AN APPLICATION OF THE THOMIFIED EILENBERG-MOORE SPECTRAL SEQUENCE TO THE SPECTRA $T(n)$

DOUGLAS C. RAVENEL
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Theorem 1. *The first nontrivial differential in the Adams–Novikov spectral sequence for the spectrum $T(1)$ at an odd prime p is*

$$d_{2p-1}(b_{3,0}) = h_{2,0}b_{2,0}^p$$

where $b_{3,0} \in E_2^{2,2p^4-2p}$.

This element is in dimension $2p^4 - 2p - 2$, which is 154 for $p = 3$ and 1238 for $p = 5$. The notation will be explained below.

$T(1)$ is the p-local spectrum with

$$BP_*(T(1)) = BP_*[t_1].$$

It is the minimal summand of the p-local Thom spectrum for the stable complex vector bundle induced by the map

$$\Omega SU(p) \to \Omega SU = BU.$$

It is also the p-local Thom spectrum for the map

$$\Omega S^{2p-1} \to BU$$

obtained by extending the generator of

$$\pi_{2p-2}(BU) = Z$$

from S^{2p-2} to ΩS^{2p-1}.

More generally for each $n \geq 0$ there is a p-local ring spectrum $T(n)$ with

$$BP_*(T(n)) = BP_*[t_1, \dots, t_n].$$

It is a minimal summand of the p-local Thom spectrum associated with the map

$$\Omega SU(m) \to \Omega SU = BU$$

for any m with $p^n \leq m < p^{n+1}$. This was proved in [Rav86] as Theorem 6.5.1.

We will now consider the Adams–Novikov spectral sequence for $T(n)$. For a spectrum let, let

$$\mathrm{Ext}(BP_*(X)) = \mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*(X))$$

Then the following was proved in [Rav86] as Proposition 6.5.9 and Theorem 6.5.11.

Theorem 2. (i) *Let $A(n) = Z_{(p)}[v_1, \dots, v_n] \subset BP_*$. Then the map*

$$\mathrm{Ext}^0(BP_*(T(n))) \to \mathrm{Ext}^0(BP_*(BP)) = BP_*$$

is monomorphic with image $A(n)$.

(ii) *Unless* $n=0$ *and* $p=2$ *(which is the subject of* [Rav86, 5.2.6]*)*, $\mathrm{Ext}^1(BP_*(T(n)))$ *is the* $A(n)$*-module generated by the images of the elements*

$$\frac{v_{n+1}}{pi} \in \mathrm{Ext}^0(BP_*(T(n)) \otimes N^1)$$

where $N^1 = BP_* \otimes \mathbf{Q}/Z_{(p)}$ *and we are mapping into* Ext^1 *with the connecting homomorphism for the short exact sequence*

$$0 \to BP_* \to BP_* \otimes \mathbf{Q} \to N^1 \to 0.$$

For the 2-line and above, we have the following, essentially proved as Theorem 7.1.13 in [Rav86].

Theorem 3. *For* $n > 0$, $\mathrm{Ext}^{2,t}(BP_*(T(n)))$ *for* $t \le 2(p^{n+3} - p)$ *is the* $A(n+1)$*-module generated by*

$$\left\{ \frac{v_{n+2}^i}{ipv_1}, \frac{v_{n+2}^p}{pv_1^i} : 1 \le i \le p \right\} \cup \left\{ \frac{v_{n+3}}{pv_1} - \frac{v_2 v_{n+2}^p}{pv_1^{1+p}} + \frac{v_2^{p^{n+1}} v_{n+1}}{p^2 v_1} \right\}$$

and each of these supports a copy of

$$E(h_{n+1,0}) \otimes P(b_{n+1,0}) \qquad \text{where} \quad h_{n+1,0} = \frac{v_{n+1}}{p} \quad \text{and} \quad b_{n+1,0} = \frac{v_{n+2}}{pv_1}.$$

We also let

$$\begin{aligned} b_{n+1,1} &= \frac{v_{n+1}^p}{pv_1} \\ \text{and} \quad b_{n+2,0} &= \frac{v_{n+3}}{pv_1} - \frac{v_2 v_{n+2}^p}{pv_1^{1+p}} + \frac{v_2^{p^{n+1}} v_{n+1}}{p^2 v_1}. \end{aligned}$$

Remark: Theorem 7.1.13 of [Rav86] omits the element $b_{3,0}$ and is only correct for $t < |v_{n+2}^p|$. The extra generator in that dimension follows easily from other results of [Rav86, Chapter 6], and we will extend the methods and resilts of Chapter 7 in a future paper.

This Ext group is illustrated for $n = 1$ and $p = 3$ in Figure .

To prove Theorem 1 we use cohomology operations in $T(1)$-theory, i.e., maps

$$T(1) \xrightarrow{\quad r_i \quad} \Sigma^{2i(p-1)} T(1)$$

derived from the splitting of $T(1) \wedge T(1)$. They have properties similar to Steenrod and Quillen operations, but *they commute with each other.* In E_2 we have

$$\begin{aligned} r_1(b_{3,0}) &= 0, \\ r_p(b_{3,0}) &= -b_{2,1}, \\ \text{and} \qquad r_1(b_{2,1}) = r_p(b_{2,1}) &= 0. \end{aligned}$$

Lemma 4. *In* $\pi_*(T(1))$,

$$r_1(b_{2,1}) = \pm b_{2,0}^p \qquad \text{and} \qquad r_p(b_{2,1}) = 0.$$

This means that in $\pi_*(T(1))$,

$$\begin{aligned} r_1 r_p(b_{3,0}) &= b_{2,0}^p \\ \text{but} \qquad r_p r_1(b_{3,0}) &= 0, \end{aligned}$$

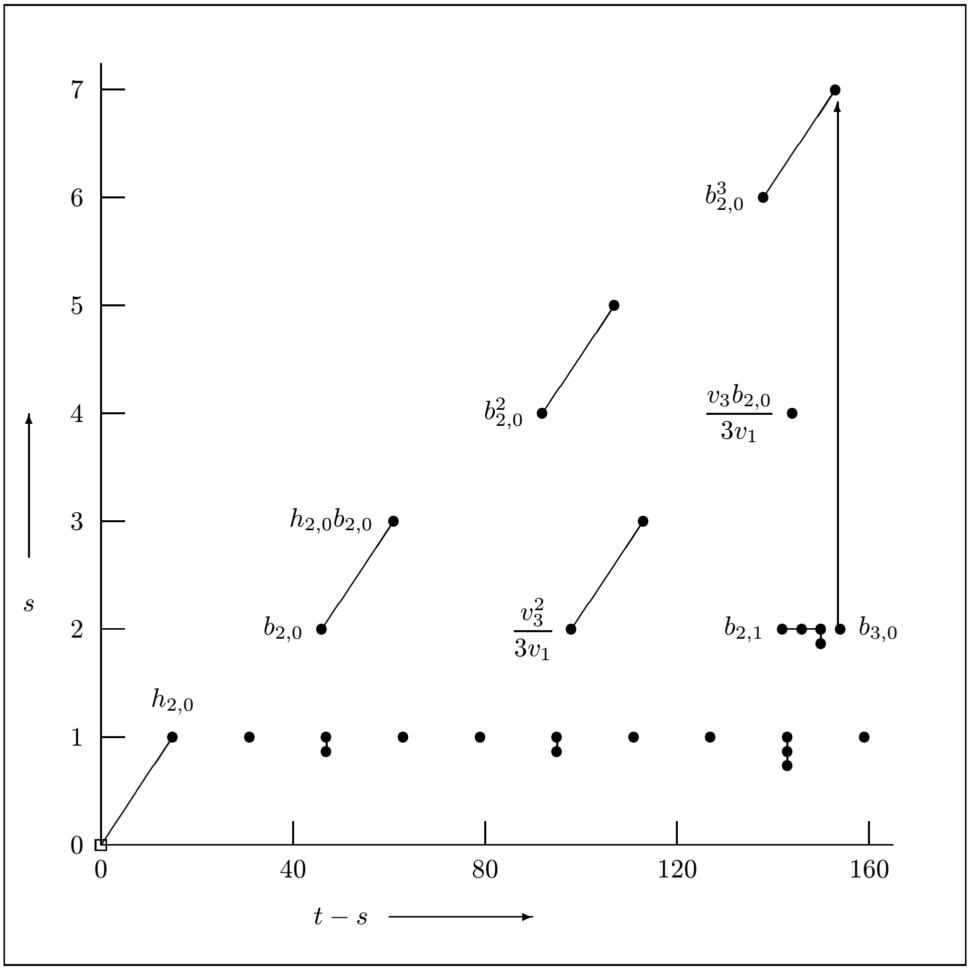

FIGURE 1. The Adams-Novikov E_2-term for $T(1)$ at $p = 3$ in dimensions ≤ 154, showing the first nontrivial differential. Elements on the 0- and 1-lines divisible by v_1 are not shown. Elements on the 2-line and above divisible by v_2 are not shown.

which is a contradicts the commutativity of the r_i. This means that $b_{3,0}$ cannot survive to represent a homotopy element. The indicated differential

$$d_{2p-1}(b_{3,0}) = h_{2,0}b_{2,0}^p$$

is the only one it can support.

To prove Lemma 4 we need the Thomified Eilenberg-Moore spectral sequence of [MRS]. Given a fibration of spaces

$$X \to E \to B$$

with a stable vector bundle over E, we get a spectral sequence converging to the homotopy of Y, the Thom spectrum for the induced bundle over X.

If the fibration is

$$\Omega SU(p^2 - 1) \to \Omega SU \to B$$

with the evident vector bundle over $\Omega SU = BU$, we get the usual Adams–Novikov spectral sequence for $X(p^2-1)$, the Thom spectrum associated with $\Omega SU(p^2-1)$, which has $T(1)$ as a summand.

If we take the Cartesian product of this fibration with

$$\Omega^2 S^{2p^2-1} \to \text{pt.} \to \Omega S^{2p^2-1},$$

the E_2-term is a subquotient of the tensor product of the one above with $H_*(\Omega^2 S^{2p^2-1})$ equipped with the Eilenberg-Moore filtration.

The map

$$\Omega^2 S^{2p^2-1} \to \Omega SU(p^2-1)$$

sends this homology to

$$E(h_{2,0}, h_{2,1}, \dots) \otimes P(b_{2,0}, b_{2,1}, \dots)$$

in the Adams–Novikov spectral sequence for $T(1)$. The lemma then follows from the fact that in $H_*(\Omega^2 S^{2p^2-1})$,

$$P_*^1(b_{2,1}) = b_{2,0}^p \qquad \text{and} \qquad P_*^p(b_{2,1}) = 0.$$

Next we will show that there are no differentials in a similar range for $T(n)$ for $n > 1$. Figure illustrates this for $p = 3$ and $n = 2$. There is no generator on the 2-line in the right position to kill $h_{3,0}b_{3,0}^3$, and there is no target for a differential supported by $b_{4,0}$.

References

[MRS] M. E. Mahowald, D. C. Ravenel, and P. Shick. The Thomified Eilenberg-Moore spectral sequence. `http://www.math.rochester.edu:8080/u/drav/temss.dvi`.

[Rav86] D. C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres.* Academic Press, New York, 1986.

University of Rochester, Rochester, NY 14627

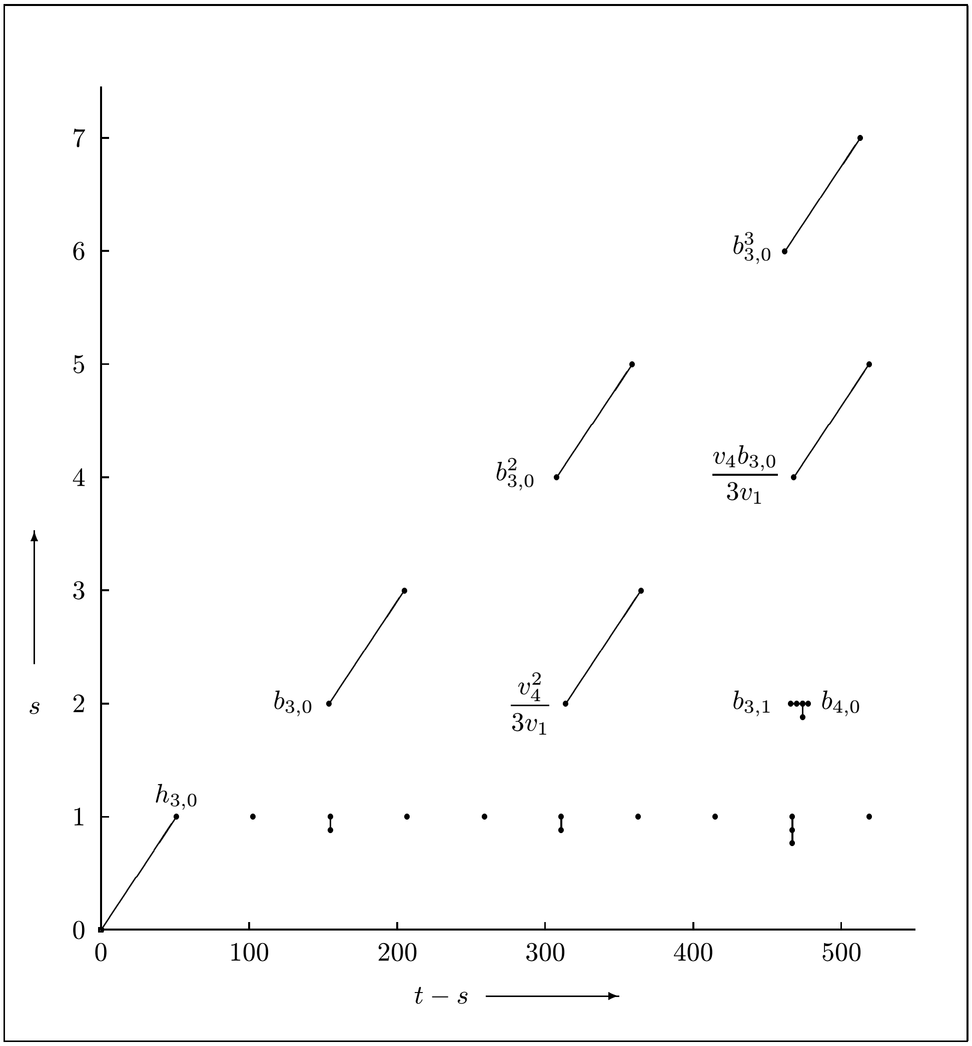

FIGURE 2. The Adams-Novikov E_2-term for $T(2)$ at $p = 3$ in dimensions ≤ 520. Elements on the 0- and 1-lines divisible by v_1 or v_2 are not shown. Elements on the 2-line and above divisible by v_2 or v_3 are not shown.