

A universality theorem for Voevodsky’s algebraic cobordism spectrum

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Abstract

An algebraic version of a theorem due to Quillen is proved. More precisely, for a ground field k we consider the motivic stable homotopy category $\mathrm{SH}(k)$ of \mathbf{P}^1 -spectra, equipped with the symmetric monoidal structure described in [PPR1]. The algebraic cobordism \mathbf{P}^1 -spectrum MGL is considered as a commutative monoid equipped with a canonical orientation $th^{\mathrm{MGL}} \in \mathrm{MGL}^{2,1}(\mathrm{Th}(\mathcal{O}(-1)))$. For a commutative monoid E in the category $\mathrm{SH}(k)$ it is proved that assignment $\varphi \mapsto \varphi(th^{\mathrm{MGL}})$ identifies the set of monoid homomorphisms $\varphi: \mathrm{MGL} \rightarrow E$ in the motivic stable homotopy category $\mathrm{SH}(k)$ with the set of all orientations of E . The result was stated originally in a slightly different form by G. Vezzosi in [Ve].

1 Oriented commutative ring spectra

We refer to [PPR1, Appendix] for the basic terminology, notation, constructions, definitions, results. For the convenience of the reader we recall the basic definitions. Let S be a Noetherian scheme of finite Krull dimension. One may think of S being the spectrum of a field or the integers. Let $\mathcal{S}m/S$ be the category of smooth quasi-projective S -schemes, and let \mathbf{sSet} be the category of simplicial sets. A *motivic space over S* is a functor

$$A: \mathcal{S}m\mathcal{O}p/S \rightarrow \mathbf{sSet}$$

(see [PPR1, A.1.1]). The category of motivic spaces over S is denoted $\mathbf{M}(S)$. This definition of a motivic space is different from the one considered by Morel and Voevodsky in [MV] – they consider only those simplicial presheaves which are sheaves in the Nisnevich topology on $\mathcal{S}m/S$. With our definition the Thomason-Trobaugh K -theory functor obtained by using big vector bundles is a motivic space on the nose. It is not a simplicial Nisnevich sheaf. This is why we prefer to work with the above notion of “space”.

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We write $H_{\bullet}^{\text{cm}}(S)$ for the pointed motivic homotopy category and $\text{SH}^{\text{cm}}(S)$ for the stable motivic homotopy category over S as constructed in [PPR1, A.3.9, A.5.6]. By [PPR1, A.3.11 resp. A.5.6] there are canonical equivalences to $H_{\bullet}(S)$ of [MV] resp. $\text{SH}(S)$ of [V1]. Both $H_{\bullet}^{\text{cm}}(S)$ and $\text{SH}_{\bullet}^{\text{cm}}(S)$ are equipped with closed symmetric monoidal structures such that the \mathbf{P}^1 -suspension spectrum functor is a strict symmetric monoidal functor

$$\Sigma_{\mathbf{P}^1}^{\infty} : H_{\bullet}^{\text{cm}}(S) \rightarrow \text{SH}^{\text{cm}}(S).$$

Here \mathbf{P}^1 is considered as a motivic space pointed by $\infty \in \mathbf{P}^1$. The symmetric monoidal structure $(\wedge, \mathbb{I}_S = \Sigma_{\mathbf{P}^1}^{\infty} S_+)$ on the homotopy category $\text{SH}^{\text{cm}}(S)$ is constructed on the model category level by employing symmetric \mathbf{P}^1 -spectra. It satisfies the properties required by Theorem 5.6 of Voevodsky congress talk [V1]. From now on we will usually omit the superscript $(-)^{\text{cm}}$.

Every \mathbf{P}^1 -spectrum $E = (E_0, E_1, \dots)$ represents a cohomology theory on the category of pointed motivic spaces. Namely, for a pointed motivic space (A, a) set

$$E^{p,q}(A, a) = \text{Hom}_{\text{SH}_{\bullet}(S)}(\Sigma_{\mathbf{P}^1}^{\infty}(A, a), \Sigma^{p,q}(E))$$

and $E^{*,*}(A, a) = \bigoplus_{p,q} E^{p,q}(A, a)$. This definition extends to motivic spaces via the functor $A \mapsto A_+$ which adds a disjoint basepoint. That is, for a non-pointed motivic space A set $E^{p,q}(A) = E^{p,q}(A_+, +)$ and $E^{*,*}(A) = \bigoplus_{p,q} E^{p,q}(A)$. Recall that there is a canonical element in $E^{2n,n}(E_n)$, denoted as $\Sigma_{\mathbf{P}^1}^{\infty} E_n(-n) \rightarrow E$. It is represented by the canonical map $(*, \dots, *, E_n, E_n \wedge \mathbf{P}^1, \dots) \rightarrow (E_0, E_1, \dots, E_n, \dots)$ of \mathbf{P}^1 -spectra.

Every $X \in \mathcal{S}m/S$ defines a representable motivic space constant in the simplicial direction taking an S -smooth scheme U to $\text{Hom}_{\mathcal{S}m/S}(U, X)$. It is not possible in general to choose a basepoint for representable motivic spaces. So we regard S -smooth varieties as motivic spaces (non-pointed) and set

$$E^{p,q}(X) = E^{p,q}(X_+, +).$$

Given a \mathbf{P}^1 -spectrum E we will reduce the double grading on the cohomology theory $E^{*,*}$ to a grading. Namely, set $E^m = \bigoplus_{m=p-2q} E^{p,q}$ and $E^* = \bigoplus_m E^m$. We often write $E^*(k)$ for $E^*(\text{Spec}(k))$ below.

To complete this section, note that for us a \mathbf{P}^1 -ring spectrum is a monoid (E, μ, e) in $(\text{SH}(S), \wedge, \mathbb{I}_S)$. A commutative \mathbf{P}^1 -ring spectrum is a commutative monoid (E, μ, e) in $(\text{SH}(S), \wedge, 1)$. The cohomology theory E^* defined by a \mathbf{P}^1 -ring spectrum is a ring cohomology theory. The cohomology theory E^* defined by a commutative \mathbf{P}^1 -ring spectrum is a ring cohomology theory, however it is not necessary graded commutative. The cohomology theory E^* defined by an oriented commutative \mathbf{P}^1 -ring spectrum is a graded commutative ring cohomology theory.

1.1 Oriented commutative ring spectra

Following Adams and Morel we define an orientation of a commutative \mathbf{P}^1 -ring spectrum. However we prefer to use Thom classes instead of Chern classes. Consider the pointed motivic space $\mathbf{P}^{\infty} = \text{colim}_{n \geq 0} \mathbf{P}^n$ having base point $g_1 : S = \mathbf{P}^0 \hookrightarrow \mathbf{P}^{\infty}$.

The tautological “vector bundle” $\mathcal{J}(1) = \mathcal{O}_{\mathbf{P}^\infty}(-1)$ is also known as the Hopf bundle. It has zero section $z: \mathbf{P}^\infty \hookrightarrow \mathcal{J}(1)$. The fiber over the point $g_1 \in \mathbf{P}^\infty$ is \mathbb{A}^1 . For a vector bundle V over a smooth S -scheme X , with zero section $z: X \hookrightarrow V$, its *Thom space* $\mathrm{Th}(V)$ is the Nisnevich sheaf associated to the presheaf $Y \mapsto V(Y)/(V \setminus z(X))(Y)$ on the Nisnevich site $\mathcal{S}m/S$. In particular, $\mathrm{Th}(V)$ is a pointed motivic space in the sense of [PPR1, Defn. A.1.1]. It coincides with Voevodsky’s Thom space [V1, p. 422], since $\mathrm{Th}(V)$ already is a Nisnevich sheaf. The Thom space of the Hopf bundle is then defined as the colimit $\mathrm{Th}(\mathcal{J}(1)) = \mathrm{colim}_{n \geq 0} \mathrm{Th}(\mathcal{O}_{\mathbf{P}^n}(-1))$. Abbreviate $T = \mathrm{Th}(\mathbf{A}_S^1)$.

Let E be a commutative \mathbf{P}^1 -ring spectrum. The unit gives rise to an element $1 \in E^{0,0}(\mathrm{Spec}(k)_+)$. Applying the \mathbf{P}^1 -suspension isomorphism to that element we get an element $\Sigma_{\mathbf{P}^1}(1) \in E^{2,1}(\mathbf{P}^1, \infty)$. The canonical covering of \mathbf{P}^1 defines motivic weak equivalences

$$\mathbf{P}^1 \xrightarrow{\sim} \mathbf{P}^1/\mathbf{A}^1 \xleftarrow{\sim} \mathbf{A}^1/\mathbf{A}^1 \setminus \{0\} = T$$

of pointed motivic spaces inducing isomorphisms $E(\mathbf{P}^1, \infty) \leftarrow E(\mathbf{A}^1/\mathbf{A}^1 \setminus \{0\}) \rightarrow E(T)$. Let $\Sigma_T(1)$ be the image of $\Sigma_{\mathbf{P}^1}(1)$ in $E^{2,1}(T)$.

Definition 1.1.1. Let E be a commutative ring \mathbf{P}^1 -spectrum. A *Thom orientation* of E is an element $th \in E^{2,1}(\mathrm{Th}(\mathcal{J}(1)))$ such that its restriction to the Thom space of the fibre over the distinguished point coincides with the element $\Sigma_T(1) \in E^{2,1}(T)$. A *Chern orientation* of E is an element $c \in E^{2,1}(\mathbf{P}^\infty)$ such that $c|_{\mathbf{P}^1} = -\Sigma_{\mathbf{P}^1}(1)$. An *orientation* of E is either a Thom orientation or a Chern orientation. One says that a Thom orientation th of E coincides with a Chern orientation c of E provided that $c = z^*(th)$ or equivalently the element th coincides with the one $th(\mathcal{O}(-1))$ given by (2) below.

Remark 1.1.2. The element th should be regarded as the Thom class of the tautological line bundle $\mathcal{J}(1) = \mathcal{O}(-1)$ over \mathbf{P}^∞ . The element c should be regarded as the Chern class of the tautological line bundle $\mathcal{J}(1) = \mathcal{O}(-1)$ over \mathbf{P}^∞ .

Example 1.1.3. The following orientations given right below are relevant for our work. Here MGL denotes the \mathbf{P}^1 -ring spectrum representing algebraic cobordism obtained below in Definition 2.1.1, and BGL denotes the \mathbf{P}^1 -ring spectrum representing algebraic K -theory constructed in [PPR1, Theorem 2.2.1].

- Let $u_1 : \Sigma_{\mathbf{P}^1}^\infty(\mathrm{Th}(\mathcal{J}(1)))(-1) \rightarrow \mathrm{MGL}$ be the canonical map of \mathbf{P}^1 -spectra. Set $th^{\mathrm{MGL}} = u_1 \in \mathrm{MGL}^{2,1}(\mathrm{Th}(\mathcal{J}(1)))$. Since $th^{\mathrm{MGL}}|_{\mathrm{Th}(\mathbf{1})} = \Sigma_{\mathbf{P}^1}(1)$ in $\mathrm{MGL}^{2,1}(\mathrm{Th}(\mathbf{1}))$, the class th^{MGL} is an orientation of MGL.
- Set $c = (-\beta) \cup ([\mathcal{O}] - [\mathcal{O}(1)]) \in \mathrm{BGL}^{2,1}(\mathbf{P}^\infty)$. The relation (11) from [PPR1] shows that the class c is an orientation of BGL.

2 Oriented ring spectra and infinite Grassmannians

Let (E, c) be an oriented commutative \mathbf{P}^1 -ring spectrum. In this section we compute the E -cohomology of infinite Grassmannians and their products. The results are the expected ones – see Theorems 2.0.6 and 2.0.7.

The oriented \mathbf{P}^1 -ring spectrum (E, c) defines an oriented cohomology theory on $\mathcal{S}m\mathcal{O}p$ in the sense of [PS1, Defn. 3.1] as follows. The restriction of the functor $E^{*,*}$ to the category $\mathcal{S}m/S$ is a ring cohomology theory. By [PS1, Th. 3.35] it remains to construct a Chern structure on $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ in the sense of [PS1, Defn. 3.2]. Let $\mathbf{H}_\bullet(k)$ be the homotopy category of pointed motivic spaces over k . The functor isomorphism $\mathrm{Hom}_{\mathbf{H}_\bullet(k)}(-, \mathbf{P}^\infty) \rightarrow \mathrm{Pic}(-)$ on the category $\mathcal{S}m/S$ provided by [MV, Thm. 4.3.8] sends the class of the identity map $\mathbf{P}^\infty \rightarrow \mathbf{P}^\infty$ to the class of the tautological line bundle $\mathcal{O}(-1)$ over \mathbf{P}^∞ . For a line bundle L over $X \in \mathcal{S}m/S$ let $[L]$ be the class of L in the group $\mathrm{Pic}(X)$. Let $f_L: X \rightarrow \mathbf{P}^\infty$ be a morphism in $\mathbf{H}(k)$ corresponding to the class $[L]$ under the functor isomorphism above. For a line bundle L over $X \in \mathcal{S}m/S$ set $c(L) = f_L^*(c) \in E^{2,1}(X)$. Clearly, $c(\mathcal{O}(-1)) = c$. The assignment $L/X \mapsto c(L)$ is a Chern structure on $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ since $c|_{\mathbf{P}^1} = -\Sigma_{\mathbf{P}^1}(1) \in E^{2,1}(\mathbf{P}^1, \infty)$. With that Chern structure $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ is an oriented ring cohomology theory in the sense of [PS1]. In particular, (BGL, c^K) defines an oriented ring cohomology theory on $\mathcal{S}m\mathcal{O}p$.

Given this Chern structure, one obtains a theory of Thom classes $V/X \mapsto th(V) \in E^{2\mathrm{rank}(V), \mathrm{rank}(V)}(\mathrm{Th}_X(V))$ on the cohomology theory $E^{*,*}|_{\mathcal{S}m\mathcal{O}p/S}$ in the sense of [PS1, Defn. 3.32] as follows. There is a unique theory of Chern classes $V \mapsto c_i(V) \in E^{2i,i}(X)$ such that for every line bundle L on X one has $c_1(L) = c(L)$. For a rank r vector bundle V over X consider the vector bundle $W := \mathbf{1} \oplus V$ and the associated projective vector bundle $\mathbf{P}(W)$ of lines in W . Set

$$\bar{th}(V) = c_r(p^*(V) \otimes \mathcal{O}_{\mathbf{P}(W)}(1)) \in E^{2r,r}(\mathbf{P}(W)). \quad (1)$$

It follows from [PS1, Cor. 3.18] that the support extension map

$$E^{2r,r}(\mathbf{P}(W)/(\mathbf{P}(W) \setminus \mathbf{P}(\mathbf{1}))) \rightarrow E^{2r,r}(\mathbf{P}(W))$$

is injective and $\bar{th}(E) \in E^{2r,r}(\mathbf{P}(W)/(\mathbf{P}(W) \setminus \mathbf{P}(\mathbf{1})))$. Set

$$th(E) = j^*(\bar{th}(E)) \in E^{2r,r}(\mathrm{Th}_X(V)), \quad (2)$$

where $j: \mathrm{Th}_X(V) \rightarrow \mathbf{P}(W)/(\mathbf{P}(W) \setminus \mathbf{P}(\mathbf{1}))$ is the canonical motivic weak equivalence of pointed motivic spaces induced by the open embedding $V \hookrightarrow \mathbf{P}(W)$. The assignment V/X to $th(V)$ is a theory of Thom classes on $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ (see the proof of [PS1, Thm. 3.35]). Hence the Thom classes are natural, multiplicative and satisfy the following Thom isomorphism property.

Theorem 2.0.4. *For a rank r vector bundle $p: V \rightarrow X$ on $X \in \mathcal{S}m/S$ with zero section $z: X \hookrightarrow V$, the map*

$$- \cup th(V): E^{*,*}(X) \rightarrow E^{*+2r, *+r}(V/(V \setminus z(X)))$$

is an isomorphism of two-sided $E^{,*}(X)$ -modules, where $- \cup th(V)$ is written for the composition map $(- \cup th(V)) \circ p^*$.*

Proof. See [PS1, Defn. 3.32.(4)]. □

Analogous to [V1, p. 422] one obtains for vector bundles $V \rightarrow X$ and $W \rightarrow Y$ in $\mathcal{S}m/S$ a canonical map of pointed motivic spaces $\mathrm{Th}(V) \wedge \mathrm{Th}(W) \rightarrow \mathrm{Th}(V \times_S W)$ which is a motivic weak equivalence as defined in [PPR1, Defn. 3.1.6]. In fact, the canonical map becomes an isomorphism after Nisnevich (even Zariski) sheafification. Taking $Y = S$ and $W = \mathbf{1}$ the trivial line bundle yields a motivic weak equivalence $\mathrm{Th}(V) \wedge T \rightarrow \mathrm{Th}(V \oplus \mathbf{1})$. The canonical covering of \mathbf{P}^1 defines motivic weak equivalences

$$T = \mathbf{A}^1/\mathbf{A}^1 \setminus \{0\} \xrightarrow{\sim} \mathbf{P}^1/\mathbf{A}^1 \xleftarrow{\sim} \mathbf{P}^1$$

and the arrow $T = \mathbf{A}^1/\mathbf{A}^1 \setminus \{0\} \rightarrow \mathbf{P}^1/\mathbf{P}^1 \setminus \{0\}$ is an isomorphism. Hence one may switch between T and \mathbf{P}^1 as desired.

Corollary 2.0.5. *For $W = V \oplus \mathbf{1}$ consider the motivic weak equivalences*

$$\epsilon: \mathrm{Th}(V) \wedge \mathbf{P}^1 \rightarrow \mathrm{Th}(V) \wedge \mathbf{P}^1/\mathbf{A}^1 \leftarrow \mathrm{Th}(V) \wedge T \rightarrow \mathrm{Th}(W)$$

of pointed motivic spaces over S . The diagram

$$\begin{array}{ccc} E^{*+2r,*+r}(\mathrm{Th}(V)) & \xrightarrow{\Sigma_{\mathbf{P}^1}} & E^{*+2r+2,*+r+1}(\mathrm{Th}(V) \wedge \mathbf{P}^1) \\ \uparrow \mathrm{id} & & \uparrow \epsilon^* \\ E^{*+2r,*+r}(\mathrm{Th}(V)) & \xrightarrow{\Sigma_T} & E^{*+2r+2,*+r+1}(\mathrm{Th}(W)) \\ \uparrow -\cup th(V) & & \uparrow -\cup th(W) \\ E^{*,*}(X) & \xrightarrow{\mathrm{id}} & E^{*,*}(X) \end{array}$$

commutes.

Let $\mathrm{Gr}(n, n+m)$ be the Grassmann scheme of n -dimensional linear subspaces of \mathbf{A}_S^{n+m} . The closed embedding $\mathbf{A}^{n+m} = \mathbf{A}^{n+m} \times \{0\} \hookrightarrow \mathbf{A}^{n+m+1}$ defines a closed embedding

$$\mathrm{Gr}(n, n+m) \hookrightarrow \mathrm{Gr}(n, n+m+1). \quad (3)$$

The tautological vector bundle is denoted $\mathcal{J}(n, n+m) \rightarrow \mathrm{Gr}(n, n+m)$. The closed embedding (3) is covered by a map of vector bundles $\mathcal{J}(n, n+m) \hookrightarrow \mathcal{J}(n, n+m+1)$. Let $\mathrm{Gr}(n) = \mathrm{colim}_{m \geq 0} \mathrm{Gr}(n, n+m)$, $\mathcal{J}(n) = \mathrm{colim}_{m \geq 0} \mathcal{J}(n, n+m)$ and $\mathrm{Th}(\mathcal{J}(n)) = \mathrm{colim}_{m \geq 0} \mathrm{Th}(\mathcal{J}(n, n+m))$. These colimits are taken in the category of motivic spaces over S .

Theorem 2.0.6. *Let E be an oriented \mathbf{P}^1 -ring spectrum. Then*

$$E^{*,*}(\mathrm{Gr}(n)) = E^{*,*}(k)[[c_1, c_2, \dots, c_n]]$$

is the formal power series ring, where $c_i := c_i(\mathcal{J}(n)) \in E^{2i,i}(\mathrm{Gr}(n))$ denotes the i -th Chern class of the tautological bundle $\mathcal{J}(n)$. The inclusion $\mathrm{inc}_n: \mathrm{Gr}(n) \hookrightarrow \mathrm{Gr}(n+1)$ satisfies $\mathrm{inc}_n^*(c_m) = c_m$ for $m < n+1$ and $\mathrm{inc}_n^*(c_{n+1}) = 0$.

Proof. The case $n = 1$ is well-known (see for instance [PS1, Thm. 3.9]). For a finite dimensional vector space W and a positive integer m let $\mathbf{F}(m, W)$ be the flag variety of flags $W_1 \subset W_2 \subset \cdots \subset W_m$ of linear subspaces of W such that the dimension of W_i is i . Let $\mathcal{T}^i(m, W)$ be the tautological rank i vector bundle on $\mathbf{F}(m, W)$.

Let $V = \mathbf{A}^\infty$ be an infinite dimensional vector bundle over S and set $e = (1, 0, \dots)$. Then V_n denotes the n -fold product of V , and $e_i^n \in V_n$ the vector $(0, \dots, 0, e, 0, \dots, 0)$ having e precisely at the i th position. Let $F(m) = \text{colim}_W \mathbf{F}(m, W)$ and let $\mathcal{T}^i(m) = \text{colim}_W \mathcal{T}^i(m, W)$, where W runs over all finite-dimensional vector subspaces of V_n . Thus we have a flag $\mathcal{T}^1(m) \subset \mathcal{T}^2(m) \subset \cdots \subset \mathcal{T}^m(m)$ of vector bundles over $F(m)$. Set $L^i(m) = \mathcal{T}^i(m)/\mathcal{T}^{i-1}(m)$. It is a line bundle over $F(m)$.

Consider the morphism $p_m: F(m) \rightarrow F(m-1)$ which takes a flag $W_1 \subset W_2 \subset \cdots \subset W_m$ to the flag $W_1 \subset W_2 \subset \cdots \subset W_{m-1}$. It is a projective vector bundle over $F(m-1)$ such that the line bundle $L^i(m)$ is its tautological line bundle. Thus there exists a tower of projective vector bundles $F(m) \rightarrow F(m-1) \rightarrow \cdots \rightarrow F(1) = \mathbf{P}(V_n)$. The projective bundle theorem implies that

$$E^{*,*}(F(n)) = E^{*,*}(k)[[t_1, t_2, \dots, t_n]]$$

(the formal power series in n variables), where $t_i = c(L^i(n))$ is the first Chern class of the line bundle $L^i(n)$ over $F(n)$.

Consider the morphism $q: F(n) \rightarrow \text{Gr}(n)$, which takes a flag $W_1 \subset W_2 \subset \cdots \subset W_n$ to the space W_n . It can be decomposed as a tower of projective vector bundles. In particular, the pull-back map $q^*: E^{*,*}(\text{Gr}(n)) \rightarrow E^{*,*}(F(n))$ is a monomorphism. It takes the class c_i to the symmetric polynomial $\sigma_i = t_1 t_2 \dots t_i + \cdots + t_{n-i+1} \dots t_{n-1} t_n$. So the image of q^* contains $E^{*,*}(k)[[\sigma_1, \sigma_2, \dots, \sigma_n]]$. It remains to check that the image of q^* is contained in $E^{*,*}(k)[[\sigma_1, \sigma_2, \dots, \sigma_n]]$. To do that consider another variety.

Namely, let V^0 be the n -dimensional subspace of V_n generated by the vectors e_i^n 's. Let l_i^n be the line generated by the vector e_i^n . Let V_i^0 be a subspace of V^0 generated by all e_j^n 's with $j \leq i$. So one has a flag $V_1^0 \subset V_2^0 \subset \cdots \subset V_n^0$. We denote this flag F^0 . For each vector subspace W in V_n containing V^0 consider three algebraic subgroups of the general linear group \mathbb{GL}_W . Namely, set

$$P_W = \text{Stab}(V^0), \quad B_W = \text{Stab}(F^0), \quad T_W = \text{Stab}(l_1^n, l_2^n, \dots, l_n^n).$$

The group T_W stabilizes each line l_i^n . Clearly, $T_W \subset B_W \subset P_W$ and $\text{Gr}(n, W) = \mathbb{GL}_W/P_W$, $\mathbf{F}(n, W) = \mathbb{GL}_W/B_W$. Set $M(n, W) = \mathbb{GL}_W/T_W$. One has a tower of obvious morphisms

$$M(n, W) \xrightarrow{r_W} \mathbf{F}(n, W) \xrightarrow{q_W} \text{Gr}(n, W).$$

Set $M(n) = \text{colim}_W M(n, W)$, where W runs over all finite dimensional subspace W of V_n containing V^0 . Now one has a tower of morphisms

$$M(n) \xrightarrow{r} F(n) \xrightarrow{q} \text{Gr}(n).$$

The morphisms r_W can be decomposed in a tower of affine bundles. Hence it induces an isomorphism on any cohomology theory. The same then holds for the morphism r and

$$E^{*,*}(M(n)) = E^{*,*}(k)[[t_1, t_2, \dots, t_n]].$$

Permuting vectors e_i^n 's yields an inclusion $\Sigma_n \subset GL(V^0)$ of the symmetric group Σ_n in $\mathbb{GL}(V^0)$. The action of Σ_n by the conjugation on \mathbb{GL}_W normalizes the subgroups T_W and P_W . Thus Σ_n acts as on $M(n)$ so on $\text{Gr}(n)$ and the morphism $q \circ r : M(n) \rightarrow \text{Gr}(n)$ respects this action. Note that the action of Σ_n on $\text{Gr}(n)$ is trivial and the action of Σ_n on $E^{*,*}(M(n))$ permutes the variable t_1, t_2, \dots, t_n . Thus the image of $(q \circ r)^*$ is contained in $E^{*,*}(k)[[\sigma_1, \sigma_2, \dots, \sigma_n]]$. Whence the same holds for the image of q^* . The Theorem is proven. \square

The projection from the product $\text{Gr}(m) \times \text{Gr}(n)$, to the j -th factor is called p_j . For every integer $i \geq 0$ set $c'_i = p_1^*(c_i(\mathcal{J}(m)))$ and $c''_i = p_2^*(c_i(\mathcal{J}(n)))$

Theorem 2.0.7. *Suppose E is an oriented commutative \mathbf{P}^1 -ring spectrum. There is an isomorphism*

$$E^{*,*}((\text{Gr}(m) \times \text{Gr}(n))) = E^{*,*}(k)[[c'_1, c'_2, \dots, c'_m, c''_1, c''_2, \dots, c''_n]]$$

is the formal power series on the c'_i 's and c''_i 's. The inclusion $i_{m,n} : G(m) \times \text{Gr}(n) \hookrightarrow G(m+1) \times G(n+1)$ satisfies $i_{m,n}^*(c'_r) = c'_r$ for $r < m+1$, $i_{m,n}^*(c'_{m+1}) = 0$, and $i_{m,n}^*(c''_r) = c''_r$ for $r < n+1$, $i_{m,n}^*(c''_{n+1}) = 0$.

Proof. Follows as in the proof of Theorem 2.0.6. \square

2.1 The symmetric ring spectrum representing algebraic cobordism

To give a construction of the symmetric \mathbf{P}^1 -ring spectrum MGL , recall the external product of Thom spaces described in [V1, p. 422]. For vector bundles $V \rightarrow X$ and $W \rightarrow Y$ in $\mathcal{S}m/S$ one obtains a canonical map of pointed motivic spaces $\text{Th}(V) \wedge \text{Th}(W) \rightarrow \text{Th}(V \times_S W)$ which is a motivic weak equivalence as defined in [PPR1, Defn. 3.1.6]. In fact, the canonical map becomes an isomorphism after Nisnevich (even Zariski) sheafification.

The algebraic cobordism spectrum appears naturally as a T -spectrum, not as a \mathbf{P}^1 -spectrum. Hence we describe it as a symmetric T -ring spectrum and obtain a symmetric \mathbf{P}^1 -ring spectrum (and in particular a \mathbf{P}^1 -ring spectrum) by switching the suspension coordinate (see [PPR1, A.6.9]). For $m, n \geq 0$ let $\mathcal{J}(n, mn) \rightarrow \text{Gr}(n, mn)$ denote the tautological vector bundle over the Grassmann scheme of n -dimensional linear subspaces of $\mathbf{A}_S^{mn} = \mathbf{A}_S^m \times_S \dots \times_S \mathbf{A}_S^m$. Permuting the copies of \mathbf{A}_S^m induces a Σ_n -action on $\mathcal{J}(n, mn)$ and $\text{Gr}(n, mn)$ such that the bundle projection is equivariant. The closed embedding $\mathbf{A}_S^m = \mathbf{A}_S^m \times \{0\} \hookrightarrow \mathbf{A}_S^{m+1}$ defines a closed Σ_n -equivariant embedding $\text{Gr}(n, mn) \hookrightarrow \text{Gr}(n, (m+1)n)$. In particular, $\text{Gr}(n, mn)$ is pointed by $g_n : S = \text{Gr}(n, n) \hookrightarrow \text{Gr}(n, mn)$. The fiber of $\text{Gr}(n, mn)$ over g_n is \mathbf{A}_S^n . Let $\text{Gr}(n)$ be the colimit of the sequence

$$\text{Gr}(n, n) \hookrightarrow \text{Gr}(n, 2n) \hookrightarrow \dots \hookrightarrow \text{Gr}(n, mn) \hookrightarrow \dots$$

in the category of pointed motivic spaces over S . The pullback diagram

$$\begin{array}{ccc} \mathcal{J}(n, mn) & \longrightarrow & \mathcal{J}(n, (m+1)n) \\ \downarrow & & \downarrow \\ \mathrm{Gr}(n, mn) & \longrightarrow & \mathrm{Gr}(n, (m+1)n) \end{array}$$

induces a Σ_n -equivariant inclusion of Thom spaces

$$\mathrm{Th}(\mathcal{J}(n, mn)) \hookrightarrow \mathrm{Th}(\mathcal{J}(n, (m+1)n)).$$

Let MGL_n denote the colimit of the resulting sequence

$$\mathrm{MGL}_n = \operatorname{colim}_{m \geq n} \mathrm{Th}(\mathcal{J}(n, mn)) \quad (4)$$

with the induced Σ_n -action. There is a closed embedding

$$\mathrm{Gr}(n, mn) \times \mathrm{Gr}(p, mp) \hookrightarrow \mathrm{Gr}(n+p, m(n+p)) \quad (5)$$

which sends the linear subspaces $V \hookrightarrow \mathbf{A}^{mn}$ and $W \hookrightarrow \mathbf{A}^{mp}$ to the product subspace $V \times W \hookrightarrow \mathbf{A}^{mn} \times \mathbf{A}^{mp} = \mathbf{A}^{m(n+p)}$. In particular (g_n, g_p) maps to g_{n+p} . The inclusion (5) is covered by a map of tautological vector bundles and thus gives a canonical map of Thom spaces

$$\mathrm{Th}(\mathcal{J}(n, mn)) \wedge \mathrm{Th}(\mathcal{J}(p, mp)) \rightarrow \mathrm{Th}(\mathcal{J}(n+p, m(n+p))) \quad (6)$$

which is compatible with the colimit (4). Furthermore, the map (6) is $\Sigma_n \times \Sigma_p$ -equivariant, where the product acts on the target via the standard inclusion $\Sigma_n \times \Sigma_p \subseteq \Sigma_{n+p}$. After taking colimits, the result is a $\Sigma_n \times \Sigma_p$ -equivariant map

$$\mu_{n,p}: \mathrm{MGL}_n \wedge \mathrm{MGL}_p \rightarrow \mathrm{MGL}_{n+p} \quad (7)$$

of pointed motivic spaces (see [V1, p. 422]). The inclusion of the fiber \mathbf{A}^p over g_p in $\mathcal{J}(p)$ induces an inclusion $\mathrm{Th}(\mathbf{A}^p) \subset \mathrm{Th}(\mathcal{J}(p)) = \mathrm{MGL}_p$. Precomposing it with the canonical Σ_p -equivariant map of pointed motivic spaces

$$\mathrm{Th}(\mathbf{A}^1) \wedge \mathrm{Th}(\mathbf{A}^1) \wedge \cdots \wedge \mathrm{Th}(\mathbf{A}^1) \rightarrow \mathrm{Th}(\mathbf{A}^p)$$

defines a family of maps $e_p: (\Sigma_T^\infty S_+)_p = T^{\wedge p} \rightarrow \mathrm{MGL}_p$. Inserting it in the inclusion (7) yields $\Sigma_n \times \Sigma_p$ -equivariant structure maps

$$\mathrm{MGL}_n \wedge \mathrm{Th}(\mathbf{A}^1) \wedge \mathrm{Th}(\mathbf{A}^1) \wedge \cdots \wedge \mathrm{Th}(\mathbf{A}^1) \rightarrow \mathrm{MGL}_{n+p} \quad (8)$$

of the symmetric T -spectrum MGL . The family of $\Sigma_n \times \Sigma_p$ -equivariant maps (7) form a commutative, associative and unital multiplication on the symmetric T -spectrum MGL (see [J, Sect. 4.3]). Regarded as a T -spectrum it coincides with Voevodsky's spectrum MGL described in [V1, 6.3].

Let \overline{T} be the Nisnevich sheaf associated to the presheaf $X \mapsto \mathbf{P}^1(X)/(\mathbf{P}^1 - \{0\})(X)$ on the Nisnevich site $\mathcal{S}m/S$. The canonical covering of \mathbf{P}^1 supplies an isomorphism

$$T = \mathrm{Th}(\mathbf{A}_S^1) \xrightarrow{\cong} \overline{T}$$

of pointed motivic spaces. This isomorphism induces an isomorphism $\mathbf{MSS}_T(S) \cong \mathbf{MSS}_{\overline{T}}(S)$ of the categories of symmetric T -spectra and symmetric \overline{T} -spectra. In particular, \mathbf{MGL} may be regarded as a symmetric \overline{T} -spectrum by just changing the structure maps up to an isomorphism. Note that the isomorphism of categories respects both the symmetric monoidal structure and the model structure. The canonical projection $p: \mathbf{P}^1 \rightarrow \overline{T}$ is a motivic weak equivalence, because \mathbf{A}^1 is contractible. It induces a Quillen equivalence

$$\mathbf{MSS}(S) = \mathbf{MSS}_{\mathbf{P}^1}(S) \begin{array}{c} \xrightarrow{p_!} \\ \xleftarrow{p^*} \end{array} \mathbf{MSS}_{\overline{T}}(S)$$

when equipped with model structures as described in [J] (see [PPR1, A.6.9]). The right adjoint p^* is very simple: it sends a symmetric \overline{T} -spectrum E to the symmetric \mathbf{P}^1 -spectrum having terms $(p^*(E))_n = E_n$ and structure maps

$$E_n \wedge \mathbf{P}^1 \xrightarrow{E_n \wedge p} E \wedge \overline{T} \xrightarrow{\text{structure map}} E_{n+1}.$$

In particular $\mathbf{MGL} := p^*\mathbf{MGL}$ is a symmetric \mathbf{P}^1 -spectrum by just changing the structure maps. Since p^* is a lax symmetric monoidal functor, \mathbf{MGL} is a commutative monoid in a canonical way. Finally, the identity is a left Quillen equivalence from the model category $\mathbf{MSS}^{\mathrm{cm}}(S)$ used in [PPR1] to Jardine's model structure by the proof of [PPR1, A.6.4]. Let $\gamma: \mathrm{Ho}(\mathbf{MSS}^{\mathrm{cm}}(S)) \rightarrow \mathrm{SH}(S)$ denote the equivalence obtained by regarding a symmetric \mathbf{P}^1 -spectrum just as a \mathbf{P}^1 -spectrum.

Definition 2.1.1. Let $(\mathbf{MGL}, \mu_{\mathbf{MGL}}, e_{\mathbf{MGL}})$ denote the commutative \mathbf{P}^1 -ring spectrum which is the image $\gamma(\mathbf{MGL})$ of the commutative symmetric \mathbf{P}^1 -ring spectrum \mathbf{MGL} in the motivic stable homotopy category $\mathrm{SH}(S)$.

2.2 A universality theorem for the algebraic cobordism spectrum

The complex cobordism spectrum, equipped with its natural orientation, is a universal oriented ring cohomology theory by Quillen's universality theorem [Q]. In this section we prove a motivic version of Quillen's universality theorem. The statement is contained already in [Ve]. Recall that the \mathbf{P}^1 -ring spectrum \mathbf{MGL} carries a canonical orientation $th^{\mathbf{MGL}}$ as defined in 1.1.3. It is the canonical map $th^{\mathbf{MGL}}: \Sigma_{\mathbf{P}^1}^\infty(\mathrm{Th}(\mathcal{O}(-1)))(-1) \rightarrow \mathbf{MGL}$ of \mathbf{P}^1 -spectra.

Theorem 2.2.1 (Universality Theorem). *Let E be a commutative \mathbf{P}^1 -ring spectrum and let $S = \mathrm{Spec}(k)$ for a field k . The assignment $\varphi \mapsto \varphi(th^{\mathrm{MGL}}) \in E^{2,1}(\mathrm{Th}(\mathcal{J}(1)))$ identifies the set of monoid homomorphisms*

$$\varphi: \mathrm{MGL} \rightarrow E \quad (9)$$

in the motivic stable homotopy category $\mathrm{SH}^{\mathrm{cm}}(S)$ with the set of orientations of E . The inverse bijection sends an orientation $th \in E^{2,1}(\mathrm{Th}(\mathcal{J}(1)))$ to the unique morphism

$$\varphi \in E^{0,0}(\mathrm{MGL}) = \mathrm{Hom}_{\mathrm{SH}(S)}(\mathrm{MGL}, E)$$

such that $u_i^(\varphi) = th(\mathcal{J}(i)) \in E^{2i,i}(\mathrm{Th}(\mathcal{J}(i)))$, where $th(\mathcal{J}(i))$ is given by (2) and $u_i: \Sigma_{\mathbf{P}^1}^\infty(\mathrm{Th}(\mathcal{J}(i)))(-i) \rightarrow \mathrm{MGL}$ is the canonical map of \mathbf{P}^1 -spectra.*

Proof. Let $\varphi: \mathrm{MGL} \rightarrow E$ be a homomorphism of monoids in $\mathrm{SH}(S)$. The class $th := \varphi(th^{\mathrm{MGL}})$ is an orientation of E , because

$$\varphi(th)|_{\mathrm{Th}(1)} = \varphi(th|_{\mathrm{Th}(1)}) = \varphi(\Sigma_{\mathbf{P}^1}(1)) = \Sigma_{\mathbf{P}^1}(\varphi(1)) = \Sigma_{\mathbf{P}^1}(1).$$

Now suppose $th^E \in E^{2i,i}(\mathrm{Th}(\mathcal{O}(-1)))$ is an orientation of E . We will construct a monoid homomorphism $\varphi: \mathrm{MGL} \rightarrow E$ in $\mathrm{SH}(S)$ such that $u_i^*(\varphi) = th(\mathcal{J}(i))$ and prove its uniqueness. To do so, we compute $E^{*,*}(\mathrm{MGL})$. By [PPR1, Cor. 2.1.4], this group fits into the short exact sequence

$$0 \rightarrow \varprojlim^1 E^{*+2i-1, *+i}(\mathrm{Th}(\mathcal{J}(i))) \rightarrow E^{*,*}(\mathrm{MGL}) \rightarrow \varprojlim E^{*+2i, *+i}(\mathrm{Th}(\mathcal{J}(i))) \rightarrow 0$$

where the connecting maps in the tower are given by the top line of the commutative diagram

$$\begin{array}{ccccc} E^{*+2i-1, *+i}(\mathrm{Th}_i) & \xleftarrow{\Sigma_{\mathbf{P}^1}^{-1}} & E^{*+2i+1, *+i+1}(\mathrm{Th}_i \wedge \mathbf{P}^1) & \xleftarrow{\quad} & E^{*+2i+1, *+i+1}(\mathrm{Th}_{i+1}) \\ \uparrow -\cup th(\mathcal{J}(i)) & & \uparrow \epsilon^* \circ (-\cup th(\mathcal{J}(i) \oplus \mathbf{1})) & & \uparrow -\cup th(\mathcal{J}(i+1)) \\ E^{*,*}(\mathrm{Gr}(i)) & \xleftarrow{\mathrm{id}} & E^{*,*}(\mathrm{Gr}(i)) & \xleftarrow{\mathrm{inc}_i^*} & E^{*,*}(\mathrm{Gr}(i+1)) \end{array}$$

Here $\epsilon: \mathrm{Th}(V) \wedge \mathbf{P}^1 \rightarrow \mathrm{Th}(V \oplus \mathbf{1})$ is the canonical map. The pull-backs inc_i^* are all surjective by Theorem 2.0.4. So we proved the following

Claim 2.2.2. *The canonical map*

$$E^{*,*}(\mathrm{MGL}) \rightarrow \varprojlim E^{*+2i, *+i}(\mathrm{Th}(\mathcal{J}(i))) = E^{*,*}(k)[[c_1, c_2, c_3, \dots]]$$

is an isomorphism of two-sided $E^{,*}(k)$ -modules.*

The family of elements $th(\mathcal{J}(i))$ is an element in the \varprojlim -group, thus there is a unique element $\varphi \in E^{0,0}(\text{MGL})$ with $u_i^*(\varphi) = th(\mathcal{J}(i))$. We claim that φ is a monoid homomorphism. To check that it respects the multiplicative structure, consider the diagram

$$\begin{array}{ccc}
\Sigma_{\mathbf{P}^1}^\infty(\text{Th}(\mathcal{J}(i)))(-i) \wedge \Sigma_{\mathbf{P}^1}^\infty(\text{Th}(\mathcal{J}(j)))(-j) & \xrightarrow{\Sigma_{\mathbf{P}^1}^\infty(\mu_{i,j})(-i-j)} & \Sigma_{\mathbf{P}^1}^\infty(\text{Th}(\mathcal{J}(i+j)))(-i-j) \\
\downarrow u_i \wedge u_j & & \downarrow u_{i+j} \\
\text{MGL} \wedge \text{MGL} & \xrightarrow{\mu_{\text{MGL}}} & \text{MGL} \\
\downarrow \varphi \wedge \varphi & & \downarrow \varphi \\
E \wedge E & \xrightarrow{\mu_E} & E.
\end{array}$$

Its enveloping square commutes in $\text{SH}(S)$ by the chain of relations

$$\begin{aligned}
\varphi \circ u_{i+j} \circ \Sigma_{\mathbf{P}^1}^\infty(\mu_{i,j})(-i-j) &= \mu_{i,j}^*(th(\mathcal{J}(i+j))) = th(\mu_{i,j}^*(\mathcal{J}(i+j))) = th(\mathcal{J}(i) \times \mathcal{J}(j)) \\
&= th(\mathcal{J}(i)) \times th(\mathcal{J}(j)) = \mu_E(th(\mathcal{J}(i)) \wedge th(\mathcal{J}(j))) \\
&= \mu_E \circ ((\varphi \circ u_i) \wedge (\varphi \circ u_j)).
\end{aligned}$$

To obtain the equality $\mu_E \circ (\varphi \wedge \varphi) = \varphi \circ \mu_{\text{MGL}}$ in $\text{SH}(k)$ consider the short exact sequence

$$\begin{aligned}
0 \rightarrow \varprojlim^1 E^{*+4i-1, *+2i}(\text{Th}(\mathcal{J}(i)) \wedge \text{Th}(\mathcal{J}(i))) &\rightarrow E^{*,*}(\text{MGL} \wedge \text{MGL}) \\
&\rightarrow \varprojlim E^{*+4i, *+2i}(\text{Th}(\mathcal{J}(i)) \wedge \text{Th}(\mathcal{J}(i))) \rightarrow 0.
\end{aligned}$$

Note that since $\text{Th}(\mathcal{J}(i)) \wedge \text{Th}(\mathcal{J}(i)) \cong \text{Th}(\mathcal{J}(i) \times \mathcal{J}(i))$, there is a Thom isomorphism $E^{*+4i-1, *+2i}(\text{Th}(\mathcal{J}(i) \times \mathcal{J}(i))) \cong E^{*-1, *}(\text{Gr}(i) \times \text{Gr}(i))$ by Theorem 2.0.4. The \varprojlim^1 -group is trivial because the connecting maps coincide with the pull-back maps

$$E^{*-1, *}(\text{Gr}(i+1) \times \text{Gr}(i+1)) \rightarrow E^{*-1, *}(\text{Gr}(i) \times \text{Gr}(i))$$

and these are surjective by Theorem 2.0.7. This implies the following

Claim 2.2.3. *The canonical map*

$$\begin{aligned}
E^{*,*}(\text{MGL} \wedge \text{MGL}) &\rightarrow \varprojlim E^{*+2i, *+i}(\text{Th}(\mathcal{J}(i)) \wedge \text{Th}(\mathcal{J}(i))) = \\
&E^{*,*}(k)[[c'_1, c''_1, c'_2, c''_2, \dots]]
\end{aligned}$$

is an isomorphism of two-sided $E^{*,*}(k)$ -modules. Here c'_i is the i -th Chern class coming from the first factor of $\text{Gr} \times \text{Gr}$ and c''_i is the i -th Chern class coming from the second factor.

Now the equality

$$\varphi \circ u_{i+i} \circ \Sigma_{\mathbf{P}^1}^\infty(\mu_{i,i})(-2i) = \mu_E \circ ((\varphi \circ u_i) \wedge (\varphi \circ u_i))$$

shows that $\mu_E \circ (\varphi \wedge \varphi) = \varphi \circ \mu_{\text{MGL}}$ in $\text{SH}(k)$.

To prove the Theorem it remains to check that the two assignments described in the Theorem are inverse to each other. An orientation $th \in E^{2,1}(\text{Th}(\mathcal{O}(-1)))$ induces a morphism φ such that for each i one has $\varphi \circ u_i = th(\mathcal{J}_i)$. And the new orientation $th' := \varphi(th^{\text{MGL}})$ coincides with the original one, due to the chain of relations

$$th' = \varphi(th^{\text{MGL}}) = \varphi(u_1) = \varphi \circ u_1 = th(\mathcal{J}_1) = th(\mathcal{O}(-1)) = th.$$

On the other hand a monoid homomorphism φ defines an orientation $th := \varphi(th^{\text{MGL}})$ of E . The monoid homomorphism φ' we obtain then satisfies $u_i^*(\varphi') = th(\mathcal{J}_i)$ for every $i \geq 0$. To check that $\varphi' = \varphi$, recall that MGL is oriented, so we may use Claim 2.2.2 with $E = \text{MGL}$ to deduce an isomorphism

$$\text{MGL}^{*,*}(\text{MGL}) \rightarrow \varprojlim \text{MGL}^{*+2i, *+i}(\text{Th}(\mathcal{J}(i))).$$

This isomorphism shows that the identity $\varphi' = \varphi$ will follow from the identities $u_i^*(\varphi') = u_i^*(\varphi)$ for every $i \geq 0$. Since $u_i^*(\varphi') = th(\mathcal{J}_i)$ it remains to check the relation $u_i^*(\varphi) = th(\mathcal{J}_i)$. It follows from the

Claim 2.2.4. *There is an equality $u_i = th^{\text{MGL}}(\mathcal{J}_i) \in \text{MGL}^{2i,i}(\text{Th}(\mathcal{J}(i)))$.*

In fact, $u_i^*(\varphi) = \varphi \circ u_i = \varphi(u_i) = \varphi(th^{\text{MGL}}(\mathcal{J}(i))) = th(\mathcal{J}(i))$. The last equality in this chain of relations holds, because φ is a monoid homomorphism sending th^{MGL} to th . It remains to prove the Claim. We will do this in the case $i = 2$. The general case can be proved similarly. The commutative diagram

$$\begin{array}{ccc} \Sigma_{\mathbf{P}^1}^\infty \text{Th}(\mathcal{J}(1))(-1) \wedge \Sigma_{\mathbf{P}^1}^\infty \text{Th}(\mathcal{J}(1))(-1) & \xrightarrow{\Sigma_{\mathbf{P}^1}^\infty (\mu_{1,1})(-2)} & \Sigma_{\mathbf{P}^1}^\infty \text{Th}(\mathcal{J}(2))(-2) \\ \downarrow u_1 \wedge u_1 & & \downarrow u_2 \\ \text{MGL} \wedge \text{MGL} & \xrightarrow{\mu_{\text{MGL}}} & \text{MGL} \end{array}$$

in $\text{SH}(k)$ implies that

$$\mu_{1,1}^*(u_2) = u_1 \times u_1 \in \text{MGL}^{4,2}(\text{Th}(\mathcal{J}(1)) \wedge \text{Th}(\mathcal{J}(1))) = \text{MGL}^{4,2}(\text{Th}(\mathcal{J}(1) \times \mathcal{J}(1))).$$

The equalities

$$\begin{aligned} \mu_{1,1}^*(th^{\text{MGL}}(\mathcal{J}(2))) &= th^{\text{MGL}}(\mu_{1,1}^*(\mathcal{J}(2))) = th^{\text{MGL}}(\mathcal{J}(1) \times \mathcal{J}(1)) \\ &= th^{\text{MGL}}(\mathcal{J}(1)) \times th^{\text{MGL}}(\mathcal{J}(1)) \end{aligned}$$

imply that it remains to prove the injectivity of the map $\mu_{1,1}^*$. Consider the commutative diagram

$$\begin{array}{ccc} \text{MGL}^{*,*}(\text{Th}(\mathcal{J}(1) \times \mathcal{J}(1))) & \xleftarrow{\mu_{1,1}^*} & \text{MGL}^{*,*}(\text{Th}(\mathcal{J}(2))) \\ \uparrow \cong_{\text{Thom}} & & \uparrow \cong_{\text{Thom}} \\ \text{MGL}^{*,*}(\text{Gr}(1) \times \text{Gr}(1)) & \xleftarrow{\nu_{1,1}^*} & \text{MGL}^{*,*}(\text{Gr}(2)) \end{array}$$

where the vertical arrows are the Thom isomorphisms from Theorem 2.0.4 and $\nu_{1,1} : \mathrm{Gr}(1) \times \mathrm{Gr}(1) \hookrightarrow \mathrm{Gr}(2)$ is the embedding described by equation (5). For an oriented commutative ring \mathbf{P}^1 -spectrum (E, th) one has $E^{*,*}(\mathrm{Gr}(2)) = E^{*,*}(k)[[c_1, c_2]]$ (the formal power series on c_1, c_2) by Theorem 2.0.6. From the other hand

$$E^{*,*}(\mathrm{Gr}(1) \times \mathrm{Gr}(1)) = E^{*,*}(k)[[t_1, t_2]]$$

(the formal power series on t_1, t_2) by Theorem 2.0.7 and the map $\nu_{1,1}^*$ takes c_1 to $t_1 + t_2$ and c_2 to $t_1 t_2$. Whence $\nu_{1,1}^*$ is injective. The proofs of the Claim and of the Theorem are completed. \square

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