BROWN-PETERSON COHOMOLOGY FROM MORAVA K-THEORY

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Abstract. We give some structure to the Brown-Peterson cohomology (or its p-completion) of a wide class of spaces. The class of spaces are those with Morava K-theory even dimensional. We can say that the Brown-Peterson cohomology is even dimensional (concentrated in even degrees) and is flat as a BP^*-module for the category of finitely presented $BP^*(BP)$-modules. At first glance this would seem to be a very restricted class of spaces but the world abounds with naturally occurring examples: Eilenberg-Mac Lane spaces, loops of finite Postnikov systems, classifying spaces of most finite groups whose Morava K-theory is known (including the symmetric groups), QS^{2n}, $BO(n)$, $MO(n)$, BO, $\operatorname{Im} J$, etc. We finish with an explicit algebraic construction of the Brown-Peterson cohomology of a product of Eilenberg-Mac Lane spaces and a general Künneth isomorphism applicable to our situation.

1. Introduction

We will be concerned with a number of cohomology theories related to Brown-Peterson cohomology and Morava K-theory. We fix a prime p for the duration of the paper. Recall that the coefficient ring for Brown-Peterson cohomology is $BP^* \simeq \mathbf{Z}_{(p)}[v_1, v_2, \dots]$ where the degree of v_n is $-2(p^n-1)$. Let I_n be the ideal $(p, v_1, \dots, v_{n-1})$. We will need the p-adic completion of BP, $BP_p\hat{}$, and the theories $P(n)$ with coefficient rings $P(n)^* \simeq BP^*/I_n$. Letting $P(0)$ be either BP or $BP_p\hat{}$ and $v_0 = p$ we have stable cofibrations

$$(1.1) \qquad \Sigma^{2(p^n-1)}P(n) \xrightarrow{v_n} P(n) \longrightarrow P(n+1),$$

which lead to long exact sequences in cohomology. Let

$$BP\langle q\rangle^* = \mathbf{Z}_{(p)}[v_1, \dots, v_q].$$

There are spectra $E(k,n)$ with coefficient rings $E(k,n)^* \simeq v_n^{-1}BP\langle n\rangle^*/I_k$ with similar long exact sequences. A special case, when $k = n > 0$, is the n^{th} Morava K-theory, $K(n)^*(X)$, with $K(n)^* \simeq \mathbf{F}_p[v_n, v_n^{-1}]$.

Before we state our main theorem we have a result which makes the statements easier to make. *Throughout this paper we assume all of our spaces to be of the homotopy type of CW complexes with $H^*(X;\mathbf{Z}_{(p)})$ of finite type.*

We will say that a graded object (such as the generalized cohomology of a space) is **even dimensional** *if it is concentrated in even degrees.*

Theorem 1.2. *If $K(n)^*(X)$, X a space, is even dimensional for an infinite number of n, then $K(n)^*(X)$ is even dimensional for all $n > 0$.*

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We say X has **even Morava K-theory** *if $K(n)^*(X)$ is even dimensional for all $n > 0$.* We use the weaker sounding assumption to prove our results, but when all is said and done, the proofs show it is equivalent to having even Morava K-theory. We get simple, but interesting, corollaries:

Corollary 1.3. *If $K(q)^*(X)$ has a non-zero odd degree element for some $q > 0$ then $K(n)^*(X)$ has an odd degree element for all but a finite number of n.*

This does not apply to the usual extension of Morava K-theories to include $K(0)^*(X) = H^*(X; \mathbf{Q})$ and there are examples ($X = K(\mathbf{Z}; 2n+1)$, [RW80]) where this is non-zero in odd degrees but X has even Morava K-theory. Originally, this led us to worry a lot about the possibility of bad low Morava K-theories with K-theory "stabilizing" to even degrees. We thought that our results or proofs would need exotic types of completion. However, such examples cannot exist. Because $K(0)$ does not fit the pattern we must sometimes go to the p-adic completion of BP for our results.

Proposition 1.4. *If X and Y have even Morava K-theory, then so does $X \times Y$.*

This follows from the Künneth isomorphism for Morava K-theories and shows that the class of spaces for which our main results hold is closed under finite products.

We can now state our main theorem. To avoid unnecessary repetition we have:

Definition 1.5. Let $P(0)$ be BP if $\lim^1 BP^*(X^m) = 0$ for each space X under discussion, and the p-adic completion of BP, $BP_p\hat{}$, if any of the spaces do not have this property. Likewise, if we have chosen $P(0)$ to be $BP_p\hat{}$ then we chose $E(0,n)$ to be the p-adic completion as well.

Remark 1.6. We make this definition so we always have the inverse limit giving the cohomology,

$$\lim^0 P(0)^*(X^m) \simeq P(0)^*(X).$$

Definition 1.7. We say a $P(k)^*$-module is **Landweber flat** if it is a flat $P(k)^*$-module for the category of $P(k)^*(P(k))$-modules which are finitely presented over $P(k)^*$.

Theorem 1.8. *Let $k \geq 0$. If a space X has even Morava K-theory then $P(k)^*(X)$ is even dimensional and is Landweber flat.*

Note that our results are strictly unstable. There are counter-examples if X is a spectrum and not a space. Note that this includes the case of BP when there are no phantom maps and $BP_p\hat{}$ if there are.

Flatness, for BP, in the sense of our theorem has been explored by Peter Landweber in [Lan76] where he proves his exact functor theorem. He shows that flat means that v_n-multiplication on M/I_nM is always injective. So, we get BP flatness from him by proving the following:

Theorem 1.9. *Let a space X have even Morava K-theory. For $k \geq 0$ we have short exact sequences (where $v_0 = p$):*

$$0 \longrightarrow P(k)^*(X) \xrightarrow{v_k} P(k)^*(X) \longrightarrow P(k+1)^*(X) \longrightarrow 0$$

and $P(k)^*(X)$ is even dimensional.

Remark 1.10. Flatness was further studied by Zen-ichi Yosimura in [Yos76] and Nobuaki Yagita in [Yag76]. Their papers show that the above result implies that $P(k)^*(X)$ is Landweber flat. This follows from Landweber's results once it is known that $P(k)^*(P(k))$ is a $BP^*(BP)$-module from [Yag77]. Taken together we see that $P(q)^*(X)$ Landweber flat is equivalent to having the short exact sequences of Theorem 1.9 for all $k \geq q$ which in turn implies that $P(k)^*(X)$ is Landweber flat for all $k \geq q$.

A Künneth isomorphism follows if one space is nice, and it **must** be a space as this is an unstable result.

Theorem 1.11. *Let $k \geq 0$ and let X be a space with $P(k)^*(X)$ Landweber flat, e.g. if X is a space with even Morava K-theory. We have a Künneth isomorphism:*

$$P(k)^*(X \times Y) \simeq P(k)^*(X) \widehat{\otimes}_{P(k)^*} P(k)^*(Y).$$

This generalizes early work of Peter Landweber. In [Lan70a] he has it for special X and in [Lan76] he has it for Y finite without the completion. This Künneth isomorphism expands the number of spaces we have "computed" the BP-cohomology for quite dramatically.

Recall that our spaces are all CW complexes of finite type and that $P(0)$ is chosen according to Definition 1.5. There are similar isomorphisms for the theories $E(k,n)^*(-)$ if $K(n)^*(X)$ is even.

Remark 1.12. By this Künneth isomorphism, if X is an H-space with even Morava K-theory then $P(k)^*(X)$ has all the structure of a Hopf algebra.

Although it is reasonable to ask for even Morava K-theory if you want all of these theories to be even dimensional, Landweber flatness is the really interesting property and it should have nothing to do with even Morava K-theory. It seems some sort of fluke that there are so many examples of spaces with even Morava K-theory around. Presumably such spaces have a significantly deeper reason for having even Morava K-theory than their association with flatness. This is just the first nontrivial place the general phenomenon of a class of spaces having Landweber flat Brown-Peterson cohomology has shown up. Having observed it here one would expect to see it frequently in the future in a more general setting. That future has arrived in the paper [Kasb] by T. Kashiwabara. For example, we can see that $P(k)^*(QS^{2n})$ is Landweber flat but we can not see that $P(k)^*(QS^{2n+1})$ is. Kashiwabara can. He pushes this type of work much further than we have gone.

Our results are a simultaneous generalization of previous observations on these two rather different concepts: even Morava K-theory and flatness. First, if X is a finite complex then the Atiyah-Hirzebruch spectral sequence must collapse for $K(n)^*(X)$ when the dimension of the space is less than $2(p^n - 1)$, i.e., for all big n. If $K(n)^*(X)$ is even dimensional for such an n and X, the mod p cohomology must also be even dimensional, which implies that there is no torsion and the integral cohomology is even dimensional. It then follows that $BP^*(X)$ is free over BP^* and is even dimensional. Our theorem generalizes this to infinite complexes. Second, over twenty years ago Peter Landweber, [Lan70a], computed the Brown-Peterson cohomology (at the time he worked with complex cobordism) of BG where G is a finitely generated abelian group and showed it was flat and even dimensional. This is a special case of our result applied to the first Eilenberg-MacLane space, $K(G,1)$. The $K(G,n)$ have many similar properties.

Our results have nothing new to say about finite complexes. Infinite complexes can have many a subtle unpleasant property in cohomology. This, and other factors, motivated J. Frank Adams to steer people in the direction of homology rather than cohomology, [Ada74] [Ada69]. What we are observing is that things are not as bad as they seemed and that looking at cohomology can be rewarding. In particular it is turning out to be easier to compute and describe the cohomology than the homology in several examples. Landweber's example for BG where G is abelian should have showed the way. It was much later that $BP_*(BG)$, for G an elementary p-group, was computed ([JW85]) and little progress has been made on more complicated abelian groups. Likewise, $BP^*(BO)$ was computed in a reasonable fashion ([Wil84]) before $BP_*(BO)$ was properly understood ([Yan95]). We can now add all Eilenberg-Mac Lane spaces to the list of spaces whose Brown-Peterson cohomology is completely described but whose Brown-Peterson homology is still a mystery.

Although for many spaces that fit our hypothesis we do not have more detailed descriptions of the cohomology, our result is still way ahead of anything we can produce for homology.

A brief description of our proof is now in order. First we show Proposition 4.12 that any given nontrivial element of $P(k)^*(X)$ maps nontrivially to $E(k,n)^*(X)$ if n is big enough. ($P(k)^*$ and $E(k,n)^*$ were defined in the opening paragraph.) Second, we show Lemma 5.1 that if $K(n)^*(X)$ is even dimensional then $E(k,n)^*(X)$ is also. (This allows us to "compute" $E(k,n)^*(X)$ for all spaces with $K(n)^*(X)$ even degree.) Thus, if X has even Morava K-theory, then $P(k)^*(X)$ is also even dimensional. This is proved using the Atiyah-Hirzebruch spectral sequence. Because all of our spaces are infinite complexes, there are technicalities to worry about. For example, we must show that for $k > 0$ there are no phantom maps in $E(k,n)^*(X)$. This, and more, is achieved using a generalization of Quillen's theorem saying that $P(k)^*(X)$, X a space, has only non-negative dimensional generators and a generalization of the Landweber exact functor theorem, which says that tensoring with $E(k,n)^*$ is exact in the category of finitely presented $P(k)^*(P(k))$-modules, i.e. that $E(k,n)^*$ is Landweber flat.

In the process of proof some subtle differences between cohomology and homology for infinite complexes come to the surface. The Morava structure theorem for complex cobordism (see [JW75]) allows one to use the Morava K-theory, $K(n)_*(X)$, to compute the v_n-torsion free part of $P(n)_*(X)$. Not so in the cohomology of infinite complexes. In fact, in all of our examples, all elements of $P(n)^*(X)$ are v_n-torsion free, but only some show up in the Morava K-theory. This partial failure of the Morava structure theorem is compensated for by the lack of infinite divisibility by v_k in $E(k,n)^*(X)$, $k < n$, whereas in $E(k,n)_*(X)$ it is commonplace and shows up in the proof of the Conner-Floyd conjecture of [RW80]. In that proof it was important that the Morava structure theorem detected all of the v_n torsion free part of homology and that one could have infinite divisibility as well. In the present work we can live without the Morava K-theory detecting all of the v_n torsion free part but we must be able to eliminate infinite divisibility.

The Morava structure theorem still tells us that we can recover $K(n)^*(X)$ from $P(n)^*(X)$ by using

$$K(n)^*(X) \simeq K(n)^* \widehat{\otimes}_{P(n)^*} P(n)^*(X).$$

for infinite complexes, except that now this doesn't pick up all of the v_n torsion free part. Since $P(n)^*(X)$ is determined by $P(0)^*(X)$ for our special spaces with even Morava K-theory, we have a result which was first suggested in papers of Tezuka and Yagita, [TY89] and [TY90], and later in a paper of A. Kono and N. Yagita, [KY93]:

$$K(n)^*(X) \simeq K(n)^* \widehat{\otimes}_{P(0)^*} P(0)^*(X).$$

In fact we can replace $K(n)$ with $E(k,n)$.

We started this project with the belief that the time had come to seriously attack the Brown-Peterson cohomology of Eilenberg-Mac Lane spaces. We tried many approaches, including the Adams spectral sequence, before we found the present one. Calculations led us to believe that it was possible everything was even dimensional; motivating our study even more. Since we began with Eilenberg-Mac Lane spaces we have a measure of satisfaction that these spaces all satisfy our conditions ([RW80]), and we are even more pleased that we can describe their BP-cohomology completely. We will give an algebraic construction of the Brown-Peterson cohomology of Eilenberg-Mac Lane spaces.

There is a BP-module spectrum $BP\langle q\rangle$, [JW73] and [Wil75], with

$$\pi_*(BP\langle q\rangle) = \mathbf{Z}_{(p)}[v_1, \dots, v_q]$$

and for each $q > 0$ there is a stable cofibre sequence

$$\Sigma^{2(p^q-1)} BP\langle q\rangle \xrightarrow{v_q} BP\langle q\rangle \longrightarrow BP\langle q-1\rangle \longrightarrow \Sigma^{2p^q-1} BP\langle q\rangle. \tag{1.13}$$

This gives rise to corresponding fibrations in the Ω-spectra for the $BP\langle q\rangle$, $\{\underline{BP\langle q\rangle}_*\}$. The following is also true using $P(n)$ cohomology in place of BP or $BP_p\hat{}$. The Künneth isomorphism, Theorem 1.11, gives us the Brown-Peterson cohomology for all (abelian) Eilenberg-Mac Lane spaces.

Theorem 1.14. *Let $g(q) = 2(p^{q+1}-1)/(p-1)$. Then $BP_p\hat{}^*(K(\mathbf{Z}_{(p)}, q+2))$ is isomorphic to:*

$$BP_p\hat{}^*(\underline{BP\langle q\rangle}_{g(q)})/(v_1^*, \dots, v_q^*) \simeq BP_p\hat{}^*(\underline{BP\langle q\rangle}_{g(q)})/(v_q^*)$$

and $BP^*(K(\mathbf{Z}/(p^i), q+1))$ is isomorphic to:

$$BP^*(\underline{BP\langle q\rangle}_{g(q)})/(p^{i^*}, v_1^*, \dots, v_q^*) \simeq BP^*(\underline{BP\langle q\rangle}_{g(q)})/(p^{i^*}, v_q^*)$$

for $q > 0$. For $q = 0$ delete the v_q^* from the ideal.

We need the p-adic completion for $K(\mathbf{Z}_{(p)}, n)$, $n > 2$, because there are phantom maps for these spaces. However, we don't need it for finite groups.

Remark 1.15. Because all of this comes from spaces and maps of spaces we have much more here than just the BP^* module structure. In fact, these things are as good as Hopf algebras and the structure maps are included in what is known. Furthermore, everything is completely understood as unstable modules over $BP^*(\underline{BP}_*)$ (or $BP_p\hat{}^*(\underline{BP_p\hat{}}_*)$) from [BJW95]. Later we will give a set of algebra generators.

The $q = 0$ version of the theorem was known to Stong and presumably others, in the 1960s. Landweber, in [Lan70a], showed these $q = 0$ cases were flat and then calculated the result for products of these spaces.

Some explanation is called for. The ideal is generated by the images of the maps in Brown-Peterson cohomology induced from the maps of spaces in the Ω-spectrum which come from the stable maps described above in 1.13. There is a map which induces this isomorphism. It comes from the iterated boundary maps of 1.13. Unstably, the boundary map is:

$$\underline{BP\langle k-1\rangle}_{j} \to \underline{BP\langle k\rangle}_{j+2p^k-1}.$$

and the iteration is:

$$K(\mathbf{Z}/(p^i), q+1) \to K(\mathbf{Z}_{(p)}, q+2) \to \underline{BP\langle 1\rangle}_{q+2p+1} \to \cdots \to \underline{BP\langle q\rangle}_{g(q)}.$$

This is the same map used by Hopkins-Ravenel [HR92] to prove that suspension spectra are harmonic.

The reason this is a satisfactory answer for us is that everything is "known" about $BP^*(\underline{BP\langle q\rangle}_{g(q)})$, $BP^*(\underline{BP\langle q\rangle}_{g(q)-2(p^q-1)})$, and the map v_q^* between them. This is because $\underline{BP\langle q\rangle}_{g(q)}$ splits off of $\underline{BP}_{g(q)}$ and all of the spaces $\underline{BP}_*$ are well understood from [RW77]. In particular, in that paper we give an algebraic construction for $BP_*(\underline{BP}_k)$. $BP_*(\underline{BP\langle q\rangle}_k)$ is a well defined quotient of this construction for $k \leq g(q)$ (just set all $[v_i] = 0$ for $i > q$ where $[v_i]$ is defined in $BP_0(\underline{BP}_{-2(p^i-1)})$ using $v_i \in \pi_{2(p^i-1)}(BP) \simeq [\mathrm{pt}, \underline{BP}_{-2(p^i-1)}]$). Since the spaces $\underline{BP\langle q\rangle}_k$, $k \leq g(q)$ are all torsion free for ordinary homology, we know that they are BP_* free and the Brown-Peterson cohomology is just the BP^* dual. Likewise, the maps are just the dual maps. This theorem gives insights into H. Tamanoi's results, [Tam83b], [Tam83a](an announcement with no proofs), (see [Yag86, Theorem 3.3]), and vice versa. H. Tamanoi has only recently written up his work in [Tam97].

Not only do we know the BP-homology of the above spaces and use it to describe the BP-cohomology of the Eilenberg-Mac Lane spaces, but the same can be done for Morava K-theory. Although in principle the maps are all known, in practice it can be difficult to compute them. Our main technical proposition about these spaces which allows us to go up to our BP cohomology answer is (see [HRW97] and [SW] for the category of $K(n)_*$-Hopf algebras):

Proposition 1.16. *Let $g(q) = 2(p^{q+1}-1)/(p-1)$. There is an exact sequence in the category of $K(n)_*$-Hopf algebras:*

$$K(n)_* \to K(n)_*(K(\mathbf{Z}_{(p)}, q+2)) \to K(n)_*(\underline{BP\langle q\rangle}_{g(q)}) \xrightarrow{v_{q*}} K(n)_*(\underline{BP\langle q\rangle}_{g(q)-2(p^q-1)}).$$

In order to translate this Morava K-theory information into information about Brown-Peterson cohomology we have to have some general results about exactness for Morava K-theory implying exactness for BP. We have theorems about injectivity, surjectivity and just enough exactness for our purposes:

Theorem 1.17. *Let spaces X_i, $i = 1, 2$, have even Morava K-theory. If $f : X_1 \longrightarrow X_2$ has $f^* : K(n)^*(X_2) \longrightarrow K(n)^*(X_1)$ surjective (injective) for all $n > 0$, then $f^* : P(k)^*(X_2) \longrightarrow P(k)^*(X_1)$ is also surjective (injective), for $k \geq 0$.*

Theorem 1.18. *Let spaces X_i, $i = 1, 2, 3$, have even Morava K-theory. If $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3$ has $f_2 \circ f_1 \simeq 0$ and gives rise to exact sequences (as $K(n)^*$ modules)*

$$0 \longleftarrow K(n)^*(X_1) \xleftarrow{f_1^*} K(n)^*(X_2) \xleftarrow{f_2^*} K(n)^*(X_3)$$

for all $n > 0$ then for all $n \geq 0$ we get exact sequences:

$$0 \longleftarrow P(n)^*(X_1) \xleftarrow{f_1^*} P(n)^*(X_2) \xleftarrow{f_2^*} P(n)^*(X_3).$$

Theorem 1.19. *Let spaces X_i, $i = 1, 2, 3$, have even Morava K-theory. Assume that*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3$$

has $f_2 \circ f_1 \simeq 0$ and all spaces are H-spaces and all maps are H-space maps. Assume also that this gives exact sequences of bicommutative Hopf algebras for all $n > 0$:

$$K(n)_* \longrightarrow K(n)_*(X_1) \xrightarrow{\quad {f_1}_* \quad} K(n)_*(X_2) \xrightarrow{\quad {f_2}_* \quad} K(n)_*(X_3).$$

Then,

$$P(n)^*(X_1) \simeq P(n)^*(X_2)/(f_2^*)$$

for all $n \geq 0$.

The above theorem will be used repeatedly for our examples. What is surprising is that, more often than not, our spaces do not come from fibrations.

Although we cannot give the level of detail for most of our examples that we give for Eilenberg-Mac Lane spaces, there are some general statements which we can make about generators and relations. Note that these results do not depend on having even Morava K-theory but only on being Landweber flat which even Morava K-theory implies.

Theorem 1.20. *Let a space X have $P(n)^*(X)$ Landweber flat for $n \geq 0$, then there is a set T_n in $P(n)^*(X)$ such that the elements T_n satisfy (a)-(c). Let R_n be the set of relations on the elements T_n in $P(n)^*(X)$. Then any set T_n which satisfies (a)-(c) also satisfies (d)-(g).*

(a) *They generate $P(n)^*(X)$ topologically as a $P(n)^*$-module,*
(b) *are all essential to generate, and*
(c) *are almost all in F^s, the s^{th} skeletal filtration of the Atiyah-Hirzebruch spectral sequence, for each $s \geq 0$.*
(d) *These elements, T_n, reduce to a set, T_q, in $P(q)^*(X)$, $q > n$, with the same properties.*
(e) *all relations must be infinite sums, in particular, the elements of T_n are linearly independent over $P(n)^*$,*
(f) *any relation, in R_q, $q > n$, on the reduced set T_q in $P(q)^*(X)$ comes from R_n, and*
(g) *any relation whose coefficients all map to zero in $P(q)^*$ can be written $\sum_{i=n}^{q-1} v_i r_i$, with r_i in R_n.*

The last statement is a nice generalization of "regular" in Landweber's paper [Lan70a]. It is clear that the image of the set T_n in $K(q)^*(X)$ generates. From the next result we see that every element in T_n must show up in some Morava K-theory (or else it would be unnecessary). In fact, it follows that every generator must be detected by an infinite number of the Morava K-theories. It seems reasonable, but we were unable to prove, that if a generator shows up in $K(q)^*(X)$, then it also shows up in $K(q+1)^*(X)$. Such is the case for Eilenberg-Mac Lane spaces. Our next result says that if elements generate the Morava K-theories then they actually generate everything. This is a strong result which allows us to prove our exactness theorems and go on to attack Eilenberg-Mac Lane spaces.

Theorem 1.21. *Let a space X have $P(n)^*(X)$ Landweber flat. Let $n \geq 0$ and let $T_n \subset P(n)^*(X)$ be such that*

(a) *the elements of T_n are almost all in F^s, the s^{th} skeletal filtration of the Atiyah-Hirzebruch spectral sequence and*
(b) *for each $q \geq n$ ($q > 0$), $K(q)^*(X)$ is generated topologically as a $K(q)^*$-module by the image of T_n.*

Then T_n generates $P(n)^*(X)$ topologically as a $P(n)^*$-module.

Remark 1.22. If T_n is multiplicatively generated by a finite subset G_n of elements of positive skeletal filtration, then T_n satisfies (a).

Corollary 1.23. *Let X and T_n be as in Theorem 1.21 with all of the elements of T_n essential. Then T_n satisfies the conditions of Theorem 1.20.*

Corollary 1.24. *Let X and T_n be as in Theorem 1.20. Every $t \in T_n$ maps non-trivially to $K(q)^*(X)$ for an infinite number of $q > n$.*

We see structure in many examples of Brown-Peterson cohomology where there was not known to be structure before. We consider what we have done as just a start. The problem of computing these examples more completely is a problem that remains, but now with more than a glimmer of hope that the answers we will find will be nice. We hope this work will inspire others to tackle these explicit computations. We leave people with the question: If these things are so nice, what are they?

Section 2 elaborates on our examples. In Section 3 we organize the preliminaries needed in the rest of the paper. After that we have Section 4 on the Atiyah-Hirzebruch spectral sequence for our theories. Then we move on to assume even Morava K-theory and deduce the main result in Section 5 (Theorems 1.2, 1.8, and 1.9). We then do our work with generators and relations, Section 6 (Theorems 1.20 and 1.21, and Corollaries 1.23 and 1.24), followed by our work with exactness in Section 7 (Theorems 1.17, 1.18, and 1.19). In Section 8 we work out the details of the Eilenberg-Mac Lane example (Theorem 1.14 and Proposition 1.16). Our final section, Section 9, deals with the Künneth isomorphism (Theorem 1.11).

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2. Examples

Before listing our examples, we need the following easy result.

Proposition 2.0.1. *Let*

$$F \xrightarrow{i} E \xrightarrow{f} B$$

be a fibration in which $K(n)_*(F)$ and $H_*(B)$ $(= H_*(B;\mathbf{Z}/(p)))$ are concentrated in even dimensions. Then $K(n)_*(i)$ is one-to-one, $K(n)_*(f)$ is onto, and $K(n)_*(E)$ is even degree. Moreover, if the fibration is one of loop spaces, then

$$K(n)_* \longrightarrow K(n)_*(F) \xrightarrow{i_*} K(n)_*(E) \xrightarrow{f_*} K(n)_*(B) \longrightarrow K(n)_*$$

is a short exact sequence of Hopf algebras.

Corollary 2.0.2. *If we have a fibration of double loop spaces as in Proposition 2.0.1, F has even Morava K-theory and $H_*(B)$ is even, then*

$$P(n)^*(F) \simeq P(n)^*(E)/(f^*).$$

Proof of Proposition. The Atiyah-Hirzebruch-Serre spectral sequence converging to $K(n)_*(E)$ with

$$E_2 = H_*(B; K(n)_*(F))$$

collapses since it is concentrated in even dimensions. The result follows. □

Proof of Corollary. The double loops implies bicommutative Hopf algebras so this follows from Proposition 2.0.1 using Theorem 1.19. □

2.1. **Finite Postnikov systems.** All Eilenberg-Mac Lane spaces for abelian groups not having the circle, S^1, as a homotopy factor, and all products of such spaces satisfy the conditions of our theorems. The first from [RW80] and the products from Proposition 1.4. Furthermore, we can describe the Brown-Peterson cohomology of all of these spaces explicitly. See the introduction and Section 8 of the paper. This is our main example.

In Hopkins-Ravenel-Wilson, [HRW97], we show that the loop space of a finite Postnikov system has even Morava K-theory, provided that it does not have an S^1 as a factor. If F is such a space, but with double loops replacing loops, then $K(n)_*(F)$ is isomorphic as a Hopf algebra to $K(n)_*(E)$, where E is the product of Eilenberg-Mac Lane spaces having the same homotopy as F. In other words, k-invariants are not seen by Morava K-theory.

2.2. **Classifying spaces of compact Lie groups.** In [HKR92] it was conjectured that the Morava K-theory of a finite group, i.e., $K(n)^*(BG)$, should be even dimensional. If this conjecture had been true, then our result would have applied to all finite groups. It is true for most groups whose Morava K-theory is known. However, the conjecture is false, [Kri97], [Lee]. As it is, our result applies only to those groups with even Morava K-theory which have had their Morava K-theory computed. That list starts with finite abelian groups. Their Brown-Peterson cohomology was known, in detail, to Landweber [Lan70a] who also knew of their flatness. Perhaps next on the list, in terms of interest, are the symmetric groups. Hopkins-Kuhn-Ravenel, [HKR92] and [HKR], and Hunton, [Hun90], independently proved that the Morava K-theory of these groups is even. These would be good examples to understand explicitly. Hopkins, Kuhn, and Ravenel give other examples where the result is known. The result is known for groups G with $\text{rank}_p G \le 2$. For groups with $\text{rank}_p G = 2$ all but one case is done by Tezuka and Yagita in [TY89] and [TY90]. All cases, including the missing one, are done in [Yag93]. M. Tanabe also has an interesting class of examples in [Tan95].

The result about symmetric groups mentioned above is a consequence of the following statement. If the conjecture is true for a finite group G, then it is also true for the wreath product $\mathbf{Z}/(p) \wr G$. By this we mean the evident semidirect product in the split group extension

$$G^p \longrightarrow \mathbf{Z}/(p) \wr G \longrightarrow \mathbf{Z}/(p)$$

in which $\mathbf{Z}/(p)$ acts on G^p by permuting the factors cyclically. The proofs given by Hunton and Hopkins-Kuhn-Ravenel differ slightly in the assumption made about $K(n)^*(BG)$. The latter assume that it is additively generated by images under the transfer map of Euler classes of representations of subgroups $H \subset G$, while Hunton assumes that there is a map $f : BG \to Y$ with $K(n)^*(f)$ onto and $K(n)^*(Y)$ a finitely generated power series ring. He calls such a map a 'unitary like embedding' because in the case where G is abelian, Y can be taken to be $BU(m)$ for a suitable unitary group $U(m)$.

We can improve on the wreath product result of [HKR92] as follows. For a group G let $\mathrm{Tre}_{K(n)}(G)$ denote the subalgebra of $K(n)^*(BG)$ generated by transferred Euler classes of irreducible representations of subgroups of G, and similarly for $\mathrm{Tre}_{BP}(G)$ and $\mathrm{Tre}_{P(n)}(G)$. (In [HKR92, Cor. 8.3] it is shown that the module generated by transferred Euler classes of *all* representations of subgroups of G is the same as this algebra.) We let Tr^* denote the transfer and $e(\rho)$ the Euler class of a representation ρ. Then $\mathrm{Tr}^*(e(\rho))$ stands for the transferred Euler class of a representation.

We will say that a group G is **good** if

$$K(n)^*(BG) = \mathrm{Tre}_{K(n)}(G).$$

for all n. We know that finite abelian groups and groups G with $\mathrm{rank}_p G \leq 2$ are good. A group G is good if its p-Sylow subgroup is. In [HKR92] it was shown that $W = \mathbf{Z}/(p) \wr G$ is good if G is. The following result is a consequence of Theorem 1.21.

Corollary 2.2.1. *Let G be a finite group which is good in the sense above, and let $W = \mathbf{Z}/(p) \wr G$. Then, with $P(0) = BP$,*

$$\begin{aligned} P(n)^*(BG) &= \mathrm{Tre}_{P(n)}(G), \\ \textit{and} \qquad P(n)^*(BW) &= \mathrm{Tre}_{P(n)}(W). \end{aligned}$$

Proof. First note that for $X = BG$ or BW, we know from [BM68] and [Lan72] that $\lim^1 BP^*(X^m) = 0$, so $P(0)$ is BP as in Definition 1.5.

Now let $T_n \subset P(n)^*(BG)$ be the subalgebra generated by the set of transferred Euler classes of irreducible representations of subgroups of G. There are finitely many such classes, so T_n is multiplicatively generated by a finite set as required by Remark 1.22 so the statements about the cohomology of BG follow from Theorem 1.21.

Let $T'_n \subset P(n)^*(BW)$ be similarly defined. Since W is good, T'_n also satisfies the hypotheses of Theorem 1.21 because of Remark 1.22 and the statements about the cohomology of BW follow. $\square$

We want to give a more detailed description.

Theorem 2.2.2. *Let G be good and $BP^*(BG) \otimes_{BP^*} \mathbf{Z}/(p) \simeq \mathbf{Z}/(p)\{b_\lambda\}$, that is, the b_λ are BP^* generators. Then*

$$BP^*(B(\mathbf{Z}/(p) \wr G)) \otimes_{BP^*} \mathbf{Z}/(p) \simeq$$

$$\mathbf{Z}/(p)\{P(\lambda),\ \sigma(\lambda_1, \dots, \lambda_p)y^s,\ y^{s'} \mid s \geq 0,\ s' > 0,\ \exists \lambda_i \neq \lambda_j\}$$

where

$$\sigma(\lambda_1, \dots, \lambda_p) = \mathrm{Tr}^*(b_{\lambda_1} \otimes \cdots \otimes b_{\lambda_p}),$$

with

$$\mathrm{Tr}^* : BP^*(BG^p) \to BP^*(B(\mathbf{Z}/(p) \wr G)),$$

$y = \pi^*(\tilde{y})$ *with*

$$\pi^* : BP^*(B\mathbf{Z}/(p)) \simeq BP^*[[\tilde{y}]]/[p](\tilde{y}) \to BP^*(B(\mathbf{Z}/(p) \wr G)),$$

$P(\lambda) = \mathrm{Tr}^*(e(\hat{\rho_\lambda}))$ *if* $b_\lambda = \mathrm{Tr}^*(e(\rho_\lambda))$ *for some representation* ρ_λ *of* $H \subset G$*, and* $\hat{\rho_\lambda}$ *is the representation of* $\mathbf{Z}/(p) \wr H$ *with* $\hat{\rho_\lambda}|H^p = \rho_\lambda \otimes \cdots \otimes \rho_\lambda$.

Proof. The exact sequence

$$1 \longrightarrow G^p \longrightarrow \mathbf{Z}/(p) \wr G \longrightarrow \mathbf{Z}/(p) \longrightarrow 1$$

induces the spectral sequence

$$H^*(B\mathbf{Z}/(p); K(n)^*(G^p)) \Longrightarrow K(n)^*(B(\mathbf{Z}/(p) \wr G)).$$

In [HKR92, between 8.3 and 8.6], the differentials are computed and they get a similar theorem as ours for $K(n)$ except that there are some restrictions. One must use a subset of the b's, $s \le p^n - 1$ and $s' \le p^n - 1$. All of the elements in the statement of the theorem can be defined for BP cohomology and we now see that their reductions generate all of the Morava K-theories. Our result follows from Theorem 1.21 and we see that all of these elements are necessary as well. □

Now let $X = BG$, where G is a compact Lie group. From Buhštaber-Miščhenko, [BM68], and Landweber, [Lan72], it is known that $\lim^1 BP^*(X^m) = 0$. In [KY93], Kono-Yagita conjecture that $BP^*(X)$ is even degree and flat in our sense. They go on to prove this for $O(n)$, $SO(2n+1)$, $PU(3)$ and F_4. The Brown-Peterson cohomology of BO, $BO(n)$ and $MO(n)$ was computed in [Wil84].

Remark 2.2.3. Our results show that the Hopkins-Kuhn-Ravenel conjecture about finite groups is equivalent to the Kono-Yagita conjecture (for finite groups). Since the first is false, so is the second.

2.3. **The sphere spectrum.** The evenly indexed spaces in the Ω-spectrum for the sphere, QS^{2n}, have even Morava K-theory as they are the limit of spaces which have even Morava K-theory. (This follows from Hunton's theorem about the Morava K-theory of wreath products [Hun90].) We want to thank Takuji Kashiwabara for bringing this example to our attention. Kashiwabara has, since we proved our basic theorems, pushed this example to its ideal conclusion in [Kasa]. There, he shows that if E is a bouquet of BP spectra and there is a map, f, from BP to E such that

$$0 \longleftarrow BP^*(S^0) \longleftarrow BP^*(BP) \overset{f^*}{\longleftarrow} BP^*(E),$$

is an exact sequence, then there is an exact sequence of $K(n)_*$-Hopf algebras

$$K(n)_* \longrightarrow K(n)_*(QS^{2k}) \longrightarrow K(n)_*(\underline{BP}_{2k}) \overset{(f_{2k})_*}{\longrightarrow} K(n)_*(\underline{E}_{2k})$$

which, by Theorem 1.19, gives $BP^*(QS^{2k})$ as a quotient,

$$BP^*(\underline{BP}_{2k})/(f_{2k}^*).$$

Note that these maps and spaces do not form a fibration.

It is easy to come up with a spectrum E and a map from BP. A minimal one, for example, is just to have E be the wedge of $\Sigma^{2(p-1)p^i} BP$ for each $i \ge 0$. The maps just cover the generators for $BP^*(BP)$. In principle, this gives complete information not just for $BP^*(QS^{2k})$ but for $K(n)_*(QS^{2k})$. Everything about the map from BP to E is known and one can use the techniques developed in [BJW95].

Kashiwabara has recently pushed his work even further, see [Kasb].

2.4. **Image of J and related spaces.** We thank Stewart Priddy and Fred Cohen for tutorials which allowed us to include this example.

We will outline a computation of $K(n)^*(J)$ for an odd prime p, where J is the fibre of

$$BU_{(p)} \xrightarrow{\psi^k - 1} BU_{(p)}$$

for a suitable choice of the integer k, namely it must be congruent to a primitive $(p-1)^{\text{th}}$ root of unity mod p but not mod p^2. It is also known that if k is a power of some prime other than p, then the fibre of the map above is the p-localization of algebraic K-theory of the field $\mathbf{F}_k$ [Qui72]. In any case this space is known to be a direct limit of the classifying spaces of finite groups studied by Tanabe in [Tan95] ([Qui72]). He shows that each of them has even Morava K-theory.

In particular J has even Morava K-theory, so the theorems of this paper apply to it. Moreover, the fibration

$$J \longrightarrow BU_{(p)} \xrightarrow{\psi^k - 1} BU_{(p)}$$

gives a short exact sequence of Hopf algebras in Morava K-theory by Proposition 2.0.1. Corollary 2.0.2 then gives us the result that

$$BP^*(J) \simeq BP^*(BU_{(p)})/((\psi^k - 1)^*)$$

because there are no $\lim^1$ problems (J is torsion and $BU_{(p)}$ has no torsion).

This discussion could be made self contained, and thus not dependent on Tanabe, by showing that $(\psi^k - 1)_*$ is surjective as a map from $K(n)_*(BU_{(p)})$ to itself. A simple argument then shows that $K(n)^*(J)$ is even. As it is, Tanabe, with the collapsing of the spectral sequence, gives us this surjectivity.

The effect of the map $\psi^k - 1$ in BP-cohomology can be computed with the help of the formal group law and the splitting principle. We have

$$BP^*(BU) = BP^*[[c_1, c_2, \cdots]],$$

the power series ring on the Chern classes. Consider the formal expression

$$c(t) = \sum_{i \geq 0} c_i t^i \qquad \text{where } c_0 = 1.$$

Under the splitting principle we can write

$$c(t) = \prod_j (1 + x_j t),$$

which should be understood to mean that c_i is the i^{th} elementary symmetric function in the x_j. Then we have

$$(\psi^k)^*(c(t)) = \prod_j (1 + [k]_{BP^*}(x_j t)),$$

where $[k]_{BP^*}(x)$ denotes the k-series for the formal group law associated with BP^* and this gives the action of $(\psi^k)^*$ on our generator for $BP^*(CP^\infty)$. The expression on the right is symmetric in the x_j, so the coefficient of t^i is a certain symmetric polynomial (with coefficients in BP^*) in the x_j, so it can be written in terms of the elementary symmetric functions. Then $(\psi^k)^*(c_i)$ is the corresponding polynomial in the Chern classes. In a similar sense we have

$$(\psi^k - 1)^*(c(t)) = \prod_j \left(\frac{1+[k]_{BP^*}(x_j t)}{1+x_j t}\right).$$

2.5. BO**.** The object of this section is to recover the second author's computation of $BP^*(BO)$ [Wil84] using the results of this paper. It is shown there that $BP^*(BO)$ is a certain quotient of

$$BP^*(BU) = BP^*[[c_1, c_2, \cdots]]$$

(the power series ring on the Chern classes of the universal complex vector bundle) under the map $Bi : BO \to BU$ induced by the complexification map $i : O \to U$. Let c_i^* denote the i^{th} Chern class of the conjugate of the universal bundle [MS74, page 167]. Then the result of [Wil84] which we want to reprove is

$$BP^*(BO) = BP^*(BU)/(c_i - c_i^* : i > 0).$$

We do not know if similar methods can be used to recover $BP^*(BO(m))$. For more on $BO(m)$ the reader should see [Kri97, Section 5].

We first observe that BO does not have a $\lim^1$ problem so that we really can use BP and not $BP_p\hat{}$. In general this is done by Landweber in [Lan72] but in this case it it quite easy to see because the rational cohomology of BU surjects to BO. The only way there can be an infinite number of differentials in the Atiyah-Hirzebruch spectral sequence, giving a phantom class, is if there are an infinite number of differentials on one of the integral classes of BO. Some multiple is in the image of the spectral sequence from BU and that spectral sequence collapses, so this cannot happen.

Next we observe that BO has no odd prime torsion so we have nothing to prove to get the result at odd primes.

For $p = 2$ we begin by showing that BO has even Morava K-theory. $K(n)^*(BO)$ can be computed as follows. We know that (the mod 2 homology)

$$H_*(BO) = P(b_1, b_2, \cdots)$$

where $b_i \in H_i(BO)$ is the image of the generator of $H_i(RP^\infty)$. The action of the Milnor primitive Q_n is given by

$$Q_n(b_i) = \begin{cases} b_{i+1-2^{n+1}} & \text{if } i \text{ is even and } i \geq 2^{n+1} \\ 0 & \text{otherwise.} \end{cases}$$

It follows that in the Atiyah-Hirzebruch spectral sequence for $K(n)_*(BO)$ we have

$$d_{2^{n+1}-1}(b_i) = \begin{cases} v_n b_{i+1-2^{n+1}} & \text{if } i \text{ is even and } i \geq 2^{n+1} \\ 0 & \text{otherwise,} \end{cases}$$

so we have

$$E_{2^{n+1}} = K(n)_*[b_2, b_4, \cdots b_{2^{n+1}-2}] \otimes K(n)_*[b_{2i}^2 \colon i \geq 2^n].$$

It follows that $K(n)_*(BO)$ and $K(n)^*(BO)$ are even dimensional.

Recall that we are trying to show that the map $Bi : BO \to BU$ induces a surjection in BP-cohomology. Bott periodicity ([Bot59]; see also Milnor's treatment in [Mil63, §24]) gives us a fibre sequence

$$\mathbf{Z} \times BO \xrightarrow{Bi} \mathbf{Z} \times BU \xrightarrow{\Omega j} Sp/U = \Omega Sp.$$

Delooping this gives

$$\Omega^2 Sp \longrightarrow U \xrightarrow{j} Sp$$

where j is the usual inclusion map. Delooping twice more gives

$$Sp \xrightarrow{f} U \longrightarrow \Omega^2 O$$

where f is the usual inclusion map.

Now ΩSp has even dimensional homology. To see this, recall [Whi78, §VII.4] that

$$H_*(Sp; \mathbf{Z}) = E(x_3, x_7, \cdots, x_{4m+3}, \cdots) \qquad \text{with } |x_{4m+3}| = 4m + 3.$$

The Eilenberg-Moore spectral sequence for the homology of its loop space collapses, giving

$$H_*(\Omega Sp; \mathbf{Z}) = P(x_2, x_6, \cdots, x_{4m+2}, \cdots) \qquad \text{with } |x_{4m+2}| = 4m + 2.$$

Likewise, the bar spectral sequence collapses giving

$$E_0 H_*(BSp; \mathbf{Z}) = \Gamma(x_4, x_8, \cdots, x_{4m}, \cdots) \qquad \text{with } |x_{4m}| = 4m$$

where Γ means the divided power Hopf algebra.

Hence Proposition 2.0.1 and Corollary 2.0.2 apply to the fibration

$$BO \xrightarrow{Bi} BU \xrightarrow{\Omega j} \Omega Sp$$

and we have

$$BP^*(BO) \simeq BP^*(BU)/((\Omega j)^*)$$

which is not quite what we want yet. It came from a short exact sequence of Hopf algebras from Proposition 2.0.1

$$K(n)_* \to K(n)_*(BO) \to K(n)_*(BU) \to K(n)_*(\Omega Sp) \to K(n)_*$$

Now consider the fibration

$$\begin{array}{ccccc} Sp/U & \longrightarrow & BU & \xrightarrow{Bj} & BSp \\ \| & & \| & & \\ \Omega Sp & \xrightarrow{\Omega f} & \Omega SU & & \end{array}$$

All three spaces have even dimensional homology, so by Proposition 2.0.1 we have another short exact sequence of Hopf algebras

$$K(n)_* \to K(n)_*(\Omega Sp) \to K(n)_*(BU) \to K(n)_*(BSp) \to K(n)_*$$

Thus we get a diagram

$$\begin{array}{ccccccc} \mathbf{Z} \times BO & \xrightarrow{Bi} & \mathbf{Z} \times BU & & \xrightarrow{\Omega(fj)} & & \mathbf{Z} \times BU \\ & & \| & & & & \| \\ & & \Omega U & \xrightarrow{\Omega j} & \Omega Sp & \xrightarrow{\Omega f} & \Omega U \end{array}$$

and we can splice together the two exact sequences to get an exact sequence

$$K(n)_* \longrightarrow K(n)_*(BO) \longrightarrow K(n)_*(BU) \longrightarrow K(n)_*(BU).$$

This no longer comes from a fibration but the result of [Wil84] can be recovered by using Theorem 1.19 after we have identified the self-map $\Omega(fj)$ on BU as the one inducing the difference between the universal complex bundle and its conjugate. To do this, consider the composite

$$U(2m) \xrightarrow{j} Sp(2m) \xrightarrow{f} U(4m).$$

To study this we suppose that we have inclusions

$$\mathbf{R}^{2m} \subset \mathbf{C}^{2m} \subset \mathbf{H}^{2m} = \mathbf{C}^{2m} + j\mathbf{C}^{2m}$$

where the quaternion $j \in \mathbf{H}$ has its usual meaning. Then for $M \in U(2m) \subset Sp(2m)$ and $a, b \in \mathbf{C}^{2m}$ we have

$$M(a + jb) = Ma + j\overline{M}b;$$

note here that conjugation in $U(2m)$ is well defined since we have chosen a real subspace of $\mathbf{C}^{2m}$. It follows that the map fj sends M to

$$\begin{pmatrix} M & 0 \\ 0 & \overline{M} \end{pmatrix} \in U(4m).$$

It follows that $\Omega(fj) = 1 \oplus \Omega c$ where $c : U \to U$ is the conjugation map. All that remains is to show $\Omega c = -Bc$. Restricting to $\Omega U(1) \simeq \mathbf{Z}$, we see that Ωc induces multiplication by -1 in π_0. To evaluate Ωc on the 0-component $BU \simeq \Omega SU$, recall [Mil63, Theorem 23.3] that this equivalence is derived from a certain map

$$G_m(\mathbf{C}^{2m}) \xrightarrow{g} \Omega SU(2m)$$

where $G_m(\mathbf{C}^{2m})$ is the Grassmannian of complex m-planes in $\mathbf{C}^{2m}$. (Bott proves the complex case of his theorem by showing that the map g is an equivalence through a range of dimensions that increases with m.) The map is defined by associating to each point in $G_m(\mathbf{C}^{2m})$ a path in $SU(2m)$ from I to $-I$ as follows. Choose a basis of $\mathbf{C}^{2m}$ such that the m-dimensional subspace in question is spanned by the first m basis elements. We parametrize the path by $\theta \in [0, \pi]$ with

$$\theta \mapsto \begin{pmatrix} e^{i\theta} I & 0 \\ 0 & e^{-i\theta} I \end{pmatrix} \in SU(2m),$$

where I here denotes the identity element in $U(m)$. In other words (independently of the choice of basis) we send θ to a unitary transformation having eigenvalue $e^{i\theta}$ on the given subspace and $e^{-i\theta}$ on its complement.

Again we note that there is a well defined conjugation map c on $SU(2m)$, given our choice of a real subspace of $\mathbf{C}^{2m}$. Applying it does two things. First it conjugates the basis, replacing each subspace by its conjugate. It also conjugates the coefficient $e^{i\theta}$, so that with respect to the conjugated basis the map above becomes

$$\theta \mapsto \begin{pmatrix} e^{-i\theta} I & 0 \\ 0 & e^{i\theta} I \end{pmatrix} \in SU(2m).$$

Thus the direction of the path gets reversed, which effectively replaces the conjugated subspace by its unitary complement.

It follows that the conjugation map c on $SU(2m)$ restricts on the subspace $G_m(\mathbf{C}^{2m})$ to the map which sends each complex m-plane through the origin in $\mathbf{C}^{2m}$ to the complement of its conjugate. Passing to the limit as $m \to \infty$, we see that

$$\mathbf{Z} \times BU \xrightarrow{\Omega c} \mathbf{Z} \times BU$$

is the map inducing the Whitney inverse of the conjugate universal bundle as required.

2.6. **Connective covers of BU and related spaces and spectra.** Let $BU\langle 2m\rangle$ denote the $(2m-1)$-connected cover of BU for $m \geq 2$ and consider the fibration

$$F \longrightarrow BU\langle 2m\rangle \longrightarrow BSU.$$

Then $K(n)_*(F)$ is even dimensional for $n > 0$ by [HRW97], as is $H_*(BSU)$. Thus Proposition 2.0.1 applies and we conclude that $K(n)_*(BU\langle 2m\rangle)$ is even dimensional. We also know that $K(0)_*(BU\langle 2m\rangle)$ (the rational homology of $BU\langle 2m\rangle$) is even dimensional. Thus the results of this paper give information about $BP_p\hat{}^*(BU\langle 2m\rangle)$, and similarly for $MU\langle 2m\rangle$, the associated Thom spectrum.

Localizing at an odd prime, we can say the same about the fibration

$$F' \longrightarrow BO\langle 4m\rangle \longrightarrow BSO,$$

so we can say a lot about $BP^*(BO\langle 4m\rangle)$ and $BP^*(MO\langle 4m\rangle)$.

3. Preliminaries

We need a large selection of theories to state and prove our results. First, there is the Brown-Peterson cohomology, $BP^*(-)$, associated with a prime, p. Some basic references for BP are Brown-Peterson, [BP66], Adams, [Ada74], Quillen, [Qui69], Ravenel, [Rav86], and Wilson, [Wil82]. Next, we need the p-adic completion of BP, $BP_p\hat{}$, defined by

$$BP_p\hat{} = \lim{}^0(BP \wedge M(p^i)) \tag{3.1}$$

where $M(p^i)$ is the mod p^i Moore spectrum. The coefficient ring for BP, BP^*, is $\mathbf{Z}_{(p)}[v_1, v_2, \dots]$ where the degree of v_n is $-2(p^n-1)$. The coefficient ring for $BP_p\hat{}$ is just the p-adic completion of this. Either of these theories can be labeled $P(0)$. Next, we need the theories introduced by Morava, $P(n)$. Their coefficient rings are BP^*/I_n where $I_n = (p, v_1, \dots, v_{n-1})$. For references, see Johnson-Wilson, [JW75], Würgler, [Wür77], and Yagita, [Yag77]. Thanks to their construction using Baas-Sullivan singularities, [Baa73], [BM71], they come equipped with stable cofibrations:

$$\Sigma^{2(p^n-1)}P(n) \xrightarrow{\quad v_n \quad} P(n) \xrightarrow{\qquad} P(n+1), \tag{3.2}$$

which give us long exact sequences in cohomology. Note that $P(0)$ can be either BP or $BP_p\hat{}$ in this cofibration (let $v_0 = p$). Letting $BP\langle n\rangle^*$ be $\mathbf{Z}_{(p)}[v_1, \dots, v_n]$, theories, $E(k,n)$, can be constructed, using Baas-Sullivan singularities and localization, which have coefficients $v_n^{-1}BP\langle n\rangle^*/I_k$, and similar stable cofibrations. These spectra are discussed by Baker-Würgler in [BW89, page 523] and in [Hun92]. The earliest reference to these theories is probably in [Yos76, Prop. 4.6] (where they go by a different name). The theories without localization play a prominent role in [Yos76], [Yag76] and [BW]. For $k = 0$ these theories are usually denoted by $E(n)$. They come in two flavors; regular and p-adically complete. They have been studied in [JW73], [Lan76], [Rav84], and others. It is proven in [JW73, Remark 5.13, p. 347], and later follows from the Landweber exact functor theorem of [Lan76], that

$$E(n)_*(X) = E(n)_* \otimes_{BP_*} BP_*(X).$$

A similar result for $k > 0$ (with BP_* replaced by $P(k)_*$) was proved in [Yag76]. As a special case, when $k = n > 0$, we have the n^{th} Morava K-theory, $K(n)$, see [JW75], [Hop87], [Rav86], [Rav92], [Wür91] and [Yag80].

One of our main tools is the Atiyah-Hirzebruch spectral sequence

$$E_r^{*,*} \Rightarrow G^*(X) \tag{3.3}$$

where

$$E_2^{s,t} \simeq H^s(X; G^t) \tag{3.4}$$

which we will denote $E_r^{*,*}$, $E_r^{*,*}(X)$, or $E_r^{*,*}(G^*(X))$, depending on the context. The differential, d_r, has bidegree $(r, 1-r)$. When G is one of our connected spectra this is a fourth quadrant spectral sequence. If not, it is a first and fourth quadrants spectral sequence. Let $F^s = \ker(G^*(X) \to G^*(X^{s-1}))$ where X^{s-1} is the $s-1$ skeleton of X. Then we have $F^s/F^{s+1} \simeq E_\infty^{s,*}$ and F^∞ gives the phantom maps.

The spectral sequence really converges to $G^*(X)/F^\infty$ so it will be important for us to be able to show that F^∞ is zero in our cases. By Milnor's theorem, [Mil62]:

Theorem 3.5 (Milnor)**.** *There is a short exact sequence*

$$0 \to \lim{}^1 G^{*-1}(X^m) \to G^*(X) \to \lim{}^0 G^*(X^m) \to 0.$$

Since the term on the right of Milnor's theorem is what the Atiyah-Hirzebruch spectral sequence converges to, the triviality of F^∞ is equivalent to the $\lim^1$ term being zero.

Remark 3.6. One way to show the $\lim^1$ term is zero is by using the Mittag-Leffler condition. In our case, we have a sequence of subgroups

$$\mathrm{Im}\{G^n(X^{m+i}) \to G^n(X^m)\}.$$

If they stabilize for big i and all n we say the Mittag-Leffler condition is satisfied. In this case, the $\lim^1$ term in Milnor's theorem is zero. See [Ada74] for more details. Various assumptions on the G or X can give us the Mittag-Leffler condition:

(i) If $E_2^{s,t}$ is always finite. This can happen if one of G^s or $H^t(X)$ is always finite and the other is finitely generated. This is the case for several of our theories; e.g. $P(n)$ and $K(n)$, $n > 0$.
(ii) If $E_*^{s,t}$ always has only a finite number of nontrivial differentials on it. We show that this is the case for $E(k,n)$ when $0 < k < n$, which is a bit surprising and our most difficult technical lemma.

Remark 3.7. When the Mittag-Leffler condition is not satisfied for BP then we need to move to our p-completion (because the $\lim^1$ term is not zero, [Lan70b]). The Mittag-Leffler condition is still not satisfied when we p-complete! However, because of the compactness of the p-adics we do have the $\lim^1$ term is zero (see [Ada74]). Thus, by our choice of $P(0)$, we always have

$$P(0)^*(X) \simeq \lim{}^0 P(0)^*(X^m).$$

The skeletal filtration of $G^*(X)$ associated with the Atiyah-Hirzebruch spectral sequence also gives a topology on $G^*(X)$ which is nontrivial if X is an infinite complex. Since all of our spaces are infinite complexes this topology is always there. For a detail reference on such things, see [Boa95]. We will not need much about the topology here except that when we look at subgroups generated by a set of elements, we will mean *topologically* generated, i.e., the closure of the literal subgroup. When G is a p-adic completion then we need a slightly different topology. Here we use finite complexes which are torsion in the sense that the identity map is stably torsion. An open neighborhood of zero is the kernel of $G^*(-)$ when we map a torsion finite complex to X. If X is a finite complex and G is $BP_p\hat{}$ then the topology is just the p-adic completion. Then, since these are finitely generated they are compact and our $\lim^1 P(0)^*(X^m)$ is always zero as above. When we look at the p-adic completion of $E(n)$ we lose our compactness but since $E(n)^*(X^m)$ is $E(n)^* \otimes P(0)^*(X^m)$ we still have our $\lim^1$ zero.

We need three theorems which generalize known results. First we need a generalization of Quillen's theorem.

Theorem 3.8 ([Qui71] for $n = 0$ and [BW] for $n > 0$)**.** *For X a finite complex, $P(n)^*(X)$, $n \geq 0$, is generated by non-negative degree elements.*

Quillen proved the $n = 0$ version of this in [Qui71]. A second proof for Quillen's result, $n = 0$ of this, was given in [Wil75]. More recently, Quillen's result follows from abstract information about unstable BP operations, [BJW95]. When we started this project we believed the result could be proven for $n > 0$ using Quillen's approach. By the time we discovered that this was not the case, there were two proofs, analogous to those in [Wil75] and [BJW95], following from the splitting theorem of [BW] and are included there.

All of our proofs go through for $p = 2$. Normally, there can be problems with this case because most of our theories that are mod 2 do not have a commutative multiplication on them. However, in our case they are always even dimensional. In [Wür77], Würgler computes the obstruction to commutativity and shows that it factors through odd degrees and is thus of no concern to us. Where it could bother us because we do have odd degree elements, is in the Atiyah-Hirzebruch spectral sequence, but that is commutative by itself so it is no problem. The rest of the arguments are no problem.

The next result that we need is a generalization of the Landweber exact functor theorem, [Lan76].

Theorem 3.9 ([Lan76] for $k = 0$. [Yos76] and [Yag76] for $k > 0$)**.** *Let $I_{n,k}$ be the ideal $(v_k, \dots, v_{n-1})$ in $P(k)^*$. M is Landweber flat, i.e. flat for the category of finitely presented $P(k)^*(P(k))$-modules, if v_n multiplication is injective on $M/I_{n,k}M$ for all $n \geq k$.*

Note that this result is *not* a cohomological version of the Landweber exact functor theorem, but merely an algebraic statement about finitely presented $P(k)^*(P(k))$-modules.

We also need a generalization of the Landweber filtration theorem.

Theorem 3.10 ([Lan73] for $k = 0$. [Yos76] and [Yag76] for $k > 0$)**.** *Let $I_{n,k}$ be the ideal $(v_k, \dots, v_{n-1})$ in $P(k)^*$. Let M be a finitely presented $P(k)^*(P(k))$-module. There exists a finite filtration of M by $P(k)^*(P(k))$-modules,*

$$M = M_0 \supset M_1 \supset \cdots \supset M_j = \{0\},$$

where

$$M_s/M_{s+1} = P(k)^*/I_{n_s,k}.$$

Theorem 3.11 (Boardman-Johnson-Wilson)**.** *Let M be an unstable BP module which is bounded above and of finite type, then there is a (finite) unstable Landweber* *filtration, as in Theorem 3.10, with the generators of the quotient modules all in non-negative degrees.*

Remark 3.12. This is from Theorem 20.11 of [BJW95]. The bounds on the degrees of the quotient modules are much more refined in [BJW95] than we give here. It is also stated quite differently in [BJW95]. There it is assumed that M is finitely presented, but the assumption is never used. In fact, finitely generated need not be assumed because of the algebraic version of Quillen's theorem, [BJW95, 20.3], which says the generators are all in non-negative degrees. The proof of this theorem inductively constructs cyclic submodules whose generators are in non-negative degrees, thus reducing the size of the non-negative part with each step. Finite presentation follows from the finiteness of the filtration. In [BW], a $P(k)$ version of this theorem is proven.

For our X of finite type, $BP^*(X)/F^s$ is an unstable module, including the case of $s = \infty$. We show later, Corollary 4.8, that $F^\infty = 0$ for $P(k)$, $k > 0$. Our work relies heavily on the unstable Quillen-Boardman-Wilson result. The unstable Landweber filtration is a stronger statement. Our interest in it is the following.

Corollary 3.13. *Let X be a space, then $BP^*(X)/F^{s+1} \subset BP^*(X^s)$ is a $BP^*(BP)$ module which is finitely presented over BP^*.*

Proof. This is an unstable module which fits the hypothesis of Theorem 3.11. ☐

Remark 3.14. In fact, we prove this corollary ourselves in Lemma 6.1 for $P(k)$ except when $k = 0$ and $P(0)$ is the p-adic completion of BP. For the p-adic completion we must resort to the unstable Landweber filtration. The only place in this paper where we need this result is when we prove the Künneth isomorphism 1.11 for $k = 0$ when X has $\lim^1 BP^*(X^i)$ non-zero.

Remark 3.15. Since $P(k)^*$ is a coherent ring, all of our finitely presented modules are coherent, and, of course, coherent implies finitely presented, see [Smi69].

Remark 3.16. The unstable Landweber filtration answers the cohomology version of an old question of Landweber's from [Lan71, Problem 4]. It shows that for X of finite type, $BP^*(X)/F^\infty$ is pseudo-coherent, i.e. every finitely generated submodule is finitely presented. This is still not known for homology despite being bounded below, unlike the cohomology which is not bounded in either direction. Likewise, similar theorems are true for $P(k)$.

4. The Atiyah-Hirzebruch spectral sequence

In this section we develop the Atiyah-Hirzebruch spectral sequence for the things which we need. In particular, we accomplish two main goals. First, we show that there are no phantom maps in $G^*(X)$ for all of the theories that we are concerned with except BP. This simplifies our life considerably. Originally it seemed that some sort of exotic completions would be necessary to state our theorem, but because of this lack of phantom maps in general, the only place we have to go to completion is occasionally with BP where we have to resort only to p-adic completion. This lack of phantom maps is just the same as having all elements of $G^*(X)$ represented in the Atiyah-Hirzebruch spectra sequence; i.e., having the infinite filtration, F^∞, equal to zero. We must eliminate the phantom maps so that in the next section we can show $K(n)^*(X)$ even implies $E(k,n)^*(X)$ is also even. With the advantage of hindsight, an alternative route to these results might be to use Yosimura's work [Yos88]. Second, we show that for any given element of $P(k)^*(X)$, $k \geq 0$, there is some N such that the element maps nontrivially to $E(k,n)^*(X)$ for every $n > N$. Our proof is somewhat technical and uses the Atiyah-Hirzebruch spectral sequence extensively. However, to see what is going on is not so difficult. Consider the $k = 0$ case where we are working with BP and $E(0,n) = E(n)$, the localization of $BP\langle n\rangle$. From [Wil75] we know that $\underline{BP\langle n\rangle}_i$ splits off of $\underline{BP}_i$ for $i \leq 2(p^{n+1}-1)/(p-1)$. It is easy to see that it splits off of $\underline{E(n)}_i$ as well. Any $0 \neq x \in BP^i(X)$ must reduce to a non-trivial element in $BP\langle n\rangle^i(X)$ for some large n. Thus we can see, quite geometrically by looking at the classifying spaces, the result we want. No such splitting was around for $k > 0$ when we wanted to generalize this. Our proof depends heavily on the theorem that $P(n)^*(X)$ is generated by non-negative degree elements for finite complexes. Since this proof, a splitting theorem has been found, [BW], which would allow us to prove the result by looking at the representing spaces for the cohomology theories. This gives an alternative approach to this part of the proof as well.

Unless otherwise stated, *let $E_r^{*,*}(X) \Rightarrow P(n)^*(X)$, $n \geq 0$, be the Atiyah-Hirzebruch spectral sequence with X a space.* Although we do not need X to be a space for the first few lemmas we will assume it anyway. X could just as well be a (-1)-connected spectra. We will point out when having X a space becomes necessary. Let R be either the integers localized at p or the p-adic integers, depending on which $P(0)$ we are using for a given X. We define

$$L(0,N) \simeq R[v_1, v_2, \ldots, v_{N-1}] \subset P(0)^*$$

and

$$L(n,N) \simeq \mathbf{Z}/(p)[v_n, \ldots, v_{N-1}] \subset P(n)^*$$

for $n > 0$.

Lemma 4.1. *Let $E_r^{*,*}(X) \Rightarrow P(n)^*(X)$, X a space, with $n \geq 0$. For each r and s there is a number $N = N(s,r)$ such that there is a finitely generated $L \equiv L(n,N)$-module, $A_r^s = A_r^{s,*}$, generated in nonpositive degrees (second degree) and satisfying*

$$E_r^{s,*} \simeq A_r^s \otimes_L P(n)^*.$$

Proof. The proof is by induction on r. For $r = 2$ we can take $A_2^s = H^s(X;\mathbf{Z}/(p))$ and $N = n$. (For $n = 0$ we use $H^s(X;R)$ and $N = 1$.) Assume the case r. Then we have integers $N(s,r)$ as in the lemma for all s. Let $\{y_i(s)\}$ be a (finite) set of generators of A_r^s with (second) degree $|y_i(s)|$. Now fix an s. Write

$$\begin{aligned} M &= \max\{N(k,r)\colon 0 \leq k \leq s+r\} \\ S &= L(n,M) \\ d &= \min\{|y_i(k)|\colon 0 \leq k \leq s+r, i > 0\}. \end{aligned}$$

Then, for $n > 0$, there is a bigraded S-module $B_r^q = B_r^{q,*}$ generated by $\{y_i(q)\}$ for $0 \leq q \leq s+r$, such that

$$E_r^{q,*} \simeq B_r^q \otimes_S P(n)^*, \qquad 0 \leq q \leq s+r,$$

with $0 \geq |y_i(q)| \geq d$ for all $0 \leq q \leq s+r$. Consider the differential

$$d_r(y_i(s')) = \sum_j c_{i,j}(s'+r) y_j(s'+r), \quad c_{i,j}(s'+r) \in P(n)^*$$

for $0 \leq s' \leq s$. Then,

$$1 - r + |y_i(s')| = |c_{i,j}(s' + r)| + |y_j(s' + r)|,$$

so

$$c_{i,j}(s' + r) \in L(n, M')$$

where M' is the smallest number such that $-2(p^{M'-1} - 1) \leq 1 - r + d$. Take $S_{r+1} = L(n, M')$. Then d_r induces an S_{r+1} map

$$d_r : B_r^s \otimes_{S_r} S_{r+1} \longrightarrow B_r^{s+r} \otimes_{S_r} S_{r+1}$$

such that

$$\ker(d_r : E_r^{s,*} \to E_r^{s+r,*}) = \ker(d_r | B_r^s \otimes_{S_r} S_{r+1}) \otimes_{S_{r+1}} P(n)^*.$$

Similarly, consider $d_r : E_r^{s-r,*} \to E_r^{s,*}$. Then $E_{r+1}^{s,*} \simeq A^s \otimes_{S_{r+1}} P(n)^*$ for some S_{r+1}-module $A^s = A^{s,*}$. A^s is a subquotient of the finitely generated S_{r+1}-module $B_r^s \otimes_{S_r} S_{r+1}$. Hence A^s is finitely generated as an S_{r+1}-module since S_{r+1} is Noetherian. Take $N = N(s, r+1)$ as M' and A_{r+1}^s as A^s. This completes the induction. □

Recall X^m is the m-skeleton of X and $i : X^m \to X$ is the inclusion.

Lemma 4.2. *Let $E_r^{*,*}(X) \Rightarrow P(n)^*(X)$, X a space, $n \geq 0$. For each r,*

$$E_r^{s,*}(X) \simeq E_r^{s,*}(X^m) \quad \text{for all} \quad 0 \leq s \leq m - r(r-1)/2.$$

Proof. Since $E_2^{s,*}(X) \simeq E_2^{s,*}(X^m)$ for $s \leq m - 1$ and $|d_r| = (r, 1 - r)$ this follows by an easy induction. □

Lemma 4.3. *Let $E_r^{*,*}(X) \Rightarrow P(n)^*(X)$, X a space, $n \geq 0$. For all $m \geq (s+1)s/2 + s$ and all r, $i^* : E_r^{s,*}(X) \longrightarrow E_r^{s,*}(X^m)$ is injective.*

Proof. By Lemma 4.2 we have an isomorphism for $r \leq s + 1$ when $0 \leq s \leq m - (s+1)s/2$, i.e., $m \geq (s+1)s/2 + s$. Then, because $|d_r| = (r, 1 - r)$, $E_r^{s,t}$ is not in the image of d_r for $r > s$ so we have

$$\begin{array}{ccc}
E_{s+1}^{s,*}(X) & = & E_{s+1}^{s,*}(X^m) \\
\uparrow & & \uparrow \\
E_r^{s,*}(X) & \xrightarrow{i^*} & E_r^{s,*}(X^m) \\
\uparrow & & \uparrow \\
E_\infty^{s,*}(X) & \xrightarrow{i^*} & E_\infty^{s,*}(X^m)
\end{array}$$

for $r > s$. This implies the result. □

The next lemma uses the Boardman-Wilson version (Theorem 3.8) of Quillen's theorem for $P(n)^*(-)$, and is the main technical lemma which makes the Atiyah-Hirzebruch spectral sequence approach work. At this stage it becomes essential that we are working with spaces and not spectra.

Lemma 4.4. *Let $E_r^{*,*}(X) \Rightarrow P(n)^*(X)$, X a space, $n > 0$ or if $P(0) = BP$, $n = 0$. For each s there is an m such that*

$$i^* : E_\infty^{s,*}(X) \simeq E_\infty^{s,*}(X^m)$$

and $d_r(E_r^{s,*}(X)) = 0$ for $r > m$, i.e., $E_{m+1}^{s,*}(X) \simeq E_\infty^{s,*}(X)$.

Remark 4.5. The result is not true if $P(0) = BP_p\hat{}$ because if the Mittag-Leffler condition does not hold for BP then it still fails for $BP_p\hat{}$ even though we have no $\lim^1$ problems there.

Proof. Since each group $E_r^{s,t}$ is finite, for $n > 0$, (if $n = 0$, each group is finitely generated over $\mathbf{Z}_{(p)}$), we can find a $T \geq s$ such that d_r restricted to $\{E_r^{s',t'} | s' \leq s, t' \geq -s'\}$ is zero for all $r \geq T$. (For $n = 0$, if we had an infinite number of differentials then the Mittag-Leffler condition would not be satisfied. We see from [Lan70b] that the Mittag-Leffler condition is equivalent to the $\lim^1$ term being non-zero. This would contradict our choice of $P(0) = BP$.) From Lemma 4.2 and Lemma 4.3 we can find $m \gg s$ such that $E_T^{s',t}(X) \simeq E_T^{s',t}(X^m)$ for all $s' \leq s$ and $i^* : E_r^{s',t}(X) \to E_r^{s',t}(X^m)$ injects for all r and $s' \leq s$. Certainly

$$E_{m-s+1}^{s,*}(X) \hookrightarrow E_{m-s+1}^{s,*}(X^m) \simeq E_\infty^{s,*}(X^m).$$

We want to show that $E_{m-s+1}^{s,*}(X) \simeq E_\infty^{s,*}(X)$. Assume there is some $0 \neq \tilde{x} \in E_r^{s,t}(X)$, and $r \geq m - s + 1$ with $d_r(\tilde{x}) \neq 0$. We have

$$i^*(\tilde{x}) = \tilde{x}^m \in E_r^{s,t}(X^m) \simeq E_\infty^{s,t}(X^m).$$

This is non-zero by our injectivity. Furthermore, $t < -s$ because our original choice says $d_r = 0$ if $t \geq -s$. Thus the total degree of $\tilde{x}$ is negative. By Boardman-Wilson's version of Quillen's Theorem for $P(n)^*(X^m)$, Theorem 3.8, we know there are non-negative degree $P(n)^*$ generators, g_i^m, for $P(n)^*(X^m)/F^{s+1}$ in $E_\infty^{s',t'}(X^m)$ with $s' \leq s$ and $t' \geq -s'$, which our starting assumption says is isomorphic to $E_\infty^{s',t'}(X)$.

(The fact that X is a space rather than a connective spectrum is crucial at this point of the proof. In the latter case, Theorem 3.8 would not give us the precise control we need on the dimensions of these generators.)

Thus $P(n)^*(X)/F^{s+1}$ surjects to $P(n)^*(X^m)/F^{s+1}$ in non-negative degrees. So we can choose generators $\{g_i \in P(n)^*(X)\}$ which reduce to the generators $\{g_i^m \in P(n)^*(X^m)/F^{s+1}\}$. This is a finite set because $E_2^{s',t'}$, $s' \leq s$ and $t' \geq -s'$, is finite (for $n = 0$, finitely generated over $\mathbf{Z}_{(p)}$).

If we have our $d_r(\tilde{x}) \neq 0$, then

$$\tilde{x}^m = i^*(\tilde{x}) \in E_r^{s,t}(X^m) = E_\infty^{s,t}(X^m)$$

is not in the image of $i^* : E_\infty^{s,t}(X) \to E_\infty^{s,t}(X^m)$. Let $x^m \in P(n)^*(X^m)/F^{s+1}$ be an element represented by $\tilde{x}^m$. Then $x^m = \sum v(i) g_i^m$. Define $z = \sum v(i) g_i \in P(n)^*(X)$. Then $i^*(z) = x^m$ so $\tilde{z}$ must be in $E_\infty^{s',t}$ with $s' < s$. This is so because $\tilde{x}^m$ is not in the image but the element it represents is; therefore the element representing the element that hits it must be in $E_\infty^{s',t}$ with $s' < s$. But, that means $\tilde{z}$ must go to zero in order to do its duty of changing filtrations to hit $\tilde{x}^m$. This contradicts the injectivity of Lemma 4.3 and we have $E_{m-s+1}^{s,*}(X) \simeq E_\infty^{s,*}(X)$. The argument just given also shows that $E_r^{s,*}(X)$ maps surjectively to $E_r^{s,*}(X^m)$ for $r \geq m - s + 1$. □

We have some quick corollaries now.

Corollary 4.6. *Let X be a space and let $n > 0$ or if $P(0) = BP$, $n = 0$. For each r, there is an m such that*

$$P(n)^*(X)/F^r \simeq P(n)^*(X^m)/F^r$$

where F^r is the r^{th} filtration of the Atiyah-Hirzebruch spectral sequence.

Corollary 4.7. *Let X be a space and let $n > 0$ or if $P(0) = BP$, $n = 0$. $P(n)^*(X)$ is (topologically) generated by non-negative degree elements.*

Corollary 4.8. *Let X be a space. Let $n \geq k \geq 0$ and $n > 0$. Let G be $P(k)$, $E(k,n)$, $v_n^{-1}P(k)$, or $K(n)$. Let $E_r^{*,*}(G^*(X)) \Rightarrow G^*(X)$ be the Atiyah-Hirzebruch spectral sequence. (For $k = 0$ we use the p-adically complete $E(k,n)$ if we use it for $P(0)$.)*

(a) *Then $F^\infty = 0$, every element in $G^*(X)$ is represented in $E_\infty^{*,*}$, $G^*(X) \simeq \lim^0 G^*(X^m) \simeq \lim^0 G^*(X)/F^{m+1}$, and $\lim^1 G^*(X^m) = 0 = \lim^1 G^*(X)/F^{m+1}$.*

(b) *Furthermore*

$$\begin{aligned} E_r^{*,*}(E(k,n)^*(X)) &\simeq E_r^{*,*}(P(k)^*(X)) \otimes_{P(k)^*} E(k,n)^* \\ &\simeq E_r^{*,*}(v_n^{-1}P(k)^*(X)) \otimes_{v_n^{-1}P(k)^*} E(k,n)^* \end{aligned}$$

and

(c)

$$E_r^{*,*}(v_n^{-1}P(k)^*(X)) \simeq E_r^{*,*}(P(k)^*(X)) \otimes_{P(k)^*} v_n^{-1}P(k)^*,$$

which is just localization.

Remark 4.9. This is not true for spectra. For example, $P(n)^*(k(n)) = 0$ but $K(n)^*(k(n))$ is not, so the first line for $k = n$ cannot hold.

Proof. The statements of (a) are all equivalent so it is enough to show any one of them. For $G = P(k)$, $k > 0$, we have that $P(k)^s$ is finite so $E_r^{s,t}$ is finite. By the Mittag-Leffler condition, Remark 3.6(i), we are done. For $k = 0$ we can use Remark 3.7 to see that $F^\infty = 0$, the main purpose of our choice of $P(0)$. There is nothing to prove in this case as it is really part of our assumptions. The other G will follow from the displayed tensor products (of (b) and (c)) and Lemma 4.4. Keep in mind that $K(n)$ is just a special case of $E(k,n)$, with $n = k$. All of the tensor products (of (b) and (c)) are true for $E_2^{*,*}$. Since the tensor product of (c) is just localization, and localization preserves exactness, we see that it is true. $E_r^{s,*}$ for $P(k)$ is always a finitely generated $P(k)^*(P(k))$ module so tensoring with $E(k,n)^*$ is exact by the generalized Landweber exact functor theorem for $P(k)^*(-)$ which says that $E(k,n)^*$ is $P(k)^*$ flat in this situation, Theorem 3.9. This gives the first equivalence of (b). Lemma 4.4 gives the first equivalence (of (a)), ($F^\infty = 0$), for $E(k,n)$ (with $k > 0$) by Remark 3.6(ii). If $k = 0$ and $P(0) = BP$ then this also follows. If $P(0) = BP_p\hat{}$ then $E(0,n)^*$ is a module over the p-adics and by compactness, see Remark 3.7, $F^\infty = 0$. The second isomorphism of (b) follows from the first (of (b)) and (c). □

Remark 4.10. Although this is a very technical result, it is an exciting one because it removes most of our $\lim^1$ problems. Since all of our theorems are about infinite complexes, worrying about phantom maps was a major concern which this corollary eliminates. For example, certain types of elements cannot exist. In the spectral sequence for $P(n)^*(X)$ it is quite possible, as we shall see in the next section (see

Remark 5.6), to have an element whose filtration is raised every time you multiply by v_n, and which is never zero. Such an element, by the corollary, can never give rise to an element in $K(n)^*(X)$ even though it is a torsion free element.

Corollary 4.11. *Let X be a space. Let $0 \leq k < n$. If x is infinitely divisible by v_k ($v_0 = p$) in $E(k,n)^*(X)$, then it is zero.*

Any such element would have to be a phantom map, and there are none.

Proof. By Corollary 4.8 we have $E(k,n)^*(X) \simeq \lim^0 E(k,n)^*(X^m)$ so if x is non-zero it maps nontrivially to some $E(k,n)^*(X^m)$ and is still infinitely divisible by v_k there. This is the difficult step, reducing the proof to looking at a finite complex. To complete the proof we use the generalized Landweber filtration of Theorem 3.10. We have

$$P(k)^*(X^m) = M = M_0 \supset M_1 \supset \cdots \supset M_j = \{0\}.$$

By the generalized Landweber exact functor Theorem 3.9, $E(k,n)^*$ is exact. We can tensor the filtration with $E(k,n)^*$ to get a filtration of $E(k,n)^*(X^m)$ with successive quotients given by $E(k,n)^*/I_{n_s,k}$. If $n_s > n$ then this is zero. Eliminating the zero quotients we have a finite filtration

$$E(k,n)^*(X^m) = M' = {M'}_0 \supset {M'}_1 \supset \cdots \supset {M'}_{j'} = \{0\}$$

with ${M'}_s/{M'}_{s+1} = E(k,n)^*/I_{n_s,k}$, $n_s \leq n$. We need to show that there is no $x \in E(k,n)^*(X^m)$ which is infinitely divisible by v_k. Assume that there is such an x. Find the maximum s with such an $x \in {M'}_s$. This x must reduce non-trivially to ${M'}_s/{M'}_{s+1} = E(k,n)^*/I_{n_s,k}$ and still be infinitely divisible by v_k here, which is impossible. (If it were not infinitely divisible by v_k then there would have to be such an $x \in {M'}_{s+1}$, contradicting the assumption on s.) □

Proposition 4.12. *Let X be a space. Let $k \geq 0$. Given $0 \neq x \in P(k)^*(X)$, there exists an N such that x maps nontrivially to $E(k,n)^*(X)$ for all $n \geq N$.*

Proof. Let $k > 0$. The element x is represented in the Atiyah-Hirzebruch spectral sequence by an element $\tilde{x}$ in $E_\infty^{s,*}$, which, by Lemma 4.4 is isomorphic to $E_r^{s,*}$ for some big r. For degree reasons we can pick an N such that $\tilde{x}$ is a $P'(N)^*$-generator where $P'(N)^*$ is a subalgebra of $P(n)^*$ isomorphic to $P(N)^*$ and $E_\infty^{s,*}$ is $P'(N)^*$ free. Using Lemma 4.1 we can assure that N is big enough so that $\tilde{x}$ survives the tensor product of Corollary 4.8 to the spectral sequence for $E(k,n)^*(X)$ for $n > N$. In this spectral sequence we still have $E_r^{s,*} \simeq E_\infty^{s,*}$ so our element x maps nontrivially. For the case of $P(0)$ we use Lemma 4.3 with $r = \infty$. For m big enough we have an injection:

$$E_\infty^{s,*}(X) \longrightarrow E_\infty^{s,*}(X^m).$$

The right hand side is very nice and we can tensor it with our (p-adically complete) $E(0,n)^*$ for big n, forcing what we need, as above, for the left hand side. □

5. Even Morava K-theory

We complete the proof of the main Theorem 1.8 in this section.

Lemma 5.1. *Let X be a space. Let $0 \leq k \leq n$ and $n > 0$. If $K(n)^*(X)$ is even dimensional, then $E(k,n)^*(X)$ is even dimensional and has no v_k torsion ($v_0 = p$).*

Remark 5.2. For $k = 0$ this is $E(n)^*(X)$ or its p-adic completion if necessary. It follows from the proof that $E(n)^*(X)$ maps onto $K(n)^*(X)$. This was proven by Hunton in [Hun92, Theorem 11], for finite complexes. This is improved in [BW91], p. 559, to give a surjection of $\widehat{E(n)}^*(X)$, the I_n-adic completion. As we only need p-adic completion our result is somewhat stronger.

Proof. We prove this by downward induction on k. Since $E(n, n) = K(n)$ our induction is grounded by our assumption. By induction, assume that $E(k+1, n)^*(X)$ is even dimensional and has no v_{k+1} torsion. We have a long exact sequence from the cofibration analogous to 3.2:

$$\begin{array}{ccc} E(k,n)^*(X) & \xrightarrow{\quad v_k \quad} & E(k,n)^*(X) \\ & {\scriptstyle\delta}\nwarrow \quad \swarrow{\scriptstyle\rho} & \\ & E(k+1,n)^*(X) & \end{array}$$

Since $E(k + 1, n)^*(X)$ is even dimensional and δ is an odd degree map, there are two possible types of odd degree elements in $E(k, n)^*(X)$:

(i) an element which never shows itself in $E(k+1, n)^*(X)$ because it is infinitely divisible by v_k and not v_k torsion;
(ii) an element which is infinitely divisible by v_k but is v_k torsion (the element that v_k kills comes by way of δ).

Either way the element is infinitely divisible by v_k, which cannot happen by Corollary 4.11. Thus δ is zero and all elements are even degree. If any even degree element were v_k torsion, then it would have to be hit by δ coming from an odd degree element, which doesn't exist by our induction assumption. Thus we get a short exact sequence and all elements are v_k torsion free. □

We can now prove Theorem 1.2.

Lemma 5.3. *Let X be a space. If $K(n)^*(X)$ is even dimensional for an infinite number of n, then $P(k)^*(X)$, $k \geq 0$, and $K(k)^*(X)$, $k > 0$, are even dimensional.*

Proof. If $0 \neq x \in P(k)^*(X)$ pick N as in Lemma 4.12 so x maps nontrivially into $E(k, n)^*(X)$ for $n > N$. Find some $n > N$ for which $K(n)^*(X)$ is even dimensional. By Lemma 5.1, $E(k, n)^*(X)$ is even dimensional so x must be even dimensional as well. This concludes the proof for $P(k)^*(X)$. By Lemma 4.8, all elements of $P(k)^*(X)$ are represented in $E_\infty^{*,*}(P(k)^*(X))$ which is even dimensional. Furthermore, $E_\infty^{*,*}(K(k)^*(X))$ is just the tensor product with $K(k)^*$ so it too is even dimensional, and it also represents elements. □

Corollary 5.4. *Let X be a space with even Morava K-theory. For $k \geq 0$ we have the short exact sequence:*

$$0 \longrightarrow P(k)^*(X) \xrightarrow{v_k} P(k)^*(X) \longrightarrow P(k+1)^*(X) \longrightarrow 0.$$

Proof. The three terms fit into a long exact sequence with odd degree connecting term, by 3.2. By Lemma 5.3 all terms are even dimensional so the boundary homomorphism must be zero. □

Corollary 5.5. *If X is a space with even Morava K-theory and $k \geq 0$ then $P(k)^*(X)$ is even degree and is Landweber flat.*

Proof. $P(k)^*(X)$ is even dimensional by Lemma 5.3. To prove flatness we need only invoke the generalized Landweber exact functor theorem for $P(k)$, Theorem 3.9, and Corollary 5.4. □

This finishes the proof of Theorem 1.9 and Theorem 1.8 follows.

Remark 5.6. This is a good time to insert a fundamental example which illustrates the phenomenon described in Remark 4.10. This is an old, well known example but it supplies useful guidance. Let $X = B\mathbf{Z}/(p)$. The mod p cohomology is $E(e_1) \otimes P(x_2)$, so E_2 of the spectral sequence is

$$E(e_1) \otimes P(x_2) \otimes P(n)^*.$$

The only nontrivial differential takes e_1 to $v_n x_2^{p^n}$ leaving E_∞ to be a copy of $P(n)^*$ for each x_2^i for $i < p^n$ and a copy of $P(n+1)^*$ for each x_2^i for $i \geq p^n$. Tensoring this with $K(n)^*$ we get the correct answer for $K(n)^*(B\mathbf{Z}/(p))$; free on generators x_2^i for $i < p^n$. However, we know from our corollary that there is no v_n-torsion. If you take an element in $P(n)^*(B\mathbf{Z}/(p))$ which is represented by x_2^i for $i \geq p^n$ and you multiply by v_n, then it is represented by $v_{n+1} x_2^{i-p^n+p^{n+1}}$. So, iterating the multiplication by v_n continues to raise filtration and give a nontrivial element. However, it does not give rise to an element of $K(n)^*(B\mathbf{Z}/(p))$.

Looking briefly at $P(n)_*(B\mathbf{Z}/(p))$ we see that E^∞ is free over $P(n+1)_*$ on elements α_i in degree $2i-1$ for $i > 0$ and free over $P(n)_*$ on β_i in degree $2i$ for $0 < i < p^n$. The relations on the α come from the p-sequence. In particular, we have $v_{n+1}\alpha_i + v_n \alpha_{i+p^n-p^{n+1}} \bmod (v_n, v_{n+1})^2$. We see that all of the α_i are infinitely divisible by v_n in $E(n, n+1)_*(X)$.

Remark 5.7. If $H^k(X, \mathbf{Z}_{(p)})$ is finite for all k then the Mittag-Leffler condition, Remark 3.6, is satisfied and the Atiyah-Hirzebruch spectral sequence for $BP^*(X)$ converges giving $\lim^1 BP^*(X^m) = 0$, so the results of Theorem 1.8 hold for $BP^*(X)$. As we shall see later, this is the case for $X = K(\mathbf{Z}/(p^i), n)$. When $H^k(X)$ is not finite we may have to resort to the p-adic completion of BP, such as with $X = K(\mathbf{Z}_{(p)}, n)$, $n > 2$, which is known to have phantom maps.

6. Generators and relations

In this section we will prove Theorems 1.20 and 1.21, and Corollaries 1.23 and 1.24.

Proof of Theorem 1.20. We know that the Atiyah-Hirzebruch spectral sequence converges and $E_r^{s,t}(X)$ is a finitely generated module. By Boardman-Wilson's and Quillen's Theorem 3.8 the generators must be represented by elements with $s+t \geq 0$. Since $t \leq 0$, there can only be a finite number of generators for $P(n)^*(X)$ represented in $E_\infty^{s,*}$. Assume inductively that we have chosen a minimal number of generators for $P(n)^*(X)/F^s$. Then pick a few more, if necessary, that are represented in $E_\infty^{s,*}$ in order to get minimal generators of $P(n)^*(X)/F^{s+1}$. The construction of T_n with properties (a), (b) and (c) is now complete.

We now show (d), that T_n reduces to a set T_q with the same properties. We do this inductively. Because we know, Remark 1.10, that $P(n)^*(X)$ surjects to $P(n+1)^*(X)$ we get part (a) that T_{n+1} generates. The map is a filtered map so part (c) follows. Part (b), that all elements remain essential, is really the only thing left to prove. If some proper subset of T_n could be used to generate $P(n+1)^*(X)$ then we could write some $t \in T_{n+1}$ in terms of the t's:

$$t = \sum c_i t_i$$

where $c_i \in P(n+1)^*$ can all be lifted to $c_i \in P(n)^*$ and we use the same notation for elements in $P(n)^*(X)$ and $P(n+1)^*(X)$. From the exact sequence

$$0 \longrightarrow P(n)^*(X) \xrightarrow{v_n} P(n)^*(X) \longrightarrow P(n+1)^*(X) \longrightarrow 0$$

we can lift this to

$$t = \sum c_i t_i + v_n \sum d_i t_i$$

in $P(n)^*(X)$, contradicting (b). The result follows for T_q by induction.

Proof of part (e). Let FR_n be the set of finite linear relations among the elements of T_n in $P(n)^*(X)$. A typical relation looks like $\sum_i c_i t_i$ where $t_i \in T_n$ and $c_i \in P(n)^*$. We can write the c_i in terms of monomials in the v_k (where we let $v_0 = p$ for $P(0)^*$). We can define the length of a monomial as the sum of the powers of v's, i.e., for $v^I = v_n^{i_n} v_{n+1}^{i_{n+1}} \dots$ we define $l(v^I) = \sum i_k$. We now extend this definition to the elements of FR_n. We take the length of a relation to be the maximal length of a monomial occurring in any of its coefficients, c_i. There is an obvious map from FR_n to FR_{n+1}. Because there are a finite number of coefficients, every element goes to zero after enough of these maps have been applied. Let us find a relation, r, which has the minimal length as defined above. Let us assume that it is in FR_n and maps to zero in FR_{n+1}. We can do this because the length of a relation can never increase under these maps. Recall from Theorem 1.9 that we have a short exact sequence. Since each coefficient, c_i, maps to zero in $P(n+1)^*$ it must be divisible by v_n. Thus we can divide the sum r by v_n to get r/v_n. This is a finite sum with a smaller length than our minimal one so it must be a non-zero element. This cannot be true as it is a v_n-torsion element in $P(n)^*(X)$ which by Theorem 1.9 is known to have no such torsion. Thus there are no finite relations anywhere.

Proof of part (f). We prove this by downward induction. Let $r \in R_q$ be written $\sum c_i t_i$ with $c_i \in P(q)^*$. Lift each c_i to $P(q-1)^*$ (and note that v_{q-1} does not divide them). Then the element $\sum c_i t_i \in P(q-1)^*(X)$ reduces to zero in $P(q)^*(X)$. If it is not zero in $P(q-1)^*(X)$, then it is divisible by v_{q-1} and we can write $0 = \sum c_i t_i + v_{q-1} r'$ where r' can be written in terms of the t_i. Thus we have a relation which reduces to our r.

Proof of part (g). This follows from (f). □

We need a couple of lemmas to prove Theorem 1.21.

Lemma 6.1. *Let X be a space and $q \geq 0$, then $P(q)^*(X)/F^{s+1}$ is coherent.*

Proof. By Lemma 4.4 we have

$$P(q)^*(X)/F^{s+1} \simeq P(q)^*(X^m)/F^{s+1}$$

for some large m except when $q = 0$ and $P(0) = BP_p\hat{}$. $P(q)^*$ is coherent. Since X^m is finite, $P(q)^*(X^m)$ is coherent. Since $P(q)^*(X^m)/F^{s+1}$ is the image of the map

$$P(q)^*(X^m) \longrightarrow P(q)^*(X^s)$$

we see that it is coherent. For the case of $q = 0$ and $P(0) = BP_p\hat{}$ we have to resort to Corollary 3.13. □

Lemma 6.2. *Let X be a space and let $q > 0$. Let $J_2 = (v_{q+1}, v_{q+2}, \dots)$. Let T be a set in $v_q^{-1}P(q)^*(X)$ such that*

(a) *all but a finite number of the elements of T are in F^s, the s filtration for the Atiyah-Hirzebruch spectral sequence and*
(b) *the image of T in $K(q)^*(X)$ generates (topologically).*

Then the image of T generates $v_q^{-1}P(q)^*(X)/(F^{s+1} + J_2^N)$ for all s and all N.

Proof. $P(q)^*(X)/F^{s+1}$ is a finitely presented $P(q)^*(P(q))$-module and, as such, it has a Landweber filtration, Theorem 3.10. When you localize at v_q such a filtration becomes a finitely generated free module over $v_q^{-1}P(q)^*$. It is then easy to see that

$$K(q)^*(X)/F^{s+1} \simeq K(q)^* \otimes_{v_q^{-1}P(q)^*} v_q^{-1}P(q)^*(X)/F^{s+1}.$$

Pick a set of generators x_i for $v_q^{-1}P(q)^*(X)/F^{s+1}$. The image of these x_i must generate $K(q)^*(X)/F^{s+1}$; as does the image of T. Thus, modulo J_2, the x_i must be in the submodule generated by T. We have (finite) sums $x_i = \sum v_q^{s_{i,k}} t_k + \sum c_{i,j} x_j$ where $c_{i,j} \in J_2$. (Note that $s_{i,k} \in \mathbf{Z}$.) Now, to show that the x_i are in the image modulo J_2^2 we just substitute the equations for the x_j into this. Iterate to get the theorem modulo J_2^N. □

Lemma 6.3. *Let $q > n \geq 0$. Let X be a space with $P(n)^*(X)$ Landweber flat. Let*

$$\begin{aligned} J_2 &= (v_{q+1}, v_{q+2}, \dots) \qquad \text{and} \\ J_1 &= (v_n, \dots, v_{q-1}). \end{aligned}$$

Let T_n be a set in $v_q^{-1}P(n)^*(X)$ such that

(a) *all but a finite number of the elements of T_n are in F^s, the s filtration for the Atiyah-Hirzebruch spectral sequence and*
(b) *the image of T_n in $K(q)^*(X)$ generates (topologically).*

Then the image of T_n generates $v_q^{-1}P(n)^*(X)/(F^{s+1} + J_1^N + J_2^N)$ for all s and N.

Proof. From the short exact sequences of Remark 1.10 we see we can localize with respect to v_q to get

$$v_q^{-1}P(q)^*(X) \simeq v_q^{-1}P(q)^* \widehat{\otimes}_{v_q^{-1}P(n)^*} v_q^{-1}P(n)^*(X)$$

Thus, if we have $x, y \in v_q^{-1}P(n)^*(X)$ which reduce to the same element in $v_q^{-1}P(q)^*(X)$, then $x = y + \sum e_i r_i$ where $e_i \in J_1$ and the sum is possibly infinite.

Fix the N and s of the Lemma. Let T be the image of T_n in $v_q^{-1}P(q)^*(X)$. We can pick generators, $\{y_i\}$, for $P(n)^*(X)$ with property (a) above by picking (a finite number of) generators for $P(n)^*(X)/F^{s'+1}$, lifting them to $P(n)^*(X)$ and extending this choice by enlarging s'. See the first part of the proof of Theorem 1.20 above for more detail. Map these generators to a set of generators, $\{x_i\}$, for $v_q^{-1}P(n)^*(X)$. Reduce these elements further to $z_i \in v_q^{-1}P(q)^*(X)$. By Lemma 6.2, we can write z_i, in $v_q^{-1}P(q)^*(X)/(F^{s'+1} + J_2^N)$, in terms of the reduction of T. Taking the limit, we can write each

$$z_i = \sum d_{i,k} t_k + \sum c_{i,j} z_j$$

where $d_{i,k} \in v_q^{-1}P(q)^*$ and $c_{i,j} \in J_2^N$. The two elements, x_i and $\sum d_{i,k} t_k + \sum c_{i,j} x_j$ both reduce to the same element and we see from the above that

$$x_i = \sum d_{i,k} t_k + \sum c_{i,j} x_j + \sum e_{i,m} x_m$$

where $e_{i,m} \in J_1$ and the sums are possibly infinite. Reduce this to $v_q^{-1}P(n)^*(X)/F^{s+1}$ and the sums are now finite. As in the proof of Lemma 6.2, substitute this formula in for the x_i and iterate in order to show that the t's generate modulo $J_1^N + J_2^N$ □

Proof of Theorem 1.21. We prove our theorem by induction on s, i.e. we show that T_n generates $P(n)^*(X)/F^s$. Assume inductively that we have this for s. Now, if we have an $x \in P(n)^*(X)/F^{s+1}$ which is not in the submodule generated by T_n then we will derive a contradiction. Since there are only a finite number of the T_n which are non-zero, the quotient of $P(n)^*(X)/F^{s+1}$ (which is coherent by Lemma 6.1) by the submodule generated by T_n must be coherent, [Smi69]. We will show that is not the case.

Our x must be represented in $E_\infty^{s,*}$. Pick an N such that we can see that our x is not in $J_1^N + J_2^N$ for strictly dimensional reasons. (For the case $n = 0$ we have to modify this a little. Put a weight on p to act as a non-trivial degree so that the previous statement still holds. Otherwise, we can just prove the result for $n > 0$ first and then lift it to $n = 0$ easily afterwards, a choice we can make to avoid the use of Lemma 6.1 in the one case which depends on [BJW95].) Pick N' such that $v_{N'}$ acts freely on $P(n)^*(X)/F^{s+1}$ by Lemmas 4.1 and 4.3. Thus $P(n)^*(X)/F^{s+1}$ injects to $v_q^{-1}P(n)^*(X)/F^{s+1}$ for $q \geq N'$. In the last group, by the previous lemma, we can write x in terms of t's modulo $J_1^N + J_2^N$. (We don't need to do this for every $q \geq N'$, only for an infinite number of such q.) However, we may use negative powers of v_q to do so. Since all sums are finite, we can multiply by some power of v_q, say s_q, so that $v_q^{s_q}x$ is in the image of the submodule generated by T_n modulo $J_1^N + J_2^N$. This is true for all $q \geq N'$. Thus we see that there are an infinite number of relations; one each with a term $v_q^{s_q}x$ in it, $q > N'$. Thus it is not coherent and we have our contradiction. x must therefore be in the submodule generated by T_n. □

Proof of Corollary 1.23. This is immediate. □

Proof of Corollary 1.24. If for some $t \in T_n$, t goes to zero in $K(q)^*(X)$ for $q \geq N$ for some large N, then t is not essential to generate $P(N)^*(X)$ by Theorem 1.21. However, Theorem 1.20 says the reduction of T_n to T_q retains property (b) of Theorem 1.20. Contradiction. □

Remark 6.4. For some of the most interesting examples which we "understand" completely, all of the generators reduce to mod p cohomology where they are still independent. This is the case for QS^{2k} and Eilenberg-MacLane spaces and it probably contributes a great deal to our being able to understand them. This is not always the case though. When all generators are of this sort, then they never change filtration when we map from the spectral sequence for $P(k)^*(X)$ to that for $P(n)^*(X)$, $n > k$. The filtration can change when we map from $P(k)^*(X)$ to $K(n)^*(X)$ though. Generators that do not map to mod p cohomology must behave quite differently. They must change filtration when we map from the spectral sequence for $P(k)^*(X)$ to that for $P(n)^*(X)$ if n is large enough because the location in the spectral sequence, $x \in E_2^{s,t}$, $t < 0$, is zero when $t > -2(p^n - 1)$. So, as n grows, the filtration of such a generator must keep changing and it never shows up in mod p cohomology. An example of this behavior was pointed out to us by Takuji Kashiwabara. The example is $BSO(4)$ which was computed in [KY93, Theorem 5.5].

7. Exactness

Once again we want properties of Morava K-theories to imply similar properties for Brown-Peterson cohomology. We have four theorems to prove in this section: one for surjectivity, one for injectivity, and two for the exactness that we need in our applications. Although we state our theorems with the assumption of even Morava K-theory and injectivity or surjectivity for all of the Morava K-theories, we can get by with only assuming these things for an infinite number of the Morava K-theories. The proofs are unchanged. The statements of the theorems are much cleaner this way and there are no examples that need our greater generality.

In this section we give the proofs for Theorems 1.17, 1.18, and 1.19 from the introduction.

Proof of surjectivity in Theorem 1.17. By Theorem 1.20 we can pick a set T_0 which generates $P(0)^*(X_2)$. We know that it reduces to generators for each $P(n)^*(X_2)$ and thus also for all $K(n)^*(X_2)$. Map these generators to $P(n)^*(X_1)$. By naturality and the fact that the Morava K-theories surject, we have that the image of T_0 in the $P(n)^*(X_1)$ satisfies the conditions of Theorem 1.21 and so we see that the image generates. □

To prove the theorem on injectivity we need a lemma.

Lemma 7.1. *Let X_1 and X_2 be spaces with even Morava K-theory. Let $f : X_1 \longrightarrow X_2$. If $f^* : K(n)^*(X_2) \longrightarrow K(n)^*(X_1)$ is injective, $n > 0$, then so is $f^* : E(k,n)^*(X_2) \longrightarrow E(k,n)^*(X_1)$, $0 \leq k \leq n$.*

Proof. The proof is by downward induction on k. By Lemma 5.1 we have short exact sequences:

$$0 \longrightarrow E(k,n)^*(X_i) \xrightarrow{v_k} E(k,n)^*(X_i) \longrightarrow E(k+1,n)^*(X_i) \longrightarrow 0.$$

Given $0 \neq x \in E(k,n)^*(X_2)$, we know it cannot be infinitely divisible by v_k by Corollary 4.11. Find a $y \in E(k,n)^*(X_2)$ and a j such that $x = v_k^j y$ and y maps non-trivially to $E(k+1,n)^*(X_2)$. By our induction, $E(k+1,n)^*(X_2)$ injects to $E(k+1,n)^*(X_1)$ so y must map non-trivially to $E(k,n)^*(X_1)$. Since this group has no v_k torsion, $x = v_k^j y$ must map non-trivially. □

Proof of injectivity in Theorem 1.17. For $k \geq 0$ and $x \in P(k)^*(X_2)$ we use Proposition 4.12 to see that x maps non-trivially to some $E(k,n)^*(X_2)$. By the injectivity of $K(n)^*(-)$ and Lemma 7.1 we have that this group injects to $E(k,n)^*(X_1)$. By the naturality of maps between all of the cohomology theories involved, we must have x mapping non-trivially to $P(k)^*(X_1)$. □

Proof of Theorem 1.18. Take the cofibre $X_1 \stackrel{f_1}{\to} X_2 \stackrel{r}{\to} C(f_1)$. This gives rise to a long exact sequence in any cohomology theory. By Theorem 1.17 we have the surjectivity of f_1^*. By this surjectivity of f_1^*, we have a short exact sequence

$$0 \leftarrow G^*(X_1) \stackrel{f_1^*}{\leftarrow} G^*(X_2) \stackrel{r^*}{\leftarrow} G^*(C(f_1)) \leftarrow 0.$$

for all $G = P(k)$. Since $f_2 \circ f_1 \simeq 0$, f_2 factors through $C(f_1)$. By the assumption of exactness for all $K(n)$ the map $C(f_1) \leftarrow X_3$ is surjective for all of the Morava K-theories. Thus, by Theorem 1.17 we have surjectivity for the $P(k)$. We then just patch up our surjectivity with the short exact sequence to get the result. □

Proof of Theorem 1.19. For bicommutative Hopf algebras we have that the cokernel can be constructed using the tensor product and the kernel from the cotensor product, see [HRW97, Section 4] or, better yet, [Bou96, Appendix, especially Theorem 10.12], so that although this is a theorem about Hopf algebras, algebras play the main role here. We want to reduce this theorem to Theorem 1.18. Define a map F by $X_2 \to (X_2 \times X_3)/(X_2 \times *)$ by

$$X_2 \xrightarrow{\text{diag}} X_2 \times X_2 \xrightarrow{(I,f_2)} X_2 \times X_3 \to (X_2 \times X_3)/(X_2 \times *).$$

We have $F \circ f_1 \simeq 0$. Our exact sequence of Hopf algebras implies an exact sequence of $K(n)_*$ modules (this is from the cotensor product model for the kernel):

$$0 \to K(n)_*(X_1) \to K(n)_*(X_2) \xrightarrow{F_*} K(n)_*((X_2 \times X_3)/(X_2 \times *))$$

which dualizes to (the tensor product model for cokernel):

$$0 \leftarrow K(n)^*(X_1) \leftarrow K(n)^*(X_2) \xleftarrow{F^*} K(n)^*((X_2 \times X_3)/(X_2 \times *)).$$

The $\lim^1$ condition of Theorem 1.18 is satisfied for the product by Landweber, [Lan70a, Lemma 6], because X_2 and X_3 both satisfy the condition. By Theorem 1.18 we now have an exact sequence:

$$0 \leftarrow P(n)^*(X_1) \leftarrow P(n)^*(X_2) \xleftarrow{F^*} P(n)^*((X_2 \times X_3)/(X_2 \times *)).$$

Let $I(-)$ be the augmentation ideal. Then

$$P(n)^*(X_2)\hat{\otimes}I(P(n)^*(X_3))$$

maps to the last module. We claim that this map is surjective. To see this, pick sequences of generators, $\{t_i\}$ and $\{s_i\}$ for $P(n)^*(X_2)$ and $I(P(n)^*(X_3))$ respectively. The elements $\{t_i \otimes s_j\}$ map to generators of $K(n)^*((X_2 \times X_3)/(X_2 \times *))$ because $K(n)^*(-)$ has a Künneth isomorphism (and our X_2 and X_3 are very nice spaces of the sort we are studying). Mapping these elements over to $P(n)^*((X_2 \times X_3)/(X_2 \times *))$ we see that since they generate all of the Morava K-theories then they must, by Theorem 1.21, generate. Since the tensor product maps onto generators, it must be surjective. The result follows. □

8. Eilenberg-Mac Lane spaces

In this section we give a purely algebraic construction for the $P(0)^*$ algebra which is isomorphic to the $P(0)$ cohomology of an Eilenberg-Mac Lane space, and then, of course, we go on to show the isomorphism of Theorem 1.14.

8.1. **Preliminaries.** From [RW77] we have a completely algebraic construction for the Hopf ring $E_*(\underline{BP}_*)$ whenever E is a complex orientable generalized homology theory. Because the answer is a free E_* module we have duality and have also given a construction for $E^*(\underline{BP}_*)$. In particular, we can use $K(n)$, BP, and $BP_p\hat{}$ for E. The nice properties all come from the fact that $H_*(\underline{BP}_*; \mathbf{Z}_{(p)})$ has no torsion, [Wil73]. For the evenly indexed spaces this is even dimensional and is a bipolynomial Hopf algebra (i.e. both it and its dual are polynomial algebras) and for the odd spaces it is an exterior algebra. The same is true for the cohomology. These properties lift to $E_*(\underline{BP}_*)$, and, by duality, give completed exterior algebras (for odd spaces) and power series algebras (for evenly indexed spaces) for cohomology.

We really need information about the $\underline{BP\langle q\rangle}_*$ and we can derive it from the above using:

Theorem 8.1.1 ([Wil75])**.** *Let $g(q) = 2(p^{q+1}-1)/(p-1)$. For $k \le g(q)$, the standard maps, $\underline{BP}_k \to \underline{BP\langle q\rangle}_k$ and $\underline{BP\langle q+1\rangle}_k \to \underline{BP\langle q\rangle}_k$, split. For $k < g(q)$ this splitting is as H-spaces. The second splitting splits the fibration:*

$$\underline{BP\langle q+1\rangle}_{k+2(p^{q+1}-1)} \xrightarrow{v_{q+1}} \underline{BP\langle q+1\rangle}_k \longrightarrow \underline{BP\langle q\rangle}_k$$

to give a homotopy equivalence:

$$\underline{BP\langle q+1\rangle}_k \simeq \underline{BP\langle q\rangle}_k \times \underline{BP\langle q+1\rangle}_{k+2(p^{q+1}-1)}.$$

It now follows that $K(n)_*(\underline{BP\langle q\rangle}_{2k})$ is a polynomial algebra for $2k < g(q)$ and is even dimensional for $2k = g(q)$. This is also true for $BP_*(-)$.

For our computations we need the bar spectral sequence (see [RW80, pages 704-5] and [HRW97, Section 2]). In our cases all of our maps are of infinite loop spaces and we only need it for Morava K-theory.

Theorem 8.1.2 (Bar spectral sequence)**.** *Let $F \to E \to B$ be a fibration of infinite loop spaces, then we have a spectral sequence of Hopf algebras, converging to $K(n)_*(B)$, with E^2 term:*

$$\mathrm{Tor}^{K(n)_*(F)}(K(n)_*(E), K(n)_*).$$

Next we need to know how this behaves in a special case that has already been computed. The following was proved in [RW80, Theorem 12.3, p. 743].

Theorem 8.1.3. *For the path space fibration:*

$$K(\mathbf{Z}_{(p)}, q+1) \to PK(\mathbf{Z}_{(p)}, q+1) \to K(\mathbf{Z}_{(p)}, q+2)$$

the bar spectral sequence for $K(n)_*(-)$ is even dimensional and collapses.

Looking at the statement of Theorem 1.14 we see that for each type of Eilenberg-Mac Lane space we really have two statements. For them to both be true we must have that the ideals (v_q^*) and $(v_1^*, v_2^*, \ldots, v_q^*)$ are equal. We will prove the theorem for the ideal (v_q^*). Since this is contained in the "bigger" ideal, it is enough to show that our map of

$$P(0)^*(K(\mathbf{Z}_{(p)}, q+2)) \leftarrow P(0)^*(\underline{BP\langle q\rangle}_{g(q)})/(v_q^*)$$

factors through

$$P(0)^*(\underline{BP\langle q\rangle}_{g(q)})/(v_1^*, v_2^*, \ldots, v_q^*).$$

The stable cofibration sequence 1.13 is one of BP module spectra, [JW73]. Thus, all of the boundary maps used to define our map

$$K(\mathbf{Z}_{(p)}, q+2) \to \underline{BP\langle q\rangle}_{g(q)}$$

commute with multiplication by v_j. Since the map

$$K(\mathbf{Z}_{(p)}, q+2) \xrightarrow{v_j} K(\mathbf{Z}_{(p)}, q+2-2(p^j-1))$$

is homotopically trivial, we have what we need.

We admit that the equality of the two ideals was quite a surprise to us which we did not notice until late in the game.

8.2. **Construction.** In [RW77], a completely algebraic construction for the Hopf ring, $E_*(\underline{BP}_*)$ is given. By the splitting above, we know that the algebraic construction of $E_*(\underline{BP}_k)$ maps surjectively to $E_*(\underline{BP\langle q\rangle}_k)$ for $k \leq g(q)$ and that it factors through the quotient given by setting all $[v_i] = [0_{-2(p^i-1)}]$ for $i > q$ (see [RW77]). There is a minor concern that maybe there could be some other relation in order to get injectivity. However, this is not the case. Note in the lemma that when we mod out by $I(q)$, we are setting elements in it equal to the $[0_i]$, not 0, although because everything is in positive degrees it is the same.

Lemma 8.2.1. *Let $E_*^R(\underline{BP}_*)$ be the algebraic construction for $E_*(\underline{BP}_*)$ from* [RW77]*. If we mod out by $I(q) = ([v_{q+1}], [v_{q+2}], \dots)$ we have $E_*^R(\underline{BP}_k)/I(q) \simeq E_*(\underline{BP\langle q\rangle}_k)$ for $0 < k \leq g(q)$.*

We should point out that neither the statement nor proof of Theorem 1.14 depends on this lemma. The theorem is given strictly in terms of spaces and we do need the splitting 8.1.1. The attraction of the theorem to us is this lemma because it gives us a purely algebraic construction for everything in the theorem.

Proof. The map $E_*(\underline{BP}_i) \xrightarrow{[v_q]} E_*(\underline{BP}_{i-2(p^q-1)})$ is just the induced algebraic map coming from multiplication by v_q. Each of the spectra $BP\langle q\rangle$ is a BP module spectra ([JW73]) so the maps between spectra commute with the maps of v_q. We prove our lemma with a multiple induction. It is enough to prove our lemma for mod p homology because all of our spaces are torsion free and everything is therefore E_* free. Our main induction is on $j-k$ in $H_j(\underline{BP\langle s\rangle}_k)$. Our second induction is downward induction on s. To ground our first induction, there is nothing to prove if $j = k$ (and $k \leq g(s)$). To ground our second induction, we see that $H_j(\underline{BP}_i) \simeq H_j(\underline{BP\langle s\rangle}_i)$ for $j - i < 2(p^{s+1}-1)$ because these spaces are homotopy equivalent in this range. For a fixed $j-k$ we must pick s such that $j-k < 2(p^{s+1}-1)$. Then we can start the second induction to prove our result for this degree. To do the second induction we need only observe that the split fibration in Theorem 8.1.1 must give rise to a short exact sequence of Hopf algebras where the first map is just $[v_{q+1}]\circ$ multiplication. □

One can go further with this and write $E^*(\underline{BP\langle q\rangle}_{2k})$ as a power series ring on generators dual to the primitives $E_*(\underline{BP\langle q\rangle}_{2k})$ for $2k \leq g(q)$, which can be written down directly from [RW77] as was done in [Sin76]. We will discuss this more after the proof.

In principle, [RW77] tells you how to compute the map

$$BP_*(\underline{BP\langle q\rangle}_{g(q)}) \xrightarrow{[v_q]\circ} BP_*(\underline{BP\langle q\rangle}_{g(q)-2(p^q-1)}).$$

What that really means is that you can train a computer to do it, because, in practice, it is very difficult although complete information is available. Because everything is BP_* free you can take the duals and the dual map and again, everything is, in principle, computable. It is certain that

$$BP^*(\underline{BP\langle q\rangle}_{g(q)})/(v_q^*)$$

is a well-defined algebraic construct, as is its p-adic completion. Likewise for

$$BP^*(\underline{BP\langle q\rangle}_{g(q)})/(p^{i^*}, v_q^*)$$

which comes from the product map (p^i, v_q)

$$\underline{BP\langle q\rangle}_{g(q)} \to \underline{BP\langle q\rangle}_{g(q)} \times \underline{BP\langle q\rangle}_{g(q)-2(p^q-1)}.$$

8.3. **Proof for $K(\mathbf{Z}_{(p)}, q+2)$.** We can now prove Theorem 1.14 for the integral spaces. It is known that $BP^*(K(\mathbf{Z}_{(p)}, q))$, $q > 2$, has phantom maps. This follows from [AH68] where they show this for complex K-theory and [Lan72] where it is shown that the situation for complex cobordism is the same as that for complex K-theory. We must work, therefore, with $P(0) = BP_p\hat{}$. In fact, because of Theorem 1.19, all we must prove is Proposition 1.16. We just let $K(\mathbf{Z}_{(p)}, q+2) = X_1$, $\underline{BP\langle q\rangle}_{g(q)} = X_2$, f_1 the iterated boundary map given in the introduction, $\underline{BP\langle q\rangle}_{g(q)-2(p^q-1)} = X_3$ and f_2 the map coming from v_q. Observe that the composition of the two maps is indeed null homotopic because the first map factors through the boundary map

$$\underline{BP\langle q-1\rangle}_{g(q-1)+1} \longrightarrow \underline{BP\langle q\rangle}_{g(q)}$$

which is just the inclusion of the fibre of the map f_2. Note that this is an example where the spaces do not form a fibration.

The proof breaks up into two pieces. First, we need to show injectivity of the map

$$K(n)_*(K(\mathbf{Z}_{(p)}, q+2)) \longrightarrow K(n)_*(\underline{BP\langle q\rangle}_{g(q)}),$$

which we do using the Steenrod algebra and a bit of Hopf algebra machinery. Second we need to show that the cokernel of this map injects to the third Hopf algebra:

$$K(n)_*(\underline{BP\langle q\rangle}_{g(q)})/\!\!/K(n)_*(K(\mathbf{Z}_{(p)}, q+2)) \longrightarrow K(n)_*(\underline{BP\langle q\rangle}_{g(q)-2(p^q-1)}).$$

We will do this using the bar spectral sequence a few times.

8.3.1. *Proof of injectivity.* This is going to reduce to a calculation over the Steenrod algebra. With apologies to the reader, understanding this proof will also require an intimacy with the Morava K-theory of Eilenberg-Mac Lane spaces from [RW80]. From that paper we know [RW80, Corollary 12.2, p. 742] that

$$\varinjlim_i K(n)_*(K(\mathbf{Z}/(p^i), q+1)) \simeq K(n)_*(K(\mathbf{Z}_{(p)}, q+2)).$$

Furthermore, we know ([RW80, Theorem 11.1(b), p. 734]) that the very first space in this limit, $K(\mathbf{Z}/(p), q+1)$, picks up all of the Hopf algebra primitives for $K(n)_*(K(\mathbf{Z}_{(p)}, q+2))$. To get our injection for this last space we want to just show that the primitives

$$PK(n)_*(K(\mathbf{Z}/(p), q+1)),$$

and thus also for $K(n)_*(K(\mathbf{Z}_{(p)}, q+2))$, inject to those for $K(n)_*(\underline{BP\langle q\rangle}_{g(q)})$. An injection on primitives automatically gives an injection on $K(n)_*(K(\mathbf{Z}_{(p)}, q+2))$, see, for example, [HRW97, Lemma 4.2].

This calculation is probably contained in H. Tamanoi's Master's Thesis, [Tam83b], and should have been deduced by us from [Yag86]. It is certainly contained in [Tam97]. Those proofs are in cohomology and we work in homology but the results are the same. H. Tamanoi computes the image of the map:

$$BP^*(K(\mathbf{Z}_{(p)}, q+2)) \to H^*(K(\mathbf{Z}_{(p)}, q+2), \mathbf{Z}/(p)).$$

In the proof one sees that these elements all come from $BP^*(\underline{BP\langle q\rangle}_{g(q)})$. As it turns out, these elements generate. At any rate, it is Tamanoi who first made the connection between $BP^*(K(\mathbf{Z}_{(p)}, q+2))$ and $BP^*(\underline{BP\langle q\rangle}_{g(q)})$ more than ten years ago!

We now assume a working knowledge of [RW80]. Let $\mathcal{A}_*$ be the dual of the Steenrod algebra. We have the usual map from $H_*(K(\mathbf{Z}/(p), q+1), \mathbf{Z}/(p))$ to $\mathcal{A}_*$ which is an $\mathcal{A}_*$ comodule map. $K(n)_*(K(\mathbf{Z}/(p), 1))$ has elements $a_{(i)}$, $0 \leq i < n$, in degrees $2p^i$ which are represented in the Atiyah-Hirzebruch spectral sequence by the elements used to define the τ_i of the Steenrod algebra ([RW80, Theorem 5.7]). Under the usual map

$$\bigotimes^{q+1} K(n)_*(K(\mathbf{Z}/(p), 1)) \longrightarrow K(n)_*(K(\mathbf{Z}/(p), q+1))$$

all elements

$$a_{(i_0)} \circ a_{(i_1)} \circ \cdots \circ a_{(i_q)}$$

with $0 \leq i_0 < i_1 < i_2 \cdots < i_q < n$ are nontrivial ([RW80, Theorem 9.2]). They are therefore represented by elements which map to

$$\tau_{i_0} \tau_{i_1} \tau_{i_2} \cdots \tau_{i_q}$$

in the Steenrod algebra. The elements which are primitive are those with $i_0 = 0$, ([RW80, Theorem 9.2]). It is not important, but note that there are only a finite number of these elements. Define a subvector space, $E(q, n)$, of $\mathcal{A}_*$ with basis

$$\tau_{i_1} \tau_{i_2} \cdots \tau_{i_j}$$

with $0 \leq i_1 < i_2 \cdots < i_j < n$ with $j \leq q$ in $\mathcal{A}_*$. This is clearly a subcomodule of $\mathcal{A}_*$ over $\mathcal{A}_*$ and we can take its quotient, $\mathcal{A}_*/E(q, n)$ which is now a comodule over $\mathcal{A}_*$. Note that the above set of elements of $H_*(K(\mathbf{Z}/(p), q+1), \mathbf{Z}/(p))$ which survive to primitives in the Atiyah-Hirzebruch spectral sequence maps injectively to a subcomodule (over $\mathcal{A}_*$) of $\mathcal{A}_*/E(q, n)$; call it $E(q)$. We have our map, $K(\mathbf{Z}/(p), q+1) \to \underline{BP\langle q\rangle}_{g(q)}$ which induces a map of $\mathcal{A}_*$ comodules in mod p homology. All of the elements in $H_*(\underline{BP\langle q\rangle}_{g(q)}, \mathbf{Z}/(p))$ survive in the Atiyah-Hirzebruch spectral sequence to the Morava K-theory because the space has no torsion. Thus, it is enough to show that our elements which represent primitives map nontrivially and independently to $H_*(\underline{BP\langle q\rangle}_{g(q)}, \mathbf{Z}/(p))$. In cohomology, the iterated boundary map,

$$K(\mathbf{Z}/(p), q+1) \to \underline{BP\langle q\rangle}_{g(q)},$$

takes the fundamental class in $H^*(\underline{BP\langle q\rangle}_{g(q)}, \mathbf{Z}/(p))$ to $Q_0 Q_1 \cdots Q_q$ times the fundamental class in $H^*(K(\mathbf{Z}/(p), q+1), \mathbf{Z}/(p))$, see [Wil75]. This tells us two things we need to know. First, it says our map is trivial on $E(q, n)$ because Q_i is dual to τ_i and there are q or fewer τ but $q+1$ Q. So, we get a map of $\mathcal{A}_*$ comodules, $E(q) \to H_*(\underline{BP\langle q\rangle}_{g(q)}, \mathbf{Z}/(p))$. Second, it says our map is non-zero on the lowest dimensional element

$$\tau_0 \tau_1 \cdots \tau_q.$$

All we have to do now is show that this element forces an injection of $E(q)$.

Recall that the coproduct on τ_i is

$$\tau_i \otimes 1 + \sum_{0 \le j \le i} \xi_{i-j}^{p^j} \otimes \tau_j.$$

We can ignore the first term in computing the comodule expansion on τ_I because it will lead to a product of the τ on the right in $E(q, n)$. Because τ_I always has τ_0 in it (recall these are the primitives), there is only one term we can use from its coproduct, $1 \otimes \tau_0$. Recall also that $\tau_j^2 = 0$, so we cannot use the τ_0 term of any of the other τ's. Thus, modulo $E(q, n)$ we have

$$\psi(\tau_I) = \sum_{0 < J \le I} \xi_{I-J}^{p^J} \otimes \tau_0 \tau_J,$$

where the sum is over all q-tuples $J = (j_1, \cdots j_q)$ with $0 < j_t \le i_t$ and $\xi_{I-J}^{p^J}$ denotes the product $\xi_{i_1-j_1}^{p^{j_1}} \cdots \xi_{i_q-j_q}^{p^{j_q}}$. It is understood that τ_J vanishes if the j_t are not all distinct, but the j_t need not increase with t as the i_t are required to do. We know that the comodule expansion on τ_I contains the term

$$\xi_{i_1-1}^{p^1} \xi_{i_2-2}^{p^2} \cdots \xi_{i_q-q}^{p^q} \otimes \tau_0 \tau_1 \tau_2 \cdots \tau_q.$$

associated with $J = (1, 2, \ldots, q)$. In order to show that τ_I maps non-trivially, we need to analyze the comodule expansion more carefully to be sure that these terms are not cancelled out by J' which are permutations of J. The subscripts of the ξs in this special term are in nondecreasing order, while their exponents are strictly increasing. We claim that no other term for such J' has this property. To see this, let $J' = (\sigma^{-1}(1), \cdots \sigma^{-1}(q))$ for σ a permutation of q letters. Then the J'-th term can be rewritten as

$$(-1)^\sigma \xi_{i_{\sigma(1)}-1}^{p^1} \xi_{i_{\sigma(2)}-2}^{p^2} \cdots \xi_{i_{\sigma(q)}-q}^{p^q} \otimes \tau_J$$

It will agree (up to sign) with the J-th term only if

$$i_{\sigma(t)} - t = i_t - t \qquad \text{for } 1 \le t \le q.$$

Since the i_t are distinct, this means that the permutation σ must be the identity.

This shows that our map is nontrivial on τ_I for all I. Since they lie in different dimensions (for different I), they are all linearly independent as well. This concludes our proof of injectivity.

8.3.2. *Proof that the cokernel injects.* Our proof will be by induction on q. Let us assume our exact sequence for $q-1$ and furthermore let us assume, inductively, that as algebras

$$K(n)_*(\underline{BP\langle q-1\rangle}_{g(q-1)}) \simeq K(n)_*(K(\mathbf{Z}_{(p)}, q+1)) \otimes PA_{q-1}$$

where PA_{q-1} is a polynomial algebra.

This induction is trivial to ground; just use

$$K(n)_*(\underline{BP\langle 0\rangle}_2) = K(n)_*(K(\mathbf{Z}_{(p)}, 2))$$

which we know because we know $K(n)_*(CP^\infty)$. Here the polynomial part is vacuous.

The first step in our induction is to compute $K(n)_*(\underline{BP\langle q-1\rangle}_{g(q-1)+1})$. To do this we use the bar spectral sequence 8.1.2 with E^2 term:

$$\mathrm{Tor}^{K(n)_*(\underline{BP\langle q-1\rangle}_{g(q-1)})}(K(n)_*, K(n)_*).$$

By induction and the Künneth isomorphism, we see that this breaks into two parts:

$$\mathrm{Tor}^{K(n)_*(K(\mathbf{Z}_{(p)},q+1))}(K(n)_*,K(n)_*)\otimes \mathrm{Tor}^{PA_{q-1}}(K(n)_*,K(n)_*)$$

where

$$\mathrm{Tor}^{PA_{q-1}}(K(n)_*,K(n)_*)\simeq EA_{q-1}$$

where EA_{q-1} is an exterior algebra on the homology suspension of the generators of the polynomial algebra PA_{q-1}. These generators lie in the first filtration in the spectral sequence so all differentials on them are trivial. We also know how to compute

$$\mathrm{Tor}^{K(n)_*(K(\mathbf{Z}_{(p)},q+1))}(K(n)_*,K(n)_*)$$

from Theorem 8.1.3. Furthermore, we have maps of fibrations:

$$\begin{array}{ccccc}
K(\mathbf{Z}_{(p)},q+1) & \longrightarrow & \mathrm{pt.} & \longrightarrow & K(\mathbf{Z}_{(p)},q+2)\\
\downarrow & & \| & & \downarrow\\
\underline{BP\langle q-1\rangle}_{g(q-1)} & \longrightarrow & \mathrm{pt.} & \longrightarrow & \underline{BP\langle q-1\rangle}_{g(q-1)+1}.
\end{array}$$

By naturality, we have no differentials on this part of the bar spectral sequence we are using to compute $K(n)_*(\underline{BP\langle q-1\rangle}_{g(q-1)+1})$. Since there can be no differentials on the exterior part, we see that the spectral sequence collapses and, as algebras, we have:

$$K(n)_*(\underline{BP\langle q-1\rangle}_{g(q-1)+1})\simeq K(n)_*(K(\mathbf{Z}_{(p)},q+2))\otimes EA_{q-1}.$$

All of the algebra extension problems in the $K(n)_*(K(\mathbf{Z}_{(p)},q+2))$ part have been solved by naturality. In case there is any question about this algebra splitting as a tensor product recall that EA_{q-1} is a free commutative algebra (if our prime is odd) on odd degree elements. We certainly have a short exact sequence with EA_{q-1} the quotient. Because it is free we can split it. If $p=2$ we must observe that the generators of PA_{q-1} come from $BP_*(\underline{BP}_{g(q-1)})$ and thus, so do the generators of EA_{q-1} come from $BP_*(\underline{BP}_{g(q-1)+1})$. Since they are exterior generators in BP there can be no extension problems where we are working.

That ends the proof of the first step of the induction and we can move on to the next (and final) step. We will study the bar spectral sequence for the fibration:

$$\underline{BP\langle q-1\rangle}_{g(q-1)+1}\to \underline{BP\langle q\rangle}_{g(q)}\to \underline{BP\langle q\rangle}_{g(q)-2(p^q-1)}.$$

We know quite a lot about things already.

(i) We know that
$$K(n)_*(\underline{BP\langle q\rangle}_{g(q)-2(p^q-1)})$$
is a polynomial algebra.

(ii) We know that
$$K(n)_*(\underline{BP\langle q\rangle}_{g(q)})$$
is even dimensional.

(iii) We have just "computed"
$$K(n)_*(\underline{BP\langle q-1\rangle}_{g(q-1)+1}).$$

(iv) We know that we have the injection part of our desired exact sequence

$$K(n)_* \to K(n)_*(K(\mathbf{Z}_{(p)}, q+2)) \to K(n)_*(\underline{BP\langle q\rangle}_{\,g(q)}).$$

Because of (ii) we see that the map

$$K(n)_*(\underline{BP\langle q-1\rangle}_{\,g(q-1)+1}) \to K(n)_*(\underline{BP\langle q\rangle}_{\,g(q)})$$

must take EA_{q-1} to zero. All of this allows us to simplify our computation of the E^2 term of the bar spectral sequence converging to

$$K(n)_*(\underline{BP\langle q\rangle}_{\,g(q)-2(p^q-1)}).$$

The E^2 term starts out as

$$\mathrm{Tor}^{K(n)_*(\underline{BP\langle q-1\rangle}_{\,g(q-1)+1})}(K(n)_*(\underline{BP\langle q\rangle}_{\,g(q)}), K(n)_*)$$

and simplifies, by [Smi70, Theorem 2.4, p. 67], to

$$K(n)_*(\underline{BP\langle q\rangle}_{\,g(q)})//K(n)_*(K(\mathbf{Z}_{(p)}, q+2)) \otimes \mathrm{Tor}^{EA_{q-1}}(K(n)_*, K(n)_*)$$

where the Tor is just a divided power Hopf algebra. In particular, it is even dimensional, as is the first part; thus this spectral sequence collapses.

We can now just read off our answers. The quotient Hopf algebra is just the coker which we wanted to inject into $K(n)_*(\underline{BP\langle q\rangle}_{\,g(q)-2(p^q-1)})$ and the map is just the edge homomorphism. This gives us the desired injection. However, to complete our induction we must show that this cokernel is polynomial. It is a sub-Hopf algebra of a polynomial Hopf algebra and so it must be polynomial as well (this follows immediately from [Bou, Theorem B.7]). Now we have a short exact sequence of Hopf algebras

$$K(n)_*(K(\mathbf{Z}_{(p)}, q+2)) \to K(n)_*(\underline{BP\langle q\rangle}_{\,g(q)}) \to PA_q.$$

Because PA_q is free we see that this splits as algebras and we have completed our induction ([Bou, Proposition B.9]). We thank S. Halperin, J. Moore and F. Peterson for help solving the above Hopf algebra problems before we found the paper by Bousfield.

8.4. **Proof for $K(\mathbf{Z}/(p^i), q+1)$.** This proof is only a slight modification of the previous proof. Our sequence of spaces is now:

$$K(\mathbf{Z}/(p^i), q+1) \longrightarrow \underline{BP\langle q\rangle}_{\,g(q)} \xrightarrow{(p^i, v_q)} \underline{BP\langle q\rangle}_{\,g(q)} \times \underline{BP\langle q\rangle}_{\,g(q)-2(p^q-1)}$$

so we need an exact sequence of Hopf algebras:

$$\begin{array}{c} K(n)_* \\ \downarrow \\ K(n)_*(K(\mathbf{Z}/(p^i), q+1)) \\ \downarrow \\ K(n)_*(\underline{BP\langle q\rangle}_{\,g(q)}) \\ {\scriptstyle (p^i_*, [v_q])}\downarrow \\ K(n)_*(\underline{BP\langle q\rangle}_{\,g(q)}) \otimes K(n)_*(\underline{BP\langle q\rangle}_{\,g(q)-2(p^q-1)}). \end{array}$$

The last tensor product is just the product in the category of Hopf algebras and we have already computed the kernel of the map $[v_q]$ to the right side. It was just $K(n)_*(K(\mathbf{Z}_{(p)}, q+2))$. All we have to do now is worry about the kernel of the map p^i_* restricted to this part. From [RW80, Corollary 13.1, p. 745], we have an extension of Hopf algebras which solves that problem:

$$\begin{array}{c} K(n)_* \\ \downarrow \\ K(n)_*(K(\mathbf{Z}/(p^i), q+1)) \\ \downarrow \\ K(n)_*(K(\mathbf{Z}_{(p)}, q+2)) \\ {\scriptstyle p^i_*}\downarrow \\ K(n)_*(K(\mathbf{Z}_{(p)}, q+2)), \end{array}$$

and we are almost done with this case. We want to use BP as opposed to $BP\hat{_p}$ in this case. Our assumptions in Theorem 1.19 require us to have $\lim^1 BP^*(X^m) = 0$ for all spaces involved. We have this for the Eilenberg-Mac Lane space by Remark 5.7. Because the other spaces have no torsion we know that the Atiyah-Hirzebruch spectral sequence collapses and the lim^1 for them is zero as well.

8.5. **Generators and relations.** If one really wants to use the construction given above to describe

$$BP\hat{_p}^*(\mathbf{Z}_{(p)}, q+2)$$

there are some simplifications which an intimacy with [RW77] can give you quite quickly but which the reader has been spared the necessity of knowing so far. We will just briefly describe here what can be done.

Both $BP^*(\underline{BP\langle q\rangle}_{g(q)})$ and $BP^*(\underline{BP\langle q\rangle}_{g(g)-2(p^q-1)})$ are power series rings on generators dual to the primitives in the BP homology. In [RW77], a basis for the primitives is written down explicitly and one can see that most of them are mapped to basis elements for primitives in the second space. The consequence in the dual is that we do not need to worry about those primitives at all. The remaining primitives for $BP_*(\underline{BP\langle q\rangle}_{g(q)})$ are identified in [RW77] as

$$b_{(0)} \circ b^{\circ p}_{(j_1)} \circ b^{\circ p^2}_{(j_2)} \circ \cdots \circ b^{\circ p^q}_{(j_q)}$$

where $0 \leq j_1 \leq \cdots \leq j_q$. This element is in degree $2(1+p^{j_1+1}+p^{j_2+2}+\cdots+p^{j_q+q})$. Note that as q goes up, the degree of the generators goes up much faster. (Recall that the Brown-Peterson cohomology of the Eilenberg-Mac Lane spectra is trivial.) When reduced to $K(n)_*(\underline{BP\langle q\rangle}_{g(q)})$ we can see that each of these elements is in the image of $K(n)_*(K(\mathbf{Z}_{(p)}, q+2))$ for some n large enough. If we take a bunch of dual generators, say c^J, we can see that $BP\hat{_p}^*(K(\mathbf{Z}_{(p)}, q+2))$ is a quotient of the power series algebra on the c^J. To see what the relations would be requires a good deal more work. For a slight check on reality, there is only one case here that is degenerate enough to be familiar. Let $q = 0$ and we are talking about $BP\hat{_p}^*(K(\mathbf{Z}_{(p)}, 2))$ and there is only one generator in degree 2. It may or may not be an interesting exercise to try to say more about the relations. These generators are those found by Tamanoi. He just did not know that he had found them all.

Because of the splitting of Theorem 8.1.1 the map

$$K(\mathbf{Z}/(p^i), q+1) \longrightarrow \underline{BP\langle q\rangle}_{g(q)}$$

is really a map to $\underline{BP}_{g(q)}$. Tamanoi ([Tam97]) calls this the fundamental class and from the above it is easy to see that using stable BP operations, the algebra structure, and topological completion, this class generates everything in $BP^*(K(\mathbf{Z}/(p^i), q+1))$.

9. The Künneth isomorphism

This section is dedicated to the proof of Theorem 1.11, which turns out to be much more involved and much more general than we expected. We thank Michael Boardman, Dan Christensen, Michael Mandell, Peter May, Jean-Pierre Meyer, and Hal Sadofsky for some help with our general education about limits. We are also indebted to the paper by Jan-Erik Roos, [Roo61].

Although the result could be proven directly just for this situation, we prefer to bring to light some very nice mathematics which we were previously unaware of. It also makes our proof shorter. The main algebraic result we need is:

Theorem 9.1 (Roos, Theorem 2, [Roo61])**.** *Let $A = \{A_\alpha\}$ be a direct system of R-modules and M be an R-module with a finite projective resolution of finite type, $\mathcal{P}_*$, then there are two spectral sequences which converge to the same thing:*

$$E_2^{p,q} = \mathrm{Tor}_{-p}(\lim{}^q A_\alpha, M) \quad \textit{and} \quad E_2^{p,q} = \lim{}^p \mathrm{Tor}_{-q}(A_\alpha, M).$$

There is some guidance for the proof in [Roo61] and much more in [Jen72] where this theorem appears as Theorem 4.4 with more discussion on pages 102-3. Dan Christensen helped us understand this approach, which gives much more insight into what is going on than our previous approach.

Proof. In the case when we are indexed over the natural numbers and we have maps $f_i : A_{i+1} \to A_i$, then we are familiar with the exact sequence of Milnor from [Mil62],

$$0 \to \lim{}^0 A_i \to \prod A_i \xrightarrow{f} \prod A_i \to \lim{}^1 A_i \to 0,$$

where the map f is given by

$$f(a_1, a_2, \dots) \to (a_1 - f_1(a_2), a_2 - f_2(a_3), \dots).$$

In this case, $\lim^* A$ is just the homology of the complex

$$\prod A_i \xrightarrow{f} \prod A_i.$$

In the general case, there is a complex whose homology gives $\lim^* A$ and whose terms are all (big) products of the A_α, see [Jen72, Theorem 1.1, page 32] and [Roo61]. Denote this complex by $\mathcal{A}^*$. We then get our two spectral sequences from the two standard filtrations of the bicomplex $\mathcal{A}^* \otimes \mathcal{P}_*$.

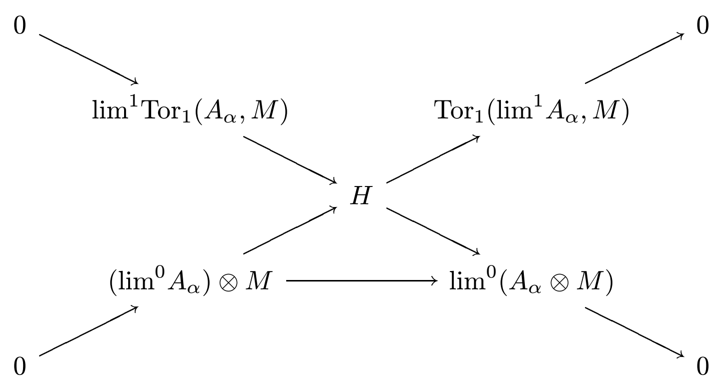
Filtering first using $\mathcal{P}_*$ we use only the differential on $\mathcal{A}^*$. Since each $\mathcal{P}_i$ is a finitely generated projective R-module, taking the tensor product with $\mathcal{A}^*$ and then taking the differential on $\mathcal{A}^*$ gives us $(\lim^* A) \otimes \mathcal{P}_*$. Taking the second differential to get our E_2 term we have $\mathrm{Tor}_*(\lim^* A, M)$, giving us our first spectral sequence.

Filtering next using $\mathcal{A}^*$ we use only the differential on $\mathcal{P}_*$. Because $\mathcal{P}_i$ is a finitely generated projective R-module. $(\prod A_\alpha) \otimes \mathcal{P}_*$ is the same as $\prod (A_\alpha \otimes \mathcal{P}_*)$ and so the homology is also the same. This is easy to evaluate as $\prod \mathrm{Tor}_*(A_\alpha, M)$, since products are exact. To take the next differential to get our E_2 term we see that this is just the complex which gives $\lim^* \mathrm{Tor}_*(A_\alpha, M)$, giving our second spectral sequence.

Both spectral sequences converge (to the same thing) because of the finiteness of the resolution $\mathcal{P}_*$. □

This spectral sequence simplifies a great deal when $\lim^i$ is always zero for $i > 1$. This is always the case for us for multiple reasons. In particular, it is the case when we are indexed over the natural numbers, which we always are. It is also true when the ground ring for the algebra R is $\mathbf{Z}/(p)$, $\mathbf{Z}_{(p)}$, or $\mathbf{Z}_p$, which is the case for us since $R = P(n)$.

Corollary 9.2 ([Roo61]). *If all* $\lim^i = 0$, $i > 1$, *and we have the conditions of Theorem 9.1, then*



where the diagonals give short exact sequences.

Proof. This follows immediately from Theorem 9.1. □

The commutativity of the tensor product and the inverse limit will be of primary importance to us. This result measures the failure to commute explicitly. We will show that in the cases we care about, both terms having $\lim^1$ will be zero, making the horizontal arrow an isomorphism. Built in to all of our assumptions is that $\lim^1 A_\alpha$ will be zero, so the $\lim^1 \mathrm{Tor}$ term is all we need to consider. At this stage we need to get more specific about our modules and rings.

In all that follows our tensor products and our Tor are over $P(k)^*$.

Lemma 9.3. *Let the* A_α *be* $P(k)^*$ *modules which are bounded above and of finite type and let* M *be a* $P(k)^*(P(k))$ *module which is finitely presented over* $P(k)^*$. *For each degree,* $\mathrm{Tor}_1^*(A_\alpha, M)$ *is finite.*

Proof. For $k > 0$ everything in sight is a finite dimensional $\mathbf{Z}/(p)$ vector space and so our result follows immediately. We prove the $k = 0$ case by induction on the Landweber filtration, Theorem 3.10. We have a long exact sequence:

$$\ldots \to \mathrm{Tor}_1(A_\alpha, M_{q+1}) \to \mathrm{Tor}_1(A_\alpha, M_q) \to \mathrm{Tor}_1(A_\alpha, P(0)^*/I_{n_q}) \to \ldots$$

where if $n_q > 0$ we have $P(0)^*/I_{n_q}$ is finite in each degree and therefore so is $\mathrm{Tor}_1(A_\alpha, P(0)^*/I_{n_q})$ (this uses the assumptions on A_α). If $n_q = 0$ then $\mathrm{Tor}_1(A_\alpha, P(0)^*) = 0$ because $P(0)^*$ is free. By induction $\mathrm{Tor}_1(A_\alpha, M_{q+1})$ is finite so since $\mathrm{Tor}_1(A_\alpha, M_q)$ is trapped between two finite groups, it too is finite. □

Corollary 9.4. *Let $\{A_i\}$ be a direct system, indexed over $\mathbf{N}$, the natural numbers, of $P(k)^*$ modules which are bounded above and of finite type, and let M be a $P(k)^*(P(k))$ module which is finitely presented over $P(k)^*$. We have* $\lim^1 \mathrm{Tor}_1^*(A_i, M) = 0$.

Proof. $\lim^1$ of finite groups is always zero. □

Corollary 9.5. *Let $\{A_i\}$ be a direct system, indexed over $\mathbf{N}$, of $P(k)^*$ modules which are bounded above and of finite type, and let M be a $P(k)^*(P(k))$ module which is finitely presented over $P(k)^*$. Assume that* $\lim^1 A_i = 0$. *Then*

$$(\lim^0 A_i) \otimes M \xrightarrow{\simeq} \lim^0(A_i \otimes M).$$

Proof. This follows immediatedly from Corollaries 9.2 and 9.4. □

Remark 9.6. This result is what we need and it can be proven directly, but not as nicely. Note that the proof as we have given it really shows that

$$\mathrm{Tor}_n(\lim^0 A_i, M) \xrightarrow{\simeq} \lim^0 \mathrm{Tor}_n(A_i, M).$$

It is time to put some topology into the argument. Recall that we have $\lim^0 P(k)^*(Z^i) \simeq P(k)^*(Z)$ and $\lim^1 P(k)^*(Z^i) = 0$ for $Z = X$ and Y. We have

$$X^{2i} \times Y^{2i} \supset (X \times Y)^{2i} \supset X^i \times Y^i$$

so they give rise to two equivalent sequences and we have

$$\begin{aligned} P(k)^*(X \times Y) &\simeq \lim^0 P(k)^*((X \times Y)^i) \\ &\simeq \lim^0 P(k)^*(X^i \times Y^i) \end{aligned}$$

and

$$\lim^1 P(k)^*(X^i \times Y^i) = 0$$

by [Lan70a, Lemma 6] (If there are no phantom maps (for MU) for X and for Y then there are none for $X \times Y$.) when $P(0) = BP$, by Remark 3.7 when we are p-adically complete, and by Corollary 4.8(a), ($F^\infty = 0$), when $k > 0$. In particular, if Y is finite, we have

$$\lim^1 P(k)^*(X^i \times Y) = 0$$

and

$$\begin{aligned} P(k)^*(X \times Y) &\simeq \lim^0 P(k)^*((X \times Y)^i) \\ &\simeq \lim^0 P(k)^*(X^i \times Y). \end{aligned}$$

We need the following to proceed. It is possible these statements don't warrant a proof but we are neophytes at this business.

Lemma 9.7. *Let X and Y be spaces and let $P(k)^*(X)$ be Landweber flat. Then*

$$\begin{aligned} P(k)^*(X \times Y) &\simeq \lim^0 P(k)^*(X \times Y^i) \\ &\simeq \lim^0 (P(k)^*(X) \otimes P(k)^*(Y^i)) \end{aligned}$$

and

$$\lim^1 P(k)^*(X \times Y^i) = 0.$$

Proof. Both $P(k)^*(X \times -)$ and $P(k)^*(X) \otimes P(k)^*(-)$ are cohomology theories for finite complexes. We have a map

$$P(k)^*(X) \otimes P(k)^*(-) \longrightarrow P(k)^*(X \times -)$$

which is an isomorphism on a point. The usual arguments by induction on the number of cells gives us

$$P(k)^*(X) \otimes P(k)^*(-) \simeq P(k)^*(X \times -)$$

for finite complexes. This proves the vertical isomorphism. What we need now is either one of the two other statements since by Milnor, [Mil62], we have

$$0 \to \lim^1 P(k)^*(X \times Y^i) \to P(k)^*(X \times Y) \to \lim^0 P(k)^*(X \times Y^i) \to 0.$$

Comparing this with the other Milnor sequence, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \lim^1 P(k)^*(X \times Y^i) & \to & P(k)^*(X \times Y) & \to & \lim^0 P(k)^*(X \times Y^i) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \lim^1 P(k)^*((X \times Y)^i) & \to & P(k)^*(X \times Y) & \to & \lim^0 P(k)^*((X \times Y)^i) & \longrightarrow & 0 \end{array}$$

and since the middle vertical arrow is an isomorphism and $\lim^1 P(k)^*((X \times Y)^i) = 0$, we must have $\lim^1 P(k)^*(X \times Y^i) = 0$. □

By Corollary 4.8(a), we have

$$P(k)^*(Z)/F^{s+1} \longrightarrow P(k)^*(Z^s)$$

induces

$$\lim^0 P(k)^*(Z)/F^{s+1} \xrightarrow{\simeq} \lim^0 P(k)^*(Z^s) \simeq P(k)^*(Z), \tag{9.8}$$

and

$$\lim^1 P(k)^*(Z)/F^{s+1} = 0 = \lim^1 P(k)^*(Z^s). \tag{9.9}$$

Remark 9.10. In our proof below of the Künneth isomorphism we use that $P(k)^*(X)/F^{i+1}$ and $P(k)^*(Y^i)$ are finitely presented and that $P(k)^*(Y)/F^{i+1}$ is bounded above and of finite type. The first one is the only difficult one and it requires that X be a space and so this is an unstable result. When $k > 0$ or if $P(0) = BP$, then our proof of Lemma 6.1 is independent of [BJW95]. The only place we need the unstable Landweber Filtration is if $P(0) = BP_p\hat{}$ because $\lim^1 BP^*(X^i) \neq 0$.

Proof of the Künneth isomorphism, Theorem 1.11.

$$
\begin{aligned}
&P(k)^*(X\times Y)\\
&\quad\simeq \lim\nolimits^0_j P(k)^*(X\times Y^j) && \text{by Lemma 9.7}\\
&\quad\simeq \lim\nolimits^0_j (P(k)^*(X)\otimes P(k)^*(Y^j)) && \text{by Lemma 9.7}\\
&\quad\simeq \lim\nolimits^0_j ((\lim\nolimits^0_i P(k)^*(X)/F^{i+1})\otimes P(k)^*(Y^j)) && \text{by Equation 9.8}\\
&\quad\simeq \lim\nolimits^0_j \lim\nolimits^0_i (P(k)^*(X)/F^{i+1}\otimes P(k)^*(Y^j)) && \text{by Proposition 9.5}\\
&\quad\simeq \lim\nolimits^0_i \lim\nolimits^0_j (P(k)^*(X)/F^{i+1}\otimes P(k)^*(Y^j)) && \\
&\quad\simeq \lim\nolimits^0_i (P(k)^*(X)/F^{i+1}\otimes (\lim\nolimits^0_j P(k)^*(Y^j))) && \text{by Proposition 9.5}\\
&\quad\simeq \lim\nolimits^0_i (P(k)^*(X)/F^{i+1}\otimes (\lim\nolimits^0_j P(k)^*(Y)/F^{j+1})) && \text{by Equation 9.8}\\
&\quad\simeq \lim\nolimits^0_i \lim\nolimits^0_j (P(k)^*(X)/F^{i+1}\otimes P(k)^*(Y)/F^{j+1}) && \text{by Proposition 9.5}\\
&\quad\simeq \lim\nolimits^0_{i,j} (P(k)^*(X)/F^{i+1}\otimes P(k)^*(Y)/F^{j+1})
\end{aligned}
$$

which, by definition, is

$$P(k)^*(X)\widehat{\otimes}_{P(k)^*}P(k)^*(Y).$$

□

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