THE MICROSTABLE ADAMS-NOVIKOV SPECTRAL SEQUENCE

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Introduction. For a fixed prime p, recall the spectra $T(m)$ (introduced in [Rav86, §6.5]) with

$$BP_*(T(m)) = BP_*[t_1, \dots, t_m] \subset BP_*(BP).$$

It is a p-local summand of the Thom spectrum associated with the map

$$\Omega SU(k) \to \Omega SU = BU$$

for any k satisfying $p^m \leq k < p^{m+1}$. These Thom spectra figure in the proof of the nilpotence theorem of [DHS88].

Let (A, Γ) denote the Hopf algebroid $(BP_*, BP_*(BP))$; see [Rav86, A1] for more information. A change-of-rings isosmorphism identifies the Adams-Novikov E_2-term for for $T(m)$ with $\mathrm{Ext}_{\Gamma(m+1)}(A, A)$ where

$$\Gamma(m+1) = \Gamma/(t_1, \dots, t_m) = BP_*[t_{m+1}, t_{m+2}, \dots]$$

This Hopf algebroid is cocommutative below the dimension of t_{2m+2}, so its Ext group (and the homotopy of $T(m)$) in this range is relatively easy to deal with. Moreover empirical evidence suggests that $\pi_*(T(m))$ for roughly $2p^{m+1} < * < 2p^{2m+2}$ is the same (up to a suitable regrading) as that of $\pi_*(T(m+1))$ for roughly $2p^{m+2} < * < 2p^{2m+3}$. *The purpose of this note is to set up an algebraic framework that allows us to study the limit of this behavior as m goes to infinity.*

This will entail defining a bigraded Hopf algebroid $(\widehat{A}, \widehat{\Gamma})$. The grading is over $\mathbf{Z} \oplus \mathbf{Z}\omega$ where ω becomes p^m when we specialize to $T(m)$. We call the corresponding Ext group the *microstable* Adams-Novikov E_2-term for the following reason. For each spectrum $T(m)$ one can set up a chromatic spectral sequence as in [Rav86, Chapter 5]. Each Morava stabilizer group S_n gets replaced by a certain open subgroup which shrinks as m increases. Thus in the limit each S_n gets replaced by an infinitesimal version of itself. We conjecture that this Ext group is the E_2-term of a trigraded spectral sequence.

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The bigraded Hopf algebroid $(\widehat{A}, \widehat{\Gamma})$. Define a Hopf algebroid $(\widehat{A}, \widehat{\Gamma})$ over $\mathbf{Z}_{(p)}$, graded over $\mathbf{Z} \oplus \mathbf{Z}\omega$, by

$$\begin{aligned} \widehat{A} &= BP_*[c_{i,m}, u_i : 0 \leq i \leq m]/(c_{i,m} - v_i^{(p-1)p^m} c_{i,m+1}) \\ &\qquad \text{with } v_0 = p,\ |c_{i,m}| = (\omega - p^m)|v_i|, \text{ and } |u_i| = 2p^i\omega - 2; \\ \widehat{\Gamma} &= \widehat{A}[s_i : i > 0] \qquad \text{with } |s_i| = 2p^i\omega - 2. \end{aligned}$$

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This definition should be compared with the usual (A, Γ) given by

$$\begin{aligned} A &= \mathbf{Z}_{(p)}[v_i : i > 0] \qquad \text{with } |v_i| = 2p^i - 2; \\ \Gamma &= A[t_i : i > 0] \qquad \text{with } |t_i| = 2p^i - 2. \end{aligned}$$

We will denote the element $v_i^{p^m} c_{i,m}$ by $v_i^\omega$for $0 \le i \le m$. Because of the relations in $\widehat{A}$, this element is independent of m. Conversely we will often write $v_i^{\omega - p^m}$ for $c_{i,m}$.

The generators $c_{0,m}$ in dimension 0 are somewhat problematic. The defining relation in $\widehat{A}$ implies that $c_{0,m} = p^{\omega - p^m} = v_0^{\omega - p^m}$, so p^ω is infinitely divisible by p. It follows that each $c_{0,m}$ maps to 0 in $\widehat{A}/(p)$. One could get rid of them by replacing (2) below (which is analogous to Araki's definition of the v_i) by a formula similar to Hazewinkel's, namely

$$pk_i = \sum_{0<j<i} k_j v_{i-j}^{\omega p^j} + \sum_{0 \le j < i} \ell_j u_{i-j}^{p^j}, \tag{1}$$

in which there is no term involving v_0^ω. The drawback of this is that, like Hazewinkel's formula in the stable case, it leads to a right unit formula which is messier before reduction mod p. The homogeneous analog of (9) below would be

$$ps_i + \sum_{0 \le j < i} \ell_j u_{i-j}^{p^j} + \sum_{0 \le k < j < i} \ell_k v_{j-k}^{p^k} s_{i-j}^{p^j} = \sum_{0 \le j < i} \ell_j \eta_R(u_{i-j}^{p^j}) + \sum_{0 \le k < j < i} \ell_k s_{j-k}^{p^k} v_{i-j}^{\omega p^j},$$

where $i > 0$ is fixed, and we sum over the indicated values of j and k. The presence of the first term on the left would lead to an error (which can be shown to be divisible by p using [Rav76, Lemma 2]) in the analog of (10).

We will adopt the following convention: *whenever possible an element in* $\widehat{\Gamma}$ *or a related Ext group will be denoted by the letter of the alphabet preceding the one usually used for the corresponding element associated with* Γ, *and algebraic objects assocaied with it will be denoted by a hat over the symbol for the corresponding object associated with* Γ, In particular, the elements u_i and s_i are the microstable analogs of v_i and t_i.

Recall the the log coefficients $\ell_i \in A \otimes \mathbf{Q}$ are related to the v_i by Araki's formula

$$p\ell_i = \sum_{0 \le j \le i} \ell_j v_{i-j}^{p^j},$$

and the right unit is defined by

$$\eta_R(\ell_i) = \sum_{0 \le j \le i} \ell_j t_{i-j}^{p^j}.$$

The microstable analogs of these formulas are

$$pk_i = \sum_{0<j \le i} k_j v_{i-j}^{\omega p^j} + \sum_{0 \le j < i} \ell_j u_{i-j}^{p^j} \tag{2}$$

and the right unit is defined by

$$\eta_R(k_i) = k_i + \sum_{0 \le j < i} \ell_j s_{i-j}^{p^j}, \tag{3}$$

while it is trivial on v_i and v_i^ω. The coproduct on $\widehat{\Gamma}$ is defined by setting

(4) $$\sum_{0\le j<i} \ell_j s_{i-j}^{p^j}$$

primitive. Note that this coproduct is cocommutative.

Maps from subalgebras of $\widehat{\Gamma}$ to the $\Gamma(m+1)$. Next define sub-Hopf algebroids

$$\begin{array}{rcl} \left(\widehat{A}(m),\, \widehat{G}(1,m)\right) & \subset & \left(\widehat{A},\, \widehat{\Gamma}\right) \\ \text{and} \qquad \left(\widehat{A}(m+n)/I_n,\, \widehat{G}(1,m,n)\right) & \subset & \left(\widehat{A}/I_n,\, \widehat{\Gamma}/I_n\right) \end{array}$$

for $m, n > 0$ by

$$\begin{array}{rcl} \widehat{A}(m) & = & \mathbf{Z}_{(p)}[v_1,\dots,v_m; v_0^{\omega-p^m}, v_1^{\omega-p^m},\dots,v_m^{\omega-p^m}; u_1,\dots,u_{m+1}] \\ \widehat{G}(1,m) & = & \widehat{A}(m)[s_1,\dots,s_{m+1}] \\ \widehat{A}(m,n) & = & \mathbf{Z}_{(p)}[v_1,\dots,v_{m+n}; v_0^{\omega-p^m}, v_1^{\omega-p^m},\dots,v_{m+n}^{\omega-p^m}; u_1,\dots,u_{m+n+1}]/I_n \\ \widehat{G}(1,m,n) & = & \widehat{A}(m,n)[s_1,\dots,s_{m+1}]. \end{array}$$

Let

$$\begin{array}{rrcl} & A(k) & = & \mathbf{Z}_{(p)}[v_1,\dots,v_k], \\ & G(m+1,m) & = & A(2m+1)[t_{m+1},\dots,t_{2m+1}] \qquad \text{as in [Rav86, §7.1]}, \\ \text{and} & G(m+1,k,n) & = & A(m+1+k+n)/I_n[t_{m+1},\dots,t_{m+1+k}]. \end{array}$$

There are maps

(5) $$\widehat{G}(1,m) \xrightarrow{\theta_m} G(m+1,m) \quad \subset \quad \Gamma(m+1)$$

and

(6) $$\widehat{G}(1,m,n) \xrightarrow{\theta_m} G(m+1,m,n) \quad \subset \quad \Gamma(m+1)/I_n$$

given by

$$\begin{array}{rcl} v_i & \mapsto & v_i \\ u_i & \mapsto & v_{i+m} \\ v_i^\omega & \mapsto & v_i^{p^m} \\ s_i & \mapsto & t_{i+m}. \end{array}$$

In addition, we have

$$\theta_m(k_i) = \overline{\ell}_{m+i}$$

where $\overline{\ell}_{m+i}$ is obtained from ℓ_{m+i} by removing all terms which are monomials in the v_j for $0 < j \le m$. The analog of Araki's formula for these elements is

$$p\overline{\ell}_{m+i} = \sum_{0\le j<i} \ell_j v_{m+i-j}^{p^j} + \sum_{0<j\le i} \overline{\ell}_{m+j} v_{i-j}^{p^{m+j}}.$$

Since $\ell_{m+i} - \overline{\ell}_{m+i}$ is invariant, we have

$$\begin{aligned} \eta_R(\overline{\ell}_{m+i}) - \overline{\ell}_{m+i} &= \eta_R(\ell_{m+i}) - \ell_{m+i} \\ &= \sum_{0 \le j < i} \ell_j t_{m+i-j}^{p^j}, \end{aligned}$$

which is compatible with our formula for $\eta_R(k_i)$.

The indexing set $\mathbf{Z} \oplus \mathbf{Z}\omega$ is mapped to $\mathbf{Z}$ by sending ω to p^m. Then each element of $\mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A},\, \widehat{A})$ can be pulled back to $\mathrm{Ext}_{\widehat{G}(1,m)}(\widehat{A}(m),\, \widehat{A}(m))$ for $m \gg 0$, and hence mapped via θ_m to $\mathrm{Ext}_{\Gamma(m+1)}(A,\, A)$, which is the Adams–Novikov spectral sequence E_2-term for the spectrum $T(m)$.

Thus we have a diagram of Hopf algebroids

$$\begin{array}{ccccccccc} \widehat{G}(1,0) & \longrightarrow & \widehat{G}(1,1) & \longrightarrow & \widehat{G}(1,2) & \longrightarrow & \cdots & \longrightarrow & \widehat{\Gamma} \\ \Big\downarrow{\scriptstyle \theta_0} & & \Big\downarrow{\scriptstyle \theta_1} & & \Big\downarrow{\scriptstyle \theta_2} & & & & \\ \Gamma(1) & & \Gamma(2) & & \Gamma(3) & & & & \end{array}$$

Conjecture 7. *There is a spectral sequence with*

$$E_2 = \mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A},\, \widehat{A})$$

which is compatible in a range of dimensions with the Adams–Novikov spectral sequence for $T(m)$. We call this the **microstable** *Adams–Novikov spectral sequence.*

Remark 8. *The map θ_m is onto below dimension $|t_{2m+2}|$, and $T(m)$ is equivalent to BP below dimension $|t_{m+1}|$. We believe the behavior of the Adams–Novikov spectral sequence in this range is essentially isomorphic (up to regrading) to that of the Adams–Novikov spectral sequence for $T(m+1)$ between dimensions $|t_{m+2}|$ and $|t_{2m+3}|$. This is the rationale for the conjecture.*

A right unit formula for $\widehat{\Gamma}$. Using (2) and (3) we can deduce a right unit formula for the u_i. We can rewrite (2) as

$$pk_i = k_i v_j^{\omega p^i} + \ell_i u_j^{p^i}$$

where it is understood that both sides are summed over all nonnegative values of the indices, and $k_0 = u_0 = v_0^{\omega} = s_0 = 0$. Applying the right unit to both sides, we get

$$pk_i + p\ell_i s_j^{p^i} = k_i v_j^{\omega p^i} + \ell_i s_j^{p^i} v_k^{\omega p^{i+j}} + \ell_i \eta_R(u_j)^{p^i}.$$

Applying Araki's formula and (2) to the left hand side, the equation becomes

$$k_i v_j^{\omega p^i} + \ell_i u_j^{p^i} + \ell_i v_j^{p^i} s_k^{p^{i+j}} = k_i v_j^{\omega p^i} + \ell_i s_j^{p^i} v_k^{\omega p^{i+j}} + \ell_i \eta_R(u_j)^{p^i},$$

which simplifies to

$$\ell_i u_j^{p^i} + \ell_i v_j^{p^i} s_k^{p^{i+j}} = \ell_i s_j^{p^i} v_k^{\omega p^{i+j}} + \ell_i \eta_R(u_j)^{p^i}. \tag{9}$$

This can be rewritten as

$$\sum_i^F u_i +_F \sum_{i,j}^F v_i s_j^{p^i} = \sum_i^F \eta_R(u_i) +_F \sum_{i,j}^F s_i v_j^{\omega p^i}. \tag{10}$$

This is the microstable analog of the right unit formula for $BP_*(BP)$,

$$\sum_{i,j}^F v_i t_j^{p^i} = \sum_{i,j}^F t_i \eta_R(v_j)^{p^i}. \tag{11}$$

Recall that (11) implies that

$$\mathrm{Ext}^0_\Gamma(BP_*,\, BP_*/I_n) = \begin{cases} \mathbf{Z}_{(p)} & \text{if } n = 0 \\ \mathbf{Z}/(p)[v_n] & \text{if } n > 0. \end{cases}$$

One can derive a similar formula for $\mathrm{Ext}^0_{\widehat{\Gamma}}(\widehat{A},\, \widehat{A}/I_n)$ from (10). Let

$$\widehat{V} = \mathbf{Z}_{(p)}[v_i, c_{i,m} : 0 \leq i \leq m] \subset \widehat{A}.$$

Then we have

$$\mathrm{Ext}^0_{\widehat{\Gamma}}(\widehat{A},\, \widehat{A}/I_n) = \begin{cases} \widehat{V} & \text{if } n = 0 \\ \widehat{V}/I_n[u_1, \ldots u_n] & \text{if } n > 0. \end{cases} \tag{12}$$

The microchromatic spectral sequence. The chromatic spectral sequence converging to $\mathrm{Ext}_\Gamma(A,\, A)$ is obtained from the resolution

$$0 \to BP_* \to M^0 \to M^1 \to M^2 \to \ldots$$

where

$$M^n = v_n^{-1} BP_*/(p^\infty, v_1^\infty, \ldots, v_{n-1}^\infty).$$

More details can be found in [Rav86, Chapter 5]. This can be tensored over A with $\widehat{A}$, leading in the same way to a spectral sequence converging to $\mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A},\, \widehat{A})$ which we call the *microchromatic spectral sequence*.

We also define

$$M_i^{n-i} = v_n^{-1} BP_*/(p, \ldots, v_{i-1}, v_i^\infty, \ldots, v_{n-1}^\infty).$$

so for each $i > 0$ there is a resolution

$$0 \to BP_*/I_i \to M_i^0 \to M_i^1 \to M_i^2 \to \ldots,$$

and there are short exact sequences

$$0 \longrightarrow M_{i+1}^{n-i-1} \longrightarrow \Sigma^{|v_i|} M_i^{n-i} \xrightarrow{v_i} M_i^{n-i} \longrightarrow 0$$

which lead to Bockstein spectral sequences. In particular there is a chain of n Bockstein spectral sequences leading from $\mathrm{Ext}_\Gamma(A,\, v_n^{-1}BP_*/I_n)$ to $\mathrm{Ext}_\Gamma(A,\, M^n)$. The former group can be identified with

$$\mathrm{Ext}_{\Sigma(n)}(K(n)_*,\, K(n)_*)$$

where

$$\begin{aligned} \Sigma(n) &= K(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} K(n)_* \\ &= K(n)_*[t_i : i > 0]/(v_n t_i^{p^n} - v_n^{p^i} t_i) \end{aligned}$$

as an algebra, with coproduct inherited from $BP_*(BP)$. The formula (11) is pivotal in the proof of this result.

The microstable analog, which can proved in a similar way using (10) is

$$\mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A},\, v_n^{-1}\widehat{A}/I_n) = \mathrm{Ext}_{\widehat{\Sigma}(n)}(\widehat{K}(n)_*,\, \widehat{K}(n)_*) \tag{13}$$

where

$$\begin{aligned} \widehat{K}(n)_* &= v_n^{-1}\widehat{V}/I_n[u_1, \ldots, u_n] \\ \text{and} \quad \widehat{\Sigma}(n) &= \widehat{K}(n)_*[s_i : i > 0]/(v_n s_i^{p^n} - v_n^{\omega p^i} s_i). \end{aligned}$$

The methods of [Rav86, Chapter 6] can be used to compute this Ext group explicitly, and the result is

$$\mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A},\, v_n^{-1}\widehat{A}/I_n) = \widehat{K}(n)_* \otimes E(g_{i+1,j} : 0 \le i, j < n) \tag{14}$$

where $g_{i,j}$ corresponds to $s_i^{p^j}$. It is also true that

$$\mathrm{Ext}_{\Gamma(m)}(A,\, v_n^{-1}A/I_n) = K(n)_*[v_{n+1}, \ldots, v_{2n}] \otimes E(h_{i+m+1,j} : 0 \le i, j < n) \text{ for } m \ge n,$$

where $h_{i+m,j}$ corresponds to $t_{i+m}^{p^j}$. Thus the microchromatic spectral sequence is simpler than the usual one.

The microstable 1-line. We can use the microchromatic spectral sequence to compute $\mathrm{Ext}^1_{\widehat{\Gamma}}(\widehat{A},\, \widehat{A})$ in the same way that we use the chromatic spectral sequence to compute $\mathrm{Ext}^1_{\Gamma}(A,\, A)$. We need to analyze the Bockstein spectral sequence going from

$$\mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A},\, \widehat{M}_1^0) = \widehat{K}(1)_* \otimes E(g_{1,0})$$

to $\mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A},\, \widehat{M}_0^1)$. This behaves in much the same way as the stable analog, i.e., the one going from

$$\mathrm{Ext}_{\widehat{\Gamma}}(A,\, \widehat{M}_1^0) = K(1)_* \otimes E(h_{1,0})$$

to $\mathrm{Ext}_{\Gamma}(A,\, M_0^1)$.

For odd primes the relevant fact about the right unit is that for all $t > 0$,

$$\eta_R(u_1^t) \equiv u_1^t + ptu_1^{t-1}s_1 \qquad \mod (p^2 t).$$

From this we deduce that $\mathrm{Ext}^1_{\widehat{\Gamma}}(\widehat{A},\, \widehat{A})$ is the $\widehat{V}$-module generated by the set

$$\left\{ \frac{u_1^t}{pt} : t > 0 \right\}.$$

For $p = 2$ let

$$w_{1,1} = u_1^2 + 2v_1^{2\omega-1}u_1 + 4v_1^{-1}u_2.$$

Then for all $s > 0$ we have

$$\begin{aligned} \eta_R(u_1^{2s-1}) &\equiv u_1^{2s-1} + 2u_1^{2s-2}s_1 \quad \mod (4) \\ \text{and} \quad \eta_R(w_{1,1}^s) &\equiv w_{1,1}^s + 4su_1^{2s-1}s_1 \quad \mod (8s). \end{aligned}$$

From this we deduce that $\mathrm{Ext}^1_{\widehat{\Gamma}}(\widehat{A},\, \widehat{A})$ is the $\widehat{V}$-module generated by the set

$$\left\{ \frac{u_1^{2s-1}}{2},\, \frac{w_{1,1}^s}{4s} : s > 0 \right\}.$$

For all primes it follows that

$$\mathrm{Ext}^1_{\widehat{\Gamma}}(\widehat{A},\, \widehat{M}^1_0) = 0,$$

unlike the stable case where $\mathrm{Ext}^1_{\Gamma}(A,\, M^1_0) \supset \mathbf{Q}/\mathbf{Z}$.

Note that for odd primes each element in Ext^1 can be pulled back to

$$\mathrm{Ext}_{\widehat{G}(1,0)}\left(\widehat{A}(0),\, \widehat{A}(0)\right),$$

so we can map them via the map θ_m of (5) to

$$\mathrm{Ext}^1_{\Gamma(m)}(A,\, A)$$

for $m \geq 0$. For $p = 2$ we can only do this for $m \geq 1$. This is to be expected since the structure of $\mathrm{Ext}^1_{\Gamma(1)}(A,\, A)$ for $p = 2$ differs from that of $\mathrm{Ext}^1_{\widehat{\Gamma}}(\widehat{A},\, \widehat{A})$ in that for $s > 1$, $\frac{v_1^{2s}}{2}$ is divisible by $4s$ while $\frac{u_1^{2s}}{2}$ is only divisible by $2s$.

The Miller-Wilson computation. In [MW76] Miller-Wilson computed

$$\mathrm{Ext}^0_{\Gamma}(A,\, M^1_{n-1})$$

in nearly all cases by studying the relvant Bockstein spectral sequence. Their result is restated as [Rav86, 5.2.13], which says that the group is a direct sum of $K(n-1)_*/k(n-1)_*$ generated by $\{\frac{1}{v_{n-1}^j} : j > 0\}$, and cyclic $k(n-1)_*$-modules generated by the set

$$\left\{ \frac{x_{n,i}^s}{v_{n-1}^{a_{n,i}}} : i \geq 0,\ s > 0,\ s \not\equiv 0 \bmod p \right\}, \tag{15}$$

where $x_{n,i}$ is a certain expression of the form $v_n^{p^i}$ modulo decomposables, and $a_{n,i}$ is a certain integer not less than p^i. For $n = 2$ we have

$$\begin{aligned}
x_{2,0} &= v_2,\\
x_{2,1} &= v_2^p - v_1^p v_2^{-1} v_3,\\
x_{2,2} &= x_{2,1}^p - v_1^{p^2-1} v_2^{p^2-p+1} - v_1^{p^2+p-1} v_2^{p^2-2p} v_3, \quad \text{and}\\
x_{2,i} &= \begin{cases} x_{2,i-1}^2 & \text{if } p = 2\\ x_{2,i-1}^p - 2 v_1^{b_{2,i}} v_2^{(p-1)p^{i-1}+1} & \text{if } p > 2 \end{cases} \qquad \text{for } i \geq 3,
\end{aligned}$$

where $b_{2,i} = (p+1)(p^{i-1} - 1)$. The exponents $a_{2,i}$ are given by

$$\begin{aligned}
a_{2,0} &= 1,\\
a_{2,1} &= p,\\
\text{and} \quad a_{2,i} &= \begin{cases} 2^i + 2^{i-1} & \text{if } p = 2\\ p^i + p^{i-1} - 1 & \text{if } p > 2 \end{cases} \qquad \text{for } i \geq 2.
\end{aligned} \tag{16}$$

A microstable analog of this computation should be feasible, and we give some partial results here. Analyzing the Bockstein spectral sequence for a fixed $n > 1$ amounts to finding elements $w_{n,i} \in v_n^{-1}\widehat{A}/I_{n-1}$ congruent to $u_n^{p^i}$ modulo I_n and integers $e_{n,i}$ so that $\mathrm{Ext}^0_{\widehat{\Gamma}}(\widehat{A},\, \widehat{M}^1_{n-1})$ is the $\widehat{k}(n-1)_*$-module generated by the set

$$\left\{ \frac{w_{n,i}^s}{v_{n-1}^{e_{n,i}}} : i \geq 0,\ s > 0,\ s \not\equiv 0 \bmod p \right\},$$

where $\widehat{k}(n-1)_* = \widehat{V}/I_{n-1}[u_1, \ldots, u_{n-1}]$. To prove such a result one needs to compute

$$\eta_R(w_{n,i}) - w_{n,i} \qquad \bmod (v_{n-1}^{1+e_{n,i}})$$

and show that these represent v_{n-1}-multiples of linearly independent (over $\widehat{K}(n)_*$) elements in the group $\mathrm{Ext}^1_{\widehat{\Gamma}}(\widehat{A}, \widehat{M}_n^0)$ given in (14).

Here is the relevant information for $n = 2$; details will appear in a forthcoming paper with Nakai. Let

$$\begin{aligned}
w_{2,0} &= u_2, \\
w_{2,1} &= w_{2,0}^p - v_1^p r_{2,3}, \\
w_{2,2} &= w_{2,1}^p - v_1^{p^2-1} v_2^{a+1} w_{2,0} + v_1^{p^2+p} v_2^{-p}(r_{2,4} - v_2^{-p\omega} v_3^{\omega p} r_{2,3}), \\
w_{2,3} &= w_{2,2}^p, \\
w_{2,4} &= w_{2,2}^{p^2} + v_1^{p^3(p+1)} v_2^{p^2 a}(r_{2,4} - v_1^{-p(p+1)} v_2^p w_{2,2} - v_2^{-p\omega} v_3^{p\omega} r_{2,3}) \\
&\quad + v_1^{p^3(p+1)+p} v_2^{p^2 a} r_{2,3} w_{2,1}^{p-1} \\
&= w_{2,2}^{p^2} + v_1^{p^3(p+1)-p} v_2^{p^2 a}(u_3^p - v_1^{-p^2} v_2^p u_2^{p^2} + v_1^{-1} v_2^{\omega p^2} u_2) \\
&\quad - v_1^{p^3(p+1)} v_2^{p^2 a} w_{2,1}^{p-1}(w_{2,1} - w_{2,0}^p), \quad \text{and} \\
w_{2,i} &= w_{2,i-1}^p - v_1^{p^{i-1}(p+1)} v_2^{p^{i-2} a} w_{2,i-3}^{p-1}(w_{2,i-3} - w_{2,i-4}^p) \\
&\quad \text{for } i > 4,
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
a &= p^2\omega - p - 1, \\
r_{2,3} &= v_2^{-1} u_3, \\
\text{and} \quad r_{2,4} &= v_2^{-1}(u_4 - v_3 r_3^p) + v_1^{-p} v_2^{-p\omega} v_3^{p\omega} w_{2,1} - v_1^{-1} v_2^a v_3 w_{2,0}.
\end{aligned}$$

The elements $r_{n,n+i} = v_n^{-1} u_{n+i} + \ldots$ are chosen so that

$$d(r_{n,n+i}) \equiv s_i^{p^n} - v_n^{p^i\omega-1} s_i \qquad \bmod I_n.$$

where $d(x) = \eta_R(x) - x$.

The exponents $e_{2,i}$ are

$$\begin{aligned}
e_{2,0} &= 1, \\
e_{2,1} &= p, \\
\text{and for } i \geq 2 \qquad e_{2,i} &= pe_{2,i-1} + \begin{cases} 0 & \text{if } i \equiv 0 \bmod 3 \\ p & \text{otherwise.} \end{cases}
\end{aligned}$$

Note that these microstable exponents $e_{2,i}$ are are in general greater than the corresponding stable exponents $a_{2,i}$ defined in (16).

Then we have

$$\begin{aligned}
d(w_{2,0}) &\equiv v_1 s_1^p && \bmod (v_1^2), \\
d(w_{2,1}) &\equiv v_1^p v_2^{p\omega-1} s_1 && \bmod (v_1^{1+p}), \\
d(w_{2,2}) &\equiv -v_1^{p(p+1)} v_2^a s_2 && \bmod (v_1^{1+p(p+1)}), \\
d(w_{2,3}) &\equiv -v_1^{p^2(p+1)} v_2^{pa} s_2^p && \bmod (v_1^{1+p^2(p+1)}), \\
d(w_{2,4}) &\equiv -v_1^{p^3(p+1)} v_2^{p^2 a} w_{2,1}^{p-1} d(w_{2,1}) && \bmod (v_1^{1+p^2(p+1)+p}), \\
\text{and} \quad d(w_{2,i}) &\equiv -v_1^{e_{2,i}-e_{2,i-3}} v_2^{p^{i-2} a} w_{2,i-3}^{p-1} d(w_{2,i-3}) && \bmod (v_1^{1+e_{2,i}}) \\
&\quad \text{for } i > 4.
\end{aligned}$$

Alternatively we can get some simpler formulas by defining

$$
\begin{array}{rrcl}
 & w_{2,0} & = & u_2, \\
 & w_{2,1} & = & w_{2,0}^p, \\
 & w_{2,2} & = & w_{2,1}^p - v_1^{p^2} r_{2,3}^p - v_1^{p^2-1} v_2^{a+1} w_{2,0} \\
 & w_{2,3} & = & w_{2,2}^p, \\
 & w_{2,4} & = & w_{2,3}^p + v_1^{p^3(p+1)} v_2^{-p^3} (r_{2,4}^{p^2} - v_1^{-p(p+1)} v_2^{\omega p^4+p-p^2} w_{2,2} - v_2^{-p^3\omega} v_3^{\omega p^3} r_{2,3}^{p^2}), \\
\text{and} & w_{2,i} & = & w_{2,i-1}^p - v_1^{p^{i-1}(p+1)} v_2^{p^{i-2}a} w_{2,i-3}^{p-1} (w_{2,i-3} - w_{2,i-4}^p) \\
 & & & \qquad \text{for } i > 4,
\end{array}
$$

which leads to

$$
\begin{array}{rrcll}
 & d(w_{2,0}) & \equiv & v_1 s_1^p & \text{mod} \quad (v_1^2), \\
 & d(w_{2,1}) & \equiv & v_1^p s_1^{p^2} & \text{mod} \quad (v_1^{1+p}), \\
 & d(w_{2,2}) & \equiv & -v_1^{p(p+1)} v_2^{-p} s_2^{p^2} & \text{mod} \quad (v_1^{1+p(p+1)}), \\
 & d(w_{2,3}) & \equiv & -v_1^{p^2(p+1)} v_2^{-p^2} s_2^{p^3} & \text{mod} \quad (v_1^{1+p^2(p+1)}), \\
 & d(w_{2,4}) & \equiv & -v_1^{p^3(p+1)} v_2^{p^2 a} w_{2,1}^{p-1} d(w_{2,1}) & \text{mod} \quad (v_1^{1+p^2(p+1)+p}), \\
\text{and} & d(w_{2,i}) & \equiv & -v_1^{e_{2,i}-e_{2,i-3}} v_2^{p^{i-2}a} w_{2,i-3}^{p-1} d(w_{2,i-3}) & \text{mod} \quad (v_1^{1+e_{2,i}}) \\
 & & & \qquad \text{for } i > 4.
\end{array}
$$

These elements all pull back to an Ext group over $\widehat{G}(1,2,1)$, so we can map them via the map θ_m of (6) to

$$\mathrm{Ext}_{\Gamma(m+1)}(A, M_1^1)$$

for $m \geq 2$.

The Thom reduction. One can ask about the image of $\mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A},\, \widehat{A})$ in $\mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A},\, \widehat{A}/I)$, where $I = (p, v_1, v_2, \dots)$, since the latter can be computed explicitly. Each s_i is primitive mod I, so we have

$$\mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A},\, \widehat{A}/I) \quad = \quad \widehat{A}/I \otimes E(g_{i,j} : i > 0, j \geq 0) \otimes E(a_{i,j} : i > 0, j \geq 0)$$

where $g_{i,j} \in \mathrm{Ext}^{1,2p^j(p^i\omega-1)}$ corresponds to $s_i^{p^j}$, and $a_{i,j} \in \mathrm{Ext}^{2,2p^{j+1}(p^i\omega-1)}$ is its transpotent, not to be confused with the exponents $a_{i,j}$ of (15).

Let ρ denote the mod I reduction in Ext. Then we have

$$\begin{aligned}
\rho\left(\frac{u_1^t}{pt}\right) &= \begin{cases} u_1^{t-1}g_{1,0} & \text{for } p \text{ odd} \\ u_1^{t-1}g_{1,0} + (t-1)u_1^{t-2}g_{1,1} & \text{for } p=2. \end{cases}\\
\rho\left(\frac{u_1^s u_2^t}{pv_1}\right) &= stu_1^{s-1}u_2^{t-1}g_{1,1}g_{1,0} + tu_1^s u_2^{t-1}a_{1,0}\\
&\qquad + t(t-1)u_1^s u_2^{t-2}g_{1,1}g_{2,0}\\
\rho\left(\frac{u_1^s u_2^{p^j t}}{pv_1^{p^j}}\right) &= stu_1^{s-1}u_2^{(t-1)p^j}g_{1,j+1}g_{1,0}\\
&\qquad + tu_1^s u_2^{p^j(t-1)}a_{1,j} \qquad \text{for } j>0\\
\rho\left(\frac{u_3^t}{pv_1v_2}\right) &= t(t-1)u_3^{t-2}(g_{1,2}a_{2,0} - g_{2,1}a_{1,1})\\
&\qquad + t(t-1)(t-2)u_3^{t-3}g_{1,2}g_{2,1}g_{3,0}
\end{aligned}$$

Hence the image appears to be rather complicated.

On the other hand, it appears likely that all of the $a_{i,j}$ are in the image. Given $x \in BP_*[s_1, \dots] \otimes \mathbf{Q}$, let $x^{(j)}$ denote the expression obtained from x by replacing each v_k and s_k by its p^jth power. Using chromatic notation, we conjecture that

$$A_{i,j} = \sum_{0 \le k < i} \frac{(p^{i-1}\ell_k s_{i-k}^{p^k})^{(j+1)}}{p^i}$$

is a cocycle that reduces to $a_{i,j}$ mod I. For example we have

$$A_{1,j} = \frac{s_1^{p^{j+1}}}{p}$$

which is cohomologous to

$$\sum_{0<k<p^{j+1}} p^{-1}\binom{p^{j+1}}{k} s_1^k \otimes s_1^{p^{j+1}-k} \equiv \sum_{0<k<p} p^{-1}\binom{p}{k} s_1^{kp^j} \otimes s_1^{(p-k)p^j} \bmod (p),$$

which is the usual definition of $a_{1,j}$.

Next we consider $A_{2,j}$. Araki's definition of the v_i gives

$$v_1 \equiv p\ell_1 \bmod (p^2),$$

so the primitivity of $s_2 + \ell_1 s_1^p$ implies that the coproduct on s_2 is congruent to

$$s_2 \otimes 1 + 1 \otimes s_2 - v_1 \sum_{0<k<p} p^{-1}\binom{p}{k} s_1^k \otimes s_1^{p-k}$$

modulo p. Now let d denote the differential in the cobar complex we have

$$\begin{aligned}
d(s_2) &\equiv -v_1 \sum_{0<k<p} p^{-1}\binom{p}{k} s_1^k \otimes s_1^{p-k} \qquad \bmod (p)\\
\text{so} \qquad d(s_2^{p^{j+1}}) &\equiv -v_1^{p^{j+1}} \sum_{0<k<p} p^{-1}\binom{p}{k} s_1^{kp^{j+1}} \otimes s_1^{(p-k)p^{j+1}}\\
\text{and} \qquad d(ps_2^{p^{j+1}} + v_1^{p^{j+1}} s_1^{p^{j+2}}) &\equiv 0 \qquad \bmod (p^2).
\end{aligned}$$

It follows that

$$A_{2,j} = \frac{ps_2 p^{j+1} + v_1^{p^{j+1}} s_1^{p^{j+2}}}{p^2}$$

is a cocycle, and it is easily seen that it is cohomologous to

$$\sum_{0<k<p} p^{-1} \binom{p}{k} s_2^{kp^j} \otimes s_2^{(p-k)p^j}$$

modulo (p, v_1).

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