat{Y})) = 0$  for all  $n, i \geq 0$
- $[\Sigma^n X_\tau, Y_\rho] = *$ ,  $\tilde{H}^n(X_\tau, \pi_i(Y_\rho)) = 0$  for all  $n, i \geq 0$

*Proof.* That  $\tilde{H}^n(X_{(0)}, \pi_i(\hat{Y})) = 0$  follows from the fact that

$$Hom(A, B) = 0, Ext(A, B) = 0$$

for rational group  $A$  and  $B = \prod_p Ext(\mathbb{Z}_{p^\infty}, B')$

Then the equation

$$[\Sigma^n X_{(0)}, \hat{Y}] = \lim_{\leftarrow n} [\Sigma^n X_{(0)}, \hat{Y}^{(n)}]$$

implies the first statement.

The equation about cohomology in second statement is true since  $\pi_n(Y_\rho)$  is rational while the proof of another equation is similar to that as in the first statement.  $\square$

**Proposition 2.4.** *Let  $X$  be nilpotent space and  $Y$  be 1-connected such that  $\pi_n(Y)$  is reduced group for  $n \geq 2$ . Then the followings hold :*

- $\tau^* : \text{map}_*(X, \hat{Y}) \xrightarrow{w} \text{map}_*(X_\tau, \hat{Y})$
- $\rho_* : \text{map}_*(X_{(0)}, Y_\rho) \xrightarrow{w} \text{map}_*(X_{(0)}, Y)$
- $e_* : \text{map}_*(X_\tau, Y) \xrightarrow{w} \text{map}_*(X_\tau, \hat{Y})$
- $l^* : \text{map}_*(X_{(0)}, Y_\rho) \xrightarrow{w} \text{map}_*(X, Y_\rho)$

*Proof.* The first and the last statements follow from the last Proposition and the well known Zabrodsky Lemma. The second and third statements follow from a  $\lim^1$  argument for  $j \geq 0$

$$* \rightarrow \lim_{\leftarrow n}^1 \pi_{j+1} \text{map}_*(Z, E_n) \rightarrow \pi_j \text{map}_*(Z, E) \rightarrow \lim_{\leftarrow n} \pi_j \text{map}_*(Z, E_n) \rightarrow *$$

where in the second statement,  $Z = X_{(0)}$  and  $E_n$  is the  $n$ -th term in the Postnikov-Moore tower of the map  $\rho : Y_\rho \rightarrow Y$  while in the third statement,  $Z = X_\tau$  and  $E_n$  is the  $n$ -th term in the Postnikov-Moore tower of the map  $e : Y \rightarrow \hat{Y}$ . In both case the sequence  $\{\pi_j \text{map}_*(Z, E_n)\}$  is a sequence consisting of isomorphisms and thus the  $\lim^1$  is trivial and the wanted isomorphisms follows immediately.  $\square$

**Proposition 2.5.** *Let  $X$  be nilpotent space and  $Y$  1-connected such that  $\pi_n(Y)$  is reduced group for  $n \geq 2$ . Let  $g : X \rightarrow \hat{Y}$  be any map. Then*

$$Ph_j^g(X, \hat{Y}) = *$$

*Proof.*  $Ph_j^g(X, \hat{Y})$  is the  $\lim^1$  of a sequence of compact groups and continuous homomorphisms which is will known to be trivial.  $\square$

**Proposition 2.6.** *Let  $X, Y$  be two nilpotent spaces with  $Y$  1-connected such that  $\pi_n(Y)$  is reduced group for  $n \geq 2$ . Then the followings hold :*

- $\text{map}_*(X_\tau, Y) \xrightarrow{w} \text{map}_*(X, \hat{Y})$
- $\text{map}_*(X_{(0)}, Y) \xrightarrow{w} \text{map}_*(X, Y_\rho)$

*Proof.* This is an easy consequence of the Proposition above.  $\square$

*Proof of Theorem 1.2.* The equivalence between the last two statements follows directly from the following commutative diagram where the

bottom horizontal homomorphism and the right side vertical homomorphism are isomorphisms by Proposition 2.4 .

$$\begin{array}{ccc} \pi_j \text{map}_*(X, Y) & \xrightarrow{(e_*)\#} & \pi_j \text{map}_*(X_\tau, Y) \\ (e_*)\# \downarrow & & (e_*)\# \downarrow \\ \pi_j \text{map}_*(X, \hat{Y}) & \xrightarrow{(e_*)\#} & \pi_j \text{map}_*(X_\tau, \hat{Y}) \end{array}$$

Now assume the first statement, then we have  $(i_n^*)\#(e_*(\alpha)) = 0$  for all  $n \geq 0$ . It follows from Proposition 2.5 that  $e_*(\alpha) = 0$ . The proof of another direction is similar to that in [11] using Lemma 2.2 instead of Sullivan's original result which is stated only for space of finite type.  $\square$

*Remark 2.7.* It is easy to see that the above proof follows the same pattern as that given by Oda and Shitanda[9]. We give a prove here because Oda informed us that there were gaps in their proof and he don't know if the result is true or not. The similar proof applies also to the equivariant case which will be discussed in future publication.

As noted in [11] , the natural question related to the application of phantom element to the forgetful map is

**Question 2.8.** *For two maps  $f, g : X \rightarrow Y$ , what is the relation between  $Ph_j^g(X, Y)$  and  $Ph_j^f(X, Y)$  ?*

**Proposition 2.9.** *Let  $X, Y$  be nilpotent CW complexes such that*

$$[\Sigma^j X_\tau, Y] = [\Sigma^{j+1} X_\tau, Y] = 0$$

*If  $g : X \rightarrow Y$  is a phantom map, then we have*

$$Ph_j^g(X, Y) = \pi_j(\text{map}_*(X, Y); g)$$

*Proof.* The proof is the same as that in [11] .  $\square$

In our application we have to be able to compute  $Ph_j^g(X, Y)$ . Before giving this kind of result, recall that a CW complex is called unstable if all the attaching maps vanish under suspension. It is Baues [2] who noted the following which is dual to Zabrodsky's integral approximation.

**Theorem 2.10.** *Let  $X$  be 1-connected CW complex. Then there is an unstable complex and a rational equivalence  $h : \bar{X} \rightarrow X$ .*

*Remark 2.11.* Let  $X$  be an unstable CW complex. Then it is easy to prove that  $Ph_j^g(X, Y) = *$  for any map  $g : X \rightarrow Y$ .

**Proposition 2.12.** *Let  $X$  be a 1-connected CW complex and  $Y$  1-connected such that  $\pi_n(Y)$  is reduced group for  $n \geq 2$ . Suppose further that the component of  $map_*(X, \hat{Y})$  consisting constant map is weakly contractible and  $g : X \rightarrow Y$  is a phantom map. Then*

$$Ph_j^g(X, Y) = \pi_j(map_*(X, Y); g) = \prod_{k>0} H^k(X, \pi_{k+j+1}(Y_\rho))$$

*Proof.* As first noted by Oda and Shitanda , similar proof as in that of Theorem B of [15] leads to the following homotopy fibration

$$\bigcup_g map_*(X, Y)_g \rightarrow map_*(\bar{X}, Y)_* \rightarrow map_*(\bar{X}, \hat{Y})_*$$

where the union is over phantom maps  $g$ . On the other hand , different components of  $\bigcup_g map_*(X, Y)_g$  are homotopy equivalent since  $\bigcup_g map_*(X, Y)_g$  is the homotopy fiber of a map between two connected spaces. It follows that

$$\begin{aligned} Ph_j^g(X, Y) &= \pi_j(map_*(X, Y); const) = \pi_j(map_*(X, Y_\rho); const) = \\ &= [\Sigma^{j-1} X, \Omega Y_\rho] = \prod_{k>0} H^k(X, \pi_{j+k+1}(Y_\rho)) \end{aligned}$$

□

We are ready to state results related to the applications. Before that we have another definition

**Definition 2.13.** Let  $A_p$  be the modp Steenrod algebra. An unstable module  $M$  over  $A_p$  is called locally finite iff , for any  $x \in M$ , only finite elements of  $A_p$  can acts nontrivially on  $M$ .

**Example 2.14.** *Let  $P$  be a space such that  $H^*(P, \mathbb{Z}_p)$  is locally finite over  $A_p$  . Then so is  $\Omega P$ . In particular, if  $P$  is finite CW , then  $H^*(\Omega P, \mathbb{Z}_p)$  is locally finite over  $A_p$ .*

**Theorem 2.15.** *Let  $X = K(H, m + 2)$ ,  $Y = \text{Baut}(P)$  such that  $\pi_n(Y)$  is reduced group for  $n \geq 2$  and  $g : X \rightarrow Y$  is any map where  $P$  is 1-connected finite dimensional CW complex or such that  $H^*(P, \mathbb{Z}_p)$  is locally finite over  $A_p$  for each prime  $p$  and  $m \geq 1$ . Then for  $j \geq 1$*

$$Ph_j^g(X, Y) = \pi_j(\text{map}_*(X, Y); g) = [\Sigma^j X, Y_\rho]$$

*Proof.* The proof is the same as that of the corresponding result in [11] using results of Zabrodsky and Miller[8] or Theorem 8.8 in [14].  $\square$

Similarly we have

**Theorem 2.16.** *Let  $X = BG$ ,  $Y = \text{Baut}(P)$  such that  $\pi_n(Y)$  is reduced group for  $n \geq 2$ . and  $g : X \rightarrow Y$  is a phantom map where  $G$  is a connected compact Lie group and  $P$  is 1-connected finite dimensional CW complex or such that  $H^*(P, \mathbb{Z}_p)$  is locally finite over  $A_p$  for each prime  $p$ . Then for  $j \geq 1$  we have*

$$Ph_j^g(X, Y) = \pi_j(\text{map}_*(X, Y); g) = [\Sigma^j X, Y_\rho]$$

### 3. APPLICATION TO THE FORGETFUL MAP

Given a principal  $G$ -bundle  $\pi : P \rightarrow B$ , Let

$$\text{aut}^G(P) = \{g | g : P \rightarrow P \text{ is a } G\text{-equivariant homotopy equivalence}\}$$

and

$$\text{aut}(P) = \{g | g : P \rightarrow P \text{ is a homotopy equivalence}\}$$

There is a natural map  $f : \text{aut}^G(P) \rightarrow \text{aut}(P)$ . Let

$$\text{Aut}^G(P) = \pi_0(\text{aut}^G(P))$$

and

$$\text{Aut}(P) = \pi_0(\text{aut}(P))$$

Then the map  $f$  induces a map

$$F : \text{Aut}^G(P) \rightarrow \text{Aut}(P)$$

which is called a Forgetful map by Tsukiyama. The question posed by Tsukiyama in [7] is the following

**Question 3.1.** *Is the forgetting map  $F$  injective?*



One of the main results in [11] is the following

**Theorem 3.2.** *Let  $\pi : P \rightarrow B$  be a principal  $G$ -bundle. Then there is an exact sequence*

$$\pi_1 \text{aut}(P) \xrightarrow{\delta} \pi_1(\text{map}_*(BG, \text{Baut}(P)), c) \rightarrow \text{Aut}^G(P) \xrightarrow{F} \text{Aut}(P)$$

where  $c : BG \rightarrow \text{Baut}(P)$  is determined by the principal bundle.

Combined with results in [11], we have

**Proposition 3.3.** *Let  $P$  be as in Theorem 2.16. If*

$$\bigoplus_{i>1} \pi_{2i}(\text{map}(P_{(0)}, P_{(0)}); id) = 0$$

then for all  $m \geq 1$ , finitely generated abelian group  $H$  and every principal  $K(H, 2m)$ -bundle with total space homotopy equivalent to  $P$ , the associated Forgetful map is injective.

We have also similar results for  $K(H, 2m + 1)$  or  $G$  bundle where  $G$  is a connected compact Lie group which will be omitted.

Unlike that in [11], there are no complete results if group  $\pi_1(\text{map}_*(BG, \text{Baut}(P)), c)$  is nontrivial although we know that it is still uncountable since group  $\pi_1 \text{aut}(P)$  itself may be uncountable too. Thus same results as in [11] can be obtained if  $\pi_1 \text{aut}(P)$  is countable. This is so if  $P = \Omega P'$  where  $P'$  is finite complex. An interesting question is

**Question 3.4.** *Study the map  $\delta$  in the exact sequence of Theorem 3.2. Is it possible that  $\text{Image}(\delta)$  is always countable group.*

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