Cofibrations in Homotopy Theory

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2000 *Mathematics Subject Classification.* Primary 18G55, 55U35, 18G10, 18G30, 55U10

Key words and phrases. cofibration category, model category, homotopy colimit

Abstract. We define Anderson-Brown-Cisisnski (ABC) cofibration categories, and construct homotopy colimits of diagrams of objects in ABC cofibraction categories. Homotopy colimits for Quillen model categories are obtained as a particular case. We attach to each ABC cofibration category a right derivator. A dual theory is developed for homotopy limits in ABC fibration categories and for left derivators. These constructions provide a natural framework for 'doing homotopy theory' in ABC (co)fibration categories.

Contents

Preface

Model categories, introduced by Daniel Quillen [**Qui67**], are a natural framework for doing homotopy theory in an axiomatic way. A Quillen model category $(\mathcal{M}, \mathcal{W}, \mathcal{C}of, \mathcal{F}ib)$ consists of a category $\mathcal{M}$, and three distinguished classes of maps: the weak equivalences $\mathcal{W}$, the cofibrations $\mathcal{C}of$ and the fibrations $\mathcal{F}ib$, subject to a list of axioms (Def. 2.2.2).

Let us fix some notation. If $\mathcal{D}$ is a small category we denote $\mathcal{M}^{\mathcal{D}}$ the category of $\mathcal{D}$-diagrams in $\mathcal{M}$. For a small functor $u : \mathcal{D}_1 \to \mathcal{D}_2$ we denote $u^* : \mathcal{M}^{\mathcal{D}_2} \to \mathcal{M}^{\mathcal{D}_1}$ the functor defined by $(u^*X)_{d_1} = X_{ud_1}$ for objects $d_1 \epsilon \mathcal{D}_1$. For a category with a class of weak equivalences $(\mathcal{M}, \mathcal{W})$ we denote $\mathbf{ho}\mathcal{M} = \mathcal{M}[\mathcal{W}^{-1}]$ its homotopy category, and we will always consider the weak equivalences on $\mathcal{M}^{\mathcal{D}}$ to be *pointwise*, i.e. f is a weak equivalence in $\mathcal{M}^{\mathcal{D}}$ if f_d is a weak equivalence in $\mathcal{M}$ for all objects $d \epsilon \mathcal{D}$.

A Quillen model category $\mathcal{M}$ admits homotopy pushouts and homotopy pullbacks. These are the total left (resp. right) derived functors of the pushout (resp. pullback) functor in $\mathcal{M}$. If $\mathcal{M}$ is pointed one can construct the homotopy cofiber and fiber of a map in $\mathbf{ho}\mathcal{M}$, and the suspension and loop space of an object in $\mathbf{ho}\mathcal{M}$. One can then form the cofibration sequence and the fibration sequence of a map in $\mathbf{ho}\mathcal{M}$.

Furthermore, a Quillen model category $\mathcal{M}$ admits all small homotopy colimits and limits (also called the homotopy left and right Kan extensions). For a small functor $u : \mathcal{D}_1 \to \mathcal{D}_2$, the homotopy colimit $\mathbf{L}\operatorname{colim}^u : \mathbf{ho}(\mathcal{M}^{\mathcal{D}_1}) \to \mathbf{ho}(\mathcal{M}^{\mathcal{D}_2})$ is the total left derived functor of the colimit functor colim^u, and is left adjoint to $\mathbf{ho}u^* : \mathbf{ho}(\mathcal{M}^{\mathcal{D}_2}) \to \mathbf{ho}(\mathcal{M}^{\mathcal{D}_1})$. Dually, the homotopy limit $\mathbf{R}\lim^u$ is the total right derived functor of $\lim^u$, and is right adjoint to $\mathbf{ho}u^*$.

For the general construction of homotopy colimits and limits, we recommend the work of Dwyer, Kan, Hirschhorn and Smith [**DKHS04**], Hirschhorn [**Hir00**], Chachólski and Scherer [**CS02**] and Cisinski [**Cis03**]. Each of these references presents a different perspective on homotopy (co)limits.

Once the construction of homotopy colimits and limits is understood, a very interesting question we can ask is: What is really the role of cofibrations and fibrations in a Quillen model category? The homotopy (co)limits are the total derived functors of the (co)limit, so the definition of homotopy (co)limits depends just on the weak equivalences. The construction of homotopy colimits on the other hand requires the presence of cofibrations and weak equivalences - but does not really involve fibrations. Dually, the construction of homotopy limits requires the presence of fibrations and weak equivalences, but not cofibrations.

It is therefore natural to ask if one can simplify Quillen's set of axioms, and separate a minimal set of axioms required by cofibrations and weak equivalences in order to still be able to construct homotopy colimits.

In a very influential paper, Ken Brown [**Bro74**] formalized some of these obsevations by defining categories of fibrant objects and working out in detail their properties. Reversing arrows, one defines categories of cofibrant objects, and Brown's work carries over by duality to categories of cofibrant objects.

A category of cofibrant objects $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ consists of a category $\mathcal{M}$, the class of weak equivalences $\mathcal{W}$ and the class of cofibrations $\mathcal{C}of$, subject to a list of axioms (the duals of the axioms of [**Bro74**]). The axioms require in particular all objects to be cofibrant.

For a pointed category of cofibrant objects, Brown was able to construct homotopy cofibers of maps, suspensions of objects and the cofibration sequence of a map. These constructions exist in dual form for categories of fibrant objects.

Building on Brown's work, Don Anderson [**And78**] extended Brown's axioms for a category of cofibrant objects by dropping the requirement that all objects be cofibrant. Anderson called the categories defined by his new axioms *left homotopical*; our text changes terminology and calls them *Anderson-Brown-Cisinski cofibration categories* (or just cofibration categories for simplicity). The cofibration category axioms we use are slightly more general than Anderson's.

Anderson's main observation was that the cofibration category axioms on $\mathcal{M}$ suffice for the construction of a left adjoint of $\mathbf{ho}u^*$, for any small functor u. It is implicit in his work that the left adjoint of $\mathbf{ho}u^*$ is a left derived of colim^u.

Unfortunately for the history of this subject, Anderson's paper [**And78**] contains statements but omits proofs, and has a title ("Fibrations and Geometric Realizations") that does not reflect the generality of his work. Also, Anderson quit mathematics shortly after his paper was published, the proofs of [**And78**] got lost and as a result his whole theory lay dormant for twenty five years.

We can be grateful to Denis-Charles Cisinski [**Cis02**], [**Cis03**] for bringing back to light Brown and Anderson's ideas. Cisinski simplifies Anderson's arguments, and provides for cofibration categories a complete construction of homotopy colimits along functors $u : \mathcal{D}_1 \to \mathcal{D}_2$ with $\mathcal{D}_1$ finite and direct.

Cisinski has also worked out the construction of homotopy colimits along arbitrary small functors u, as well as the end result regarding the derivability of cofibration categories (our Chap. 7). While this part of his work remains unpublished, he was kind enough to share with me its outline. I would like to thank him for suggesting the correct formulation of axioms CF5-CF6, and for patiently explaining to me the finer points of excision.

The goal of these notes is then to work out a self-contained account of homotopy colimits from the axioms of an Anderson-Brown-Cisinski cofibration category, and show that they satisfy the axioms of a right derivator. There are a number of properties of homotopy colimits that are a formal consequence of the right derivator axioms, but they are outside of the scope of our text. We will instead try to investigate the relation with the better-known Quillen model categories, and compare with other axiomatizations that have been proposed for cofibration categories.

While some of the proofs we propose may be new, the credit for this theory should go entirely to Brown, Anderson and Cisinski. It was our choice in this text to make use of approximation functors and abstract Quillen equivalences, and for that we were influenced by the work of Dwyer, Kan, Hirschhorn and Smith [**DKHS04**].

Our treatment of direct and inverse categories bears the influence of Daniel Kan's theory of Reedy categories outlined in [**DKH97**], [**Hov99**] and [**Hir00**].

I would like to thank Haynes Miller and Daniel Kan, my mentors in abstract homotopy, for their gracious support and encouragement. I am grateful to Denis-Charles Cisinski, Philip Hirschhorn and Haynes Miller for the conversations we had on the subject of this text.

Outline. We start our exposition with the axioms of an Anderson-Brown-Cisinski (ABC) cofibration category, and we proceed in Chap. 1 with their elementary properties. We will make a distinction between ABC precofibration categories (satisfying axioms CF1-CF4) and ABC cofibration categories (satisfying axioms CF1-CF6).

This allows us going to Chap. 2 to compare ABC cofibration categories with other axiomatic systems that have been proposed for categories with cofibrations. Most notably, we will show that any Quillen model category is an ABC cofibration (and fibration) category.

In Chap. 3, we recall the theory of cylinders and homotopic maps in a precofibration category. This allows us to abstract out the properties of *cofibrant approximation* functors between precofibration categories, and later in Chap. 4 we abstract the cofibrant approximation functors to yet another level - these are the *left approximation* functors.

In Chap. 4, we recall the definition of Kan extensions and total derived functors. The purpose of *left approximation* functors will be to construct total derived functors, and to prove a Quillen-type adjunction property between the total derived functors of an adjoint pair of functors.

In Chap. 5, we recall over and under categories, and elementary properties of limits and colimits.

In Chap. 6, we reach our main objective. We define the homotopy colimit $\mathbf{L}\operatorname{colim}^{u} : \mathbf{ho}(\mathcal{M}^{\mathcal{D}_1}) \to \mathbf{ho}(\mathcal{M}^{\mathcal{D}_2})$ as a left derived functor, and show that it exists and it is left adjoint to $\mathbf{ho}u^* : \mathbf{ho}(\mathcal{M}^{\mathcal{D}_2}) \to \mathbf{ho}(\mathcal{M}^{\mathcal{D}_1})$. We show that if $\mathcal{D}$ is a small category and $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ is a cofibration category, then the diagram category $\mathcal{M}^{\mathcal{D}}$ is again a cofibration category, with pointwise weak equivalences and pointwise cofibrations.

Finally, in Chap. 7 we recall the notion of a *right derivator*, and show that the homotopy colimits in a cofibration category satisfy the axioms of a right derivator. The purpose of the last chapter is simply to assert the results of Chap. 6 within the axiomatic language of derivators.

CHAPTER 1

Cofibration categories

This chapter defines Anderson-Brown-Cisinski (or ABC) cofibration, fibration and model categories. For simplicity, we refer to Anderson-Brown-Cisinski cofibration (fibration, model) categories as just cofibration (fibration, model) categories, when no confusion with Quillen model categories or Baues cofibration categories is possible.

What is an ABC cofibration category? It is a category $\mathcal{M}$ with two distinguished classes of maps, the weak equivalences and the cofibrations, satisfying a set of six axioms which are denoted CF1-CF6. An ABC fibration category is a category $\mathcal{M}$ with weak equivalences and fibrations, satisfying the dual axioms F1-F6. Am ABC model category is a category $\mathcal{M}$ with weak equivalences, cofibrations and fibrations that is at the same time a cofibration and a fibration category.

Any Quillen model category is an ABC model category (Prop. 2.2.3). Diagrams indexed by a small category in an ABC cofibration category form again a cofibration category (Thm. 6.5.5), a property not enjoyed in general by Quillen model categories.

The ultimate goal using the cofibration category axioms CF1-CF6 is to construct (in Chap. 6) homotopy colimits in $\mathcal{M}$ indexed by small categories $\mathcal{D}$, and more generally to construct 'relative' homotopy colimits along small functors $u : \mathcal{D}_1 \to \mathcal{D}_2$.

The category $\mathcal{M}$ will not be assumed in general to be cocomplete. Under the simplifying assumption that $\mathcal{M}$ is cocomplete, however, the homotopy colimit along a functor $u : \mathcal{D}_1 \to \mathcal{D}_2$ is the left Kan extension along $\operatorname{colim}^{u} : \mathcal{M}^{\mathcal{D}_1} \to \mathcal{M}^{\mathcal{D}_2}$. It is a known fact that a colimit indexed by a small category $\mathcal{D}$ can be constructed in terms of pushouts of small sums of objects in $\mathcal{M}$. Furthermore, a 'relative' colimit colim^{u} can be described in terms of absolute colimits indexed by the over categories $(u \downarrow d_2)$ for $d_2 \epsilon \mathcal{D}_2$ (see Lemma 5.2.1).

It is perhaps not surprising then that the cofibration category axioms specify an approximation property of maps by cofibrations (axiom CF4), as well as the behaviour of cofibrations under pushouts (axiom CF3) and under small sums (axiom CF5).

A large part of the theory can be developed actually from a subset of the axioms, namely the axioms CF1-CF4. The theory of homotopic maps, of homotopy calculus of fractions and of cofibrant approxiomation functors of Chap. 3 only requires this smaller set of axioms. A category with weak equivalences and cofibrations satisfying the axioms CF1-CF4 will be called a *precofibration* category. From the precofibration category axioms, it turns out that one can construct all the homotopy colimits indexed by *finite, direct* categories [**Cis02**].

One of the ideas worth repeating is that while we need both the cofibrations and the weak equivalences to *construct* homotopy colimits, the homotopy colimits are *characterized* in the end just by the weak equivalences. So when working with a cofibration category with a fixed class of weak equivalences, it would be desirable to try to work with a class of cofibrations as large as possible. This is where the concept of *left proper* maps becomes useful (see Section 1.7). In a *left proper* precofibration category ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{C}of$), any cofibration $A \to B$ with A cofibrant is a left proper map, and the class of left proper maps denoted $\mathcal{P}rCof$ yields again a precofibration category structure ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{P}rCof$). But if $\mathcal{M}$ is a left proper CF1-CF6 cofibration category, then ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{P}rCof$) may not satisfy the cofibration category axioms CF5 and CF6. Dual results hold for right proper prefibration categories.

1.1. The axioms

Definition 1.1.1 (Anderson-Brown-Cisinski cofibration categories).
An ABC cofibration category ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{C}of$) consists of a category $\mathcal{M}$, and two distinguished classes of maps of $\mathcal{M}$ - the weak equivalences (or trivial maps) $\mathcal{W}$ and the cofibrations $\mathcal{C}of$, subject to the axioms CF1-CF6. The initial object $\mathbf{0}$ of $\mathcal{M}$ exists by axiom CF1, and an object A is called *cofibrant* if the map $\mathbf{0} \to A$ is a cofibration.

The axioms are:

CF1: All isomorphisms of $\mathcal{M}$ are weak equivalences, and all isomorphisms with the domain a cofibrant object are trivial cofibrations. $\mathcal{M}$ has an initial object $\mathbf{0}$, which is cofibrant. Cofibrations are stable under composition.

CF2: (Two out of three axiom) Suppose f and g are maps such that gf is defined. If two of f, g, gf are a weak equivalence, then so is the third.

CF3: (Pushout axiom) Given a solid diagram in $\mathcal{M}$, with i a cofibration and A, C cofibrant

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail, "i"'] & C \arrow[d, tail, dashed, "j"] \\
B \arrow[r, dashed] & D
\end{tikzcd}$$

then

1. the pushout exists in $\mathcal{M}$ and j is a cofibration, and
2. if additionally i is a trivial cofibration, then so is j.

CF4: (Factorization axiom) Any map $f : A \to B$ in $\mathcal{M}$ with A cofibrant factors as $f = rf'$, with f' a cofibration and r a weak equivalence

CF5: If $f_i : A_i \to B_i$ for $i \epsilon I$ is a set of cofibrations with A_i cofibrant, then

1. $\sqcup A_i$, $\sqcup B_i$ exist and are cofibrant, and $\sqcup f_i$ is a cofibration.
2. if additionally all f_i are trivial cofibrations, then so is $\sqcup f_i$.

CF6: For any countable direct sequence of cofibrations with A_0 cofibrant

$$\begin{tikzcd}
A_0 \arrow[r, tail, "a_0"] & A_1 \arrow[r, tail, "a_1"] & A_2 \arrow[r, tail, "a_2"] & \cdots
\end{tikzcd}$$

1. the colimit object colim A_n exists and the transfinite composition $A_0 \to \text{colim } A_n$ is a cofibration.
2. if additionally all a_i are trivial cofibrations, then so is $A_0 \to \text{colim } A_n$.

If ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{C}of$) only satisfies the axioms CF1-CF4, it is called a precofibration category.

Pushouts are defined by an universal property, and are only defined up to an unique isomorphism. Since all isomorphisms with cofibrant domain are trivial cofibrations, it does not matter which isomorphic representative of the pushout we choose in CF3.

In the axiom CF4, the map f' is sometimes called a *cofibrant replacement* of f. If $r : A' \to A$ is a weak equivalence with A' cofibrant, the object A' is called a *cofibrant replacement* of A.

We would like to stress that the class of weak equivalences is not necessarily assumed to be saturated, although a cofibration category is still a cofibration category with respect to the saturation of weak equivalences (Thm. 3.7.2).

If $A \to B$ is a cofibration with A cofibrant then B is cofibrant. But there may exist cofibrations $A \to B$ with A not cofibrant. If we denote $\mathcal{C}of'$ the class of cofibrations $A \to B$ with A cofibrant, then $(\mathcal{M}, \mathcal{W}, \mathcal{C}of')$ is again a cofibration category.

We will sometimes refer to a cofibration category as just $\mathcal{M}$. We will also denote $\mathcal{M}_{cof}$ the full subcategory of cofibrant objects of $\mathcal{M}$.

The category $\mathcal{M}_{cof}$ is a cofibration category, and in fact so is any full subcategory $\mathcal{M}'$ of $\mathcal{M}$ that includes $\mathcal{M}_{cof}$, with the induced structure $(\mathcal{M}', \mathcal{W} \cap \mathcal{M}', \mathcal{C}of \cap \mathcal{M}')$.

If $\mathcal{M}$ is a precofibration category, then $(\mathcal{M}, \mathcal{W}, \mathcal{C}of')$, $\mathcal{M}_{cof}$ and any $\mathcal{M}'$ as above are precofibration categories. We will sometimes refer to precofibration categories as CF1-CF4 cofibration categories.

Definition 1.1.2 (Anderson-Brown-Cisinski fibration categories).
An ABC fibration category $(\mathcal{M}, \mathcal{W}, \mathcal{F}ib)$ consists of a category $\mathcal{M}$, and two distinguished classes of maps - the weak equivalences $\mathcal{W}$ and the fibrations $\mathcal{F}ib$, subject to the axioms F1-F6. The terminal object $\mathbf{1}$ of $\mathcal{M}$ exists by axiom F1, and an object A of $\mathcal{F}ib$ is called *fibrant* if the map $A \to \mathbf{1}$ is a fibration.

The axioms are:

F1: All isomorphisms of $\mathcal{M}$ are weak equivalences, and all isomorphisms with fibrant codomain are trivial fibrations. $\mathcal{M}$ has a final object $\mathbf{1}$, which is fibrant. Fibrations are stable under composition.

F2: (Two out of three axiom) Suppose f and g are maps such that gf is defined. If two of f, g, gf are a weak equivalence, then so is the third.

F3: (Pullback axiom) Given a solid diagram in $\mathcal{M}$, with p a fibration and A, C fibrant,

$$\begin{tikzcd}
D \arrow[r, dashed] \arrow[d, dashed, two heads, "q"'] & B \arrow[d, two heads, "p"] \\
C \arrow[r] & A
\end{tikzcd}$$

then
(1) the pullback exists in $\mathcal{M}$ and q is a fibration, and
(2) if additionally p is a trivial prefibration, then so is q.

F4: (Factorization axiom) Any map $f : A \to B$ in $\mathcal{M}$ with B fibrant factors as $f = f's$, with s a weak equivalence and f' a fibration.

F5: If $f_i : A_i \to B_i$ for $i \epsilon I$ is a set of fibrations with B_i fibrant, then
(1) $\times A_i$, $\times B_i$ exist and are fibrant, and $\times f_i$ is a fibration
(2) if additionally all f_i are trivial fibrations, then so is $\times f_i$.

F6: For any countable inverse sequence of fibrations with A_0 fibrant

$$\cdots \xrightarrow{a_2} A_2 \xrightarrow{a_1} A_1 \xrightarrow{a_0} A_0$$

(1) the limit object $\lim A_i$ exists and the transfinite composition $\lim A_n \to A_0$ is a fibration

(2) if additionally all a_i are trivial fibrations, then so is $\lim A_n \to A_0$.

If ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{F}ib$) only satisfies the axioms F1-F4, it is called a prefibration category.

The axioms are dual in the sense that ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{C}of$) is a cofibration category if and only if $(\mathcal{M}^{op}, \mathcal{W}^{op}, \mathcal{C}of^{op})$ is a fibration category.

In the axiom F4, the map $f^{'}$ is called a *fibrant replacement* of f. If $r : A \to A^{'}$ is a weak equivalence with $A^{'}$ fibrant, the object $A^{'}$ is called a *fibrant replacement* of A.

If we denote $\mathcal{F}ib^{'}$ the class of fibrations $A \to B$ with B fibrant then ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{F}ib^{'}$) again is a fibration category.

We will denote $\mathcal{M}_{fib}$ to be the full subcategory of fibrant objects of a fibration category $\mathcal{M}$. The category $\mathcal{M}_{fib}$ as well as any full subcategory $\mathcal{M}^{'}$ of $\mathcal{M}$ that includes $\mathcal{M}_{fib}$ satisfy again the axioms of a fibration category.

If $\mathcal{M}$ is a prefibration category, then so are ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{F}ib^{'}$), $\mathcal{M}_{fib}$ and any $\mathcal{M}^{'}$ as above.

Definition 1.1.3 (Anderson-Brown-Cisinski model categories).
An ABC model category ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{C}of, \mathcal{F}ib$) consists of a category $\mathcal{M}$ and three distinguished classes of maps $\mathcal{W}, \mathcal{C}of, \mathcal{F}ib$ with the property that ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{C}of$) is an ABC cofibration category and that ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{F}ib$) is an ABC fibration category.

In all sections of this chapter except Section 1.6, we will do our work assuming only the precofibration category axioms CF1-CF4 (and dually the prefibration category axioms F1-F4). In Section 1.6, we will assume that the full set of axioms is verified.

1.2. Sums of objects in a cofibration category

In general, the objects of a precofibration category are not closed under finite sums. But finite sums of cofibrant objects exist and are cofibrant. Dually, in a prefibration category finite products of fibrant objects exist and are fibrant. In fact we can prove the slightly more general statement:

Lemma 1.2.1.

(1) *Suppose that* $\mathcal{M}$ *is a precofibration category. If* $f_i : A_i \to B_i$ *for* $i = 0, ..., n$ *are cofibrations with* A_i *cofibrant, then* $\sqcup A_i$, $\sqcup B_i$ *exist and are cofibrant, and* $\sqcup f_i$ *is a cofibration which is trivial if all* f_i *are trivial.*

(2) *Suppose that* $\mathcal{M}$ *is a prefibration category. If* $f_i : A_i \to B_i$ *for* $i = 0, ..., n$ *are fibrations with* B_i *fibrant, then* $\times A_i$, $\times B_i$ *exist and are fibrant, and* $\times f_i$ *is a fibration which is trivial if all* f_i *are trivial.*

Proof. We will prove (1), and observe that statement (2) is dual to (1). Using induction on n, we can reduce the problem to two maps $f_0 : A_0 \to B_0$ and $f_1 ; A_1 \to B_1$. If we prove the statement for $f_0, 1_{A_1}$ and $1_{B_0}, f_1$ then the statement follows for f_0, f_1. So it suffices to show that if $f : A \to B$ is a (trivial) cofibration and A, C are cofibrant, then $A \sqcup C$, $B \sqcup C$ exist and are cofibrant and $f \sqcup 1_C$ is a (trivial) cofibration. From axiom CF3 (1) applied to

$$\begin{tikzcd}
\mathbf{0} \arrow[r, tail] \arrow[d, tail] & C \arrow[d, tail] \\
A \arrow[r, tail] & A \sqcup C
\end{tikzcd}$$

we see that $A \sqcup C$ exists and is cofibrant, and $A \to A \sqcup C$ is a cofibration. Similarly, $B \sqcup C$ exists and is cofibrant, and from CF3 applied to

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, tail, "f"'] & A \sqcup C \arrow[d, tail, "f \sqcup 1_C"] \\
B \arrow[r, tail] & B \sqcup C
\end{tikzcd}$$

we see that $f \sqcup 1_C$ is a cofibration which is trivial if f is trivial. ☐

1.3. Factorization lemmas

The Brown Factorization Lemma is an improvement of the factorization axiom CF4 for maps between cofibrant objects.

Lemma 1.3.1 (Brown factorization, [**Bro74**]).

(1) *Let $\mathcal{M}$ be a precofibration category, and $f : A \to B$ be a map between cofibrant objects. Then f factors as $f = rf'$, where f' is a cofibration and r is a left inverse to a trivial cofibration.*

(2) *Let $\mathcal{M}$ be a prefibration category, and $f : A \to B$ be a map between fibrant objects. Then f factors as $f = f's$, where f' is a fibration and s is a right inverse to a trivial fibration.*

Proof. The statements are dual, so it suffices to prove (1). We need to construct f', r and s with $f = rf'$ and $rs = 1_B$.

If we apply the factorization axiom to $f + 1_B$, we get a diagram

$$\begin{tikzcd}
A \sqcup B \arrow[rr, "f+1_B"] \arrow[dr, tail, "f^{'}+s"'] & & B \\
& B^{'} \arrow[ur, "\sim", "r"'] &
\end{tikzcd}$$

Since $f^{'} + s$ is a cofibration and A, B are cofibrant, the maps $f^{'}$ and s are cofibrations. The map r is a weak equivalence, and from the commutativity of the diagram we have $rs = 1_B$, therefore s is also a weak equivalence. ☐

Remark 1.3.2. We have in fact proved a stronger statement. We have shown that any map $f : A \to B$ between cofibrant objects in a precofibration category factors as $f = rf'$, with $rs = 1_B$ where f', $f^{'} + s$ are cofibrations and s is a trivial cofibration.

Dually, any map $f : A \to B$ between fibrant objects in a prefibration category factors as $f = f's$, with $rs = 1_A$ where f' and $(f^{'}, r)$ are fibrations and r is a trivial fibration.

Next lemma is a relative version of the factorization axiom CF4 (resp. F4).

Lemma 1.3.3 (Relative factorization of maps).

(1) *Let $\mathcal{M}$ be a precofibration category, and let*

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "a"'] & B_1 \arrow[d, "b"] \\
A_2 \arrow[r, "f_2"] & B_2
\end{tikzcd}$$

be a commutative diagram with A_1, A_2 *cofibrant. Suppose that* $f_1 = r_1 f_1^{'}$ *is a factorization of* f_1 *as a cofibration followed by a weak equivalence. Then there exists a commutative diagram*

$$\begin{tikzcd}
A_1 \arrow[r, tail, "f_1^{'}"] \arrow[d, "a"'] & A_1^{'} \arrow[r, "r_1", "\sim"'] \arrow[d, "a^{'}"'] & B_1 \arrow[d, "b"] \\
A_2 \arrow[r, tail, "f_2^{'}"] & A_2^{'} \arrow[r, "r_2", "\sim"'] & B_2
\end{tikzcd}$$

where $r_2 f_2^{'}$ *is a factorization of* f_2 *as a cofibration followed by a weak equivalence and such that* $A_2 \sqcup_{A_1} A_1^{'} \to A_2^{'}$ *is a cofibration.*

(2) *The dual of (1) holds for prefibration categories.*

Proof. To prove (1), in the commutative diagram

$$\begin{tikzcd}
A_1 \arrow[r, tail, "f_1^{'}"] \arrow[d, "a"'] & A_1^{'} \arrow[rr, "r_1", "\sim"'] \arrow[d] & & B_1 \arrow[d, "b"] \\
A_2 \arrow[r, tail] & A_2 \sqcup_{A_1} A_1^{'} \arrow[r, tail, "s"] & A_2^{'} \arrow[r, "r_2", "\sim"'] & B_2
\end{tikzcd}$$

the pushout $A_2 \sqcup_{A_1} A_1^{'}$ exists by CF3, and we construct the cofibration s and the weak equivalence r_2 using the factorization axiom CF4 applied to $A_2 \sqcup_{A_1} A_1^{'} \longrightarrow B_2$. ☐

The Brown Factorization Lemma has the following relative version:

Lemma 1.3.4 (Relative Brown factorization).

(1) *Suppose that* $\mathcal{M}$ *is a precofibration category, and that*

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "a"'] & B_1 \arrow[d, "b"] \\
A_2 \arrow[r, "f_2"] & B_2
\end{tikzcd}$$

is a commutative diagram with cofibrant objects. Suppose that $f_1 = r_1 f_1^{'}$, $r_1 s_1 = 1$ *is a Brown factorization of* f_1, *with* $f_1^{'}$, $f_1^{'} + s_1$ *cofibrations and* s_1 *a trivial cofibration. Then there exists a Brown factorization* $f_2 = r_2 f_2^{'}$, $r_2 s_2 = 1$ *with* $f_2^{'}$, $f_2^{'} + s_2$ *cofibrations and* s_2 *a trivial cofibration and a map* $b^{'}$ *such that in the diagram*

$$\begin{tikzcd}
A_1 \arrow[r, tail, "f_1'"] \arrow[d, "a"'] & B_1' \arrow[r, shift left, "r_1"] \arrow[r, phantom, "\sim"] \arrow[d, "b'"] & B_1 \arrow[l, tail, shift left, "s_1"] \arrow[d, "b"] \\
A_2 \arrow[r, tail, "f_2'"] & B_2' \arrow[r, shift left, "r_2"] \arrow[r, phantom, "\sim"] & B_2 \arrow[l, tail, shift left, "s_2"]
\end{tikzcd}$$

we have that $b'f_1' = f_2'a$, $br_1 = r_2b'$, $b's_1 = s_2b$, *and that*

$$A_2 \sqcup_{A_1} B_1' \to B_2' \text{ and } B_2 \sqcup_{B_1} B_1' \to B_2'$$

are a cofibration (resp. a trivial cofibration).

(2) *The dual of (1) holds for prefibration categories.*

Proof. To prove (1), denote $A_3 = A_2 \sqcup_{A_1} B_1'$ and $B_3 = B_2 \sqcup_{B_1} B_1'$.

$$\begin{tikzcd}
A_1 \arrow[rr, tail, "f_1'"] \arrow[d, "a"'] & & B_1' \arrow[dl] \arrow[dr] \arrow[rr, shift left, "r_1"] \arrow[rr, phantom, "\sim"] & & B_1 \arrow[ll, tail, shift left, "s_1"] \arrow[d, "b"] \\
A_2 \arrow[r, tail, "f'"] & A_3 \arrow[rr, "f_3"] & & B_3 \arrow[r, shift left, "r"] \arrow[r, phantom, "\sim"] & B_2 \arrow[l, tail, shift left, "s"]
\end{tikzcd}$$

We apply Lemma 1.3.3 to the commutative diagram

$$\begin{tikzcd}
A_1 \sqcup B_1 \arrow[r, tail, "f_1'+s_1"] \arrow[d, "f'a \sqcup sb"'] & B_1' \arrow[r, "r_1", "\sim"'] & B_1 \arrow[d, "sb"] \\
A_3 \sqcup B_3 \arrow[rr, "f_3+1_{B_3}"] & & B_3
\end{tikzcd}$$

and we construct a commutative diagram

$$\begin{tikzcd}
A_1 \sqcup B_1 \arrow[rr, "f_1+1_{B_1}"] \arrow[dr, tail, "f_1'+s_1"'] \arrow[dd, "f'a \sqcup sb"'] & & B_1 \arrow[dd, "sb"] \\
& B_1' \arrow[ur, "\sim", "r_1"'] \arrow[dd, "b'", crossing over] & \\
A_3 \sqcup B_3 \arrow[rr] \arrow[dr, tail, "f_3'+s_3"'] & & B_3 \\
& B_2' \arrow[ur, "\sim", "r_3"'] &
\end{tikzcd}$$

We now set $f_2' = f_3'f'$, $r_2 = rr_3$ and $s_2 = s_3s$. □

1.4. Extension lemmas

The Gluing Lemma describes the behavior of cofibrations and weak equivalences under pushouts, and is one of the basic building blocks we will employ in the construction of homotopy colimits.

Lemma 1.4.1 (Gluing Lemma).

(1) *Let $\mathcal{M}$ be a precofibration category. In the diagram*

$$\begin{tikzcd}
A_1 \arrow[rr, tail, "f_{12}"] \arrow[dr, "f_{13}"'] \arrow[dd, "u_1"'] & & A_2 \arrow[dr] \arrow[dd, "u_2"] & \\
& A_3 \arrow[rr, tail] \arrow[dd, "u_3"] & & A_4 \arrow[dd, "u_4"] \\
B_1 \arrow[rr, tail, "g_{12}"'] \arrow[dr, "g_{13}"'] & & B_2 \arrow[dr] & \\
& B_3 \arrow[rr, tail] & & B_4
\end{tikzcd}$$

suppose that A_1, A_3, B_1, B_3 are cofibrant, that f_{12}, g_{12} are cofibrations, and that the top and bottom faces are pushouts.

(a) *If u_1, u_3 are cofibrations and the natural map $B_1 \sqcup_{A_1} A_2 \longrightarrow B_2$ is a cofibration, then u_2, u_4 and the natural map $B_3 \sqcup_{A_3} A_4 \longrightarrow B_4$ are cofibrations.*

(b) *If u_1, u_2, u_3 are weak equivalences, then u_4 is a weak equivalence.*

(2) *Let $\mathcal{M}$ be a prefibration category. In the diagram*

$$\begin{tikzcd}
B_4 \arrow[rr, two heads] \arrow[dr] \arrow[dd, "u_4"'] & & B_3 \arrow[dr, "g_{31}"] \arrow[dd, "u_3"] & \\
& B_2 \arrow[rr, two heads, "g_{21}"] \arrow[dd, "u_2"] & & B_1 \arrow[dd, "u_1"] \\
A_4 \arrow[rr, two heads] \arrow[dr] & & A_3 \arrow[dr, "f_{31}"] & \\
& A_2 \arrow[rr, two heads, "f_{21}"'] & & A_1
\end{tikzcd}$$

suppose that A_1, A_3, B_1, B_3 are fibrant, that f_{21}, g_{21} are fibrations, and that the top and bottom faces are pullbacks.

(a) *If u_1, u_3 are fibrations and the natural map $B_2 \longrightarrow B_1 \times_{A_1} A_2$ is a fibration, then u_2, u_4 and the natural map $B_4 \longrightarrow B_3 \times_{A_3} A_4$ are fibrations.*

(b) *If u_1, u_2, u_3 are weak equivalences, then u_4 is a weak equivalence.*

Proof. The statements are dual, and we will prove only (1). Let $B_2^{'} = B_1 \sqcup_{A_1} A_2$ and $B_4^{'} = B_3 \sqcup_{A_3} A_4$ be the pushout of the front and back faces of the diagram of (1). These pushouts exist because of the pushout axiom CF3.

$$\begin{tikzcd}
A_1 \arrow[rr, tail, "f_{12}"] \arrow[dr, "f_{13}"'] \arrow[ddd, "u_1"'] & & A_2 \arrow[dr] \arrow[dd, "u_2^{'}"] & \\
& A_3 \arrow[rr, tail] \arrow[ddd, "u_3"] & & A_4 \arrow[dd, "u_4^{'}"] \\
& & B_2^{'} \arrow[d, "u_2^{''}"] \arrow[dr] & \\
B_1 \arrow[urr, tail, "g_{12}^{'}"] \arrow[rr, tail, "g_{12}"'] \arrow[dr, "g_{13}"'] & & B_2 \arrow[dr] & B_4^{'} \arrow[d, "u_4^{''}"] \\
& B_3 \arrow[urr, tail, "g_{34}^{'}"] \arrow[rr, tail] & & B_4
\end{tikzcd}$$

The maps $g_{12}^{'}$ and $g_{34}^{'}$ are pushouts of cofibrations, therefore cofibrations. Furthermore, we observe that $B_4^{'} = B_3 \sqcup_{B_1} B_2^{'}$, and therefore $u_4^{''}$ is the pushout of $u_2^{''}$ along $B_2^{'} \to B_4^{'}$.

Let's prove (a). If u_1, u_3 and $u_2^{''}$ are cofibrations, then by the pushout axiom $u_2^{'}$ and $u_4^{'}$ are cofibrations. $u_4^{''}$ is a pushout of $u_2^{''}$, therefore also a cofibration. It follows that u_2, u_4 are cofibrations.

Let's now prove (b), first under the assumption that

$$u_1, u_3 \text{ and } u_2^{''} \text{ are cofibrations} \tag{1.1}$$

If they are, since u_1, u_2, u_3 are weak equivalences we see that u_1, u_3 and their pushouts $u_2^{'}$, $u_4^{'}$ must be trivial cofibrations. From the two out of three axiom, $u_2^{''}$ is a weak equivalence, therefore a trivial cofibration, and so its pushout $u_4^{''}$ also is a trivial cofibration, which shows that $u_4 = u_4^{''} u_4^{'}$ is a weak equivalence.

For general weak equivalences u_1, u_2, u_3, we use the relative Brown factorization lemma to construct the diagram

$$\begin{tikzcd}
A_2 \arrow[r, tail, "v_2"] & B_2^{'} \arrow[r, shift left, "r_2"] & B_2 \arrow[l, tail, shift left, "w_2"] \\
A_1 \arrow[r, tail, "v_1"] \arrow[u, tail, "f_{12}"] \arrow[d, "f_{13}"'] & B_1^{'} \arrow[r, shift left, "r_1"] \arrow[u, tail, "h_{12}"] \arrow[d, "h_{13}"'] & B_1 \arrow[l, tail, shift left, "w_1"] \arrow[u, tail, "g_{12}"'] \arrow[d, "g_{13}"] \\
A_3 \arrow[r, tail, "v_3"'] & B_3^{'} \arrow[r, shift left, "r_3"] & B_3 \arrow[l, tail, shift left, "w_3"]
\end{tikzcd}$$

In this diagram:

1. v_1, w_1, r_1 are constructed as a Brown factorization of u_1 as in Remark 1.3.2
2. v_i, w_i, r_i for $i = 2, 3$ are constructed as relative Brown factorizations of u_2 resp. u_3 over the Brown factorization v_1, w_1, r_1.

The maps (w_1, w_2, w_3) are trivial cofibrations and (u_1, u_2, u_3) are weak equivalences, so (v_1, v_2, v_3) are trivial cofibrations.

Statement (b) is true for

- (v_1, v_2, v_3) resp. (w_1, w_2, w_3) because they satisfy property (1.1)
- therefore true for (r_1, r_2, r_3) as a left inverse to (w_1, w_2, w_3)
- therefore true for (u_1, u_2, u_3) as the composition of (v_1, v_2, v_3) and (r_1, r_2, r_3).

□

As a corollary we have

Lemma 1.4.2 (Excision).

(1) *Let $\mathcal{M}$ be a precofibration category. In the diagram below*

$$\begin{tikzcd} A \arrow[r, "f", "\sim"'] \arrow[d, tail, "i"'] & C \\ B & \end{tikzcd}$$

suppose that A, C are cofibrant, i is a cofibration and f is a weak equivalence. Then the pushout of f along i is again a weak equivalence.

(2) *Let $\mathcal{M}$ be a prefibration category. In the diagram below*

$$\begin{tikzcd} & B \arrow[d, two heads, "p"] \\ C \arrow[r, "f", "\sim"'] & A \end{tikzcd}$$

suppose that A, C are fibrant, p is a fibration and f is a weak equivalence. Then the pullback of f along p is again a weak equivalence.

Proof. Part (1) is a particular case of the Gluing Lemma (1) for $f_{12} = g_{12} = i$, $g_{13} = u_3 = f$, $f_{13} = u_3 = 1_A$ and $u_2 = 1_B$. Part (2) is dual. □

It is now not hard to see that in the presence of the rest of the axioms, the axiom CF3 (2) is equivalent to the Gluing Lemma (1) (b) and to excision. A dual statement holds for the fibration axioms.

Lemma 1.4.3 (Equivalent formulation of CF3).

(1) *If $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ satisfies the axioms CF1-CF2, CF3 (1) and CF4, then the following are equivalent:*
  (a) *It satisfies CF3 (2)*
  (b) *It satisfies the Gluing Lemma (1) (b)*
  (c) *It satisfies the Excision Lemma 1.4.2 (1).*

(2) *If $(\mathcal{M}, \mathcal{W}, \mathcal{F}ib)$ satisfies the axioms F1-F2, F3 (1) and F4, then the following are equivalent:*
  (a) *It satisfies F3 (2)*
  (b) *It satisfies the Gluing Lemma (2) (b)*
  (c) *It satisfies the Excision Lemma 1.4.2 (2).*

Proof. It suffices to prove (1). We have proved (a) ⇒ (b) as Lemma 1.4.1, and we have seen (b) ⇒ (c) in the proof of Lemma 1.4.2.

For (a) ⇐ (c), suppose that A, C are cofibrant, that $i : A \rightarrowtail B$ is a trivial cofibration and that $f : A \to C$ is a map. Factor f as a cofibration $f^{'}$ followed by a weak equivalence r, and using axiom CF3 (1) construct the pushouts $g^{'}, s$ of $f^{'}, r$

$$\begin{tikzcd}
A \arrow[r, tail, "f'"] \arrow[d, tail, "i"', "\sim"] & C' \arrow[r, "r", "\sim"'] \arrow[d, tail, "\sim"', "i'"] & C \arrow[d, tail, "j"] \\
B \arrow[r, tail, "g'"] & B' \arrow[r, "s", "\sim"'] & D
\end{tikzcd}$$

The maps $g^{'}, i^{'}$ and j are cofibrations by axiom CF3 (1). Using excision, $i^{'}$ and s are weak equivalences, so by the 2 out of 3 axiom CF2 the map j is also a weak equivalence. ☐

1.5. Cylinder and path objects

We next define cylinder objects in a precofibration category, and show that cylinder objects exist. Dually, we define and prove existence of path objects in a prefibration category.

Definition 1.5.1 (Cylinder and path objects).

(1) Let $\mathcal{M}$ be a precofibration category, and A a cofibrant object of $\mathcal{M}$. A cylinder object for A consists of an object IA and a factorization of the codiagonal $\nabla :\ A \sqcup A \xrightarrow{i_0+i_1} IA \xrightarrow[\sim]{p} A$, with $i_0 + i_1$ a cofibration and p a weak equivalence.

(2) Let $\mathcal{M}$ be a prefibration category, and A a fibrant object of $\mathcal{M}$. A path object for A consists of an object A^I and a factorization of the diagonal $\Delta :\ A \xrightarrow[\sim]{i} A^I \xrightarrow{(p_0,p_1)} A \times A$, with (p_0, p_1) a fibration and i a weak equivalence.

Lemma 1.5.2 (Existence of cylinder and path objects).

(1) *Let $\mathcal{M}$ be a precofibration category, and A a cofibrant object of $\mathcal{M}$. Then A admits a (non-functorial) cylinder object.*

(2) *Let $\mathcal{M}$ be a prefibration category, and A a fibrant object of $\mathcal{M}$. Then A admits a (non-functorial) path object.*

Proof. To prove (1), observe that if A is cofibrant then the sum $A \sqcup A$ exists and is cofibrant by Lemma 1.2.1, and we can then use the factorization axiom CF4 to construct a cylinder object $A \sqcup A \xrightarrow{i_0+i_1} IA \xrightarrow[\sim]{p} A$. The statement (2) follows from duality. ☐

Observe that for cylinder objects, the inclusion maps $i_0, i_1 : A \longrightarrow IA$ are trivial cofibrations. For path objects, the projection maps $p_0, p_1 : A^I \longrightarrow A$ are trivial fibrations.

Lemma 1.5.3 (Relative cylinder and path objects).

(1) *Let $\mathcal{M}$ be a precofibration category, and $f : A \longrightarrow B$ a map with A, B cofibrant objects. Let IA be a cylinder of A. Then there exists a cylinder IB and a commutative diagram*

$$\begin{tikzcd}
A \sqcup A \arrow[r, tail] \arrow[d, "f \sqcup f"'] & IA \arrow[r, "\sim"] \arrow[d, "If"'] & A \arrow[d, "f"] \\
B \sqcup B \arrow[r, tail] & IB \arrow[r, "\sim"] & B
\end{tikzcd}$$

with $(B \sqcup B) \sqcup_{A \sqcup A} IA \rightarrowtail IB$ *a cofibration.*

(2) *Let* $\mathcal{M}$ *be a prefibration category, and* $f : A \longrightarrow B$ *a map with* A, B *fibrant objects. Let* B^I *be a path object for of* B*. Then there exists a path object* A^I *and a commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "\sim"] \arrow[d, "f"'] & A^I \arrow[r, two heads] \arrow[d, "f^I"'] & A \times A \arrow[d, "f \times f"] \\
B \arrow[r, "\sim"] & B^I \arrow[r, two heads] & B \times B
\end{tikzcd}$$

with $A^I \twoheadrightarrow B^I \times_{B \times B} (A \times A)$ *a fibration.*

Proof. To prove (1), apply Lemma 1.3.3 to the diagram

$$\begin{tikzcd}
A \sqcup A \arrow[r, tail] \arrow[d, "f \sqcup f"'] & IA \arrow[r, "\sim"] & A \arrow[d, "f"] \\
B \sqcup B \arrow[rr, "\nabla"] & & B
\end{tikzcd}$$

Statement (2) is dual to (1). □

1.6. Elementary consequences of CF5 and CF6

In the previous sections we have proved a number of elementary lemmas that are consequences of the precofibration category axioms CF1-CF4. In this section, we will do the same bringing in one by one the cofibration category axioms CF5 and CF6.

A word on the motivation behind the two additional axioms CF5-CF6. In the construction of homotopy colimits indexed by small diagrams in a cofibration category, it turns out that the role of small *direct categories* (Def. 6.1.1) is essential, because an arbitrary small diagram can be approximated by a diagram indexed by a small direct category (Section 6.5).

For a small direct category $\mathcal{D}$, its degreewise filtration can be used to show that colimits indexed by $\mathcal{D}$ may be constructed using small sums, pushouts and countable direct transfinite compositions (at least if the base category is cocomplete). To put things in perspective, the axiom CF3 is a property of pushouts, the axiom CF5 is a property of small sums of maps and the axiom CF6 is a property of countable direct transfinite compositions of maps.

Let us clarify for a moment what we mean in CF6 and F6 by transfinite direct and inverse compositions of maps.

Definition 1.6.1. Let $\mathcal{M}$ be a category, and let k be an ordinal.

(1) A *direct k-sequence* of maps (or a *direct sequence of length k*)

$$A_0 \xrightarrow{a_{01}} A_1 \xrightarrow{a_{12}} \cdots \longrightarrow A_i \longrightarrow \cdots \qquad (i < k)$$

consists of a collection of objects A_i for $i < k$ and maps $a_{i_1 i_2} : A_{i_1} \to A_{i_2}$ for $i_1 < i_2 < k$, such that $a_{i_2 i_3} a_{i_1 i_2} = a_{i_1 i_3}$ for all $i_1 < i_2 < i_3 < k$. The map $A_0 \to \operatorname{colim}^{i<k} A_i$, if the colimit exists, is called the *transfinite composition* of the direct k-sequence.

(2) An *inverse k-sequence* of maps (or an *inverse sequence of maps of length k*)

$$\cdots \longrightarrow A_i \longrightarrow \cdots \xrightarrow{a_{21}} A_1 \xrightarrow{a_{10}} A_0 \qquad (i < l)$$

consists of a collection of objects A_i for $i < k$ and maps $a_{i_2 i_1} : A_{i_2} \to A_{i_1}$ for $i_1 < i_2 < k$, such that $a_{i_2 i_1} a_{i_3 i_2} = a_{i_3 i_1}$ for all $i_1 < i_2 < i_3 < k$. The map $\lim^{i<k} A_i \to A_0$, if the limit exists, is called the transfinite composition of the inverse k-sequence.

A direct k-sequence of maps is nothing but a diagram in $\mathcal{M}$ indexed by ordinals $< k$. A map of direct k-sequences is a map of such diagrams.

If $\mathcal{M}$ is a precofibration category, a direct k-sequence of (trivial) cofibrations is a direct k-sequence in which all maps a_{ij}, $i < j < k$ are (trivial) cofibrations. If $\mathcal{M}$ is a fibration category, an inverse k-sequence of (trivial) fibrations is an inverse k-sequence in which all maps a_{ji}, $i < j < k$ are (trivial) fibrations.

A sufficient condition for the additional axioms CF5-CF6 to be satisfied is that cofibrations and trivial cofibrations with cofibrant domain are stable under all *small* transfinite direct compositions.

LEMMA 1.6.2.

(1) *If* $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ *satisfies axioms CF1-CF4, and if cofibrations (resp. trivial cofibrations) with cofibrant domain are stable under transfinite compositions of direct k-sequences for any small ordinal k, then* $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ *also satisfies axioms CF5-CF6.*

(2) *If* $(\mathcal{M}, \mathcal{W}, \mathcal{F}ib)$ *satisfies axioms F1-F4, and if fibrations (resp. trivial fibrations) with fibrant codomain are stable under transfinite compositions of inverse k-sequences for any small ordinal k, then* $(\mathcal{M}, \mathcal{W}, \mathcal{F}ib)$ *also satisfies axioms F5-F6.*

PROOF. We only prove (1). Axiom CF6 is clearly verified for $\mathcal{M}$, since it states that cofibrations (resp. trivial cofibrations) with cofibrant domain are stable under *countable* transfinite direct compositions. For the axiom CF5, given a set of (trivial) cofibrations $f_i : A_i \to B_i$ for $i \epsilon I$ with A_i cofibrant, we choose a well ordering of I. Denote I^+ the well ordered set I with a maximal element adjoined. Parenthetically, I^+ can be viewed as the succesor ordinal of I, and all elements $i \epsilon I$ can be viewed as the ordinals smaller than I.

We show that for any $i \epsilon I^+$, we have that $\sqcup_{k<i} f_k : \sqcup_{k<i} A_k \to \sqcup_{k<i} B_k$ is well defined and is a (trivial) cofibration with a cofibrant domain. We use transfinite induction, and the initial step is trivial. Suppose the statement is true for all elements $< i$, and let's prove it for i.

If i is a succesor ordinal, the statement for i follows from Lemma 1.2.1. Suppose that i is a limit ordinal.

For any $i^{''} < i^{'} < i$, the inclusion

$$\sqcup_{k<i^{''}} A_k \to \sqcup_{k<i^{'}} A_k$$

is a cofibration, using the inductive hypothesis. The transfinite composition of these cofibrations defines $\sqcup_{k<i} A_k$, which therefore exists and is cofibrant. Similarly, $\sqcup_{k<i} B_k$ exists and is cofibrant.

For any $i^{''} < i^{'} < i$, the map

$$(\sqcup_{k<i^{''}} 1_{B_k}) \sqcup (\sqcup_{i^{''} \leq k < i^{'}} f_k) \sqcup (\sqcup_{i^{'} \leq k < i} 1_{A_k})$$

defined on

$$\sqcup_{k<i^{''}} B_k \sqcup_{i^{''} \leq k < i} A_k \to \sqcup_{k<i^{'}} B_k \sqcup_{i^{'} \leq k < i} A_k$$

is a well defined (trivial) cofibration with cofibrant domain, using the inductive hypothesis. The transfinite composition of these (trivial) cofibrations with cofibrant domain defines $\sqcup_{k<i} f_k$, which is therefore a (trivial) cofibration. The statement of our lemma now follows if we take i to be the maximal element of I^+. ☐

The next two lemmas describe properties of the additional axiom CF5 (resp. F5).

Lemma 1.6.3.

1. *Suppose that $\mathcal{M}$ is a CF1-CF4 (resp. CF1-CF5) cofibration category. If $f_i : A_i \to B_i$ for $i \epsilon I$ is a finite (resp. small) set of weak equivalences between cofibrant objects, then $\sqcup f_i$ is a weak equivalence.*
2. *Suppose that $\mathcal{M}$ is an F1-F4 (resp. F1-F5) fibration category. If $f_i : A_i \to B_i$ for $i \epsilon I$ is a finite (resp. small) set of weak equivalences between fibrant objects, then $\times f_i$ is a weak equivalence.*

Proof. We only prove (1). Using the Brown Factorization Lemma, write $f_i = r_i f_i^{'}$, where $f_i^{'}$ is a trivial cofibration and r_i is a left inverse to a trivial cofibration s_i. Under both alternative hypotheses, the maps $\sqcup f_i^{'}$ and $\sqcup s_i$ are trivial cofibrations, so $\sqcup r_i$ and therefore $\sqcup f_i$ are weak equivalences. ☐

Lemma 1.6.4 (Equivalent formulation of CF5).

1. *Suppose that $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ satisfies axioms CF1-CF4 and CF5 (1). Then the following are equivalent:*
   - (a) *It satisfies axiom CF5 (2)*
   - (b) *The class of weak equivalences between cofibrant objest is stable under small sums.*
2. *Suppose that $(\mathcal{M}, \mathcal{W}, \mathcal{F}ib)$ satisfies axioms F1-F4 and F5 (1). Then the following are equivalent:*
   - (a) *It satisfies axiom F5 (2)*
   - (b) *The class of weak equivalences between fibrant objest is stable under small products.*

Proof. For part (1), the implication $\Leftarrow$ is trivial, and the implication $\Rightarrow$ is a consequence of Lemma 1.6.3. Part (2) has a dual proof. ☐

Here is a consequence of the axiom CF6 (resp. F6).

Lemma 1.6.5.

1. *Let $\mathcal{M}$ be a CF1-CF4 cofibration category satisfying CF6. For any map of countable direct sequences of cofibrations with A_0, B_0 cofibrant and all f_n weak equivalences*

$$\begin{tikzcd}
A_0 \arrow[r, tail, "a_0"] \arrow[d, "f_0"', "\sim"] & A_1 \arrow[r, tail, "a_1"] \arrow[d, "f_1"', "\sim"] & A_2 \arrow[r, tail, "a_2"] \arrow[d, "f_2"', "\sim"] & \cdots \\
B_0 \arrow[r, tail, "b_0"] & B_1 \arrow[r, tail, "b_1"] & B_2 \arrow[r, tail, "b_2"] & \cdots
\end{tikzcd}$$

   the colimit map $\operatorname{colim} f_n : \operatorname{colim} A_n \to \operatorname{colim} B_n$ *is a weak equivalence between cofibrant objects.*

2. *Let $\mathcal{M}$ be an F1-F4 fibration category satisfying CF6. For any map of countable inverse sequences of fibrations with A_0, B_0 fibrant and all f_n weak equivalences*

$$\begin{tikzcd}
\cdots \arrow[r, two heads, "a_2"] & A_2 \arrow[r, two heads, "a_1"] \arrow[d, "\sim"', "f_2"] & A_1 \arrow[r, two heads, "a_0"] \arrow[d, "\sim"', "f_1"] & A_0 \arrow[d, "\sim"', "f_0"] \\
\cdots \arrow[r, two heads, "b_2"] & B_2 \arrow[r, two heads, "b_1"] & B_1 \arrow[r, two heads, "b_0"] & B_0
\end{tikzcd}$$

the limit map $\lim f_n : \lim A_n \to \lim B_n$ *is a weak equivalence between fibrant objects.*

Proof. To prove (1), observe that using Rem. 1.3.2 and Lemma 1.3.4, we can inductively construct Brown factorizations of f_n that make the diagram

$$\begin{tikzcd}
A_0 \arrow[r, tail, "a_0"] \arrow[d, tail, "f_0^{'}"', "\sim"] & A_1 \arrow[r, tail, "a_1"] \arrow[d, tail, "f_1^{'}"', "\sim"] & A_2 \arrow[r, tail, "a_2"] \arrow[d, tail, "f_2^{'}"', "\sim"] & \cdots \\
B_0^{'} \arrow[r, tail, "b_0^{'}"] \arrow[d, shift left, "\sim"', "r_0"] & B_1^{'} \arrow[r, tail, "b_1^{'}"] \arrow[d, shift left, "\sim"', "r_1"] & B_2^{'} \arrow[r, tail, "b_2^{'}"] \arrow[d, shift left, "\sim"', "r_2"] & \cdots \\
B_0 \arrow[r, tail, "b_0"] \arrow[u, tail, shift left, "s_0"] & B_1 \arrow[r, tail, "b_1"] \arrow[u, tail, shift left, "s_1"] & B_2 \arrow[r, tail, "b_2"] \arrow[u, tail, shift left, "s_2"] & \cdots
\end{tikzcd}$$

commutative, such that additionally all maps

$$B_{n-1}^{'} \sqcup_{A_{n-1}} A_n \to B_n^{'} \text{ and } B_{n-1}^{'} \sqcup_{B_{n-1}} B_n \to B_n^{'}$$

are trivial cofibrations. If we show that colim $f_n^{'}$ and colim s_n are weak equivalences, it will follow that colim f_n is a weak equivalence.

It suffices therefore to prove that colim f_n is a weak equivalence under the additional assumption that f_0 and all $B_{n-1} \sqcup_{A_{n-1}} A_n \to B_n$ are trivial cofibrations. The objects denoted $A =$ colim A_n and $B =$ colim B_n are cofibrant by CF6 (1).

$$\begin{tikzcd}
A_0 \arrow[r, tail, "a_0"] \arrow[d, tail, "f_0"'] & A_1 \arrow[r, tail, "a_1"] \arrow[d, tail] & \cdots \arrow[r, tail] & A \arrow[d, tail] \\
B_0 \arrow[r, tail] \arrow[dr, tail, "b_0"'] & B_0 \sqcup_{A_0} A_1 \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] & B_0 \sqcup_{A_0} A \arrow[d, tail] \\
& B_1 \arrow[r, tail] \arrow[dr, tail, "b_1"'] & \cdots \arrow[r, tail] & B_1 \sqcup_{A_1} A \arrow[d, tail] \\
& & \cdots \arrow[dr, tail] & \vdots \arrow[d, tail] \\
& & & B
\end{tikzcd}$$

The map $A \to B$ factors as the composition of the direct sequence of maps $A \to B_0 \sqcup_{A_0} A$ followed by $B_{n-1} \sqcup_{A_{n-1}} A \to B_n \sqcup_{A_n} A$ for $n \geq 1$, and each map in the sequence is a trivial cofibration as the pushout of the trivial cofibrations f_0 resp. $B_{n-1} \sqcup_{A_{n-1}} A_n \to B_n$, so by CF6 (2) the map $A \to B$ is a trivial cofibration.

The proof of statement (2) is dual to the proof of (1). □

Lemma 1.6.6 (Equivalent formulation of CF6).

1. *Suppose that* $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ *satisfies axioms CF1-CF4 and CF6 (1). Then the following are equivalent:*
   1. (a) *It satisfies axiom CF6 (2)*
   2. (b) *It satisfies the conclusion of Lemma 1.6.5 (1)*
   3. (c) *It satisfies the conclusion of Lemma 1.6.5 (1) for any map* f_n *of countable direct sequences of cofibrations, with each* f_n *a trivial cofibration.*
2. *Suppose that* $(\mathcal{M}, \mathcal{W}, \mathcal{F}ib)$ *satisfies axioms F1-F4 and F6 (1). Then the following are equivalent:*
   1. (a) *It satisfies axiom F6 (2)*
   2. (b) *It satisfies the conclusion of Lemma 1.6.5 (2)*
   3. (c) *It satisfies the conclusion of Lemma 1.6.5 (2) for any map* f_n *of countable inverse sequences of fibrations, with each* f_n *a trivial fibration.*

Proof. Let us prove part (1). The implication (i) ⇒ (ii) is a consequence of Lemma 1.6.5, and the implication (ii) ⇒ (iii) is trivial.

Let us prove the implication (iii) ⇒ (i). Suppose we have a countable direct sequence of trivial cofibrations with A_0 cofibrant

$$\begin{tikzcd} A_0 \arrow[r, tail, "a_0", "\sim"'] & A_1 \arrow[r, tail, "a_1", "\sim"'] & A_2 \arrow[r, tail, "a_2", "\sim"'] & \cdots \end{tikzcd}$$

The colimit colim A_n exists and $A_0 \to$ colim A_n is a cofibration by axiom CF6 (1). If we view A_0 as a constant, countable direct sequence of identity maps, we get a map $f_n = a_{n-1}...a_0$ of countable direct sequences of cofibrations, with each f_n a trivial cofibration. From the conclusion of Lemma 1.6.5 (1) we see that $A_0 \to$ colim A_n is a weak equivalence.

Part (2) is proved in a dual fashion. □

1.7. Properness

Sometimes, a precofibration category $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ admits more than one precofibration structure with weak equivalences $\mathcal{W}$. Under an additional condition (left properness) one can show that $(\mathcal{M}, \mathcal{W})$ admits an intrinsic structure of a CF1-CF4 precofibration category, larger than $\mathcal{C}of$, defined in terms of what we will call *left proper* maps.

Definition 1.7.1.

1. A precofibration category $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ is *left proper* if it satisfies
   PCF: Given a solid diagram in $\mathcal{M}$ with i a cofibration and A cofibrant,

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[d, tail, "i"'] & C \arrow[d, tail, dashed] \\ B \arrow[r, dashed, "r'"] & D \end{tikzcd}$$

   then the pushout exists in $\mathcal{M}$. Moreover, if r is a weak equivalence then so is r'.
2. A prefibration category $(\mathcal{M}, \mathcal{W}, \mathcal{F}ib)$ is *right proper* if
   PF: Given a solid diagram in $\mathcal{M}$ with p a fibration and A fibrant,

$$\begin{tikzcd}
D \arrow[r, dashed, "r'"] \arrow[d, dashed] & B \arrow[d, two heads, "p"] \\
C \arrow[r, "r", "\sim"'] & A
\end{tikzcd}$$

then the pullback exists in $\mathcal{M}$. Moreover, if r is a weak equivalence then so is r'.

(3) An ABC model category is proper if its underlying precofibration and prefibration categories are left resp. right proper.

From the Excision Lemma, a precofibration category with all objects cofibrant is left proper. A prefibration category with all objects fibrant is right proper.

We will define proper maps in the context of what we call *categories with weak equivalences.*

Definition 1.7.2. A category with weak equivalences ($\mathcal{M}$, $\mathcal{W}$) consists of a category $\mathcal{M}$ with a class of weak equivalence maps $\mathcal{W}$, where $\mathcal{W}$ is stable under composition and includes the identity maps of $\mathcal{M}$.

From the definition, in a category with weak equivalences $(\mathcal{M}, \mathcal{W})$ the weak equivalences $\mathcal{W}$ define a subcategory with the same objects as $\mathcal{M}$.

Definition 1.7.3. Suppose that $(\mathcal{M}, \mathcal{W})$ is a category with weak equivalences.

(1) A map $f : A \to B$ is called *left proper* if for any diagram of full maps with r a weak equivalence

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & C_1 \arrow[r, "r", "\sim"'] \arrow[d, dashed] & C_2 \arrow[d, dashed] \\
B \arrow[r, dashed] & D_1 \arrow[r, dashed, "r'"] & D_2
\end{tikzcd}$$

the pushouts exist, and the map r' is again a weak equivalence.

(2) A map $f : B \to A$ is called *right proper* if for any diagram of full maps with r a weak equivalence

$$\begin{tikzcd}
D_2 \arrow[r, dashed, "r'"] \arrow[d, dashed] & D_1 \arrow[r, dashed] \arrow[d, dashed] & B \arrow[d, "f"] \\
C_2 \arrow[r, "r", "\sim"'] & C_1 \arrow[r] & A
\end{tikzcd}$$

the pullbacks exist, and the map r' is again a weak equivalence.

We say that an object A is *left proper* if the map $\mathbf{0} \to A$ is left proper. An object A is *right proper* if the map $A \to \mathbf{1}$ is right proper.

The class of left proper maps of $(\mathcal{M}, \mathcal{W})$ will be denoted $\mathcal{P}rCof$. The class of right proper maps will be denoted $\mathcal{P}rFib$. It is easy to see that the left proper maps are stable under composition and under pushout. The right proper maps are stable under composition and under pullback. All isomorphisms are left and right proper.

The left proper weak equivalences will be called *trivial left proper* maps, and the right proper weak equivalences will be called *trivial right proper* maps.

Theorem 1.7.4.

(1) *If* $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ *is a left proper precofibration category, then*

   (a) *Any cofibration $A \to B$ with A cofibrant is left proper.*
   (b) *Any map of $\mathcal{M}$ factors as a left proper map followed by a weak equivalence.*
   (c) *Trivial left proper maps are stable under pushout.*
   (d) $(\mathcal{M}, \mathcal{W}, \mathcal{P}rCof)$ *is a precofibration category.*
(2) *If* $(\mathcal{M}, \mathcal{W}, \mathcal{F}ib)$ *is a right proper prefibration category, then*
   (a) *Any fibration $A \to B$ with B fibrant is right proper.*
   (b) *Any map of $\mathcal{M}$ factors as a weak equivalence followed by a right proper map.*
   (c) *Trivial right proper maps are stable under pushout.*
   (d) $(\mathcal{M}, \mathcal{W}, \mathcal{P}rFib)$ *is a prefibration category.*

Proof. We only prove (1). For (1) (a), suppose we have a diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, tail, "i"'] & C_1 \arrow[r, "r", "\sim"'] \arrow[d] & C_2 \arrow[d] \\
B \arrow[r] & D_1 \arrow[r, "r^{'}"] & D_2
\end{tikzcd}$$

with A cofibrant, i a cofibration and r a weak equivalence, such that both squares are pushouts. We'd like to show that $r^{'}$ is a weak equivalence.

Denote $f = f^{'} s$ a factorization of f as a cofibration $f^{'}$ followed by a weak equivalence s.

$$\begin{tikzcd}
A \arrow[r, tail, "f_1"] \arrow[d, tail, "i"'] & C_1^{'} \arrow[r, "s", "\sim"'] \arrow[d, tail, "i^{'}"] & C_1 \arrow[r, "r", "\sim"'] \arrow[d] & C_2 \arrow[d] \\
B \arrow[r, tail] & D_1^{'} \arrow[r, "s^{'}"] & D_1 \arrow[r, "r^{'}"] & D_2
\end{tikzcd}$$

Denote $i^{'}$ the pushout of i, and $s^{'}$ the pushout of s. Since $i^{'}$ is a cofibration and s, rs are weak equivalences, from the left properness of $\mathcal{M}$ we see that $s^{'}$, $r^{'}s^{'}$ and therefore $r^{'}$ are weak equivalences.

For (1) (b), suppose $f : A \to B$ is a map in $\mathcal{M}$. We'd like to construct a factorization $f = rf^{'}$ with $f^{'}$ left proper and r a weak equivalence.

Let $a : A^{'} \to A$ be a cofibrant replacement of A, and $fa = r_1^{'} f_1$ factorization of fa as a cofibration f_1 followed by a weak equivalence r_1.

$$\begin{tikzcd}
 & & B \\
A \arrow[urr, "f"] \arrow[r, "f^{'}"'] & Y^{'} \arrow[ur, "r", "\sim"'] & \\
A^{'} \arrow[u, "a^{'}", "\sim"'] \arrow[r, tail, "f_1"'] & A_1 \arrow[u, "b^{'}", "\sim"'] \arrow[uur, "r_1^{'}"'] &
\end{tikzcd}$$

Define $f^{'}$ as the pushout of f_1. The map $b^{'}$ is a weak equivalence as the pushout of $a^{'}$, since $\mathcal{M}$ is left proper. The map r is a weak equivalence by the two out of three axiom, and the map $f^{'}$ is left proper as the pushout of f_1 which is a cofibration with cofibrant domain (therefore left proper).

For (1) (c), let $i : A \to B$ be a trivial left proper map and let $f : A \to C$ be a map. In the diagram

$$\begin{tikzcd}
A \arrow[r, "f^{'}"] \arrow[d, "i"', "\sim"] & C^{'} \arrow[r, "r", "\sim"'] \arrow[d, "i^{'}"] & C \arrow[d, "j"] \\
B \arrow[r] & D^{'} \arrow[r, "r^{'}"] & D
\end{tikzcd}$$

using part (b) we have factored f as $rf^{'}$, with $f^{'}$ left proper and r a weak equivalence. We define $i^{'}$ and j as the pushouts of i. We therefore have that $i^{'}$ and j are left proper. The map $i^{'}$ is a weak equivalence since $f^{'}$ is left proper. The map $r^{'}$ is a weak equivalence since i is proper. From the two out of three axiom, the map j is also a weak equivalence, therefore a trivial left proper map.

For (1) (d), the axioms CF1, CF2 and CF3 (1) are trivially verified. Part (c) proves axiom CF3 (2), and part (b) proves axiom CF4. ☐

It does not appear to be the case that $(\mathcal{M}, \mathcal{W}, \mathcal{P}rCof)$ necessarily satisfies CF5 or CF6 if $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ does. The left proper maps appear to satisfy only the axioms CF1-CF4 with respect to the given class of weak equivalences $\mathcal{W}$.

In the rest of the section, we will review briefly a number of elementary properties of proper maps.

PROPOSITION 1.7.5.

(1) *Suppose that* $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ *is a left proper precofibration category. Then a weak equivalence is a trivial left proper map iff all its pushouts exist and are weak equivalences.*
(2) *Suppose that* $(\mathcal{M}, \mathcal{W}, \mathcal{F}ib)$ *is a right proper prefibration category. Then a weak equivalence is a trivial right proper map iff all its pullbacks exist and are weak equivalences.*

PROOF. We only prove (1).

Implication ⇒ is a consequence of Thm. 1.7.4 (1) (c).

For ⇐, suppose that $i : A \to B$ is a weak equivalence whose pushouts remain weak equivalences. We'd like to show that i is left proper. For any map f and weak equivalence r, we construct the diagram with pushout squares

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"', "\sim"] & C_1 \arrow[r, "r", "\sim"'] \arrow[d, "\sim"', "i_1"] & C_2 \arrow[d, "\sim"', "i_2"] \\
B \arrow[r] & D_1 \arrow[r, "r^{'}"] & D_2
\end{tikzcd}$$

The maps i_1, i_2 and are weak equivalences as pushouts of i. From the 2 out of 3 axiom, the map $r^{'}$ is a weak equivalence, which shows that i is left proper. ☐

PROPOSITION 1.7.6.

(1) *Suppose that* $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ *is a cocomplete, left proper precofibration category. If* f, g *are two composable maps such that* gf *is left proper and* g *is trivial left proper. Then* f *is left proper.*

(2) *Suppose that* $(\mathcal{M}, \mathcal{W}, \mathcal{F}ib)$ *is a complete, right proper prefibration category. If* f, g *are two composable maps such that* gf *is left proper and* g *is trivial left proper. Then* f *is left proper.*

Proof. We only prove (1). In the diagram

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f"'] & B_1 \arrow[r, "r", "\sim"'] \arrow[d] & C_1 \arrow[d] \\
A_2 \arrow[r] \arrow[d, "g"', "\sim"] & B_2 \arrow[r, "r^{'}"] \arrow[d, "g^{'}"] & C_2 \arrow[d, "g^{''}"] \\
A_3 \arrow[r] & B_3 \arrow[r, "r^{''}"] & C_3
\end{tikzcd}$$

r is a weak equivalence and all squares are pushouts. To prove that f is left proper, we need to show that $r^{'}$ is a weak equivalence. But $r^{''}$ is a weak equivalence since gf is proper. $g^{'}$ and $g^{''}$ are trivial left proper as pushouts of g. It follows from the two out of three axiom that $r^{'}$ is a weak equivalence. ☐

Recall the definition of the retract of a map

Definition 1.7.7. A map $f : A \to B$ in a category $\mathcal{M}$ is a retract of $g : C \to D$ if there exists a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "1_A"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r] \arrow[rr, bend right, "1_B"'] & D \arrow[r] & B
\end{tikzcd}$$

Note that the saturation $\overline{\mathcal{W}}$ of the class of weak equivalences $\mathcal{W}$ is closed under retracts.

Proposition 1.7.8. *Suppose that* $(\mathcal{M}, \mathcal{W})$ *is a category with weak equivalences and that* $\mathcal{W}$ *is closed under retracts.*

(1) *Assume that* $\mathcal{M}$ *is cocomplete. Then the class of left proper maps and that of trivial left proper maps are both closed under retracts.*

(2) *Assume that* $\mathcal{M}$ *is complete. Then the class of right proper maps and that of trivial right proper maps are both closed under retracts.*

Proof. Follows directly from the definitions. ☐

CHAPTER 2

Relation with other axiomatic systems

We would like to describe in this chapter how ABC cofibration categories categories relate to Brown's categories of cofibrant objects, to Quillen model categories and to other axiomatizations that have been proposed for categories with cofibrations.

Aside from the goal of bringing together and comparing various axiomatizations that have been proposed for (co)fibrations and weak equivalences, this allows us to tap into a large class of examples of ABC model categories.

For example, simplicial sets *sSets* form a Quillen model category [**Qui67**], with inclusions as cofibrations, with maps satisfying the Kan extension property as fibrations and with maps whose geometric realization is a topological homotopy equivalence as weak equivalences. Any Quillen model category is an ABC model category, and therefore *sSets* is an ABC model category. By Thm. 6.5.5, so is any diagram category $sSets^{\mathcal{D}}$ for a small category $\mathcal{D}$, and by Def. 6.3.8 and Thm. 6.5.6 so are the $\mathcal{D}_2$-reduced $\mathcal{D}_1$-diagrams $sSets^{(\mathcal{D}_1,\mathcal{D}_2)}$ for a small category pair $(\mathcal{D}_1, \mathcal{D}_2)$.

For considerations of space, we will not talk about categories with a natural cylinder ([**Kam72**], [**Shi89**] and [**KP96**], which are a natural development of the ideas introduced in [**Kan56**]), although categories with a natural cylinder are an important part of the story of ABC cofibration categories.

2.1. Brown's categories of cofibrant objects

In his paper [**Bro74**], Brown defines categories of fibrant objects (and dually categories of cofibrant objectss). We list below Brown's axioms, stated in the cofibration setting, slightly modified but equivalent to the actual axioms of [**Bro74**].

Definition 2.1.1 (Categories of cofibrant objects). A category of cofibrant objects ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{C}of$) consists of a category $\mathcal{M}$ and two distinguished classes of maps $\mathcal{W}, \mathcal{C}of$ - the weak equivalences and respectively cofibrations of $\mathcal{M}$, subject to the axioms below:

CFObj1: All isomophisms of $\mathcal{M}$ are trivial cofibrations. $\mathcal{M}$ has an initial object $\mathbf{0}$, and all objects of $\mathcal{M}$ are cofibrant. Cofibrations are stable under composition.

CFObj2: (Two out of three axiom) If f, g are maps of $\mathcal{M}$ such that gf is defined, and if two of f, g, gf are weak equivalences, then so is the third.

CFObj3: (Pushout axiom) Given a solid diagram in $\mathcal{M}$, with i a cofibration,

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail, "i"'] & C \arrow[d, tail, "j"] \\
B \arrow[r, dashed] & D
\end{tikzcd}$$

then the pushout exists in $\mathcal{M}$ and j is a cofibration. If additionally i is a trivial cofibration, then j is a trivial cofibration.

CFObj4: (Cylinder axiom) For any object A of $\mathcal{M}$, the codiagonal $\nabla : A \sqcup A \to A$ admits a factorization as a cofibration followed by a weak equivalence.

The axioms for categories of fibrant objects are dual to those of Def. 2.1.1, and are denoted FObj1-FObj4.

The precofibration categories (satisfying the minimal axioms CF1-CF4 but not the additional axioms CF5-CF6) are essentialy a modification of Brown's categories of cofibrant objects - in the sense that we allow objects to be non-cofibrant. The following lemma explains the precise relationship between precofibration categories and categories of cofibrant objects.

Proposition 2.1.2.

(1) *Any category of cofibrant objects is a precofibration category. Conversely, if $\mathcal{M}$ is a precofibration category, then $\mathcal{M}_{cof}$ is a category of cofibrant objects.*
(2) *Any category of fibrant objects is a prefibration category. Conversely, if $\mathcal{M}$ is a prefibration category, then $\mathcal{M}_{fib}$ is a category of fibrant objects.*

Proof. We only prove (1). Implication $\Leftarrow$ is an easy consequence of the axioms. For the other direction $\Rightarrow$, the only axiom that needs to be proved is the factorization axiom CF4.

To prove axiom CF4, we need to prove Brown's factorization lemma 1.3.1 in the context of the axioms CFObj1-CFObj4. We want to show that any map $f : A \to B$ factors as $f = rf'$, where f' is a cofibration and $rs = 1_B$ for a trivial cofibration s.

Choose a cylinder IA, and construct s as the pushout of the trivial cofibration i_0.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, tail, "i_0"', "\sim"] & B \arrow[d, tail, "s", "\sim"'] \\
IA \arrow[r, "F"] & B'
\end{tikzcd}$$

Notice that i_0 has $p : IA \longrightarrow A$ as a left inverse, and it follows that s has a left inverse r. Let f' be Fi_1, which satisfies $f = rf'$, and to complete the proof it remains to prove that f' is a cofibration.

We notice that B' is also the pushout of the diagram below

$$\begin{tikzcd}
A \sqcup A \arrow[r, "f \sqcup 1"] \arrow[d, tail, "i_0+i_1"'] & B \sqcup A \arrow[d, tail, "s+f'"] \\
IA \arrow[r, "F"] & B'
\end{tikzcd}$$

so f' is $A \rightarrowtail B \sqcup A \overset{s+f'}{\rightarrowtail} B'$, therefore a cofibration. $\square$

2.2. Quillen model categories

Quillen's model categories involve both cofibrations and fibrations, and come with built-in Eckman-Hilton duality between cofibrations and fibrations.

In preparation, recall the definition of the left (and right) lifting property of maps.

Definition 2.2.1. For a solid commutative diagram in a category $\mathcal{M}$

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & C \arrow[d, "p"] \\
B \arrow[r] \arrow[ur, dashed] & D
\end{tikzcd}$$

if a dotted arrow exists making the diagram commutative we say that i has the LLP (left lifting property) with respect to p, and that p has the RLP (right lifting property) with respect to i.

Also, recall the definition Def. 1.7.7 of the retract of a map. We will use the (closed) Quillen model category axiom formulation from Hirschhorn's monograph [**Hir00**].

Definition 2.2.2 (Quillen model categories). A Quillen model category ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{C}of$, $\mathcal{F}ib$) consists of a category $\mathcal{M}$ and three distinguished classes of maps $\mathcal{W}$, $\mathcal{C}of$, $\mathcal{F}ib$ - the weak equivalences, the cofibrations and respectively the fibrations of $\mathcal{M}$, subject to the axioms below:

- **M1:** $\mathcal{M}$ is complete and cocomplete.
- **M2:** (Two out of three axiom) If f, g are maps of $\mathcal{M}$ such that gf is defined, and if two of f, g, gf are weak equivalences, then so is the third.
- **M3:** (Retract axiom) Weak equivalences, cofibrations and fibrations are closed under retracts.
- **M4:** (Lifting axiom) Cofibrations have the LLP with respect to trivial fibrations, and trivial cofibrations have the LLP with respect to fibrations.
- **M5:** (Factorization axiom) Any map of $\mathcal{M}$ admits a factorization as a cofibration followed by a trivial fibration, and a factorization as a trivial cofibration followed by a fibration.

The retract axiom M3 states that given a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "1_A"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r] \arrow[rr, bend right, "1_B"'] & D \arrow[r] & B
\end{tikzcd}$$

if g is a weak equivalence (resp. cofibration, resp. fibration) then so is f.

The factorizations in M5 are not assumed to be functorial.

From the axioms M1-M5 one can show that in a Quillen model category any two of the following classes of maps of $\mathcal{M}$ - the cofibrations, the trivial cofibrations, the fibrations and the trivial fibrations - determine each other by the following rules:

- A map is a cofibration $\Leftrightarrow$ it has the LLP with respect to all trivial fibrations

- A map is a trivial cofibration ⇔ it has the LLP with respect to all fibrations
- A map is a fibration ⇔ it has the RLP with respect to all trivial cofibrations
- A map is a trivial fibration ⇔ it has the RLP with respect to all cofibrations

Proposition 2.2.3. *Any Quillen model category is an ABC model category.*

Proof. A Quillen model category trivially satisfies the axioms CF2 and CF4. The axioms CF1, CF3, CF5 and CF6 are satisfied since a map is a cofibration (resp. trivial cofibration) iff it has the LLP with respect to all trivial fibrations (resp. fibrations). A dual argument shows that a Quillen model category satisfies the axioms F1-F6. □

If $\mathcal{M}$ is a Quillen model category and $\mathcal{D}$ is a small category, then the category of diagrams $\mathcal{M}^{\mathcal{D}}$ does not generally form a Quillen model category (except in important particular cases, for example when $\mathcal{M}$ is cofibrantly generated or when $\mathcal{D}$ is a Reedy category). But we will see further down (Thm. 6.5.5) that $\mathcal{M}^{\mathcal{D}}$ carries an ABC model category structure, and in that sense one can always 'do homotopy theory' on $\mathcal{M}^{\mathcal{D}}$.

A Quillen model category $\mathcal{M}$ is called *left proper* if for any pushout diagram

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[d, tail, "i"'] & C \arrow[d, tail, dashed] \\
B \arrow[r, dashed, "j'"] & D
\end{tikzcd}$$

with i a cofibration and j a weak equivalence, the map j' is a weak equivalence. $\mathcal{M}$ is called *right proper* if for any pullback diagram

$$\begin{tikzcd}
D \arrow[r, dashed, "q'"] \arrow[d, two heads, dashed] & B \arrow[d, two heads, "p"] \\
C \arrow[r, "q"] & A
\end{tikzcd}$$

with p a fibration and q a weak equivalence, the map q' is a weak equivalence. $\mathcal{M}$ is called *proper* if it is left and right proper.

From this definition and from Prop. 2.2.3 we immediately get

Proposition 2.2.4. *Any proper Quillen model category is a proper ABC model category.* □

2.3. Baues cofibration categories

We next turn our attention to the notion of (co)fibration category as defined by Baues. We state below the axioms of a Baues cofibration category, in a slightly modified but equivalent form to [**Bau88**], Sec. 1.1.

Definition 2.3.1. A Baues cofibration category ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{C}of$) consists of a category $\mathcal{M}$ and two distingushed classes of maps $\mathcal{W}$ and $\mathcal{C}of$ - the weak equivalences and the cofibrations of $\mathcal{M}$ - subject to the axioms below:

BCF1: All isomophisms of $\mathcal{M}$ are trivial cofibrations. Cofibrations are stable under composition.

BCF2: (Two out of three axiom) If f, g are maps of $\mathcal{M}$ such that gf is defined, and if two of f, g, gf are weak equivalences, then so is the third.

BCF3: (Pushout and excision axiom) Given a solid diagram in $\mathcal{M}$, with i a cofibration,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, tail, "i"'] & C \arrow[d, tail, dashed, "j"] \\
B \arrow[r, dashed, "g"] & D
\end{tikzcd}$$

then the pushout exists in $\mathcal{M}$ and j is a cofibration. Moreover:

(a) If i is a trivial cofibration, then j is a trivial cofibration
(b) If f is a weak equivalence, then g is a weak equivalence.

BCF4: (Factorization axiom) Any map of $\mathcal{M}$ admits a factorization as a cofibration followed by a weak equivalence.

BCF6: (Axiom on fibrant models) For each object A of $\mathcal{M}$ there is a trivial cofibration $A \to B$, with B satisfying the property that each trivial cofibration $C \to B$ admits a left inverse.

A Baues cofibration category may not have an initial object, but if it does then by BCF1 the initial object is cofibrant. We will not state the axioms for a Baues fibration category - they are dual to the above axioms.

Proposition 2.3.2 (Relation with Baues cofibration categories).

(1) *Any Baues cofibration category with an initial object is a left proper precofibration category.*
(2) *Any Baues fibration category with a terminal object is a right proper prefibration category.*

Proof. Easy consequence of the axioms and of Lemma 1.4.3. ☐

2.4. Waldhausen categories

We list below the axioms we'll use for a Waldhausen cofibration category. These axioms are equivalent to the axioms Cof1-Cof3, Weq1 and Weq2 of [**Wal85**]. Axioms for a Waldhausen fibration category will of couse be dual to the axioms below.

Definition 2.4.1. A Waldhausen cofibration category $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ consists of a category $\mathcal{M}$ and two distinguished classes of maps $\mathcal{W}$ and $\mathcal{C}of$ - the weak equivalences and the cofibrations of $\mathcal{M}$, subject to the axioms below:

WCF1: $\mathcal{M}$ is pointed. All isomophisms of $\mathcal{M}$ are trivial cofibrations. All objects of $\mathcal{M}$ are cofibrant. Cofibrations are stable under composition.

WCF2: (Pushout axiom) Given a solid diagram in $\mathcal{M}$, with i a cofibration,

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail, "i"'] & C \arrow[d, tail, dashed, "j"] \\
B \arrow[r, dashed] & D
\end{tikzcd}$$

then the pushout exists in $\mathcal{M}$ and j is a cofibration

WCF3: (Gluing axiom) In the diagram below

$$\begin{tikzcd}
A_1 \arrow[rr, tail, "f_{12}"] \arrow[dr, "f_{13}"'] \arrow[dd, "u_1"'] & & A_2 \arrow[dr] \arrow[dd, "u_2"] & \\
& A_3 \arrow[rr, tail] \arrow[dd, "u_3"] & & A_4 \arrow[dd, "u_4"] \\
B_1 \arrow[rr, tail, "g_{12}"'] \arrow[dr, "g_{13}"'] & & B_2 \arrow[dr] & \\
& B_3 \arrow[rr, tail] & & B_4
\end{tikzcd}$$

if f_{12}, g_{12} are cofibrations, u_1, u_2, u_3 are weak equivalences and the top and bottom faces are pushouts, then u_4 is a weak equivalence.

We have the following

PROPOSITION 2.4.2 (Relation with Waldhausen cofibration categories).

(1) (a) *If $\mathcal{M}$ is a pointed precofibration category, then $\mathcal{M}_{cof}$ is a Waldhausen cofibration category.*
    (b) *If $\mathcal{M}$ is a Waldhausen cofibration category satisfying the 2 out of 3 axiom CF2 and the cylinder axiom CFObj4, then it is a precofibration category.*

(2) (a) *If $\mathcal{M}$ is a pointed prefibration category, then $\mathcal{M}_{fib}$ is a Waldhausen fibration category.*
    (b) *If $\mathcal{M}$ is a Waldhausen fibration category satisfying the 2 out of 3 axiom F2 and the path object axiom FObj4, then it is an prefibration category.*

PROOF. Part (1) (a) follows from the Gluing Lemma 1.4.1, part (1) (b) from Lemma 1.4.3 and Prop. 2.1.2, and part (2) is dual to the above. ☐

CHAPTER 3

The homotopy category of a cofibration category

Our goal in this chapter is to describe the homotopy category of a cofibration category. All the definition and the results of this chapter actually only require the smaller set of precofibration category axioms CF1-CF4.

After a small discussion about categories and universes, we will recall the definition of the homotopy category $\mathbf{ho}\mathcal{M}$ of a category $\mathcal{M}$ with a class of weak equivalences $\mathcal{W}$. The homotopy category $\mathbf{ho}\mathcal{M}$ is defined by a universal property, but admits a description in terms of generators and relations, starting with the objects and maps of $\mathcal{M}$ and formally adding inverses of the maps in $\mathcal{W}$.

For a precofibration category $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$, recall that $\mathcal{M}_{cof}$ denotes the full subcategory of cofibrant objects of $\mathcal{M}$. In $\mathcal{M}_{cof}$ we define the homotopy relation $\simeq$ on maps, and show that it is an equivalence relation. The localization of $\mathcal{M}_{cof}$ modulo homotopy is denoted $\pi\mathcal{M}_{cof}$. We show that the class of weak equivalences between cofibrant objects admits a calculus of fractions in $\pi\mathcal{M}_{cof}$. As a consequence we obtain a description of $\mathbf{ho}\mathcal{M}_{cof}$ whereby any map in $\mathbf{ho}\mathcal{M}_{cof}$ can be written (up to homotopy!) as a 'left fraction' ft^{-1}, with t a weak equivalence. The theory of homotopic maps and calculus of fractions up to homotopy as described here is due to Brown [**Bro74**].

We then show that the functor $\mathbf{ho}\mathcal{M}_{cof} \to \mathbf{ho}\mathcal{M}$ is an equivalence of categories (Anderson, [**And78**]). In fact, we will develop an axiomatic description of *cofibrant approximation* functors $t : \mathcal{M}^{'} \to \mathcal{M}$, which are modelled on the properties of the inclusion $\mathcal{M}_{cof} \to \mathcal{M}$. A cofibrant approximation functor induces an isomorphism at the level of the homotopy category.

Cofibrant aproximation functors will resurface later in Section 6.5, when we will reduce the construction of homotopy colimits indexed by arbitrary small diagrams to the construction of homotopy colimits indexed by small direct categories.

We will show that the homotopy relation extends to maps of $\mathcal{M}$, by mandating that $f \simeq g$ if and only if f, g become equal in the homotopy category $\mathbf{ho}\mathcal{M}$. We prove that the new equivalence relation is backward compatible to homotopy relation on maps of $\mathcal{M}_{cof}$.

As an application of the theory developed in this chapter, we show that if $\mathcal{M}_k$ is a small set of precofibration categories, then the functor $\mathbf{ho}(\times\mathcal{M}_k) \to \times\mathbf{ho}\mathcal{M}_k$ is an *isomorphism* of categories.

We will also prove for a cofibration category $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ that the saturation $\overline{\mathcal{W}}$ of the weak equivalences yields again a cofibration category $(\mathcal{M}, \overline{\mathcal{W}}, \mathcal{C}of)$. This is a result due to Cisinski, [**Cis02**].

3.1. Universes and smallness

If $\mathcal{U} \subset \mathcal{U}^+$ are two universes, a $(\mathcal{U}^+, \mathcal{U})$-category $\mathcal{C}$ has by definition a $\mathcal{U}^+$-small set of objects $Ob\mathcal{C}$ and $\mathcal{U}$-small Hom-sets. A category $\mathcal{C}$ is $\mathcal{U}$-small if it is a $(\mathcal{U}, \mathcal{U})$-category. A functor $f : \mathcal{C}_1 \to \mathcal{C}_2$ is $\mathcal{U}$-small if $\mathcal{C}_1, \mathcal{C}_2$ are $\mathcal{U}$-small.

For a fixed universe pair $\mathcal{U} \subset \mathcal{U}^+$, the $\mathcal{U}^+$-small sets are also referred to as *classes*, while the $\mathcal{U}$-small sets are referred to simply as *sets*, or *small sets*. We will denote $\mathcal{C}at$ (resp. $\mathcal{C}AT$) the category of $\mathcal{U}$-small categories and functors (resp. $(\mathcal{U}^+, \mathcal{U})$ categories and functors).

3.2. The homotopy category

Suppose we have a $(\mathcal{U}^+, \mathcal{U})$-category $\mathcal{M}$ with a class of weak equivalences $\mathcal{W}$.

The homotopy category $\mathbf{ho}\mathcal{M}$ by definition is a $\mathcal{U}^+$-small category which comes with a localization functor $\gamma_{\mathcal{M}} : \mathcal{M} \to \mathbf{ho}\mathcal{M}$, with the properties that

1. $\gamma_{\mathcal{M}}$ sends weak equivalences to isomorphisms
2. for any other $\mathcal{U}^+$-small functor $\gamma^{'} : \mathcal{M} \to \mathcal{M}^{'}$ that sends weak equivalences to isomorphisms, there exists a unique $\mathcal{U}^+$-small functor $\delta : \mathbf{ho}\mathcal{M} \to \mathcal{M}^{'}$ such that $\delta\gamma_{\mathcal{M}} = \gamma^{'}$

The homotopy category is sometimes also denoted $\mathcal{M}[\mathcal{W}^{-1}]$. A theorem of Gabriel-Zisman [**GZ67**] states that the homotopy category exists, and is *isomorphic* to the category with

1. objects $Ob\mathcal{M}$
2. maps between X and Y the equivalence classes of zig-zags

$$X = A_0 \overset{f_1}{\text{———}} A_1 \overset{f_2}{\text{———}} \cdots \overset{f_n}{\text{———}} A_n = Y$$

   where f_i are maps in $\mathcal{M}$ going either forward or backward, and the maps going backward are in $\mathcal{W}$, where
3. two zig-zags are equivalent if they can be obtained one from another by a finite number of the following three operations and their inverses:
   (a) skipping elements $A \overset{1_A}{\to} A$ or $A \overset{1_A}{\leftarrow} A$
   (b) replacing $A \overset{f}{\to} B \overset{g}{\to} C$ with $A \overset{gf}{\to} C$ or $A \overset{f}{\leftarrow} B \overset{g}{\leftarrow} C$ with $A \overset{fg}{\leftarrow} C$
   (c) skipping elements $A \overset{f}{\to} B \overset{f}{\leftarrow} A$ or $A \overset{f}{\leftarrow} B \overset{f}{\to} A$
4. composition of maps is induced by the concatenation of zig-zags

The saturation $\overline{\mathcal{W}}$ of the class of weak equivalences by definition is the class of maps of $\mathcal{M}$ that become isomorphisms in $\mathbf{ho}\mathcal{M}$. The saturation $\overline{\mathcal{W}}$ is closed under composition, and includes the isomorphisms of $\mathcal{M}$.

If $\mathcal{M}$ is a $(\mathcal{U}^+, \mathcal{U})$-category with a class of weak equivalences $\mathcal{W}$, then $\mathbf{ho}\mathcal{M} = \mathcal{M}[\mathcal{W}^{-1}]$ is not necessarily a $(\mathcal{U}^+, \mathcal{U})$-small category. For example, if $\mathcal{M}$ is the $(\mathcal{U}^+, \mathcal{U})$-category of sets $\mathcal{S}et$, then $\mathcal{W} = \mathcal{C}of = \mathcal{F}ib = \mathcal{S}et$ gives $\mathcal{M}$ a structure of both a cofibration and a fibration category, but $\mathcal{S}et[\mathcal{S}et^{-1}]$ is only $\mathcal{U}^+$-small.

If a $(\mathcal{U}^+, \mathcal{U})$-category $\mathcal{M}$ admits a Quillen model category structure on the other hand, it is a known result that $\mathbf{ho}\mathcal{M}$ is again a $(\mathcal{U}^+, \mathcal{U})$-category.

One benefit of the intrinsic description of the homotopy category in terms of zig-zags is that it shows that the homotopy category remains the same *independent of the universe pair* $(\mathcal{U}^+, \mathcal{U})$ that we start with.

For the rest of the text, it is convenient to restrict ourselves to the case when $\mathcal{W}$ is closed under composition and includes the identity maps of $\mathcal{M}$. In the language of Def. 1.7.2, these are the *categories with weak equivalences* ($\mathcal{M}$, $\mathcal{W}$).

3.3. Homotopic maps and $\pi\mathcal{M}$

From this point on, we will restrict our focus to precofibration categories and (dually) to prefibration categories. Let $\mathcal{M}$ be a precofibration category, and let $f, g : A \rightrightarrows B$ be two maps with A, B cofibrant. A *homotopy* from f to g is a commutative diagram

$$\begin{tikzcd}
A \sqcup A \arrow[r, "f+g"] \arrow[d, tail, "i_0+i_1"'] & B \arrow[d, tail, "\sim"', "b"] \\
IA \arrow[r, "H"] & B^{'}
\end{tikzcd}$$

with IA a cylinder of A and b a trivial cofibration. We thus have that $Hi_0 = bf$ and $Hi_1 = bg$. The map H is called the *homtopy map* between f and g, and we say that the homotopy goes through the cylinder IA and through the trivial cofibration b . We write $f \simeq g$ to say that f, g are homotopic.

A homotopy $f \simeq g$ with $B = B^{'}$ and $b = 1_B$ is called a *strict homotopy.* We should be careful to point out that Brown uses the notation $\simeq$ differently - to denote what we call strict homotopy.

We have defined the homotopy relation in terms of the weak equivalences and cofibrations of $\mathcal{M}$, but it turns out that the homotopy relation depends only on the weak equivalences. In fact, we will show that $f \simeq g$ if and only if the maps f, g become equal in **ho**$\mathcal{M}$.

In view of this, we will make an abuse of language and use the same notation $\simeq$ to denote homotopy in prefibration categories. Let $\mathcal{M}$ be a prefibration category, and let $f, g : B \rightrightarrows A$ be two maps with A, B fibrant. A homotopy from f to g is a commutative diagram

$$\begin{tikzcd}
B^{'} \arrow[r, "H"] \arrow[d, two heads, "b"', "\sim"] & A^I \arrow[d, two heads, "{(p_0, p_1)}"] \\
B \arrow[r, "{(f, g)}"] & A \times A
\end{tikzcd}$$

with A^I a path object of A and b a trivial fibration. If f, g are homotopic we therefore also write $f \simeq g$. A strict homotopy is a homotopy with $B = B^{'}$ and $b = 1_B$.

Let A be a cofibrant object in the precofibration category $\mathcal{M}$. If IA is a cylinder of A, a *refinement* of IA consists of a cylinder $I^{'}A$ and a trivial cofibration $j : IA \to I^{'}A$ such that the diagram below commutes

$$\begin{tikzcd}
 & A \sqcup A \arrow[dl, tail, "i_0+i_1"'] \arrow[dr, tail, "i_0^{'}+i_1^{'}"] & \\
IA \arrow[rr, tail, "j"] \arrow[dr, "p"', "\sim"] & & I^{'}A \arrow[dl, "p^{'}", "\sim"'] \\
 & A &
\end{tikzcd}$$

If $f \simeq g : A \to B$ through $I'A$ with homotopy map H', then $H'j$ defines a homotopy through IA.

Note that given any two cylinders IA, $I'A$, we can construct a common refinement $I''A$ by factoring $IA \sqcup_{A \sqcup A} I'A \to A$ as a cofibration $IA \sqcup_{A\sqcup A} I'A \to I''A$ followed by a weak equivalence $I''A \to A$.

This allows us to prove the following lemma:

Lemma 3.3.1.

(1) *Let $\mathcal{M}$ be a precofibration category, and $f \simeq g : A \to B$ with A, B cofibrant. Let IA be a cylinder of A. Then one can construct a homotopy $f \simeq g$ through IA.*

(2) *Let $\mathcal{M}$ be a prefibration category, and $f \simeq g : B \to A$ with A, B fibrant. Let A^I be a path object for A. Then one can construct a homotopy $f \simeq g$ through A^I.*

Proof. We only prove (1). Assume that there exists a homotopy $f \simeq g$ through another cylinder $I'A$. Construct a common refinement $I''A$ of IA and $I'A$. To prove (1), it suffices to construct a homotopy through the refinement $I''A$. In the commutative diagram below

$$\begin{tikzcd}
A \sqcup A \arrow[r, "f+g"] \arrow[d, tail, "i'_0+i'_1"'] & B \arrow[d, tail, "b", "\sim"'] \\
I'A \arrow[r, "H'"] \arrow[d, tail, "j"', "\sim"] & B' \arrow[d, tail, "b'", "\sim"'] \\
I''A \arrow[r, "H''"] & B''
\end{tikzcd}$$

H', B' and b define the homotopy $f \simeq g$, and j is the cylinder refinement map from $I'A$ to $I''A$. We construct B'' as the pushout of j and H', and we have the desired homotopy H'', B'' and $b'b$ through $I''A$. □

We can now prove

Theorem 3.3.2.

(1) *If $\mathcal{M}$ is a precofibration category, then $\simeq$ is an equivalence relation in $\mathcal{M}_{cof}$. Furthermore if $f \simeq g : A \to B$ with A, B cofibrant then*
    (a) *If $h : B \to C$ with C cofibrant then $hf \simeq hg$*
    (b) *If $h : C \to A$ with C cofibrant then $fh \simeq gh$*

(2) *If $\mathcal{M}$ is a prefibration category, then $\simeq$ is an equivalence relation in $\mathcal{M}_{fib}$. Furthermore if $f \simeq g : A \to B$ with A, B fibrant then*
    (a) *If $h : B \to C$ with C fibrant then $hf \simeq hg$*
    (b) *If $h : C \to A$ with C fibrant then $fh \simeq gh$*

Proof. We only prove (1). Clearly $\simeq$ is symmetric and reflexive.

To see that $\simeq$ is transitive, assume $f \simeq g \simeq h : A \to B$. From the previous lemma, we may assume that both homotopies go through the same cylinder IA. Denote these homotopies H_1, B'_1, b_1 and H_2, B'_2, b_2. Taking the pushout of b_1 and b_2, we obtain homotopies H'_1, B', b and H'_2, B', b.

Notice that both diagrams below are pushouts

$$\begin{tikzcd} A \arrow[r, tail, "i_0", "\sim"'] \arrow[d, tail, "i_1"', "\sim"] & IA \arrow[d, tail, "\sim"] \\ IA \arrow[r, tail, "\sim"] & IA \sqcup_A IA \end{tikzcd} \qquad \begin{tikzcd} A \sqcup A \sqcup A \arrow[r, tail, "A\sqcup(i_0+i_1)"] \arrow[d, tail, "(i_0+i_1)\sqcup A"'] & A \sqcup IA \arrow[d, tail, "i_0\sqcup IA"] \\ IA \sqcup A \arrow[r, tail, "IA\sqcup i_1"] & IA \sqcup_A IA \end{tikzcd}$$

The factorization $\nabla : \ A \sqcup A \overset{i_0\sqcup i_1}{\rightarrowtail} IA \sqcup_A IA \xrightarrow[\sim]{p+p} A$ is a cylinder: the map $p+p$ is a weak equivalence because of the first diagram, and the map $i_0 \sqcup i_1$ is a cofibration because it is the second diagram precomposed with the cofibration $(i_0, i_2) : A \sqcup A \to A \sqcup A \sqcup A$. The commutative diagram below then defines a homotopy from f to h.

$$\begin{tikzcd} A \sqcup A \arrow[r, "f+h"] \arrow[d, tail, "i_0\sqcup i_1"'] & B \arrow[d, tail, "\sim"', "b"] \\ IA \sqcup_A IA \arrow[r, "H_1^{'}+H_2^{'}"] & B^{'} \end{tikzcd}$$

To prove (a), let $IA, H, B^{'}, b$ define a homotopy $f \simeq g$. In the diagram below let $C^{'}$ be the pushout of b, h.

$$\begin{tikzcd} A \sqcup A \arrow[r, "f+g"] \arrow[d, tail, "i_0\sqcup i_1"'] & B \arrow[r, "h"] \arrow[d, tail, "\sim"', "b"] & C \arrow[d, tail, "\sim"', "c"] \\ IA \arrow[r, "H"] & B^{'} \arrow[r, "h^{'}"] & C^{'} \end{tikzcd}$$

The outer rectangle defines a homotopy $hf \simeq hg$.

For (b), use Lemma 1.5.3 to construct relative cylinders IC, IA along h. From Lemma 3.3.1, we can construct a homotopy $f \simeq g$ through IA. Precomposing this homotopy with $IC \to IA$ yields a homotopy $fh \simeq gh$. □

As a consequence of the previous theorem, for a precofibration category we can factor $\mathcal{M}_{cof}$ modulo $\simeq$ and obtain a category $\pi\mathcal{M}_{cof}$, with same objects as $\mathcal{M}_{cof}$ and with $Hom_{\pi\mathcal{M}_{cof}}(A,B) = Hom_{\mathcal{M}_{cof}}(A,B)_{/\simeq}$. For weak equivalences in $\pi\mathcal{M}_{cof}$ we consider homotopy classes of maps that have one (and hence all) representatives weak equivalece maps of $\mathcal{M}$.

For a prefibration category, $\pi\mathcal{M}_{fib}$ denotes the factorization of $\mathcal{M}_{fib}$ modulo $\simeq$. Weak equivalences in $\pi\mathcal{M}_{fib}$ are by definition homotopy classes of maps that have one (and hence all) representatives weak equivalence maps of $\mathcal{M}$.

3.4. Homotopy calculus of fractions

We will show that for a precofibration category $\mathcal{M}$, the category $\pi\mathcal{M}_{cof}$ admits a calculus of left fractions in the sense of Gabriel-Zisman with respect to weak equivalences. A nice way to phrase this is to say that $\mathcal{M}_{cof}$ admits a *homotopy* calculus of left fractions. Dually, given a prefibration category $\mathcal{M}$, its category of fibrant objects $\mathcal{M}_{fib}$ admits a homotopy calculus of right fractions.

We say that a category with weak equivalences ($\mathcal{M}$, $\mathcal{W}$) satisfies the 2 out of 3 axiom provided that for any composable morphisms f, g of $\mathcal{M}$, if two of f, g, gf are in $\mathcal{W}$ then so is the third. Weak equivalences in a precofibration category $\mathcal{M}$ satisfy the 2 out of 3 axiom, and so do weak equivalences in $\pi\mathcal{M}_{cof}$.

For categories with weak equivalences $(\mathcal{M}, \mathcal{W})$ satisfying the 2 out of 3 axiom, the Gabriel-Zisman calculus of fractions takes a simplified form that we recall below. The general case of calculus of fractions - when weak equivalences do not necessarily satisfy the 2 out of 3 axiom - is described in [**GZ67**] at pag. 12.

THEOREM 3.4.1 (Simplified calculus of left fractions). *Let $(\mathcal{M}, \mathcal{W})$ be a category with weak equivalences satisfying the 2 out of 3 axiom. If any full diagram with weak equivalence map a*

(3.1)
$$\begin{tikzcd}
A \arrow[r] \arrow[d, "a"', "\sim"] & B \arrow[d, dashed, "\sim"', "b"] \\
A' \arrow[r, dashed] & B'
\end{tikzcd}$$

extends to a commutative diagram with b a weak equivalence, then

(1) *Each map in $Hom_{\mathbf{ho}\mathcal{M}}(A, B)$ can be written as a left fraction $s^{-1}f$*

$$A \xrightarrow{f} B' \xleftarrow[\sim]{s} B$$

with s a weak equivalence.

(2) *Two fractions $s^{-1}f$, $t^{-1}g$ are equal in $\mathbf{ho}\mathcal{M}$ if and only if there exist weak equivalences $s^{'}, t^{'}$ as in the diagram below*

(3.2)
$$\begin{tikzcd}
 & & B''' & & \\
 & B' \arrow[ur, "s'", "\sim"'] & & B'' \arrow[ul, "t'"', "\sim"] & \\
A \arrow[ur, "f"] \arrow[urrr, "g"' near end] & & & & B \arrow[ul, "t"', "\sim"] \arrow[ulll, "s", "\sim"' near end]
\end{tikzcd}$$

so that $s^{'}s = t^{'}t$ and $s^{'}f = t^{'}g$.

If furthermore weak equivalences are left cancellable, in the sense that for any pair of maps $f, g : A \rightrightarrows B$ and weak equivalence $h : B \to B^{'}$ with $hf = hg$ we have $f = g$, then

(3) *Two maps $f, g : A \rightrightarrows B$ are equal in $\mathbf{ho}\mathcal{M}$ if and only if $f = g$.*

The dual result for right fractions is

THEOREM 3.4.2 (Simplified calculus of right fractions). *Let $(\mathcal{M}, \mathcal{W})$ be a category with weak equivalences satisfying the 2 out of 3 axiom. If any full diagram with weak equivalence map a*

(3.3)
$$\begin{tikzcd}
B' \arrow[r, dashed] \arrow[d, dashed, "b"', "\sim"] & A' \arrow[d, "\sim"', "a"] \\
B \arrow[r] & A
\end{tikzcd}$$

extends to a commutative diagram with b a weak equivalence, then

(1) *Each map in $Hom_{\mathbf{ho}\mathcal{M}}(A, B)$ can be written as a right fraction fs^{-1}*

$$A \xleftarrow[\sim]{s} A' \xrightarrow{f} B$$

with s a weak equivalence.

(2) *Two fractions* fs^{-1}, gt^{-1} *are equal in* $\mathbf{ho}\mathcal{M}_{fib}$ *if and only if there exist weak equivalences* $s^{'}, t^{'}$ *as in the diagram below*

(3.4)
$$\begin{tikzcd}
 & & A^{'''} \arrow[dl, "s^{'}"', "\sim"] \arrow[dr, "t^{'}", "\sim"'] & & \\
 & A^{'} \arrow[dl, "s"', "\sim"] \arrow[drrr, "f"' near end] & & A^{''} \arrow[dlll, "t" near start, "\sim"'] \arrow[dr, "g"] & \\
A & & & & B
\end{tikzcd}$$

so that $ss^{'} = tt^{'}$ *and* $fs^{'} = gt^{'}$.

If furthermore weak equivalences are right cancellable, in the sense that for any pair of maps $f, g : A \rightrightarrows B$ *and weak equivalence* $h : A^{'} \to A$ *with* $fh = gh$ *we have* $f = g$, *then*

(3) *Two maps* $f, g : A \rightrightarrows B$ *are equal in* $\mathbf{ho}\mathcal{M}$ *if and only if* $f = g$.

The two statements being dual, we only need to supply a proof for the first Thm. 3.4.1.

Proof. We construct a category $\mathcal{C}$ with objects $Ob\mathcal{M}$, and maps defined in terms of fractions as explained below.

Fix two objects A and B. Consider the set of fractions $s^{-1}f$ as in (1). Denote $\sim$ the relation defined by (2) on the set of fractions $s^{-1}f$ from A to B. The relation $\sim$ is clearly reflexive and symmetric.

To see that the relation is transitive, assume ${s_1}^{-1}f_1 \sim {s_2}^{-1}f_2 \sim {s_3}^{-1}f_3$. We get a commutative diagram

$$\begin{tikzcd}
B_1^{''} \arrow[rr, "u_1"] & & B^{'''} & & B_2^{''} \arrow[ll, "u_2"'] \\
B_1^{'} \arrow[u, "t_1"] & & B_2^{'} \arrow[ull, "t_2"] \arrow[urr, "t_3"'] & & B_3^{'} \arrow[u, "t_4"'] \\
A \arrow[u, "f_1"] \arrow[urr, "f_2"] \arrow[urrrr, "f_3"'] & & & & B \arrow[ullll, "s_1"] \arrow[ull, "s_2"'] \arrow[u, "s_3"']
\end{tikzcd}$$

where the weak equivalences t_1, t_2 exist since ${s_1}^{-1}f_1 \sim {s_2}^{-1}f_2$, the weak equivalences t_3, t_4 exist since ${s_2}^{-1}f_2 \sim {s_3}^{-1}f_3$ and the weak equivalences u_1, u_2 exist from the hypothesis (3.1) applied to t_2, t_3. The compositions u_1t_1, u_2t_4 satisfy $u_1t_1f_1 = u_2t_4f_3$ and $u_1t_1s_1 = u_2t_4s_3$ which shows that ${s_1}^{-1}f_1 \sim {s_3}^{-1}f_3$, and we have proved that $\sim$ is transitive.

We let $Hom_{\mathcal{C}}(A, B)$ denote the set of fractions from A to B modulo the equivalence relation $\sim$. Given three objects A, B, C, we define composition $Hom_{\mathcal{C}}(A, B) \times Hom_{\mathcal{C}}(B, C) \to Hom_{\mathcal{C}}(A, C)$ as follows. Given fractions $s^{-1}f$, $t^{-1}g$

$$\begin{tikzcd}
& & C'' & & \\
& B' \arrow[ur, "g'"] & & C' \arrow[ul, "t'"', "\sim"] & \\
A \arrow[ur, "f"] & & B \arrow[ul, "s"', "\sim"] \arrow[ur, "g"'] & & C \arrow[ul, "t"', "\sim"]
\end{tikzcd}$$

we use hypothesis (3.1) to construct an object C'', a map g' and a weak equivalence t' such that $g's = t'g$, and then we define $t^{-1}g \circ s^{-1}f$ as $(t't)^{-1}(g'f)$. The proof that the definition of composition does not depend on the choices involved is entirely similar to the proof that $\sim$ is transitive (we leave this verification to the reader). Given an object A, the fraction $(1_A)^{-1}1_A$ is an identity element for the composition.

Define a functor $F : \mathcal{M} \to \mathcal{C}$, by $F(A) = A$ and by sending $f : A \to B$ to the fraction $F(f) = (1_B)^{-1}f$. It is not hard to see that F is compatible with composition, and that if $s : A \to B$ is a weak equivalence in $\mathcal{M}$ then $F(s)$ has $s^{-1}1_B$ as an inverse.

Since F sends weak equivalences to isomorphisms, it descends to a functor $\overline{F} : \mathbf{ho}\mathcal{M} \to \mathcal{C}$, and it is straightforward to check that any other functor $\mathbf{ho}\mathcal{M} \to \mathcal{C}'$ factors uniquely through $\overline{F}$. It follows that the category $\mathcal{C}$ we constructed is equivalent to $\mathbf{ho}\mathcal{M}$, and the proof of (1) and (2) is complete.

If $f, g : A \rightrightarrows B$ are equal in $\mathbf{ho}\mathcal{M}$, then by (2) there exists a weak equivalence $h : B \to B'$ such that $hf = hg$. If weak equivalences are left cancellable, then $f = g$, and this proves (3). ☐

The next result shows that given a precofibration category $\mathcal{M}$, the category $\pi\mathcal{M}_{cof}$ satisfies the hypotheses of Thm. 3.4.1 up to homotopy.

THEOREM 3.4.3.

(1) *In a precofibration category*
   (a) *Any full diagram with cofibrant objects and weak equivalence a*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "a"', "\sim"] & B \arrow[d, dashed, "\sim"', "b"] \\
A' \arrow[r, dashed] & B'
\end{tikzcd}$$

   extends to a (strictly) homotopy commutative diagram with b a weak equivalence and B' cofibrant
   (b) *For any $f, g : A \rightrightarrows B$ with A, B cofibrant*
      (i) *If there is a weak equivalence $a : A' \to A$ with A' cofibrant such that $fa \simeq ga$, then $f \simeq g$*
      (ii) *If there is a weak equivalence $b : B \to B'$ with B' cofibrant such that $bf \simeq bg$, then $f \simeq g$*

(2) *In a prefibration category*
   (a) *Any full diagram with fibrant objects and weak equivalence a*

$$\begin{tikzcd}
B' \arrow[r, dashed] \arrow[d, dashed, "b"', "\sim"] & A' \arrow[d, "\sim"', "a"] \\
B \arrow[r] & A
\end{tikzcd}$$

extends to a (strictly) homotopy commutative diagram with b a weak equivalence and $B^{'}$ fibrant

(b) *For any $f, g : A \rightrightarrows B$ with A, B fibrant*
  (i) *If there is a weak equivalence $a : A^{'} \to A$ with $A^{'}$ fibrant such that $fa \simeq ga$, then $f \simeq g$*
  (ii) *If there is a weak equivalence $b : B \to B^{'}$ with $B^{'}$ fibrant such that $bf \simeq bg$, then $f \simeq g$*

Proof. To prove (1) (a), denote $f : A \to B$ and let IA be a cylinder of A. The diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"', "\sim"] & B \arrow[d, "b"] \\
A^{'} \arrow[r] & A^{'} \sqcup_A IA \sqcup_A B
\end{tikzcd}$$

is strictly homotopy commutative. It remains to show that b is a weak equivalence.

Denote $F : IA \to IA \sqcup_A B$. The diagram

$$\begin{tikzcd}
A \sqcup A \arrow[r, "1 \sqcup f"] \arrow[d, tail, "i_0 + i_1"'] & A \sqcup B \arrow[d, tail, "\pi_{IA \sqcup_A B} \circ (i_0 \sqcup 1_B)"] \\
IA \arrow[r, "F"] & IA \sqcup_A B
\end{tikzcd}$$

is a pushout, so the right vertical map is a cofibration and therefore $Fi_0 : A \to IA \sqcup_A B$ is a cofibration.

The map b factors as $B \to IA \sqcup_A B \to A^{'} \sqcup_A IA \sqcup_A B$. The first factor is a pushout of $i_1 : A \to IA$, therefore a trivial cofibration. The second factor is a pushout of the weak equivalence a by the cofibration Fi_0, therefore a weak equivalence by excision.

To prove (1) (b) (i), pick (Lemma 1.5.3) relative cylinders $IA^{'}, IA$ over a. By Lemma 3.3.1, there exists a homotopy $fa \simeq ga$ through $IA^{'}$ and through a trivial cofibration $b^{''}$. We get a commutative diagram

$$\begin{tikzcd}
A^{'} \sqcup A^{'} \arrow[r, "a \sqcup a", "\sim"'] \arrow[d, tail, "i_0^{'} + i_1^{'}"'] & A \sqcup A \arrow[r, "f+g"] \arrow[d, tail, "i_0 + i_1"'] & B \arrow[d, tail, "\sim"', "b^{''}"] \\
IA^{'} \arrow[r, "h_1", "\sim"'] & IA^{'} \sqcup_{A^{'} \sqcup A^{'}} A \sqcup A \arrow[r, "h_2"] \arrow[d, tail, "\sim"', "j"] & B^{''} \arrow[d, tail, "\sim"', "b^{'}"] \\
 & IA \arrow[r, "H"] & B^{'}
\end{tikzcd}$$

where the map j is a cofibration because $IA^{'}, IA$ are relative cylinders. But j is actualy a trivial cofibration. To see that, notice that since a is a weak equivalence and $A, A^{'}$ are cofibrant, $a \sqcup a$ is also a weak equivalence and by excision so is h_1. The map $IA^{'} \to IA$ is a weak equivalence since a is, and by the 2 out of 3 Axiom the map j is a weak equivalence.

We define B' as the pushout of j by h_2. The map b' is therefore a trivial cofibration. We let $b = b'b''$, and $IA, H, B', 1_{B'}$ defines a homotopy $bf \simeq bg$.

Let us now prove (1) (b) (ii). Pick a homotopy $bf \simeq bg$ going through the cylinder IA and through the trivial cofibration b'. In the diagram below

$$\begin{tikzcd}
A \sqcup A \arrow[r, "f+g"] \arrow[d, tail, "i_0+i_1"'] & B \arrow[rr, "b", "\sim"'] \arrow[d, tail, "b_1"] & & B' \arrow[d, tail, "b'", "\sim"'] \\
IA \arrow[r, "h_1"] & B_1 \arrow[r, tail, "h_2"] & B_2 \arrow[r, "h_3", "\sim"'] & B''
\end{tikzcd}$$

construct b_1 as the pushout of $i_0 + i_1$ and h_2, h_3 as the factorization of $B_1 \to B''$ as a cofibration followed by a weak equivalence. Notice that $h_2 b_1$ is a trivial cofibration, and we have constructed a homotopy $f \simeq g$ with homotopy map $h_2 h_1$.

The proof of (2) is dual and is omitted.

□

The Gabriel-Zisman left calculus of fractions applies therefore to the case of $\pi\mathcal{M}_{cof}$, if $\mathcal{M}$ is a precofibration category.

Theorem 3.4.4.

(1) *Let $\mathcal{M}$ be a precofibration category, and A, B be two cofibrant objects.*
   (a) *Each map in $Hom_{\mathbf{ho}\mathcal{M}_{cof}}(A, B)$ can be written as a left fraction $s^{-1}f$*

$$\begin{tikzcd}
A \arrow[r, "f"] & B' & B \arrow[l, "s"', "\sim"]
\end{tikzcd}$$

   with s a weak equivalence and B' cofibrant.
   (b) *Two fractions $s^{-1}f$, $t^{-1}g$ are equal in $\mathbf{ho}\mathcal{M}_{cof}$ if and only if there exist weak equivalences s', t' as in the diagram (3.2) with B''' cofibrant so that $s's \simeq t't$ and $s'f \simeq t'g$.*
   (c) *Two maps $f, g : A \rightrightarrows B$ are equal in $\mathbf{ho}\mathcal{M}_{cof}$ if and only if they are homotopic.*

(2) *Let $\mathcal{M}$ be a prefibration category, and A, B be two fibrant objects.*
   (a) *Each map in $Hom_{\mathbf{ho}\mathcal{M}_{fib}}(A, B)$ can be written as a right fraction fs^{-1}*

$$\begin{tikzcd}
A & A' \arrow[l, "s"', "\sim"] \arrow[r, "f"] & B
\end{tikzcd}$$

   with s a weak equivalence and A' fibrant.
   (b) *Two fractions fs^{-1}, gt^{-1} are equal in $\mathbf{ho}\mathcal{M}_{fib}$ if and only if there exist weak equivalences s', t' as in the diagram (3.4) with A''' fibrant so that $ss' \simeq tt'$ and $fs' \simeq gt'$.*
   (c) *Two maps $f, g : A \rightrightarrows B$ are equal in $\mathbf{ho}\mathcal{M}_{fib}$ if and only if they are homotopic.*

Proof. This is a consequence of Thm. 3.4.1 and Thm. 3.4.3. □

We can also prove a version Thm. 3.4.4 that describes $\mathbf{ho}\mathcal{M}_{cof}$ in terms of fractions $s^{-1}f$ with f a cofibration and s a trivial cofibration:

Theorem 3.4.5.

(1) *Let $\mathcal{M}$ be a precofibration category, and A, B be two cofibrant objects.*
   (a) *Each map in $Hom_{\mathbf{ho}\mathcal{M}_{cof}}(A, B)$ can be written as a left fraction $s^{-1}f$*

$$\begin{tikzcd} A \arrow[r, tail, "f"] & B' & B \arrow[l, tail, "s"', "\sim"] \end{tikzcd}$$

with f a cofibration and s a trivial cofibration

   (b) *Two such fractions $s^{-1}f$, $t^{-1}g$ are equal in $\mathbf{ho}\mathcal{M}_{cof}$ if and only if there exist trivial cofibrations s', t' as in the diagram (3.2) such that $s's \simeq t't$ and $s'f \simeq t'g$.*

(2) *Let $\mathcal{M}$ be a prefibration category, and A, B be two fibrant objects.*

   (a) *Each map in $Hom_{\mathbf{ho}\mathcal{M}_{fib}}(A, B)$ can be written as a right fraction fs^{-1}*

$$\begin{tikzcd} A & A' \arrow[l, two heads, "s"', "\sim"] \arrow[r, two heads, "f"] & B \end{tikzcd}$$

with f a fibration and s a trivial fibration

   (b) *Two fractions fs^{-1}, gt^{-1} are equal in $\mathbf{ho}\mathcal{M}_{fib}$ if and only if there exist trivial fibrations s', t' as in the diagram (3.4) such that $ss' \simeq tt'$ and $fs' \simeq gt'$.*

Proof. We only prove (1). Denote $\sim$ the equivalence relation defined by Thm. 3.4.4 (1) (b), and $\sim_{cof}$ the equivalence relation defined by the current Theorem's (1) (b). It suffices to show that:

1. Any fraction $s^{-1}f$ with s a weak equivalence is $\sim$ equivalent to a fraction $t^{-1}g$, with g a cofibration and t a trivial cofibration
2. If two fractions $s^{-1}f, t^{-1}g$ with f, g cofibrations and s, t trivial cofibrations are $\sim$ equivalent, then they are $\sim_{cof}$ equivalent.

To prove (1), construct the commutative diagram

$$\begin{tikzcd}
 & B' & \\
A \arrow[ur, "f"] \arrow[dr, tail, "i_A"'] & B'' \arrow[u, "v"', "\sim"] & B \arrow[ul, "s"', "\sim"] \arrow[dl, tail, "i_B"] \\
 & A \sqcup B \arrow[u, tail, "u"'] &
\end{tikzcd}$$

where vu is the factorization of $f+s$ as a cofibration followed by a weak equivalence. Define $g = ui_A$ and $t = ui_b$. The map g is a cofibration, the map t is a trivial cofibration and the trivial map v yields the desired $\sim$ equivalence between the fractions $s^{-1}f$ and $t^{-1}g$.

To prove (2), in the diagram below

$$\begin{tikzcd}
 & & B''' & & \\
 & B' \arrow[ur, "s'", "\sim"'] & & B'' \arrow[ul, "t'"', "\sim"] & \\
A \arrow[ur, "f"] \arrow[urrr, "g"] & & & & B \arrow[ulll, "s", "\sim"'] \arrow[ul, "t"', "\sim"]
\end{tikzcd}$$

construct s', t' as the pushouts of t, s. We therefore have $s's = t't$, and therefore by Thm. 3.4.4 we get that $s'f$ and $t'g$ are equal in $\mathbf{ho}\mathcal{M}_{cof}$. By Thm. 3.4.4 (1) (c), we therefore get that $s^{'}f \simeq t^{'}g$, which means that the fractions $s^{-1}f$, $t^{-1}g$ are $\sim_{cof}$ equivalent. ☐

We have proved that in a precofibration category $\mathcal{M}$, two maps between cofibrant objects $f, g : A \rightrightarrows B$ are homotopic if and only if they become equal in $\mathbf{ho}\mathcal{M}_{cof}$. But more is true - in fact they are homotopic if and only if they become equal in $\mathbf{ho}\mathcal{M}$. This will become apparent once we prove further down that $\mathbf{ho}\mathcal{M}_{cof} \to \mathbf{ho}\mathcal{M}$ is an equivalence of categories.

3.5. Fibrant and cofibrant approximations

We are interested in a class of precofibration category functors which induce isomorphisms when passed to the homotopy category, and which serve as resolutions for computing left derived functors. These are the *cofibrant approximation* functors, defined below. The cofibrant approximation functors should be thought of as an axiomatization of the inclusion $\mathcal{M}_{cof} \to \mathcal{M}$, where $\mathcal{M}$ is a precofibration category.

Definition 3.5.1. (Cofibrant approximation functors) Let $\mathcal{M}$ be a precofibration category. A functor $t : \mathcal{M}^{'} \to \mathcal{M}$ is a *cofibrant approximation* if $\mathcal{M}^{'}$ is a precofibration category with all objects cofibrant and

CFA1: t preserves the initial object and cofibrations

CFA2: A map f of $\mathcal{M}^{'}$ is a weak equivalence if and only if tf is a weak equivalence

CFA3: If $A \to B$, $A \to C$ are cofibrations of $\mathcal{M}^{'}$ then the natural map $tB \sqcup_{tA} tC \to t(B \sqcup_A C)$ is an isomorphism

CFA4: Any map $f : tA \to B$ factors as $f = r \circ tf^{'}$ with $f^{'}$ a cofibration of $\mathcal{M}^{'}$ and r a weak equivalence of $\mathcal{M}$.

A cofibrant approximation functor in particular sends any object to a cofibrant object, and sends trivial cofibrations to trivial cofibrations. If $\mathcal{M}$ is a precofibration category, then the inclusion $\mathcal{M}_{cof} \to \mathcal{M}$ is a cofibrant approximation.

The dual definition for prefibration categories is

Definition 3.5.2. (Fibrant approximation functors) Let $\mathcal{M}$ be a prefibration category. A functor $t : \mathcal{M}^{'} \to \mathcal{M}$ is a *fibrant approximation* if $\mathcal{M}^{'}$ is a prefibration category with all objects fibrant and

FA1: t preserves the final object and fibrations

FA2: A map f of $\mathcal{M}^{'}$ is a weak equivalence if and only if tf is a weak equivalence

FA3: If $B \to A$, $C \to A$ are fibrations of $\mathcal{M}^{'}$ then the natural map $t(B \times_A C) \to tB \times_{tA} tC$ is an isomorphism

FA4: Any map $f : A \to tB$ factors as $f = tf^{'} \circ s$ with $f^{'}$ a fibration of $\mathcal{M}^{'}$ and s a weak equivalence of $\mathcal{M}$.

To investigate properties of cofibrant and fibrant approximations we need the notion of cofibrant (resp. fibrant) splitting. The notion is due to Cisinski, and the term is a translation of the french *clivage*.

Definition 3.5.3. Let $t : \mathcal{M}^{'} \to \mathcal{M}$ be a cofibrant approximation of a precofibration category. A *cofibrant splitting along t* consists of the following data:

(1) For any object A of $\mathcal{M}$, an object $C(A)$ of $\mathcal{M}^{'}$ and a weak equivalence $p_A : tC(A) \to A$

(2) For any map $f : A \to B$, a commutative diagram

$$\begin{tikzcd}
tC(A) \arrow[r, "p_A", "\sim"'] \arrow[d, tail, "ti(f)"'] & A \arrow[dd, "f"] \\
tD(f) \arrow[dr, "\sigma(f)"] & \\
tC(B) \arrow[u, tail, "tj(f)", "\sim"'] \arrow[r, "p_B", "\sim"'] & B
\end{tikzcd}$$

with $i(f)$ a cofibration and $j(f)$ a trivial cofibration of $\mathcal{M}^{'}$ (and therefore $\sigma(f)$ a weak equivalence in $\mathcal{M}$).

The cofibrant splitting along t is *normalized* if for any object A of $\mathcal{M}$ we have $D(1_A) = C(A)$, $i(1_A) = j(1_A) = 1_{C(A)}$ and $\sigma(1_A) = p_A$.

A cofibrant splitting along $\mathcal{M}_{cof} \to \mathcal{M}$ is simply called a cofibrant splitting of $\mathcal{M}$.

The dual definition for prefibration categories is

DEFINITION 3.5.4. Let $t : \mathcal{M}^{'} \to \mathcal{M}$ be a fibrant approximation of a prefibration category. A *fibrant splitting along* t consists of the following data:

(1) For any object A of $\mathcal{M}$, an object $R(A)$ of $\mathcal{M}^{'}$ and a weak equivalence $i_A : A \to tR(A)$
(2) For any map $f : A \to B$, a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "i_A", "\sim"'] \arrow[dr, "\sigma(f)"'] \arrow[dd, "f"'] & tR(A) \\
& tS(f) \arrow[u, two heads, "tq(f)"', "\sim"] \arrow[d, two heads, "tp(f)"] \\
B \arrow[r, "i_B", "\sim"'] & tR(B)
\end{tikzcd}$$

with $p(f)$ a fibration and $q(f)$ a trivial fibration of $\mathcal{M}^{'}$ (and therefore $\sigma(f)$ a weak equivalence).

The fibrant splitting along t is *normalized* if for any object A of $\mathcal{M}$ we have $S(1_A) = R(A)$, $p(1_A) = q(1_A) = 1_{R(A)}$ and $\sigma(1_A) = i_A$.

A fibrant splitting along $\mathcal{M}_{fib} \to \mathcal{M}$ is just called a fibrant splitting of $\mathcal{M}$.

LEMMA 3.5.5.

(1) *Any cofibrant approximation admits a normalized cofibrant splitting.*
(2) *Any fibrant approximation admits a normalized fibrant splitting.*

PROOF. We only prove (1). Let us construct a normalized cofibrant splitting along a cofibrant approximation $t : \mathcal{M}^{'} \to \mathcal{M}$.

For an object A of $\mathcal{M}$, we use the axiom CFA4 applied to $t\mathbf{0} \to A$ to construct the object $C(A)$ of $\mathcal{M}^{'}$ and the weak equivalence $p_A : tC(A) \to A$.

For a map $f : A \to B$ of $\mathcal{M}$, if $A = B$ and $f = 1_A$ then we just define $D(1_A) = C(A)$, $i(1_A) = j(1_A) = 1_{C(A)}$ and $\sigma(1_A) = p_A$.

If $f : A \to B$ is not an identity, we consider the map

$$fp_A + p_B : tC(A) \sqcup tC(B) \to B$$

We can use CFA3 and CFA4 to factor the map $fp_A + p_B$ into

$$tC(A) \sqcup tC(B) \rightarrowtail^{\!\!\!\!\!\!\!\!ti(f)+tj(f)} tD(f) \xrightarrow[\sim]{\sigma(f)} B$$

where $i(f) + j(f)$ is a cofibration in $\mathcal{M}^{'}$ and $\sigma(f)$ is a weak equivalence in $\mathcal{M}$. In particular $i(f)$ and $j(f)$ are cofibrations of $\mathcal{M}^{'}$. But $\sigma(f) \circ tj(f) = p_B$ and from the 2 out of 3 axiom $tj(f)$ and therefore $j(f)$ are weak equivalences. The construction of the normalized splitting is complete. ☐

Lemma 3.5.6.

1. *Any precofibration category* $\mathcal{M}$ *admits a normalized cofibrant splitting along* $\mathcal{M}_{cof} \to \mathcal{M}$ *such that for* A *cofibrant we have* $C(A) = A$ *and* $p_A = 1_A$*, and for* $f : A \to B$ *a cofibration between cofibrant objects we have* $D(f) = B$*,* $i(f) = f$ *and* $j(f) = \sigma(f) = 1_B$*.*
2. *Any prefibration category* $\mathcal{M}$ *admits a normalized fibrant splitting along* $\mathcal{M}_{fib} \to \mathcal{M}$ *such that for* A *fibrant we have* $R(A) = A$ *and* $i_A = 1_A$ *and for* $f : A \to B$ *a fibration between fibrant objects we have* $S(f) = A$*,* $p(f) = f$ *and* $q(f) = \sigma(f) = 1_A$*.*

Proof. This proof is a slight variation on the proof of the previous lemma. To prove (1), for cofibrant objects A of $\mathcal{M}$ just define $C(A) = A$ and $p_A = 1_A$. For non-cofibrant objects A, use the factorization axtim CF4 to construct a weak equivalence $p_A : C(A) \to A$ with $C(A)$ cofibrant.

Let $f : A \to B$ be a map in $\mathcal{M}$. If $A = B$ and $f = 1_A$ then we just define $D(1_A) = C(A)$, $i(1_A) = j(1_A) = 1_{C(A)}$ and $\sigma(1_A) = p_A$.

If f is a non-identity cofibration between cofibrant objects, then $C(A) = A$ and $C(B) = B$. We define $D(f) = B$, $i(f) = f$ and $j(f) = \sigma(f) = 1_B$.

If f is neither an identity nor a cofibration between cofibrant objects, we construct $D(f)$, $i(f)$, $j(f)$, $\sigma(f)$ factoring $fp_A + p_B$ as a cofibration followed by a weak equivalence

$$fp_A + p_B : C(A) \sqcup C(B) \rightarrowtail^{\!\!\!\!\!\!\!\!i(f)+j(f)} D(f) \xrightarrow[\sim]{\sigma(f)} B$$

Since $\sigma(f)j(f) = p_B$, by the 2 out of 3 axiom $j(f)$ is a weak equivalence and the construction of the normalized cofibrant splitting is complete. The proof of (2) is dual. ☐

We will also need the lemmas below.

Lemma 3.5.7.

1. *Let* $t : \mathcal{M}^{'} \to \mathcal{M}$ *be a cofibrant approximation of a precofibration category, and let* $f, g : A \rightrightarrows B$ *be maps of* $\mathcal{M}^{'}$*. We then have* $f \simeq g$ *in* $\mathcal{M}^{'}$ *if and only if* $tf \simeq tg$ *in* $\mathcal{M}_{cof}$*.*
2. *Let* $t : \mathcal{M}^{'} \to \mathcal{M}$ *be a fibrant approximation of a prefibration category, and let* $f, g : A \rightrightarrows B$ *be maps of* $\mathcal{M}^{'}$*. We then have* $f \simeq g$ *in* $\mathcal{M}^{'}$ *if and only if* $tf \simeq tg$ *in* $\mathcal{M}_{fib}$*.*

Proof. The statement (2) is dual to (1), and we only prove (1). The maps tf, tg are inside $\mathcal{M}_{cof}$. Any cylinder $A \sqcup A \overset{i_0+i_1}{\rightarrowtail} IA \xrightarrow[\sim]{p} A$ in $\mathcal{M}^{'}$ yields a cylinder $tA \sqcup tA \overset{ti_0+ti_1}{\rightarrowtail} tIA \xrightarrow[\sim]{tp} tA$ in $\mathcal{M}_{cof}$. If we have a homotopy $f \simeq g$ through IA, $B^{'}$, H and b then we get a homotopy $tf \simeq tg$ through tIA, $tB^{'}$, tH and tb.

Conversely, assume $tf \simeq tg$ and let us prove that $f \simeq g$. We may assume that $f + g : A \sqcup A \to B$ is a cofibration. Indeed, for general f, g we factor $f + g$ as a cofibration $f^{'} + g^{'}$ followed by a weak equivalence r. The map tr is a weak equivalence, and $tr \circ tf^{'} \simeq tr \circ tg^{'}$ therefore from Thm. 3.4.3 we have $tf^{'} \simeq tg^{'}$. If we proved $f^{'} \simeq g^{'}$ then by Thm. 3.3.2 it would follow that $f \simeq g$.

So assume that $f + g : A \sqcup A \to B$ is a cofibration. Pick a cylinder IA, and construct using Lemma 3.3.1 a homotopy $tf \simeq tg$ through a map $H : tIA \to B^{'}$ and a weak equivalence $b : tB \to B^{'}$. We construct step by step the following commutative diagram, where the map b is the bottom composition $tB \to B^{'}$.

$$\begin{tikzcd}
tA \sqcup tA \arrow[r, tail, "ti_0+ti_1"] \arrow[d, tail, "tf+tg"'] & tIA \arrow[d, tail, "th"'] \arrow[drr, "H"] & & \\
tB \arrow[r, tail, "tb_1"] & tB_1 \arrow[r, tail, "tb_2"] & tB_2 \arrow[r, "b_3", "\sim"'] & B^{'}
\end{tikzcd}$$

Construct $B_1 = B \sqcup_{A \sqcup A} IA$ with component maps b_1 and h. From CFA3, tB_1 is the pushout of the left square of the diagram, and using the pushout property we construct the map $tB_1 \to B^{'}$. Finally we construct $b_3 \circ tb_2$ as the CFA4 factorization of $tB_1 \to B^{'}$.

The maps b_1, b_2, h are cofibrations. The map $tb_2 \circ tb_1$ is a weak equivalence by the 2 out of 3 axiom CF2, and by CFA2 so is $b_2 b_1$.

We get $f \simeq g$ with homotopy map $b_2 h$ through the trivial cofibration $b_2 b_1$. ☐

Recall that a functor $u : \mathcal{M}_1 \to \mathcal{M}_2$ is

1. *essentially surjective* if any object of $\mathcal{M}_2$ is isomorphic to an object in the image of u
2. *full* if any map in $Hom_{\mathcal{M}_2}(uA, uB)$ is in the image of u, for all objects A, B of $\mathcal{M}_1$
3. *faithful* if u is injective on $Hom_{\mathcal{M}_1}(A, B)$ for all objects A, B of $\mathcal{M}_1$

The functor u is an equivalence of categories if and only if it is essentially surjective, full and faithful.

Lemma 3.5.8.

1. *A cofibrant approximation* $t : \mathcal{M}^{'} \to \mathcal{M}$ *of a precofibration category* $\mathcal{M}$ *induces a faithful functor* $\mathbf{ho}\mathcal{M}^{'} \to \mathbf{ho}\mathcal{M}_{cof}$
2. *A fibrant approximation* $t : \mathcal{M}^{'} \to \mathcal{M}$ *of a prefibration category* $\mathcal{M}$ *induces a faithful functor* $\mathbf{ho}\mathcal{M}^{'} \to \mathbf{ho}\mathcal{M}_{fib}$

Proof. The statements are dual, and we only prove (1). The image of t lies inside $\mathcal{M}_{cof}$, and t sends weak equivalences to weak equivalences, therefore t induces a functor $\mathbf{ho}\mathcal{M}^{'} \to \mathbf{ho}\mathcal{M}_{cof}$. To show that the induced functor is faithful, we have to show cf. Thm. 3.4.4 that given two left fractions in $\mathbf{ho}\mathcal{M}^{'}$

$$A \xrightarrow{f_1} B^{'} \xleftarrow[\sim]{s_1} B \text{ and } A \xrightarrow{f_2} B^{''} \xleftarrow[\sim]{s_2} B$$

if $(ts_1)^{-1} \circ tf_1 = (ts_2)^{-1} \circ tf_2$ in $\mathbf{ho}\mathcal{M}_{cof}$ then $s_1^{-1} f_1 = s_2^{-1} f_2$ in $\mathbf{ho}\mathcal{M}^{'}$.

Assume there exist weak equivalences $\overline{s_1}, \overline{s_2}$ in $\mathcal{M}_{cof}$ such that

$$\begin{tikzcd}
 & & \overline{B} & & \\
 & tB^{'} \arrow[ur, "\overline{s_1}", "\sim"'] & & tB^{''} \arrow[ul, "\overline{s_2}"', "\sim"] & \\
tA \arrow[ur, "tf_1"] \arrow[urrr, "tf_2"] & & & & tB \arrow[ulll, "ts_1", "\sim"'] \arrow[ul, "ts_2"', "\sim"]
\end{tikzcd}$$

$\overline{s_1} \circ ts_1 \simeq \overline{s_2} \circ ts_2$ and $\overline{s_1} \circ tf_1 \simeq \overline{s_2} \circ tf_2$. We use the CFA3 and CFA4 axioms to factor $\overline{s_1} + \overline{s_2}$ as

$$\begin{tikzcd}
tB^{'} \sqcup tB^{''} \arrow[r, tail, "ts_1^{'} + ts_2^{'}"] & tB^{'''} \arrow[r, "\overline{s}", "\sim"'] & \overline{B}
\end{tikzcd}$$

Since $\overline{s_1}, \overline{s_2}$ are weak equivalences and $\overline{s_1} = \overline{s} \circ ts_1^{'}$, $\overline{s_2} = \overline{s} \circ ts_2^{'}$ it follows that $ts_1^{'}, ts_2^{'}$ and therefore $s_1^{'}, s_2^{'}$ are weak equivalences.

On the other hand, using Thm. 3.4.3 we get that $ts_1^{'} \circ ts_1 \simeq ts_2^{'} \circ ts_2$ and $ts_1^{'} \circ tf_1 \simeq ts_2^{'} \circ tf_2$. We apply Lemma 3.5.7 and get that $s_1^{'} s_1 \simeq s_2^{'} s_2$ and $s_1^{'} f_1 \simeq s_2^{'} f_2$. In other words, $s_1^{-1} f_1 = s_2^{-1} f_2$ in $\mathbf{ho}\mathcal{M}^{'}$ and the proof is complete. ☐

LEMMA 3.5.9.

(1) *Let* $t : \mathcal{M}^{'} \to \mathcal{M}$ *be a cofibrant approximation of a precofibration category. For any maps* $f, g : A \rightrightarrows B$ *of* $\mathcal{M}^{'}$ *and weak equivalence* $b : tB \to B^{'}$ *of* $\mathcal{M}$ *with* $b \circ tf = b \circ tg$*, we have that* $f \simeq g$*.*

(2) *Let* $t : \mathcal{M}^{'} \to \mathcal{M}$ *be a fibrant approximation of a prefibration category. For any maps* $f, g : A \rightrightarrows B$ *of* $\mathcal{M}^{'}$ *and weak equivalence* $a : A^{'} \to tA$ *of* $\mathcal{M}$ *with* $tf \circ a = tg \circ a$*, we have that* $f \simeq g$*.*

PROOF. We only prove (1) - statement (2) is dual. The proof will resemble part of the proof of Lemma 3.5.7. Notice though that tA, tB are cofibrant, but $B^{'}$ is not necessarily cofibrant.

Just as in the proof of Lemma 3.5.7, we may assume that $f + g : A \sqcup A \to B$ is a cofibration. Pick a cylinder IA, and construct step by step the commutative diagram below

$$\begin{tikzcd}
tA \sqcup tA \arrow[r, tail, "ti_0 + ti_1"] \arrow[d, tail, "tf+tg"'] & tIA \arrow[rr, "tp", "\sim"'] \arrow[d, tail, "th"] & & A \arrow[d, "b \circ tf = b \circ tg"] \\
tB \arrow[r, tail, "tb_1"] & tB_1 \arrow[r, tail, "tb_2"] & tB_2 \arrow[r, "b_3", "\sim"'] & B^{'}
\end{tikzcd}$$

In this diagram, the bottom horizontal composition is $b : tB \to B^{'}$. We construct $B_1 = B \sqcup_{A \sqcup A} IA$ with component maps b_1 and h. tB_1 is the pushout of the left square of our diagram, and using the pushout property we construct the map $tB_1 \to B^{'}$. We then construct $b_3 \circ tb_2$ as the CFA4 factorization of $tB_1 \to B^{'}$.

The maps b_1, b_2, h are cofibrations. By the 2 out of 3 axiom CF2, the maps $tb_2 \circ tb_1$ and therefore $b_2 b_1$ are weak equivalences.

We get $f \simeq g$ with homotopy map $b_2 h$ through the trivial cofibration $b_2 b_1$. $\square$

As a consequence, we can prove for a cofibrant splitting along a cofibrant approximation $t : \mathcal{M}' \to \mathcal{M}$ that given a map $f : A \to B$ in $\mathcal{M}$ and assuming the data $C(A), C(B), p_A, p_B$ fixed, then the fraction $j(f)^{-1}i(f)$ in $\mathbf{ho}\mathcal{M}'$ is independent of the choice of $D(f), i(f), j(f)$ and $\sigma(f)$. More precisely:

Lemma 3.5.10.

(1) *Let $t : \mathcal{M}' \to \mathcal{M}$ be a cofibrant approximation of a precofibration category. For any commutative diagram*

$$\begin{tikzcd}
 & tA' \arrow[dl, tail, "ti'"'] \arrow[dr, tail, "ti''"] & & \\
tD' \arrow[rrrd, "\sigma'"] & & tD'' \arrow[dr, "\sigma''"] & \\
 & tB' \arrow[ul, tail, "tj'"] \arrow[ur, tail, "tj''"'] & & B
\end{tikzcd}$$

with i', i'' cofibrations and j', j'' trivial cofibrations of $\mathcal{M}'$ and σ', σ'' weak equivalences of $\mathcal{M}$ we have that $j'^{-1}i' = j''^{-1}i''$ in $\mathbf{ho}\mathcal{M}'$

(2) *Let $t : \mathcal{M}' \to \mathcal{M}$ be a fibrant approximation of a prefibration category. For any commutative diagram*

$$\begin{tikzcd}
 & tA' & & A \\
tD' \arrow[ur, two heads, "tq'"] \arrow[rrru, "\sigma'"] \arrow[dr, two heads, "tp'"'] & & tD'' \arrow[ul, two heads, "tq''"'] \arrow[ur, "\sigma''"'] \arrow[dl, two heads, "tp''"] & \\
 & tB' & &
\end{tikzcd}$$

with p', p'' fibrations and q', q'' trivial fibrations of $\mathcal{M}'$ and σ', σ'' weak equivalences of $\mathcal{M}$ we have that $p'q'^{-1} = p''q''^{-1}$ in $\mathbf{ho}\mathcal{M}'$

Proof. Statement (2) is dual to (1), and we only prove (1). Construct the sum $D' \sqcup_{B'} D''$ with component maps the trivial cofibrations h' and h''. In the commutative diagram below

$$\begin{tikzcd}
 & tA' \arrow[dl, tail, "ti'"'] \arrow[dr, tail, "ti''"] & & \\
tD' \arrow[r, tail, "th'"] & t(D' \sqcup_{B'} D'') \arrow[drr, "\sigma"] & tD'' \arrow[l, tail, "th''"'] & \\
 & tB' \arrow[ul, tail, "tj'"] \arrow[ur, tail, "tj''"'] & & B
\end{tikzcd}$$

the bottom triangle is a pushout by CFA3. The map σ exists by the universal property of the pushout, since $\sigma' \circ tj' = \sigma'' \circ tj''$, and is a weak equivalence by the

2 out of 3 axiom. Lemma 3.5.9 (1) applied to $h^{'}i^{'}, h^{''}i^{''}, \sigma$ implies that $h^{'}i^{'} \simeq h^{''}i^{''}$ in $\mathcal{M}^{'}$, and we conclude that $j^{'-1}i^{'} = j^{''-1}i^{''}$ in $\mathbf{ho}\mathcal{M}^{'}$. ☐

We can now state the all-important

Theorem 3.5.11 (Anderson).

(1) *Given a precofibration category $\mathcal{M}$, the inclusion $i_{\mathcal{M}} : \mathcal{M}_{cof} \to \mathcal{M}$ induces an equivalence of categories $\mathbf{ho}i_{\mathcal{M}} : \mathbf{ho}\mathcal{M}_{cof} \to \mathbf{ho}\mathcal{M}$*
(2) *Given a prefibration category $\mathcal{M}$, the inclusion $j_{\mathcal{M}} : \mathcal{M}_{fib} \to \mathcal{M}$ induces an equivalence of categories $\mathbf{ho}j_{\mathcal{M}} : \mathbf{ho}\mathcal{M}_{fib} \to \mathbf{ho}\mathcal{M}$*

More generally we have

Theorem 3.5.12.

(1) *A cofibrant approximation $t : \mathcal{M}^{'} \to \mathcal{M}$ of a precofibration category induces an equivalence of categories $\mathbf{ho}\mathcal{M}^{'} \to \mathbf{ho}\mathcal{M}$.*
(2) *A fibrant approximation $t : \mathcal{M}^{'} \to \mathcal{M}$ of a prefibration category induces an equivalence of categories $\mathbf{ho}\mathcal{M}^{'} \to \mathbf{ho}\mathcal{M}$.*

It should be noted that the last theorem is actually a particular case of an even more general result of Cisinski, for which we refer the reader to [**Cis02**], 3.12.

Proof of Thm. 3.5.12. Let us prove (1). By CFA2 the functor t sends weak equivalences to weak equivalences and therefore descends to a functor $\mathbf{ho}t : \mathbf{ho}\mathcal{M}^{'} \to \mathbf{ho}\mathcal{M}$. Pick a normalized cofibrant splitting along t, and construct a functor $s : \mathcal{M} \to \mathbf{ho}\mathcal{M}^{'}$ as follows:

(i) On objects, $s(X) = C(X)$
(ii) On maps, $s(f) = j(f)^{-1}i(f)$

Because the splitting is normalized, s sends identity maps to identity maps. To see that s preserves composition, let $f : A \to B$ and $g : B \to C$. In the commutative diagram below

$$\begin{tikzcd}
 & & tC(A) \arrow[dl, tail, "ti(f)"'] \arrow[r, "p_A", "\sim"'] & A \arrow[dd, "f"] \\
 & tD(f) \arrow[dl, tail, "ti^{'}(g)"'] \arrow[drr, "\sigma(f)"] & & \\
t(D(f) \sqcup_{C(B)} D(g)) & & tC(B) \arrow[ul, tail, "tj(f)", "\sim"'] \arrow[dl, tail, "ti(g)"'] \arrow[r, "p_B", "\sim"'] & B \arrow[dd, "g"] \\
 & tD(g) \arrow[ul, tail, "tj^{'}(f)", "\sim"'] \arrow[drr, "\sigma(g)", "\sim"'] & & \\
 & & tC(C) \arrow[ul, tail, "tj(g)", "\sim"'] \arrow[r, "p_C", "\sim"'] & C
\end{tikzcd}$$

the map $j^{'}(f)$ is a pushout of $j(f)$, therefore a trivial cofibration, and $i^{'}(g)$ is a pushout of $i(g)$ therefore a cofibration. By CFA3 the object $t(D(f) \sqcup_{C(B)} D(g))$ is a pushout, and given that $g\sigma(f)tj(f) = gp_B = \sigma(g)ti(g)$, from the universal property of the pushout there exists a map $h : t(D(f) \sqcup_{C(B)} D(g)) \to C$ that keeps the diagram commutative. The map h is a weak equivalence by the 2 out of 3 axiom. From Lemma 3.5.10, we get $j(gf)^{-1}i(gf) = (j^{'}(f)j(g))^{-1}i^{'}(g)i(f) = j(g)^{-1}i(g)j(f)^{-1}i(f)$ in $\mathbf{ho}\mathcal{M}^{'}$, therefore $s(gf) = s(g)s(f)$.

We conclude that $s : \mathcal{M} \to \mathbf{ho}\mathcal{M}^{'}$ is a functor. But s sends weak equivalences to isomorphisms, therefore descends to a functor $\mathbf{ho}s : \mathbf{ho}\mathcal{M} \to \mathbf{ho}\mathcal{M}^{'}$.

The maps p_A define a natural isomorphism $\mathbf{ho}t\,\mathbf{ho}s \to 1_{\mathbf{ho}\mathcal{M}}$. Therefore, the functor $\mathbf{ho}t$ is essentially surjective and full.

To construct a natural isomorphism $\mathbf{ho}s\,\mathbf{ho}t \to 1_{\mathbf{ho}\mathcal{M}'}$ it suffices to show that $\mathbf{ho}t$ is faithful. But $\mathbf{ho}t$ factors as $\mathbf{ho}\mathcal{M}^{'} \to \mathbf{ho}\mathcal{M}_{cof}$, which is faithful by Lemma 3.5.8, followed by $\mathbf{ho}i_{\mathcal{M}} : \mathbf{ho}\mathcal{M}_{cof} \to \mathbf{ho}\mathcal{M}$. So it suffices to prove that $\mathbf{ho}i_{\mathcal{M}}$ is faithful.

In case $t = i_{\mathcal{M}}$, we may apply the construction of the functor s for a normalized cofibrant splitting satisfying the additional properties of Lemma 3.5.6. In that case, $\mathbf{ho}s\,\mathbf{ho}i_{\mathcal{M}}$ is the identity on the objects of $\mathcal{M}^{'}$ and on the cofibration maps of $\mathcal{M}^{'}$. By Thm. 3.4.5, any map in $\mathbf{ho}\mathcal{M}^{'}$ can be written as a fraction of cofibrations, and therefore $\mathbf{ho}s\,\mathbf{ho}i_{\mathcal{M}}$ is the identity on any map of $\mathbf{ho}\mathcal{M}^{'}$. In particular, $\mathbf{ho}i_{\mathcal{M}}$ is faithful, and the proof is complete.

Statement (2) follows from duality. □

REMARK 3.5.13. If $t : \mathcal{M}^{'} \to \mathcal{M}$ is a cofibrant approximation of a precofibration category $\mathcal{M}$, suppose that $\mathcal{M}_1 \subset \mathcal{M}$ is a sub-cofibration category that includes the image of t. Then the corestriction $t_1 : \mathcal{M}^{'} \to \mathcal{M}_1$ of t defines a cofibrant approximation, and by Thm. 3.5.12 both $\mathbf{ho}t$, $\mathbf{ho}t_1$ are equivalences of categories, therefore $\mathbf{ho}\mathcal{M}_1 \to \mathbf{ho}\mathcal{M}$ is an equivalence of categories.

REMARK 3.5.14. Suppose that $t : \mathcal{M}^{'} \to \mathcal{M}$ is a functor between precofibration categories such that t restricted to $\mathcal{M}^{'}_{cof}$ is a cofibrant approximation. In view of Thm. 3.5.11 and Thm. 3.5.12, it is not hard to see that t induces a composite equivalence of categories $\mathbf{ho}\mathcal{M}^{'} \leftarrow \mathbf{ho}\mathcal{M}^{'}_{cof} \to \mathbf{ho}\mathcal{M}$. The proper way to formulate this is to say that the *total left derived* functor of t is an equivalence, which we prove as Thm. 4.6.3 in the next section.

Given a pair $(\mathcal{M}, \mathcal{W})$, we denote $\gamma_{\mathcal{M}}$ the localization functor $\gamma_{\mathcal{M}} : \mathcal{M} \to \mathbf{ho}\mathcal{M}$. From Thm. 3.5.11 and Thm. 3.4.4 (c) we get:

COROLLARY 3.5.15.

(1) *In a precofibration category $\mathcal{M}$, two maps $f, g : A \rightrightarrows B$ with A, B cofibrant are homotopic if and only if $\gamma_{\mathcal{M}} f = \gamma_{\mathcal{M}} g$.*

(2) *In a prefibration category $\mathcal{M}$, two maps $f, g : A \rightrightarrows B$ with A, B fibrant are homotopic if and only if $\gamma_{\mathcal{M}} f = \gamma_{\mathcal{M}} g$.* □

As a consequence, the homotopy relation in a precofibration category depends only on the weak equivalences - and not on the choice of cofibrations. Dually, in a prefibration category the homotopy relation depends only on the weak equivalences.

To see that, let $(\mathcal{M}, \mathcal{W})$ be a category with weak equivalences. Given two maps $f, g : A \rightrightarrows B$ we say that $f \simeq g$ if $\gamma_{\mathcal{M}} f = \gamma_{\mathcal{M}} g$. This defines an equivalence relation compatible with composition of maps. In case $\mathcal{M}$ is either a cofibration or a prefibration category, the new definiton of homotopic maps is compatible with the old one.

We will denote $\pi\mathcal{M}$ the factor category - with same objects as $\mathcal{M}$, and with maps $Hom_{\pi\mathcal{M}}(A,B) = Hom_{\mathcal{M}}(A,B)_{/\simeq}$. The new definition of $\pi\mathcal{M}$ is again backward compatible. With these notations we have:

Corollary 3.5.16.

(1) *In a precofibration category $\mathcal{M}$, the functors in the commutative diagram*

$$\begin{tikzcd}
\mathcal{M}_{cof} \arrow[r, "p_{\mathcal{M}_{cof}}"] \arrow[d, "i_{\mathcal{M}}"'] & \pi\mathcal{M}_{cof} \arrow[r, "q_{\mathcal{M}_{cof}}"] \arrow[d, "\pi i_{\mathcal{M}}"'] & \mathbf{ho}\mathcal{M}_{cof} \arrow[d, "\mathbf{ho}i_{\mathcal{M}}"'] \\
\mathcal{M} \arrow[r, "p_{\mathcal{M}}"] & \pi\mathcal{M} \arrow[r, "q_{\mathcal{M}}"] & \mathbf{ho}\mathcal{M}
\end{tikzcd}$$

have the property that $q_{\mathcal{M}_{cof}}, q_{\mathcal{M}}, \pi i_{\mathcal{M}}$ are faithful and $\mathbf{ho}i_{\mathcal{M}}$ is an equivalence.

(2) *In a prefibration category $\mathcal{M}$, the functors in the commutative diagram*

$$\begin{tikzcd}
\mathcal{M}_{fib} \arrow[r, "p_{\mathcal{M}_{fib}}"] \arrow[d, "j_{\mathcal{M}}"'] & \pi\mathcal{M}_{fib} \arrow[r, "q_{\mathcal{M}_{fib}}"] \arrow[d, "\pi j_{\mathcal{M}}"'] & \mathbf{ho}\mathcal{M}_{fib} \arrow[d, "\mathbf{ho}j_{\mathcal{M}}"'] \\
\mathcal{M} \arrow[r, "p_{\mathcal{M}}"] & \pi\mathcal{M} \arrow[r, "q_{\mathcal{M}}"] & \mathbf{ho}\mathcal{M}
\end{tikzcd}$$

have the property that $q_{\mathcal{M}_{fib}}, q_{\mathcal{M}}, \pi j_{\mathcal{M}}$ are faithful and $\mathbf{ho}j_{\mathcal{M}}$ is an equivalence. □

3.6. Products of cofibration categories

Here is an application of homotopy calculus of fractions. If $(\mathcal{M}_k, \mathcal{W}_k)$ for $k\epsilon K$ is a set of categories with weak equivalences, one can form the product $(\times_{k\epsilon K}\mathcal{M}_k, \times_{k\epsilon K}\mathcal{W}_k)$ and its homotopy category denoted $\mathbf{ho}(\times\mathcal{M}_k)$. Denote $p_k : \times_{k\epsilon K}\mathcal{M}_k \to \mathcal{M}_k$ the projection. The components $(\mathbf{ho}p_k)_{k\epsilon K}$ define a functor

$$P : \mathbf{ho}(\times\mathcal{M}_k) \to \times\mathbf{ho}\mathcal{M}_k$$

If each category $\mathcal{M}_k$ carries a (pre)cofibration category structure $(\mathcal{M}_k, \mathcal{W}_k, \mathcal{C}of_k)$, then $(\times\mathcal{M}_k, \times\mathcal{W}_k, \times\mathcal{C}of_k)$ defines the product (pre)cofibration category structure on $\times\mathcal{M}_k$. Dually, if each $\mathcal{M}_k$ carries a (pre)fibration category structure, then $\times\mathcal{M}_k$ carries a product (pre)fibration category structure.

Suppose that $\mathcal{M}_k$ are precofibration categories, and that $A = (A_k)_{k\epsilon K}$ is a cofibrant object of $\times\mathcal{M}_k$. Any factorization $A \sqcup A \to IA \to A$ defines a cylinder in $\times\mathcal{M}_k$ iff each component $A_k \sqcup A_k \to (IA)_k \to A_k$ is a cylinder in $\mathcal{M}_k$.

If $B = (B_k)_{k\epsilon K}$ is a second cofibrant object and $f, g : A \rightrightarrows B$ is a pair of maps in $\times\mathcal{M}_k$, then any homotopy $f \simeq g$ induces componentwise homotopies $f_k \simeq g_k$ in $\mathcal{M}_k$. Conversely, any set of homotopies $f_k \simeq g_k$ induces a homotopy $f \simeq g$.

Theorem 3.6.1. *If $\mathcal{M}_k$ for $k\epsilon K$ are each precofibration categories, or are each prefibration categories, then the functor*

$$P : \mathbf{ho}(\times\mathcal{M}_k) \to \times\mathbf{ho}\mathcal{M}_k$$

is an isomorphism of categories.

Proof. Assume that each $\mathcal{M}_k$ is a precofibration category (the proof for prefibration categories is dual). Our functor P is a bijection on objects, and we'd like to show that it is also fully faithful.

By Thm. 3.5.11 we have equivalences of categories $\mathbf{ho}\mathcal{M}_k \cong \mathbf{ho}(\mathcal{M}_k)_{cof}$ and $\mathbf{ho}(\times\mathcal{M}_k) \cong \mathbf{ho}(\times(\mathcal{M}_k)_{cof})$. It suffices therefore in our proof to assume that $\mathcal{M}_k = (\mathcal{M}_k)_{cof}$ for all k.

To prove fullness of P, let A_k and B_k be objects of $\mathcal{M}_k$. Denote $A = (A_k)_k$ and $B = (B_k)_k$. Any map $\phi : A \to B$ in $\times\mathbf{ho}\mathcal{M}_k$ can be expressed on components, using Thm. 3.4.4 (a) for each $\mathcal{M}_k$, as a left fraction $\phi_k = s_k^{-1} f_k$, with weak equivalences s_k. The map ϕ therefore is the image via P of $(s_k)^{-1}(f_k)$.

To prove faithfulness of P, suppose that $\phi, \psi : A \to B$ are maps in $\mathbf{ho}(\times\mathcal{M}_k)$ which have the same image via P. Using Thm. 3.4.4 (a) applied to $\times\mathcal{M}_k$, we can write ϕ as $(s_k)^{-1}(f_k)$ and ψ as $(t_k)^{-1}(g_k)$, with s_k, t_k weak equivalences. From Thm. 3.4.4 (b) applied to each $\mathcal{M}_k$, we can find weak equivalences $s_k^{'}, t_k^{'}$ such that we have componentwise homotopies $s_k^{'} s_k \simeq t_k^{'} t_k$ and $s_k^{'} f_k \simeq t_k^{'} g_k$. The componentwise homotopies induce homotopies $s^{'} s \simeq t^{'} t$ and $s^{'} f \simeq t^{'} g$ in $\times\mathcal{M}_k$, so $\phi = \psi$ and we have shown that P is faithful. □

3.7. Saturation

Another important consequence of homotopy calculus of fractions is the next result, due to Cisinski [**Cis02**], which shows that the saturation of weak equivalences in a cofibration category yields again a cofibration category.

Given a category with weak equivalences $(\mathcal{M}, \mathcal{W})$, recall that $\overline{\mathcal{W}}$ denotes the saturation of $\mathcal{W}$, i.e. the class of maps of $\mathcal{M}$ that become isomorphisms in $\mathbf{ho}\mathcal{M}$.

LEMMA 3.7.1.

(1) *Suppose that* $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ *is a precofibration category, and that* $f : A \to B$ *is a map with* A, B *cofibrant.*
   (a) f *has a left inverse in* $\mathbf{ho}\mathcal{M}$ *if and only if there exists a cofibration* $f^{'} : B \to B^{'}$ *such that* $f^{'} f$ *is a weak equivalence.*
   (b) f *is an isomorphism in* $\mathbf{ho}\mathcal{M}$ *if and only if there exist cofibrations* $f^{'} : B \to B^{'}$, $f^{''} : B^{'} \to B^{''}$ *such that* $f^{'} f, f^{''} f^{'}$ *are weak equivalences.*

(2) *Suppose that* $(\mathcal{M}, \mathcal{W}, \mathcal{F}ib)$ *is a prefibration category, and that* $f : A \to B$ *is a map with* A, B *fibrant.*
   (a) f *has a right inverse in* $\mathbf{ho}\mathcal{M}$ *if and only if there exists a fibration* $f^{'} : A^{'} \to A$ *such that* $f f^{'}$ *is a weak equivalence.*
   (b) f *is an isomorphism in* $\mathbf{ho}\mathcal{M}$ *if and only if there exist fibrations* $f^{'} : A^{'} \to A$, $f^{''} : A^{''} \to A^{'}$ *such that* $f f^{'}, f^{'} f^{''}$ *are weak equivalences.*

PROOF. We only prove (1). The implications (a) ($\Leftarrow$), (b) ($\Leftarrow$) are clear.

To prove (a) ($\Rightarrow$), using Thm. 3.5.11 f has a left inverse in $\mathbf{ho}\mathcal{M}$ if and only if f has a left inverse in $\mathbf{ho}\mathcal{M}_{cof}$. From Thm. 3.4.5, write the left inverse of f in $\mathbf{ho}\mathcal{M}_{cof}$ as a left fraction $s^{-1} f^{'}$ with $f^{'}$ a cofibration and s a trivial cofibration. We get $1 = s^{-1} f^{'} f$ in $\mathbf{ho}\mathcal{M}_{cof}$, therefore $s = f^{'} f$ in $\mathbf{ho}\mathcal{M}_{cof}$ which means $s \simeq f^{'} f$. Since s is a weak equivalence, $f^{'} f$ must be a weak equivalence.

Part (b) ($\Rightarrow$) is a corollary of (a) applied first to the map f to construct $f^{'}$ then to the map $f^{'}$ to construct $f^{''}$. □

This allows us to prove the following result.

THEOREM 3.7.2 (Cisinski).

(1) *If* $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ *is a (pre)cofibration category, then so is* $(\mathcal{M}, \overline{\mathcal{W}}, \mathcal{C}of)$.

(2) *If* $(\mathcal{M}, \mathcal{W}, \mathcal{F}ib)$ *is a (pre)fibration category, then so is* $(\mathcal{M}, \overline{\mathcal{W}}, \mathcal{F}ib)$.
(3) *If* $(\mathcal{M}, \mathcal{W}, \mathcal{C}of, \mathcal{F}ib)$ *is an ABC model category, then so is* $(\mathcal{M}, \overline{\mathcal{W}}, \mathcal{C}of, \mathcal{F}ib)$.

Proof. Part (2) is dual to (1), and part (3) is a consequence of (1) and (2). Let's prove (1) (a) for $(\mathcal{M}, \overline{\mathcal{W}}, \mathcal{F}ib)$ in the case when $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ is a precofibration category.

(i) *The axioms CF1, CF2, CF3 (1), CF4* are clearly satisfied.

(ii) *The axiom CF3 (2).* Given a solid diagram in $\mathcal{M}$, with A, C cofibrant and i a $\overline{\mathcal{W}}$-trivial cofibration,

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail, "i"'] & C \arrow[d, dashed, "j"] \\
B \arrow[r, dashed] & D
\end{tikzcd}$$

then by the previous lemma there exist cofibrations i', i'' such that $i'i, i''i'$ are $\mathcal{W}$-trivial cofibrations. Denote j', j'' the pushouts of the cofibrations i', i''. We get that $j'j, j''j'$ are $\mathcal{W}$-trivial cofibrations, and thereore j is a $\overline{\mathcal{W}}$-trivial cofibration.

Assume now that $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ satisfies CF5 and CF6.

(iii) *The axioms CF5 (1) and CF6 (1).* If $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ satisfies CF5 (1), resp. CF6 (1), then clearly so does $(\mathcal{M}, \overline{\mathcal{W}}, \mathcal{C}of)$..

(iv) *The axiom CF5 (2).* Suppose that the cofibration category $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ satisfies CF5. Assume that $f_i : A_i \to B_i$ for $i \epsilon I$ is a set of $\overline{\mathcal{W}}$-trivial cofibrations with A_i cofibrant. The map $\sqcup f_i$ is a cofibration by axiom CF5 (1). By the previous lemma, there exist cofibrations $f_i' : B_i \to B_i'$, $f_i'' : B_i' \to B_i''$ such that $f_i' f_i$, $f_i'' f_i'$ are $\mathcal{W}$-trivial cofibrations. It follows that $\sqcup f_i' f_i$, $\sqcup f_i'' f_i'$ are $\mathcal{W}$-trivial cofibrations, therefore $\sqcup f_i$ is a $\overline{\mathcal{W}}$-trivial cofibration.

(v) *The axiom CF6 (2).* Suppose that the cofibration category $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ satisfies CF6. We will use the equivalent formulation of CF6 (2) given by Lemma 1.6.6 (1) (iii). Consider a map of countable direct sequences of cofibrations, with A_0 cofibrant and all a_n, b_n, f_n cofibrations, such that f_n are $\overline{\mathcal{W}}$-trivial cofibrations. These are the full maps in the diagram below.

$$\begin{tikzcd}
A_0 \arrow[r, tail, "a_0"] \arrow[d, tail, "f_0"'] & A_1 \arrow[r, tail, "a_1"] \arrow[d, tail, "f_1"'] & A_2 \arrow[r, tail, "a_2"] \arrow[d, tail, "f_2"'] & \cdots \\
B_0 \arrow[r, tail, "b_0"] \arrow[d, tail, dashed, "f_0'"'] & B_1 \arrow[r, tail, "b_1"] \arrow[d, tail, dashed, "f_1'"'] & B_2 \arrow[r, tail, "b_2"] \arrow[d, tail, dashed, "f_2'"'] & \cdots \\
B_0' \arrow[r, tail, dashed, "b_0'"] & B_1' \arrow[r, tail, dashed, "b_1'"] & B_2' \arrow[r, tail, dashed, "b_2'"] & \cdots
\end{tikzcd}$$

We know from CF6 (1) that colim A_n, colim B_n exist and are cofibrant. We want to show that colim f_n is a $\overline{\mathcal{W}}$-weak equivalence.

By induction, we will construct the cofibrant objects B_n' and the cofibrations f_n', b_n' with the property that $f_n' f_n$ are $\mathcal{W}$-trivial cofibrations. Once the construction is complete, it will follow that colim f_n'colim f_n is a $\mathcal{W}$-weak equivalence, and applying the construction again to f_n' we get cofibrant objects B_n'' and cofibrations

$f^{''}_n, b^{''}_n$ such that colim $f^{''}_n$ colim $f^{'}_n$ is a $\mathcal{W}$-weak equivalence. It will follow that colim f_n is a $\overline{\mathcal{W}}$-weak equivalence.

To complete the proof, it remains to construct the cofibrant objects $B^{'}_n$ and the cofibrations $f^{'}_n, b^{'}_n$. We use induction on n. For $n = 0$, we use Lemma 3.7.1 to construct B_0 and $f^{'}_0$.

Assume we constructed the object $B^{'}_n$ and the cofibrations $f^{'}_n, b^{'}_{n-1}$ such that $f^{'}_n f_n$ is a $\mathcal{W}$-weak equivalence.

We construct the following diagram

$$\begin{tikzcd}
A_n \arrow[r, tail, "a_n"] \arrow[d, tail, "f^{'}_n f_n"', "\sim"] & A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, tail, "\beta_n"', "\sim"] & B_{n+1} \arrow[r, tail, "\overline{f^{'}_{n+1}}"] & \overline{B_{n+1}} \arrow[d, tail, "\overline{\overline{f^{'}_{n+1}}}"', "\sim"] \\
B^{'}_n \arrow[r, tail, "\gamma_n"] & \overline{B^{'}_{n+1}} \arrow[rr, tail, "\delta_n", "\sim"'] & & B^{'}_{n+1}
\end{tikzcd}$$

where:

1. $f^{'}_n f_n$ is a $\mathcal{W}$-trivial cofibration
2. β_n is constructed as the pushout of $f^{'}_n f_n$, therefore is a $\mathcal{W}$-trivial cofibration. γ_n is a pushout of a_n, therefore is a cofibration.
3. $\overline{f^{'}_{n+1}}$ is a cofibration constructed by Lemma 3.7.1 applied to f_{n+1}, so that $\overline{f^{'}_{n+1}} f_{n+1}$ is a $\mathcal{W}$-trivial cofibration
4. δ_n is constructed as the pushout of $\overline{f^{'}}_{n+1} f_{n+1}$, therefore a $\mathcal{W}$-trivial cofibration. $\overline{\overline{f^{'}_{n+1}}}$ is a pushout of β_n, therefore a $\mathcal{W}$-trivial cofibration.

We define $f^{'}_{n+1} = \overline{\overline{f^{'}_{n+1}}}\, \overline{f^{'}_{n+1}}$, which is a cofibration, and $f^{'}_{n+1} f_{n+1}$ is a $\mathcal{W}$-trivial cofibration. We also define $b^{'}_n = \delta_n \gamma_n$, which is a cofibration. The inductive step is now complete, and with it the proof of CF6 (2).

□

Let us adapt this discussion to the case of left proper cofibration categories, and show that if ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{C}of$) is a left proper cofibration category then so is ($\mathcal{M}$, $\overline{\mathcal{W}}$, $\mathcal{C}of$).

Lemma 3.7.3.

1. *Suppose that* ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{C}of$) *is a left proper precofibration category, and that* $f : A \to B$ *be a map*
   - (a) f *has a left inverse in* **ho**$\mathcal{M}$ *if and only if there exists a left proper map* $f^{'} : B \to B^{'}$ *such that* $f^{'} f$ *is a weak equivalence.*
   - (b) f *is an isomorphism in* **ho**$\mathcal{M}$ *if and only if there exist left proper maps* $f^{'} : B \to B^{'}$, $f^{''} : B^{'} \to B^{''}$ *such that* $f^{'} f, f^{''} f^{'}$ *are weak equivalences.*
2. *Suppose that* ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{F}ib$) *is a right proper prefibration category, and that* $f : A \to B$ *is a map.*
   - (a) f *has a right inverse in* **ho**$\mathcal{M}$ *if and only if there exists a right proper map* $f^{'} : A^{'} \to A$ *such that* $f f^{'}$ *is a weak equivalence.*
   - (b) f *is an isomorphism in* **ho**$\mathcal{M}$ *if and only if there exist right proper maps* $f^{'} : A^{'} \to A$, $f^{''} : A^{''} \to A^{'}$ *such that* $f f^{'}, f^{'} f^{''}$ *are weak equivalences.*

Proof. We only prove (1). The implications (a) (⇐), (b) (⇐) are trivial. Let us prove (a) (⇒). Construct the diagram

$$\begin{tikzcd}
 & B \arrow[r, "f'"] & B' \\
A \arrow[ru, "f"] \arrow[r, "f_2"] & B_2 \arrow[u, "s_2"'] \arrow[r, "f_2'"] & B_2' \arrow[u] \\
A_1 \arrow[u, "r_1", "\sim"'] \arrow[r, tail, "f_1"] & B_1 \arrow[u, "s_1"'] \arrow[r, tail, "f_1'"] & B_1' \arrow[u]
\end{tikzcd}$$

as follows:

1. A_1 is a cofibrant replacement of A, f_1 is a cofibrant replacement of fr_1.
2. It follows that f_1 has a left inverse in $\mathbf{ho}\mathcal{M}$. We use Lemma 3.7.1 to construct a cofibration f_1' such that $f_1'f_1$ is a $\mathcal{W}$-weak equivalence.
3. The maps f_2, resp. f_2' and f' are pushouts of f_1, resp. f_1'. These pushouts can be constructed because $\mathcal{M}$ is left proper.

The maps r_1 and s_2s_1 are $\mathcal{W}$-weak equivalences, and all horizontal maps are left proper. It follows that all vertical maps are $\mathcal{W}$-weak equivalences. Since $f_1'f_1$ is a $\mathcal{W}$-weak equivalence, so is $f'f$

Part (b) is proved applying (a) first to the map f to construct f' and then a second time to the map f' to construct f''. □

Using the previous lemma, the statement below is immediate:

Proposition 3.7.4.

1. *In a left proper precofibration category* ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{C}of$), *all the* $\mathcal{W}$*-left proper maps are* $\overline{\mathcal{W}}$*-left proper*
2. *In a right proper prefibration category* ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{F}ib$), *all the* $\mathcal{W}$*-right proper maps are* $\overline{\mathcal{W}}$*-right proper*

Proof. We only prove (1). Suppose that $f : A \to B$ is a $\mathcal{W}$-left proper map. In the diagram with full maps

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & C_1 \arrow[r, "r"] \arrow[d, dashed] & C_2 \arrow[r, dashed, "r_1"] \arrow[d, dashed] & C_3 \arrow[r, dashed, "r_2"] \arrow[d, dashed] & C_4 \arrow[d, dashed] \\
B \arrow[r, dashed] & D_1 \arrow[r, dashed, "r'"] & D_2 \arrow[r, dashed, "r_1'"] & D_3 \arrow[r, dashed, "r_2'"] & D_4
\end{tikzcd}$$

suppose that r is a $\overline{\mathcal{W}}$-weak equivalence. Use Lemma 3.7.3 to construct the maps r_1, r_2 such that r_1r and r_2r_1 are $\mathcal{W}$-weak equivalences. Denote r', r_1', r_2' the pushouts along f. The maps $r_1'r'$ and $r_2'r_1'$ are $\mathcal{W}$-weak equivalences, therefore r' is a $\overline{\mathcal{W}}$-weak equivalence. □

We can now show the following result.

Theorem 3.7.5.

1. *If* ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{C}of$) *is a left proper (pre)cofibration category, then so is* ($\mathcal{M}$, $\overline{\mathcal{W}}$, $\mathcal{C}of$).

(2) *If* $(\mathcal{M}, \mathcal{W}, \mathcal{F}ib)$ *is a right proper (pre)fibration category, then so is* $(\mathcal{M}, \overline{\mathcal{W}}, \mathcal{F}ib)$.

(3) *If* $(\mathcal{M}, \mathcal{W}, \mathcal{C}of, \mathcal{F}ib)$ *is a proper ABC model category, then so is* $(\mathcal{M}, \overline{\mathcal{W}}, \mathcal{C}of, \mathcal{F}ib)$.

PROOF. Consequence of Thm. 3.7.2 and Prop. 3.7.4. □

CHAPTER 4

Kan extensions. Total derived functors.

The purpose of this chapter is to introduce the apparatus of Kan extensions and total derived functors, which will be used later to prove the existence and basic properties of homotopy colimits in cofibration categories.

We will use *approximation functors* (Def. 4.5.1) as a tool for stating and proving an existence theorem for total derived functors (Thm. 4.5.6), and a rather technical adjointness property of total derived functors (Thm. 4.5.10).

The approximation functors defined in this chapter should be thought of as an axiomatization at the level of categories with weak equivalences $(\mathcal{M}, \mathcal{W})$ of the (co)fibrant approximation functors defined in Section 3.5.

4.1. The language of 2-categories

Recall that a 2-category $\mathcal{C}$ is a category enriched over categories. This means by definition that for each two objects A, B of $\mathcal{C}$ the hom-set $Hom(A, B)$ forms the objects of a category $\mathcal{H}om(A, B)$. The the composition functor $c_{ABC} : \mathcal{H}om(A, B) \times \mathcal{H}om(B, C) \to \mathcal{H}om(A, C)$ is required to be associative and to have $1_A : \star \to \mathcal{H}om(A, A)$ as a left and right unit, where $\star$ denotes the point-category.

The objects of a 2-category $\mathcal{C}$ are called 0-cells, the objects of $\mathcal{H}om(A, B)$ are called 1-cells and the morphisms of $\mathcal{H}om(A, B)$ are called 2-cells. A good reference on 2-categories is the Borceux monography [**Bor94**].

This section describes the notation we use for compositions of 1-cells and 2-cells in a 2-category. Each notation has a full form and a simple form. The simple form of the notation is ambiguous, and is only used if it is clear from the context which functor or natural map operation we refer to.

We denote as usual 1-cells $f : A \to B$ with a single arrow. Between 1-cells $f, g : A \to B$, we denote 2-cells as $\alpha : f \Rightarrow g$, or just $\alpha : f \to g$ if no confusion can occur.

The composition of 1-cells f, g

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{4.1}$$

is just the composition at the level of unenriched hom-sets $Hom(-, -)$ and is denoted $g \circ f : A \to C$, or in simple form gf.

The composition of 2-cells α, β

$$\begin{tikzcd}[column sep=large] A \arrow[r, shift left=4, "f"] \arrow[r, "g" description] \arrow[r, shift right=4, "h"'] & B \end{tikzcd} \qquad \Downarrow\alpha \quad \Downarrow\beta \tag{4.2}$$

is composition at the level of $\mathcal{H}om(-, -)$ and is denoted $\beta \odot \alpha : f \Rightarrow h$, or in simple form $\beta\alpha$.

The composition of a 2-cell α with a 1-cell f

(4.3)
$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\alpha"] & C \end{tikzcd}$$

is just $c_{ABC}(1_f, \alpha)$, and we denote it $\alpha.f : gf \Rightarrow hf$ or in simple form αf. The composition in the other direction

(4.4)
$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, "h"] & C \end{tikzcd}$$

is $c_{ABC}(\alpha, 1_h)$ and we denote it $h.\alpha : hf \Rightarrow hg$, or in simple form $h\alpha$.

The notations we have established up until now can be used to completely describe compositions of 1- and 2-cells. However it is convenient to introduce the $\star$ notation to denote thecomposition of adjacent of 2-cells of planar diagrams.

We will denote the composition of 2-cells $\alpha : jf \Rightarrow g$ and $\beta : h \Rightarrow ij$

(4.5)
$$\begin{tikzcd} & & B \arrow[dd, "j"] \arrow[drr, "h"] & & \\ A \arrow[urr, "f"] \arrow[drr, "g"'] & \Downarrow\alpha & & \Downarrow\beta & D \\ & & C \arrow[urr, "i"'] & & \end{tikzcd}$$

as $\beta \star \alpha = (i.\alpha) \odot (\beta.f)$.

We use the same $\star$ notation to denote the composition of 2-cells $\alpha : f \Rightarrow jg$ and $\beta : hj \Rightarrow i$

(4.6)
$$\begin{tikzcd} & & B \arrow[drr, "h"] & & \\ A \arrow[urr, "f"] \arrow[drr, "g"'] & \Downarrow\alpha & & \Downarrow\beta & D \\ & & C \arrow[uu, "j"] \arrow[urr, "i"'] & & \end{tikzcd}$$

as $\beta \star \alpha = (\beta.g) \odot (h.\alpha)$.

In particular, taking j to be an identity map in (4.5) or (4.6) we get the composition of 2-cells $\beta \star \alpha = c_{ABC}(\alpha, \beta)$

(4.7)
$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "h"] \arrow[r, bend right, "i"'] \arrow[r, phantom, "\Downarrow\beta"] & C \end{tikzcd}$$

Since c_{ABC} is a functor, $(i.\alpha) \odot (\beta.f) = (\beta.g) \odot (h.\alpha)$ and the $\star$ notation is consistent in (4.7) with (4.5) and (4.6).

In simple form, if no confusion is possible we denote $\beta\alpha$ for $\beta \star \alpha$.

There is a general theorem called the Pasting Theorem regarding compositions of 2-cells of planar diagrams, for which we refer the reader for example to [**Pow90**]
.

Let us also recall that a (strict) 2-functor $f : \mathcal{C}_1 \to \mathcal{C}_2$ is a function that sends 1, 2 and 3-cells of $\mathcal{C}_1$ to 1, 2 resp. 3-cells of $\mathcal{C}_2$, with the property that it preserves the various types of units and compositions of cells. If $\mathcal{C}^{op}$ denotes the opposite of a 2-category (with composition of 1- and 2-cells having direction reversed), then a *contravariant* 2-functor $f : \mathcal{C}_1 \to \mathcal{C}_2$ is just a (covariant) 2-functor $f : \mathcal{C}_1^{op} \to \mathcal{C}_2$.

A 2-subcategory $\mathcal{C}'$ of a 2-category $\mathcal{C}$ consists of a subclass of objects $Ob\mathcal{C}' \subset Ob\mathcal{C}$ along with subcategories $\mathcal{H}om_{\mathcal{C}'}(A, B) \subset \mathcal{H}om_{\mathcal{C}}(A, B)$ for objects $A, B \epsilon Ob\mathcal{C}'$ that are stable under the composition rule c_{ABC} for $A, B, C \epsilon Ob\mathcal{C}'$ and include the image of the unit $1_A : \star \to \mathcal{H}om_{\mathcal{C}}(A, A)$ for $A \epsilon Ob\mathcal{C}'$. A 2-subcategory is 2-*full* if $\mathcal{H}om_{\mathcal{C}'}(A, B) = \mathcal{H}om_{\mathcal{C}}(A, B)$ for any $A, B \epsilon Ob\mathcal{C}'$.

The category of categories forms a 2-category, with categories as 0-cells, functors as 1-cells and natural maps as 2-cells. We will use the notation introduced in this section to denote compositions of functors and natural maps. In particular, natural maps will be denoted with a double arrow $\Rightarrow$ (or simply with $\to$ if no confusion is possible), and composition of natural maps viewed as adjacent 2-cells will be denoted with $\star$.

4.2. Adjoint functors

Recall that an adjunction $u_1 \dashv u_2$ between two functors $u_1 : \mathcal{A} \rightleftarrows \mathcal{B} : u_2$ is a bijection of sets

$$\zeta : Hom_{\mathcal{B}}(u_1 A, B) \cong Hom_{\mathcal{A}}(A, u_2 B)$$

natural in objects $A \epsilon \mathcal{A}$ and $B \epsilon \mathcal{B}$. For example, if u_1, u_2 are equivalences of categories, then $u_1 \dashv u_2$ (and $u_2 \dashv u_1$).

Whenever we say that u_1, u_2 is an adjoint pair we refer to a particular bijection ζ. The following proposition encodes the basic properties of adjoint functors that we will need.

Proposition 4.2.1. *Suppose that $u_1 : \mathcal{A} \rightleftarrows \mathcal{B} : u_2$ is a pair of functors.*

(1) *An adjunction $u_1 \dashv u_2$ is uniquely determined by natural maps $\phi : 1_{\mathcal{A}} \Rightarrow u_2 u_1$ (the unit) and $\psi : u_1 u_2 \Rightarrow 1_{\mathcal{B}}$ (the counit of the adjunction) with the property that both the following compositions are identities*

$$u_1 \xRightarrow{u_1 \phi} u_1 u_2 u_1 \xRightarrow{\psi u_1} u_1 \qquad u_2 \xRightarrow{\phi u_2} u_2 u_1 u_2 \xRightarrow{u_2 \psi} u_2$$

(2) *If $u_1 \dashv u_2$ is an adjunction with unit ϕ and counit ψ, then*
   (a) *u_1 (resp. u_2) is fully faithful iff ϕ (resp. ψ) is a natural isomorphism.*
   (b) *u_1 and u_2 are inverse equivalences of categories iff both ϕ and ψ are natural isomorphisms.*

Proof. See for example Mac Lane [**Lan98**]. ☐

Here is another way to state part (2) of the previous proposition. The proof is left to the reader.

Proposition 4.2.2. *Suppose that $u_1 \dashv u_2$ is an adjoint pair of functors as above. Then the following statements are equivalent:*

(1) *(resp. (1r), resp. (1l)). For any objects $A \epsilon \mathcal{A}$, $B \epsilon \mathcal{B}$, a map $u_1 A \to B$ is an isomorphism iff (resp. if, resp. only if) its adjoint $A \to u_2 B$ is an isomorphism*
(2) *(resp. (2r), resp. (2l)). u_1 and u_2 are inverse equivalences of categories (resp. u_2 is fully faithful, resp u_1 is fully faithful).*

4.3. Kan extensions

For a category with weak equivalences $(\mathcal{M}, \mathcal{W})$, cf. Def. 1.7.2, we denote $\gamma_{\mathcal{M}} : \mathcal{M} \to \mathbf{ho}\mathcal{M}$ the localization functor.

A functor $u : \mathcal{M}_1 \to \mathcal{M}_2$ of categories with weak equivalences $(\mathcal{M}_1, \mathcal{W}_1), (\mathcal{M}_2, \mathcal{W}_2)$ descends to a functor $\mathbf{ho}u : \mathbf{ho}\mathcal{M}_1 \to \mathbf{ho}\mathcal{M}_2$ if and only if $u(\mathcal{W}_1) \subset \overline{\mathcal{W}_2}$, where $\overline{\mathcal{W}_2}$ denotes the saturation of $\mathcal{W}_2$ in $\mathcal{M}_2$.

In the general case however $\mathbf{ho}u$ does not exist, and the best we can hope for is the existence of a left or right Kan extension of $\gamma_{\mathcal{M}_2} u$ along $\gamma_{\mathcal{M}_1}$.

Definition 4.3.1. Consider two functors $u : \mathcal{A} \to \mathcal{B}$ and $\gamma : \mathcal{A} \to \mathcal{A}'$

(1) A left Kan extension of u along γ is a pair $(\mathbf{L}_\gamma u, \epsilon)$ where $\mathbf{L}_\gamma u : \mathcal{A}' \to \mathcal{B}$ is a functor and $\epsilon : \mathbf{L}_\gamma u \circ \gamma \Rightarrow u$ is a natural map

$$\begin{tikzcd}
\mathcal{A} \arrow[d, "\gamma"'] \arrow[rd, "u"] \arrow[d, phantom, "\epsilon \Uparrow", shift left=4ex] & \\
\mathcal{A}' \arrow[r, "\mathbf{L}_\gamma u"'] & \mathcal{B}
\end{tikzcd}$$

satisfying the following universal property: if (Λ, λ) is another pair of a functor $\Lambda : \mathcal{A}' \to \mathcal{B}$ and natural map $\lambda : \Lambda \circ \gamma \Rightarrow u$

$$\begin{tikzcd}
\mathcal{A} \arrow[d, "\gamma"'] \arrow[rd, "u"] \arrow[d, phantom, "\epsilon \Uparrow", shift left=4ex] & \\
\mathcal{A}' \arrow[r, bend left, "\mathbf{L}_\gamma u"] \arrow[r, bend right, "\Lambda"'] \arrow[r, phantom, "\delta \Uparrow"] & \mathcal{B}
\end{tikzcd}$$

then there exists a unique natural map $\delta : \Lambda \Rightarrow \mathbf{L}_\gamma u$ with $\epsilon \star \delta = \lambda$

(2) A right Kan extension of u along γ is a pair $(\mathbf{R}_\gamma u, \nu)$ where $\mathbf{R}_\gamma u : \mathcal{A}' \to \mathcal{B}$ is a functor and $\nu : u \Rightarrow \mathbf{R}_\gamma u \circ \gamma$ is a natural map

$$\begin{tikzcd}
\mathcal{A} \arrow[d, "\gamma"'] \arrow[rd, "u"] \arrow[d, phantom, "\nu \Downarrow", shift left=4ex] & \\
\mathcal{A}' \arrow[r, "\mathbf{R}_\gamma u"'] & \mathcal{B}
\end{tikzcd}$$

satisfying the following universal property: if (Λ, λ) is another pair of a functor $\Lambda : \mathcal{A}' \to \mathcal{B}$ and a natural map $\lambda : u \Rightarrow \Lambda \circ \gamma$

$$\begin{tikzcd}
\mathcal{A} \arrow[d, "\gamma"'] \arrow[rd, "u"] \arrow[d, phantom, "\nu \Downarrow", shift left=4ex] & \\
\mathcal{A}' \arrow[r, bend left, "\mathbf{R}_\gamma u"] \arrow[r, bend right, "\Lambda"'] \arrow[r, phantom, "\delta \Downarrow"] & \mathcal{B}
\end{tikzcd}$$

then there exists a unique natural map $\delta : \mathbf{R}_\gamma u \Rightarrow \Lambda$ with $\delta \star \nu = \lambda$

The left Kan extension $(\mathbf{L}_\gamma u, \epsilon)$ is also called in some references the *left derived* of u along γ and the right Kan extension $(\mathbf{R}_\gamma u, \epsilon)$ the *right derived* of u along γ. Since it is defined by an universal property, if the left (or right) Kan extension exists then it is unique up to a unique isomorphism.

We next state a few simple properties of Kan extensions. First, we note that the Kan extension of u along γ is independent on the choice of u and γ within a *natural isomorphism* class.

PROPOSITION 4.3.2. *Consider two functor pairs* $u, u^{'} : \mathcal{A} \rightrightarrows \mathcal{B}$ *and* $\gamma, \gamma^{'} : \mathcal{A} \rightrightarrows \mathcal{A}^{'}$ *and natural isomorphisms* $u \cong u^{'}$, $\gamma \cong \gamma^{'}$.

1. *The left Kan extension* $(\mathbf{L}_\gamma u, \epsilon)$ *exists iff the left Kan extension* $(\mathbf{L}_{\gamma'} u^{'}, \epsilon^{'})$ *exists, and if they both exist they are naturally isomorphic.*
2. *The right Kan extension* $(\mathbf{R}_\gamma u, \nu)$ *exists iff the right Kan extension* $(\mathbf{R}_{\gamma'} u^{'}, \nu^{'})$ *exists, and if they both exist they are naturally isomorphic.*

PROOF. Immediate using the definitions. ☐

Second, we prove an existence criterion for Kan extensions. If γ admits a left (resp. right) adjoint, then we show that u admits a left (resp. right) Kan extension along γ.

PROPOSITION 4.3.3. *Consider two functors* $u : \mathcal{A} \to \mathcal{B}$ *and* $\gamma : \mathcal{A} \to \mathcal{A}^{'}$.

1. *If* γ *admits a left adjoint* $\gamma^{'}$ *with adjunction unit* $\phi : 1_{\mathcal{A}'} \Rightarrow \gamma\gamma^{'}$ *and counit* $\psi : \gamma^{'}\gamma \Rightarrow 1_{\mathcal{A}}$, *then* $(u\gamma^{'}, u\psi)$ *is a left Kan extension of* u *along* γ
2. *If* γ *admits a right adjoint* $\gamma^{'}$ *with adjunction unit* $\phi : 1_{\mathcal{A}} \Rightarrow \gamma^{'}\gamma$ *and counit* $\psi : \gamma\gamma^{'} \Rightarrow 1_{\mathcal{A}'}$, *then* $(u\gamma^{'}, u\phi)$ *is a right Kan extension of* u *along* γ

PROOF. We only prove (1).

$$\begin{tikzcd}
\mathcal{A} \arrow[rd, "u"] \arrow[d, shift left, "\gamma"] & \\
\mathcal{A}^{'} \arrow[u, shift left, "\gamma^{'}"] \arrow[r, bend left, "u\gamma^{'}"'] \arrow[r, bend right, "\Lambda"'] & \mathcal{B}
\arrow[from=1-1, to=2-2, phantom, "u\psi \Uparrow", near end, swap]
\arrow[from=2-1, to=2-2, phantom, "\delta \Uparrow"]
\end{tikzcd}$$

For any pair (Λ, λ) with $\Lambda : \mathcal{A}^{'} \to \mathcal{B}$ and $\lambda : \Lambda\gamma \Rightarrow u$, we'd like to show that there exists a unique natural map $\delta : \Lambda \Rightarrow u\gamma^{'}$ with

$$\lambda = (u\psi) \circ (\delta\gamma) \tag{4.8}$$

To show existence, we define $\delta : \Lambda \Rightarrow \Lambda\gamma\gamma^{'} \Rightarrow u\gamma^{'}$ as the composition

$$\delta = (\lambda\gamma^{'}) \circ (\Lambda\phi) \tag{4.9}$$

δ defined by (4.9) satisfies (4.8), using the commutative diagram

$$\begin{tikzcd}
\Lambda\gamma \arrow[r, Rightarrow, "\Lambda\phi\gamma"] \arrow[rd, Rightarrow, "1_{\Lambda\gamma}"'] & \Lambda\gamma\gamma^{'}\gamma \arrow[r, Rightarrow, "\lambda\gamma^{'}\gamma"] \arrow[d, Rightarrow, "\Lambda\gamma\psi"'] & u\gamma^{'}\gamma \arrow[d, Rightarrow, "u\psi"] \\
& \Lambda\gamma \arrow[r, Rightarrow, "\lambda"] & u
\end{tikzcd}$$

To show uniqueness, if a map δ satisfies (4.8) then δ satisfies (4.9) because of the commutative diagram

$$\begin{tikzcd}
\Lambda \arrow[r, Rightarrow, "\delta"] \arrow[d, Rightarrow, "\Lambda\phi"'] & u\gamma' \arrow[d, Rightarrow, "u\gamma'\phi"'] \arrow[dr, Rightarrow, "1_{u\gamma'}"] & \\
\Lambda\gamma\gamma' \arrow[r, Rightarrow, "\delta\gamma\gamma'"'] & u\gamma'\gamma\gamma' \arrow[r, Rightarrow, "u\psi\gamma'"'] & u\gamma'
\end{tikzcd}$$

☐

We next show that Kan extensions commute with composition *along the base functor* γ.

Proposition 4.3.4. *Consider three functors* $u : \mathcal{A} \to \mathcal{B}$, $\gamma : \mathcal{A} \to \mathcal{A}'$ *and* $\gamma' : \mathcal{A}' \to \mathcal{A}''$.

1. *If* $(\mathbf{L}_\gamma u, \epsilon)$ *and* $(\mathbf{L}_{\gamma'}\mathbf{L}_\gamma u, \epsilon')$ *exist, then* $(\mathbf{L}_{\gamma'}\mathbf{L}_\gamma u,\ \epsilon \star \epsilon')$ *is a left Kan extension of* u *along* $\gamma'\gamma$.
2. *If* $(\mathbf{R}_\gamma u, \nu)$ *and* $(\mathbf{R}_{\gamma'}\mathbf{R}_\gamma u, \nu')$ *exist, then* $(\mathbf{R}_{\gamma'}\mathbf{R}_\gamma u,\ \nu' \star \nu)$ *is a left Kan extension of* u *along* $\gamma'\gamma$.

Proof. Immediate using the universal property of the left (resp. right) Kan extensions. ☐

We state a corollary needed in the proof of Thm. 6.6.3.

Corollary 4.3.5. *Consider two functors* $u : \mathcal{A} \to \mathcal{B}$ *and* $\gamma : \mathcal{A} \to \mathcal{A}'$, *and a pair of inverse equivalences of categories* $\gamma' : \mathcal{A}' \leftrightarrows \mathcal{A}'' : \gamma''$, *with natural isomorphisms* $\phi : 1_{\mathcal{A}'} \Rightarrow \gamma''\gamma'$ *and* $\psi : 1_{\mathcal{A}''} \Rightarrow \gamma'\gamma''$.

1. *The left Kan extension* $(\mathbf{L}_\gamma u, \epsilon)$ *exists iff the left Kan extension* $(\mathbf{L}_{\gamma'\gamma}u, \epsilon')$ *exists. If they both exist, then the latter is isomorphic to* $((\mathbf{L}_\gamma u)\gamma'', \epsilon\star\psi)$.
2. *The right Kan extension* $(\mathbf{R}_\gamma u, \nu)$ *exists iff the right Kan extension* $(\mathbf{R}_{\gamma'\gamma}u, \nu')$ *exists. If they both exist, then the latter is isomorphic to* $((\mathbf{R}_\gamma u)\gamma'', \phi \star \nu)$.

Proof. Consequence of Prop. 4.3.3 and Prop. 4.3.4. ☐

4.4. Total derived functors

If we specialize the definition of the left and right Kan extensions to the context of categories with weak equivalences, we obtain the notion of *total left and right derived functors*.

Definition 4.4.1. Suppose that $(\mathcal{M}_1, \mathcal{W}_1), (\mathcal{M}_2, \mathcal{W}_2)$ are two categories with weak equivalences, with localization functors denoted $\gamma_{\mathcal{M}_1}$ and respectively $\gamma_{\mathcal{M}_1}$, and suppose that $u : \mathcal{M}_1 \to \mathcal{M}_2$ is a functor.

1. The *total left derived* functor of u, denoted $(\mathbf{L}u, \epsilon)$, is the left Kan extension $(\mathbf{L}_{\gamma_{\mathcal{M}_1}}(\gamma_{\mathcal{M}_2}u), \epsilon)$ of $\gamma_{\mathcal{M}_2}u$ along $\gamma_{\mathcal{M}_1}$

$$\begin{tikzcd}
\mathcal{M}_1 \arrow[r, "u"] \arrow[d, "\gamma_{\mathcal{M}_1}"'] \arrow[dr, phantom, "\epsilon\Nearrow"] & \mathcal{M}_2 \arrow[d, "\gamma_{\mathcal{M}_2}"] \\
\mathbf{ho}\mathcal{M}_1 \arrow[r, "\mathbf{L}u"'] & \mathbf{ho}\mathcal{M}_2
\end{tikzcd}$$

(2) The *total right derived* functor of u, denoted $(\mathbf{R}u, \nu)$, is the right Kan extension $(\mathbf{R}_{\gamma_{\mathcal{M}_1}}(\gamma_{\mathcal{M}_2}u), \nu)$ of $\gamma_{\mathcal{M}_2}u$ along $\gamma_{\mathcal{M}_1}$

$$\begin{tikzcd}
\mathcal{M}_1 \arrow[r, "u"] \arrow[d, "\gamma_{\mathcal{M}_1}"'] \arrow[dr, phantom, "\not\swarrow_{\eta}"] & \mathcal{M}_2 \arrow[d, "\gamma_{\mathcal{M}_2}"] \\
\mathbf{ho}\mathcal{M}_1 \arrow[r, "\mathbf{R}u"'] & \mathbf{ho}\mathcal{M}_2
\end{tikzcd}$$

The total left and derived functors $(\mathbf{L}u, \epsilon), (\mathbf{R}u, \nu)$ are defined in terms of the localization functors $\gamma_{\mathcal{M}_1}, \gamma_{\mathcal{M}_2}$ and therefore will not change if we replace in the definition $\mathcal{W}_1$, $\mathcal{W}_2$ with their saturations $\overline{\mathcal{W}_1}$, $\overline{\mathcal{W}_2}$.

Note that if $u(\mathcal{W}_1) \subset \overline{\mathcal{W}_2}$ then $\mathbf{L}u = \mathbf{R}u = \mathbf{ho}u$.

4.5. Left and right approximation functors.

Our next goal is to provide an existence result for total left (and right) derived functors, which will be applied in the next section to the case of (co)fibration categories.

Definition 4.5.1. Consider a functor $t : \mathcal{M}^{'} \to \mathcal{M}$ between two categories with weak equivalences $(\mathcal{M}^{'}, \mathcal{W}^{'})$, $(\mathcal{M}, \mathcal{W})$ with the property that

(a) t sends weak equivalences to weak equivalences and induces an equivalence of categories $\mathbf{ho}t : \mathbf{ho}\mathcal{M}^{'} \to \mathbf{ho}\mathcal{M}$.

The functor t is called a *left approximation* if

(1b) For any object A of $\mathcal{M}$ there exists a weak equivalence map $tA^{'} \to A$ with $A^{'}$ an object of $\mathcal{M}^{'}$

(1c) Any diagram of full maps with p a weak equivalence

$$\begin{tikzcd}
tA^{'} \arrow[d, dashed, "ti^{'}"'] \arrow[ddr, "f"] & \\
tD^{'} \arrow[dr, dashed, "\sigma"] & \\
tB^{'} \arrow[u, dashed, "tj^{'}", "\sim"'] \arrow[r, "\sim", "p"'] & B
\end{tikzcd}$$

can be completed to a dotted commutative diagram with $i^{'}, j^{'}$ maps of $\mathcal{M}^{'}$ and $j^{'}$ a weak equivalence (and therefore $tj^{'}$ a weak equivalence).

Dually, the functor t is called a *right approximation* if

(2b) For any object A of $\mathcal{M}$ there exists a weak equivalence map $A \to tA^{'}$ with $A^{'}$ an object of $\mathcal{M}^{'}$

(2c) Any diagram of full maps with i a weak equivalence

$$\begin{tikzcd}
A \arrow[r, "i", "\sim"'] \arrow[dr, dashed, "\sigma"'] \arrow[ddr, "f"'] & tA^{'} \\
& tS^{'} \arrow[u, dashed, "tq^{'}"', "\sim"] \arrow[d, dashed, "tp^{'}"'] \\
& tB^{'}
\end{tikzcd}$$

can be completed to a dotted commutative diagram with $p^{'}, q^{'}$ maps of $\mathcal{M}^{'}$ and $q^{'}$ a weak equivalence (and therefore $tq^{'}$ a weak equivalence).

A family of examples of left and right approximations comes from

Lemma 4.5.2.

(1) *Cofibrant approximation functors are left approximations.*
(2) *Fibrant approximation functors are right approximations.*

Proof. We only prove (1). If $t : \mathcal{M}^{'} \to \mathcal{M}$ is a cofibrant approximation of a precofibration category $\mathcal{M}$, then t sends weak equivalences to weak equivalences by CFA2. $\mathbf{ho}t$ is an equivalence of categories by Thm. 3.5.12. Axiom (1b) follows from CFA4. To prove axiom (1c), use CFA4 to construct a factorization $f + p : tA^{'} \sqcup tB^{'} \to B$ as a cofibration $ti^{'} + tj^{'}$ followed by a weak equivalence σ. Since $\sigma \circ tj^{'} = p$, the maps $tj^{'}$ and therefore $j^{'}$ are weak equivalences. It follows that the functor t is a left approximation. ☐

The analogue of (co)fibrant splitting in this context is given by

Definition 4.5.3. Let $t : \mathcal{M}^{'} \to \mathcal{M}$ be a left approximation of categories with weak equivalences $(\mathcal{M}^{'}, \mathcal{W}^{'})$ and $(\mathcal{M}, \mathcal{W})$. A *left splitting along* t consists of the following data:

(1) For any object A of $\mathcal{M}$, an object $C(A)$ of $\mathcal{M}^{'}$ and a weak equivalence $p_A : tC(A) \to A$
(2) For any map $f : A \to B$, a commutative diagram

$$\begin{tikzcd}
tC(A) \arrow[r, "p_A", "\sim"'] \arrow[d, "ti(f)"'] & A \arrow[dd, "f"] \\
tD(f) \arrow[dr, "\sigma(f)"] & \\
tC(B) \arrow[u, "tj(f)", "\sim"'] \arrow[r, "p_B", "\sim"'] & B
\end{tikzcd}$$

with $j(f)$ a weak equivalence of $\mathcal{M}^{'}$

The left splitting along t is *normalized* if for any object A of $\mathcal{M}$ we have $D(1_A) = C(A)$, $i(1_A) = j(1_A) = 1_{C(A)}$ and $\sigma(1_A) = p_A$.

The dual definition for right approximations is

Definition 4.5.4. Let $t : \mathcal{M}^{'} \to \mathcal{M}$ be a right approximation of categories with weak equivalences $(\mathcal{M}^{'}, \mathcal{W}^{'})$ and $(\mathcal{M}, \mathcal{W})$. A *right splitting along* t consists of the following data:

(1) For any object A of $\mathcal{M}$, an object $R(A)$ of $\mathcal{M}^{'}$ and a weak equivalence $i_A : A \to tR(A)$
(2) For any map $f : A \to B$, a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "i_A", "\sim"'] \arrow[dr, "\sigma(f)"'] \arrow[dd, "f"'] & tR(A) \\
 & tS(f) \arrow[u, "\sim", "tq(f)"'] \arrow[d, "tp(f)"] \\
B \arrow[r, "i_B", "\sim"'] & tR(B)
\end{tikzcd}$$

with $q(f)$ a weak equivalence of $\mathcal{M}^{'}$.

The right splitting along t is *normalized* if for any object A of $\mathcal{M}$ we have $S(1_A) = R(A)$, $p(1_A) = q(1_A) = 1_{R(A)}$ and $\sigma(1_A) = i_A$.

Clearly any left approximation admits a normalized left splitting, and any right approximation admits a normalized right splitting. A cofibrant splitting is a left splitting, and a normalized cofibrant splitting is a normalized left splitting. Dually a (normalized) fibrant splitting is a (normalized) right splitting.

We also need the following

DEFINITION 4.5.5. Suppose that $(\mathcal{M}^{'}, \mathcal{W}^{'})$ and $(\mathcal{M}, \mathcal{W})$ are two categories with weak equivalences.

(1) Let $t : \mathcal{M}^{'} \to \mathcal{M}$ be a left approximation functor.
    (a) An object $A^{'} \epsilon \mathcal{M}^{'}$ is *regular* with respect to a left splitting along t (referred to simply as *regular* if no confusion is possible) if $C(tA^{'}) = A^{'}$ and $p_{tA^{'}} = 1_{tA^{'}}$.
    (b) A *regular left splitting* along t is a normalized left splitting with the additional property that any object $A^{'} \epsilon \mathcal{M}^{'}$ admits an object $A^{''} \epsilon \mathcal{M}^{'}$, regular with respect to the left splitting, such that $A^{''}$ is isomorphic to $A^{'}$ in $\mathbf{ho}\mathcal{M}^{'}$.
(2) Let $t : \mathcal{M}^{'} \to \mathcal{M}$ be a right approximation functor.
    (a) An object $A^{'} \epsilon \mathcal{M}^{'}$ is *regular* with respect to a right splitting along t if $R(tA^{'}) = A^{'}$ and $i_{tA^{'}} = 1_{tA^{'}}$.
    (b) A *regular right splitting* along t is a normalized right splitting with the additional property that any object $A^{'} \epsilon \mathcal{M}^{'}$ admits an object $A^{''} \epsilon \mathcal{M}^{'}$, regular with respect to the right splitting, with $A^{''}$ isomorphic to $A^{'}$ in $\mathbf{ho}\mathcal{M}^{'}$.

We have seen that a left approximation functor $t : \mathcal{M}^{'} \to \mathcal{M}$ by definition induces an equivalence of categories $\mathbf{ho}t : \mathbf{ho}\mathcal{M}^{'} \to \mathbf{ho}\mathcal{M}$. As a consequence, a normalized left splitting along t is regular iff any object $A \epsilon \mathcal{M}$ admits a regular object $A^{''} \epsilon \mathcal{M}^{'}$ such that $tA^{''}$ is isomorphic to A in $\mathbf{ho}\mathcal{M}$.

It is not hard to see that any left (resp. right) approximation functor admits a regular left (resp. right) splitting. If a left (resp. right) approximation $t : \mathcal{M}^{'} \to \mathcal{M}$ is *injective on objects*, we could actually construct a normalized left (resp. right) splitting such that all objects of $\mathcal{M}^{'}$ are regular.

With these definitions at hand, we can proceed to prove an existence property for total derived functors.

THEOREM 4.5.6. *Given three categories with weak equivalences* $(\mathcal{M}'_1, \mathcal{W}'_1)$, $(\mathcal{M}_1, \mathcal{W}_1)$ *and* $(\mathcal{M}_2, \mathcal{W}_2)$ *and two functors* $\mathcal{M}'_1 \xrightarrow{t} \mathcal{M}_1 \xrightarrow{u} \mathcal{M}_2$.

1. *If t is a left approximation and ut preserves weak equivalences, then u admits a total left derived functor $(\mathbf{L}u, \epsilon)$. The natural map $\epsilon : (\mathbf{L}u)(tA') \Rightarrow utA'$ is an isomorphism in objects A' of $\mathcal{M}'_1$.*
2. *If t is a right approximation and ut preserves weak equivalences, then u admits a total right derived functor $(\mathbf{R}u, \eta)$. The natural map $\eta : utA' \Rightarrow (\mathbf{R}u)(tA')$ is an isomorphism in A'.*

PROOF. We only prove (1). Pick a regular left splitting along t.

$$\begin{tikzcd}
tC(A) \arrow[r, "p_A", "\sim"'] \arrow[d, "ti(f)"'] & A \arrow[dd, "f"] \\
tD(f) \arrow[dr, "\sigma(f)"] & \\
tC(B) \arrow[u, "tj(f)", "\sim"'] \arrow[r, "p_B", "\sim"'] & B
\end{tikzcd} \tag{4.10}$$

We know that $\mathbf{ho}t$ is an equivalence of categories. To construct a quasi-inverse of $\mathbf{ho}t$, we need to pick for each object A of $\mathcal{M}_1$ an object A' of $\mathcal{M}'_1$ such that tA' is weakly equivalent to A. Denote $\mathbf{s} : \mathbf{ho}\mathcal{M}_1 \to \mathbf{ho}\mathcal{M}'_1$ the quasi-inverse to $\mathbf{ho}t$ such that $\mathbf{s}(A) = C(A)$. We define $\mathbf{L}u = \mathbf{ho}(ut) \circ \mathbf{s}$.

Let us spell out in detail the functor $\mathbf{L}u$. For an object A of $\mathcal{M}_1$, we have $(\mathbf{L}u)(A) = utC(A)$. On maps $f : A \to B$ of $\mathcal{M}_1$, we have $\mathbf{L}u(f) = (utj(f))^{-1}uti(f)$.

We define $\epsilon : (\mathbf{L}u)(A) \to uA$ as $u(p_A)$. The commutativity of the diagram (4.10) implies that $\epsilon : (\mathbf{L}u)\gamma_{\mathcal{M}_1} \Rightarrow \gamma_{\mathcal{M}_2}u$ is a natural map.

We need to show that the pair $(\mathbf{L}u, \epsilon)$ is *terminal* among pairs (Λ, λ) where $\Lambda : \mathbf{ho}\mathcal{M}_1 \to \mathbf{ho}\mathcal{M}_2$ is a functor and $\lambda : \Lambda\gamma_{\mathcal{M}_1} \Rightarrow \gamma_{\mathcal{M}_2}u$ is a natural transformation. For any object A of $\mathcal{M}_1$ the sequence of full maps in $\mathbf{ho}\mathcal{M}_2$ and their inverses

$$\begin{tikzcd}
\Lambda tC(A) \arrow[r, "\lambda(tC(A))"] \arrow[d, "\Lambda(p_A)"', "\sim"] & utC(A) = \mathbf{L}u(A) \arrow[d, dashed, "u(p_A)=\epsilon(A)"] \\
\Lambda(A) \arrow[r, dashed, "\lambda(A)"] & uA
\end{tikzcd} \tag{4.11}$$

defines a map $\delta : \Lambda(A) \to \mathbf{L}u(A)$. For maps $f : A \to B$ of $\mathcal{M}_1$ we have a commutative diagram in $\mathbf{ho}\mathcal{M}_2$

$$\begin{tikzcd}
\Lambda(A) \arrow[dd, "\Lambda(f)"'] & \Lambda tC(A) \arrow[l, "\Lambda(p_A)"', "\sim"] \arrow[r, "\lambda(tC(A))"] \arrow[d, "\Lambda ti(f)"] & utC(A) \arrow[d, "uti(f)"] \\
 & \Lambda tD(f) \arrow[r, "\lambda(tD(f))"] \arrow[dl, "\Lambda\sigma(f)"'] & utD(f) \\
\Lambda(B) & \Lambda tC(B) \arrow[l, "\sim", "\Lambda(p_B)"'] \arrow[r, "\lambda(tC(B))"] \arrow[u, "\Lambda tj(f)", "\sim"'] & utC(B) \arrow[u, "utj(f)", "\sim"']
\end{tikzcd} \tag{4.12}$$

where $utj(f)$ is a weak equivalence since $j(f)$ is. The commutativity of (4.12) implies that $\delta : \Lambda \Rightarrow \mathbf{L}u$ is natural in maps of $\mathcal{M}_1$, therefore natural in maps of $\mathbf{ho}\mathcal{M}_1$.

The commutativity of diagram (4.11) shows that we have $\epsilon \star \delta = \lambda$.

The natural map $\epsilon : (\mathbf{L}u)(tA') \Rightarrow utA'$ is an isomorphism for *regular* objects $A'\epsilon\mathcal{M}_1'$, therefore for any object $A'\epsilon\mathcal{M}_1'$. To see that, note that for A' regular the map $\epsilon : (\mathbf{L}u)(tA') \Rightarrow utA'$ can be identified with $1_{utA'}$. Furthermore, our left splitting was assumed to be regular so any object in $\mathcal{M}_1'$ is isomorphic in $\mathbf{ho}\mathcal{M}_1'$ to a regular object A'. ☐

Corollary 4.5.7.

(1) *If* t, u *are as in Thm. 4.5.6 (1) and* $\mathbf{s}$ *denotes a quasi-inverse of* $\mathbf{ho}t$*, then* $\mathbf{ho}(ut)\,\mathbf{s}$ *is naturally isomorphic to* $\mathbf{L}u$*.*
(2) *If* t, u *are as in Thm. 4.5.6 (2) and* $\mathbf{s}$ *denotes a quasi-inverse of* $\mathbf{ho}t$*, then* $\mathbf{ho}(ut)\,\mathbf{s}$ *is naturally isomorphic to* $\mathbf{R}u$*.* ☐

We next state an adjunction property of total derived functors.

Theorem 4.5.8 (Abstract Quillen adjunction). *Given four categories with weak equivalences* $(\mathcal{M}_1', \mathcal{W}_1')$, $(\mathcal{M}_1, \mathcal{W}_1)$, $(\mathcal{M}_2', \mathcal{W}_2')$, $(\mathcal{M}_2, \mathcal{W}_2)$ *and four functors*

$$\begin{tikzcd}
\mathcal{M}_1' \arrow[r, "t_1"] & \mathcal{M}_1 \arrow[r, shift left, "u_1"] & \mathcal{M}_2 \arrow[l, shift left, "u_2"] & \mathcal{M}_2' \arrow[l, "t_2"']
\end{tikzcd}$$

where

(1) $u_1 \dashv u_2$ *is an adjoint pair*
(2) t_1 *is a left approximation,* t_2 *is a right approximation*
(3) u_1t_1 *and* u_2t_2 *preserve weak equivalences*

then $\mathbf{L}u_1 \dashv \mathbf{R}u_2$ *is a naturally adjoint pair*

$$\begin{tikzcd}
\mathbf{ho}\mathcal{M}_1 \arrow[r, shift left, "\mathbf{L}u_1"] & \mathbf{ho}\mathcal{M}_2 \arrow[l, shift left, "\mathbf{R}u_2"]
\end{tikzcd}$$

If additionally

(4) *(resp. (4r), resp. (4l)). For any objects* $A'\epsilon\mathcal{M}_1'$, $B'\epsilon\mathcal{M}_2'$*, a map* $u_1t_1A' \to t_2B'$ *is a weak equivalence iff (resp. if, resp. only if) its adjoint* $t_1A' \to u_2t_2B'$ *is a weak equivalence*

then $\mathbf{L}u_1$ *and* $\mathbf{R}u_2$ *are inverse equivalences of categories (resp.* $\mathbf{R}u_2$ *is fully faithful, resp* $\mathbf{L}u_1$ *is fully faithful).*

This theorem suggests the following

Definition 4.5.9. We will call the functors u_1, u_2 satisfying the properties (1), (2), (3) of Thm. 4.5.8 an *abstract Quillen adjoint* pair with respect to t_1, t_2. If the additional property (4) is satisfied, we will call u_1, u_2 an *abstract Quillen equivalence* pair with respect to t_1, t_2.

Thm. 4.5.8 will be a consequence of the more general theorem below. But before stating next theorem, let us introduce more notations and definitions. Given a diagram of functors

(4.13)
$$\begin{tikzcd}
\mathcal{A} \arrow[r, "v_1"] \arrow[d, "t_1"'] & \mathcal{B} \\
\mathcal{C} & \mathcal{D} \arrow[l, "v_2"] \arrow[u, "t_2"']
\end{tikzcd}$$

a *partial adjunction* between v_1, v_2 with respect to t_1, t_2 is a bijection

$$\zeta : Hom_{\mathcal{B}}(v_1 A, t_2 D) \cong Hom_{\mathcal{C}}(t_1 A, v_2 D)$$

natural in objects $A \epsilon \mathcal{A}_1, D \epsilon \mathcal{D}$. Whenever we say that v_1, v_2 is an adjoint pair with respect to t_1, t_2 we refer to a particular bijection ζ. Note that ζ^{-1} defines a partial adjunction between t_1, t_2 with respect to v_1, v_2.

If in addition we have adjoint pairs $t_1 \dashv s_1$, $s_2 \dashv t_2$

(4.14)
$$\begin{tikzcd}
\mathcal{A} \arrow[r, "v_1"] \arrow[d, shift right, "t_1"'] & \mathcal{B} \arrow[d, shift right, "s_2"'] \\
\mathcal{C} \arrow[u, shift right, "s_1"'] & \mathcal{D} \arrow[l, "v_2"] \arrow[u, shift right, "t_2"']
\end{tikzcd}$$

then the partial adjunctions of v_1, v_2 with respect to t_1, t_2 are in a one to one correspondence with adjunctions $s_2 v_1 \dashv s_1 v_2$. In the diagram (4.14), assuming that s_1, t_1 and s_2, t_2 are inverse equivalences of categories, then the partial adjuctions of v_1, v_2 with respect to t_1, t_2 are in a one to one correspondence with adjunctions $s_2 v_1 \dashv s_1 v_2$, and furthermore in a one to one correspondence with adjunctions $v_1 s_1 \dashv v_2 s_2$.

In a diagram of the form

(4.15)
$$\begin{tikzcd}
\mathcal{A}^{'} \arrow[r, "a"] & \mathcal{A} \arrow[r, "v_1"] \arrow[d, "t_1"'] & \mathcal{B} & \\
& \mathcal{C} & \mathcal{D} \arrow[l, "v_2"] \arrow[u, "t_2"'] & \mathcal{D}^{'} \arrow[l, "d"']
\end{tikzcd}$$

a partial adjunction between v_1, v_2 with respect to t_1, t_2 will induce a partial adjunction between $v_1 a, v_2 d$ with respect to $t_1 a, t_2 d$, given by

$$\zeta : Hom_{\mathcal{B}}(v_1 a A^{'}, t_2 d D^{'}) \cong Hom_{\mathcal{C}}(t_1 a A^{'}, v_2 d D^{'})$$

We can now state

Theorem 4.5.10 (Abstract Quillen partial adjunction). *Given four categories with weak equivalences* $(\mathcal{M}_1^{'}, \mathcal{W}_1^{'})$, $(\mathcal{M}_1, \mathcal{W}_1)$, $(\mathcal{M}_2^{'}, \mathcal{W}_2^{'})$, $(\mathcal{M}_2, \mathcal{W}_2)$ *and four functors* t_1, t_2, v_1, v_2

$$\begin{tikzcd}
\mathcal{M}_1^{'} \arrow[r, "v_1"] \arrow[d, "t_1"'] & \mathcal{M}_2 \\
\mathcal{M}_1 & \mathcal{M}_2^{'} \arrow[l, "v_2"'] \arrow[u, "t_2"']
\end{tikzcd}$$

such that:

(1) *v_1, v_2 are partially adjoint with respect to t_1, t_2, meaning that there exists a bijection*

$$\zeta : Hom_{\mathcal{M}_2}(v_1 A^{'}, t_2 B^{'}) \cong Hom_{\mathcal{M}_1}(t_1 A^{'}, v_2 B^{'})$$

natural in $A^{'} \epsilon \mathcal{M}_1^{'}$ and $B^{'} \epsilon \mathcal{M}_2^{'}$

(2) *t_1 is a left approximation, t_2 is a right approximation*

(3) *v_1 and v_2 preserve weak equivalences*

Then $\mathbf{ho}v_1$, $\mathbf{ho}v_2$ are naturally partial adjoint with respect to $\mathbf{ho}t_1$, $\mathbf{ho}t_2$. Equivalently, denote $\mathbf{s}_i$ a quasi-inverse of $\mathbf{ho}t_i$, and let $\mathbf{V}_i = \mathbf{ho}(v_i)\mathbf{s}_i$ for $i = 1, 2$. Then $\mathbf{V}_1 \dashv \mathbf{V}_2$

$$\begin{tikzcd}
\mathbf{ho}\mathcal{M}_1 \arrow[r, shift left, "\mathbf{V}_1"] & \mathbf{ho}\mathcal{M}_2 \arrow[l, shift left, "\mathbf{V}_2"]
\end{tikzcd}$$

is a naturally adjoint pair.

If additionally

(4) *(resp. (4r), resp. (4l)). For any objects $A^{'} \epsilon \mathcal{M}_1^{'}$, $B^{'} \epsilon \mathcal{M}_2^{'}$, a map $v_1 A^{'} \to t_2 B^{'}$ is a weak equivalence iff (resp. if, resp. only if) its partial adjoint $t_1 A^{'} \to v_2 B^{'}$ is a weak equivalence*

then $\mathbf{V}_1$ and $\mathbf{V}_2$ are inverse equivalences of categories (resp. $\mathbf{V}_2$ is fully faithful, resp $\mathbf{V}_1$ is fully faithful).

The following definition is suggested:

Definition 4.5.11. We will call the functors v_1, v_2 satisfying the properties (1), (2), (3) of Thm. 4.5.10 an *abstract Quillen partially adjoint* pair with respect to t_1, t_2. If the additional property (4) is satisfied, we will call v_1, v_2 an *abstract Quillen partial equivalence* pair with respect to t_1, t_2.

Proof of Thm. 4.5.8 assuming Thm. 4.5.10. Since $u_1 \dashv u_2$ is an adjoint pair, we see that $v_1 = u_1 t_1$, $v_2 = u_2 t_2$ is partially adjoint with respect to t_1, t_2. From Cor. 4.5.7, we have natural isomorphisms $\mathbf{L}u_1 \cong \mathbf{ho}(u_1 t_1)\,\mathbf{s}_1 = \mathbf{V}_1$ and $\mathbf{R}u_2 \cong \mathbf{ho}(u_2 t_2)\,\mathbf{s}_2 = \mathbf{V}_2$. The statement now follows. ☐

Proof of Thm. 4.5.10. If we can prove the conclusion for a particular choice of $\mathbf{s}_1 : \mathbf{ho}\mathcal{M}_1 \to \mathbf{ho}\mathcal{M}_1^{'}$ and $\mathbf{s}_2 : \mathbf{ho}\mathcal{M}_2 \to \mathbf{ho}\mathcal{M}_2^{'}$, then the conclusion follows for any $\mathbf{s}_1$ and $\mathbf{s}_2$.

We pick a regular left splitting along t_1, and a regular right splitting along t_2. As in the proof of Thm. 4.5.6, the normalized left splitting along t_1 determines a choice of $\mathbf{s}_1$ such that $\mathbf{s}_1(A) = C(A)$ for all objects A of $\mathcal{M}_1$. The normalized right splitting along t_2 determines a choice of $\mathbf{s}_2$ such that $\mathbf{s}_2(B) = R(B)$ for all objects B of $\mathcal{M}_2$. We will work with these particular choices of $\mathbf{s}_1$ and $\mathbf{s}_2$.

Denote $\mathbf{V}_1 = \mathbf{ho}(v_1)\mathbf{s}_1$ and $\mathbf{V}_2 = \mathbf{ho}(v_2)\mathbf{s}_2$. We will construct natural maps $\mathbf{\Phi} : 1_{\mathbf{ho}\mathcal{M}_1} \Rightarrow \mathbf{V}_2\mathbf{V}_1$ and $\mathbf{\Psi} : \mathbf{V}_1\mathbf{V}_2 \Rightarrow 1_{\mathbf{ho}\mathcal{M}_2}$, and show that they are the unit and counit of an adjunction between $\mathbf{V}_1$ and $\mathbf{V}_2$. We start by constructing a natural map

$$\overline{\mathbf{\Phi}} : \mathbf{ho}t_1 \Rightarrow \mathbf{ho}(v_2)\,\mathbf{s}_2\,\mathbf{ho}(v_1)$$

by defining $\overline{\mathbf{\Phi}}(A^{'}) = \zeta i_{v_1A'}$ for any object $A^{'}$ of $\mathbf{ho}\mathcal{M}_1^{'}$, where $i_{v_1A'}$ and $\zeta i_{v_1A'}$ are the maps

$$v_1A^{'} \xrightarrow[\sim]{i_{v_1A'}} t_2R(v_1A^{'}) \qquad t_1A^{'} \xrightarrow{\zeta i_{v_1A'}} v_2R(v_1A^{'})$$

Given a map $f : A^{'} \to B^{'}$ in $\mathcal{M}_1^{'}$ we get a commutative diagram

$$\begin{tikzcd}
v_1A^{'} \arrow[r, "i_{v_1A'}", "\sim"'] \arrow[dd, "v_1f"'] \arrow[dr, "\sigma(v_1f)"'] & t_2R(v_1A^{'}) \\
 & t_2S(v_1f) \arrow[u, "t_2q(v_1f)"', "\sim"] \arrow[d, "t_2p(v_1f)"] \\
v_1B^{'} \arrow[r, "i_{v_1B'}", "\sim"'] & t_2R(v_1B^{'})
\end{tikzcd}$$

in $\mathcal{M}_2$, where $q(v_1f)$ is a weak equivalence. Applying the natural bijection ζ we get a commutative diagram in $\mathcal{M}_1$

$$\begin{tikzcd}
t_1A^{'} \arrow[r, "\zeta i_{v_1A'}"] \arrow[dd, "t_1f"'] \arrow[dr, "\zeta\sigma(v_1f)"'] & v_2R(v_1A^{'}) \\
 & v_2S(v_1f) \arrow[u, "v_2q(v_1f)"', "\sim"] \arrow[d, "v_2p(v_1f)"] \\
t_1B^{'} \arrow[r, "\zeta i_{v_1B'}"] & v_2R(v_1B^{'})
\end{tikzcd}$$

where $v_2q(v_1f)$ is a weak equivalence since $q(v_1f)$ is. The commutativity of the second diagram shows that $\overline{\mathbf{\Phi}}$ is natural in maps of $\mathcal{M}_1^{'}$, and therefore natural in maps of $\mathbf{ho}\mathcal{M}_1^{'}$.

Since $\mathbf{ho}t_1$ and $\mathbf{s}_1$ are quasi-inverses of each other, the natural map $\overline{\mathbf{\Phi}} : \mathbf{ho}t_1 \Rightarrow \mathbf{ho}(v_2)\,\mathbf{s}_2\,\mathbf{ho}(v_1)$ yields the desired natural map

$$\mathbf{\Phi} : 1_{\mathbf{ho}\mathcal{M}_1} \Rightarrow \mathbf{ho}(v_2)\,\mathbf{s}_2\,\mathbf{ho}(v_1)\,\mathbf{s}_1 = \mathbf{V}_2\mathbf{V}_1$$

We dually construct a natural map

$$\overline{\mathbf{\Psi}} : \mathbf{ho}(v_1)\,\mathbf{s}_1\,\mathbf{ho}(v_2) \Rightarrow \mathbf{ho}t_2$$

by defining $\overline{\mathbf{\Psi}}(B^{'}) = \zeta^{-1}p_{v_2B'}$ for any object $B^{'}$ of $\mathbf{ho}\mathcal{M}_2^{'}$, where the maps $p_{v_2B'}$ and $\zeta^{-1}p_{v_2B'}$ are the maps

$$t_1C(v_2B^{'}) \xrightarrow[\sim]{p_{v_2B'}} v_2B^{'} \qquad v_1C(v_2B^{'}) \xrightarrow{\zeta^{-1}p_{v_2B'}} t_2B^{'}$$

The proof that $\overline{\mathbf{\Psi}}$ is a natural map is dual to the proof that $\overline{\mathbf{\Phi}}$ is a natural map. Since $\mathbf{ho}t_2$ and $\mathbf{s}_2$ are quasi-inverses of each other, the natural map $\overline{\mathbf{\Psi}}$ yields the desired natural map

$$\mathbf{\Psi} : \mathbf{V}_1\mathbf{V}_2 = \mathbf{ho}(v_1)\,\mathbf{s}_1\,\mathbf{ho}(v_2)\,\mathbf{s}_2 \Rightarrow 1_{\mathbf{ho}\mathcal{M}_2}$$

It remains to show that the natural maps $\mathbf{\Phi}, \mathbf{\Psi}$ are the unit resp. the counit of an adjunction between the functors $\mathbf{V}_1, \mathbf{V}_2$. In other words, we need to prove that the following composites are identities.

$$\mathbf{V}_1 \xRightarrow{\mathbf{V}_1\mathbf{\Phi}} \mathbf{V}_1\mathbf{V}_2\mathbf{V}_1 \xRightarrow{\mathbf{\Psi}\mathbf{V}_1} \mathbf{V}_1 \tag{4.16}$$

$$\mathbf{V}_2 \xRightarrow{\mathbf{\Phi}\mathbf{V}_2} \mathbf{V}_2\mathbf{V}_1\mathbf{V}_2 \xRightarrow{\mathbf{V}_2\mathbf{\Psi}} \mathbf{V}_2 \tag{4.17}$$

We only prove that (4.16) is an identity, since the proof for (4.17) is dual. It suffices to prove that (4.16) is an identity on objects of the form $A = t_1A^{'}$, with $A^{'}\epsilon\mathcal{M}_1^{'}$ regular with respect to the left splitting along t_1.

If $A^{'}$ is a regular object of $\mathcal{M}_1^{'}$, we have that $C(t_1A^{'}) = A^{'}$ and $p_{tA^{'}} = 1_{t_1A^{'}} : t_1C(t_1A^{'}) \to t_1A^{'}$.

Denote $h = \zeta i_{v_1A^{'}}$. The left splitting diagram associated to h yields a commutative diagram in $\mathcal{M}_1$

$$\begin{tikzcd}
t_1C(t_1A^{'}) = t_1A^{'} \arrow[r, "p_{t_1A^{'}}=1_{t_1A^{'}}"] \arrow[d, "t_1i(h)"'] & t_1A^{'} \arrow[dd, "h=\zeta i_{v_1A^{'}}"] \\
t_1D(h) \arrow[dr, "\sigma(h)"] & \\
t_1C(v_2R(v_1A^{'})) \arrow[u, "t_1j(h)", "\sim"'] \arrow[r, "\sim", "p_{v_2R(v_1A^{'})}"'] & v_2R(v_1A^{'})
\end{tikzcd}$$

Applying the natural bijection ζ^{-1} to the diagram we get a commutative diagram in $\mathcal{M}_2$

$$\begin{tikzcd}
v_1A^{'} \arrow[r, "1_{v_1A^{'}}"] \arrow[d, "v_1i(h)"'] & v_1A^{'} \arrow[dd, "\sim"', "\zeta^{-1}h=i_{v_1A^{'}}"] \\
v_1D(h) \arrow[dr, "\zeta^{-1}\sigma(h)"] & \\
v_1C(v_2R(v_1A^{'})) \arrow[u, "v_1j(h)", "\sim"'] \arrow[r, "\zeta^{-1}p_{v_2R(v_1A^{'})}"'] & t_2R(v_1A^{'})
\end{tikzcd}$$

In this diagram, $j(h)$ and therefore $v_1j(h)$ are weak equivalences.

Since $A = t_1A^{'}$, in $\mathbf{ho}\mathcal{M}_1$ we can identify $v_1A^{'}$ with $\mathbf{V}_1A$ and $v_1C(v_2R(v_1A^{'}))$ with $\mathbf{V}_1\mathbf{V}_2\mathbf{V}_1A$. Under this identification the composition $(v_1j(h))^{-1} \circ v_1i(h)$ becomes $\mathbf{V}_1\mathbf{\Phi}(A)$, and the composition $(i_{v_1A^{'}})^{-1} \circ (\zeta^{-1}p_{v_2R(v_1A^{'})})$ becomes $\mathbf{\Psi}\mathbf{V}_1(A)$. It follows that the composition (4.16) is an identity. Dually, (4.17) is an identity and we have proved that $\mathbf{V}_1 \dashv \mathbf{V}_2$ are adjoint with adjunction unit $\mathbf{\Phi}$ and counit $\mathbf{\Psi}$.

For the second part of the theorem, we assume hypothesis (4l) and we will show that $\mathbf{\Phi}$ is a natural isomorphisms. From Prop. 4.2.1, this will imply that $\mathbf{V}_1$ is fully faithful.

For any object $A^{'}$ of $\mathbf{ho}\mathcal{M}_1^{'}$, the map

$$v_1A^{'} \xrightarrow[\sim]{i_{v_1A^{'}}} t_2R(v_1A^{'})$$

and therefore from hypothesis (4) the map

$$t_1A^{'} \xrightarrow[\sim]{\zeta i_{v_1A^{'}}} v_2R(v_1A^{'})$$

are weak equivalences. It follows that the natural maps $\overline{\mathbf{\Phi}}$ and therefore $\mathbf{\Phi}$ are isomorphisms. A dual proof shows that hypothesis (4l) implies that $\mathbf{\Psi}$ is a natural isomorphism, and therefore $\mathbf{V}_2$ is fully faithful.

If hypothesis (4) is satisfied, then both $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are isomorphisms, therefore $\mathbf{V}_1, \mathbf{V}_2$ are inverse equivalences of categories (cf. Prop. 4.2.1). ☐

4.6. Total derived functors in cofibration categories

Let us now turn to the context of cofibration and fibration categories. The following result describes a sufficient condition for the existence of a total left resp. right derived functor:

THEOREM 4.6.1. *Let $(\mathcal{M}_2, \mathcal{W}_2)$ be a category with weak equivalences.*

1. *If $\mathcal{M}_1$ is a precofibration category and $u : \mathcal{M}_1 \to \mathcal{M}_2$ is a functor that sends trivial cofibrations between cofibrant objects to weak equivalences, then u admits a total left derived functor $(\mathbf{L}u, \epsilon)$. The natural map $\epsilon : (\mathbf{L}u)(A) \Rightarrow uA$ is an isomorphism for A cofibrant.*
2. *If $\mathcal{M}_1$ is a prefibration category and $u : \mathcal{M}_1 \to \mathcal{M}_2$ is a functor that sends trivial fibrations between fibrant objects to weak equivalences, then u admits a total right derived functor $(\mathbf{R}u, \eta)$. The natural map $\eta : uA \Rightarrow (\mathbf{R}u)(A)$ is an isomorphism for A fibrant.*

More generally we will prove

THEOREM 4.6.2. *Let $(\mathcal{M}_2, \mathcal{W}_2)$ be a category with weak equivalences.*

1. *If $t : \mathcal{M}_1^{'} \to \mathcal{M}_1$ is a cofibrant approximation of a precofibration category $\mathcal{M}_1$ and $u : \mathcal{M}_1 \to \mathcal{M}_2$ is a functor such that ut sends trivial cofibrations to weak equivalences, then u admits a total left derived functor $(\mathbf{L}u, \epsilon)$. The natural map $\epsilon : (\mathbf{L}u)(tA) \Rightarrow utA$ is an isomorphism for objects A of $\mathcal{M}_1^{'}$.*
2. *If $t : \mathcal{M}_1^{'} \to \mathcal{M}_1$ is a fibrant approximation of a prefibration category $\mathcal{M}_1$ and $u : \mathcal{M}_1 \to \mathcal{M}_2$ is a functor that sends trivial fibrations to weak equivalences, then u admits a total right derived functor $(\mathbf{R}u, \eta)$. The natural map $\eta : utA \Rightarrow (\mathbf{R}u)(tA)$ is an isomorphism for objects A of $\mathcal{M}_1^{'}$.*

PROOF. Let us prove part (1). We may assume that $\mathcal{W}_2$ is saturated. The composition ut sends trivial cofibrations to weak equivalences, therefore using the Brown Factorization Lemma 1.3.1 ut sends weak equivalences to weak equivalences since $\mathcal{W}_2$ is saturated. The result follows from Thm. 4.5.6 applied to the cofibrant approximation t and the functor u.

Part (2) is proved dually. ☐

And the next result describes a sufficient condition for the total left (resp. right) derived functor to be an equivalence of categories:

THEOREM 4.6.3.

(1) *If* $t : \mathcal{M}_1 \to \mathcal{M}_2$ *is a functor between precofibration categories such that its restriction* $(\mathcal{M}_1)_{cof} \to \mathcal{M}_2$ *is a cofibrant approximation, then* t *admits a total left derived functor* $(\mathbf{L}t, \epsilon)$ *and* $\mathbf{L}t$ *is an equivalence of categories. The natural map* $\epsilon : (\mathbf{L}t)(A) \Rightarrow tA$ *is an isomorphism for* A *cofibrant.*

(2) *If* $t : \mathcal{M}_1 \to \mathcal{M}_2$ *is a functor between prefibration categories such that its restriction* $(\mathcal{M}_1)_{fib} \to \mathcal{M}_2$ *is a fibrant approximation, then* t *admits a total right derived functor* $(\mathbf{R}t, \epsilon)$ *and* $\mathbf{R}t$ *is an equivalence of categories. The natural map* $\eta : tA \Rightarrow (\mathbf{R}t)(A)$ *is an isomorphism* A *fibrant.*

PROOF. For part (1), denote $i_{\mathcal{M}_1} : (\mathcal{M}_1)_{cof} \to \mathcal{M}_1$ the inclusion. The functors $i_{\mathcal{M}_1}, ti_{\mathcal{M}_1}$ are cofibrant approximations and induce equivalences of categories $\mathbf{ho}i_{\mathcal{M}_1}$, $\mathbf{ho}(ti_{\mathcal{M}_1})$ by Thm. 3.5.12.

From Thm. 4.6.2 applied to the cofibrant approximation $i_{\mathcal{M}_1}$ followed by t we see that t admits a total left derived functor $(\mathbf{L}t, \epsilon)$. Furthermore, $\epsilon : (\mathbf{L}t)(A) \Rightarrow tA$ is an isomorphism for A cofibrant, therefore $\mathbf{L}t\mathbf{ho}i_{\mathcal{M}_1} = \mathbf{ho}(ti_{\mathcal{M}_1})$ and $\mathbf{L}t$ is an equivalence of categories since $\mathbf{ho}i_{\mathcal{M}_1}$ and $\mathbf{ho}(ti_{\mathcal{M}_1})$ are equivalences.

Part (2) is proved dually. ☐

We will also introduce in the context of (co)fibration categories the notion of Quillen adjoint functors and of Quillen equivalences.

DEFINITION 4.6.4. Consider four functors

$$\begin{tikzcd} \mathcal{M}_1' \arrow[r, "t_1"] & \mathcal{M}_1 \arrow[r, shift left, "u_1"] & \mathcal{M}_2 \arrow[l, shift left, "u_2"] & \mathcal{M}_2' \arrow[l, "t_2"'] \end{tikzcd}$$

where

(1) $u_1 \dashv u_2$ is an adjoint pair.

(2) t_1 is a cofibrant approximation of a precofibration category $\mathcal{M}_1$ and t_2 is a fibrant approximation of a prefibration category $\mathcal{M}_2$

(3) u_1t_1 sends trivial cofibrations to weak equivalences and u_2t_2 sends trivial fibrations to weak equivalences

We then say that u_1, u_2 is a *Quillen adjoint* pair with respect to t_1 and t_2. If additionally

(4) For any objects $A^{'} \epsilon \mathcal{M}_1^{'}$, $B^{'} \epsilon \mathcal{M}_2^{'}$, a map $u_1t_1A^{'} \to t_2B^{'}$ is a weak equivalence iff its adjoint $t_1A^{'} \to u_2t_2B^{'}$ is a weak equivalence

we say that u_1, u_2 is a *Quillen equivalence* pair with respect to t_1 and t_2.

If the functors t_1, t_2 are implied by the context, we may refer to u_1, u_2 as a Quillen pair of adjoint functors (resp. equivalences) without direct reference to t_1 and t_2.

THEOREM 4.6.5 (Quillen adjunction).

(1) *A Quillen adjoint pair* $u_1 \dashv u_2$ *with respect to* t_1, t_2

$$\begin{tikzcd} \mathcal{M}_1' \arrow[r, "t_1"] & \mathcal{M}_1 \arrow[r, shift left, "u_1"] & \mathcal{M}_2 \arrow[l, shift left, "u_2"] & \mathcal{M}_2' \arrow[l, "t_2"'] \end{tikzcd}$$

induces a pair of adjoint functors $\mathbf{L}u_1 \dashv \mathbf{R}u_2$

$$\mathbf{L}u_1 : \mathbf{ho}\mathcal{M}_1 \rightleftarrows \mathbf{ho}\mathcal{M}_2 : \mathbf{R}u_2$$

(2) *If additionally* u_1, u_2 *satisfy (4l) (resp. (4r)) of Thm. 4.5.8 with respect to* t_1, t_2*, then* $\mathbf{L}u_1$ *(resp.* $\mathbf{R}u_2$*) are fully faithful.*

(3) *If* u_1, u_2 *is a pair of Quillen equivalences with respect to* t_1, t_2 *then*

$$\mathbf{L}u_1 : \mathbf{ho}\mathcal{M}_1 \rightleftarrows \mathbf{ho}\mathcal{M}_2 : \mathbf{R}u_2$$

is a pair of equivalences of categories.

PROOF. This is a corollary of Thm. 4.5.8. □

We leave it to the reader to formulate the definition of abstract Quillen partially adjoint functors in the context of cofibration and fibration categories, and to state the analogue in this context of Thm. 4.5.10

CHAPTER 5

Review of category theory

This chapter recalls classic constructions of category theory. We discuss over and under categories, inverse image categories, and diagram categories. The construction of limits and colimits is recalled. We discuss cofinal functors, the Grothendieck construction and some of its basic properties.

5.1. Basic definitions and notations

5.1.1. Initial and terminal categories. The initial object in a category is denoted $\mathbf{0}$, and the terminal object $\mathbf{1}$. The initial category (with an empty set of objects) is denoted $\emptyset$, and the terminal category (with one object and one identity map) is denoted e.

For a category $\mathcal{D}$ we denote $e_d : e \to \mathcal{D}$ the functor that embeds e as the object $d \epsilon \mathcal{D}$, and $p_{\mathcal{D}} : \mathcal{D} \to e$ the terminal category projection.

5.1.2. Over and under categories. If $u : \mathcal{A} \to \mathcal{B}$ is a functor and b is an object of $\mathcal{B}$, the over category $(u \downarrow b)$ has:

(1) as objects, pairs (a, g) with $a \epsilon \mathcal{A}$ and $g : ua \to b$
(2) as maps $(a_1, g_1) \to (a_2, g_2)$, the maps $f : a_1 \to a_2$ such that $g_2 \circ uf = g_1$.

The under category $(b \downarrow u)$ by definition has:

(1) as objects, pairs (a, g) with $a \epsilon \mathcal{A}$ and $g : b \to ua$
(2) as maps $(a_1, g_1) \to (a_2, g_2)$, the maps $f : a_1 \to a_2$ such that $uf \circ g_1 = g_2$.

The two definitions are dual in the sense that $(b \downarrow u) \cong (u^{op} \downarrow b)^{op}$.

We have a canonical functor $i_{u,b} : (u \downarrow b) \to A$ that sends an object $(a, g : ua \to b)$ to $a \epsilon A$ and a map $(a_1, g_1) \to (a_2, g_2)$ to the component map $a_1 \to a_2$. Dually, we have a canonical functor $i_{b,u} : (b \downarrow u) \to A$ defined by $i_{b,u} = (i_{u^{op},b})^{op}$.

If $a \epsilon \mathcal{A}$ is an object, for simplicity we denote $(\mathcal{A} \downarrow a)$ for $(1_{\mathcal{A}} \downarrow a)$ and $(a \downarrow \mathcal{A})$ for $(a \downarrow 1_{\mathcal{A}})$.

5.1.3. Inverse image category. For a functor $u : \mathcal{A} \to \mathcal{B}$ and an object b of $\mathcal{B}$, the *inverse image* of u is the subcategory $u^{-1}b$ of $\mathcal{A}$ consisting of objects a with $ua = b$ and maps $f : a \to a^{'}$ with $uf = 1_b$.

5.1.4. Categories of diagrams. If $\mathcal{D}$ and $\mathcal{M}$ are categories, the category of functors $\mathcal{D} \to \mathcal{M}$ (or $\mathcal{D}$-diagrams of $\mathcal{M}$) is denoted $\mathcal{M}^{\mathcal{D}}$.

A functor $u : \mathcal{D}_1 \to \mathcal{D}_2$ induces a functor of diagram categories denoted $u^* : \mathcal{M}^{\mathcal{D}_2} \to \mathcal{M}^{\mathcal{D}_1}$. If two functors u, v are composable, then $(uv)^* = v^* u^*$. If two functors $u \dashv v$ are adjoint, then $v^* \dashv u^*$ are adjoint.

If $\mathcal{C}$ is a class of maps of $\mathcal{M}$ (for example the weak equivalences or the cofibrations in a cofibration category), we denote $\mathcal{C}^{\mathcal{D}}$ the class of maps f of $\mathcal{M}^{\mathcal{D}}$ such that $f_d \epsilon C$ for any object $d \epsilon \mathcal{D}$.

5.2. Limits and colimits

Assume $\mathcal{D}_1$ and $\mathcal{D}_2$ are categories, $u : \mathcal{D}_1 \to \mathcal{D}_2$ is a functor and $\mathcal{M}$ is a category. A colimit functor along u, denoted $\operatorname{colim}^u : \mathcal{M}^{\mathcal{D}_1} \to \mathcal{M}^{\mathcal{D}_2}$, is by definition a left adjoint of $u^* : \mathcal{M}^{\mathcal{D}_2} \to \mathcal{M}^{\mathcal{D}_1}$. A limit functor along u, denoted $\lim^u : \mathcal{M}^{\mathcal{D}_1} \to \mathcal{M}^{\mathcal{D}_2}$, is by definition a right adjoint of u^*. colim^u is also called the left Kan extension along u, and $\lim^u$ is called the right Kan extension along u.

Being a left adjoint, if colim^u exists then it is unique up to a unique natural isomorphism. As a right adjoint, if $\lim^u$ exists it is unique up to a unique natural isomorphism.

We will also consider the case when $\operatorname{colim}^u X$ (resp. $\lim^u X$) exists for some, but not all objects X of $\mathcal{M}^{\mathcal{D}_1}$. Each of $\operatorname{colim}^u X$ and $\lim^u X$ are defined by an universal property, and if $\operatorname{colim}^u X$ or $\lim^u X$ exist, then they are unique up to unique isomorphism.

If $\mathcal{D}_1 = \mathcal{D}$ and $\mathcal{D}_2$ is the point category, colim^u is the well-known 'absolute' colimit of $\mathcal{D}$-diagrams, denoted $\operatorname{colim}^{\mathcal{D}}$ (or simply colim if there is no confusion). Dually, in this case $\lim^u$ is the 'absolute' limit $\lim^{\mathcal{D}}$. If $\mathcal{D}$ has a terminal object $\mathbf{1}$, then $\operatorname{colim}^{\mathcal{D}} X$ always exists and the natural map $X_{\mathbf{1}} \to \operatorname{colim}^{\mathcal{D}} X$ is an isomorphism. If $\mathcal{D}$ has an initial object $\mathbf{0}$ then $\lim^{\mathcal{D}} X$ always exists and the natural map $\lim^{\mathcal{D}} X \to X_{\mathbf{0}}$ is an isomorphism.

Next lemma presents a base change formula for relative (co)limits. In the case of colimits, the lemma states that the relative colimit along a functor $u : \mathcal{D}_1 \to \mathcal{D}_2$ can be pointwisely computed in terms of absolute colimits over $(u \downarrow d_2)$ for all objects $d_2 \epsilon \mathcal{D}_2$.

Lemma 5.2.1. *Suppose that $u : \mathcal{D}_1 \to \mathcal{D}_2$ is a functor, and suppose that X is a $\mathcal{D}_1$ diagram in a category $\mathcal{M}$.*

(1) *If* $\operatorname{colim}^{(u\downarrow d_2)} X$ *exists for all $d_2 \epsilon \mathcal{D}_2$, then* $\operatorname{colim}^u X$ *exists and*

$$\operatorname{colim}^{(u\downarrow d_2)} X \cong (\operatorname{colim}^u X)_{d_2}$$

(2) *If* $\lim^{(d_2\downarrow u)} X$ *exists for all $d_2 \epsilon \mathcal{D}_2$, then* $\lim^u X$ *exists and*

$$(\lim^u X)_{d_2} \cong \lim^{(d_2\downarrow u)} X$$

Proof. We only prove (1). We show that $(\operatorname{colim}^{(u\downarrow d_2)} X)_{d_2 \epsilon \mathcal{D}_2}$ satisfies the universal property that defines $\operatorname{colim}^u X$. Maps from X to a diagram $u^* Y$ can be identified with maps $X_{d_1} \to Y_{ud_1}$ that make the diagram below commutative for all maps $f : d_1 \to d_1^{'}$

$$\begin{tikzcd}
X_{d_1} \arrow[r] \arrow[d, "X_f"'] & Y_{ud_1} \arrow[d, "Y_{uf}"] \\
X_{d_1'} \arrow[r] & Y_{ud_1'}
\end{tikzcd}$$

They can be further identified with maps $X_{d_1} \to Y_{d_2}$, defined for all $\phi : ud_1 \to d_2$, that make the diagram below commutative

$$\begin{tikzcd}
X_{d_1} \arrow[r] \arrow[d, "X_f"'] & Y_{d_2} \arrow[d, "Y_g"] \\
X_{d_1'} \arrow[r] & Y_{d_2'}
\end{tikzcd}$$

for all maps f, g that satisfy $\phi' uf = g\phi$. They can finally be identified with maps $\operatorname{colim}^{(u\downarrow d_2)} X \to Y_{d_2}$ compatible in d_2, and from the universal property of the colimit we see that $\operatorname{colim}^{u} X$ exists and $\operatorname{colim}^{(u\downarrow d_2)} X \cong (\operatorname{colim}^{u} X)_{d_2}$. ☐

In particular, if $\mathcal{M}$ is closed under absolute colimits, it is also closed under relative colimits. By definition, if $\mathcal{M}$ is closed under small colimits we say that $\mathcal{M}$ is *cocomplete*. Dually, if $\mathcal{M}$ is closed under absolute limits, it is also closed under relative limits. If $\mathcal{M}$ is closed under small limits we say by definition that $\mathcal{M}$ is *complete*.

The following lemma presents another well known result - that the composition of (co)limits is the (co)limit of the composition.

LEMMA 5.2.2. *Suppose that $u : \mathcal{D}_1 \to \mathcal{D}_2$ and $v : \mathcal{D}_2 \to \mathcal{D}_3$ are two functors, and suppose that X is a $\mathcal{D}_1$ diagram in a category $\mathcal{M}$.*

1. *Assume that* $\operatorname{colim}^{u} X$ *exists. If either* $\operatorname{colim}^{v} \operatorname{colim}^{u} X$ *or* $\operatorname{colim}^{vu} X$ *exist, then they both exist and are canonically isomorphic.*
2. *Assume that* $\lim^{u} X$ *exists. If either* $\lim^{v} \lim^{u} X$ *or* $\lim^{vu} X$ *exist, then they both exist and are canonically isomorphic.*

PROOF. We only prove (1). Since $\operatorname{colim}^{u} X$ exists, we have a bijection of sets natural in $Y \epsilon \mathcal{M}^{\mathcal{D}_3}$

$$Hom(X, u^* v^* Y) \cong Hom(\operatorname{colim}^{u} X, v^* Y)$$

If $\operatorname{colim}^{vu} X$ exists, we also have a natural bijection

$$Hom(X, u^* v^* Y) \cong Hom(\operatorname{colim}^{vu} X, Y)$$

therefore $\operatorname{colim}^{vu} X$ satisfies the universal property of $\operatorname{colim}^{v} \operatorname{colim}^{u} X$. If on the other hand $\operatorname{colim}^{v} \operatorname{colim}^{u} X$ exists, we have a natural bijection

$$Hom(\operatorname{colim}^{u} X, v^* Y) \cong Hom(\operatorname{colim}^{v} \operatorname{colim}^{u} X, Y)$$

and $\operatorname{colim}^{v} \operatorname{colim}^{u} X$ satisfies the universal property of $\operatorname{colim}^{vu} X$. ☐

In particular, if $\mathcal{M}$ is cocomplete then $\operatorname{colim}^{vu}$ and $\operatorname{colim}^{v} \operatorname{colim}^{u}$ are naturally isomorphic. Dually, if $\mathcal{M}$ is complete then $\lim^{vu}$ and $\lim^{v} \lim^{u}$ are naturally isomorphic.

5.3. Simplicial sets

This text assumes familiarity with simplicial sets, and the reader may refer for example to [**GJ99**] or [**Hov99**] for a treatment of the standard theory of simplicial sets.

We will denote by $\mathbf{n}$ the poset $\{0, 1, ..., n\}$, for $n \geq 0$, with the natural order. The cosimplicial indexing category Δ is the category with objects $\mathbf{0}, \mathbf{1}, ..., \mathbf{n}, ...$ and maps the order-preserving maps $\mathbf{n}_1 \to \mathbf{n}_2$. If $\mathcal{C}$ is a category, a simplicial object in $\mathcal{C}$ is then a functor $\Delta^{op} \to \mathcal{C}$, and a cosimplicial object in $\mathcal{C}$ is a functor $\Delta \to \mathcal{C}$.

Taking in particular simplicial objects in *Sets* we get the category of simplicial sets, denoted *sSets*. The representable functor $Hom_{\Delta^{op}}(-, \mathbf{n})$ determines a simplicial set denoted $\Delta[n]$, called the standard n-simplex (for $n \geq 0$). For any map $\mathbf{m} \to \mathbf{n}$ we have a simplicial set map $\Delta[m] \to \Delta[n]$, thus $\Delta[-]$ determines a cosimplicial object in *sSets*.

In particular, we have simplicial set inclusions $d^0, d^1 : \Delta[0] \rightrightarrows \Delta[1]$, which induce respectively functors $i_0, i_1 : X_\bullet \cong X_\bullet \times \Delta[0] \to X_\bullet \times \Delta[1]$. We say that two maps $f, g : X_\bullet \to Y_\bullet$ are simplcially homotopic if there exists a simplicial set map $h : X_\bullet \times \Delta[1] \to Y_\bullet$ such that $f = hi_0, g = hi_1$.

In general, the simplicial homotopy relation is not reflexive nor transitive.

5.4. The nerve of a category

Given a category $\mathcal{C}$, its nerve (or classifying space) $B\mathcal{C}$ is a simplicial set having as n-simplices the composable strings $A_0 \to A_1 \to ... \to A_n$ of maps in $\mathcal{C}$. The poset $\mathbf{n}$ can be viewed as a category with objects $\{0, 1, ..., n\}$ and maps $n_1 \to n_2$ for all $n_1 \leq n_2$. $B\mathcal{C}_n$ can then be identified with the set of functors $\mathbf{n} \to \mathcal{C}$, and we define the maps $(\mathbf{n_1} \to \mathcal{C}) \longrightarrow (\mathbf{n_2} \to \mathcal{C})$ to be the commutative diagrams

$$\begin{tikzcd}
\mathbf{n_2} \arrow[rr] \arrow[dr] & & \mathbf{n_1} \arrow[dl] \\
& \mathcal{C} &
\end{tikzcd}$$

The nerve functor $B\mathcal{C}$ commutes with arbitrary limits (has a left adjoint).

Denote I the category $\mathbf{1}$, with two objects and only one non-identity map. Its nerve BI is isomorphic to $\Delta[1]$.

A functor $f : \mathcal{C}_1 \to \mathcal{C}_2$ induces a map of simplicial sets $Bf : B\mathcal{C}_1 \to B\mathcal{C}_2$. A natural map between two functors $h : f \Rightarrow g$ induces a functor $H : \mathcal{C}_1 \times I \to \mathcal{C}_2$, therefore a simplicial homotopy $BH : B\mathcal{C}_1 \times \Delta[1] \to B\mathcal{C}_2$ between Bf and Bg.

5.5. Cofinal functors

This section is a short primer on cofinal and homotopy cofinal functors. We only prove the minimal set of properties that we will need for the rest of the text. For more information on cofinal and homotopy cofinal functors, please refer to Hirschhorn [**Hir00**].

Definition 5.5.1. A small functor $u : \mathcal{D}_1 \to \mathcal{D}_2$ is

1. *left cofinal* if for any object d_2 of $\mathcal{D}_2$ the space $B(u \downarrow d_2)$ is connected and non-empty,
2. *homotopy left cofinal* if for any object d_2 of $\mathcal{D}_2$ the space $B(u \downarrow d_2)$ is contractible,
3. *right cofinal* if for any object d_2 of $\mathcal{D}_2$ the space $B(d_2 \downarrow u)$ is connected and non-empty
4. *homotopy right cofinal* if for any object d_2 of $\mathcal{D}_2$ the space $B(d_2 \downarrow u)$ is contractible

A homotopy left (resp. right) cofinal functor is in particular left (resp. right) cofinal.

Lemma 5.5.2. *If $u : \mathcal{D}_1 \rightleftarrows \mathcal{D}_2 : v$ is an adjoint functor pair, then u is homotopy left cofinal and v is homotopy right cofinal.*

Proof. For any object d_2 of $\mathcal{D}_2$, the under category $(u \downarrow d_2)$ is isomorphic to $(\mathcal{D}_1 \downarrow vd_2)$, and the latter has $id : vd_2 \to vd_2$ as a terminal object. $B(u \downarrow d_2)$ is therefore contractible for any d_2, so u is homotopy left cofinal.

A dual proof shows that v is homotopy right cofinal. □

The following lemma gives a characterization of left (resp. right) cofinal functors in terms of their preservation of limits (resp. colimits).

PROPOSITION 5.5.3. *Suppose that* $u : \mathcal{D}_1 \to \mathcal{D}_2$ *is a small functor. Then:*

1. *u is right cofinal iff for any cocomplete category $\mathcal{M}$ and any diagram $X \epsilon \mathcal{M}^{\mathcal{D}_2}$ the natural map* $\operatorname{colim}^{\mathcal{D}_1} u^* X \to \operatorname{colim}^{\mathcal{D}_2} X$ *is an isomorphism in $\mathcal{M}$.*
2. *u is left cofinal iff for any complete category $\mathcal{M}$ and any diagram $X \epsilon \mathcal{M}^{\mathcal{D}_2}$ the natural map* $\lim^{\mathcal{D}_2} X \to \lim^{\mathcal{D}_1} u^* X$ *is an isomorphism in $\mathcal{M}$.*

Note the subtle positional difference of u between the formulas of Lemma 5.2.1 (1) and Prop. 5.5.3 (2). For a diagram $Y \epsilon \mathcal{M}^{\mathcal{D}_1}$ and an object $d_2 \epsilon \mathcal{D}_2$ we have that $(\operatorname{colim}^{u} Y)_{d_2} \cong \operatorname{colim}^{(u \downarrow d_2)} Y$. But for a diagram $X \epsilon \mathcal{M}^{\mathcal{D}_2}$, the natural map $\operatorname{colim}^{\mathcal{D}_1} u^* X \to \operatorname{colim}^{\mathcal{D}_2} X$ is an isomorphism iff $(d_2 \downarrow u)$ is connected and non-empty for all objects $d_2 \epsilon \mathcal{D}_2$.

PROOF OF PROP. 5.5.3.

(1) ($\Leftarrow$). Let $\mathcal{M}$ be $\mathcal{S}et$ and X be the $\mathcal{D}_2$-diagram $Hom_{\mathcal{D}_2}(d_2, -)$ for $d_2 \epsilon \mathcal{D}_2$. We notice that $\operatorname{colim}^{\mathcal{D}_2} X \cong \operatorname{colim}^{d_2' \epsilon \mathcal{D}_2} Hom_{\mathcal{D}_2}(d_2, d_2')$ is the one-point set, and $\operatorname{colim}^{\mathcal{D}_1} u^* X \cong \operatorname{colim}^{d_1 \epsilon \mathcal{D}_1} Hom_{\mathcal{D}_2}(d_2, ud_1)$ is the set of connected components of $B(d_2 \downarrow u)$, which must therefore be isomorphic with the one-point set. The conclusion is now proved.

(1) ($\Rightarrow$). This will be a consequence of Prop. 5.5.4 below.

The proof of (2) is dual to the proof of (1). ☐

We next prove something a bit stronger than the right to left implication of Prop. 5.5.3 (1).

PROPOSITION 5.5.4. *Suppose that* $u : \mathcal{D}_1 \to \mathcal{D}_2$ *is a functor, that $\mathcal{M}$ is a category and that X an object of $\mathcal{M}^{\mathcal{D}_2}$.*

1. *Assume that the functor u is right cofinal. If either* $\operatorname{colim}^{\mathcal{D}_1} u^* X$ *or* $\operatorname{colim}^{\mathcal{D}_2} X$ *exist, then they both exist and the natural map* $\operatorname{colim}^{\mathcal{D}_1} u^* X \to \operatorname{colim}^{\mathcal{D}_2} X$ *is an isomorphism.*
2. *Assume that the functor u is left cofinal. If either* $\lim^{\mathcal{D}_1} u^* X$ *or* $\lim^{\mathcal{D}_2} X$ *exist, then they both exist and the natural map* $\lim^{\mathcal{D}_2} X \to \lim^{\mathcal{D}_1} u^* X$ *is an isomorphism.*

PROOF. We will prove (1); the statement (2) follows from duality. We denote $p_{\mathcal{D}_i} : \mathcal{D}_i \to e$, $i = 1, 2$ the terminal category projections.

We start by constructing a natural isomorphism in $X \epsilon \mathcal{M}^{\mathcal{D}_2}, Y \epsilon \mathcal{M}$

$$Hom_{\mathcal{M}^{\mathcal{D}_1}}(u^* X, p_{\mathcal{D}_1}^* Y) \cong Hom_{\mathcal{M}^{\mathcal{D}_2}}(X, p_{\mathcal{D}_2}^* Y) \tag{5.1}$$

From right to left, if $f : X \to p_{\mathcal{D}_2}^* Y \epsilon \mathcal{M}^{\mathcal{D}_2}$ then we define $F(f) : u^* X \to p_{\mathcal{D}_1}^* Y \epsilon \mathcal{M}^{\mathcal{D}_1}$ on component $d_1 \epsilon \mathcal{D}_1$ as $F(f)_{d_1} = f_{ud_1} : X_{ud_1} \to Y$.

From left to right, if $f : u^* X \to p_{\mathcal{D}_1}^* Y \epsilon \mathcal{M}^{\mathcal{D}_1}$ then we define $G(f) : X \to p_{\mathcal{D}_2}^* Y \epsilon \mathcal{M}^{\mathcal{D}_2}$ on component $d_2 \epsilon \mathcal{D}_2$ as follows. Since $(d_2 \downarrow u)$ is non-empty we can pick a map $\alpha : d_2 \to ud_1$, and we define $G(f)_{d_2} : X_{d_2} \to X_{ud_1} \to Y$ to be the composition of X_α with f_{d1}. Since $(d_2 \downarrow u)$ is connected, the map $G(f)$ does not depend on the choice involved, $G(f)$ is indeed a diagram map from X to $p_{\mathcal{D}_2}^* Y$ and furthermore F, G are inverses of each other.

If we view both terms of 5.1 as functors of Y, then $\operatorname{colim}^{\mathcal{D}_1} u^*X$ exists iff the left side of 5.1 is a representable functor of Y iff the right side of 5.1 is a representable functor of Y iff $\operatorname{colim}^{\mathcal{D}_2} X$ exists. Under these conditions it also follows that the natural map $\operatorname{colim}^{\mathcal{D}_1} u^*X \to \operatorname{colim}^{\mathcal{D}_2} X$ is an isomorphism. ☐

As a consequence of Prop. 5.5.3, it is not hard to see that left (resp. right) cofinal functors are stable under composition.

5.6. The Grothendieck construction

Denote $\mathcal{C}at$ the category of small categories and functors. If $\mathcal{D}$ is a small category and H is a functor $\mathcal{D} \to \mathcal{C}at$, the Grothendieck construction of H, denoted $\int_{\mathcal{D}} H$, is the category for which:

1. objects are pairs (d, x) with d an object of $\mathcal{D}$ and x an object of the category $H(d)$
2. maps $(d_1, x_1) \to (d_2, x_2)$ are pairs of maps (f, ϕ) with $f : d_1 \to d_2$ and $\phi : H(f)x_1 \to x_2$
3. the composition of maps $(g, \psi)(f, \phi)$ is the map $(gf, \psi \circ H(g)\phi)$
4. the identity of (d, x) is $(1_d, 1_x)$

The Grothendieck construction comes with a projection functor $p : \int_{\mathcal{D}} H \to \mathcal{D}$, defined by $(d, x) \to d$. If d is an object of $\mathcal{D}$, then the inverse image category $p^{-1}d$ may be identified with the category $H(d)$.

Dually if H is a functor $\mathcal{D}^{op} \to \mathcal{C}at$, the contravariant Grothendieck construction of H is defined as $\int^{\mathcal{D}} H = (\int_{\mathcal{D}^{op}} H^{op})^{op}$, and comes with a projection functor $p : \int^{\mathcal{D}} H \to \mathcal{D}$.

Proposition 5.6.1.

1. *Suppose that* $H : \mathcal{D} \to \mathcal{C}at$ *is a functor from a small category to the category of small categories, with projection functor* $p : \int_{\mathcal{D}} H \to \mathcal{D}$. *Then for any object* $d \epsilon \mathcal{D}$, *the inclusion* $p^{-1}d \to (p \downarrow d)$ *admits a left adjoint.*
2. *Suppose that* $H : \mathcal{D}^{op} \to \mathcal{C}at$ *is a functor from a small category to the category of small categories, with projection functor* $p : \int^{\mathcal{D}} H \to \mathcal{D}$. *Then for any object* $d \epsilon \mathcal{D}$, *the inclusion* $p^{-1}d \to (d \downarrow p)$ *admits a right adjoint.*

Proof. To prove (1), denote $i_d : p^{-1}d \to (p \downarrow d)$ the inclusion. A left adjoint $j_d : (p \downarrow d) \to p^{-1}d$ of i_d can be constructed such that:

(a) j_d sends the object $((d_1, x_1), f_1 : d_1 \to d)$ of $(p \downarrow d)$ to the object $H(f_1)x_1$ of $H(d) \cong p^{-1}d$

(b) For objects $((d_1, x_1), f_1 : d_1 \to d)$ and $((d_2, x_2), f_2 : d_2 \to d)$ of $(p \downarrow d)$, j_d sends a map $(f, \phi) : (d_1, x_1) \to (d_2, x_2)$ with $f_2 f = f_1$ to a map $H(f_1)x_1 \to H(f_2)x_2$ in $H(d)$ defined by $H(f_2)\phi$

(c) The adjunction counit is $id : j_d i_d \Rightarrow 1_{p^{-1}d}$

(d) The adjunction unit $1_{(p\downarrow d)} \Rightarrow i_d j_d$ maps
$$((d_1, x_1), f_1 : d_1 \to d) \Rightarrow ((d, H(f_1)x_1), 1_d : d \to d)$$
via the map $(d_1, x_1) \to (d, H(f_1)x_1)$ in $\int_{\mathcal{D}} H$ defined by $(f_1, 1_{H(f_1)x_1})$.

The proof of (2) is dual. ☐

As a consequence we can prove a Fubini-type formula for the computation of (co)limits indexed by Grothendieck constructions.

PROPOSITION 5.6.2.

1. *Suppose that* $H : \mathcal{D} \to \mathcal{C}at$ *is a functor from a small category to the category of small categories. Let* $\mathcal{M}$ *be a category, and* X *be a* $\int_{\mathcal{D}} H$*-diagram of* $\mathcal{M}$*. Assume that the inner colimit of the right side of the equation below exists for all* $d \epsilon \mathcal{D}$*. If either the left side or the outer right side colimits exist, then both exist and we have a natural isomorphism*
$$\operatorname{colim}^{\int_{\mathcal{D}} H} X \cong \operatorname{colim}^{d \epsilon \mathcal{D}} \operatorname{colim}^{H(d)} X$$
2. *Suppose that* $H : \mathcal{D}^{op} \to \mathcal{C}at$ *is a functor from a small category to the category of small categories. Let* $\mathcal{M}$ *be a category, and* X *be a* $\int^{\mathcal{D}} H$*-diagram of* $\mathcal{M}$*. Assume that the inner limit of the right side of the equation below exists for all* $d \epsilon \mathcal{D}$*. If either the left side or the outer right side limits exist, then both exist and we have a natural isomorphism*
$$\lim^{\int^{\mathcal{D}} H} X \cong \lim^{d \epsilon \mathcal{D}} \lim^{H(d)^{op}} X$$

PROOF. We only prove (1) - statement (2) uses a dual proof.

For any object d of $\mathcal{D}$, the inclusion $p^{-1}d \to (p \downarrow d)$ has a left adjoint, therefore by Lemma 5.5.2 it is homotopy right cofinal, and in particular right cofinal. We identify $p^{-1}d \cong H(d)$. Since $\operatorname{colim}^{H(d)} X$ exists for all d, by Prop. 5.5.4 $\operatorname{colim}^{(p \downarrow d)} X \cong \operatorname{colim}^{H(d)} X$ exists for all d, and by Lemma 5.2.1 $\operatorname{colim}^{p} X$ exists. By Lemma 5.2.2, if either $\operatorname{colim}^{d \epsilon \mathcal{D}} \operatorname{colim}^{H(d)} X \cong \operatorname{colim}^{\mathcal{D}} \operatorname{colim}^{p} X$ or $\operatorname{colim}^{\int_{\mathcal{D}} H} X$ exist, then they both exist and are canonically isomorphic. □

If $H : \mathcal{D}_1 \to \mathcal{C}at$ is a constant functor with value $\mathcal{D}_2$, then the Grothendieck construction $\int_{\mathcal{D}} H$ is *isomorphic* to the product of categories $\mathcal{D}_1 \times \mathcal{D}_2$. In this case, Prop. 5.6.2 yields

COROLLARY 5.6.3. *Suppose that* $\mathcal{D}_1$*,* $\mathcal{D}_2$ *are two small categories and suppose that* X *is a* $\mathcal{D}_1 \times \mathcal{D}_2$ *diagram in a category* $\mathcal{M}$*.*

1. *Assume that the inner colimit of the right side of the equation below exists for all* $d_1 \epsilon \mathcal{D}_1$*. If either the left side or the outer right side colimits exist, then both exist and we have a natural isomorphism*
$$\operatorname{colim}^{\mathcal{D}_1 \times \mathcal{D}_2} X \cong \operatorname{colim}^{d_1 \epsilon \mathcal{D}_1} \operatorname{colim}^{\{d_1\} \times \mathcal{D}_2} X$$
2. *Assume that the inner limit of the right side of the equation below exists for all* $d_1 \epsilon \mathcal{D}_1$*. If either the left side or the outer right side limits exist, then both exist and we have a natural isomorphism*
$$\lim^{\mathcal{D}_1 \times \mathcal{D}_2} X \cong \lim^{d_1 \epsilon \mathcal{D}_1} \lim^{\{d_1\} \times \mathcal{D}_2} X \ \square$$

REMARK 5.6.4. A typical application of Cor. 5.6.3 (1) will be for the case when $\operatorname{colim}^{\{d_1\} \times \mathcal{D}_2} X$ and $\operatorname{colim}^{\mathcal{D}_1 \times \{d_2\}} X$ exist for all objects $d_1 \epsilon \mathcal{D}_1$ and $d_2 \epsilon \mathcal{D}_2$. Then in the equation below if either the left or the right outer side colimits exist, they both exist and we have a natural isomorphism
$$\operatorname{colim}^{d_1 \epsilon \mathcal{D}_1} \operatorname{colim}^{\{d_1\} \times \mathcal{D}_2} X \cong \operatorname{colim}^{d_2 \epsilon \mathcal{D}_2} \operatorname{colim}^{\mathcal{D}_1 \times \{d_2\}} X$$

CHAPTER 6

Homotopy colimits in a cofibration category

In this chapter we prove the existence of homotopy colimits in cofibration categories, and dually the existence of homotopy limits in fibration categories. The homotopy colimit should be thought of as the total left derived functor of the colimit - at least if the base cofibration category is cocomplete. The actual construction of the homotopy colimit proceeds in two steps - first, for diagrams of direct categories, and second, reducing the general case to the case of diagrams of direct categories. This essentially follows Anderson's original argument [**And78**], simplified by Cisinski [**Cis02**], [**Cis03**].

Throughout this chapter, we will assume the entire set of cofibration category axioms CF1-CF6. An important exercise left for the reader is to see how the results can be reformulated and proved within the restricted cofibration category axioms CF1-CF5, or even within the precofibration category axioms CF1-CF4.

We show that given a cofibration category $\mathcal{M}$ and a small direct category $\mathcal{D}$, the diagram category $\mathcal{M}^{\mathcal{D}}$ with pointwise weak equivalences $\mathcal{W}^{\mathcal{D}}$ admits two cofibration category structures - the Reedy $(\mathcal{M}^{\mathcal{D}}, \mathcal{W}^{\mathcal{D}}, \mathcal{C}of^{\mathcal{D}}_{reedy})$ and the pointwise structure $(\mathcal{M}^{\mathcal{D}}, \mathcal{W}^{\mathcal{D}}, \mathcal{C}of^{\mathcal{D}})$. A general small category $\mathcal{D}$ only yields a pointwise cofibration structure $(\mathcal{M}^{\mathcal{D}}, \mathcal{W}^{\mathcal{D}}, \mathcal{C}of^{\mathcal{D}})$, but $\mathcal{M}^{\mathcal{D}}$ admits a certain cofibrant approximation functor that allows us to reduce the construction of homotopy colimits in $\mathcal{M}^{\mathcal{D}}$ to the construction of homotopy colimits indexed by direct diagrams in $\mathcal{M}$.

We then proceed to prove a number of properties of homotopy limits and of $\mathbf{ho}\mathcal{M}^{\mathcal{D}}$ that will allow us to show in Chap. 7 that that homotopy colimits in a cofibration category satisfy the axioms of a right derivator. Dually, homotopy limits in a fibration category satisfy the axioms of a left derivator.

6.1. Direct and inverse categories

Diagram categories that are *direct* are very well suited for computing homotopy colimits. Homotopy colimits can be essentially described in terms of iterated pushouts in case the indexing diagram is a small direct category.

Dually, homotopy limits can be described in terms of iterated pullbacks if the indexing diagram is a small *inverse* category.

We start with the definition of direct and inverse categories.

DEFINITION 6.1.1. Let $\mathcal{D}$ be a category. A non-negative degree function on its objects is a function $deg : Ob\mathcal{D} \to \mathbb{Z}_+$.

(1) The category $\mathcal{D}$ is *direct* if there is a non-negative degree function deg on the objects of $\mathcal{D}$ such that any non-identity map $d_1 \to d_2$ satisfies $deg(d_1) < deg(d_2)$.

(2) The category $\mathcal{D}$ is *inverse* if there is a non-negative degree function deg on the objects of $\mathcal{D}$ such that any non-identity map $d_1 \to d_2$ satisfies $deg(d_1) > deg(d_2)$.

Let us recall the definition of the latching object of a diagram indexed by a direct category, and dually the definition of the matching object of a diagram indexed by an inverse category.

First we give the definition of the latching and matching categories:

Definition 6.1.2. Let $\mathcal{D}$ be a category, and d be an object of $\mathcal{D}$.

(1) If $\mathcal{D}$ is direct, the *latching category* $\partial(\mathcal{D} \downarrow d)$ is the full subcategory of the over category $(\mathcal{D} \downarrow d)$ consisting of all objects except the identity of d.

(2) If $\mathcal{D}$ is inverse, the *matching category* $\partial(d \downarrow \mathcal{D})$ is the full subcategory of the under category $(d \downarrow \mathcal{D})$ consisting of all objects except the identity of d.

It is an easy consequence of the definitions that if $\mathcal{D}$ is direct, then $\partial(\mathcal{D} \downarrow d)$ is also a direct category, with $deg(d^{'} \to d) = deg(d^{'})$. Dually, if the category $\mathcal{D}$ is inverse, then $\partial(d \downarrow \mathcal{D})$ is an inverse category.

We next recall the definition of the latching (resp. matching) objects of a diagram indexed by a direct (resp. inverse) category.

Definition 6.1.3. Let $\mathcal{D}$ be a category, and d be an object of $\mathcal{D}$. Assume that $\mathcal{M}$ is a category, and that X is a $\mathcal{D}$ -diagram of $\mathcal{M}$.

(1) If $\mathcal{D}$ is direct, the *latching object* of X at d is by definition

$$LX_d = \operatorname{colim}^{\partial(\mathcal{D}\downarrow d)} X$$

(2) If $\mathcal{D}$ is inverse, the *matching object* of X at d is by definition

$$MX_d = \lim^{\partial(d\downarrow\mathcal{D})} X$$

The latching object LX_d may not exist in general for all $\mathcal{D}$ -diagrams of $\mathcal{M}$; but if the category $\mathcal{M}$ is cocomplete, then the latching object LX_d always exists.

If $\mathcal{D}$ is direct or inverse and $n \epsilon \mathbb{Z}_+$ then we will denote $\mathcal{D}^{<n}, \mathcal{D}^{\leq n}$ the full subcategories with objects of degree $< n$ respectively $\leq n$. It is an easy consequence of the definitions that if $\mathcal{D}$ is direct, then $\partial(\mathcal{D} \downarrow d) \cong \partial(\mathcal{D}^{\leq deg(d)} \downarrow d) \cong (\partial(\mathcal{D} \downarrow d))^{<deg(d)}$. Furthermore, the latching category at $d^{'} \to d$ is isomorphic to $\partial(\mathcal{D} \downarrow d^{'})$, and therefore $LX_{d' \to d} \cong LX_{d'}$.

Latching objects are related to Kan extensions in the following sense. Denote $\delta_d : \mathcal{D}^{<deg(d)} \to \mathcal{D}$ the inclusion functor. If $\mathcal{M}$ is a cocomplete category and X is a $\mathcal{D}$ -diagram of $\mathcal{M}$, then the latching object LX_d is isomorphic to $(\operatorname{colim}^{\delta_d} X)_d$.

Dually, if the category $\mathcal{D}$ is inverse, then $\partial(d \downarrow \mathcal{D})$ is an inverse category and $\partial(d \downarrow \mathcal{D}) \cong \partial(d \downarrow \mathcal{D}^{\leq deg(d)}) \cong (\partial(d \downarrow \mathcal{D}))^{<deg(d)}$. If $\mathcal{M}$ is a complete category and X is a $\mathcal{D}$ -diagram of $\mathcal{M}$, then $MX_{d \to d'} \cong MX_{d'}$ and $MX_d \cong (\lim^{\delta_d} X)_d$.

6.2. Reedy and pointwise cofibration structures for direct diagrams

Gived a cofibration category $\mathcal{M}$ and a small direct category $\mathcal{D}$, we define two cofibration category structures on $\mathcal{M}^{\mathcal{D}}$ - the Reedy and the pointwise cofibration structure. The pointwise cofibration structure has pointwise weak equivalences and pointwise cofibrations. The Reedy cofibration structure also has pointwise weak equivalences, but has a more restrictive set of cofibrations - these are called the Reedy cofibrations. In particular, Reedy cofibrations are pointwise cofibrations.

The pointwise cofibration structure is very easily constructed, and all axioms are quickly verified - except for the factorization axiom, which has a non-trivial proof. To prove the factorization axiom for the pointwise cofibration structure, we will construct the Reedy cofibration structure, prove all the axioms (including the factorization axiom) for the Reedy cofibration structure, and obtain as a corollary the factorization axiom for the pointwise cofibration structure.

Dually, if $\mathcal{M}$ is a fibration category and $\mathcal{D}$ is a small inverse category then we define two fibration category structures on $\mathcal{M}^{\mathcal{D}}$ - the pointwise and the Reedy fibration category structures.

We start with the definition of pointwise weak equivalences, pointwise cofibrations and pointwise fibrations:

Definition 6.2.1. Let $\mathcal{D}$ be a small category. Given a cofibration (respectively fibration) category $\mathcal{M}$, a map of $\mathcal{D}$-diagrams $X \to Y$ in $\mathcal{M}^{\mathcal{D}}$ is a pointwise weak equivalence (resp. a pointwise cofibration, resp. a pointwise fibration) if all maps $X_d \to Y_d$ are weak equivalences (resp. cofibrations, resp. fibrations) for all objects d of $\mathcal{D}$.

A diagram X is therefore pointwise cofibrant (resp. pointwise fibrant) if all X_d are cofibrant (resp. fibrant) for all objects d of $\mathcal{D}$.

The category of pointwise weak equivalences is then just $\mathcal{W}^{\mathcal{D}}$. The category of pointwise cofibrations is $\mathcal{C}of^{\mathcal{D}}$. The category of pointwise fibrations is $\mathcal{F}ib^{\mathcal{D}}$.

We continue with the definition of Reedy cofibrations and Reedy fibrations:

Definition 6.2.2.

1. Let $\mathcal{M}$ be a cofibration category and $\mathcal{D}$ be a small direct category.
    - (a) A $\mathcal{D}$-diagram X of $\mathcal{M}$ is called *Reedy cofibrant* if for any object d of $\mathcal{D}$, the latching object LX_d exists and is cofibrant, and the natural map $i_d : LX_d \to X_d$ is a cofibration.
    - (b) A map of Reedy cofibrant $\mathcal{D}$-diagrams $f : X \to Y$ is called a *Reedy cofibration* if for any object d of $\mathcal{D}$, the natural map $X_d \sqcup_{LX_d} LY_d \to Y_d$ is a cofibrantion. (Notice that the pushout $X_d \sqcup_{LX_d} LY_d$ always exists if X, Y are Reedy cofibrant because of the pushout axiom.) The class of Reedy cofibrations will be denoted $\mathcal{C}of^{\mathcal{D}}_{reedy}$.
2. Let $\mathcal{M}$ be a fibration category and $\mathcal{D}$ be a small inverse category.
    - (a) A $\mathcal{D}$-diagram X of $\mathcal{M}$ is called *Reedy fibrant* if for any object d of $\mathcal{D}$, the matching object MX_d exists and is fibrant, and the natural map $p_d : X_d \to MX_d$ is a fibration.
    - (b) A map of Reedy fibrant $\mathcal{D}$-diagrams $f : X \to Y$ is called a *Reedy fibration* if for any object d of $\mathcal{D}$, the natural map $X_d \to MX_d \times_{MY_d} Y_d$ is a fibrantion. The class of Reedy cofibrations will be denoted $\mathcal{F}ib^{\mathcal{D}}_{reedy}$.

We'd like to stress that a Reedy cofibration $X \to Y$ has by definition a Reedy cofibrant domain X. A Reedy fibration $X \to Y$ has by definition a Reedy fibrant codomain Y. In this regard, our definition of Reedy (co)fibrations is different than the usual one in Quillen model categories, where Reedy cofibrations (resp. fibrations) are allowed to have non-Reedy cofibrant domain (resp. non-Reedy fibrant codomain).

On the other hand, a pointwise cofibration $X \to Y$ does not necessarily have a pointwise cofibrant domain X.

If $\mathcal{M}$ is a cofibration category and $\mathcal{D}$ is a small direct category, notice that the constant initial-object $\mathcal{D}$-diagram $c\,\mathbf{0}$ is Reedy cofibrant (as well as pointwise cofibrant). A $\mathcal{D}$-diagram X is Reedy cofibrant iff the map $c\,\mathbf{0} \to X$ is a Reedy cofibration.

Proposition 6.2.3.

(1) *Let* $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ *be a cofibration category and* $\mathcal{D}$ *be a small direct category. Then the Reedy cofibrations* $\mathcal{C}of^{\mathcal{D}}_{reedy}$ *are stable under composition, and include any isomorphism with a Reedy cofibrant domain.*

(2) *Let* $(\mathcal{M}, \mathcal{W}, \mathcal{F}ib)$ *be a fibration category and* $\mathcal{D}$ *be a small inverse category. Then the Reedy fibrations* $\mathcal{F}ib^{\mathcal{D}}_{reedy}$ *are stable under compositions, and include any isomorphism with a Reedy fibrant codomain.*

Proof. We only prove (1) - statement (2) follows from duality.

Clearly Reedy cofibrations include any isomorphism with a Reedy cofibrant domain. Let us show that Reedy cofibrations are stable under compositions.

If $X \to Y \to Z$ is a composition of Reedy cofibrations, then for any object d of $\mathcal{D}$ we can factor $X_d \sqcup_{LX_d} LZ_d \to Z_d$ as the composition

$$X_d \sqcup_{LX_d} LZ_d \to Y_d \sqcup_{LY_d} LZ_d \to Z_d$$

The second map is a cofibration, and the first map is a pushout of the cofibration $X_d \sqcup_{LX_d} LY_d \to Y_d$, therefore also a cofibration. We deduce that $X \to Z$ is a Reedy cofibration.

□

Theorem 6.2.4 (Reedy and pointwise (co)fibration structures).

(1) *If* $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ *is a cofibration category and* $\mathcal{D}$ *is a small direct category, then*
   (a) $(\mathcal{M}^{\mathcal{D}}, \mathcal{W}^{\mathcal{D}}, \mathcal{C}of^{\mathcal{D}}_{reedy})$ *is a cofibration category - called the Reedy cofibration structure on* $\mathcal{M}^{\mathcal{D}}$.
   (b) $(\mathcal{M}^{\mathcal{D}}, \mathcal{W}^{\mathcal{D}}, \mathcal{C}of^{\mathcal{D}})$ *is a cofibration category - called the pointwise cofibration structure on* $\mathcal{M}^{\mathcal{D}}$.

(2) *If* $(\mathcal{M}, \mathcal{W}, \mathcal{F}ib)$ *is a fibration category and* $\mathcal{D}$ *is a small inverse category, then*
   (a) $(\mathcal{M}^{\mathcal{D}}, \mathcal{W}^{\mathcal{D}}, \mathcal{F}ib^{\mathcal{D}}_{reedy})$ *is a fibration category - called the Reedy fibration structure on* $\mathcal{M}^{\mathcal{D}}$.
   (b) $(\mathcal{M}^{\mathcal{D}}, \mathcal{W}^{\mathcal{D}}, \mathcal{F}ib^{\mathcal{D}})$ *is a fibration category - called the pointwise fibration structure on* $\mathcal{M}^{\mathcal{D}}$.

The proof of this theorem is deferred until next section. In order to prove it, we will need first to work out some basic properties of colimits and limits.

6.3. Colimits in direct categories (the absolute case)

Recall that a functor $u : \mathcal{D} \to \mathcal{D}^{'}$ is an *open embedding* (or a *crible*) if u is a full embedding with the property that for any map $f : d^{'} \to ud$ with $d \epsilon \mathcal{D}, d^{'} \epsilon \mathcal{D}^{'}$ we have that $d^{'}$ (and f) are in the image of u. Dually, u is a *closed embedding* (or a *cocrible*) if u^{op} is an open embedding, meaning that for any map $f : ud \to d^{'}$ with $d \epsilon \mathcal{D}, d^{'} \epsilon \mathcal{D}^{'}$ we have that $d^{'}$ (and f) are in the image of u.

Open and closed embedding functors are stable under compositions.

If $\mathcal{D}$ is a direct category, note that $\mathcal{D}^{\leq n} \to \mathcal{D}^{\leq n+1}$ and $\mathcal{D}^{\leq n} \to \mathcal{D}$ are open embeddings. If $\mathcal{D}$ is an inverse category, then $\mathcal{D}^{\leq n} \to \mathcal{D}^{\leq n+1}$ and $\mathcal{D}^{\leq n} \to \mathcal{D}$ are closed embeddings.

The Lemmas 6.3.1, 6.3.2 that we prove in this section provide the essential inductive step required for building colimits of Reedy cofibrant diagrams in a direct category.

The colimits of any two weakly equivalent Reedy cofibrant diagrams are weakly equivalent, cf. Thm. 6.3.5 below. In that sense, the colimit of a Reedy cofibrant diagram actually computes its *homotopy* colimit. In view of this, the meaning of Lemma 6.3.1 below is that homotopy colimits over a direct category can be constructed as iterated pushouts of latching object maps.

If $\{d_k\}_{k \in K}$ is a set of objects of a category $\mathcal{D}$, denote $\mathcal{D}\backslash\{d_k\}$ the maximal full subcategory of a category $\mathcal{D}$ without the objects d_k. Our first lemma applies to the case of a direct (resp. inverse) category $\mathcal{D}$ and an object d of $\mathcal{D}$ such that $\mathcal{D}\backslash\{d\} \to \mathcal{D}$ is an open (resp. closed) embedding.

For example, if $\mathcal{D}$ is a direct category with objects of degree $\leq n$ and d is an object of $\mathcal{D}$ of degree n then $\mathcal{D}\backslash\{d\} \to \mathcal{D}$ is an open embedding. Dually, if $\mathcal{D}$ an inverse category with objects of degree $\leq n$ and d is an object of $\mathcal{D}$ of degree n then $\mathcal{D}\backslash\{d\} \to \mathcal{D}$ is a closed embedding.

LEMMA 6.3.1.

(1) *Let $\mathcal{M}$ be a cofibration category. Let $\mathcal{D}$ be a direct category, d an object of $\mathcal{D}$ such that $\mathcal{D}\backslash\{d\} \to \mathcal{D}$ is an open embedding, and X be a $\mathcal{D}$-diagram of $\mathcal{M}$. Assume that LX_d and* $\operatorname{colim}^{\mathcal{D}\backslash\{d\}} X$ *exist and are cofibrant, and that $i_d : LX_d \to X_d$ is a cofibration.*

$$\begin{tikzcd}
LX_d \arrow[r, "f_d"] \arrow[d, tail, "i_d"'] & \operatorname{colim}^{\mathcal{D}\backslash\{d\}} X \arrow[d, tail, dashed, "j_d"] \\
X_d \arrow[r, dashed] & \operatorname{colim}^{\mathcal{D}} X
\end{tikzcd}$$

Then the pushout of (i_d, f_d) is isomorphic to $\operatorname{colim}^{\mathcal{D}} X$.

In particular $\operatorname{colim}^{\mathcal{D}} X$ *exists and is cofibrant, and j_d is a cofibration.*

(2) *Let $\mathcal{M}$ be a fibration category. Let $\mathcal{D}$ be an inverse category, d an object of $\mathcal{D}$ such that $\mathcal{D}\backslash\{d\} \to \mathcal{D}$ is a closed embedding, and X be a $\mathcal{D}$-diagram of $\mathcal{M}$. Assume that MX_d and* $\lim^{\mathcal{D}\backslash\{d\}} X$ *exist and are fibrant, and that $p_d : X_d \to MX_d$ is a fibration.*

$$\begin{tikzcd}
\lim^{\mathcal{D}} X \arrow[r, dashed] \arrow[d, two heads, dashed, "q_d"'] & X_d \arrow[d, two heads, "p_d"] \\
\lim^{\mathcal{D}\backslash\{d\}} \arrow[r, "g_d"'] & MX_d
\end{tikzcd}$$

Then the pullback of (p_d, g_d) is isomorphic to $\lim^{\mathcal{D}} X$.

In particular $\lim^{\mathcal{D}} X$ *exists and is fibrant, and q_d is a fibration.*

PROOF. We will prove (1) - the proof of (2) is dual.

The pushout of (1) exists from axiom CF3. The latching object LX_d is by definiton $\operatorname{colim}^{\partial(\mathcal{D}\downarrow d)} X$. Since $\mathcal{D}\backslash\{d\} \to \mathcal{D}$ is an open embedding it is not hard to see that the pushout of (i_d, f_d) satisfies the universal property that defines $\operatorname{colim}^{\mathcal{D}} X$.

The map j_d is a cofibration as the pushout of i_d, and therefore $\operatorname{colim}^{\mathcal{D}} X$ is cofibrant. $\square$

We will need a slight reformulation of Lemma 6.3.2 for the more restrictive context of cofibration categories.

LEMMA 6.3.2.

(1) *Let $\mathcal{M}$ be a cofibration category. Let $\mathcal{D}$ be a direct category, $\{d_k\}_{k\in K}$ a set object of $\mathcal{D}$ of the same degree such that $\mathcal{D}\backslash\{d_k\}_{k\in K} \to \mathcal{D}$ is an open embedding, and X be a $\mathcal{D}$-diagram of $\mathcal{M}$. Assume that LX_{d_k} for all k and $\operatorname{colim}^{\mathcal{D}\backslash\{d_k\}} X$ exist and are cofibrant, and that $i_k : LX_{d_k} \to X_{d_k}$ is a cofibration for all k.*

$$\begin{tikzcd}
\sqcup LX_{d_k} \arrow[r, "f_K"] \arrow[d, tail, "\sqcup i_k"'] & \operatorname{colim}^{\mathcal{D}\backslash\{d_k\}} X \arrow[d, tail, dashed, "j_K"] \\
\sqcup X_{d_k} \arrow[r, dashed] & \operatorname{colim}^{\mathcal{D}} X
\end{tikzcd}$$

Then the pushout of $(\sqcup i_k, f_K)$ is isomorphic to $\operatorname{colim}^{\mathcal{D}} X$.

In particular $\operatorname{colim}^{\mathcal{D}} X$ exists and is cofibrant, and j_K is a cofibration.

(2) *Let $\mathcal{M}$ be a fibration category. Let $\mathcal{D}$ be an inverse category, $\{d_k\}_{k\in K}$ a set object of $\mathcal{D}$ of the same degree such that $\mathcal{D}\backslash\{d_k\}_{k\in K} \to \mathcal{D}$ is a closed embedding, and X be a $\mathcal{D}$-diagram of $\mathcal{M}$. Assume that MX_{d_k} and $\lim^{\mathcal{D}\backslash\{d_k\}} X$ exist and are fibrant, and that $p_k : X_{d_k} \to MX_{d_k}$ is a fibration.*

$$\begin{tikzcd}
\lim^{\mathcal{D}} X \arrow[r, dashed] \arrow[d, two heads, dashed, "q_K"'] & \times X_{d_k} \arrow[d, two heads, "\times p_k"] \\
\lim^{\mathcal{D}\backslash\{d_k\}} \arrow[r, "g_K"'] & \times MX_{d_k}
\end{tikzcd}$$

Then the pullback of $(\times p_k, g_K)$ is isomorphic to $\lim^{\mathcal{D}} X$.

In particular $\lim^{\mathcal{D}} X$ exists and is fibrant, and q_K is a fibration.

PROOF. We only prove (1). By axiom CF5, the map $\sqcup i_k$ is a cofibration, and the pushout of $(\sqcup i_k, f_K)$ exists. Since $\mathcal{D}\backslash\{d_k\}_{k\in K} \to \mathcal{D}$ is an open embedding, the pushout satisfies the universal property that defines $\operatorname{colim}^{\mathcal{D}} X$. $\square$

We will also need the following easily proved two lemmas

LEMMA 6.3.3.

(1) *Let $\mathcal{M}$ be a cofibration category, and consider a commutative diagram*

$$\begin{tikzcd}
A_0 \arrow[r, tail, "a_{01}"] \arrow[d, "f_0"'] & A_1 \arrow[r, tail, "a_{12}"] \arrow[d, "f_1"'] & A_2 \arrow[d, "f_2"'] \\
B_0 \arrow[r, "b_{01}"] & B_1 \arrow[r, "b_{12}"] & B_2
\end{tikzcd}$$

with a_{01}, a_{12} cofibrations and A_0, B_0, B_1, B_2 cofibrant.

(a) *If $B_0 \sqcup_{A_0} A_1 \to B_1$ and $B_1 \sqcup_{A_1} A_2 \to B_2$ are cofibrations, then so is $B_0 \sqcup_{A_0} A_2 \to B_2$*

(b) *If $B_0 \sqcup_{A_0} A_1 \to B_1$ and $B_1 \sqcup_{A_1} A_2 \to B_2$ are weak equivalences, then so is $B_0 \sqcup_{A_0} A_2 \to B_2$*

(2) *Let $\mathcal{M}$ be a fibration category, and consider a commutative diagram*

$$\begin{tikzcd} A_2 \arrow[r, "a_{21}"] \arrow[d, "f_2"'] & A_1 \arrow[r, "a_{10}"] \arrow[d, "f_1"'] & A_0 \arrow[d, "f_0"'] \\ B_2 \arrow[r, two heads, "b_{21}"] & B_1 \arrow[r, two heads, "b_{10}"] & B_0 \end{tikzcd}$$

with b_{10}, b_{21} fibrations and A_0, A_1, A_2, B_0 fibrant.

(a) *If $A_1 \to B_1 \times_{B_0} A_0$, $A_2 \to B_2 \times_{B_1} A_1$ are fibrations, then so is $A_2 \to B_2 \times_{B_0} A_0$*

(b) *If $A_1 \to B_1 \times_{B_0} A_0$, $A_2 \to B_2 \times_{B_1} A_1$ are weak equivalences, then so is $A_2 \to B_2 \times_{B_0} A_0$*

Proof. We only prove (1). The pushouts $B_0 \sqcup_{A_0} A_1$, $B_1 \sqcup_{A_1} A_2$ and $B_0 \sqcup_{A_0} A_2$ exist since a_{01} and a_{12} are cofibrations and $A_0, , B_0, B_1, B_2$ are cofibrant. In the diagram

$$\begin{tikzcd} A_0 \arrow[r, tail, "a_{01}"] \arrow[d, "f_0"'] & A_1 \arrow[r, tail, "a_{12}"] \arrow[d] & A_2 \arrow[d] \\ B_0 \arrow[r, tail] \arrow[dr, "b_{01}"'] & B_0 \sqcup_{A_0} A_1 \arrow[r, tail, "b"] \arrow[d, "a"'] & B_0 \sqcup_{A_0} A_2 \arrow[d, "c"] \\ & B_1 \arrow[r, tail] \arrow[dr, "b_{12}"'] & B_1 \sqcup_{A_1} A_2 \arrow[d, "d"] \\ & & B_2 \end{tikzcd}$$

the map b is a cofibration, since it is the pushout of a_{12}. The map c is a pushout of the map a.

If a and d are cofibrations, then c is a cofibration as the pushout of a, therefore dc is a cofibration. This proves part (a).

If a and d are weak equivalences, then by excision c is a weak eqivalence, and therefore dc is a weak equivalence. This proves part (b). □

Lemma 6.3.4.

(1) *Let $\mathcal{M}$ be a cofibration category, and consider a map of countable direct sequences of cofibrations*

$$\begin{tikzcd} A_0 \arrow[r, tail, "a_0"] \arrow[d, "f_0"'] & A_1 \arrow[r, tail, "a_1"] \arrow[d, "f_1"'] & \cdots \arrow[r, tail] & A_n \arrow[r, tail, "a_n"] \arrow[d, "f_n"'] & \cdots \\ B_0 \arrow[r, tail, "b_0"] & B_1 \arrow[r, tail, "b_1"] & \cdots \arrow[r, tail] & B_n \arrow[r, tail, "b_n"] & \cdots \end{tikzcd}$$

with A_0, B_0 cofibrant and a_n, b_n cofibrations for $n \geq 0$.

(a) *If all maps $B_{n-1} \sqcup_{A_{n-1}} A_n \to B_n$ are cofibrations, then*

$$B_0 \sqcup_{A_0} \operatorname{colim} A_n \to \operatorname{colim} B_n$$

is a cofibration.

(b) *If all maps $B_{n-1} \sqcup_{A_{n-1}} A_n \to B_n$ are weak equivalences, then*

$$B_0 \sqcup_{A_0} \operatorname{colim} A_n \to \operatorname{colim} B_n$$

is a weak equivalence.

(2) *Let $\mathcal{M}$ be a fibration category and consider a map of countable inverse sequences of fibrations*

$$\begin{tikzcd}
\cdots \arrow[r, two heads, "a_n"] & A_n \arrow[r, two heads] \arrow[d, "f_n"] & \cdots \arrow[r, two heads, "a_1"] & A_1 \arrow[r, two heads, "a_0"] \arrow[d, "f_1"] & A_0 \arrow[d, "f_0"] \\
\cdots \arrow[r, two heads, "b_n"] & B_n \arrow[r, two heads] & \cdots \arrow[r, two heads, "b_1"] & B_1 \arrow[r, two heads, "b_0"] & B_0
\end{tikzcd}$$

with A_0, B_0 fibrant and a_n, b_n fibrations for $n \geq 0$.

(a) *If all maps $A_n \to B_n \times_{B_{n-1}} A_{n-1}$ are fibrations, then*
$$\lim A_n \to \lim B_n \times_{B_0} A_0$$
is a fibration.

(b) *If all maps $A_n \to B_n \times_{B_{n-1}} A_{n-1}$ are weak equivalences, then*
$$\lim A_n \to (\lim B_n) \times_{B_0} A_0$$
is a weak equivalence.

Proof. We only prove(1), since part (2) is dual. Denote $A = \operatorname{colim} A_n$ and $B = \operatorname{colim} B_n$.

The map $B_0 \sqcup_{A_0} A \to B$ factors as the composition of the direct sequence of maps $B_{n-1} \sqcup_{A_{n-1}} A \to B_n \sqcup_{A_n} A$, and each map in the sequence is a pushout of $B_{n-1} \sqcup_{A_{n-1}} A_n \to B_n$ along the cofibration $B_{n-1} \sqcup_{A_{n-1}} A_n \to B_{n-1} \sqcup_{A_{n-1}} A$.

$$\begin{tikzcd}
A_0 \arrow[r, tail, "a_0"] \arrow[d, "f_0"'] & A_1 \arrow[r, tail, "a_1"] \arrow[d] & \cdots \arrow[r] & A \arrow[d] \\
B_0 \arrow[r, tail] \arrow[dr, "b_0"'] & B_0 \sqcup_{A_0} A_1 \arrow[r, tail] \arrow[d] & \cdots \arrow[r] & B_0 \sqcup_{A_0} A \arrow[d] \\
& B_1 \arrow[r, tail] \arrow[dr, tail, "b_1"'] & \cdots \arrow[r] & B_1 \sqcup_{A_1} A \arrow[d] \\
& & \cdots \arrow[dr] & \cdots \arrow[d] \\
& & & B
\end{tikzcd}$$

For part (a), each map $B_{n-1} \sqcup_{A_{n-1}} A_n \to B_n$ is a cofibration, therefore by CF3 so is its pushout $B_{n-1} \sqcup_{A_{n-1}} A \to B_n \sqcup_{A_n} A$, and by CF6 so is the sequence composition $B_0 \sqcup_{A_0} A \to B$.

For part (b), consider the map of direct sequences of cofibrations $\phi_n : B_0 \sqcup_{A_0} A_n \to B_n$. Each map ϕ_n is a weak equivalence by excision, and the result folows from Lemma 1.6.5. □

We are ready to state the following result.

Theorem 6.3.5.

(1) *Let $\mathcal{M}$ be a cofibration category and $\mathcal{D}$ be a small direct category. Then:*

(a) *If X is Reedy cofibrant in $\mathcal{M}^{\mathcal{D}}$, then* $\operatorname{colim}^{\mathcal{D}} X$ *exists and is cofibrant in $\mathcal{M}$*

(b) *If $f : X \to Y$ is a Reedy cofibration in $\mathcal{M}^{\mathcal{D}}$, then* $\operatorname{colim}^{\mathcal{D}} f$ *is a cofibration in $\mathcal{M}$*

   (c) *If* $f : X \to Y$ *is a pointwise weak equivalence of Reedy cofibrant objects in* $\mathcal{M}^{\mathcal{D}}$*, then* $\operatorname{colim}^{\mathcal{D}} f$ *is a weak equivalence in* $\mathcal{M}$*.*

(2) *Let* $\mathcal{M}$ *be a fibration category and* $\mathcal{D}$ *be a small inverse category. Then:*
   (a) *If* X *is Reedy fibrant in* $\mathcal{M}^{\mathcal{D}}$*, then* $\lim^{\mathcal{D}} X$ *exists and is fibrant in* $\mathcal{M}$
   (b) *If* $f : X \to Y$ *is a Reedy fibration in* $\mathcal{M}^{\mathcal{D}}$*, then* $\lim^{\mathcal{D}} f$ *is a fibration in* $\mathcal{M}$
   (c) *If* $f : X \to Y$ *is a pointwise weak equivalence of Reedy fibrant objects in* $\mathcal{M}^{\mathcal{D}}$*, then* $\lim^{\mathcal{D}} f$ *is a weak equivalence in* $\mathcal{M}$*.*

More generally we will prove

THEOREM 6.3.6.

(1) *Let* $\mathcal{M}$ *be a cofibration category and* $\mathcal{D}^{'} \to \mathcal{D}$ *be an open embedding of small direct categories. Then:*
   (a) *If* X *is Reedy cofibrant in* $\mathcal{M}^{\mathcal{D}}$*, then* $\operatorname{colim}^{\mathcal{D}^{'}} X, \operatorname{colim}^{\mathcal{D}} X$ *exist and* $\operatorname{colim}^{\mathcal{D}^{'}} X \to \operatorname{colim}^{\mathcal{D}} X$ *is a cofibration in* $\mathcal{M}$
   (b) *If* $f : X \to Y$ *is a Reedy cofibration in* $\mathcal{M}^{\mathcal{D}}$*, then*
   $$\operatorname{colim}^{\mathcal{D}} X \sqcup_{\operatorname{colim}^{\mathcal{D}'} X} \operatorname{colim}^{\mathcal{D}^{'}} Y \to \operatorname{colim}^{\mathcal{D}} Y$$
   is a cofibration in $\mathcal{M}$
   (c) *If* $f : X \to Y$ *is a pointwise weak equivalence between Reedy cofibrant diagrams in* $\mathcal{M}^{\mathcal{D}}$*, then*
   $$\operatorname{colim}^{\mathcal{D}} X \sqcup_{\operatorname{colim}^{\mathcal{D}'} X} \operatorname{colim}^{\mathcal{D}^{'}} Y \to \operatorname{colim}^{\mathcal{D}} Y$$
   is a weak equivalence in $\mathcal{M}$

(2) *Let* $\mathcal{M}$ *be a fibration category and* $\mathcal{D}^{'} \to \mathcal{D}$ *be a closed embedding of small inverse categories. Then:*
   (a) *If* X *is Reedy fibrant in* $\mathcal{M}^{\mathcal{D}}$*, then* $\lim^{\mathcal{D}^{'}} X, \lim^{\mathcal{D}} X$ *exist and* $\lim^{\mathcal{D}} X \to \lim^{\mathcal{D}^{'}} X$ *is a fibration in* $\mathcal{M}$
   (b) *If* $f : X \to Y$ *is a Reedy fibration in* $\mathcal{M}^{\mathcal{D}}$*, then*
   $$\operatorname{colim}^{\mathcal{D}^{'}} X \to \lim^{\mathcal{D}^{'}} Y \times_{\lim^{\mathcal{D}} Y} \lim^{\mathcal{D}} X$$
   is a fibration in $\mathcal{M}$
   (c) *If* $f : X \to Y$ *is a pointwise weak equivalence between Reedy fibrant diagrams in* $\mathcal{M}^{\mathcal{D}}$*, then*
   $$\operatorname{colim}^{\mathcal{D}^{'}} X \to \lim^{\mathcal{D}^{'}} Y \times_{\lim^{\mathcal{D}} Y} \lim^{\mathcal{D}} X$$
   is a weak equivalence in $\mathcal{M}$

If we take $\mathcal{D}^{'}$ to be empty in Thm. 6.3.6, then we get the statement of Thm. 6.3.5. Let us then prove the more general Thm. 6.3.6.

PROOF. Part (2) is dual to part (1), so we only need to prove part (1). We will use on $\mathcal{D}^{'}$ the degree induced from $\mathcal{D}$.

Denote $\mathcal{D}_{-1} = \mathcal{D}^{'}$, and let $\mathcal{D}_n$ be the full subcategory of $\mathcal{D}$ having as objects all the objects of $\mathcal{D}^{'}$ and all the objects of degree $\leq n$ of $\mathcal{D}$. Notice that all inclusions $\mathcal{D}_{n-1} \to \mathcal{D}_n$ are open embeddings. Also, all inclusions $\mathcal{D}^{'<n} \to \mathcal{D}^{'\leq n}$ are open embeddings.

Let us prove (1) (a). An inductive argument using Lemma 6.3.2 shows that each $\operatorname{colim}^{\mathcal{D}^{'\leq n}} X$ exists and that all maps $\operatorname{colim}^{\mathcal{D}^{'<n}} X \to \operatorname{colim}^{\mathcal{D}^{'\leq n}} X$ are cofibrations. Using CF6 (1), we see that $\operatorname{colim}^{\mathcal{D}^{'}} X$ exists and is cofibrant.

The same inductive argument using Lemma 6.3.2 shows that each $\operatorname{colim}^{\mathcal{D}_n} X$ exists and that all maps $\operatorname{colim}^{\mathcal{D}_{n-1}} X \to \operatorname{colim}^{\mathcal{D}_n} X$ are cofibrations. Using CF6 (1), we see that $\operatorname{colim}^{\mathcal{D}} X$ exists and is cofibrant, and that $\operatorname{colim}^{\mathcal{D}'} X \to \operatorname{colim}^{\mathcal{D}} X$ is a cofibration.

We now prove (1) (b). We want to show that for any open embedding of small direct categories $\mathcal{D}' \to \mathcal{D}$, any Reedy cofibrantion $f : X \to Y$ in $\mathcal{M}^{\mathcal{D}}$ and any $n_1 \leq n_2$

$$\operatorname{colim}^{\mathcal{D}_{n_2}} X \sqcup_{\operatorname{colim}^{\mathcal{D}_{n_1}} X} \operatorname{colim}^{\mathcal{D}_{n_1}} Y \to \operatorname{colim}^{\mathcal{D}_{n_2}} Y \tag{6.1}$$

is a cofibration in $\mathcal{M}$. The sum of the left term of (6.1) exists and is cofibrant because of (1) (a) and CF3.

Using Lemma 6.3.3 (a), it suffices to show that the map

$$\operatorname{colim}^{\mathcal{D}_n} X \sqcup_{\operatorname{colim}^{\mathcal{D}_{n-1}} X} \operatorname{colim}^{\mathcal{D}_{n-1}} Y \to \operatorname{colim}^{\mathcal{D}_n} Y \tag{6.2}$$

is a cofibration in $\mathcal{M}$ for any n. We prove this by induction on n.

Assume that (6.2) is a cofibration for indices $< n$. From Lemma 6.3.3 (a), the map (6.1) is a cofibration for any indices $n_1, n_2 < n$. Using Lemma 6.3.4 (a), we see that

$$\operatorname{colim}^{\mathcal{D}_{n-1}} X \sqcup_{\operatorname{colim}^{\mathcal{D}'} X} \operatorname{colim}^{\mathcal{D}'} Y \to \operatorname{colim}^{\mathcal{D}_{n-1}} Y \tag{6.3}$$

is a cofibration for all $\mathcal{D}'$, $\mathcal{D}$.

As in Lemma 6.3.2, we denoted $\{d_k\}_{k \epsilon K}$ the set of objects of $\mathcal{D} \backslash \mathcal{D}'$ of degree n. Consider the diagram below

$$\begin{tikzcd}
\sqcup LX_{d_k} \arrow[rr] \arrow[dr, tail, "\sqcup i_k(X)"'] \arrow[dd, "u_1"'] & & \operatorname{colim}^{\mathcal{D}_{n-1}} X \arrow[dr, tail] \arrow[dd, "u_3"] & \\
& \sqcup X_{d_k} \arrow[rr] \arrow[dd, "u_2"] & & \operatorname{colim}^{\mathcal{D}_n} X \arrow[dd, "u_4"] \\
\sqcup LY_{d_k} \arrow[rr] \arrow[dr, tail, "\sqcup i_k(Y)"'] & & \operatorname{colim}^{\mathcal{D}_{n-1}} Y \arrow[dr, tail] & \\
& \sqcup Y_{d_k} \arrow[rr] & & \operatorname{colim}^{\mathcal{D}_n} Y
\end{tikzcd}$$

The maps $\sqcup i_k(X), \sqcup i_k(Y)$ are cofibrations because X, Y are Reedy cofibrant. The top and bottom faces are pushouts by Lemma 6.3.2.

The vertical map u_3 is a cofibration from statement (6.3).

The vertical map u_1 is a cofibration also from statement (6.3). This is because the latching space at d_k is a colimit over the direct category $\partial(\mathcal{D} \downarrow d_k)$. All objects of $\partial(\mathcal{D} \downarrow d_k)$ have degree $< n$, and the restriction of $f : X \to Y$ to $\partial(\mathcal{D} \downarrow d_k)$ is Reedy cofibrant, so from (6.3) we get that each $\operatorname{colim}^{\partial(\mathcal{D}\downarrow d_k)} f$ is a cofibration in $\mathcal{M}$.

At last, the map $(\sqcup X_{d_k}) \sqcup_{\sqcup LX_{d_k}} (\sqcup LY_{d_k}) \to \sqcup Y_{d_k}$ is a cofibration because f is a Reedy cofibration.

The hypothesis of the Gluing Lemma 1.4.1 (1) (a) then applies, and we get that the map from the pushout of the opposite face

$$\operatorname{colim}^{\mathcal{D}_n} X \sqcup_{\operatorname{colim}^{\mathcal{D}_{n-1}} X} \operatorname{colim}^{\mathcal{D}_{n-1}} Y \to \operatorname{colim}^{\mathcal{D}_n} Y$$

is therefore a cofibration, and this is just our map (6.2). The proof of the induction step is finally completed, and an application of Lemma 6.3.4 (a) yields the desired statement (1)(b) in full generality.

The proof of (1) (c) follows the same exact steps as the proof of (1) (b) - only we use

- The Gluing Lemma 1.4.1 (b) instead of the Gluing Lemma 1.4.1 (a)
- Lemma 6.3.3 (b) instead of (a)
- Lemma 6.3.4 (b) instead of (a)

The reader is invited to verify the details. ☐

In order to go back and complete the missing proof of the previous section, we need to state the following corollary - whose proof is implicit in the proof of Thm. 6.3.6.

COROLLARY 6.3.7.

(1) *Let $\mathcal{M}$ be a cofibration category and $\mathcal{D}$ be a small direct category. If $X \to Y$ is a Reedy cofibrantion in $\mathcal{M}^{\mathcal{D}}$, then both $LX_d \to LY_d$ and $X_d \to Y_d$ are cofibrations in $\mathcal{M}$ for any object d of $\mathcal{D}$.*

(2) *Let $\mathcal{M}$ be a fibration category and $\mathcal{D}$ be a small inverse category. If $X \to Y$ is a Reedy fibrantion in $\mathcal{M}^{\mathcal{D}}$, then both $MX_d \to MY_d$ and $X_d \to Y_d$ are fibrations in $\mathcal{M}$ for any object d of $\mathcal{D}$.*

PROOF. We only prove (1). The latching category $\partial(\mathcal{D} \downarrow d)$ is a direct category, and the restriction of $X \to Y$ to $\partial(\mathcal{D} \downarrow d)$ is Reedy cofibrant. It follows from Thm. 6.3.5 (1) (b) that $LX_d \to LY_d$ is a cofibrantion. The map $X_d \to Y_d$ factors as $X_d \to X_d \sqcup_{LX_d} LY_d \to Y_d$; the second factor is a cofibration since $X \to Y$ is a Reedy cofibration, and the first factor is a pushout of $LX_d \to LY_d$, therefore a cofibration. It follows that $X_d \to Y_d$ is a cofibration. ☐

In particular, we have proved that Reedy cofibrations are pointwise cofibrations. Dually, Reedy fibrations are pointwise fibrations. We can finally go back and provide a

PROOF OF THM. 6.2.4. Given a cofibration category ($\mathcal{M}$, $\mathcal{W}$, $\mathcal{C}of$) and $\mathcal{D}$ a small direct category, we want to show that $(\mathcal{M}^{\mathcal{D}}, \mathcal{W}^{\mathcal{D}}, \mathcal{C}of^{\mathcal{D}}_{reedy})$ forms a cofibration category. Recall that any Reedy cofibration $X \to Y$ has X *Reedy cofibrant* by definition.

(i) *Axiom CF1* is verified in view of Lemma 6.2.3.

(ii) *Axiom CF2* is easily verified.

(iii) *Pushout axiom CF3 (1).* Let $f : X \to Y$ be a Reedy cofibration, and $g : X \to Z$ be a map with Z Reedy cofibrant. The pushout $Y \sqcup_X Z$ exists since $X_d \to Y_d$ is a cofibration by Cor. 6.3.7, and since Z_d is cofibrant. Denote $T = Y \sqcup_X Z$, and $f^{'}_d$ the cofibration $LX_d \to LY_d$

Consider the diagram below:

$$\begin{tikzcd}
LX_d \arrow[rr] \arrow[dd, tail, "u_1"'] \arrow[dr, tail, "f_d^{'}"'] & & LZ_d \arrow[dd, tail, "u_3"] \arrow[dr, tail] & \\
& LY_d \arrow[rr] \arrow[dd, tail, "u_2"] & & LT_d \arrow[dd, "u_4"] \\
X_d \arrow[rr] \arrow[dr, tail, "f_d"'] & & Z_d \arrow[dr, tail] & \\
& Y_d \arrow[rr] & & T_d
\end{tikzcd}$$

From the universal property of LT_d, it follows that LT_d exists and is isomorphic to $LY_d \sqcup_{LX_d} LZ_d$, so the top face is a pushout. The bottom face is a pushout as well, and the hypothesis of the Gluing Lemma 1.4.1 (1) (a) applies: namely, u_1, u_3 are cofibrations since X, Z are Reedy cofibrant, $f_d, f_d^{'}$ are cofibrations by Cor. 6.3.7, and $X_d \sqcup_{LX_d} LY_d \to Y_d$ is a cofibration since f is a Reedy cofibration .

It follows from the Gluing Lemma that u_4 is a cofibration and $LT_d \sqcup_{LZ_d} Z_d \to T_d$ is a cofibration, which proves that T is Reedy cofibrant and that $Z \to T$ is a Reedy cofibration. This completes the proof that Reedy cofibrations are stable under pushout.

(iv) *Axiom CF3 (2)* follows from a pointwise application of CF3 (2) in $\mathcal{M}$, since Reedy cofibrations are pointwise cofibrations by Cor. 6.3.7.

(v) *Factorization axiom CF4.* Let $f : X \to Y$ be a map with X Reedy cofibrant. We construct a factorization $f = rf^{'}$ with r a weak equivalence and $f^{'}$ a Reedy cofibration.

$$f :\ X \xrightarrow{f^{'}} Y^{'} \underset{\sim}{\xrightarrow{r}} Y$$

We employ induction on the degree n. Assume $Y^{'}, f^{'}, r$ constructed in degree $< n$. Let d be an object of $\mathcal{D}$ of degree n, and let us define $Y_d^{'}, f_d^{'}, r_d$. Define $LY_d^{'}$ as colim $^{\partial(\mathcal{D}\downarrow d)} Y^{'}$, which exists and is cofibrant by Thm. 6.3.5 applied to $\partial(\mathcal{D} \downarrow d)$. In the diagram below

$$\begin{tikzcd}
LX_d \arrow[r, tail] \arrow[d, tail] & LY_d^{'} \arrow[d, tail] & & \\
X_d \arrow[r, tail, "g_d"] & X_d \sqcup_{LX_d} LY_d^{'} \arrow[r, tail, "h_d"] & Y_d^{'} \arrow[r, "r_d", "\sim"'] & Y_d
\end{tikzcd}$$

the pushout exists by axiom CF3, and $Y_d^{'}, h_d, r_d$ is defined as the CF4 factorization of $X_d \sqcup_{LX_d} LY_d^{'} \to Y_d$. Define $f_d^{'} = h_d g_d$, and the inductive step is complete.

(vi) *The axiom CF5.* Suppose that $f_i : X_i \to Y_i$, $i \epsilon I$ is a set of Reedy cofibrations. The objects X_i are Reedy cofibrant, therefore pointwise cofibrant by Cor. 6.3.7. The maps f_i are in particular pointwise cofibrations by Cor. 6.3.7, and a pointwise application of CF5 shows that $\sqcup X_i$, $\sqcup Y_i$ exist and $\sqcup(f_i)$ is a pointwise cofibration. Furthermore, latching spaces commute with direct sums, from which one easily sees that $\sqcup f_i$ is actually a Reedy cofibration. If each f_i is a trivial Reedy cofibration, a pointwise application of CF5 yields that $\sqcup f_i$ is a weak equivalence.

(vii) *The axiom CF6.* Consider a countable direct sequence of Reedy cofibrations

$$\begin{tikzcd}
X_0 \arrow[r, tail, "a_0"] & X_1 \arrow[r, tail, "a_1"] & X_2 \arrow[r, tail, "a_2"] & \cdots
\end{tikzcd}$$

The object X_0 is Reedy cofibrant, therefore pointwise cofibrant by Cor. 6.3.7. The maps a_n are in particular pointwise cofibrations by Cor. 6.3.7, and a pointwise application of CF6 shows that colim X_n exists and $X_0 \to$ colim X_n is a pointwise cofibration. We'd like to show that $X_0 \to$ colim X_n is a Reedy cofibration.

For any object d of $\mathcal{D}$, in the diagram below

$$\begin{tikzcd}
L(X_0)_d \arrow[r, tail] \arrow[d, tail] & L(X_1)_d \arrow[r, tail] \arrow[d, tail] & L(X_2)_d \arrow[r, tail] \arrow[d, tail] & \cdots \\
(X_0)_d \arrow[r, tail] & (X_1)_d \arrow[r, tail] & (X_2)_d \arrow[r, tail] & \cdots
\end{tikzcd}$$

the vertical maps are cofibrations in $\mathcal{M}$ because the diagrams X_n are Reedy cofibrant. By Cor. 6.3.7 the top and bottom horizontal maps are cofibrations. A pointwise application of CF6 shows colim^n $L(X_n)_d$ exists, and one easily sees that it satisfies the universal property that defines $L(\text{colim}^n(X_n))_d$.

Furthermore, each map $(X_{n-1})_d \sqcup_{L(X_{n-1})_d} L(X_n)_d \to (X_n)_d$ is a cofibration, since a_{n-1} is a Reedy cofibration. From Lemma 6.3.4 we see that each map $(X_0)_d \sqcup_{L(X_0)_d} L(\text{colim } X_n)_d \to (\text{colim } X_n)_d$ is a cofibration, which implies that colim X_n is Reedy cofibrant and that $X_0 \to$ colim X_n is a Reedy cofibration. This proves CF6 (1).

If additionally all a_n are trivial Reedy cofibrations, a pointwise application of CF6 (2) shows that $X_0 \to$ colim X_n is a pointwise weak equivalence, therefore a trivial Reedy cofibration.

We have completed the proof that $(\mathcal{M}^{\mathcal{D}}, \mathcal{W}^{\mathcal{D}}, \mathcal{C}of^{\mathcal{D}}_{reedy})$ is a cofibration category. Let's prove now that $(\mathcal{M}^{\mathcal{D}}, \mathcal{W}^{\mathcal{D}}, \mathcal{C}of^{\mathcal{D}})$ is also a cofibration category.

(i) *Axioms CF1-CF3, CF5-CF6* are trivially verified.

(ii) *The factorization axiom CF4.* Let $f : X \to Y$ be a map of $\mathcal{D}$-diagrams with X pointwise cofibrant. Consider the commutative diagram

$$\begin{tikzcd}
 & & Y \\
X \arrow[r, "f'"'] \arrow[urr, "f"] & Y' \arrow[ur, "r", "\sim"'] & \\
X_1 \arrow[u, "a", "\sim"'] \arrow[r, "f_1"'] & Y'_1 \arrow[u, "b", "\sim"'] \arrow[uur, "r_1"'] &
\end{tikzcd}$$

where X_1 is a Reedy cofibrant replacement of X, $r_1 f_1$ is a factorization of fa as a Reedy cofibration f_1 followed by a weak equivalence r_1, and $Y^{'} = X \sqcup_{X_1} Y^{'}_1$. The map f_1 is in particular a pointwise cofibration. Its pushout f_1 is therefore a pointwise cofibration, and by excision the map b is a weak equivalence, so r is a weak equivalence by the 2 out of 3 axiom. We have thus constructed a factorization $f = rf^{'}$ as a pointwise cofibration $f^{'}$ followed by a pointwise weak equivalence r.

The proof of 6.2.4 part (1) is now complete, and part (2) is proved by duality. ☐

As a corollary of Thm. 6.2.4, we can construct the Reedy and the pointwise cofibration category structures on *restricted* small direct diagrams in a cofibration category.

Definition 6.3.8. If $(\mathcal{M}, \mathcal{W})$ is a category with weak equivalences and $(\mathcal{D}_1, \mathcal{D}_2)$ is a category pair, a $\mathcal{D}_1$ diagram X is called *restricted* with respect to $\mathcal{D}_2$ if for any map $d \to d^{'}$ of $\mathcal{D}_2$ the map $X_d \to X_{d'}$ is a weak equivalence.
We will denote $\mathcal{M}^{(\mathcal{D}_1, \mathcal{D}_2)}$ the full subcategory of $\mathcal{D}_2$-restricted diagrams in $\mathcal{M}^{\mathcal{D}_1}$.

With this definition we have

Theorem 6.3.9.

1. *If* $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ *is a cofibration category and* $(\mathcal{D}_1, \mathcal{D}_2)$ *is a small direct category pair, then*
   - (a) $(\mathcal{M}^{(\mathcal{D}_1, \mathcal{D}_2)}, \mathcal{W}^{\mathcal{D}_1}, \mathcal{C}of^{\mathcal{D}_1}_{reedy} \cap \mathcal{M}^{(\mathcal{D}_1, \mathcal{D}_2)})$ *is a cofibration category - called the* $\mathcal{D}_2$*-restricted Reedy cofibration structure on* $\mathcal{M}^{(\mathcal{D}_1, \mathcal{D}_2)}$.
   - (b) $(\mathcal{M}^{(\mathcal{D}_1, \mathcal{D}_2)}, \mathcal{W}^{\mathcal{D}_1}, \mathcal{C}of^{(\mathcal{D}_1, \mathcal{D}_2)})$ *is a cofibration category - called the* $\mathcal{D}_2$*-restricted pointwise cofibration structure on* $\mathcal{M}^{(\mathcal{D}_1, \mathcal{D}_2)}$.
2. *If* $(\mathcal{M}, \mathcal{W}, \mathcal{F}ib)$ *is a fibration category and* $(\mathcal{D}_1, \mathcal{D}_2)$ *is a small inverse category pair, then*
   - (a) $(\mathcal{M}^{(\mathcal{D}_1, \mathcal{D}_2)}, \mathcal{W}^{\mathcal{D}_1}, \mathcal{F}ib^{\mathcal{D}_1}_{reedy} \cap \mathcal{M}^{(\mathcal{D}_1, \mathcal{D}_2)})$ *is a fibration category - called the* $\mathcal{D}_2$*-restricted Reedy fibration structure.on* $\mathcal{M}^{(\mathcal{D}_1, \mathcal{D}_2)}$
   - (b) $(\mathcal{M}^{(\mathcal{D}_1, \mathcal{D}_2)}, \mathcal{W}^{\mathcal{D}_1}, \mathcal{F}ib^{(\mathcal{D}_1, \mathcal{D}_2)})$ *is a fibration category - called the* $\mathcal{D}_2$*-restricted pointwise fibration structure on* $\mathcal{M}^{(\mathcal{D}_1, \mathcal{D}_2)}$.

Proof. We only prove part (1) - part (2) is dual.
(i) *Axioms CF1 and CF2* are clearly verified for both the pointwise and the Reedy restricted cofibration structures.
(ii) *The pushout axiom CF3 (1).* Given a pointwise cofibration i and a map f with X, Y, Z pointwise cofibrant in $\mathcal{M}^{(\mathcal{D}_1, \mathcal{D}_2)}$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, tail, "i"'] & Z \arrow[d, tail, dashed, "j"] \\
Y \arrow[r, dashed, "g"'] & T
\end{tikzcd}$$

then the pushout j of i exists in $\mathcal{M}^{\mathcal{D}_1}$, and j is a pointwise cofibration. For any map $d \to d^{'}$ of $\mathcal{D}_2$ using the Gluing Lemma 1.4.1 applied to the diagram

$$\begin{tikzcd}
X_d \arrow[rr, "f_d"] \arrow[dr, tail, "i_d"'] \arrow[dd, "\sim"'] & & Z_d \arrow[dr, tail] \arrow[dd, "\sim"] & \\
& Y_d \arrow[rr] \arrow[dd, "\sim"] & & T_d \arrow[dd] \\
X_{d'} \arrow[rr, "f_{d'}"'] \arrow[dr, tail, "i_{d'}"'] & & Z_{d'} \arrow[dr, tail] & \\
& Y_{d'} \arrow[rr] & & T_{d'}
\end{tikzcd}$$

it follows that $T_d \to T_{d'}$ is an equivalence, therefore T is a $\mathcal{D}_2$ restricted diagram.

Furthermore, j is a Reedy cofibration if i is one and X, Z are Reedy cofibrant, by Thm. 6.2.4. This shows that the pushout axiom is satisfied for both the pointwise and the Reedy restricted cofibration structures.

(iii) *The axiom CF3 (2)* is clearly verified for both the pointwise and the Reedy restricted cofibration structures.

(iv) *The factorization axiom CF4.* Let $f : X \to Y$ be a map in $\mathcal{M}^{(\mathcal{D}_1,\mathcal{D}_2)}$. If X is a pointwise (resp. Reedy) cofibrant diagram, by Thm. 6.2.4 f factors as $f = rf^{'}$ with $f^{'} : X \to Y^{'}$ a pointwise (resp. Reedy) cofibration and $r : Y^{'} \to Y$ a pointwise weak equivalence. In both cases $Y^{'}$ is restricted, and CF4 is satisfied for both the pointwise and the Reedy restricted cofibration structures.

(v) *Axiom CF5* for both the restricted pointwise and restricted Reedy cofibration structures follows from the fact that if X_i, $i\epsilon I$ is a set of restricted pointwise (resp. Reedy) cofibrant diagrams, then $\sqcup X_i$ is a restricted pointwise (resp. Reedy) cofibrant diagram by Lemma 1.6.3.

(vi) *Axiom CF6.* Given a countable direct sequence of pointwise (resp. Reedy) cofibrations with X_0 pointwise (resp. Reedy) cofibrant

$$X_0 \overset{a_{01}}{\rightarrowtail} X_1 \overset{a_{12}}{\rightarrowtail} X_2 \overset{a_{23}}{\rightarrowtail} \cdots$$

the colimit colim X_n exists and is pointwise (resp. Reedy) cofibrant. If all X_n are restricted, from Lemma 1.6.5 colim X_n is restricted. The axiom CF6 now follows for both the restricted pointwise and restricted Reedy cofibration structures. ☐

6.4. Colimits in direct categories (the relative case)

We have shown that given a cofibration category $\mathcal{M}$ and a small direct category $\mathcal{D}$, then colim $^{\mathcal{D}}$ carries Reedy cofibrations (resp. weak equivalences between Reedy cofibrant diagrams) in $\mathcal{M}^{\mathcal{D}}$ to cofibrations (resp. weak equivalences between cofibrant objects) in $\mathcal{M}$. This result is extended below to the case of relative colimits from a small direct category to an arbitrary small category (Thm. 6.4.1) and between two small direct categories (Thm. 6.4.2).

THEOREM 6.4.1.

(1) *Let $\mathcal{M}$ be a cofibration category and $u : \mathcal{D}_1 \to \mathcal{D}_2$ be a functor between small categories with $\mathcal{D}_1$ direct.*
  (a) *If X is Reedy cofibrant in $\mathcal{M}^{\mathcal{D}_1}$, then* colim $^u X$ *exists and is pointwise cofibrant in $\mathcal{M}^{\mathcal{D}_2}$*

   (b) *If $f : X \to Y$ is a Reedy cofibration in $\mathcal{M}^{\mathcal{D}_1}$, then* $\operatorname{colim}^u f$ *is a pointwise cofibration in $\mathcal{M}^{\mathcal{D}_2}$*
   (c) *If $f : X \to Y$ is a pointwise weak equivalence of Reedy cofibrant objects in $\mathcal{M}^{\mathcal{D}_1}$, then* $\operatorname{colim}^u f$ *is a pointwise weak equivalence in $\mathcal{M}^{\mathcal{D}_2}$.*

(2) *Let $\mathcal{M}$ be a fibration category and $u : \mathcal{D}_1 \to \mathcal{D}_2$ be a functor between small categories with $\mathcal{D}_1$ inverse.*
   (a) *If X is Reedy fibrant in $\mathcal{M}^{\mathcal{D}_1}$, then* $\lim^u X$ *exists and is pointwise fibrant in $\mathcal{M}^{\mathcal{D}_2}$*
   (b) *If $f : X \to Y$ is a Reedy fibration in $\mathcal{M}^{\mathcal{D}_1}$, then* $\operatorname{colim}^u f$ *is a pointwise fibration in $\mathcal{M}^{\mathcal{D}_2}$*
   (c) *If $f : X \to Y$ is a pointwise weak equivalence of Reedy fibrant objects in $\mathcal{M}^{\mathcal{D}_1}$, then* $\lim^u f$ *is a pointwise weak equivalence in $\mathcal{M}^{\mathcal{D}_2}$.*

Proof. We only prove statement (1) - statement (2) follows from duality.

Let us prove (a). To prove that $\operatorname{colim}^u X$ exists, cf. Lemma 5.2.1 it suffices to show for any object $d_2 \epsilon \mathcal{D}_2$ that $\operatorname{colim}^{(u \downarrow d_2)} X$ exists. But the over category $(u \downarrow d_2)$ is direct and the restriction of X to $(u \downarrow d_2)$ is Reedy cofibrant, therefore by Thm. 6.3.5 $\operatorname{colim}^{(u \downarrow d_2)} X$ exists and is cofibrant in $\mathcal{M}$. It follows that $\operatorname{colim}^u X$ exists, and since $\operatorname{colim}^{(u \downarrow d_2)} X \cong (\operatorname{colim}^u X)_{d_2}$ we have that $\operatorname{colim}^u X$ is pointwise cofibrant in $\mathcal{M}^{\mathcal{D}_2}$.

We now prove (b). If $f : X \to Y$ is a Reedy cofibration in $\mathcal{M}^{\mathcal{D}_1}$, then the restriction of f to $(u \downarrow d_2)$ is Reedy cofibrant for any object $d_2 \epsilon \mathcal{D}_2$, therefore by Thm. 6.3.5 $\operatorname{colim}^{(u \downarrow d_2)} f$ is a cofibration in $\mathcal{M}$. Since $(\operatorname{colim}^u f)_{d_2} \cong \operatorname{colim}^{(u \downarrow d_2)} f$ by the naturality of the isomorphism in Thm. 6.3.5, it follows that $\operatorname{colim}^u f$ is pointwise cofibrant in $\mathcal{M}^{\mathcal{D}_2}$.

To prove (c), assume that $f : X \to Y$ is a pointwise weak equivalence between Reedy cofibrant diagrams in $\mathcal{M}^{\mathcal{D}_1}$. The restrictions of X and Y to $(u \downarrow d_2)$ are Reedy cofibrant for any object $d_2 \epsilon \mathcal{D}_2$, and by Thm. 6.3.5 (1) (c) the map $\operatorname{colim}^{(u \downarrow d_2)} X \to \operatorname{colim}^{(u \downarrow d_2)} Y$ is a weak equivalence. In conclusion, the map $\operatorname{colim}^u X \to \operatorname{colim}^u Y$ is a pointwise weak equivalence in $\mathcal{M}^{\mathcal{D}_2}$. ☐

Theorem 6.4.2.

(1) *Let $\mathcal{M}$ be a cofibration category and $u : \mathcal{D}_1 \to \mathcal{D}_2$ be a functor between small direct categories.*
   (a) *If X is Reedy cofibrant in $\mathcal{M}^{\mathcal{D}_1}$, then* $\operatorname{colim}^u X$ *exists and is Reedy cofibrant in $\mathcal{M}^{\mathcal{D}_2}$*
   (b) *If $f : X \to Y$ is a Reedy cofibration in $\mathcal{M}^{\mathcal{D}_1}$, then* $\operatorname{colim}^u f$ *is a Reedy cofibration in $\mathcal{M}^{\mathcal{D}_2}$*
   (c) *If $f : X \to Y$ is a pointwise weak equivalence of Reedy cofibrant objects in $\mathcal{M}^{\mathcal{D}_1}$, then* $\operatorname{colim}^u f$ *is a pointwise weak equivalence in $\mathcal{M}^{\mathcal{D}_2}$.*

(2) *Let $\mathcal{M}$ be a fibration category and $u : \mathcal{D}_1 \to \mathcal{D}_2$ be a functor between small inverse categories.*
   (a) *If X is Reedy fibrant in $\mathcal{M}^{\mathcal{D}_1}$, then* $\lim^u X$ *exists and is Reedy fibrant in $\mathcal{M}^{\mathcal{D}_2}$*
   (b) *If $f : X \to Y$ is a Reedy fibration in $\mathcal{M}^{\mathcal{D}_1}$, then* $\lim^u f$ *is a Reedy fibration in $\mathcal{M}^{\mathcal{D}_2}$*

(c) *If* $f : X \to Y$ *is a pointwise weak equivalence of Reedy fibrant objects in* $\mathcal{M}^{\mathcal{D}_1}$*, then* $\lim^{u} f$ *is a pointwise weak equivalence in* $\mathcal{M}^{\mathcal{D}_2}$.

PROOF. We only prove statement (1) - statement (2) follows from duality.

Let us prove (a). We know from Thm. 6.4.1 (1) (a) that $\operatorname{colim}^{u} X$ exists and is pointwise cofibrant in $\mathcal{M}^{\mathcal{D}_2}$, and we would like to show that $\operatorname{colim}^{u} X$ is Reedy cofibrant. For that, fix an object d_2 of $\mathcal{D}_2$, and let us try to identify the latching object $\operatorname{colim}^{\partial(\mathcal{D}_2 \downarrow d_2)} \operatorname{colim}^{u} X$.

The functor $H : \mathcal{D}_2 \to \mathcal{C}at$, $Hd_2' = (u \downarrow d_2')$ restricts to a functor $\partial(\mathcal{D}_2 \downarrow d_2) \to \mathcal{C}at$. The Grothendieck construction $\int_{\partial(\mathcal{D}_2 \downarrow d_2)} H$ has as objects 4-tuples $(d_1', d_2', d_2, ud_1' \to d_2' \to d_2)$ of objects $d_1' \epsilon \mathcal{D}_1, d_2', d_2 \epsilon \mathcal{D}_2$ and maps $ud_1' \to d_2' \to d_2$ such that $d_2' \to d_2$ is a non-identity map.

Denote $\partial(u \downarrow d_2)$ the full subcategory of the over category $(u \downarrow d_2)$ consisting of triples $(d_1', d_2, ud_1' \to d_2)$ with a non-identity map $ud_1' \to d_2$. We have an adjoint pair of functors $F : \partial(u \downarrow d_2) \rightleftarrows \int_{\partial(\mathcal{D}_2 \downarrow d_2)} H : G$, defined as follows:

$$F(d_1', d_2, ud_1' \to d_2) = (d_1', ud_1', d_2, ud_1' \to ud_1' \to d_2)$$

$$G(d_1', d_2', d_2, ud_1' \to d_2' \to d_2) = (d_1', d_2, ud_1' \to d_2)$$

$$id : 1_{\partial(u \downarrow d_2)} \Rightarrow GF$$

$$FG(d_1', d_2', d_2, ud_1' \to d_2' \to d_2) = (d_1', ud_1', d_2, ud_1' \to ud_1' \to d_2) \Rightarrow$$
$$(d_1', d_2', d_2, ud_1' \to d_2' \to d_2) \text{ given on } 2^{nd} \text{ component by } ud_1' \to d_2'.$$

The category $\partial(u \downarrow d_2)$ is direct, and the restriction of X to $\partial(u \downarrow d_2)$ is Reedy cofibrant, therefore $\operatorname{colim}^{\partial(u \downarrow d_2)} X$ exists and is cofibrant.

G is a right adjoint functor, therefore right cofinal, and by Prop. 5.5.4 we have that $\operatorname{colim}^{\int_{\partial(\mathcal{D}_2 \downarrow d_2)} H} X$ exists and is $\cong \operatorname{colim}^{\partial(u \downarrow d_2)} X$. From Prop. 5.6.2, $\operatorname{colim}^{(d_2' \to d_2) \epsilon \partial(\mathcal{D}_2 \downarrow d_2)} \operatorname{colim}^{(u \downarrow d_2')} X$ exists and is $\cong \operatorname{colim}^{\partial(u \downarrow d_2)} X$. In conclusion, the latching object $\operatorname{colim}^{\partial(\mathcal{D}_2 \downarrow d_2)} \operatorname{colim}^{u} X$ exists and is $\cong \operatorname{colim}^{\partial(u \downarrow d_2)} X$. In particular the latching object is cofibrant.

The inclusion functor $\partial(u \downarrow d_2) \to (u \downarrow d_2)$ is an open embedding, and from Thm. 6.3.6 (1) (a) the latching map $\operatorname{colim}^{\partial(u \downarrow d_2)} X \to \operatorname{colim}^{(u \downarrow d_2)} X$ is a cofibration, therefore $\operatorname{colim}^{u} X$ is Reedy cofibrant in $\mathcal{M}^{\mathcal{D}_2}$. The proof of statement (a) of our theorem is complete.

Let us prove (b). If $f : X \to Y$ is a Reedy cofibration, by Thm. 6.3.6 (1) (b) the map

$$\operatorname{colim}^{(u \downarrow d_2)} X \sqcup_{\operatorname{colim}^{\partial(u \downarrow d_2)} X} \operatorname{colim}^{\partial(u \downarrow d_2)} Y \to \operatorname{colim}^{(u \downarrow d_2)} Y$$

is a cofibration, therefore $\operatorname{colim}^{u} X \to \operatorname{colim}^{u} Y$ is a Reedy cofibration in $\mathcal{M}^{\mathcal{D}_2}$.

Part (c) has been already proved as Thm. 6.4.1 (1) (c). ☐

6.5. Colimits in arbitrary categories

Denote Δ' the subcategory of the cosimplicial indexing category Δ, with same objects as Δ and with maps the order-preserving *injective* maps $\mathbf{n_1} \to \mathbf{n_2}$.

If $\mathcal{D}$ is a category, we define $\Delta' \mathcal{D}$ to be the category with objects the functors $\mathbf{n} \to \mathcal{D}$, and with maps $(\mathbf{n_1} \to \mathcal{D}) \longrightarrow (\mathbf{n_2} \to \mathcal{D})$ the commutative diagrams

$$\begin{tikzcd}
\mathbf{n_1} \arrow[rr, "f"] \arrow[dr] & & \mathbf{n_2} \arrow[dl] \\
& \mathcal{D} &
\end{tikzcd}$$

where f is injective and order-preserving.

The category $\Delta^{'}\mathcal{D}$ is direct, and comes equipped with a *terminal projection* functor $p_t : \Delta^{'}\mathcal{D} \to \mathcal{D}$ that sends $\mathbf{n} \to \mathcal{D}$ to the image of the terminal object n of the poset $\mathbf{n}$.

The opposite category of $\Delta^{'}\mathcal{D}$ is denoted $\Delta^{'op}\mathcal{D}$. It is an inverse category, and comes equipped with an *initial projection* functor $p_i : \Delta^{'op}\mathcal{D} \to \mathcal{D}$ that sends $\mathbf{n} \to \mathcal{D}$ to the image of the initial object 0 of the poset $\mathbf{n}$.

LEMMA 6.5.1. *If $u : \mathcal{D}_1 \to \mathcal{D}_2$ is a functor and $d_2 \epsilon \mathcal{D}_2$ is an object*

1. *There is a natural isomorphism $(up_t \downarrow d_2) \cong \Delta^{'}(u \downarrow d_2)$*
2. *There is a natural isomorphism $(d_2 \downarrow up_i) \cong \Delta^{'op}(d_2 \downarrow u)$*

PROOF. Left to the reader. □

LEMMA 6.5.2. *If $\mathcal{D}$ is a category and $d \epsilon \mathcal{D}$ is an object*

1. (a) *The category $p_t^{-1}d$ has an initial object and hence has a contractible nerve.*
   (b) *The category $(p_t \downarrow d)$ has a contractible nerve.*
   (c) *The inclusion $p_t^{-1}d \to (p_t \downarrow d)$ is homotopy right cofinal.*
2. (a) *The category $p_i^{-1}d$ has a terminal object and hence has a contractible nerve.*
   (b) *The category $(d \downarrow p_t)$ has a contractible nerve.*
   (c) *The inclusion $p_i^{-1}d \to (d \downarrow p_i)$ is homotopy left cofinal.*

PROOF. We denote an object $\mathbf{n} \to \mathcal{D}$ of $\Delta^{'}\mathcal{D}$ as $(d_0 \to ... \to d_n)$, where d_i is the image of $i \epsilon \mathbf{n}$ and $d_i \to d_{i+1}$ is the image of $i \to i+1$. Each map $i : \mathbf{k} \to \mathbf{n}$ determines a map $(d_{i0} \to ... \to d_{ik}) \to (d_0 \to ... \to d_n)$ in the category $\Delta^{'}\mathcal{D}$.

We denote an object of $(p_t \downarrow d)$ as $(d_0 \to ... \to d_n) \to d$, where $d_n \to d$ is the map $p_t(d_0 \to ... \to d_n) \to d$.

The category $p_t^{-1}d$ has the initial object $(d) \epsilon \Delta^{'}\mathcal{D}$, which proves part (1) (a). For (1) (b), a contraction of the nerve of $(p_t \downarrow d)$ is defined by

$$\begin{tikzcd}
(p_t \downarrow d) \arrow[r, bend left=50, "1_d"{name=U}] \arrow[r, "a_d"{name=M}] \arrow[r, bend right=50, "c_d"'{name=D}] & (p_t \downarrow d)
\arrow[Rightarrow, from=U, to=M, "\alpha"] \arrow[Rightarrow, from=D, to=M, "\beta"']
\end{tikzcd}$$

In this diagram, c_d is the constant functor that takes as value the object $(d) \xrightarrow{1_d} d$. The functor a_d sends an object $(d_0 \to ... \to d_n) \xrightarrow{f} d$ to $(d_0 \to ... \to d_n \xrightarrow{f} d) \xrightarrow{1_d} d$, and a map defined by $i : \mathbf{k} \to \mathbf{n}$ to its extension $\bar{i} : \mathbf{k+1} \to \mathbf{n+1}$ with $\bar{i}(k+1) = n+1$.

On an object $(d_0 \to ... \to d_n) \to d$, the natural map α is given by the map $i : \mathbf{n} \to \mathbf{n+1}$, $ik = k$ and the natural map β is given by the map $i : \mathbf{0} \to \mathbf{n+1}$, $i0 = n+1$.

Let us prove (1) (c). Denote $i_d : p_t^{-1}d \to (p_t \downarrow d)$ the full inclusion functor. Note that the images of a_d and c_d are inside $i_d(p_t^{-1}d)$. Let x be an object of $(p_t \downarrow d)$ of the form $(d_0 \to ... \to d_n) \xrightarrow{f} d$.

If $d_n \xrightarrow{f} d$ is the identity map, then x is in the image of i_d therefore $(x, x \xrightarrow{1_x} x)$ is an initial object of $(x \downarrow i_d)$. If $d_n \xrightarrow{f} d$ is not the identity map, then $(a_d x, x \to a_d x)$ defined by the map $i : \mathbf{n} \to \mathbf{n+1}$, $ik = k$ is an initial object in $(x \downarrow i_d)$. In both cases, $(x \downarrow i_d)$ is contractible therefore i_d is homotopy right cofinal.

The statements of part (2) follow from duality. $\square$

The category $p_t^{-1}d$ is direct for any object $d\epsilon\mathcal{D}$, since it is a subcategory of the direct category $\Delta^{'}\mathcal{D}$. Dually, the category $p_i^{-1}d$ is inverse for any object $d\epsilon\mathcal{D}$.

Lemma 6.5.3.

(1) *Let $\mathcal{M}$ be a cofibration category and $\mathcal{D}$ be a small category. Then for any Reedy cofibrant diagram $X\epsilon\mathcal{M}^{\Delta^{'}\mathcal{D}}$ and any object $d\epsilon\mathcal{D}$, the restriction $X|_{p_t^{-1}d}$ is Reedy cofibrant in $\mathcal{M}^{p_t^{-1}d}$.*

(2) *Let $\mathcal{M}$ be a fibration category and $\mathcal{D}$ be a small category. Then for any Reedy fibrant diagram $X\epsilon\mathcal{M}^{\Delta^{'op}\mathcal{D}}$ and any object $d\epsilon\mathcal{D}$, the restriction $X|_{p_i^{-1}d}$ is Reedy fibrant in $\mathcal{M}^{p_i^{-1}d}$.*

Proof. We only prove (1). For $d\epsilon\mathcal{D}$, fix an object $\underline{d} = (d_0 \to ... \to d_n)\ \epsilon\Delta^{'}\mathcal{D}$ with $d_n = d$. We need some notations. Assume that

$$\underline{i} = \{i_1, ..., i_u\}, \quad \underline{j} = \{j_1, ..., j_v\}, \quad \underline{k} = \{k_1, ..., k_w\}$$

is a partition of $\{0, ..., n\}$ into three (possibly empty) subsets with $u+v+w = n+1$. Denote $\underline{n}_{\underline{i},\widehat{\underline{j}}}$ the full subcategory of $\Delta^{'}\mathcal{D}$. with objects $d_{l_0} \to ... \to d_{l_x}$ with $\underline{i} \subset \underline{l}$ and $\underline{j} \cap \underline{l} = \emptyset$, where $\underline{l} = \{l_0, .., l_x\}$. Although not apparent from the notation, the category $\underline{n}_{\underline{i},\widehat{\underline{j}}}$ depends on the choice of $\underline{d}\epsilon\Delta^{'}\mathcal{D}$.

The category $\underline{n}_{\underline{i},\widehat{\underline{j}}}$ is direct, and has a terminal object denoted $\underline{d}_{\widehat{\underline{j}}}$. Denote $\partial\underline{n}_{\underline{i},\widehat{\underline{j}}}$ the maximal full subcategory of $\underline{n}_{\underline{i},\widehat{\underline{j}}}$ without its terminal element.

Claim. The restriction of X to $\underline{n}_{\underline{i},\widehat{\underline{j}}}$ is Reedy cofibrant.

Taking in particular $\underline{i} = \{n\}$ and $\underline{j} = \emptyset$, the Claim implies that the Reedy condition is satisfied for $X|_{p_t^{-1}d}$ at $\underline{d}$. It remains to prove the Claim, and we will proceed by induction on n.

- The Claim can be directly verified for $n = 0$. Assume that the Claim was proved for $n^{'} < n$, and let's prove it for $n \geq 1$.
- If $\underline{j} \neq \emptyset$, the claim follows from the inductive hypothesis for smaller n. It remains to prove the Claim for $\underline{n}_{\underline{i},\widehat{\underline{\emptyset}}}$.
- We only need to prove the Reedy condition for X at the terminal object $\underline{d}_{\widehat{\underline{\emptyset}}} = \underline{d}$ of $\underline{n}_{\underline{i},\widehat{\underline{\emptyset}}}$. The Reedy condition at any other object of $\underline{n}_{\underline{i},\widehat{\underline{\emptyset}}}$ follows from the inductive hypothesis for smaller n.

- If $\underline{i} = \{i_1, ..., i_u\}$ is nonempty, in the diagram

$$\begin{tikzcd} \operatorname{colim}^{\partial \underline{n}_{\{i_1,\ldots,i_u\}\setminus\{i_s\},\widehat{\{i_s\}}}} X \arrow[r] \arrow[d, tail] & \operatorname{colim}^{\partial \underline{n}_{\{i_1,\ldots,i_u\},\widehat{\emptyset}}} X \arrow[d, tail] \\ X_{\underline{d}_{\widehat{\{i_s\}}}} \arrow[r] & \operatorname{colim}^{\partial \underline{n}_{\{i_1,\ldots,i_u\}\setminus\{i_s\},\widehat{\emptyset}}} X \end{tikzcd} \tag{6.4}$$

  the left vertical map is a cofibration with cofibrant domain, from the inductive hypothesis for smaller n. If the top right colimit in the diagram exists and is cofibrant, then the bottom right colimit exists, the diagram is a pushout and therefore the right vertical map is a cofibration.
- The colimit $\operatorname{colim}^{\partial \underline{n}_{\{0,\ldots,n\},\widehat{\emptyset}}} X$ exists and is the initial object of $\mathcal{M}$
- The map $\operatorname{colim}^{\partial \underline{n}_{\emptyset,\widehat{\emptyset}}} X \to X_{\underline{d}}$ is a cofibration with cofibrant domain since X is Reedy cofibrant in $\Delta^{'}\mathcal{D}$.
- An iterated use of (6.4) shows that the colimit below exists and

$$\operatorname{colim}^{\partial \underline{n}_{\{i_1,\ldots,i_u\},\widehat{\emptyset}}} X \to X_{\underline{d}}$$

  is a cofibration with cofibrant domain. This completes the proof of the Claim.

□

For a category $\mathcal{D}$, the subcategories $p_t^{-1}d \subset \Delta^{'}\mathcal{D}$ are disjoint for $d\epsilon\mathcal{D}$, and their union $\cup_{d\epsilon\mathcal{D}}\, p_t^{-1}d$ forms a category that we will denote $\Delta^{'}_{res}\mathcal{D}$. We will also denote $\Delta^{'op}_{res}\mathcal{D}$ the opposite of $\Delta^{'}_{res}\mathcal{D}$.

Given a cofibration category $\mathcal{M}$ we use the shorthand notation $\mathcal{M}^{\Delta^{'}\mathcal{D}}_{res}$ for the category $\mathcal{M}^{(\Delta^{'}\mathcal{D},\Delta^{'}_{res}\mathcal{D})}$ of $\Delta^{'}_{res}\mathcal{D}$ restricted $\Delta^{'}\mathcal{D}$ diagrams in $\mathcal{M}$. $\mathcal{M}^{\Delta^{'}\mathcal{D}}_{res}$ is a full subcategory of $\mathcal{M}^{\Delta^{'}\mathcal{D}}$, and (Thm. 6.3.9) it carries a restricted Reedy cofibration structure as well as a restricted pointwise cofibration structure. Furthermore, the functor $p_t^* : \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\Delta^{'}\mathcal{D}}$ has its image inside $\mathcal{M}^{\Delta^{'}\mathcal{D}}_{res}$.

Dually, for a fibration category $\mathcal{M}$ we denote $\mathcal{M}^{\Delta^{'op}\mathcal{D}}_{res}$ the category $\mathcal{M}^{(\Delta^{'op}\mathcal{D},\Delta^{'op}_{res}\mathcal{D})}$ of restricted diagrams. $\mathcal{M}^{\Delta^{'op}\mathcal{D}}_{res}$ carries a restricted Reedy fibration structure as well as a restricted pointwise fibration structure. The functor $p_i^* : \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\Delta^{'op}\mathcal{D}}$ has its image inside $\mathcal{M}^{\Delta^{'op}\mathcal{D}}_{res}$.

Proposition 6.5.4.

(1) *Let $\mathcal{M}$ be a cofibration category and $\mathcal{D}$ be a small category. Then for every restricted Reedy cofibrant diagrams $X, X^{'}\epsilon\mathcal{M}^{\Delta^{'}\mathcal{D}}_{res}$ and every diagram $Y\epsilon\mathcal{M}^{\mathcal{D}}$*
    (a) *A map $X \to p_t^*Y$ is a pointwise weak equivalence iff its adjoint* $\operatorname{colim}^{p_t} X \to Y$ *is a pointwise weak equivalence.*
    (b) *A map $X \to X^{'}$ is a pointwise weak equivalence iff the map* $\operatorname{colim}^{p_t} X \to \operatorname{colim}^{p_t} X^{'}$ *is a pointwise weak equivalence.*

(2) *Let $\mathcal{M}$ be a fibration category and $\mathcal{D}$ be a small category. Then for every restricted Reedy fibrant diagrams $X, X^{'}\epsilon\mathcal{M}^{\Delta^{'op}\mathcal{D}}_{res}$ and every diagram $Y\epsilon\mathcal{M}^{\mathcal{D}}$*
    (a) *A map $p_i^*Y \to X$ is a pointwise weak equivalence iff its adjoint $Y \to$* $\lim^{p_i} X$ *is a pointwise weak equivalence.*

(b) *A map* $X \to X^{'}$ *is a pointwise weak equivalence iff the map* $\lim^{p_t} X \to \lim^{p_t} X^{'}$ *is a pointwise weak equivalence.*

Proof. We will prove (1), and (2) will follow from duality. Let d be an object of $\mathcal{D}$. The categories $p_t^{-1}d$ and $(p_t \downarrow d)$ are direct, and since X is Reedy cofibrant so are its restrictions to $p_t^{-1}d$ (by Lemma 6.5.3) and to $(p_t \downarrow d)$.

The colimits $\operatorname{colim}^{p_t^{-1}d} X$, $\operatorname{colim}^{(p_t\downarrow d)} X$ therefore exist. Since $i_d : p_t^{-1}d \to (p_t \downarrow d)$ is right cofinal, we have $\operatorname{colim}^{p_t^{-1}d} X \cong \operatorname{colim}^{(p_t\downarrow d)} X$. We also conclude that $\operatorname{colim}^{p_t} X$ exists and $\operatorname{colim}^{(p_t\downarrow d)} X \cong (\operatorname{colim}^{p_t} X)_d$.

The diagram X is restricted and $p_t^{-1}d$ has an initial object that we will denote $e(d)$. We get a pointwise weak equivalence $cX_{e(d)} \to X|_{p_t^{-1}d}$ in $\mathcal{M}^{p_t^{-1}d}$ from the constant diagram to the restriction of X. But the diagram $cX_{e(d)}$ is Reedy cofibrant since $e(d)$ is initial in $p_t^{-1}d$. The map $cX_{e(d)} \to X|_{p_t^{-1}d}$ is a pointwise weak equivalence between Reedy cofibrant diagrams in $\mathcal{M}^{p_t^{-1}d}$, therefore $X_{e(d)} \cong \operatorname{colim}^{p_t^{-1}d} cX_{e(d)} \to \operatorname{colim}^{p_t^{-1}d} X$ is a weak equivalence.

In summary, we have showed that the composition

$$X_{e(d)} \to \operatorname{colim}^{p_t^{-1}d} X \cong \operatorname{colim}^{(p_t\downarrow d)} X \cong (\operatorname{colim}^{p_t} X)_d$$

is a weak equivalence. We can now complete our proof.

To prove (a), the map $\operatorname{colim}^{p_t} X \to Y$ is a pointwise weak equivalence iff the map $X_{e(d)} \to Y_d$ is a weak equivalence for all objects d of $\mathcal{D}$. Since X is restricted, this last statement is true iff the map $X \to p_t^* Y$ is a pointwise weak equivalence.

To prove (b), the map $\operatorname{colim}^{p_t} X \to \operatorname{colim}^{p_t} X^{'}$ is a pointwise weak equivalence iff the map $X_{e(d)} \to X^{'}_{e(d)}$ is a weak equivalence for all objects d of $\mathcal{D}$. Since $X, X^{'}$ are restricted, this last statement is true iff the map $X \to X^{'}$ is a pointwise weak equivalence. ☐

As an application, we can now prove that the category of diagrams in a cofibration (resp. fibration) category admits a pointwise cofibration (resp. fibration) structure. This result is due to Cisinski. We should note that the statement below is not true if we replace "cofibration category" with "Quillen model category".

Theorem 6.5.5 (Pointwise (co)fibration structure).

(1) *If* $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ *is a cofibration category and* $\mathcal{D}$ *is a small category then* $(\mathcal{M}^{\mathcal{D}}, \mathcal{W}^{\mathcal{D}}, \mathcal{C}of^{\mathcal{D}})$ *is a cofibration category.*

(2) *If* $(\mathcal{M}, \mathcal{W}, \mathcal{F}ib)$ *is a fibration category and* $\mathcal{D}$ *is a small category then* $(\mathcal{M}^{\mathcal{D}}, \mathcal{W}^{\mathcal{D}}, \mathcal{F}ib^{\mathcal{D}})$ *is a fibration category.*

Proof. To prove (1), axioms CF1-CF3 and CF5-CF6 are easily verified. To prove axiom CF4, we replay the argument in the proof of Thm. 6.2.4 (1) (b) part (iv). Let $f : X \to Y$ be a map of $\mathcal{D}$-diagrams with X pointwise cofibrant. Let $a : X_1 \to p_t^* X$ be a Reedy cofibrant replacement in $\mathcal{M}_{res}^{\Delta^{'}\mathcal{D}}$. We factor $X_1 \to p_t^* Y$ as a Reedy cofibration f_1 followed by a pointwise weak equivalence r_1

$$\begin{tikzcd} X_1 \arrow[r, tail, "f_1"] & Y_1 \arrow[r, "r_1", "\sim"'] & p_t^* Y \end{tikzcd}$$

We then construct a commutative diagram

$$\begin{tikzcd}
 & & Y \\
X \arrow[rru, tail, "f"] \arrow[r, tail, "f'"'] & Y' \arrow[ru, "r"] & \\
\operatorname{colim}^{p_t} X_1 \arrow[u, "a'", "\sim"'] \arrow[r, tail, "\operatorname{colim}^{p_t} f_1"'] & \operatorname{colim}^{p_t} Y_1 \arrow[u, "b'", "\sim"'] \arrow[ruu, "r_1'"', "\sim"] &
\end{tikzcd}$$

In this diagram a' resp. r_1' are the adjoints of a resp. r_1, therefore by Prop. 6.5.4 (1) (a) a' and r_1' are weak equivalences. Since f_1 is a Reedy cofibration, $\operatorname{colim}^{p_t} f_1$ is a pointwise cofibration, and we construct f' as the pushout of $\operatorname{colim}^{p_t} f_1$. It follows that f' is a pointwise cofibration. By pointwise excision, b' and therefore r are pointwise weak equivalences. The factorization $f = rf'$ is the desired decomposition of f as a pointwise cofibration followed by a weak equivalence, and CF4 is proved.

The proof of (2) is dual. ☐

As an immediate corollary, we can show that for a small category pair $(\mathcal{D}_1, \mathcal{D}_2)$, the category of reduced diagrams $\mathcal{M}^{(\mathcal{D}_1,\mathcal{D}_2)}$ carries a pointwise cofibration structure if $\mathcal{M}$ is a cofibration category.

THEOREM 6.5.6 (Reduced pointwise (co)fibration structure).

(1) *If* $(\mathcal{M},\ \mathcal{W},\ \mathcal{C}of)$ *is a cofibration category and* $(\mathcal{D}_1, \mathcal{D}_2)$ *is a small category pair, then* $(\mathcal{M}^{(\mathcal{D}_1,\mathcal{D}_2)}, \mathcal{W}^{\mathcal{D}_1}, \mathcal{C}of^{(\mathcal{D}_1,\mathcal{D}_2)})$ *is a cofibration category - called the* $\mathcal{D}_2$*-restricted pointwise cofibration structure on* $\mathcal{M}^{(\mathcal{D}_1,\mathcal{D}_2)}$.

(2) *If* $(\mathcal{M},\ \mathcal{W},\ \mathcal{F}ib)$ *is a fibration category and* $(\mathcal{D}_1, \mathcal{D}_2)$ *is a small category pair, then* $(\mathcal{M}^{(\mathcal{D}_1,\mathcal{D}_2)}, \mathcal{W}^{\mathcal{D}_1}, \mathcal{F}ib^{(\mathcal{D}_1,\mathcal{D}_2)})$ *is a fibration category - called the* $\mathcal{D}_2$*- restricted pointwise fibration structure on* $\mathcal{M}^{(\mathcal{D}_1,\mathcal{D}_2)}$.

PROOF. Entirely sililar to that of Thm. 6.3.9. ☐

Denote $\mathcal{M}^{\Delta'\mathcal{D}}_{res,rcof}$ the full subcategory of $\mathcal{M}^{\Delta'\mathcal{D}}_{res}$ of restricted Reedy cofibrant diagrams for a cofibration category $\mathcal{M}$. The next proposition states that restricted Reedy cofibrant diagrams in $\mathcal{M}^{\Delta'\mathcal{D}}$ form a cofibrant approximation of $\mathcal{M}^{\mathcal{D}}$.

Dually for a fibration category $\mathcal{M}$ denote $\mathcal{M}^{\Delta'^{op}\mathcal{D}}_{res,rfib}$ the full subcategory of $\mathcal{M}^{\Delta'^{op}\mathcal{D}}_{res}$ of restricted Reedy fibrant diagrams. We show that restricted Reedy fibrant diagrams in $\mathcal{M}^{\Delta'^{op}\mathcal{D}}$ form a fibrant approximation of $\mathcal{M}^{\mathcal{D}}$.

PROPOSITION 6.5.7. *Let* $\mathcal{D}$ *be a small category.*

(1) *If* $\mathcal{M}$ *is a cofibration category then* $\operatorname{colim}^{p_t} : \mathcal{M}^{\Delta'\mathcal{D}}_{res,rcof} \to \mathcal{M}^{\mathcal{D}}$ *is a cofibrant approximation for the pointwise cofibration structure on* $\mathcal{M}^{\mathcal{D}}$.

(2) *If* $\mathcal{M}$ *is a fibration category then* $\lim^{p_i} : \mathcal{M}^{\Delta'^{op}\mathcal{D}}_{res,rfib} \to \mathcal{M}^{\mathcal{D}}$ *is a fibrant approximation for the pointwise fibration structure on* $\mathcal{M}^{\mathcal{D}}$.

PROOF. We only prove (1), since the proof of (2) is dual.

The functor $\operatorname{colim}^{p_t}$ sends Reedy cofibrations to pointwise cofibrations by Thm. 6.3.5 and preserves the initial object, which proves CFA1. CFA2 is a consequence of Prop. 6.5.4 (1) (b). CFA3 is a consequence of Remark 5.6.4.

For CFA4, let $f : \operatorname{colim}^{p_t} X \to Y$ be a map in $\mathcal{M}^{\mathcal{D}}$ with X restricted and Reedy cofibrant. Factor its adjoint $f' : X \to p_t^* Y$ as a Reedy cofibration f_1 followed by a pointwise weak equivalence r_1

$$\begin{tikzcd} X \arrow[r, tail, "f_1"] & Y' \arrow[r, "r_1", "\sim"'] & p_t^* Y \end{tikzcd}$$

We get the following CFA4 factorization of f

$$\begin{tikzcd} \operatorname{colim}^{p_t} X \arrow[r, tail, "\operatorname{colim}^{p_t} f_1"] & \operatorname{colim}^{p_t} Y' \arrow[r, "r_1'", "\sim"'] & Y \end{tikzcd}$$

where r_1' is the adjoint of r_1, therefore a weak equivalence. ☐

Here is another application of Prop. 6.5.4:

THEOREM 6.5.8. *Let $\mathcal{D}$ be a small category.*

(1) *If $\mathcal{M}$ is a cofibration category then*

$$\mathbf{ho}p_t^* : \mathbf{ho}\mathcal{M}^{\mathcal{D}} \to \mathbf{ho}\mathcal{M}_{res}^{\Delta' \mathcal{D}}$$

is an equivalence of categories.

(2) *If $\mathcal{M}$ is a fibration category then*

$$\mathbf{ho}p_i^* : \mathbf{ho}\mathcal{M}^{\mathcal{D}} \to \mathbf{ho}\mathcal{M}_{res}^{\Delta'^{op} \mathcal{D}}$$

is an equivalence of categories.

PROOF. Let us prove (1). We will apply the Abstract Partial Quillen Adjunction Thm. 4.5.10 to

$$\begin{tikzcd} \mathcal{M}_{res,rcof}^{\Delta' \mathcal{D}} \arrow[r, "v_1 = \operatorname{colim}^{pt}"] \arrow[d, hook, "t_1"'] & \mathcal{M}^{\mathcal{D}} \\ \mathcal{M}_{res}^{\Delta' \mathcal{D}} & \mathcal{M}^{\mathcal{D}} \arrow[l, "v_2 = p_t^*"'] \arrow[u, "t_2 = 1_{\mathcal{M}^{\mathcal{D}}}"'] \end{tikzcd}$$

The functor t_1 is the full inclusion of $\mathcal{M}_{res,rcof}^{\Delta' \mathcal{D}}$ in $\mathcal{M}_{res}^{\Delta' \mathcal{D}}$. We have that v_1, v_2 is an abstract Quillen partially equivalent pair with respect to t_1, t_2. Indeed:

(1) The functor pair v_1, v_2 is partially adjoint with respect to t_1, t_2.
(2) t_1 is a cofibrant approximation of the cofibration category $\mathcal{M}_{res}^{\Delta' \mathcal{D}}$ with the Reedy reduced structure. In particular t_1 is a left approximation. The functor $t_2 = 1_{\mathcal{M}^{\mathcal{D}}}$ is a right approximation.
(3) v_1 preserves weak equivalences from Thm. 6.4.1, and so does $v_2 = p_t^*$.
(4) Prop. 6.5.4 (1) (a) states that for any objects $X \epsilon \mathcal{M}_{res,rcof}^{\Delta' \mathcal{D}}$, $Y \epsilon \mathcal{M}^{\mathcal{D}}$, a map $v_1 X \to t_2 Y$ is a weak equivalence iff its partial adjoint $t_1 X \to v_2 Y$ is a weak equivalence

In conclusion we have a pair of equivalences of categories

$$\begin{tikzcd} \mathbf{ho}\mathcal{M}_{res}^{\Delta' \mathcal{D}} \arrow[r, shift left, "\mathbf{ho}(v_1)\,\mathbf{s}_1"] & \mathbf{ho}\mathcal{M}^{\mathcal{D}} \arrow[l, shift left, "\mathbf{ho}(v_2)\,\mathbf{s}_2 \cong \mathbf{ho}(p_t^*)"] \end{tikzcd}$$

where $\mathbf{s}_1$ is a quasi-inverse of $\mathbf{ho}t_1$ and $\mathbf{s}_2$ is a quasi-inverse of $\mathbf{ho}t_2$, and therefore $\mathbf{ho}(v_2)\,\mathbf{s}_2$ is naturally isomorphic to $\mathbf{ho}(p_t^*)$. This proves that $\mathbf{ho}(p_t^*)$ is an equivalence of categories.

The proof of part (2) is dual. □

6.6. Homotopy colimits

Suppose that $(\mathcal{M}, \mathcal{W})$ is a category with weak equivalences and suppose that $u : \mathcal{D}_1 \to \mathcal{D}_2$ is a small functor. We denote $\gamma_{\mathcal{D}_i} : \mathcal{M}^{\mathcal{D}_i} \to \mathbf{ho}\mathcal{M}^{\mathcal{D}_i}$, $i = 1, 2$ the localization functors.

We define $\mathcal{M}^{\mathcal{D}_1}_{\text{colim}^u}$ to be the full subcategory of $\mathcal{M}^{\mathcal{D}_1}$ of $\mathcal{D}_1$ diagrams X with the property that $\operatorname{colim}^u X$ exists in $\mathcal{M}^{\mathcal{D}_2}$. Denote $i_{\text{colim}^u} : \mathcal{M}^{\mathcal{D}_1}_{\text{colim}^u} \to \mathcal{M}^{\mathcal{D}_1}$ the inclusion. In general $\mathcal{M}^{\mathcal{D}_1}_{\text{colim}^u}$ may be empty, but if $\mathcal{M}$ is cocomplete then $\mathcal{M}^{\mathcal{D}_1}_{\text{colim}^u} = \mathcal{M}^{\mathcal{D}_1}$. Let $\mathcal{W}^{\mathcal{D}_1}_{\text{colim}^u}$ be the class of pointwise weak equivalences of $\mathcal{M}^{\mathcal{D}_1}_{\text{colim}^u}$.

Dually, let $\mathcal{M}^{\mathcal{D}_1}_{\lim^u}$ be the full subcategory of $\mathcal{M}^{\mathcal{D}_1}$ of $\mathcal{D}_1$ diagrams X with the property that $\lim^u X$ exists in $\mathcal{M}^{\mathcal{D}_2}$, let $i_{\lim^u} : \mathcal{M}^{\mathcal{D}_1}_{\lim^u} \to \mathcal{M}^{\mathcal{D}_1}$ denote the inclusion, and let $\mathcal{W}^{\mathcal{D}_1}_{\lim^u}$ be the class of pointwise weak equivalences of $\mathcal{M}^{\mathcal{D}_1}_{\lim^u}$.

DEFINITION 6.6.1.

(1) The *homotopy colimit* of u, if it exists, is the left Kan extension of $\gamma_{\mathcal{D}_2} \operatorname{colim}^u$ along $\gamma_{\mathcal{D}_1} i_{\text{colim}^u}$

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{D}_1}_{\text{colim}^u} \arrow[r, "\operatorname{colim}^u"] \arrow[d, "\gamma_{\mathcal{D}_1} i_{\text{colim}^u}"'] \arrow[dr, phantom, "\epsilon_u \Nearrow"] & \mathcal{M}^{\mathcal{D}_2} \arrow[d, "\gamma_{\mathcal{D}_2}"] \\
\mathbf{ho}\mathcal{M}^{\mathcal{D}_1} \arrow[r, "\mathbf{L}\operatorname{colim}^u"'] & \mathbf{ho}\mathcal{M}^{\mathcal{D}_2}
\end{tikzcd}$$

and is denoted for simplicity $(\mathbf{L}\operatorname{colim}^u, \epsilon_u)$ instead of the more complete notation $(\mathbf{L}_{\gamma_{\mathcal{D}_1} i_{\text{colim}^u}} \operatorname{colim}^u, \epsilon_u)$

(2) The *homotopy limit* of u, if it exists, is the right Kan extension of $\gamma_{\mathcal{D}_2} \lim^u$ along $\gamma_{\mathcal{D}_1} i_{\lim^u}$

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{D}_1}_{\lim^u} \arrow[r, "\lim^u"] \arrow[d, "\gamma_{\mathcal{D}_1} i_{\lim^u}"'] \arrow[dr, phantom, "\Swarrow \nu_u"] & \mathcal{M}^{\mathcal{D}_2} \arrow[d, "\gamma_{\mathcal{D}_2}"] \\
\mathbf{ho}\mathcal{M}^{\mathcal{D}_1} \arrow[r, "\mathbf{R}\lim^u"'] & \mathbf{ho}\mathcal{M}^{\mathcal{D}_2}
\end{tikzcd}$$

and is denoted for simplicity $(\mathbf{R}\lim^u, \nu_u)$

If $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ is a cofibration category, then we also define the class $\mathcal{C}of^{\mathcal{D}_1}_{\text{colim}^u}$ of pointwise cofibrations $f : X \to Y$ in $\mathcal{M}^{\mathcal{D}_1}_{\text{colim}^u}$ with the property that $\operatorname{colim}^u f : \operatorname{colim}^u X \to \operatorname{colim}^u Y$ is well defined and pointwise cofibrant in $\mathcal{M}^{\mathcal{D}_2}$.

Dually, if $(\mathcal{M}, \mathcal{W}, \mathcal{F}ib)$ is a fibration category, then we define the class $\mathcal{F}ib^{\mathcal{D}_1}_{\lim^u}$ of pointwise fibration maps f in $\mathcal{M}^{\mathcal{D}_1}_{\lim^u}$ with the property that $\lim^u f$ is well defined and pointwise fibrant in $\mathcal{M}^{\mathcal{D}_2}$.

LEMMA 6.6.2. *Let $u : \mathcal{D}_1 \to \mathcal{D}_2$ be a small functor.*

(1) *If $\mathcal{M}$ is a cofibration category, then*
   (a) $(\mathcal{M}^{\mathcal{D}_1}_{\text{colim}^u}, \mathcal{W}^{\mathcal{D}_1}_{\text{colim}^u}, \mathcal{C}of^{\mathcal{D}_1}_{\text{colim}^u})$ *is a cofibration category*
   (b) $\operatorname{colim}^{p_t} : \mathcal{M}^{\Delta' \mathcal{D}_1}_{res,rcof} \to \mathcal{M}^{\mathcal{D}_1}_{\text{colim}^u}$ *is a cofibrant approximation*

(c) $\mathbf{ho}\mathcal{M}^{\mathcal{D}_1}_{\operatorname{colim}^u} \to \mathbf{ho}\mathcal{M}^{\mathcal{D}_1}$ *is an equivalence of categories.*

(2) *If* $\mathcal{M}$ *is a fibration category, then*
  (a) $(\mathcal{M}^{\mathcal{D}_1}_{\lim^u}, \mathcal{W}^{\mathcal{D}_1}_{\lim^u}, \mathcal{F}ib^{\mathcal{D}_1}_{\lim^u})$ *is a cofibration category*
  (b) $\lim^{p_i} : \mathcal{M}^{\Delta'^{op}\mathcal{D}_1}_{res,rfib} \to \mathcal{M}^{\mathcal{D}_1}_{\lim^u}$ *is a fibrant approximation*
  (c) $\mathbf{ho}\mathcal{M}^{\mathcal{D}_1}_{\lim^u} \to \mathbf{ho}\mathcal{M}^{\mathcal{D}_1}$ *is an equivalence of categories.*

PROOF. We only prove (1), since the proof of (2) is dual.

For an object $X \epsilon \mathcal{M}^{\Delta'\mathcal{D}_1}_{res,rcof}$, we have that the colimits $\operatorname{colim}^{p_t} X$, $\operatorname{colim}^{up_t} X$ exist and are pointwise cofibrant by Thm. 6.4.1 since X is Reedy cofibrant in $\mathcal{M}^{\Delta'\mathcal{D}_1}$. Since $\operatorname{colim}^{up_t} X \cong \operatorname{colim}^u \operatorname{colim}^{p_t} X$, we conclude that the functor $\operatorname{colim}^{p_t} : \mathcal{M}^{\Delta'\mathcal{D}_1}_{res,rcof} \to \mathcal{M}^{\mathcal{D}_1}$ has its image inside $\mathcal{M}^{\mathcal{D}_1}_{\operatorname{colim}^u}$.

Let us prove (a). Axioms CF1-CF2 and CF5-CF6 are easily verified for $\mathcal{M}^{\mathcal{D}_1}_{\operatorname{colim}^u}$. The pushout axiom CF3 (1) follows from the fact that if

$$\begin{tikzcd}
X \arrow[r] \arrow[d, tail, "i"'] & Z \arrow[d, tail, dashed, "j"] \\
Y \arrow[r, dashed] & T
\end{tikzcd}$$

is a pushout in $\mathcal{M}^{\mathcal{D}_1}$ with X, Y, Z cofibrant in $\mathcal{M}^{\mathcal{D}_1}_{\operatorname{colim}^u}$ and i a cofibration in $\mathcal{M}^{\mathcal{D}_1}_{\operatorname{colim}^u}$, then $\operatorname{colim}^u X, \operatorname{colim}^u Y, \operatorname{colim}^u Z$ are pointwise cofibrant and $\operatorname{colim}^u i$ is a pointwise cofibration in $\mathcal{M}^{\mathcal{D}_2}$, therefore by Remark 5.6.4 we have that $\operatorname{colim}^u T$ exists and is the pushout in $\mathcal{M}^{\mathcal{D}_2}$ of

$$\begin{tikzcd}
\operatorname{colim}^u X \arrow[r] \arrow[d, tail, "\operatorname{colim}^u i"'] & \operatorname{colim}^u Z \arrow[d, tail, dashed, "\operatorname{colim}^u j"] \\
\operatorname{colim}^u Y \arrow[r, dashed] & \operatorname{colim}^u T
\end{tikzcd}$$

The axiom CF3 (2) follows from a pointwise application of CF3 (2) in $\mathcal{M}$.

Let us prove the factorization axiom CF4. We repeat the argument in the proof of Thm. 6.5.5.

Let $f : X \to Y$ be a map in $\mathcal{M}^{\mathcal{D}_1}_{\operatorname{colim}^u}$ with X cofibrant. Let $a : X_1 \to p_t^* X$ be a Reedy cofibrant replacement in $\mathcal{M}^{\Delta'\mathcal{D}_1}_{red}$. We factor $X_1 \to p_t^* Y$ as a Reedy cofibration f_1 followed by a pointwise weak equivalence r_1

$$\begin{tikzcd}
X_1 \arrow[r, tail, "f_1"] & Y_1 \arrow[r, "r_1", "\sim"'] & p_t^* Y
\end{tikzcd}$$

We then construct a commutative diagram

$$\begin{tikzcd}
 & & Y \\
X \arrow[urr, tail, "f"] \arrow[r, tail, "f'"'] & Y' \arrow[ur, "r", "\sim"'] & \\
\operatorname{colim}^{p_t} X_1 \arrow[u, "a'", "\sim"'] \arrow[r, tail, "\operatorname{colim}^{p_t} f_1"'] & \operatorname{colim}^{p_t} Y_1 \arrow[u, "b'", "\sim"'] \arrow[uur, "r_1'"'] &
\end{tikzcd}$$

In this diagram $a^{'}$ resp. $r_1^{'}$ are the adjoints of a resp. r_1, therefore by Prop. 6.5.4 (1) (a) $a^{'}$ and $r_1^{'}$ are weak equivalences. Since f_1 is a Reedy cofibration, $\operatorname{colim}^{p_t} f_1$ is a cofibration in $\mathcal{M}^{\mathcal{D}_1}_{\operatorname{colim}^u}$, and we construct $f^{'}$ as the pushout of $\operatorname{colim}^{p_t} f_1$. It follows that $f^{'}$ is a cofibration of $\mathcal{M}^{\mathcal{D}_1}_{\operatorname{colim}^u}$. By pointwise excision, $b^{'}$ and therefore r are pointwise weak equivalences. The factorization $f = rf^{'}$ is the desired decomposition of f as a pointwise cofibration followed by a weak equivalence in $\mathcal{M}^{\mathcal{D}_1}_{\operatorname{colim}^u}$, and CF4 is proved.

Part (b) follows from Prop. 6.5.7 and the fact that $\operatorname{colim}^{p_t} : \mathcal{M}^{\Delta^{'}\mathcal{D}_1}_{res,rcof} \to \mathcal{M}^{\mathcal{D}_1}$ has its image inside $\mathcal{M}^{\mathcal{D}_1}_{\operatorname{colim}^u}$.

To prove part (c), since both functors $\operatorname{colim}^{p_t} : \mathcal{M}^{\Delta^{'}\mathcal{D}_1}_{res,rcof} \to \mathcal{M}^{\mathcal{D}_1}$ and its corestriction $\operatorname{colim}^{p_t} : \mathcal{M}^{\Delta^{'}\mathcal{D}_1}_{res,rcof} \to \mathcal{M}^{\mathcal{D}_1}_{\operatorname{colim}^u}$ are cofibrant approximations it follows (Thm. 4.6.3) that both induced functors $\mathbf{ho}\mathcal{M}^{\Delta^{'}\mathcal{D}_1}_{res,rcof} \to \mathbf{ho}\mathcal{M}^{\mathcal{D}_1}$ and $\mathbf{ho}\mathcal{M}^{\Delta^{'}\mathcal{D}_1}_{res,rcof} \to \mathbf{ho}\mathcal{M}^{\mathcal{D}_1}_{\operatorname{colim}^u}$ are equivalences of categories. The functor $\mathbf{ho}\mathcal{M}^{\mathcal{D}_1}_{\operatorname{colim}^u} \to \mathbf{ho}\mathcal{M}^{\mathcal{D}_1}$ is therefore an equivalence of categories. $\square$

We now state the main result of this section.

THEOREM 6.6.3 (Existence of homotopy (co)limits).

(1) *Let $\mathcal{M}$ be a cofibration category and $u : \mathcal{D}_1 \to \mathcal{D}_2$ be a small functor. Then the homotopy colimit $(\mathbf{L}\operatorname{colim}^u, \epsilon_u)$ exists and*
$$\mathbf{L}\operatorname{colim}^u : \mathbf{ho}\mathcal{M}^{\mathcal{D}_1} \rightleftarrows \mathbf{ho}\mathcal{M}^{\mathcal{D}_2} : \mathbf{ho}u^*$$
forms a naturally adjoint pair.

(2) *Let $\mathcal{M}$ be a fibration category and $u : \mathcal{D}_1 \to \mathcal{D}_2$ be a small functor. Then the homotopy limit $(\mathbf{R}\lim^u, \nu_u)$ exists and*
$$\mathbf{ho}u^* : \mathbf{ho}\mathcal{M}^{\mathcal{D}_2} \rightleftarrows \mathbf{ho}\mathcal{M}^{\mathcal{D}_1} : \mathbf{R}\lim^u$$
forms a naturally adjoint pair.

PROOF. Parts (1) and (2) are dual, and we will only prove part (1).

From Lemma 6.6.2 and Cor. 4.3.5, to prove the existence of the left Kan extension $(\mathbf{L}\operatorname{colim}^u, \epsilon_u)$ it suffices to prove the existence of the total left derived of $\operatorname{colim}^u : \mathcal{M}^{\mathcal{D}_1}_{\operatorname{colim}^u} \to \mathcal{M}^{\mathcal{D}_2}$. But the latter is a consequence of Thm. 4.6.2 applied to the cofibrant approximation $\operatorname{colim}^{p_t} : \mathcal{M}^{\Delta^{'}\mathcal{D}_1}_{res,rcof} \to \mathcal{M}^{\mathcal{D}_1}_{\operatorname{colim}^u}$.

To prove that $\mathbf{L}\operatorname{colim}^u \dashv \mathbf{ho}u^*$ forms a naturally adjoint pair, we will apply the Abstract Quillen Partial Adjunction Thm. 4.5.10 to

$$\begin{tikzcd}
\mathcal{M}^{\Delta^{'}\mathcal{D}_1}_{res,rcof} \arrow[r, "v_1=\operatorname{colim}^{up_t}"] \arrow[d, "t_1=\operatorname{colim}^{p_t}"'] & \mathcal{M}^{\mathcal{D}_2} \\
\mathcal{M}^{\mathcal{D}_1} & \mathcal{M}^{\mathcal{D}_2} \arrow[l, "v_2=u^*"'] \arrow[u, "t_2=1_{\mathcal{M}^{\mathcal{D}_2}}"']
\end{tikzcd}$$

We have that v_1, v_2 is an abstract Quillen partially adjoint pair with respect to t_1, t_2:

(1) The functor pair v_1, v_2 is partially adjoint with respect to t_1, t_2.

(2) t_1 is a cofibrant approximation of the cofibration category $\mathcal{M}^{\mathcal{D}_1}$ by Prop. 6.5.7. In particular t_1 is a left approximation. The functor $t_2 = 1_{\mathcal{M}^{\mathcal{D}_2}}$ is a right approximation.

(3) v_1 preserves weak equivalences by Thm. 6.4.1, and so does $v_2 = p_t^*$.

In conclusion we have a naturally adjoint pair

$$\begin{tikzcd}
\mathbf{ho}\mathcal{M}^{\mathcal{D}_1} \arrow[r, shift left, "\mathbf{ho}(v1)\,\mathbf{s}_1 \cong \mathbf{L}\operatorname{colim}^{u}"] & \mathbf{ho}\mathcal{M}^{\mathcal{D}_2} \arrow[l, shift left, "\mathbf{ho}(v_2)\,\mathbf{s}_2 \cong \mathbf{ho}(u^*)"]
\end{tikzcd}$$

where $\mathbf{s}_1$ is a quasi-inverse of $\mathbf{ho}t_1$ and $\mathbf{s}_2$ is a quasi-inverse of $\mathbf{ho}t_2$. ☐

As a consequence of Thm. 6.6.3 we can verify that

COROLLARY 6.6.4. *Suppose that* $u : \mathcal{D}_1 \to \mathcal{D}_2$ *and* $v : \mathcal{D}_2 \to \mathcal{D}_3$ *are two small functors.*

(1) *If* $\mathcal{M}$ *is a cofibration category, then* $\mathbf{L}\operatorname{colim}^{vu} \cong \mathbf{L}\operatorname{colim}^{v}\mathbf{L}\operatorname{colim}^{u}$.

(2) *If* $\mathcal{M}$ *is a fibration category, then* $\mathbf{R}\lim^{vu} \cong \mathbf{R}\lim^{v}\mathbf{R}\lim^{u}$.

PROOF. This is a consequence of the adjunction property of the homotopy (co)limit and the fact that $\mathbf{ho}(vu)^* \cong \mathbf{ho}u^*\mathbf{ho}v^*$. ☐

Consider a small diagram

(6.5)
$$\begin{tikzcd}
\mathcal{D}_1 \arrow[r, "u"] \arrow[d, "f"'] \arrow[rd, phantom, "\phi\Nearrow"] & \mathcal{D}_2 \arrow[d, "g"] \\
\mathcal{D}_3 \arrow[r, "v"'] & \mathcal{D}_4
\end{tikzcd}$$

If $\mathcal{M}$ is a cofibration category, from the adjunction property of the homotopy colimit we get a natural map denoted

$$\phi_{\mathbf{L}\operatorname{colim}} : \mathbf{L}\operatorname{colim}^{u}\mathbf{ho}f^* \Rightarrow \mathbf{ho}g^*\mathbf{L}\operatorname{colim}^{v}$$

and dually if $\mathcal{M}$ is a fibration category we get a natural map denoted

$$\phi_{\mathbf{R}\operatorname{colim}} : \mathbf{ho}v^*\mathbf{R}\lim^{g} \Rightarrow \mathbf{R}\lim^{f}\mathbf{ho}u^*$$

Suppose that $u : \mathcal{D}_1 \to \mathcal{D}_2$ is a small functor. For any object $d_2\epsilon\mathcal{D}_2$, the *standard over 2-category diagram* of u at d_2 is defined as

(6.6)
$$\begin{tikzcd}
(u \downarrow d_2) \arrow[r, "p_{(u\downarrow d_2)}"] \arrow[d, "i_{u,d_2}"'] \arrow[rd, phantom, "\phi_{u,d_2}\Nearrow"] & e \arrow[d, "e_{d_2}"] \\
\mathcal{D}_1 \arrow[r, "u"'] & \mathcal{D}_2
\end{tikzcd}$$

In this diagram, the functor i_{u,d_2} is as defined in Section 5.1.2, and $p_{\mathcal{D}} : \mathcal{D} \to e$ denotes the terminal category projection. The functor e_{d_2} embeds the terminal category e as the object $d_2\epsilon\mathcal{D}_2$, and for an object $(d_1\epsilon\mathcal{D}_1, f : ud_1 \to d_2)$ of $(u \downarrow d_2)$ the natural map $\phi_{u,d_2}(d_1, ud_1 \to d_2)$ is $f : ud_1 \to d_2$. If $\mathcal{M}$ is a cofibration category, we obtain a natural map

(6.7)
$$\mathbf{L}\operatorname{colim}^{(u\downarrow d_2)}X \Rightarrow (\mathbf{L}\operatorname{colim}^{u}X)_{d_2}$$

Dually, the *standard under 2-category diagram* of u at d_2 is

(6.8)
$$\begin{tikzcd}
(d_2 \downarrow u) \arrow[r, "i_{d_2,u}"] \arrow[d, "p_{(d_2\downarrow u)}"'] \arrow[rd, phantom, "\phi_{d_2,u}\Nearrow"] & \mathcal{D}_1 \arrow[d, "u"] \\
e \arrow[r, "e_{d_2}"'] & \mathcal{D}_2
\end{tikzcd}$$

If $\mathcal{M}$ is a fibration category, we obtain a natural map

$$(\mathbf{R}\mathrm{lim}^{u}X)_{d_2} \Rightarrow \mathbf{R}\mathrm{lim}^{(d_2\downarrow u)}X \tag{6.9}$$

The next theorem proves a base change formula for homotopy (co)limits. This lemma is a homotopy colimit analogue to the well known base change formula for ordinary colimits Lemma 5.2.1.

Theorem 6.6.5 (Base change property).

1. *If $\mathcal{M}$ is a cofibration category and $u : \mathcal{D}_1 \to \mathcal{D}_2$ is a small functor, then the natural map (6.7) induces an isomorphism*
$$\mathbf{L}\mathrm{colim}^{(u\downarrow d_2)}X \cong (\mathbf{L}\mathrm{colim}^{u}X)_{d_2}$$
for objects $X \epsilon \mathcal{M}^{\mathcal{D}_1}$ and $d_2 \epsilon \mathcal{D}_2$.
2. *If $\mathcal{M}$ is a fibration category and $u : \mathcal{D}_1 \to \mathcal{D}_2$ is a small functor, then the natural map (6.9) induces an isomorphism*
$$(\mathbf{R}\mathrm{lim}^{u}X)_{d_2} \cong \mathbf{R}\mathrm{lim}^{(d_2\downarrow u)}X$$
for objects $X \epsilon \mathcal{M}^{\mathcal{D}_1}$ and $d_2 \epsilon \mathcal{D}_2$.

Proof. We only prove (1). For a diagram $X \epsilon \mathcal{M}^{\mathcal{D}_1}$, pick a reduced Reedy cofibrant replacement $X' \epsilon \Delta'_{res}\mathcal{D}_1$ of $p_t^* X$. We have $\mathrm{colim}^{up_t}X' \cong \mathbf{L}\mathrm{colim}^{u}X$. By Lemma 6.5.1, $(up_t \downarrow d_2) \cong \Delta'(u \downarrow d_2)$. The restriction of X' to the direct category $\Delta'(u \downarrow d_2)$ is a Reedy cofibrant replacement of the restriction of X to $\Delta'(u \downarrow d_2)$, so $\mathrm{colim}^{(up_t\downarrow d_2)}X' \cong \mathrm{colim}^{\Delta'(u\downarrow d_2)}X' \cong \mathbf{L}\mathrm{colim}^{(u\downarrow d_2)}X$. Using Lemma 5.2.1, the top map and therefore all maps in the commutative diagram

$$\begin{tikzcd}
\mathrm{colim}^{(up_t\downarrow d_2)}X' \arrow[r] \arrow[d] & (\mathrm{colim}^{up_t}X')_{d_2} \arrow[d] \\
\mathbf{L}\mathrm{colim}^{(u\downarrow d_2)}X \arrow[r] & (\mathbf{L}\mathrm{colim}^{u}X)_{d_2}
\end{tikzcd}$$

are isomorphisms, and the conclusion is proved. □

Suppose that $(\mathcal{M}, \mathcal{W})$ is a category with weak equivalences, and that $u : \mathcal{D}_1 \to \mathcal{D}_2$ is a small functor. The next result describes a sufficient condition for $(\mathbf{L}\,\mathrm{colim}^{u}, \epsilon_u)$ to exist - without requiring $\mathcal{M}$ to carry a cofibration category structure.

In preparation, notice that the natural map (6.7) actually exists under the weaker assumption that $(\mathcal{M}, \mathcal{W})$ is a category with weak equivalences, that $\mathbf{L}\mathrm{colim}^{u}$ exists and is a left adjoint of $\mathbf{ho}u^*$, and that $\mathbf{L}\mathrm{colim}^{(u\downarrow d_2)}$ exists and is a left adjoint of $\mathbf{ho}p^*_{(u\downarrow d_2)}$. A dual statement holds for the map (6.9).

Theorem 6.6.6. *Suppose that $(\mathcal{M}, \mathcal{W})$ is a pointed category with weak equivalences.*

1. *If $u : \mathcal{D}_1 \to \mathcal{D}_2$ is a small closed embedding functor, then*
    - (a) colim^{u} *and its left Kan extension $(\mathbf{L}\,\mathrm{colim}^{u}, \epsilon_u)$ exist, and $\mathbf{L}\,\mathrm{colim}^{u}$ is a fully faithful left adjoint to $\mathbf{ho}u^*$*
    - (b) *For any object d_2 of $\mathcal{D}_2$, the functors* $\mathrm{colim}^{(u\downarrow d_2)}$ *and* $\mathbf{L}\mathrm{colim}^{(u\downarrow d_2)}$ *are well defined and the natural map (6.7) induces an isomorphism*
    $$\mathbf{L}\mathrm{colim}^{(u\downarrow d_2)}X \cong (\mathbf{L}\mathrm{colim}^{u}X)_{d_2}$$
    for objects $X \epsilon \mathcal{M}^{\mathcal{D}_1}$ and $d_2 \epsilon \mathcal{D}_2$.
2. *If $u : \mathcal{D}_1 \to \mathcal{D}_2$ is a small open embedding functor, then*

(a) $\lim^{u}$ *and its right Kan extension* $(\mathbf{R}\lim^{u}, \nu_u)$ *exist, and* $\mathbf{R}\lim^{u}$ *is a fully faithful left adjoint to* $\mathbf{ho}u^*$

(b) *For any object* d_2 *of* $\mathcal{D}_2$*, the functors* $\lim^{(d_2 \downarrow u)}$ *and* $\mathbf{R}\lim^{(d_2 \downarrow u)}$ *are well defined and the natural map (6.9) induces an isomorphism*

$$(\mathbf{R}\lim^{u} X)_{d_2} \cong \mathbf{R}\lim^{(d_2 \downarrow u)} X$$

for objects $X \epsilon \mathcal{M}^{\mathcal{D}_1}$ *and* $d_2 \epsilon \mathcal{D}_2$*.*

Proof. We start with (1) (a). Denote $u_! : \mathcal{M}^{\mathcal{D}_1} \to \mathcal{M}^{\mathcal{D}_2}$ the 'extension by zero' functor, that sends a diagram $X \epsilon \mathcal{M}^{\mathcal{D}_1}$ to the diagram given by $(u_! X)_{d_2} = X_{d_2}$ for $d_2 \epsilon u \mathcal{D}_1$ and $(u_! X)_{d_2} = \mathbf{0}$ otherwise. The functor $u_!$ is a fully faithful left adjoint to u^*, and sends weak equivalences to weak equivalences. In particular, $\operatorname{colim}^{u} \cong u_1$ exists and is defined on the entire $\mathcal{M}^{\mathcal{D}_1}$, and $\mathbf{L}\operatorname{colim}^{u}$ as in Def. 6.6.1 exists and is isomorphic to $\mathbf{ho}u_!$. We apply the Abstract Quillen Adjunction Thm. 4.5.8 to

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{D}_2} \arrow[r, "t_1 = id"] & \mathcal{M}^{\mathcal{D}_2} \arrow[r, shift left, "u_1 = u_!"] & \mathcal{M}^{\mathcal{D}_1} \arrow[l, shift left, "u_2 = u^*"] & \mathcal{M}^{\mathcal{D}_1} \arrow[l, "t_2 = id"']
\end{tikzcd}$$

observing that its hypotheses (1)-(3) and (4l) apply. We deduce that $\mathbf{L}\operatorname{colim}^{u} \cong \mathbf{ho}u_!$ is a fully faithful right adjoint to $\mathbf{ho}u^*$.

To prove the isomorphism (1) (b), observe that $(u \downarrow d_2)$ has d_2 as a terminal object if $d_2 \epsilon u \mathcal{D}_1$ and is empty otherwise, so $\operatorname{colim}^{(u \downarrow d_2)} X \cong X_{d_2}$ if $d_2 \epsilon u \mathcal{D}_1$ and $\cong \mathbf{0}$ otherwise. Both functors $\operatorname{colim}^{(u \downarrow d_2)}$ and colim^{u} preserve weak equivalences, and we have adjoint pairs $\mathbf{L}\operatorname{colim}^{(u \downarrow d_2)} \dashv \mathbf{ho}p^*_{(u \downarrow d_2)}$ and $\mathbf{L}\operatorname{colim}^{u} \dashv \mathbf{ho}u^*$. The natural isomorphism $\operatorname{colim}^{(u \downarrow d_2)} X \cong (\operatorname{colim}^{u} X)_{d_2}$ yields the desired isomorphism (6.7).

The proof of part (2) is dual. □

The two lemmas below are part of the proof of Thm. 6.6.9 below. We keep the notations used in Thm. 6.6.6 and its proof, and introduce a few new ones. $\mathcal{M}$ denotes a pointed cofibration category, and $u : \mathcal{D}_1 \to \mathcal{D}_2$ a small closed embedding functor. $v : \mathcal{D}_2 \backslash \mathcal{D}_1 \to \mathcal{D}_2$ denotes the inclusion functor - it is an *open* embedding. We also denote $V = \Delta' v : \Delta'(\mathcal{D}_2 \backslash \mathcal{D}_1) \to \Delta' \mathcal{D}_2$.

We denote $u_! = \operatorname{colim}^{u} : \mathcal{M}^{\mathcal{D}_1} \to \mathcal{M}^{\mathcal{D}_2}$ and $v_* = \lim^{u} : \mathcal{M}^{\mathcal{D}_2 \backslash \mathcal{D}_1} \to \mathcal{M}^{\mathcal{D}_2}$ - these are the 'extension by zero' functors. We also denote $v_! = \operatorname{colim}^{v}$ and $V_! = \operatorname{colim}^{V}$, and we will keep in mind that they are defined only on a *full subcategory* of $\mathcal{M}^{\mathcal{D}_2 \backslash \mathcal{D}_1}$ resp. $\mathcal{M}^{\Delta'(\mathcal{D}_2 \backslash \mathcal{D}_1)}$.

We denote $(\mathcal{M}^{\mathcal{D}_2})_0$ the full subcategory of $\mathcal{M}^{\mathcal{D}_2}$ consisting of objects X with the property that $v_! v^* X$ exists and is pointwise cofibrant, and the map $v_! v^* X \to X$ is a pointwise cofibration. $\mathbf{0}$ is a cofibrant object, and we define the functor $u_1' : (\mathcal{M}^{\mathcal{D}_2})_0 \to \mathcal{M}^{\mathcal{D}_2}$ as the pushout

$$\begin{tikzcd}
v_! v^* X \arrow[r, tail] \arrow[d] & X \arrow[d, dashed] \\
\mathbf{0} \arrow[r, tail, dashed] & u_1' X
\end{tikzcd}$$

Lemma 6.6.7.

(1) *There exists a canonical partial adjunction*

$$\begin{tikzcd}
(\mathcal{M}^{\mathcal{D}_2})_0 \arrow[r, "u_1^{'}"] \arrow[d, hook] & \mathcal{M}^{\mathcal{D}_2} \\
\mathcal{M}^{\mathcal{D}_2} & \mathcal{M}^{\mathcal{D}_2} \arrow[l, "u_! u^*"] \arrow[u, "1_{\mathcal{M}^{\mathcal{D}_2}}"']
\end{tikzcd}$$

(2) *There exists a canonical partial adjunction*

$$\begin{tikzcd}
(\mathcal{M}^{\mathcal{D}_2})_0 \arrow[r, "u^* u_1^{'}"] \arrow[d, hook] & \mathcal{M}^{\mathcal{D}_1} \\
\mathcal{M}^{\mathcal{D}_2} & \mathcal{M}^{\mathcal{D}_1} \arrow[l, "u_!"] \arrow[u, "1_{\mathcal{M}^{\mathcal{D}_1}}"']
\end{tikzcd}$$

PROOF. Denote $\epsilon_X : v_! v^* X \rightarrowtail X$ the natural map defined for $X \epsilon (\mathcal{M}^{\mathcal{D}_2})_0$. For any object $Z \epsilon \mathcal{M}^{\mathcal{D}_2}$, the diagram

$$\begin{tikzcd}
u_! u^* Z \arrow[r] \arrow[d] & Z \arrow[d, "\nu_Z"] \\
\mathbf{0} \arrow[r] & v_* v^* Z
\end{tikzcd}$$

is both a pushout and a pullback. For $X \epsilon (\mathcal{M}^{\mathcal{D}_2})_0$, the maps $X \to u_! u^* Z$ are in a 1-1 correspondence with maps $f : X \to Z$ such that $\nu_Z f : X \to v_* v^* Z$ is null, therefore in 1-1 correspondence with maps $f : X \to Z$ such that $f \epsilon_X : v_! v^* X \to Z$ is null, therefore in 1-1 correspondence with maps $u_1^{'} X \to Z$. This shows that we have a natural bijection $Hom(u_1^{'} X, Z) \cong Hom(X, u_! u^* Z)$, which proves (1).

For an object $Y \epsilon \mathcal{M}^{\mathcal{D}_1}$ and a map $u^* u_1^{'} X \to Y$, denote $\overline{Y}^X \epsilon \mathcal{M}^{\mathcal{D}_2}$ the object with $\overline{Y}^X_d = Y_d$ for $d \epsilon \mathcal{D}_1$ and $\overline{Y}^X_d = (u_1^{'} X)_d$ otherwise. We see that $Hom(u^* u_1^{'} X, Y) \cong Hom(u_1^{'} X, \overline{Y}^X) \cong Hom(X, u_! u^* \overline{Y}^X) \cong Hom(X, u_! Y)$, which completes the proof of (2). $\square$

LEMMA 6.6.8. *For any diagrams $Y, Y^{'} \epsilon \mathcal{M}^{\Delta^{'} \mathcal{D}_2}$ that are Reedy cofibrant, denote $X = \operatorname{colim}^{p_t} Y$ and $X^{'} = \operatorname{colim}^{p_t} Y^{'}$ in $\mathcal{M}^{\mathcal{D}_2}$. Then*

1. *The colimit $v_! v^* X$ exists and is pointwise cofibrant, and $v_! v^* X \rightarrowtail X$ is a pointwise cofibration*
2. *If $Y \to Y^{'}$ is a pointwise weak equivalence, then so is $v_! v^* X \to v_! v^* X^{'}$*

PROOF. The objects of $\Delta^{'} \mathcal{D}_2$ are all of the form

$$\underline{d} = (d_0 \to \ldots \to d_i \to d_{i+1}^{'} \to \ldots \to d_n^{'})$$

where $d_0, \ldots, d_i \epsilon \mathcal{D}_2 \backslash \mathcal{D}_1$ and $d_{i+1}^{'}, \ldots, d_n^{'} \epsilon \mathcal{D}_1$. As a consequence, given $Y \epsilon \mathcal{M}^{\Delta^{'} (\mathcal{D}_2 \backslash \mathcal{D}_1)}$ by Lemma 5.2.1 we have $(V_! Y)_{\underline{d}} \cong Y_{d_0 \to \ldots \to d_i}$. The functor $V_!$ is thus defined on the entire $\mathcal{M}^{\Delta^{'} (\mathcal{D}_2 \backslash \mathcal{D}_1)}$ - note that $v_!$ may not be defined on the entire $\mathcal{M}^{\mathcal{D}_2 \backslash \mathcal{D}_1}$.

The latching map of $V_! Y$ at $\underline{d}$ is $L Y_{\underline{d}} \to Y_{\underline{d}}$ if $i = n$, and $Y_{d_0 \to \ldots \to d_i} \overset{id}{\to} Y_{d_0 \to \ldots \to d_i}$ if $i < n$. Based on this, we see that if $Y \epsilon \mathcal{M}^{\Delta^{'} \mathcal{D}_2}$ is Reedy cofibrant then $V^* Y$ is Reedy cofibrant and $V_! V^* Y \to Y$ is a Reedy cofibration.

Pick $Y \epsilon \mathcal{M}^{\Delta' \mathcal{D}_2}_{res,rcof}$ with $\operatorname{colim}^{pt} Y = X$. Using Lemma 5.2.1 we see that $v^*X \cong \operatorname{colim}^{pt} V^*Y$.

$$\begin{tikzcd}
\mathcal{M}^{\Delta' \mathcal{D}_2 \backslash \mathcal{D}_1}_{rcof} \arrow[r, "V"] \arrow[d, "\operatorname{colim}^{pt}"'] & \mathcal{M}^{\Delta' \mathcal{D}_2}_{rcof} \arrow[d, "\operatorname{colim}^{pt}"] \\
\mathcal{M}^{\mathcal{D}_2 \backslash \mathcal{D}_1} \arrow[r, "v"] & \mathcal{M}^{\mathcal{D}_2}
\end{tikzcd}$$

The colimits $\operatorname{colim}^{pt} V^*Y \cong v^*X$ and $\operatorname{colim}^{pt} V_!V^*Y$ exist (the latter since $V_!V^*Y$ is Reedy cofibrant). By Lemma 5.2.2 we have that $v_! \operatorname{colim}^{pt} V^*Y \cong v_!v^*X$ exists and is $\cong \operatorname{colim}^{pt} V_!V^*Y$. Applying $\operatorname{colim}^{pt}$ to the Reedy cofibration $V_!V^*Y \to Y$ yields $v_!v^*X \to X$, which is a pointwise cofibration between pointwise cofibrant objects by Thm. 6.4.1. This proves part (1).

If $Y \to Y'$ is a pointwise weak equivalence between Reedy cofibrant objects, then so is $V_!V^*Y \to V_!V^*Y'$, so by Thm. 6.4.1 the map $\operatorname{colim}^{pt} V_!V^*Y \to \operatorname{colim}^{pt} V_!V^*Y'$ is a weak equivalence. This proves part (2). $\square$

Theorem 6.6.9.

1. *If $\mathcal{M}$ is a pointed cofibration category and $u : \mathcal{D}_1 \to \mathcal{D}_2$ is a small closed embedding functor, then $\mathbf{L} \operatorname{colim}^u$ admits a left adjoint.*
2. *If $\mathcal{M}$ is a pointed fibration category and $u : \mathcal{D}_1 \to \mathcal{D}_2$ is a small open embedding functor, then $\mathbf{R} \lim^u$ admits a right adjoint.*

Proof. We only prove (1). We will apply the Abstract Partial Quillen Adjunction Thm. 4.5.10 to

$$\begin{tikzcd}
\mathcal{M}^{\Delta' \mathcal{D}_2}_{res,rcof} \arrow[r, "v_1 = u^* u_1' \operatorname{colim}^{pt}"] \arrow[d, "t_1 = \operatorname{colim}^{pt}"'] & \mathcal{M}^{\mathcal{D}_1} \\
\mathcal{M}^{\mathcal{D}_2} & \mathcal{M}^{\mathcal{D}_1} \arrow[l, "v_2 = u_!"'] \arrow[u, "t_2 = 1_{\mathcal{M}^{\mathcal{D}_1}}"']
\end{tikzcd}$$

We have that v_1, v_2 is an abstract Quillen partially adjoint pair with respect to t_1, t_2:

1. From Lemma 6.6.8 (1) we have $Im \operatorname{colim}^{pt} \subset (\mathcal{M}^{\mathcal{D}_2})_0$, so the functor v_1 is correctly defined. Using Lemma 6.6.7 (2) we see that v_1, v_2 are partially adjoint with respect to t_1, t_2.
2. The functor t_1 is a cofibrant approximation of $\mathcal{M}^{\mathcal{D}_2}$ by Prop. 6.5.7, therefore a left approximation. The functor $t_2 = 1_{\mathcal{M}^{\mathcal{D}_1}}$ is a right approximation, and $u_2 t_2$ preserves weak equivalences.
3. By Lemma 6.6.8 (2), the functor $v_!v^*t_1$ preserves weak equivalences. But $u_1' t_1 Y$ is the pushout of $v_!v^*t_1Y \rightarrowtail t_1Y$ by $v_!v^*t_1Y \to 0$, and an application of the Gluing Lemma shows that $u_1' t_1$ and therefore v_1 also preserve weak equivalences.
4. The functor v_2 preserves weak equivalences.

We conclude that $\mathbf{R}v_2 \cong \mathbf{ho}u_! \cong \mathbf{L} \operatorname{colim}^u$ admits a left adjoint. $\square$

6.7. The conservation property

Recall that a family of functors $u_i : \mathcal{A} \to \mathcal{B}_i, i\epsilon I$ is *conservative* if for any map $f\epsilon\mathcal{A}$ with $u_i f$ an isomorphism in $\mathcal{B}_i$ for all $i\epsilon I$ we have that f is an isomorphism in $\mathcal{A}$. A family of functors $\{u_i\}$ is conservative iff the functor $(u_i)_i : \mathcal{A} \to \times_i\mathcal{B}_i$ is conservative.

THEOREM 6.7.1. *Let $\mathcal{D}$ be a small category, and suppose that $\mathcal{M}$ is either a cofibration or a fibration category. The projections $p_d : \mathcal{M}^{\mathcal{D}} \to \mathcal{M}$ on the d componenent for all objects $d\epsilon\mathcal{D}$ then induce a conservative family of functors $\mathbf{ho}(p_d) : \mathbf{ho}\mathcal{M}^{\mathcal{D}} \to \mathbf{ho}\mathcal{M}$.*

PROOF. Assume that $\mathcal{M}$ is a cofibration category (the proof for fibration categories is dual).

Let $\overline{f} : A \to B$ be a map in $\mathbf{ho}\mathcal{M}^{\mathcal{D}}$ such that $\mathbf{ho}(p_d)\overline{f}\epsilon\mathbf{ho}\mathcal{M}$ are isomorphisms for all objects $d\epsilon\mathcal{D}$. We want to show that $\overline{f}$ is an isomorphism in $\mathbf{ho}\mathcal{M}^{\mathcal{D}}$.

Using the factorization axiom CF4 applied to the pointwise cofibration structure $(\mathcal{M}^{\mathcal{D}}, \mathcal{W}^{\mathcal{D}}, \mathcal{C}of^{\mathcal{D}})$, we may assume that A, B are pointwise cofibrant. From Thm. 3.4.5 applied to $(\mathcal{M}^{\mathcal{D}}, \mathcal{W}^{\mathcal{D}}, \mathcal{C}of^{\mathcal{D}})$, we may further assume that $\overline{f}$ is the image of a pointwise cofibration f of $\mathcal{M}^{\mathcal{D}}$.

Assume we proved our theorem for all *direct* small categories. The map $p_t^* f$ in $\mathcal{M}^{\Delta^{'}\mathcal{D}}$ satisfies the hypothesis of our theorem and $\Delta^{'}\mathcal{D}$ is direct. It follows that $p_t^* f$ is an isomorphism in $\mathbf{ho}\mathcal{M}^{\Delta^{'}\mathcal{D}}$, so by Lemma 3.7.1 there exist pointwise cofibrations $f^{'}, f^{''}\epsilon\mathcal{M}^{\Delta^{'}\mathcal{D}}$ such that $f^{'}p^*f, f^{''}f^{'}$ are pointwise weak equivalences in $\mathcal{M}^{\Delta^{'}\mathcal{D}}$.

But p^*f is $\Delta^{'}_{res}\mathcal{D}$ restricted, and therefore so are $f^{'}$ and $f^{''}$. By the same Lemma 3.7.1 p^*f is an isomorphism in $\mathbf{ho}\mathcal{M}^{\Delta^{'}_{res}\mathcal{D}}$, and by Thm. 6.5.8 f is an isomorphism in $\mathbf{ho}\mathcal{M}^{\mathcal{D}}$

It remains to prove our theorem in the case when $\mathcal{D}$ is direct.

As before, let $\overline{f} : A \to B$ be a map in $\mathbf{ho}\mathcal{M}^{\mathcal{D}}$ such that $\mathbf{ho}(p_d)\overline{f}\epsilon\mathbf{ho}\mathcal{M}$ are isomorphisms for all objects $d\epsilon\mathcal{D}$, and we want to show that $\overline{f}$ is an isomorphism in $\mathbf{ho}\mathcal{M}^{\mathcal{D}}$. Repeating the previous argument applied to the Reedy cofibration structure $(\mathcal{M}^{\mathcal{D}}, \mathcal{W}^{\mathcal{D}}, \mathcal{C}of^{\mathcal{D}}_{reedy})$, we may further assume that $\overline{f}$ is the image of a Reedy cofibration $f : A \to B$.

We will construct a Reedy cofibration $f^{'} : B \to B^{'}$ such that $f^{'}f\epsilon\mathcal{W}^{\mathcal{D}}$. Once the construction is complete, we will be able to apply the same construction to $f^{'}$ and obtain a Reedy cofibration $f^{''} : B^{'} \to B^{''}$ such that $f^{''}f^{'}\epsilon\mathcal{W}^{\mathcal{D}}$. As a consequence, it will follow that f is an isomorphism in $\mathbf{ho}\mathcal{M}^{\mathcal{D}}$.

To summarize, given a Reedy cofibration $f : A \to B$ with the property that f_d is an isomorphism in $\mathbf{ho}\mathcal{M}$, it remains to construct a Reedy cofibration $f^{'} : B \to B^{'}$ such that $f^{'}f\epsilon\mathcal{W}^{\mathcal{D}}$. We will construct $f^{'}$ by induction on degree.

For $n = 0$, $\mathcal{D}^0$ is discrete and the existence of $f^{'\,0}$ follows from Lemma 3.7.1. Assume now $f^{'\,<n} : B^{<n} \to B^{'\,<n}$ constructed.

For any object $d\epsilon\mathcal{D}^n$, we construct the following diagram:

$$\begin{tikzcd}
LA_d \arrow[r, tail, "i_d"] \arrow[d, tail, "L(f^{'}f)_d"', "\sim"] & A_d \arrow[r, tail, "f_d"] \arrow[d, tail, "\beta_d"', "\sim"] & B_d \arrow[r, tail, "\overline{f^{'}_d}"] & \overline{B_d} \arrow[d, tail, "\overline{\overline{f^{'}_d}}"', "\sim"] \\
LB^{'}_d \arrow[r, tail, "\gamma_d"] & \overline{B^{'}_d} \arrow[rr, tail, "\delta_d", "\sim"'] & & B^{'}_d
\end{tikzcd}$$

where:

1. LA_d, $LB^{'}_d$ exist and are cofibrant because $A, B^{'\,<n}$ are Reedy cofibrant
2. Since $f^{'\,<n} f^{<n}$ is a trivial Reedy cofibration, $L(f^{'} f)_d$ is a trivial cofibration
3. The map i_d is a cofibration since A is Reedy cofibrant
4. β_d is constructed as the pushout of $L(f^{'} f)_d$, therefore a trivial cofibration. γ_d is a pushout of i_d, therefore a cofibration.
5. $\overline{f^{'}_d}$ is a cofibration constructed by Lemma 3.7.1 applied to f_d, so that $\overline{f^{'}_d} f_d$ is a trivial cofibration
6. δ_d is constructed as the pushout of $\overline{f^{'}}_d f_d$, therefore a trivial cofibration. $\overline{\overline{f^{'}_d}}$ is a pushout of β_d, therefore a trivial cofibration.

We define $f^{'}_d = \overline{\overline{f^{'}_d}}\,\overline{f^{'}_d}$, for all objects $d \epsilon \mathcal{D}^n$. The map $f^{'\,\leq n}$ is a Reedy cofibration, and $(f^{'} f)^{\leq n} \epsilon \mathcal{W}^{\leq n}$. The inductive step is now complete, and the proof is finished. □

For each category with weak equivalences $(\mathcal{M}, \mathcal{W})$ and small category $\mathcal{D}$, the functor $\mathcal{M}^{\mathcal{D}} \to (\mathbf{ho}\mathcal{M})^{\mathcal{D}}$ induces a functor denoted $\mathrm{dgm}_{\mathcal{D},\mathcal{M}} : \mathbf{ho}(\mathcal{M}^{\mathcal{D}}) \to (\mathbf{ho}\mathcal{M})^{\mathcal{D}}$, or simply

$$\mathrm{dgm}_{\mathcal{D}} : \mathbf{ho}(\mathcal{M}^{\mathcal{D}}) \to (\mathbf{ho}\mathcal{M})^{\mathcal{D}} \tag{6.10}$$

when $(\mathcal{M}, \mathcal{W})$ is inferred from the context.

Corollary 6.7.2. *If* $\mathcal{M}$ *is either a cofibration or a fibration category, and if* $\mathcal{D}$ *is a small category, then the functor* $\mathrm{dgm}_{\mathcal{D}}$ *of (6.10) is a conservative functor.*

Proof. Consequence of the fact that a map f of $\mathcal{D}$ diagrams is an isomorphism iff each f_d for $d \epsilon \mathcal{D}$ is an isomorphism. □

6.8. Realizing diagrams

Start with a category with weak equivalences $(\mathcal{M}, \mathcal{W})$ and a small category $\mathcal{D}$. Given a $\mathcal{D}$-diagram X in $\mathbf{ho}\mathcal{M}$, we would like to know under what conditions the diagram X is isomorphic to the image of a diagram $X^{'} \epsilon \mathbf{ho}(\mathcal{M}^{\mathcal{D}})$ under the functor $\mathrm{dgm}_{\mathcal{D}}$ of (6.10). If such a diagram $X^{'}$ exists, it is called a *realization* of the diagram X in the homotopy category $\mathbf{ho}(\mathcal{M}^{\mathcal{D}})$.

We will show that if $\mathcal{M}$ is a cofibration category and $\mathcal{D}$ is a small, direct, free category then any diagram $X \epsilon (\mathbf{ho}\mathcal{M})^{\mathcal{D}}$ admits a realization $X^{'} \epsilon \mathbf{ho}(\mathcal{M}^{\mathcal{D}})$. Furthermore any two such realizations $X^{'}, X^{''}$ of X are non-canonically isomorphic in $\mathbf{ho}(\mathcal{M}^{\mathcal{D}})$. Denis-Charles Cisinski [**Cis02**] proves this result for *finite* direct, free categories, but his techniques extend to our situation.

Dually, if $\mathcal{M}$ is a fibration category and $\mathcal{D}$ is a small, inverse, free category then any diagram $X \epsilon (\mathbf{ho}\mathcal{M})^{\mathcal{D}}$ admits a realization $X^{'} \epsilon \mathbf{ho}(\mathcal{M}^{\mathcal{D}})$, and $X^{'}$ is unique up to a non-canonical isomorphism in $\mathbf{ho}(\mathcal{M}^{\mathcal{D}})$.

Let us recall the definition of a small free category. A *directed graph* $\mathcal{G}$ consists of a set of vertices $\mathcal{G}_0$, a set of arrows $\mathcal{G}_1$ along with two functions $s, t : \mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ giving each arrow a source respectively a target. A *map* of directed graphs $u : \mathcal{G} \to \mathcal{G}^{'}$ consists of two functions $u_i : \mathcal{G}_i \to \mathcal{G}^{'}_i$ for $i = 0, 1$ that commute with the source and destination maps. We denote $\mathcal{G}$*raph* the category of directed graphs.

There is a functor $F : \mathcal{C}at \to \mathcal{G}raph$ which sends a small category $\mathcal{D}$ to the underlying graph $F\mathcal{D}$ - with the objects of $\mathcal{D}$ as vertices and the maps of $\mathcal{D}$ as arrows, forgetting the composition of maps. The functor F has a left adjoint $G : \mathcal{G}raph \to \mathcal{C}at$, which sends a graph $\mathcal{G}$ to the small category $G\mathcal{G}$ with objects $\mathcal{G}_0$ and with maps between $x, x' \epsilon \mathcal{G}_0$ defined as all formal compositions $f_n f_{n-1} ... f_0$ of arrows f_i, for $i \geq 0$, such that $sf_0 = x, tf_n = x'$ and $tf_i = sf_{i+1}$ for $0 \leq i < n$.

A small category $\mathcal{D}$ is *free* if it is *isomorphic* to $G\mathcal{G}$ for a graph $\mathcal{G}$. The generators of a category are its undecomposable maps, i.e. maps that cannot be written as compositions of two non-identity maps. For a small category $\mathcal{D}$, one can form the graph $\mathcal{G}$ of undecomposable maps, with the objects of $\mathcal{D}$ as vertices and the undecomposable maps of $\mathcal{D}$ as arrows. Since $\mathcal{G}$ is a subgraph of $F\mathcal{D}$, there is a functor $G\mathcal{G} \to \mathcal{D}$ which is an *isomorphism* iff the category $\mathcal{D}$ is free.

A vertex x_0 of a graph $\mathcal{G}$ has *uniformly bounded ascending chains* if there is a positive integer k such that any sequence of arrows $x_0 \to x_1 \to x_2 ...$ has at most length k. The vertex x_0 has *uniformly bounded descending chains* if there is a positive integer k such that any sequence of arrows $... \to x_2 \to x_1 \to x_0$ has at most length k. We leave the proof of the following lemma to the reader:

LEMMA 6.8.1.

(1) *The following statements are equivalent for a small category* $\mathcal{D}$.
  (a) $\mathcal{D}$ *is direct and free*
  (b) $\mathcal{D}$ *is free, and all vertices of its graph of undecomposable maps have uniformly bounded descending chains (in particular, its graph of undecomposable maps has no loops)*
  (c) $\mathcal{D}$ *is direct and for any* $d \epsilon \mathcal{D}$ *the latching category* $\partial(\mathcal{D} \downarrow d)$ *is a disjoint sum of categories with a terminal object.*

(2) *The following statements are equivalent for a small category* $\mathcal{D}$.
  (a) $\mathcal{D}$ *is inverse and free*
  (b) $\mathcal{D}$ *is free, and all vertices of its graph of undecomposable maps have uniformly bounded ascending chains*
  (c) $\mathcal{D}$ *is inverse and for any* $d \epsilon \mathcal{D}$ *the matching category* $\partial(d \downarrow \mathcal{D})$ *is a disjoint sum of categories with an initial object.*

(3) *A finite, free category is both direct and inverse.* □

For example, the categories $0 \rightrightarrows 1$, $1 \leftarrow 0 \to 2$ and $1 \to 0 \leftarrow 2$ (given by their subgraph of non-identity maps) are at the same time direct, inverse and free. The category $\mathbf{N}$ of nonnegative integers $0 \to 1 \to ...$ is free and direct, and its opposite $\mathbf{N}^{op}$ is free and inverse.

Let us now investigate the more trivial problem of realizing $\mathcal{D}$-diagrams for a *discrete* small category $\mathcal{D}$.

LEMMA 6.8.2. *Suppose that* $\mathcal{D}_k$, $k \epsilon K$ *is a set of small categories. If* $\mathcal{M}$ *is either a cofibration or a fibration category, then the functor*

$$\mathbf{ho}\mathcal{M}^{\sqcup_{k\epsilon K}\mathcal{D}_k} \xrightarrow{\mathbf{ho}(i_k^*)_{k\epsilon K}} \times_{k\epsilon K}\mathbf{ho}\mathcal{M}^{\mathcal{D}_k}$$

is an isomorphism of categories, where $i_k : \mathcal{D}_k \to \sqcup_{k'\epsilon K}\mathcal{D}_{k'}$ *denotes the component inclusion for* $k \epsilon K$.

PROOF. Consequence of Thm. 3.6.1 for $\mathcal{M}_k = \mathcal{M}^{\mathcal{D}_k}$, which is a cofibration category by Thm. 6.5.5. □

We can apply the existence of the homotopy (co)limit functor for the particular case of discrete diagrams to prove:

LEMMA 6.8.3.

(1) *If* $\mathcal{M}$ *is a cofibration category, then* $\mathbf{ho}\mathcal{M}$ *admits all (small) sums of objects. The functor* $\mathcal{M}_{cof} \to \mathbf{ho}\mathcal{M}$ *commutes with sums of objects.*
(2) *If* $\mathcal{M}$ *is a fibration category, then* $\mathbf{ho}\mathcal{M}$ *admits all (small) products of objects. The functor* $\mathcal{M}_{fib} \to \mathbf{ho}\mathcal{M}$ *commutes with products of objects.*

PROOF. We only prove (1). Given a set of objects $X_k, k \epsilon K$ of a cofibration category $\mathcal{M}$, if we view K as a discrete category then by Thm. 3.6.1 $\mathbf{L}\operatorname{colim}^K X_k$ satisfies the universal property of the sum of X_k in $\mathbf{ho}\mathcal{M}$. If all X_k are cofibrant, then $\sqcup_{k \epsilon K} X_k$ exists in $\mathcal{M}$ and computes $\mathbf{L}\operatorname{colim}^K X_k$, which proves the second part. □

We now turn to the problem of realizing $\mathcal{D}$-diagrams in a cofibration category for a small direct category $\mathcal{D}$.

LEMMA 6.8.4. *Suppose that either* $\mathcal{M}$ *is a cofibration category and* $\mathcal{D}$ *is a small direct category, or* $\mathcal{M}$ *is a fibration category and* $\mathcal{D}$ *is a small inverse category.*

Suppose that X, Y *are objects of* $\mathcal{M}^{\mathcal{D}}$*. If for each* $n \geq 0$ *there exists*

$$f_n \epsilon Hom_{\mathbf{ho}\mathcal{M}^{\mathcal{D}^{\leq n}}}(X^{\leq n}, Y^{\leq n})$$

such that f_n *restricts to* f_{n-1} *for all* $n > 0$*, then there exists* $f \epsilon Hom_{\mathbf{ho}\mathcal{M}^{\mathcal{D}}}(X, Y)$ *such that* f *restricts to* f_n *for all* $n \geq 0$*.*

PROOF. Assume that $\mathcal{M}$ is a cofibration category and that $\mathcal{D}$ is a small direct category. (The proof using the alternative hypothesis is dual).

We may assume that X, Y are Reedy cofibrant. We fix a sequence of cylinders with respect to the Reedy cofibration structure $I^n X$ and $I^n Y$, and denote $I^\infty X = \operatorname{colim}(X \overset{i_0}{\to} IX \overset{i_0}{\to} I^2 X ...)$ and $I^\infty Y = \operatorname{colim}(Y \overset{i_0}{\to} IY \overset{i_0}{\to} I^2 Y ...)$. Denote $j_n : X \to I^n X$ and $k_n : Y \to I^n Y$ the trivial cofibrations given by iterated compositions of i_0. Using axiom CF6, the maps $j_\infty = \operatorname{colim} j_n : X \to I^\infty X$ and $k_\infty = \operatorname{colim} k_n : Y \to I^\infty Y$ are trivial cofibrations.

By induction on n, we will construct a factorization in $\mathcal{D}^{\leq n}$

$$\begin{tikzcd} I^n X^{\leq n} \arrow[r, "a_n"] & Z_n & I^n Y^{\leq n} \arrow[l, "b_n", "\sim"'] \end{tikzcd}$$

with Z_n Reedy cofibrant and b_n a weak equivalence in $\mathcal{D}^{\leq n}$, such that $b_n^{-1} a_n$ has the homotopy type of f_n, and a trivial Reedy cofibration c_{n-1} in $\mathcal{D}^{<n}$ that make the next diagram commutative

$$\begin{tikzcd} I^{n-1} X^{<n} \arrow[r, "a_{n-1}"] \arrow[d, tail, "i_0"', "\sim"] & Z_{n-1} \arrow[d, tail, "c_{n-1}"', "\sim"] & I^{n-1} Y^{<n} \arrow[l, "b_{n-1}", "\sim"'] \arrow[d, tail, "i_0"', "\sim"] \\ I^n X^{<n} \arrow[r, "a_n^{<n}"'] & Z_n^{<n} & I^n Y^{<n} \arrow[l, "\sim", "b_n^{<n}"'] \end{tikzcd}$$

Once the inductive construction is complete, we get a factorization

$$\begin{tikzcd} X \arrow[r, tail, "j_\infty", "\sim"'] & I^\infty X \arrow[r, "a"] & Z & I^\infty Y \arrow[l, "b", "\sim"'] & Y \arrow[l, tail, "k_\infty", "\sim"'] \end{tikzcd}$$

defined by $a = \operatorname{colim}^n a_n$ and $b = \operatorname{colim}^n b_n$. Using axiom CF6 and Lemma 1.6.5, Z is Reedy cofibrant and b is a weak equivalence. The map $k_\infty^{-1} b^{-1} a j_\infty$ in $\mathbf{ho}\mathcal{M}^{\mathcal{D}}$ restricts to f_n for all $n \geq 0$.

To complete the proof, let us perform the inductive construction of Z_n, a_n, b_n, c_{n-1} for $n \geq 0$. The initial step construction of Z_0, a_0, b_0 folows from Lemma 6.8.2. Assume that $Z_{n-1}, a_{n-1}, b_{n-1}$ have been constructed.

From Thm. 3.4.5, there exists a factorization in $\mathcal{D}^{\leq n}$

$$\begin{tikzcd}
I^{n-1}X^{\leq n} \arrow[r, tail, "a'_n"] & Z'_n & I^{n-1}Y^{\leq n} \arrow[l, tail, "b'_n"', "\sim"]
\end{tikzcd}$$

with a'_n a Reedy cofibration and b'_n a trivial Reedy cofibration in $\mathcal{D}^{\leq n}$, such that $b'^{-1}_n a'_n$ has the homotopy type of f_n in $\mathbf{ho}\mathcal{M}^{\mathcal{D}^{\leq n}}$. Using the inductive hypothesis and Thm. 3.4.4, there exist a Reedy cofibrant diagram $Z''^{<n}_n$ and weak equivalences α, β in $\mathcal{D}^{<n}$ that make the next diagram *homotopy* commutative

$$\begin{tikzcd}
& Z_{n-1} \arrow[d, dashed, "\alpha"] & \\
I^{n-1}X^{<n} \arrow[ur, tail, "a_{n-1}"] \arrow[dr, tail, "a'^{<n}_n"'] & Z''^{<n}_n & I^{n-1}Y^{<n} \arrow[ul, tail, "b_{n-1}"'] \arrow[dl, tail, "b'^{<n}_n"] \\
& Z'^{<n}_n \arrow[u, dashed, "\beta"] &
\end{tikzcd}$$

We may assume that α, β are trivial Reedy cofibrations - if they are not, we may replace them with α', β' defined by a cofibrant replacement

$$\alpha + \beta : Z_{n-1} \sqcup Z'^{<n}_n \overset{\alpha'+\beta'}{\rightarrowtail} Z'''^{<n}_n \xrightarrow{\sim} Z''^{<n}_n$$

Using Lemma 3.3.1 twice, we can embed our homotopy commutative diagram in a diagram commutative on the nose with the maps c_{n-1}, γ trivial Reedy cofibrations

$$\begin{tikzcd}
I^{n-1}X^{<n} \arrow[r, tail, "a_{n-1}"] \arrow[d, tail, "i_0"'] & Z_{n-1} \arrow[d, tail, dashed, "c_{n-1}"] & I^{n-1}Y^{<n} \arrow[l, tail, "b_{n-1}"'] \arrow[d, tail, "i_0"] \\
I^nX^{<n} \arrow[r, dashed, "a^{<n}_n"] & Z^{<n}_n & I^nY^{<n} \arrow[l, dashed, "b^{<n}_n"'] \\
I^{n-1}X^{<n} \arrow[u, tail, "i_1"] \arrow[r, tail, "a'^{<n}_n"'] & Z'^{<n}_n \arrow[u, tail, dashed, "\gamma"] & I^{n-1}Y^{<n} \arrow[u, tail, "i_1"'] \arrow[l, tail, "b'^{<n}_n"]
\end{tikzcd}$$

This diagram defines a Reedy cofibrant object $Z^{<n}_n$, a map $a^{<n}_n$, a weak equivalence $b^{<n}_n$ and the trivial Reedy cofibration c_{n-1}. It remains to extend $Z^{<n}_n, a^{<n}_n, b^{<n}_n$ in degree n such that $b_n^{-1} a_n = f_n$ in $\mathbf{ho}\mathcal{M}^{\mathcal{D}^{\leq n}}$.

For each object $d\epsilon\mathcal{D}^n$, we construct Z'''_d as the colimit of the full diagram

$$\begin{tikzcd}
LI^n X_d \arrow[rr, "\text{colim } a_n^{<n}"] \arrow[dr, tail] & & \text{colim}^{\partial(\mathcal{D}\downarrow d)} Z_n^{<n} \arrow[dr, tail, dashed, "\epsilon"] & \\
& I^n X_d \arrow[rr, dashed, "\delta"] & & Z'''_d \\
LI^{n-1} X_d \arrow[uu, tail, "Li_{1d}"] \arrow[rr, tail] \arrow[dr, tail] & & \text{colim}^{\partial(\mathcal{D}\downarrow d)} Z'^{<n}_n \arrow[uu, tail, "\text{colim } \gamma"'] \arrow[dr, tail] & \\
& I^{n-1} X_d \arrow[uu, tail, "i_{1d}"'] \arrow[rr, tail, "a'_{nd}"] & & Z'_d \arrow[uu, dashed, "\zeta"']
\end{tikzcd}$$

The bottom and left faces are Reedy cofibrant, and the maps Li_{1d}, i_{1d} and colim γ are trivial Reedy cofibrations. A quick computation shows that the colimit Z'''_d actually exists, that ϵ is a Reedy cofibration and that the map ζ is a weak equivalence. In consequence δ has the homotopy type of f_{nd}.

We also construct Z''''_d as the colimit of the full diagram

$$\begin{tikzcd}
\text{colim}^{\partial(\mathcal{D}\downarrow d)} Z_n^{<n} \arrow[dr, tail, dashed, "\mu"'] & & LI^n Y_d \arrow[ll, "\text{colim } b_n^{<n}"'] \arrow[dr, tail] & \\
& Z''''_d & & I^n Y_d \arrow[ll, dashed, "\nu"] \\
\text{colim}^{\partial(\mathcal{D}\downarrow d)} Z'^{<n}_n \arrow[uu, tail, "\text{colim } \gamma"] \arrow[dr, tail] & & LI^{n-1} Y_d \arrow[uu, tail, "Li_{1d}"'] \arrow[ll, tail, "\text{colim } b_{n-1}"] \arrow[dr, tail] & \\
& Z'_d \arrow[uu, dashed, "\theta"] & & I^{n-1} Y_d \arrow[uu, tail, "i_{1d}"'] \arrow[ll, tail, "b_{n-1,d}"']
\end{tikzcd}$$

The bottom and right faces are Reedy cofibrant, the maps Li_{1d}, i_{1d}, colim γ, colim b_{n-1} and $b_{n-1,d}$ are trivial Reedy cofibrations and the map colim $b_n^{<n}$ is a weak equivalence. By computation one shows that the colimit Z''''_d actually exists, that μ is a Reedy cofibration and that ν, θ are weak equivalences. We now construct Z_d as the pushout

$$\begin{tikzcd}
\text{colim}^{\partial(\mathcal{D}\downarrow d)} Z_n^{<n} \arrow[r, tail, "\mu"] \arrow[d, tail, "\epsilon"'] & Z''''_d \arrow[d, tail, dashed, "\epsilon'"'] \\
Z'''_d \arrow[r, tail, dashed, "\mu'"] & Z_d
\end{tikzcd}$$

From the Gluing Lemma applied to the diagram

$$\begin{tikzcd}
\operatorname{colim}^{\partial(\mathcal{D}\downarrow d)} Z_n^{<n} \arrow[rr, tail, "\mu"] \arrow[dr, tail, "\epsilon"] & & Z_d^{''''} \arrow[dr, tail, "\epsilon^{'}"] & \\
& Z_d^{'''} \arrow[rr, tail, "\mu^{'}"] & & Z_d \\
\operatorname{colim}^{\partial(\mathcal{D}\downarrow d)} Z_n^{'<n} \arrow[uu, "\operatorname{colim} \gamma"] \arrow[rr, tail] \arrow[dr, tail] & & Z_d^{'} \arrow[uu, "\theta"] \arrow[dr, "id"] & \\
& Z_d^{'} \arrow[uu, "\zeta"] \arrow[rr, "id"] & & Z_d^{'} \arrow[uu, tail, "\iota"]
\end{tikzcd}$$

where the top and bottom faces are pushouts and the vertical maps colim γ, ζ, θ are weak equivalences we see that ι and therefore $\mu^{'}, \epsilon^{'}$ are weak equivalences.

We define $a_{nd} = \mu^{'}\delta$, $b_{nd} = \epsilon^{'}\nu$. Repeating our construction for any $d \epsilon \mathcal{D}^n$ allows us to define Z_n, a_n and b_n.

For all objects $d \epsilon \mathcal{D}^n$, the latching map $\epsilon^{'}\mu$ is a cofibration, therefore Z_n is Reedy cofibrant. The map b_n is a weak equivalence because $b_n^{<n}$ is a weak equivalence and for all objects $d \epsilon \mathcal{D}^n$ the maps $\epsilon^{'}, \nu$ are weak equivalences. The map $b_n^{-1}a_n$ has the homotopy type of $f_n \epsilon Hom_{\mathbf{ho}\mathcal{M}^{\mathcal{D}^{\leq n}}}(X^{\leq n}, Y^{\leq n})$ because $b_n^{'-1}a_n^{'}$ has the homotopy type of f_n and for all $d \epsilon \mathcal{D}^n$ the maps ζ, θ, ν, $\mu^{'}$ and $\epsilon^{'}$ are weak equivalences. $\square$

We now turn to the main theorem of this section.

THEOREM 6.8.5 (Cisinski).

(1) *If $\mathcal{M}$ is a cofibration category and $\mathcal{D}$ is a small, direct and free, category, then the functor* $\operatorname{dgm}_{\mathcal{D}} : \mathbf{ho}(\mathcal{M}^{\mathcal{D}}) \to (\mathbf{ho}\mathcal{M})^{\mathcal{D}}$ *is full and essentially surjective.*

(2) *If $\mathcal{M}$ is a fibration category and $\mathcal{D}$ is small, inverse and free, then the functor* $\operatorname{dgm}_{\mathcal{D}} : \mathbf{ho}(\mathcal{M}^{\mathcal{D}}) \to (\mathbf{ho}\mathcal{M})^{\mathcal{D}}$ *is full and essentially surjective.*

PROOF. We only prove part (1). The category $\mathcal{M}^{\mathcal{D}}$ is endowed with a pointwise cofibration structure. Since $\mathbf{ho}(\mathcal{M}^{\mathcal{D}}) \cong \mathbf{ho}(\mathcal{M}^{\mathcal{D}}_{cof})$ and $\mathbf{ho}\mathcal{M} \cong \mathbf{ho}\mathcal{M}_{cof}$, we may assume for simplicity that $\mathcal{M} = \mathcal{M}_{cof}$.

Let us first show that $\operatorname{dgm}_{\mathcal{D}}$ is essentially surjective. For an object X of $(\mathbf{ho}\mathcal{M})^{\mathcal{D}}$, we will construct by induction on degree a Reedy cofibrant diagram $X^{'} \epsilon \mathcal{M}^{\mathcal{D}}$ and an isomorphism $f : \operatorname{dgm}_{\mathcal{D}}(X^{'}) \overset{\cong}{\to} X$. The inductive hypothesis is that $X^{'\leq n}$ is Reedy cofibrant and that $f^{\leq n} : \operatorname{dgm}_{\mathcal{D}^{\leq n}}(X^{'\leq n}) \overset{\cong}{\to} X^{\leq n}$ is an isomorphism.

The initial step $n = 0$ follows from Lemma 6.8.2, since $\mathcal{D}^{\leq 0}$ is discrete. Assume that $X^{'<n}, f^{<n}$ have been constructed, and let's try to extend them over each object $d \epsilon \mathcal{D}^n$.

By Lemma 6.8.1, the latching category $\partial(\mathcal{D} \downarrow d)$ is a disjoint sum of categories with terminal objects denoted $d_i \to d$. We therefore have $\operatorname{colim}^{\partial(\mathcal{D}\downarrow d)} X^{'} \cong \sqcup_i X^{'}_{d_i}$, and $\sqcup_i X^{'}_{d_i}$ computes the sum of $X^{'}_{d_i}$ also in $\mathbf{ho}\mathcal{M}$. We can thus construct a map $\operatorname{colim}^{\partial(\mathcal{D}\downarrow d)} X^{'} \to X_d$ in $\mathbf{ho}\mathcal{M}$, compatible with $f^{<n}$. This map yields a factorization $\operatorname{colim}^{\partial(\mathcal{D}\downarrow d)} X^{'} \rightarrowtail X^{'}_d \overset{\sim}{\leftarrowtail} X_d$ in $\mathcal{M}$, since we assumed that $\mathcal{M} = \mathcal{M}_{cof}$.

Define f_d as the inverse of $X'_d \overset{\sim}{\leftarrowtail} X_d$. Repeating the construction of X'_d, f_d for each $d \epsilon \mathcal{D}^n$ yields the desired extension $X'^{\leq n}, f^{\leq n}$. The map $f : \operatorname{dgm}_{\mathcal{D}}(X') \overset{\cong}{\to} X$ we constructed is a degreewise isomorphism, therefore an isomorphism.

We have shown that $\operatorname{dgm}_{\mathcal{D}}$ is essentially surjective, let us now show that it is full. Using Lemma 6.8.4, it suffices to show that for any Reedy cofibrant diagrams $X', Y' \epsilon \mathcal{M}^{\mathcal{D}}$ and map $f : \operatorname{dgm}_{\mathcal{D}}(X') \to \operatorname{dgm}_{\mathcal{D}}(Y')$, we can construct a set of maps $f'_n \epsilon Hom_{\mathbf{ho}\mathcal{M}^{\mathcal{D}^{\leq n}}}(X^{\leq n}, Y^{\leq n})$ for $n \geq 0$ such that f'_n restricts to f'_{n-1} and f restricts to $\operatorname{dgm}_{\mathcal{D}^{\leq n}}(f'_n)$.

We will construct such a map f'_n by induction on n. The initial step map f'_0 exists as a consequence of Lemma 6.8.2. Assume that f'_{n-1} has been constructed.

Using Thm. 3.4.5 we can construct a factorization in $\mathcal{D}^{<n}$

$$\begin{tikzcd}
X'^{<n} \arrow[r, tail, "a_{n-1}"] & Z'_{n-1} & Y'^{<n} \arrow[l, tail, "\sim", "b_{n-1}"']
\end{tikzcd}$$

with a_{n-1} a Reedy cofibration and b_{n-1} a trivial Reedy cofibration in $\mathcal{D}^{<n}$, such that $b^{-1}_{n-1}a_{n-1} = f'_{n-1}$.

For any object $d \epsilon \mathcal{D}^n$, the latching category $\partial(\mathcal{D} \downarrow d)$ is a disjoint sum of categories with terminal objects, and the functor $\mathcal{M} = \mathcal{M}_{cof} \to \mathbf{ho}\mathcal{M}$ commutes with sums of objects. As a consequence, the diagram

$$\begin{tikzcd}
LX'_d \arrow[r, tail, "\operatorname{colim}^{\partial(\mathcal{D}\downarrow d)} a_{n-1}"] \arrow[d] & \operatorname{colim}^{\partial(\mathcal{D}\downarrow d)} Z'_{n-1} & LY'_d \arrow[l, tail, "\sim", "\operatorname{colim}^{\partial(\mathcal{D}\downarrow d)} b_{n-1}"'] \arrow[d] \\
X'_d \arrow[rr, "f_d"] & & Y'_d
\end{tikzcd}$$

commutes in $\mathbf{ho}\mathcal{M}$. All maps of this diagram are maps of $\mathcal{M}$, with the exception of f_d which is a map of $\mathbf{ho}\mathcal{M}$. Since $\mathcal{M}$ admits a homotopy calculus of left fractions, a quick computation shows that this diagram can be embedded in a homotopy commutative diagram with all maps in $\mathcal{M}$

$$\begin{tikzcd}
LX'_d \arrow[r, tail, "\operatorname{colim}^{\partial(\mathcal{D}\downarrow d)} a_{n-1}"] \arrow[d] & \operatorname{colim}^{\partial(\mathcal{D}\downarrow d)} Z'_{n-1} \arrow[d, tail, dashed] & LY'_d \arrow[l, tail, "\sim", "\operatorname{colim}^{\partial(\mathcal{D}\downarrow d)} b_{n-1}"'] \arrow[d] \\
X'_d \arrow[r, tail, dashed] & S_d & Y'_d \arrow[l, tail, dashed, "\sim"]
\end{tikzcd}$$

with the composition in $\mathbf{ho}\mathcal{M}$ of the bottom edge having the homotopy type of f_d.

Since the cylinders are with respect to the Reedy cofibration structure, notice that LIX'_d, LIY'_d are cylinders of $(LX')_d$ and $(LY')_d$. Using Lemma 3.3.1 twice, we can embed our homotopy commutative diagram into a diagram commutative on the nose

$$\begin{tikzcd}
LIX'_d \arrow[rrdd, dashed, "t_d"'] & LX'_d \arrow[l, tail, "i_0"', "\sim"] \arrow[r, tail] & \operatorname{colim}^{\partial(\mathcal{D}\downarrow d)} Z'_{n-1} \arrow[d, tail] & LY'_d \arrow[l, tail, "\sim"] \arrow[r, tail, "i_0", "\sim"'] & LIY'_d \arrow[lldd, dashed, "t'_d"] \\
LX'_d \arrow[u, tail, "i_1", "\sim"'] \arrow[d] & & S_d \arrow[d, tail, dashed, "\sim"] & & LY'_d \arrow[u, tail, "\sim", "i_1"'] \arrow[d] \\
X'_d \arrow[rr, tail, dashed] & & T_d & & Y'_d \arrow[ll, tail, dashed, "\sim"]
\end{tikzcd}$$

where the bottom edge has the homotopy type of f_d.

Denote z_d the map $\operatorname{colim}^{\partial(\mathcal{D}\downarrow d)} Z'_{n-1} \to T_d$. We repeat the construction above for all $d\epsilon\mathcal{D}^n$. The maps $t_d, t_d i_0, z_d, t'_d i_0, t'_d$ define Reedy cofibrant extensions $\overline{LIX'^{\leq n}}$, $\overline{LX'^{\leq n}}$, $\overline{Z'_{n-1}}$, $\overline{LY'^{\leq n}}$, $\overline{LIY'^{\leq n}}$ over $\mathcal{D}^{\leq n}$ and a zig-zag of maps

$$X'^{\leq n} \to \overline{LIX'^{\leq n}} \overset{\sim}{\leftarrow} \overline{LX'^{\leq n}} \to \overline{Z'_{n-1}} \overset{\sim}{\leftarrow} \overline{LY'^{\leq n}} \to \overline{LIY'^{\leq n}} \overset{\sim}{\leftarrow} Y'^{\leq n}$$

which defines the desired f'_n. □

Suppose that $\mathcal{C}$ is a category and $u : \mathcal{D}_1 \to \mathcal{D}_2$ is a functor. Let $X\epsilon\mathcal{C}_1^{\mathcal{D}}$ be a diagram. A *weak colimit* of X along u, if it exists, is a diagram $\operatorname{wcolim}^u X\epsilon\mathcal{C}_2^{\mathcal{D}}$ along with a map $\epsilon : X \to u^* \operatorname{wcolim}^u X$ with the property that for any other diagram $Y\epsilon\mathcal{D}_2$ and map $f : X \to u^*Y$ there exists a (not necessarily unique) map $g : \operatorname{wcolim}^u X \to Y$ such that $f = u^*g \circ \epsilon$. A weak colimit $(\operatorname{wcolim}^u X\epsilon)$, if it exists, is not necessarily unique. If the base category $\mathcal{C}$ is cocomplete, then the colimits along u are weak colimits. Weak limits are defined in a dual fashion, and are denoted $\operatorname{wlim}^u X$.

The homotopy category of a cofibration category is not cocomplete, in general. As a benefit of Thm. 6.8.5, the homotopy category of a cofibration category admits weak colimits indexed by diagrams that satisfy the conclusion of Thm. 6.8.5. Furthermore, there is a way to make these weak colimits *unique* up to a *non-canonical* isomorphism. This is an idea we learned from Alex Heller, [**Hel88**].

Suppose that $\mathcal{M}$ is a cofibration category and that $u : \mathcal{D}_1 \to \mathcal{D}_2$ is a small functor where $\mathcal{D}_1$ has the property that $\operatorname{dgm}_{\mathcal{D}_1}$ is full and essentially surjective. Let $X\epsilon(\mathbf{ho}\mathcal{M})^{\mathcal{D}_1}$ be a diagram in the homotopy category. There exists $X'\epsilon\mathbf{ho}(\mathcal{M}^{\mathcal{D}_1})$ with $\operatorname{dgm}_{\mathcal{D}_1}X' \cong X$, using the essential surjectivity of $\operatorname{dgm}_{\mathcal{D}_1}$. Denote $\epsilon' : X' \to u^*\mathbf{L}\operatorname{colim}^u X'$ the canonical map. Since $\operatorname{dgm}_{\mathcal{D}_1}$ is full, we see that $(\operatorname{dgm}_{\mathcal{D}_1}(\mathbf{L}\operatorname{colim}^u X'), \operatorname{dgm}_{\mathcal{D}_1}\epsilon')$ is a weak colimit of X along u.

We denote $\mathbf{W}\operatorname{colim}^u X$ the weak colimit that we constructed. If $X''\epsilon\mathbf{ho}(\mathcal{M}^{\mathcal{D}_1})$ also satisfies $\operatorname{dgm}_{\mathcal{D}_1}X'' \cong X$, since $\operatorname{dgm}_{\mathcal{D}_1}$ is full we can construct a *non-canonical* map $f : X' \to X''$ with $\operatorname{dgm}_{\mathcal{D}_1}f$ the isomorphism $\operatorname{dgm}_{\mathcal{D}_1}X' \cong \operatorname{dgm}_{\mathcal{D}_1}X''$. By Cor. 6.7.2, $\operatorname{dgm}_{\mathcal{D}_1}$ is conservative therefore f is an isomorphism. This shows that the weak colimit $\mathbf{W}\operatorname{colim}^u X$ is unique *up to a non-canonical isomorphism.*

The weak colimit $\mathbf{W}\operatorname{colim}^u X$ will be called a *privileged weak colimit*, following Alex Heller's terminology. Dually, if $\mathcal{M}$ is a fibration category and $u : \mathcal{D}_1 \to \mathcal{D}_2$ is a small functor with $\operatorname{dgm}_{\mathcal{D}_1}$ full and essentially surjective, we can construct privileged weak limits of $X\epsilon(\mathbf{ho}\mathcal{M})^{\mathcal{D}_1}$ along u, which are denoted $\mathbf{W}\lim^u X$.

THEOREM 6.8.6.

(1) *Suppose that* $\mathcal{M}$ *is a cofibration category and that* $u : \mathcal{D}_1 \to \mathcal{D}_2$ *is a small functor with* $\mathcal{D}_1$ *direct and free. Then the privileged weak colimits* $\mathbf{W}\operatorname{colim}^u X$ *exist for* $X \epsilon (\mathbf{ho}\mathcal{M})^{\mathcal{D}_1}$, *and are unique up to a non-canonical isomorphism.*

(2) *Suppose that* $\mathcal{M}$ *is a fibration category and that* $u : \mathcal{D}_1 \to \mathcal{D}_2$ *is a small functor with* $\mathcal{D}_1$ *inverse and free. Then the privileged weak limits* $\mathbf{W}\lim^u X$ *exist for* $X \epsilon (\mathbf{ho}\mathcal{M})^{\mathcal{D}_1}$, *and are unique up to a non-canonical isomorphism.*

PROOF. Consequence of the fact that by Thm. 6.8.5, for both (1) and (2) the functor $\mathrm{dgm}_{\mathcal{D}_1}$ is full and essentially surjective. ☐

CHAPTER 7

Derivators

In this chapter, we collect all our previous results and interpret them to say that a cofibration category has a canonically associated *right derivator*, and dually a fibration category has a canonically associated *left derivator*. Derivators were introduced by Alexandre Grothendieck in his manuscripts [**Gro83**], [**Gro90**]. Don Anderson [**And79**] and Alex Heller [**Hel88**] present alternative definitions of derivators. For an elementary introduction to the theory of derivators, the reader is invited to refer to Georges Maltsiniotis [**Mal01**].

We will recall the basic definitions we need. Fix a pair of universes $\mathcal{U} \subset \mathcal{U}^+$. We denote $\mathcal{C}at$ the 2-category of $\mathcal{U}$-small categories, and $\mathcal{C}AT$ the 2-category of $\mathcal{U}^+$-small categories. We denote $\emptyset$ the initial category (having an empty set of objects), and e the terminal category (having one object and the identity of that object as the only map). We write $p_{\mathcal{D}} : \mathcal{D} \to e$ for the functor to the terminal category and $e_d : e \to \mathcal{D}$ for the functor that embeds the terminal category e as the object $d \epsilon \mathcal{D}$.

We will denote by $\mathcal{D}ia$ any 2-full 2-subcategory of $\mathcal{C}at$, with the property that its objects (viewed as small categories) include the set of finite direct categories, and are stable under the following category operations:

1. small disjoint sums of categories
2. finite products of categories
3. stable under taking overcategories and undercategories, i.e. if $f : \mathcal{D}_1 \to \mathcal{D}_2$ is a functor with $\mathcal{D}_1, \mathcal{D}_2 \epsilon \mathcal{D}ia$ then $(f \downarrow d_2)$, $(d_2 \downarrow f)$ $\epsilon \mathcal{D}ia$ for any object $d_2 \epsilon \mathcal{D}_2$

In particular from (1) and (2), we will assume that $\emptyset$, e $\epsilon \mathcal{D}ia$.

For example, we can take $\mathcal{D}ia$ to be

1. the entire $\mathcal{C}at$
2. the 2-full subcategory $\mathcal{F}inDirCat$ whose objects are small disjoint sums of finite direct categories
3. the 2-full subcategory $\mathcal{F}inCat$ whose objects are small disjoint sums of categories with countable objects and maps

7.1. Prederivators

A *prederivator* of domain $\mathcal{D}ia$ is a (strict) 2-functor $\mathbb{D} : \mathcal{D}ia^{op} \to \mathcal{C}AT$. Note that $\mathbb{D}$ is contravariant when viewed as a functor on $\mathcal{D}ia$.

For any morphism $u : \mathcal{D}_1 \to \mathcal{D}_2$ in $\mathcal{D}ia$ we denote $u^* = \mathbb{D}u : \mathbb{D}(\mathcal{D}_2) \to \mathbb{D}(\mathcal{D}_1)$, and for any natural map $\alpha : u \Rightarrow v$ in $\mathcal{D}ia$ we denote $\alpha^* = \mathbb{D}\alpha : v^* \to u^*$.

A morphism of prederivators $\mathbb{D}_1 \to \mathbb{D}_2$ is by definition a natural transformation of 2-functors.

Let us consider an example. Any category with weak equivalences $(\mathcal{M}, \mathcal{W})$ gives rise to a canonical prederivator $\mathbb{D}_{(\mathcal{M},\mathcal{W})}$ of domain $\mathcal{C}at$ defined by $\mathbb{D}_{(\mathcal{M},\mathcal{W})}(\mathcal{D}) =$

$\mathbf{ho}\mathcal{M}^{(\mathcal{D}^{op})}$, where the homotopy category is taken with respect to pointwise weak equivalences $\mathcal{W}^{(\mathcal{D}^{op})}$. Any natural map $\alpha : u \Rightarrow v$ in $\mathcal{D}ia$ yields a natural map $\alpha^* : v^* \Rightarrow u^*$

$$\begin{tikzcd}
\mathcal{D}_1 \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D}_2
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbf{ho}\mathcal{M}^{(\mathcal{D}_2^{op})} \arrow[rr, bend left, "u^*"] \arrow[rr, bend right, "v^*"'] \arrow[rr, phantom, "\Uparrow\alpha^*"] & & \mathbf{ho}\mathcal{M}^{(\mathcal{D}_1^{op})}
\end{tikzcd}$$

where α^* is defined on components as $X_{\alpha d_1} : X_{vd_1} \to X_{ud_1}$, for diagrams $X\epsilon\mathcal{M}^{(\mathcal{D}_1^{op})}$.

7.2. Derivators

Suppose that $u : \mathcal{D}_1 \to \mathcal{D}_2$ is a functor in $\mathcal{D}ia$. A right adjoint for u^*, if it exists, is denoted $u_* : \mathbb{D}(\mathcal{D}_1) \to \mathbb{D}(\mathcal{D}_2)$. A left adjoint for $u^* : \mathbb{D}(\mathcal{D}_2) \to \mathbb{D}(\mathcal{D}_1)$, if it exists, is denoted $u_! : \mathbb{D}(\mathcal{D}_1) \to \mathbb{D}(\mathcal{D}_2)$. If the functors u_* and $u_!$ exist, they are only defined up to unique isomorphism.

Suppose we have a diagram in $\mathcal{D}ia$ with $\phi : vf \Rightarrow gu$ a natural map

$$\begin{tikzcd}
\mathcal{D}_1 \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\phi\Nearrow"] & \mathcal{D}_2 \arrow[d, "g"] \\
\mathcal{D}_3 \arrow[r, "v"'] & \mathcal{D}_4
\end{tikzcd} \tag{7.1}$$

We apply the 2-functor $\mathbb{D}$ and we get a natural map $\phi^* : u^*g^* \Rightarrow f^*v^*$. If the right adjoints u_* and v_* exist, the natural map ϕ^* yields by adjunction the *base change* natural map denoted

$$\phi_* : g^*v_* \Rightarrow u_*f^* \tag{7.2}$$

If ϕ_* is an isomorphism for a choice of right adjoints u_*, v_*, then ϕ_* is an isomorphism for any such choice of right adjoints u_*, v_*. Dually, if we assume that the left adjoints $f_!$ and $g_!$ exist, the *cobase change* morphism associated to ϕ is

$$\phi_! : f_!u^* \Rightarrow v^*g_! \tag{7.3}$$

and if $\phi_!$ is an isomorphism for a choice of $f_!, g_!$ then $\phi_!$ is an isomorphism for any such choice.

Given a functor $u : \mathcal{D}_1 \to \mathcal{D}_2$ and an object $d_2\epsilon\mathcal{D}_2$, in the standard over 2-category diagram of u at d_2 (see (6.6))

$$\begin{tikzcd}
(u \downarrow d_2) \arrow[r, "p_{(u\downarrow d_2)}"] \arrow[d, "i_{u,d_2}"'] \arrow[dr, phantom, "\phi_{u,d_2}\Nearrow"] & e \arrow[d, "e_{d_2}"] \\
\mathcal{D}_1 \arrow[r, "u"'] & \mathcal{D}_2
\end{tikzcd}$$

if the right adjoints u_* and $p_{(u\downarrow d_2)*}$ exist, the associated base change morphism is denoted

$$(\phi_{u,d_2})_* : e_{d_2}^* u_* \Rightarrow p_{(u\downarrow d_2)*} i_{u,d_2}^* \tag{7.4}$$

Dually, for a functor $u : \mathcal{D}_1 \to \mathcal{D}_2$ and an object $d_2\epsilon\mathcal{D}_2$, in the standard under 2-category diagram of u at d_2 (see (6.8))

$$\begin{tikzcd}
(d_2 \downarrow u) \arrow[r, "i_{d_2,u}"] \arrow[d, "p_{(d_2\downarrow u)}"'] & \mathcal{D}_1 \arrow[d, "u"] \\
e \arrow[r, "e_{d_2}"'] \arrow[ur, Rightarrow, shorten=12pt, "\phi_{d_2,u}"] & \mathcal{D}_2
\end{tikzcd}$$

if the left adjoints $u_!$ and $p_{(d_2 \downarrow u)!}$ exist, the associated cobase change morphism is denoted

$$(\phi_{d_2,u})_! : p_{(d_2\downarrow u)!} i^*_{d_2,u} \Rightarrow e^*_{d_2} u_! \tag{7.5}$$

Definition 7.2.1 (One-sided weak derivators).
A prederivator $\mathbb{D} : \mathcal{D}ia^{op} \to \mathcal{C}AT$ is a *left weak derivator* if it satisfies the following axioms:

Der1: For any set of small categories $\mathcal{D}_k$, $k \epsilon K$, the functor

$$\mathbb{D}(\sqcup_{k\epsilon K}\mathcal{D}_k) \xrightarrow{(i^*_k)_{k\epsilon K}} \times_{k\epsilon K}\mathbb{D}(\mathcal{D}_k)$$

is an equivalence of categories, where $i_k : \mathcal{D}_k \to \sqcup_{k'\epsilon K}\mathcal{D}_{k'}$ denotes the component inclusions for $k\epsilon K$.

Der2: For any $\mathcal{D}$ in $\mathcal{D}ia$, the family of functors $e^*_d : \mathbb{D}(\mathcal{D}) \to \mathbb{D}(e)$ for all objects $d\epsilon\mathcal{D}$ is conservative.

Der3l: For any $u : \mathcal{D}_1 \to \mathcal{D}_2$ in $\mathcal{D}ia$, the functor $u^* : \mathbb{D}(\mathcal{D}_2) \to \mathbb{D}(\mathcal{D}_1)$ admits a *right* adjoint $u_* : \mathbb{D}(\mathcal{D}_1) \to \mathbb{D}(\mathcal{D}_2)$

Der4l: For any $u : \mathcal{D}_1 \to \mathcal{D}_2$ in $\mathcal{D}ia$ and any object $d_2\epsilon\mathcal{D}_2$, the base change morphism $(\phi_{u,d_2})_* : e^*_{d_2}u_* \Rightarrow p_{(u\downarrow d_2)*}i^*_{u,d_2}$ of (7.4) associated to the standard over 2-category diagram (6.6) is an isomorphism.

A prederivator $\mathbb{D} : \mathcal{D}ia^{op} \to \mathcal{C}AT$ is a *right weak derivator* if it satisfies axioms Der1 and Der2 above and:

Der3r: For any $u : \mathcal{D}_1 \to \mathcal{D}_2$ in $\mathcal{D}ia$, the functor $u^* : \mathbb{D}(\mathcal{D}_2) \to \mathbb{D}(\mathcal{D}_1)$ admits a *left* adjoint $u_! : \mathbb{D}(\mathcal{D}_1) \to \mathbb{D}(\mathcal{D}_2)$

Der4r: For any $u : \mathcal{D}_1 \to \mathcal{D}_2$ in $\mathcal{D}ia$ and any object $d_2\epsilon\mathcal{D}_2$, the cobase change morphism $(\phi_{d_2,u})_! : p_{(d_2\downarrow u)!}i^*_{d_2,u} \Rightarrow e^*_{d_2}u_!$ of (7.5) associated to the standard under 2-category diagram (6.8) is an isomorphism.

For $\mathcal{D}_1$ and $\mathcal{D}_2$ in $\mathcal{D}ia$, we define the functor

$$\begin{gathered}
\mathrm{dgm}_{\mathcal{D}_1,\mathcal{D}_2} : \mathbb{D}(\mathcal{D}_1 \times \mathcal{D}_2) \to \mathbb{D}(\mathcal{D}_1)^{\mathcal{D}_2^{op}} \\
(\mathrm{dgm}_{\mathcal{D}_1,\mathcal{D}_2}X)_{d_2} = i^*_{d_2}X \\
(\mathrm{dgm}_{\mathcal{D}_1,\mathcal{D}_2}X)_{f^{op}:d'_2\to d_2} = i^*_{f:d_2\to d'_2}X
\end{gathered}$$

where $i_{d_2} : \mathcal{D}_1 \to \mathcal{D}_1 \times \mathcal{D}_2$ is the functor $d_1 \mapsto (d_1, d_2)$ and $i_f : i_{d_2} \Rightarrow i_{d'_2}$ is the natural map given on components by $(1_{\mathcal{D}_1}, f)$.

Definition 7.2.2 (One-sided derivators).
A prederivator $\mathbb{D} : \mathcal{D}ia^{op} \to \mathcal{C}AT$ is a *left derivator* (resp. right derivator) if it is a left weak (resp. right weak) derivator that satisfies the additional axiom Der5

Der5: For any small category $\mathcal{D}_1$ and finite, free category $\mathcal{D}_2$, the functor $\mathrm{dgm}_{\mathcal{D}_1,\mathcal{D}_2}$ is full and essentially surjective.

Note that in the literature the axiom Der5 is sometimes stated in the weaker form

Der5w: For any small category $\mathcal{D}_1$ the functor $\mathrm{dgm}_{\mathcal{D}_1,I}$ is full and essentially surjective, where I denotes the category $0 \to 1$, with two objects and one non-identity map.

DEFINITION 7.2.3 (One-sided pointed derivators).
A prederivator $\mathbb{D} : \mathcal{D}ia^{op} \to \mathcal{C}AT$ is a *pointed left derivator* (resp. pointed right derivator) if it is a left derivator (resp. right derivator) that satisfies the additional axiom Der6l (resp. Der6r).

Der6l: For any closed embedding $u : \mathcal{D}_1 \to \mathcal{D}_2$ in $\mathcal{D}ia$, the functor u_* admits a right adjoint denoted $u^!$

Der6r: For any open embedding $u : \mathcal{D}_1 \to \mathcal{D}_2$ in $\mathcal{D}ia$, the functor $u_!$ admits a left adjoint denoted $u^?$

DEFINITION 7.2.4 (Two-sided derivators).
A *weak derivator* is a prederivator that is at once a left weak and a right weak derivator. A *derivator* is a prederivator that is a left and a right derivator. A *pointed derivator* is a prederivator that is a pointed left and a pointed right derivator.

A morphism of one-sided or two-sided derivators is just a morphism of the underlying prederivators.

7.3. Derivability of cofibration categories

Let us turn back to the prederivator $\mathbb{D}_{(\mathcal{M},\mathcal{W})}$ associated to a category with weak equivalences $(\mathcal{M},\mathcal{W})$ and introduce the following

DEFINITION 7.3.1. A category with weak equivalences $(\mathcal{M},\mathcal{W})$ is *left derivable* (resp. left weakly derivable, pointed left derivable, right weakly derivable, etc through all nine permutations) over $\mathcal{D}ia$ if the prederivator $\mathbb{D}_{(\mathcal{M},\mathcal{W})}$ is a *left derivator* (left weak derivator, pointed left derivator, right weak derivator etc.) over $\mathcal{D}ia$.

A left derivable (resp. left weakly derivable, etc.) category by convention is a left derivable over the entire $\mathcal{C}at$. With the definitions of derivators and of derivable categories at hand, we can state

THEOREM 7.3.2 (Cisinski).

(1) *Any ABC cofibration category is right derivable. Any pointed ABC cofibration category is pointed right derivable.*
(2) *Any ABC fibration category is left derivable. Any pointed ABC fibration category is pointed left derivable.*
(3) *Any ABC model category is derivable. Any pointed ABC model category is pointed derivable.*

PROOF. Part (2) is dual to (1) and part (3) is a consequence of (1) and (2). Therefore it suffices to prove part (1).

Part (1) is proved by verifying the derivator axioms as follows:

- Der1 is proved by Lemma 6.8.2
- Der2 by Thm. 6.7.1
- Der3r by Thm. 6.6.3
- Der4r by Thm. 6.6.5
- Der5 by Thm. 6.5.5 and Thm. 6.8.5
- Der6r for a pointed cofibration category by Thm. 6.6.9

□

As a corollary of Thm. 7.3.2 and Prop. 2.2.3 we obtain

THEOREM 7.3.3. *Any Quillen model category is derivable. Any pointed Quillen model category is pointed derivable.* □

We should mention that our terminology for derivable categories is different than the one used by Denis-Charles Cisinski - his paper [**Cis02**] calls left derivable categories what we call F1-F4 Anderson-Brown-Cisinski fibration categories.

7.4. Odds and ends

The author has not been able to complete his research project as thoroughly as he would have liked, but it may be worthwhile to at least state how this body of work may be improved and extended. First of all, it is proved by Denis-Charles Cisinski that

THEOREM 7.4.1 ([**Cis02**]).

1. *Any CF1-CF4 cofibration category is right derivable over $\mathcal{F}inDirCat$. Any pointed CF1-CF4 cofibration category is pointed right derivable over $\mathcal{F}inDirCat$.*
2. *Any F1-F4 fibration category is left derivable over $\mathcal{F}inDirCat$. Any pointed F1-F4 fibration category is pointed left derivable over $\mathcal{F}inDirCat$.*

Indeed, we have modeled our proofs in a very large part on the arguments of [**Cis02**].

Second, consider a weaker replacement of axiom CF6.

CF6': For any sequence of weak equivalence maps $A_0 \leftarrow B_0 \rightarrow A_1 \leftarrow B_1 \rightarrow A_2 ...$ such that each B_n is cofibrant, and such that $B_0 \rightarrow A_0$ and all $B_n \sqcup B_{n+1} \rightarrow A_n$ are cofibrations, we have that

$$A_0 \rightarrow \operatorname{colim}(A_0 \leftarrow B_0 \rightarrow A_1 \leftarrow B_1 \rightarrow A_2 ...)$$

is a weak equivalence.

The colimit in CF6' can be shown to always exist using a modified version of Lemma 6.3.1. If $A_0^{'} \leftarrow B_0^{'} \rightarrow A_1^{'} ...$ is another sequence of weak equivalence maps such that each B_n is cofibrant, and such that $B_0^{'} \rightarrow A_0^{'}$ and all $B_n^{'} \sqcup B_{n+1}^{'} \rightarrow A_n^{'}$ are cofibrations, and if $a_n : A_n \rightarrow A_n^{'}$, $b_n : B_n \rightarrow B_n^{'}$ form a *diagram* map, then using a modified version of Thm. 6.3.5 one can show that the induced map

$$\operatorname{colim}(A_0 \leftarrow B_0 \rightarrow A_1 \leftarrow B_1 \rightarrow A_2 ...) \rightarrow \operatorname{colim}(A_0^{'} \leftarrow B_0^{'} \rightarrow A_1^{'} \leftarrow B_1^{'} \rightarrow A_2^{'} ...)$$

is a weak equivalence.

We conjecture that a precofibration category $(\mathcal{M}, \mathcal{W}, \mathcal{C}of)$ satisfying CF5 and CF6' is right derivable. It is an open question actually if axiom CF6' is needed at all - so an even stronger conjecture would be that axioms CF1-CF5 imply right derivability.

The dual axiom for fibration categories is

F6': For any sequence of weak equivalence maps $A_0 \rightarrow B_0 \leftarrow A_1 \rightarrow B_1 \leftarrow A_2 ...$ such that each B_n is fibrant, and such that $A_0 \rightarrow B_0$ and all $A_{n+1} \rightarrow B_n \times B_{n+1}$ are fibrations, we have that

$$\lim(A_0 \rightarrow B_0 \leftarrow A_1 \rightarrow B_1 \leftarrow A_2 ...) \rightarrow A_0$$

is a weak equivalence.

The dual conjecture is that a prefibration category $(\mathcal{M}, \mathcal{W}, \mathcal{F}ib)$ satisfying F5 and F6' is left derivable.

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