HIGH DIMENSIONAL KNOTS WITH $\pi_1 \cong \mathbb{Z}$ ARE DETERMINED BY THEIR COMPLEMENTS IN ONE MORE DIMENSION THAN FARBER'S RANGE

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Abstract. The surgery theory of Browder, Lashof and Shaneson reduces the study of high-dimensional smooth knots $\Sigma^n \hookrightarrow S^{n+2}$ with $\pi_1 \cong \mathbb{Z}$ to homotopy theory. We apply Williams's Poincaré embedding theorem to the unstable normal invariant $\rho\colon S^{n+2} \to \Sigma(M/\partial M)$ of a Seifert surface $M^{n+1} \hookrightarrow S^{n+2}$. Then a knot is determined by its complement if the $\mathbb{Z}$-cover of the complement is $[(n+2)/3]$-connected; we improve Farber's work by one dimension.

0. Introduction

A high-dimensional n-knot will mean a smooth, oriented, codimension two embedding $\Sigma^n \hookrightarrow S^{n+2}$ of an exotic sphere, with $n \geq 5$. See the survey of Kervaire and Weber [K-W] for more details. For our purposes, two knots $\Sigma_i^n \hookrightarrow S^{n+2}$ are said to be *equivalent* if there is diffeomorphism $\phi\colon S^{n+2} \xrightarrow{\cong} S^{n+2}$ such that $\phi(\Sigma_1^n) = \Sigma_2^n$. A knot $\Sigma^n \hookrightarrow S^{n+2}$ has a *complement* $X = \overline{S^{n+2} - \Sigma^n \times D^2}$, and is *determined by its complement* if it is equivalent to any other knot with diffeomorphic complement. The orientation of Σ^n and the trivialization of the normal bundle neighborhood give a preferred diffeomorphism $\beta\colon \Sigma^n \times S^1 \xrightarrow{\cong} \partial X$. The Poincaré conjecture gives an oriented homeomorphism $\varpi\colon S^n \to \Sigma^n$. We will call the composite

$$\alpha\colon S^n \times S^1 \xrightarrow{\varpi \times \mathrm{id}} \Sigma^n \times S^1 \xrightarrow{\beta} \partial X \xrightarrow{\iota} X$$

the *attaching map* of the knot. Let $\tau\colon S^n \times S^1 \xrightarrow{\simeq} S^n \times S^1$ be the homotopy equivalence (diffeomorphism) given by the generator of $\pi_1(SO(n+1)) \cong \mathbb{Z}/2$.

A knot $\Sigma^n \hookrightarrow S^{n+2}$ is called *r-simple* [Ke2, Fa1] if the $\mathbb{Z}$-cover $\widetilde{X}$ of the complement is r-connected. Levine and Browder's [Le3, B-L, Le2] work shows that $[(n+1)/2]$-simple n-knots are trivial. Following Kervaire-Milnor [K-M], Levine [Le1] used ambient surgery on the Seifert surface to show that PL $[(n-1)/2]$-simple n-knots were determined by their complements for n odd. Levine's work was partially extended to the case n even by Kearton [Ke1] and Kojima [Ko]. Using Wall's thickening theory [Wa3], Farber [Fa2] showed this for $([n/3]+1)$-simple n-knots.

Theorem A. *For $n+3 \leq 3q$ and $n \geq 5$, $(q-1)$-simple smooth knots $\Sigma^n \hookrightarrow S^{n+2}$ are determined by their complements.*

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Theorem B. *Let $\Sigma^n \hookrightarrow S^{n+2}$ be a knot with complement X and attaching map $\alpha\colon S^n \times S^1 \to X$. There exists a homotopy equivalence $\zeta\colon X \xrightarrow{\simeq} X$ so the diagram*

$$\begin{array}{ccc} S^n \times S^1 & \xrightarrow{\alpha} & X \\ \tau\big\downarrow & & \big\downarrow\zeta \\ S^n \times S^1 & \xrightarrow{\alpha} & X \end{array} \tag{1}$$

commutes up to homotopy, if the knot is $(q-1)$-simple, $n+3 \leq 3q$ and $n \geq 5$.

We prove Theorem B via Williams's Theorem 1.7. In our metastable range the Poincaré embedding $M \times I \hookrightarrow S^{n+2}$ of a Seifert surface is determined by its *unstable normal invariant* $\rho \in \pi_{n+2}(\Sigma(M/\partial M))$. We construct another Poincaré embedding $M \times I \hookrightarrow S^{n+2}$ suggested by $\alpha \cdot \tau$ (Lemma 1.4), with the same unstable normal invariant ρ (Theorem 1.6). Theorem 1.7 implies the two Poincaré embeddings are *concordant.* We use this concordance to construct the homotopy automorphism ζ.

By the work of Browder, Lashof, Shaneson and Gluck [Br1, L-S, Gl], it is well known that Theorem B implies Theorem A for piecewise linear (PL) knots. In §2 we extend their work to smooth knots.

Our proof requires Levine's result [Le3], that there exists highly connected Seifert surfaces. Together with Barratt [B-R], circa '82, we have a purely homotopy proof which uses $\mathbb{Z}$-equivariant Hopf invariants and Ranicki's [Ra] equivariant S-duality.

We conjecture that high-dimensional knots with $\pi_1 \cong \mathbb{Z}$ are determined by their complements. If $\pi_1 \not\cong \mathbb{Z}$, there exist counterexamples due to Cappell and Shaneson, Gordon, and Suciu [C-S, Go, Su]. Theorem 1.6, which is true outside our range $n+3 \leq 3q$, and the Appendix provide some evidence for this conjecture.

We have a homotopy theoretic proof [Ri] of Theorem 1.7, completing a program of Williams's [Wi2], to prove the result using Browder-Quinn Poincaré surgery [Br4, Qu]. The present paper (except §3) is independent of [Ri], but not [Wi2].

Given a subspace $A \subset X$ and a map $f\colon A \to Y$, we will write the identification space $X \cup_f Y$ as $\varinjlim(X \xleftarrow{\iota} A \xrightarrow{f} Y)$, the *colimit* or *pushout* of the diagram.

I want to thank William Browder and Andrew Ranicki for sparking my interest in Poincaré embeddings and knots with $\pi_1 \cong \mathbb{Z}$, and Sylvain Cappell for many enlightening conversations about whether such knots are determined by their complements. I want to thank Bruce Williams for explaining Farber's work to me, and encouraging me to finish his program [Wi2, Ri]. I want to thank Michael Barratt, Neal Stolzfus, Fred Cohen, Derek Hacon and Nigel Ray for helpful conversations about Farber's work and Hopf invariants. The proof of Theorem 1.6 is largely due to Jeff Smith. I want to thank to the referee for helpful comments about organizing the paper. Thanks to Paul Burchard for developing the LaTeX commutative diagram package diagram.sty, and Michael Spivak for his LAMS-TeX fonts.

1. Poincaré embeddings and the proof of Theorem B

In Theorem B we can eliminate the condition that ζ be a homotopy equivalence.

Lemma 1.1. *Any selfmap ζ of the knot complement X making diagram (1) commute up to homotopy is a homotopy equivalence.*

Proof. Diagram (1) implies that ζ is a selfmap of the pair Poincaré $(X, \partial X)$, and $\zeta_*[X] = [X] \in H_{n+2}(X, \partial X)$. Furthermore ζ induces the identity on $\pi_1(X) \cong \mathbb{Z}$. By naturality of $\mathbb{Z}$-equivariant Poincaré duality [Le2, Wa2], the composite

$$H^{n+2-*}_\Lambda\left(\widetilde{X}, \partial\widetilde{X}; \Lambda\right) \xrightarrow{\zeta^*} H^{n+2-*}_\Lambda\left(\widetilde{X}, \partial\widetilde{X}; \Lambda\right) \xrightarrow{[X]\cap\cdot} H_*\left(\widetilde{X}\right) \xrightarrow{\zeta_*} H_*\left(\widetilde{X}\right)$$

is the cap product isomorphism $[X] \cap \cdot$. Hence ζ_* is surjective. Since the group ring $\Lambda = \mathbb{Z}[\mathbb{Z}]$ is Noetherian and X is a finite complex, $\zeta_* \colon H_*(\widetilde{X}) \to H_*(\widetilde{X})$ must be an isomorphism, hence ζ is a homotopy equivalence by the Whitehead theorem. $\square$

For a smooth knot $\Sigma^n \hookrightarrow S^{n+2}$ with complement $(X, \partial X)^{n+2}$, Alexander duality and relative transversality give a map $h \colon X \to S^1$ which is transverse to the point $1 \in S^1$, with inverse image $h^{-1}(1) = (M, \partial M)$, and $\partial M = \Sigma^n$. M^{n+1} is called a *Seifert surface* for the knot. By the relative tubular neighborhood theorem there is a codimension zero embedding $M \times I \subset X$ extending the embedding $\Sigma^n \times I \subset \partial X = \Sigma^n \times S^1 = \Sigma^n \times I \cup \Sigma^n \times I$, where I is the interval $[-1, 1]$. Let $A = \partial(M \times I)$.

Let $W = \overline{X - M \times I}$ be the *Seifert surface complement*. The knot complement X is then obtained by glueing together $M \times [-1, 1]$ and W along their common boundary $M \times \{-1, 1\}$. By writing X as the union of the Seifert surface and its complement, we obtain the decomposition $S^{n+2} = M \times I \cup W \cup \Sigma^n \times D^2$. Let $\hat{W}$ be the manifold with corners $\hat{W} = W \cup \Sigma^n \times D^2$, so that $A = \partial \hat{W}$. Let $\epsilon \colon \hat{W} \twoheadrightarrow W$ be the deformation retraction which maps $\Sigma^n \times D^2$ onto $\Sigma^n \times I$, using a map $D^2 \twoheadrightarrow I$. Let $f \colon A \to W$ be the composite of the inclusion $A = \partial(M \times I) = \partial\hat{W} \subset \hat{W}$ and the retraction $\epsilon \colon \hat{W} \to W$. Note that f is a cofibration, since ϵ restricts to a homeomorphism $\epsilon \colon \partial\hat{W} \to \partial W$.

Williams [Wi1] studies codimension zero *Poincaré embeddings* [Br1, Br2, Br3, Wa1] of an m-dimensional oriented finite Poincaré pair $(Y, \partial Y)$ in the sphere S^m, which consist of a *complement* Z along with an *attaching map* $f \colon \partial Y \to Z$, such that the pushout $Y \cup_f Z$ is homotopy equivalent to S^m. Williams [Wi1] defines two Poincaré embeddings $(Y, \partial Y) \hookrightarrow S^{n+2}$ with attaching maps $f_1 \colon \partial Y \to Z_1$ and $f_2 \colon \partial Y \to Z_2$ to be *concordant* if there exists a homotopy equivalence $\xi \colon Z_1 \to Z_2$ so that $f_2 \simeq \xi \cdot f_1 \colon \partial Y \to Z_2$.

For the above Seifert surface embedding, $(M, \partial M)$ is an $(n+1)$-dimensional oriented Poincaré pair, with an oriented homeomorphism $\varpi \colon S^n \to \partial M$ given by the orientation of the knot. $(M \times I, A)$ is an $(n+2)$-dimensional Poincaré pair, with $A = M \times \{\pm 1\} \cup S^n \times I$, and the attaching map $f \colon A \to W$ gives a Poincaré embedding $(M \times I, A) \hookrightarrow S^{n+2}$, by the homotopy equivalence $S^{n+2} = M \times I \cup \hat{W} \xrightarrow{1 \cup \epsilon} M \times I \cup_f W$. Let $\omega \colon S^n \to M$ be the composite of ϖ and the inclusion $\iota \colon \partial M \to M$. Let $v_\pm \colon M \to W$ be the restriction of f to $M \times \pm 1 \subset A$. Let $\iota_\pm \colon M \to A$ be the inclusions $M \times \{\pm 1\} \subset A$. For any map $g \colon A \to W$ we denote by $g_\pm$ the restrictions $g_+ : M \times \{\pm 1\} \subset A \xrightarrow{f} W$ and $g_- : S^n \times I \subset A \xrightarrow{f} W$.

Definition 1.2. Define $\delta \colon S^n \times I \xrightarrow{\cong} S^n \times I$ by $\delta(x, t) = \left(e^{i\pi(t+1)} \cdot x, t\right)$, using the standard action of $S^1 = SO(2) \subset SO(n+1)$ on S^n. We define the diffeomorphism τ of $S^n \times S^1 = \varinjlim(S^n \times I \xleftarrow{\iota} S^n \times \{\pm 1\} \xrightarrow{\iota} S^n \times I)$ to be the identity on the left $S^n \times I$ and δ on the right $S^n \times I$. We define the selfmap γ of $A = \varinjlim(M \times \{\pm 1\} \xleftarrow{\omega \times \mathrm{id}} S^n \times \{\pm 1\} \xrightarrow{\iota} S^n \times I)$ to be the identity on $M \times \{\pm 1\}$ and δ on $S^n \times I$.

Lemma 1.3. *Let* $S^n \times S^1 \xrightarrow{\text{coaction}} S^n \times S^1 \vee S^{n+1}$ *and* $A \xrightarrow{\text{coaction}} A \vee S^{n+1}$ *be the* coaction maps [B-B, Ga, Ba] *onto the top cells. Then*

(1) *The selfmap* τ *of* $S^n \times S^1$ *is homotopic to the composite*

$$\tau' \colon S^n \times S^1 \xrightarrow{\text{coaction}} S^n \times S^1 \vee S^{n+1} \xrightarrow{\text{id} \vee \eta} S^n \times S^1 \vee S^n \xrightarrow{\text{id} \vee \iota_1} S^n \times S^1.$$

(2) *The selfmap* γ *of* A *is homotopic to the composite*

$$A \xrightarrow{\text{coaction}} A \vee S^{n+1} \xrightarrow{\text{id} \vee \eta} A \vee S^n \xrightarrow{\text{id} \vee \omega} A \vee M \xrightarrow{\text{id} \vee \iota_+ \cdot} A.$$

Proof. By the Barratt-Puppe sequence of the CW-complex $S^n \times S^1 = S^n \vee S^1 \cup e^{n+1}$, the selfmap τ' is characterized up to homotopy by the property that: τ' induces is the identity in homology; and the Hopf construction of the composite $S^n \times S^1 \xrightarrow{\tau'} S^n \times S^1 \xrightarrow{\pi_1} S^n$ is the generator $\eta \in \pi_{n+2}(S^{n+1}) \cong \mathbb{Z}/2$. But τ clearly satisfies both of these properties; hence (1). Now consider the rel boundary coaction map

$$S^n \times I \xrightarrow{\text{coaction}} S^n \times I \vee S^{n+1}$$

of $S^n \times I$. We can define the coaction map of $A = M \times I \cup S^n \times I$ onto its top cell by glueing the identity map on the left half $M \times I$ to the above rel boundary coaction map. Furthermore the selfmap δ of $S^n \times I$ is the identity on $S^n \times \{\pm 1\} \cup N \times I$ for some point N. Therefore δ is homotopic, rel boundary, to the composite

$$S^n \times I \xrightarrow{\text{coaction}} S^n \times I \vee S^{n+1} \xrightarrow{\text{id} \vee g} S^n \times I \vee S^n \xrightarrow{\text{id} \vee \iota_1} S^n \times I$$

for some map $g \in \pi_{n+1}(S^n)$. By part (1), we see that $g = \eta \in \pi_{n+1}(S^n)$. By glueing in this rel boundary homotopy, the second assertion follows. □

Lemma 1.4. *Let* $f \colon A \to W$ *be the attaching map of a Poincaré embedding* $(M \times I, A) \hookrightarrow S^{n+2}$*. Then the composite* $A \xrightarrow{\gamma} A \xrightarrow{f} W$ *is also the attaching map of a Poincaré embedding* $(M \times I, A) \hookrightarrow S^{n+2}$*. Furthermore* $(f \cdot \gamma)_+ = f_+ \colon M \times \{\pm 1\} \to W$ *and* $(f \cdot \gamma)_- = f_- \cdot \delta \colon S^n \times I \to W$*.*

Consider the general case of a Poincaré embedding $(M, A) \hookrightarrow S^m$ of an oriented, finite m-dimensional Poincaré pair (M, A), with complement W and attaching map $f \colon A \to W$. Let $\nu \colon S^m \xrightarrow{\simeq} M \cup_f W$ be the homotopy equivalence so the composite

$$\rho \colon S^m \xrightarrow{\nu} M \cup_f W \to M/A$$

is orientation preserving. Williams [Wi1] calls $\rho \in \pi_m(M/A)$ the *unstable normal invariant* of $(M, A) \hookrightarrow S^m$. Williams [Wi2] shows that Browder's cofibration [Br3]

$$S^m \xrightarrow{\rho} M/A \xrightarrow{\Sigma f \cdot \partial} \Sigma W$$

is split by the degree one map $M/A \xrightarrow{\text{pinch}} S^m$, that the composite

$$M/A \xrightarrow{\partial} \Sigma A \xrightarrow{\Sigma(f) \cdot \partial \vee \text{pinch}} \Sigma W \vee S^m \tag{2}$$

is a homotopy equivalence. From this we deduce

Lemma 1.5. *Let* $\rho, \rho' \in \pi_m(M/A)$ *be the unstable normal invariants of the Poincaré embeddings* $(M, A) \hookrightarrow S^m$ *with attaching maps* $f \colon A \to W$ *and* $f' \colon A \to W'$*. Then*

(1) *$\rho = \rho'$ iff the composite $S^m \xrightarrow{\rho'} M/A \xrightarrow{\Sigma f \cdot \partial} \Sigma W$ is nullhomotopic.*
(2) *Suppose $W' = W$, and assume the suspensions of the attaching maps $f, f' \colon A \to W$ are homotopic; $\Sigma f = \Sigma f' \in [\Sigma A, \Sigma W]$. Then the unstable normal invariants are equal; $\rho = \rho' \in \pi_m(M/A)$.*

Now consider our Seifert surface Poincaré embedding $(M \times I, A) \hookrightarrow S^{n+2}$, with attaching map $f \colon A \to W$, and unstable normal invariant $\rho \in \pi_{n+2}((M \times I)/A)$.

Theorem 1.6. *The Poincaré embeddings $(M \times I, A) \hookrightarrow S^{n+2}$ with attaching maps $f \cdot \gamma, f \colon A \to W$ have equal unstable normal invariant.*

Proof. This follows from the codimension one framed embedding of the Seifert surface, which implies the vanishing of $\Sigma\omega \in \pi_{n+1}(\Sigma M)$. The cofibration sequence

$$S^{n+1} = \Sigma S^n \xrightarrow{\Sigma\omega} \Sigma M \xrightarrow{\Sigma\iota} \Sigma(M/S^n) \xrightarrow{\Sigma\partial} \Sigma^2 S^n = S^{n+2}$$

splits; we have a homotopy equivalence $\Sigma M \vee S^{n+2} \xrightarrow{\Sigma\iota \vee \rho} \Sigma(M/S^n)$. Hence $\Sigma\iota \colon \Sigma M \to \Sigma(M/S^n)$ has a left homotopy inverse, which implies that $S^{n+1} \xrightarrow{\Sigma\omega} \Sigma M$ is nullhomotopic. By Lemma 1.3 (2), $\Sigma\gamma \simeq \mathrm{id} \colon \Sigma A \to \Sigma A$. Hence the maps $\Sigma f, \Sigma(f \cdot \gamma) \colon \Sigma A \to \Sigma W$ are homotopic. The result follows from Lemma 1.5. □

We recall the uniqueness part of Williams's [Wi1] Poincaré embedding theorem.

Theorem 1.7. *Let (M, A) be an oriented, finite, m-dimensional Poincaré pair, with $\pi_1(A) = \pi_1(M) = 0$ and $m \geq 6$. Suppose M is n-dimensional as a CW-complex, and let $q = m - n - 1$. If $m < 3q$, then any two Poincaré embeddings of (M, A) in S^m whose unstable normal invariants are equal are concordant.*

Proof of Theorem B. Let $\Sigma^n \hookrightarrow S^{n+2}$ be a $(q-1)$-simple knot, $n+3 \leq 3q$, with knot complement X^{n+2}, and attaching map $\alpha \colon S^n \times S^1 \to X$. Let M^{n+1} be a Seifert surface with resulting Poincaré embedding $(M \times I, A) \hookrightarrow S^{n+2}$, with attaching map $f \colon A \to W$. By a theorem of Levine [Le3] we can assume that M^{n+1} is $(q-1)$-connected. By Poincaré duality of the pair $(M, \partial M)^{n+1}$, M is then $(n+1-q)$-dimensional. Theorem 1.6 and Theorem 1.7 imply the Poincaré embeddings with attaching maps $f \cdot \gamma, f \colon A \to W$ are concordant. Let $\xi \colon W \xrightarrow{\simeq} W$ be a concordance, so $\xi \cdot f \simeq f \cdot \gamma \colon A \to W$. Since the geometric map f is a cofibration, we may assume that $\xi \cdot f = f \cdot \gamma \colon A \to W$. By Lemma 1.4 we have $\xi \cdot f_+ = f_+$ and $\xi \cdot f_- = f_- \cdot \delta$. The knot complement X is the pushout $X = \varinjlim(M \times I \xleftarrow{\iota} M \times \{-1, 1\} \xrightarrow{f_+} W)$, so we can define a selfmap $\zeta \colon X \to X$ to be the identity on $M \times I$ and ξ on W. But the attaching map $\alpha \colon S^n \times S^1 \to X$ and the composite $\alpha \cdot \tau \colon S^n \times S^1 \to X$ are the induced maps of colimits of the following strictly commutative diagrams

$$\begin{array}{ccccc} M \times I & \xleftarrow{\iota} & M \times \{\pm 1\} & \xrightarrow{f_+} & W \\ \uparrow{\scriptstyle \omega\times\mathrm{id}} & & {\scriptstyle \omega\times\mathrm{id}}\uparrow & & {\scriptstyle f_-}\uparrow \\ S^n \times I & \xleftarrow{\iota} & S^n \times \{\pm 1\} & \xrightarrow{\iota} & S^n \times I \end{array} \qquad \begin{array}{ccccc} M \times I & \xleftarrow{\iota} & M \times \{\pm 1\} & \xrightarrow{f_+} & W \\ \uparrow{\scriptstyle \omega\times\mathrm{id}} & & {\scriptstyle \omega\times\mathrm{id}}\uparrow & & {\scriptstyle f_-\cdot\delta}\uparrow \\ S^n \times I & \xleftarrow{\iota} & S^n \times \{\pm 1\} & \xrightarrow{\iota} & S^n \times I. \end{array}$$

Using Definition (1.2), we have $\zeta \cdot \alpha = \alpha \cdot \tau \colon S^n \times S^1 \to X$, and thus diagram (1) commutes. By Lemma 1.1 $\zeta \colon X \to X$ is a homotopy equivalence. □

2. Proof of Theorem A; smooth knots and surgery

Lashof and Shaneson [L-S, Thm. 2.1] show that any self homotopy equivalence of a knot complement pair $(X, \partial X)^{n+2}$ is homotopic to a diffeomorphism, if $n \geq 4$ and $\pi_1(X) \cong \mathbb{Z}$. This follows from the Sullivan-Wall exact sequence [Wa2, §10]

$$0 = L_{n+3}\left(\mathbb{Z}[\mathbb{Z}] \xrightarrow{\text{id}} \mathbb{Z}[\mathbb{Z}]\right) \to \mathcal{S}^O(X) \to [X, G/O] = 0.$$

Let $\phi\colon X_1 \xrightarrow{\cong} X_2$ be a diffeomorphism between the knot complements of two smooth $(q-1)$-simple knots $\Sigma_i^n \hookrightarrow S^{n+2}$ with $n+3 \leq 3q$, $n \geq 5$, for $i = 1, 2$. The homotopy equivalence $\zeta\colon X_1 \xrightarrow{\simeq} X_1$ of Theorem B is thus homotopic to a diffeomorphism $\theta\colon X_1 \xrightarrow{\cong} X_1$. Following Browder [Br1, Cor. 2], we have we have an exact sequence

$$\Gamma^{n+1} \oplus \Gamma^{n+2} \xrightarrow{\iota} \mathcal{D}\text{iff}(\Sigma_1^n \times S^1) \xrightarrow{\mathcal{F}} \mathcal{E}(S^n \times S^1) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2, \tag{3}$$

involving the pseudo-isotopy and homotopy automorphism groups. The first two $\mathbb{Z}/2$ summands are given by the degree -1 maps of S^n and S^1, and the third summand is given by the selfmap τ, which is detected by the Hopf construction (cf. Lemma 1.3). We have to modify Browder's argument slightly, since $\mathcal{F}$ will not be surjective if the exotic sphere Σ_1^n does not possess an orientation-preserving diffeomorphism (cf. [K-M]).

Let $\beta_i\colon \Sigma_i^n \times S^1 \xrightarrow{\cong} \partial X_i$ be the preferred diffeomorphisms, for $i = 1, 2$. The restriction of ϕ to the boundary gives a diffeomorphism $\partial\phi\colon \Sigma_1^n \times S^1 \xrightarrow{\cong} \Sigma_2^n \times S^1$. If the Hopf construction of the composite $\pi_2 \cdot \partial\phi\colon \Sigma_1^n \times S^1 \to \Sigma_2^n$ is $\eta \in \pi_{n+2}\left(S^{n+1}\right) \cong \mathbb{Z}/2$, then replace the diffeomorphism ϕ by the composite $\phi \cdot \theta$. By Browder's application of the Browder-Levine fibering theorem [Br1, Lem. 2], we can assume that $\partial\phi$ restricts to a diffeomorphism $\phi_0\colon \Sigma_1^n \xrightarrow{\cong} \Sigma_2^n$. Let $\epsilon = \pm 1$ be the degree of $\partial\phi$ on the S^1 factor. Consider the diffeomorphism $\psi = (\phi_0 \times \epsilon)^{-1} \cdot \partial\phi \in \mathcal{D}\text{iff}(\Sigma_1^n \times S^1)$, which induces the identity in homology. Since its Hopf construction is zero, $\psi \simeq \text{id}$, so $\mathcal{F}(\psi) = \text{id}$, and (3) shows that $\partial\phi = (\phi_0 \times \epsilon) \cdot \psi$ is pseudoisotopic to the composite

$$\Sigma_1^n \times S^1 \xrightarrow{e} \Sigma_1^n \times S^1 \xrightarrow{d \times 1} \Sigma_1^n \times S^1 \xrightarrow{\phi_0 \times \epsilon} \Sigma_2^n \times S^1,$$

where $d \in \Gamma^{n+1}$ is a diffeomorphism of Σ_1^n, and $e \in \Gamma^{n+2}$ is obtained from the identity map of $\Sigma_1^n \times S^1$ by "connecting sum" with a diffeomorphism of an $(n+1)$-disk. We claim that $\partial\phi$ extends to a diffeomorphism of $\widetilde{\partial\phi}\colon \Sigma_1^n \times D^2 \xrightarrow{\cong} \Sigma_2^n \times D^2$. Certainly $(\phi_0 \cdot d) \times \epsilon$ extends. But e must be pseudoisotopic to the identity, otherwise we could glue in the tubular neighborhoods to get a diffeomorphism between the standard sphere S^{n+2} and the exotic sphere represented by $e \in \Gamma^{n+2}$. By glueing together ϕ with this extension $\widetilde{\partial\phi}$ we have an equivalence of the two knots. □

3. Appendix

Farber [Fa2] shows that PL $(q-1)$-simple knots with $n+3 < 3q$ are determined by the homotopy class $v_+ \in [M, W]$, which is stable by the Freudenthal suspension theorem. Let $\rho_0 = \partial \cdot \rho\colon S^m \to \Sigma A$ be the composite. Using the S-duality map [Ri]

$$\mathcal{D}\colon S^{n+2} \xrightarrow{\rho_0} \Sigma A \xrightarrow{\Sigma\Delta} \Sigma A \wedge A \xrightarrow{\Sigma(f \wedge \iota)} \Sigma W \wedge M$$

we have a bijection $[M, W] \xrightarrow{\mathfrak{D}\cap\cdot} \pi^s_{n+2}\left(\Sigma W^{[2]}\right)$ in our range. We note that Farber uses a dual S-duality map $M \wedge W \to S^{n+1}$. We show that Farber's stable homotopy invariant is essentially the second Hopf invariant $\lambda_2(\rho)$ of our unstable normal invariant $\rho\colon S^{n+2} \to (M \times I)/A = \Sigma(M/\partial M)$.

Theorem 3.1. *The 2^{nd} Hopf invariant of the unstable normal invariant $\rho\colon S^{n+2} \to \Sigma(M/\partial M)$ is the S-dual of the map $v_+\colon M \to W$:*

$$\lambda_2(\rho) = \left[\Sigma\iota \cdot (\Sigma v_- - \Sigma v_+)^{-1}\right]^{[2]} (\mathrm{id} \wedge v_+) \cdot \Sigma\mathfrak{D} \in \pi^s_{n+3}\left((\Sigma(M/\partial M))^{[2]}\right),$$

using the isomorphisms

$$v_+ \in [M, W] \underset{\cong}{\xrightarrow{\Sigma\mathfrak{D}\cap\cdot}} \pi^s_{n+3}\left((\Sigma W)^{[2]}\right) \underset{\cong}{\xrightarrow{\left[\Sigma\iota\cdot(\Sigma v_- - \Sigma v_+)^{-1}\right]^{[2]}}} \pi^s_{n+3}\left((\Sigma(M/\partial M))^{[2]}\right).$$

Consider the general case of a Poincaré embedding $(M, A) \hookrightarrow S^m$, with complement W and attaching map $f\colon A \to W$ as in [Ri]. The *boundary map* $\partial\colon M/A \to \Sigma A$ is defined to be the homotopy class making the diagram

$$\begin{array}{ccc} M \cup_\iota CA & \xrightarrow{\text{pinch}} & \Sigma A \\ {\scriptstyle\text{pinch}}\downarrow{\scriptstyle\simeq} & \nearrow{\scriptstyle\partial} & \\ M/A & & \end{array} \qquad \left(\begin{array}{l} CA = A \times [0,1]/A \times 0, \\ \Sigma A = CA/A = A \wedge ([0,1]/\{0,1\}) \end{array}\right)$$

commute up to homotopy. Extending the splitting (2), from Williams's work [Wi2] we have a homotopy equivalence

$$\Pi = x \cdot \Sigma f + y \cdot \text{pinch} + z \cdot \Sigma\iota\colon \Sigma A \xrightarrow{\simeq} \Sigma W \vee S^m \vee \Sigma M,$$

where x, y and z are the inclusions of the three factors, and pinch: $\Sigma A \to S^m$ is the unique degree one homotopy class. The two maps $\Pi, y\colon \Sigma A \to \Sigma W \vee S^m \vee \Sigma M$ are equalized up to homotopy by the collapse map $M \cup_{A\times 1} A \times [0,1] \cup_f W \to \Sigma A$: the first and third maps $x \cdot \Sigma f$ and $z \cdot \Sigma\iota$ can be nullhomotoped when restricted to $M \cup_{A\times 1} A \times [0,1] \cup_f W$ since the ends M and W are "free" (as in the proof of the equivalence of Whitehead products and Samelson products [Wh]). Thus the diagram

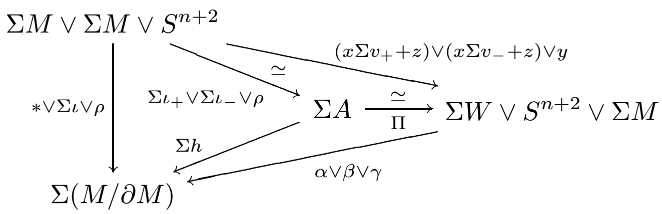
$$\begin{array}{ccccc} M/A & \xleftarrow{\rho} & S^m & \xrightarrow{y} & \Sigma W \vee S^m \vee \Sigma M \\ {\scriptstyle\text{pinch}}\uparrow & & {\scriptstyle\simeq}\downarrow & \searrow{\scriptstyle\rho_0} & {\scriptstyle\simeq}\uparrow{\scriptstyle\Pi} \\ M \cup_\iota CA & \xleftarrow{\text{pinch}} & M \cup_{A\times 1} A \times [0,1] \cup_f W & \xrightarrow{\text{pinch}} & \Sigma A \end{array}$$

is homotopy commutative. Now apply Boardman and Steer's Cartan formula and composition formula [B-S] to the equation $\Pi \cdot \rho_0 \simeq y\colon S^m \to \Sigma W \vee S^m \vee \Sigma M$. We obtain $\Pi \wedge \Pi \cdot \lambda_2(\rho_0) + \lambda_2(\Pi) \cdot \Sigma\rho_0 = 0$ and $\lambda_2(\Pi) = x\Sigma f \smile z\Sigma\iota$, which implies

$$\Pi \wedge \Pi \cdot \lambda_2(\rho_0) = -\left(x\Sigma f \smile z\Sigma\iota\right) \cdot \Sigma\rho_0 \in \pi_{m+1}\left((\Sigma W \vee S^m \vee \Sigma M)^{[2]}\right).$$

Proof of Theorem 3.1. Now consider a Seifert surface $(M \times I, A) \hookrightarrow S^{n+2}$, with complement W and attaching map $f\colon A \to W$. Let $h\colon A \to M/\partial M$ be defined by collapsing the subspace $M \times 1 \cup \partial M \times I$. Then we see that Σh is a homotopy retraction of $\partial\colon \Sigma(M/\partial M) \to \Sigma A$. Thus $\rho \simeq \Sigma h \cdot \rho_0\colon S^m \to \Sigma(M/\partial M)$, and $\lambda_2\rho = (\Sigma h)^{[2]} \cdot \lambda_2\rho_0$.

We factor $\Sigma h\colon \Sigma A \to \Sigma(M/\partial M)$ through the homotopy equivalence Π, by a map $\alpha \vee \beta \vee \gamma\colon \Sigma W \vee S^{n+2} \vee \Sigma M \to \Sigma(M/\partial M)$. The homotopy commutative diagram



yields $\alpha = \Sigma\iota \cdot (\Sigma v_- - \Sigma v_+)^{-1} \in [\Sigma W, \Sigma(M/\partial M)]$, $\gamma = -\Sigma\iota \cdot (\Sigma v_- - \Sigma v_+)^{-1} \cdot \Sigma v_+ \in [\Sigma M, \Sigma(M/\partial M)]$, and $\beta = \rho$. Recall that $\Pi^{[2]} \cdot \lambda_2(\rho_0) = -(x\Sigma f \smile z\Sigma\iota) \cdot \Sigma\rho_0$, which is the composite of $-x \wedge z$ with the suspension of $\Delta\colon S^{n+2} \to \Sigma W \wedge M$. Thus

$$\Sigma h \simeq \left[\Sigma\iota \cdot (\Sigma v_- - \Sigma v_+)^{-1} \vee \rho \vee -\Sigma\iota \cdot (\Sigma v_- - \Sigma v_+)^{-1} \cdot \Sigma v_+\right] \cdot \Pi,$$

$$(\Sigma h)^{[2]} \cdot \lambda_2(\rho_0) = \left[\Sigma\iota \cdot (\Sigma v_- - \Sigma v_+)^{-1}\right]^{[2]} (\mathrm{id} \wedge v_+) \cdot \Sigma\Delta \in \pi^s_{n+3}\left((\Sigma(M/\partial M))^{[2]}\right).$$

□

The reason that the knot can be determined by both the unstable homotopy class ρ and it's Hopf invariant $\lambda_2(\rho)$ is that Williams's relation of concordance is stricter than the usual relation of isotopy, where one also allows diffeomorphisms of $M \times I$ arising from diffeomorphisms of M. There is an EHP sequence interpretation of this fact, but we will not give it here. Theorem 3.1 thus provides some further evidence for our conjecture: the two Seifert surface Poincaré embeddings we consider are equivalent under a stronger equivalence relation than Farber's relation of isotopy.

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