LAMBDA ALGEBRA UNSTABLE COMPOSITION PRODUCTS AND THE Λ EHP SEQUENCE

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ABSTRACT. Simple combinatorial proofs are given of various Lambda algebra results, mostly due to the MIT school [B-C-K&, Cu1, Pr], but also the unstable Λ composition formulas of Wang, Mahowald and Singer, which imply the folklore Λ EHP sequence.

1. INTRODUCTION

Mahowald [Ma1, Ma2] initiated a "low-tech" approach to the unstable Adams spectral sequence, using a purely algebraic treatment of the Lambda algebra Λ, and ad-hoc tower constructions. However, full details have not yet appeared. A few such details, Λ combinatorial proofs, are given here.

The power of Mahowald's approach is shown by his [Ma1] Λ EHP sequence chain-level map $P\colon \Lambda(2n+1) \to \Lambda(n)$, defined by composing with $d(\lambda_n) \in \Lambda^{2,n+1}(n)$. But the geometric analogue $P\colon \Omega^2 S^{2n+1} \to S^n$ is only composition with the Whitehead product $[\iota_n, \iota_n]$ under the double suspension. His computation [Ma1, Prop. 3.1] of the Hopf invariant of P is the the Λ analogue of the author's result [Ri1]. Mahowald's P uses "Adams-filtration better" unstable Λ compositions, due to Wang [Wa], and codified (without proof) by Singer [Si]. Singer's formulas are proved here, first:

Proposition 1.1 (Singer)**.** *Composition in Λ restricts to an unstable composition pairing, written as a cup product:*

$$\Lambda^{s,t}(n) \otimes \Lambda(n+t) \to \Lambda(n),$$
$$\alpha \otimes \beta \mapsto \alpha \smile \beta.$$

Singer's result follows by easy induction from the special case $s = 1$ of Mahowald [Ma2, Lem. 3.5], or Wang's [Wa, Lem. 1.8.1] special case involving $\Lambda^{1,*}(n+t)$. Wang deduced [Wa, Thm. 1.8.4] the MIT school's result [B-C-K&], that $\Lambda(n) \subset \Lambda$ is a subcomplex, and his proof showed the folklore (see Remark 3.3) result that $H\colon \Lambda(n+1) \to \Lambda(2n+1)$ is a chain map. Curtis [Cu2] first stated without proof the Λ EHP sequence, which any careful reader could've deduced from these papers [Wa, Ma2, Si]:

1991 *Mathematics Subject Classification.* 55T15, 55Q40, 55Q25.

Thanks to Paul Burchard for the diagram package, which uses XY-pic arrows.

Theorem 1.2. *There's an exact sequence of complexes and a chain map P*

$$\Lambda(n) \rightarrowtail^{E} \Lambda(n+1) \xrightarrow{H} \twoheadrightarrow \Lambda(2n+1), \qquad \Lambda(2n+1) \xrightarrow{P} \Lambda(n),$$

where H and P are defined by $H(\lambda_n \alpha) = \alpha$ and $P(\alpha) = d(\lambda_n) \smile \alpha$, for $\alpha \in \Lambda(2a+1)$, and $H(\Lambda(n)) = 0$. P induces the cohomology boundary.

Bousfield and Kan [B-K] construct unstable cohomology compositions[1], which they proved are compatible with the geometric compositions:

$$H^{s,t-s}\Lambda(n) \otimes H^*\Lambda(n+t-s) \to H^*\Lambda(n), \tag{1}$$
$$\pi_{t-s+n}S^n \otimes \pi_* S^{n+t-s} \to \pi_{t-s+n}S^n.$$

Since the differential d of Λ preserves the t-degree, Proposition 1.1 immediately implies an "Adams-filtration better" improvement of (1):

Corollary 1.3 (Singer)**.** *Unstable Λ composition induces the pairing*

$$H^{s,t}\Lambda(n) \otimes H^*\Lambda(n+t) \to H^*\Lambda(n).$$

An EHPss approach to [Wa] yields the 3-lines and relations on the 4-lines for $H^*(\Lambda(n))$. That's basically how Wang (who never mentions H) proves (cf. [Ko]) the Adams differential $d(h_n) = h_0 h_{n-1}^2$ for $n > 3$. This systematization of Wang's work will appear later, as part of the author's work with Mahowald on 3-cell Poincare complexes and Unell's theorem.

The Λ admissible monomial basis of MIT school [B-C-K&, Pr] is proved in section 5, verifying Mahowald's conjecture (cf. [Ma1, p. 78]) that a combinatorial proof exists. Also proved (section 4) is the MIT school's related result, that d is well-defined (i.e. preserves Adem relations). In section 3, we prove Theorem 1.2, and in section 6, we prove the Mahowald-Singer Hopf invariant formula, and explain how [Ma1, Prop. 3.1] motivates [Ri1]. In section 7, we reprove Wang's result on the equivalence of the admissible and symmetric Adem relations, by Tangora's recursion relation.

This paper is part of an investigation of geometric applications of [Ma3] with Mark Mahowald, who I'd like to thank, especially for his guidance on the Λ basis. Thanks to Paul Goerss for many helpful tutorials about Λ and the uAss. Thanks to Pete Bousfield for 2 very interesting and encouraging discussions. Thanks to Halvard Fausk, who listened to an early version of the paper and encouraged me to write it up. Thanks to Charles Rezk, for explaining that [Pr] is a purely algebraic treatment, using nothing of the simplicial Lie algebras of [B-C-K&]. Thank to Stewart Priddy, who explained that genealogy [C-M] strongly indicates that unstable Lambda composition should be in the same order as composition in unstable homotopy groups.

[1] Actually somewhat less, due to "fringing" problems, which Bousfield says were later overcome. Bousfield and Kan work for all spaces, not just spheres, and not actually use Λ, but a description of $H^*\Lambda(n)$ as an Ext group in a category of unstable $\mathcal{A}$-modules.

2. UNSTABLE LAMBDA ALGEBRA COMPOSITION PRODUCTS

The Lambda algebra Λ is generated by $\{\lambda_i : i \geq 0\}$, and has relations the admissible Adem relations

$$(2) \qquad \lambda_p \lambda_{2p+1+n} + \sum_{k \geq 0} \binom{n-k-1}{k} \lambda_{p+n-k} \lambda_{2p+1+k}, \qquad p, n \geq 0.$$

A monomial $\lambda(a_1, \ldots, a_s)$ is *admissible* iff $a_i \leq 2a_{i-1}$ for $1 < i \leq s$. Adem relations reduce the right-lexicographical order, while fixing s and the t-degree $a_1 + \cdots + a_s + s$, so by induction the admissible monomials span. The MIT school [B-C-K&] showed the harder fact that Λ has a basis of admissible monomials, and also that $\Lambda(n) \subset \Lambda$ is a subcomplex, where $\Lambda(n)$ has the basis of admissible monomials $\lambda_I(a_1, \ldots, a_s)$ with $a_1 < n$.

To motivate Proposition 1.1, let's ask how we could construct Bousfield and Kan's unstable compositions (1) in Λ. Their geometric compatibility (same order e.g.) shows us we need $\lambda(a_1, \ldots, a_s)$ to belong to $\Lambda(n)$, for any sequence $(a_1, \ldots, a_s)$ satisfying the inequalities

$$a_1 < n \quad a_2 < n + a_1 \quad \ldots \quad a_s < n + a_1 + \cdots + a_{s-1}.$$

And it's not hard to see this is true, because Adem relations preserve these inequalities, and then we'd have $\Lambda(n) \cdot \Lambda(n + t - s) \subset \Lambda(n)$. But we note:

(1) Left multiplication by λ_{-1} is more or less d, and this [B-K] type unstable composition isn't enough to prove $\Lambda(n)$ is a subcomplex.
(2) Performing an Adem relation improves the above inequalities.

This leads us to stronger inequalities:

Definition 2.1. A monomial $\lambda(a_1, \ldots, a_s)$ is called n-pseudo-admissible if $a_i < n + i - 1 + \sum_{j<i} a_j$ for all $1 \leq i \leq s$.

The argument sketched above now gives a proof of Proposition 1.1 above. Formally, the proof follows easily from a definition and two lemmas. Our main workhorse is

Lemma 2.2. *If the monomial $\lambda(a_1, \ldots, a_s)$ is n-pseudo-admissible, then $\lambda(a_1, \ldots, a_s)$ belongs to $\Lambda(n)$.*

This wouldn't be useful without the converse, which we prove first:

Lemma 2.3. *If $I = \lambda(a_1, \ldots, a_s)$ is admissible with $a_1 < n$, then I is n-pseudo-admissible.*

Proof. We show by induction that $a_i < n + i - 1 + \sum_{j<i} a_j$ for $i = 1, \ldots, s$. We're given this inequality for $i = 1$. Assuming it's true for $i < s$, we have

$$a_{i+1} \leq 2a_i = a_i + a_i < n + i - 1 + \sum_{j \leq i} a_j < n + i + \sum_{j \leq i} a_j. \quad \square$$

Proof of Lemma 2.2. We'll first show Adem relations preserves n-pseudo-admissibility. That is, performing an Adem relation on any inadmissible pair in the monomial $I = \lambda(a_1, \ldots, a_s)$ writes I as a sum of terms (possibly none), each of which is still n-pseudo-admissible.

Suppose for some $i < s$ that $(a, b) := (a_i, a_{i+1})$ is inadmissible. The Adem relations (2) write $\lambda_a \lambda_b$ as a sum of admissibles $\lambda_{a'} \lambda_{b'}$, where each $a' < b - a$. We'll show that all these monomials are n-pseudo-admissible:

$$I' = (a_1, \ldots, a_{i-1}, a', b', a_{i+1}, \ldots, a_s)$$

Let $D = n + i - 1 + a_1 + \cdots + a_{i-1}$. Now $b < D + a + 1$, since (a, b) is D-pseudo-admissible, and $a' < b - a$, so we have $a' < b - a \leq D$. By Lemma 2.3, (a', b') is D-pseudo-admissible.

So Adem relations preserve n-pseudo-admissibility. Eventually we'll reach admissible form, and have $I = \sum J$ where each monomial J is both admissible and n-pseudo-admissible, Hence each J belongs to $\Lambda(n)$. $\square$

We just used a version of the next result in the proof above:

Lemma 2.4. *Suppose $I = \lambda(a_1, \ldots, a_s)$ is n-pseudo-admissible, and let $t = a_1 + \cdots + a_s + s$. Then for any monomial J, the product $I \cdot J$ is n-pseudo-admissible iff J is $(n + t)$-pseudo-admissible.*

Proof. Write $I \cdot J = \lambda(a_1, \ldots, a_r)$, for $r \geq s$. Then $I \cdot J$ is n-pseudo-admissible iff for all i such that $s < i \leq r$, we have

$$a_i < n + i - 1 + \sum_{j<i} a_j = n + t + i - s - 1 + \sum_{s<j<i} a_j. \quad \square$$

Singer's result now follows immediately from Lemmas 2.3, 2.2, and 2.4:

Proof of Proposition 1.1. It suffices of course to show that

$$\Lambda^{s,t}(n) \cdot \Lambda(n + t) \subset \Lambda(n)$$

Take admissible monomials $\lambda_I \in \Lambda^{s,t}(n)$ and $\lambda_J \in \Lambda(n{+}t)$. By Lemma 2.3, I and J are n- and $(n + t)$-pseudo-admissible respectively. Then $I \cdot J$ is n-pseudo-admissible, by Lemma 2.4. Hence $I \cdot J \in \Lambda(n)$ by Lemma 2.2. $\square$

Remark 2.5. Mahowald and Wang's special cases of Singer's Proposition 1.1 were proved by tricky double inductions. Our induction is the simple right-lex induction that proves the admissibles span Λ. Harper and Miller [H-M] port Mahowald's argument to odd primes, and their $s = 1$ result easily implies by induction on s a result they unfortunately do not state, that

$$\Lambda^{s,t,c}(2n + 1) \cdot \Lambda(2n + 1 + t + c) \subset \Lambda(2n + 1),$$

where c is the Cartan degree (the number of μ's). They make an error, trying to deduce Corollary 3.1 below from $\Lambda^{s,t}(n) \cdot \Lambda(n + t - s) \subset \Lambda(n)$, their [H-M, (1.17)], and this weakening of Proposition 1.1 isn't good enough. Their (1.17) doesn't even imply Bousfield and Kan's cohomology compositions (1), as d doesn't preserve $t - s$, but only t. But even with a mistake, Harper and Miller's "Adams-filtration better" proof that $\Lambda(n)$ is a subcomplex should establish the depth of the result, and also the Λ EHP sequence.

3. The Lambda algebra EHP sequence

We now give our combinatorial proof that $\Lambda(n)$ is a subcomplex. The proof is the same as Wang's [Wa] and Harper and Miller's [H-M], except they leave to the reader the statement and proof of Proposition 1.1.

Corollary 3.1. *For $n \geq 0$, we have $d\Lambda(n+1) \subset \Lambda(n+1)$.*

Proof. We must show that $d(\lambda_n J) \in \Lambda(n+1)$, for any admissible monomial $J \in \Lambda(2n+1)$. We have $d(\lambda_n J) = d(\lambda_n)J + \lambda_n d(J)$, by the Leibniz rule. We can assume that $d(J) \in \Lambda(2n+1)$, by induction on s, and hence $\lambda_n d(J) \in \Lambda(n+1)$. So it suffices to show that

$$d(\lambda_n) \cdot \Lambda(2n+1) \subset \Lambda(n+1).$$

We'll show one dimension better. Clearly

$$d(\lambda_n) = \sum_{k>0} \binom{n-k}{k} \lambda_{n-k}\lambda_{k-1} \in \Lambda^{2,n+1}(n) \tag{3}$$

Then $d(\lambda_n)J \in \Lambda(n)$ by Proposition 1.1:

$$\Lambda^{2,n+1}(n) \cdot \Lambda(2n+1) \subset \Lambda(n) \quad \square$$

Now we develop the Λ EHP sequence of the MIT school [Cu1]. Note that this follows from our proof of Corollary 3.1, but not the statement itself.

Clearly the inclusion $E\colon \Lambda(n) \to \Lambda(n+1)$ is a chain map, so we have *some* EHP sequence, but we want to a better grip on the quotient complex $\Lambda(n+1)/\Lambda(n)$. Recall that the *Hopf invariant*

$$H\colon \Lambda(n+1) \to \Lambda(2n+1)$$

is defined to annihilate E, and $H(\lambda_n J) = J$, for admissible monomials $J \in \Lambda(2n+1)$. Now we have

Corollary 3.2. *The linear map $H\colon \Lambda(n+1) \to \Lambda(2n+1)$ is a chain map.*

Proof. It suffices to show $dH = Hd$ holds for an element $\lambda_n J$, for an admissible monomial $J \in \Lambda(2n+1)$, since $d\Lambda(n) \subset \Lambda(n)$, by Corollary 3.1. But replicating the proof of Corollary 3.1, we have

$$Hd(\lambda_n J) = H(d(\lambda_n)J + \lambda_n d(J)) = 0 + d(J) = dH(\lambda_n J)$$

since $d(\lambda_n)J \in \Lambda(n)$, and $d(J) \in \Lambda(2n+1)$. $\square$

Mahowald's description [Ma2] of P is now immediate, and we've proved Proposition 1.2 of the introduction.

Remark 3.3. Ravenel and Kochman [Ra, Ko] implausibly assert that Corollaries 3.2 and 3.1 follow immediately from Formulas (2) and (3). Curtis and Mahowald [C-M, p. 128] implausibly offer no proof or citation for these 2 results. Curtis [Cu1, sec. 11] fails to prove that H is a chain map, first by merely citing [Cu2], and second by an error in his proof that H is induced by the geometric Hopf invariant. Curtis claims that a sum of maps induce an isomorphism on E_1 terms, but clearly each map induces zero, as they're all Whitehead product, with positive Adams filtration. I give Curtis credit for his bold attempt, and I think a version of his argument works with Mahowald's [Ma1] "mapping cone" construction for an unstable Adams resolution over the fiber of a map, in this case $E\colon S^n \to \Omega S^{n+1}$, although I think we'd have to abandon the Lower Central Series filtration.

Bousfield and Curtis [B-C, Rem. 5.3] construct a *long* exact cohomology EHP sequence, using unstable $\mathcal{A}$-modules, but I believe that one cannot glean a proof of Theorem 1.2 from their argument, but instead, that they use [B-C, Lem. 3.5] Corollary 3.2. Singer [Si, top p. 380] reconstructs the long exact cohomology EHP sequence, and it's clear that his proof that $h\colon \Omega L^s(S^n) \to L^{s-1}(S^{2n-1})$ is a chain map uses Corollary 3.2, which of course he could've proved himself. Wang [Wa, Prop. 1.8.3] "immediately" deduced that $d(\lambda_n)\cdot\Lambda(2n+1) \subset \Lambda(n)$, and therefore Corollary 3.1, from his special case $\Lambda^{s,t}(n)\cdot\lambda_m \subset \Lambda(n)$ for $m < n+t$ of Proposition 1.1. I contend that Wang's leap shows the importance of stating Singer's result, from which his result *does* follows immediately. Wang could easily have deduced Corollary 3.2 from his Prop. 1.8.3, and he point out its obvious corollary, that $H(x) \in \Lambda(2n+1)$ is a cycle if $x \in \Lambda(n+1)$ is a cycle.

4. d PRESERVES THE ADEM RELATIONS

Before proving the Λ admissible monomial basis, we'll prove an easier result of the MIT school [B-C-K&, Pr]:

Proposition 4.1. *The differential $d\colon \Lambda \to \Lambda$ is well-defined.*

Proof. Λ is a tensor algebra modulo the 2-sided ideal generated by the Adem relations. The Leibniz rule defines d on the tensor algebra, but we must show that d sends Adem relations to the 2-sided ideal. To prove this, we'll expand the tensor algebra to include λ_{-1}, well-known to be related to d, and use what Pete Bousfield calls "pension operators", i.e. selfmaps of tensor powers which preserve Adem relations.

Let V be the $Z/2$ vector space with basis $\{\lambda_p : p \geq -1\}$. Let e be the selfmap of V given by $e(\lambda_p) = \lambda_{p+1}$. Define the selfmap of $V^{\otimes 2}$ by

$D = e\otimes 1 + 1\otimes e$. As Mahowald recommends (cf. [Ma1, p. 78]), we'll use the original [B-C-K&] symmetric Adem relations, for $p \geq -1$, $n \geq 0$:

$$(4) \qquad [p,n] := D^n(\lambda_p \otimes \lambda_{2p+1}) = \sum_{i+j=n} \binom{n}{i} \lambda_{p+i} \otimes \lambda_{2p+1+j} \in V^{\otimes 2}.$$

The (original [B-C-K&] symmetric) differential on Λ comes from $p = -1$:

$$(5) \qquad d(\lambda_n) = [-1, n+1] + \lambda_{-1}\lambda_n + \lambda_n\lambda_{-1} \in V^{\otimes 2}, \qquad n \geq 0.$$

Now define the selfmap $C = e \otimes e^2$ of $V^{\otimes 2}$. Then C preserves Adem relations as well, and we have

$$C([p,n]) = [p+1, n], \qquad D([p,n]) = [p, n+1].$$

It's well known that all the Adem relations are obtained from $[-1,-1]$ by applying powers of C and D. Call $I_2 = 1\otimes 1$ the identity selfmap of $V^{\otimes 2}$. Now we'll define selfmaps of $V^{\otimes 3}$, and we'll apply them to

$$\lambda_p \otimes \lambda_{2p+1} \otimes \lambda_{4p+3} = [p,0] \otimes \lambda_{4p+3} = \lambda_p \otimes [2p+1, 0].$$

We'll also call D the selfmap $D = e\otimes I_2 + 1\otimes e\otimes 1 + I_2 \otimes e$ of $V^{\otimes 3}$, so

$$D = D\otimes 1 + I_2 \otimes e = 1 \otimes D + e \otimes I_2.$$

We venture into new territory with the selfmap E of $V^{\otimes 3}$ defined by

$$\begin{aligned} E &= e\otimes e^2 \otimes 1 + e\otimes 1\otimes e^2 + 1\otimes e\otimes e^2 \\ &= C\otimes 1 + D\otimes e^2 = 1\otimes C + e\otimes D^2. \end{aligned}$$

We've written both E and D as the sum of 2 commuting operators on $V^{\otimes 3}$, in 2 different ways, so the binomial theorem computes powers of D^m and E^n, just as with $[p,n]$ above. Let's define, for $p \geq -1$, $n, m \geq 0$, elements

$$[p,n,m] := D^m E^n([p,0]\otimes\lambda_{4p+3}) = D^m E^n(\lambda_p \otimes [2p+1,0]) \in V^{\otimes 3}.$$

By the binomial theorem, $[p,n,m]$ has 2 expressions. Equating them gives

$$(6) \qquad \sum_{i+j=n, s+t=m} \binom{n}{i}\binom{m}{s} \begin{pmatrix} [p+i, j+s] \otimes \lambda_{4p+3+2j+t} \\ +\lambda_{p+j+t} \otimes [2p+1+i, 2j+s] \end{pmatrix} = 0.$$

Now we specialize to $p = -1$, and assume $n > 0$, and project this equation onto the positive part of $V^{\otimes 3}$. I.e. we throw out the terms containing λ_{-1}. This will be our equation for why d preserves Adem relations.

The terms in Equation (6) containing λ_{-1} come from either $j = t = 0$ or $i = 0$, and add up to

$$\lambda_{-1}\otimes[n-1,m] + [n-1,m]\otimes\lambda_{-1} + \sum_{s+t=m} \binom{m}{s} \begin{pmatrix} [-1, n+s]\otimes\lambda_{2n+t-1} \\ +\lambda_{n+t-1}\otimes[-1, 2n+s] \end{pmatrix}.$$

By formula (5), the Leibniz rule, and switching s and t in the second part, the positive projection of this expression is

$$\sum_{s+t=m} \binom{m}{s} \begin{pmatrix} d(\lambda_{n+s-1}) \otimes \lambda_{2n+t-1} \\ +\lambda_{n+t-1} \otimes d(\lambda_{2n+s-1}) \end{pmatrix} = d([n-1, m]),$$

So the positive projection of Equation (6) shows, for $n > 0, m \geq 0$, that $d([n-1, m])$ is the sum of the positive Adem relations

$$\sum_{\substack{i+j=n,\, s+t=m \\ i>0,\, (j,t)\neq(0,0)}} \binom{n}{i} \binom{m}{s} \begin{pmatrix} [i-1, j+s] \otimes \lambda_{2j+t-1} \\ +\lambda_{j+t-1} \otimes [i-1, 2j+s] \end{pmatrix} \tag{7}$$ $\square$

5. THE Λ ADMISSIBLE MONOMIAL BASIS

Let $W \subset V$ be the subvectorspace with basis $\{\lambda_p : p \geq 0\}$, and let $R \subset W^{\otimes 2}$ be the subvectorspace $Z/2\{[p, n] : p, n \geq 0\}$. Then with I the 2-sided ideal generated by R, we have

$$\Lambda = T(W)/I, \qquad I = T(W) \cdot R \cdot T(W)$$

We now prove the MIT school's result [B-C-K&, Pr]

Proposition 5.1. *Λ has a basis of the admissible monomials.*

First we prove an analogue of Proposition 4.1:

Lemma 5.2. *For $p, n, m \geq 0$, we can rewrite $\lambda_p \otimes [2p+1+n, m]$ as a sum*

$$\lambda_p \otimes [2p+1+n, m] = \sum_i \lambda_{x_i} \otimes [p_i, n_i] + \sum_j [q_j, m_j] \otimes \lambda_{y_j} \in W^{\otimes 3}$$

where for each i, the triple $(x_i, p_i, 2p_i + 1 + n_i)$ has lower right-lex order than $(p, 2p+1+n, 4p+3+2n+m)$.

Proof. Equation (6) simplifies to

$$\sum_{i+j=n, s+t=m} \binom{n}{i} \binom{m}{s} \lambda_{p+j+t} \otimes [2p+1+i, 2j+s] \in R \otimes W \subset W^{\otimes 3}. \tag{8}$$

The (i, s)-term produces the triple $(p+j+t, 2p+1+i, 4p+3+2n+s)$, and the maximum right-lex order occurs uniquely at $s = m$ and $i = n$, which corresponds to the term $\lambda_p \otimes [2p+1+n, m]$. $\square$

Remark 5.3. We proved what we will use below, but here's a more straightforward analogue of Proposition 4.1. Define the *excess* of (a, b) to be $b - 2a - 1$. Then the excess of $(p+j+t, 2p+1+i)$ is $i - 2j - 2t \leq n$, and the maximum n is achieved only for $j = t = 0$. So Formula (8) rewrites $\lambda_p \otimes [2p+1+n, m]$ as an element of $R \otimes W$ plus a sum of elements $\lambda_a \otimes [b, c]$

with $b - 2a - 1 < n$. By induction $\lambda_p \otimes [2p + 1 + n, m]$ is an element of $R \otimes W$ plus a sum of elements $\lambda_a \otimes [b, c]$ with each (a, b) is admissible.

Proof of Proposition 5.1. The problem is that 2-sided ideal I is "too big". We first define a sub-vectorspace J of I so that $T(W)/J$ has a basis of the admissible monomials. J will be the sub-vectorspace I that's defined by the algorithm of performing an Adem relation on the left-most inadmissible pair of a monomial. Formally, let $K \subset I$ be the subvectorspace with basis

$$\{\lambda(a_1, ..., a_s)[p, n] : p, n, s \geq 0, \lambda(a_1, ..., a_s) \text{ admissible}, p \leq 2a_s \text{ if } s > 0\},$$

and define $J = K \cdot T(W)$. It's obvious that $T(W)/J$ has a basis of the admissible monomials. We'll use to Lemma 5.2 to show $I = J$.

I is spanned by *spanning elements*

$$\alpha = \lambda(a_1, ..., a_s)[p, n]\phi, \qquad a_i, p, n, s \geq 0, \phi \in T(W).$$

By abuse of notation, let's call s the *Adams filtration* of α. We'll say that α is an *admissible* spanning element if $(a_1, \ldots, a_s, p)$ is admissible. Of course, $\alpha \in J$ if α is admissible. If α is inadmissible, we'll perform reductions until α is a sum of admissible spanning elements, and then $\alpha \in J$.

We need an ordering on the spanning element, derived from the orderings Priddy [Pr] and Mahowald [Pr, Ma1, Prop. 5.5] used in their cohomological proofs of this Λ basis result. We order the spanning elements α of a given word-length $N = s + 2 + r$ and a given stem degree

$$a_1 + \cdots + a_s + p + (2p + 1 + n) + b_1 + \cdots + b_r$$

first by the Adams filtration s and then by right lexicographical order on the N-tuple $(a_1, \ldots, a_s, p, 2p + 1 + n, b_1, \ldots, b_r)$. We can now induct because there are only a finite number of elements with lower filtration than α. We're going to perform a sequence of reductions until α is a sum of admissible spanning elements, and then $\alpha \in J$. Our two reduction moves are:

(1) Apply a symmetric Adem relations $[q, n]$ to any inadmissible pair in $\lambda(a_1, \ldots, a_s)$
(2) Apply a higher Adem relations $D^m E^n(q \otimes 2q + 1 \otimes 4q + 3)$ to $a_s[p, n]$, if (a_s, p) is inadmissible.

We'll see that both moves strictly lower the filtration order. It will be obvious that both moves preserve the word-length and the stem degree.Then $I = J$ by the same inductive argument that proves why admissibles span Λ: keep applying moves in any order until (by finiteness) we have a sum of admissible spanning elements.

Let's illustrate the type (1) move for $s = 2$. If $(a_1, a_2) = (q, 2q+1+m)$ is inadmissible, then

$$\alpha = [q,m]([p,n]\phi) + \sum_{i+j=m,\, j<m} \binom{m}{i} \lambda(q+i, 2q+1+j)[p,n]\phi,$$

so α is a spanning elements Adams filtration 0, plus a sum of terms with lower right-lex order.

Our type (2) move uses Lemma 5.2 to rewrite $\alpha = \lambda(a_1, ..., a_s)[p,n]\phi$ as

$$\alpha = \sum \lambda(a_1, ..., a_{s-1}, x)[q,m]\phi$$

with each triple $(x, q, 2q+1+m)$ lower than $(a_s, p, 2p+1+n)$ in the right-lex order. Hence each term on the RHS has lower filtration than α, and our type (2) lowers filtration.

By the above inductive argument, we use moves of type (1) or (2) in any order to write α as a sum of admissible spanning elements, so $\alpha \in J$. $\square$

Note that we could've proved Proposition 4.1 by the technique of this section, by merely replacing W with V. I think that the proof of Proposition 4.1 is nicer, even though it's combinatorially more challenging.

Remark 5.4. The existence of a combinatorial proof of Proposition 5.1 was first raised by Mahowald [Ma1, p. 78]), where he asserted that it followed from the symmetric Adem relations (4). I interpret this as a conjecture, which is finally solved here. Ravenel [Ra], and Curtis & Mahowald [C-M, p. 128] assert implausibly that this result follows "immediately" from the admissible Adem relations (2). Miller & Harper [H-M] also assert this deduction, but not "immediately." The proof here was hard enough, and I can't imagine a proof using the admissible Adem relations. Kochman [Ko] gives a short false proof (false proof for span, no proof for linear independence).

6. THE MAHOWALD-SINGER FORMULA FOR THE HOPF INVARIANT OF AN UNSTABLE LAMBDA COMPOSITION

Recall Boardman and Steer's formula [B-S, thm. 3.16] for the suspended James-Hopf invariant λ_2 of a composition formula, which the author [Ri2, Thm. 2.7] found useful. For $\xi \in [\Sigma X, \Sigma W]$, and $\rho \in [\Sigma A, \Sigma X]$, we have

$$\lambda_2(\xi \circ \rho) = \lambda_2(\xi) \circ \Sigma\rho + (\xi \wedge \xi) \circ \lambda_2(\rho) \in [\Sigma^2 A, \Sigma W \wedge \Sigma W].$$

It's natural to ask for a Λ composition formula for H. Mahowald explains that the second term $(\xi \wedge \xi) \circ \lambda_2(\rho)$ vanishes in Λ because it has higher Adams filtration. But unstable Λ composition, which is Adams-filtration better than geometric composition, adds a different second term:

Proposition 6.1 (Singer). *If $\alpha \in \Lambda^{s,t}(n+1)$ and $\beta \in \Lambda(n+t+1)$, then*

$$EH(\alpha \smile \beta) = EH(\alpha) \smile \beta + Sq^0(\alpha) \smile EH(\beta) \in \Lambda(2n+2). \tag{9}$$

That is, the composite

$$\Lambda^{s,t}(n+1) \otimes \Lambda(n+t+1) \xrightarrow{\smile} \Lambda(n+1) \xrightarrow{EH} \Lambda(2n+2)$$

is the sum of the 2 composites in the diagram

$$\begin{tikzcd}
\Lambda^{s,t}(n+1) \otimes \Lambda(n+t+1) \arrow[r, "EH\otimes 1"] \arrow[d, "Sq^0 \otimes EH"'] & \Lambda^{s-1,t-n-1}(2n+2) \otimes \Lambda(n+t+1) \arrow[d, "\smile"] \\
\Lambda^{s,2t}(2n+2) \otimes \Lambda(2n+2t+2) \arrow[r, "\smile"'] & \Lambda(2n+2)
\end{tikzcd}$$

We'll prove this below after some preliminaries. Mahowald proved the special case of $\alpha = d(\lambda_n) \in \Lambda^{2,n+1}(n)$ and n even [Ma1, Prop. 3.1]. Singer states the general case [Si, Prop. 5.3] without proof, and indeed a proof (and the statement) follows by a straightforward modification of Mahowald's proof. Our proof might seem more elegant. The $s = 1$ case is easy, and $s > 1$ follows by easy induction by the strictly associativity of the formula.

First some obvious properties of unstable Λ composition, involving associativity, suspension naturality, Sq^0 and admissible concatenation:

Lemma 6.2. *If $\alpha \in \Lambda^{s,t}(n)$, $\beta \in \Lambda^{s',t'}(n+t)$, and $\gamma \in \Lambda(n+t+t')$, then*

$$\begin{aligned}
\alpha \smile (\beta \smile \gamma) &= (\alpha \smile \beta) \smile \gamma \in \Lambda(n),\\
E(\alpha \smile \beta) &= E(\alpha) \smile E(\beta) \in \Lambda(n+1),\\
Sq^0(\alpha \smile \beta) &= Sq^0(\alpha) \smile Sq^0(\beta) \in \Lambda^{s+s',2(t+t')}(2n).
\end{aligned}$$

For $\beta \in \Lambda(2n+1)$, we have $\lambda_n \beta = \lambda_n \smile E(\beta) \in \Lambda(n+1)$.

Proof. We must only check that all of the unstable compositions are defined, since unstable Λ composition is just the Λ multiplication desuspended to the appropriate sub-vectorspace $\Lambda(i) \subset \Lambda$. □

Proof of Proposition 6.1. We'll prove Equation (9) by induction on s, the Adams filtration of the first argument α.

First we'll do $s = 1$, and be very pedantic about unstable Λ products. So $\alpha = \lambda_a$, with $0 \le a \le n$. For $\beta \in \Lambda(n+a+2)$, we need

$$EH(\lambda_a \smile \beta) = \delta_{a,n}\beta + \lambda_{2a+1} \smile EH(\beta) \in \Lambda(2n+2) \tag{10}$$

Let's write $m = n+a+1$, so $\beta \in \Lambda(m+1)$.

Assume $a < n$. Write $\lambda_a \in \Lambda^{1,a+1}(n)$, and $\lambda_{2a+1} \in \Lambda^{1,2a+2}(2n+1)$. Then $\lambda_{2a+1} \smile H(\beta) \in \Lambda(2n+1)$, since $(2n+1)+(2a+2) = 2m+1$, and $H(\beta) \in \Lambda(2m+1)$. So Equation (10) desuspends to

$$H(E(\lambda_a) \smile \beta) = \lambda_{2a+1} \smile H(\beta) \in \Lambda(2n+1).$$

Let's write $\beta = \lambda_m \smile E(x) + E(y)$ in admissible form, for $x \in \Lambda(2m+1)$, and $y \in \Lambda(m)$. Now let's write the Adem relation for $\lambda_a \lambda_m$ as

$$E(\lambda_a) \smile \lambda_m = \lambda_n \lambda_{2a+1} + E(R_{a,m}) \in \Lambda(n+1), \quad \text{for } R_{a,m} \in \Lambda^{2,m+a+2}(n).$$

Then $R_{a,m} \smile x \in \Lambda(n)$, since $n+m+a+2 = 2m+1$, and $\lambda_a \smile y \in \Lambda(n)$, since $n + a + 1 = m$. Then we have

$$E(\lambda_a) \smile \beta = \lambda_n \smile E(\lambda_{2a+1} \smile x) + E(R_{a,m} \smile x + \lambda_a \smile y), \quad \text{so}$$
$$H(E(\lambda_a) \smile \beta) = \lambda_{2a+1} \smile x = \lambda_{2a+1} \smile H(\beta) \in \Lambda(2m+1).$$

This finishes the case $a < n$.

Now assume $a = n$. Then write $\beta \in \Lambda(2n+2)$ in admissible form as $\beta = \lambda_{2n+1} \smile EH(\beta) + E(y)$, for $y \in \Lambda(2n+1)$. Since $\lambda_n \lambda_{2n+1} = 0$, we have $\lambda_n \smile \beta = \lambda_n y$, and the case $s = 1$ is concluded by

$$EH(\lambda_n \smile \beta) = E(y) = \beta + \lambda_{2n+1} \smile EH(\beta) \in \Lambda(2n+2).$$

The induction step follows from the strict associativity of the RHS. Take

$$\alpha \otimes \beta \otimes \gamma \in \Lambda^{s,t}(n+1) \otimes \Lambda^{s',t'}(n+t+1) \otimes \Lambda(n+t+t'+1).$$

Assuming the result for s and s', the Adams filtrations of α and β, we'll show it's true for $\alpha \smile \beta$ in the first argument. Using Lemma 6.2, we have

$$\begin{aligned}
&EH((\alpha \smile \beta) \smile \gamma) = EH(\alpha \smile (\beta \smile \gamma)) \\
=&EH(\alpha) \smile \beta \smile \gamma + Sq^0(\alpha) \smile EH(\beta \smile \gamma) \\
=&EH(\alpha) \smile \beta \smile \gamma + Sq^0(\alpha) \smile \big(EH(\beta) \smile \gamma + Sq^0(\beta) \smile EH(\gamma)\big) \\
=&\big(EH(\alpha) \smile \beta + Sq^0(\alpha) \smile EH(\beta)\big) \smile \gamma + Sq^0(\alpha) \smile Sq^0(\beta) \smile EH(\gamma) \\
=&EH(\alpha \smile \beta) \smile \gamma + Sq^0(\alpha \smile \beta) \smile EH(\gamma).
\end{aligned}$$

So Equation (9) is true with $\alpha \smile \beta$ in the first argument. This completes our induction, since every $\alpha \in \Lambda^{s,t}(n+1)$ is a sum of such products. Just write α admissibly as $\alpha = \sum_{i=0}^{n} \lambda_i \smile E(x_i)$, for $x_i \in \Lambda^{s-1,t-i-1}(2i+1)$, and we've proved the result for Adams filtration 1 and $s - 1$. $\square$

There are two important special cases when Proposition 6.1 desuspends. First, when the second argument β desuspends, we have [Si, Prop. 5.2]

Corollary 6.3 (Singer). *For $\alpha \in \Lambda^{s,t}(n+1)$ and $\beta \in \Lambda(n+t)$, we have*

$$H(\alpha \smile E(\beta)) = H(\alpha) \smile \beta \in \Lambda(2n+1).$$

That is, letting $m = n + t$, the diagram commutes:

$$\begin{tikzcd}
\Lambda^{s,t}(n+1) \otimes \Lambda(m) \arrow[rr, "H \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes E"'] & & \Lambda^{s-1,t-n-1}(2n+1) \otimes \Lambda(m) \arrow[d, "\smile"] \\
\Lambda^{s,t}(n+1) \otimes \Lambda(m+1) \arrow[r, "\smile"] & \Lambda(n+1) \arrow[r, "H"'] & \Lambda(2n+1)
\end{tikzcd}$$

We only need observe that both sides actually desuspend. Proposition 6.1 also implies the desuspension when the first argument α desuspends:

Corollary 6.4. *For $\alpha \in \Lambda^{s,t}(n)$ and $\beta \in \Lambda(n+t+1)$, we have*

$$H(E(\alpha) \smile \beta) = E(Sq^0(\alpha)) \smile H(\beta) \in \Lambda(2n+1).$$

As Singer explains [Si], we can now perform Λ analogues of various geometric EHP construction that Toda, Barratt and others used.

Consider Toda's calculation [To] of $\pi_7^s = \mathbb{Z}/16$, generated by $\sigma \in \pi_{15}S^8$. The problem is to construct his elements σ', σ'', σ''' which are born on S^7, S^6, and S^5 respectively, and are stably 2σ, 4σ, and 8σ respectively, with Hopf invariants η, η^2 & η^3 respectively. σ''' is a Toda bracket $\langle \nu, 8\iota, \nu \rangle$, used in the constructing the Adams selfmap [Ad]. But σ' and σ'' are more mysterious, not expressed as Toda brackets.

In Λ, the σ, σ', σ'', σ''' story is easy. Starting with the cycle $\lambda_7 \in \Lambda(8)$, with $H(\lambda_7) = * \in \Lambda(15)$, Proposition 1.1 and Corollary 6.4 imply

$$\begin{aligned}
\lambda_0\lambda_7 &\in \Lambda(7), & H(\lambda_0\lambda_7) &= \lambda_1, \\
\lambda_0^2\lambda_7 &\in \Lambda(6), & H(\lambda_0^2\lambda_7) &= \lambda_1^2, \\
\lambda_0^3\lambda_7 &\in \Lambda(5), & H(\lambda_0^3\lambda_7) &= \lambda_1^3.
\end{aligned}$$

Of course, these equations are trivial to verify by hand. Note that $\lambda_0^3\lambda_7$ therefore is a cycle with leading term 4111. Compare [Ra, Ex. 3.3.11], where 4111 is completed to a cycle by the Curtis algorithm.

$\lambda_0^3\lambda_7$ brings up an obvious corollary of Proposition 1.1 and Corollary 3.1:

Corollary 6.5. *For $\alpha \in \Lambda^{s,t}(n)$ and $\beta \in \Lambda(n+t+1)$, we have*

$$d(\alpha \smile \beta) = d(\alpha) \smile \beta + \alpha \smile d(\beta) \in \Lambda(n).$$

Now take $\alpha = d\lambda_n \in \Lambda^{2,n+1}(n)$ and note $H(\alpha) = (n-1)\lambda_0$. Proposition 6.1 and Theorem 1.2 immediately imply Mahowald's result [Ma1, Prop. 3.1]: The composition

$$\Lambda(2n+1) \xrightarrow{P} \Lambda(n) \xrightarrow{H} \Lambda(2n-1) \xrightarrow{E} \Lambda(2n)$$

sends β to $(n-1)\lambda_0 \smile \beta + Sq^0(d\lambda_n) \smile H(\beta)$. Then $d\lambda_{2n+1} = ESq^0(d\lambda_n)$, and specializing to n even, Mahowald observed that the composition

$$\Lambda(4n+1) \xrightarrow{P} \Lambda(2n) \xrightarrow{H} \Lambda(4n-1) \xrightarrow{E^2} \Lambda(4n+1)$$

sends β to $\lambda_0 \smile \beta + d(\lambda_{4n+1}) \smile H(\beta)$. Recall the Hilton-Hopf expansion [B-S, Wh] James used for his 2-primary exponent [Ja, Co, B-C-G&]:

$$2\iota \cdot \alpha = \alpha \cdot 2\iota + [\iota_n, \iota_n] \cdot H(\alpha), \text{ for } \quad \alpha \in \pi_*(S^n). \tag{11}$$

It's well-known that $d(\lambda_n)$ corresponds to $[\iota_n, \iota_n]$, and λ_0 corresponds to 2ι. Assuming this, Bousfield and Kan's (1), leads us to expect that left/right composition with λ_0 corresponds to left/right geometric composition by 2ι. Mahowald then observed the following result:

Proposition 6.6 (Mahowald)**.** *The composition*

$$\Lambda(4n+1) \xrightarrow{P} \Lambda(2n) \xrightarrow{H} \Lambda(4n-1) \xrightarrow{E^2} \Lambda(4n+1)$$

induces a selfmap of $H^*\Lambda(4n+1)$*, which is* $E^2 \cdot H \cdot P(\beta) = \beta\lambda_0$.

Proof. We only need to prove the Λ analogue of Equation (11). Singer [Si, Thm. 4.1] proves the full Λ analogue of the Barratt-Toda commutation formula [To]: for $f \in \pi_{m+a}S^a$ and $g \in \pi_{n+a}S^b$ we have

$$f \cdot g - (-1)^{nm} g \cdot f = [1_{a+b-1}, 1_{a+b-1}] \cdot H(f) \wedge H(g) \in \pi_{m+n+a+b}S^{a+b-1}.$$

We'll only prove a special case. For a cycle $f \in \Lambda(p+1)$, we'll show

$$\lambda_0 \smile f + f \smile \lambda_0 = d(\lambda_{p+1}) \smile H(f) \in H^*\Lambda(p+1)$$

To prove this, write f admissibly as $f = \lambda_p A + B$, for $B \in \Lambda(p)$ and $A \in \Lambda(2p+1)$. Since f is a cycle, A must be a cycle, as Wang (who didn't use H) observed [Wa, Thm. 1.8.4]: $dA = dH(f) = Hd(f) = H(0) = 0$.

By Equation (5), commutation with λ_{-1} is the boundary map d:

$$df = [f, \lambda_{-1}] \in T(V).$$

Now we'll extend our operator D to $T(V)$, so D satisfies the Leibniz rule, and $D(\lambda_p) = \lambda_{p+1}$. Writing $D(\alpha) = \alpha'$, we have

$$(df)' = [f, \lambda_{-1}]' = [f', \lambda_{-1}] + [f, \lambda_0] = d(f') + [f, \lambda_0] \in T(W).$$

We now pass to Λ, since D, as an operator on $T(W)$, preserves Adem relations, i.e. $D([a, m]) = [a, m+1]$. Since f is a cycle, i.e. $df = 0$, that's

$$d(f') = [f, \lambda_0] = \lambda_0 f + f\lambda_0 \in \Lambda.$$

We need to show $d(f')$ is cohomologous to $d(\lambda_{p+1}) \smile H(f) \in \Lambda(p+1)$.

Differentiate the defining equation for f and apply the boundary map d:

$$f = \lambda_p A + B$$

$$f' = \lambda_{p+1} A + \lambda_p A' + B'$$

$$\lambda_0 f + f\lambda_0 = d(f') = d(\lambda_{p+1})A + d(\lambda_p A' + B') \in \Lambda, \tag{12}$$

since $d(A) = 0$. We'll show that $(\lambda_p A' + B') \in \Lambda(p+1)$, because

$$C \in \Lambda(k) \implies C' \in \Lambda(k+1) \tag{13}$$

To see this, take an admissible monomial $C = \lambda(a_1, \ldots, a_s) \in \Lambda(k)$. Then C' is a sum of s terms, each of which is either admissible or zero, since $\lambda_a \lambda_{2a+1} = 0$. So the first term $\lambda(a_1+1, a_2, \ldots, a_s) \in \Lambda(k+1)$, and the remaining terms of the sum C' belong to $\Lambda(k)$. This proves implication (13).

Thus $B' \in \Lambda(p+1)$ and $A' \in \Lambda(2p+2)$, so $\lambda_p A' = \lambda_p \smile A' \in \Lambda(p+1)$, by Proposition 1.1. So Equation (12) now reads

$$\lambda_0 f + f\lambda_0 \equiv d(\lambda_{p+1})A \in \Lambda(p+1).$$

Since $H(f) = A$, we've proved our formula: For any cycle $f \in \Lambda(p+1)$,

$$\lambda_0 \smile f + f \smile \lambda_0 = d(\lambda_{p+1}) \smile H(f) \in H^*\Lambda(p+1).$$

Now recall $E^2 \cdot H \cdot P(\beta) = \lambda_0 \smile \beta + d(\lambda_{4n+1}) \smile H(\beta) \in \Lambda(4n+1)$. $\square$

Mahowald then conjectured the geometric analogue of Proposition 6.6: the composite $\Omega^3 S^{4n+1} \xrightarrow{\Omega(P)} \Omega S^{2n} \xrightarrow{H} \Omega S^{4n-1} \xrightarrow{E^2} \Omega^3 S^{4n+1}$ is homotopic to the H-space squaring map on Ω^3. The author [Ri1] proved this conjecture, which implies the following infinite statement in homotopy groups:

$$2\pi_k S^{4n+1} \subset E^2\left(\pi_{k-2}S^{4n-1}\right), \qquad \text{for } k \geq 3. \tag{14}$$

[B-C-G&] shows that (14) is not due to James or Selick [Ja, Se], even though (14) does not improve on the James-Selick 2-primary exponent.

7. SYMMETRIC AND ADMISSIBLE ADEM RELATIONS

We'll prove Wang's result [Wa, Thm. 1.6.1] that the admissible Adem relations (2) are equivalent to the original [B-C-K&] symmetric Adem relations (4). First We'll prove the MIT school's result [B-C-K&] that $d^2 = 0$.

Lemma 7.1. $d^2(\lambda_n) = 0 \in \Lambda$, *for* $n \geq 0$.

Proof. We'll use the symmetric Adem relations, and show even more, that $d^2(\lambda_a)$ vanishes in the tensor algebra. By formula (5), for $n \geq 0$, we have

$$d(\lambda_n) = \sum_{i+j=n+1,\, ij>0} \binom{n+1}{i} \lambda_{i-1}\lambda_{j-1}.$$

Then we instantly derive $d^2(\lambda_n) = 0$. Using the Leibniz rule

$$d(\lambda_{i-1}\lambda_{j-1}) = d(\lambda_{i-1})\lambda_{j-1} + \lambda_{i-1}d(\lambda_{j-1}),$$

$d^2(\lambda_n)$ is the sum of two terms, the first of which is

$$\sum_{s+t+j=n+1,\, stj>0} \binom{n+1}{s, t, j} \lambda_{s-1}\lambda_{t-1}\lambda_{j-1},$$

as we see by using the binomial identity $\binom{n+1}{i}\binom{i}{s} = \binom{n+1}{s,t,j}$, where as usual, $\binom{n+1}{s,t,j} = (n+1)!/(s!\,t!\,j!)$. But the other term, arising from $\lambda_{i-1}d(\lambda_{j-1})$, is equal, so the sum $d^2(\lambda_n)$ is zero. □

Remark 7.2. E^n arose in a way showing the power of the symmetric formulas: Suppose (p,b) is inadmissible. Then $\lambda_p\lambda_b\lambda_{2b+1} = 0$ since $\lambda_b\lambda_{2b+1} = 0$, but perform the Adem relation $\lambda_p\lambda_b = \sum \lambda_x\lambda_y$ first. Each pair $(y, 2b+1)$ is inadmissible, so perform an Adem relation on each one. The Λ basis requires this sum to vanish in Λ, but why? Using the admissible formulas, this isn't at all clear. But using the symmetric formulas, it's easy to rewrite this sum as a sum $\sum[q,r]s$, using identities like $\binom{n}{i}\binom{2i}{2a} = \binom{n}{i,a,b}$. So we avoided a relation, and the calculation basically hands us the operator E^n.

Wang [Wa, Thm. 1.6.1] used formal power series to "admissify" the symmetric formulas. We'll use a simple recursion formula due to Tangora [Ta2, Ta1]. Define $C_{a,k} \in \mathbb{Z}/2$, for $a \geq 0$, $k \in \mathbb{Z}$, recursively by

$$(15)\quad C_{0,k} = 0, \quad C_{1,k} = \delta_{k,0}, \text{ and for } a \geq 2, \quad C_{a,k} = C_{a-1,k} + C_{a-2,k-1}.$$

Then for $p \geq -1$, and $a \geq 0$, let's define

$$(16)\qquad (p,a) := \lambda_p \otimes \lambda_{2p+1+a} + \sum_k C_{a,k}\,\lambda_{p+a-k} \otimes \lambda_{2p+1+k} \in V^{\otimes 2}.$$

By easy induction on a, we see that $\sum_k C_{a,k}\,\lambda_{p+a-k} \otimes \lambda_{2p+1+k}$ is a finite sum of admissibles: $C_{a,k} = 0$ for either $k < 0$ or $2k+1 > a$. This justifies calling Formula (16) the *admissible Adem formulas*.

Now we obtain relations between the symmetric and admissible Adem relations, by the usual procedure of applying D to formula (16):

Lemma 7.3. *Assume $p \geq -1$. Then $(p,a) = [p,a]$ for $a = 0,1,2$, and*

$$(17)\qquad (p,a+1) = D(p,a) + (p+1,a-2) \in V^{\otimes 2}, \quad \textit{for } a \geq 2.$$

Proof. The case $a = 0,1,2$ is obvious, and we'll deduce formula (17) by induction on $a \geq 2$. First, by replacing k by $k-2$, we have

$$(p+1,a-2) = \lambda_{p+1}\lambda_{2p+1+a} + \sum_k C_{a-2,k-2}\lambda_{p+1+a-k}\lambda_{2p+1+k}.$$

Then $D(p,a) + (p+1,a-2)$ equals $\lambda_p\lambda_{2p+1+a+1}$ plus the sum

$$\begin{aligned}
&\sum_k \left(C_{a,k} + C_{a,k-1} + C_{a-2,k-2}\right)\lambda_{p+1+a-k}\lambda_{2p+1+k} \\
&= \sum_k \left(C_{a,k} + C_{a-1,k-1}\right)\lambda_{p+1+a-k}\lambda_{2p+1+k}, && \text{by (15) for } k-1 \\
&= \sum_k C_{a+1,k}\,\lambda_{p+(a+1)-k}\lambda_{2p+1+k}, && \text{by (15) for } k
\end{aligned}$$

So by this double application of the Tangora recursion formula (15), we have $D(p,a) + (p+1, a-2) = (p, a+1)$. $\square$

Now we have

Lemma 7.4. *For all* $p, a \geq 0$*,* (p,a) *is the admissible Adem relation* (2).

Proof. We must only show that $C_{a,k} = \binom{a-k-1}{k}$, for $k \geq 0$ and $2k+1 \leq a$. Again this follows from induction:

$$C_{a+1,k} = C_{a,k} + C_{a-1,k-1} = \binom{a-k-1}{k} + \binom{a-k-1}{k-1} = \binom{a-k}{k}$$

by Pascal's triangle. $\square$

Now we'll show that the admissible and symmetric Adem relations imply each other. Let $A \subset W^{\otimes 2}$ be the admissible analogue of R, so A has basis $\{(p,a) : p, a \geq 0\}$. Then we have an immediate corollary of Lemma 7.3:

Lemma 7.5. $A = R$*, so* Λ *can be defined either by the admissible Adem relations* (2) *or the symmetric Adem relations* (4).

Proof. Since $D(R) \subset R$, Lemma 7.3 implies that $(p,a) \in R$ by induction on a. So $A \subset R$. But Lemma 7.3 also implies that $D(p,a) \in A$. Thus $D(A) \subset A$. Since $[p,0] = (p,0)$, and $D[p,n] = [p,n+1]$, we have $R \subset A$. Hence $A = R$. $\square$

Lemma 7.6. *The differential* $d\colon \Lambda \to \Lambda$ *can be defined either by the admissible Adem relations* (3) *or the symmetric Adem relations* (5).

Proof. For $p \geq -1$ and $a \geq 0$, we can measure the difference between (p,a) and $[p,a]$ as follows. Let $X_{p,a} = [p,a] + (p,a) \in V^{\otimes 2}$. We can restate Lemma 7.3 as $X_{p,a} = 0$ for $a = 0, 1, 2$, and

$$X_{p,a+1} = DX_{p,a} + (p+1, a-2) \in V^{\otimes 2}, \quad \text{for } a \geq 2. \tag{18}$$

Let $[\hat{p,a}]$ and $(\hat{p,a})$ be $[p,a]$ and (p,a) plus $\lambda_p \otimes \lambda_{2p+1+a} + \lambda_{p+a} \otimes \lambda_{2p+1}$. Then clearly $X_{p,a} = [\hat{p,a}] + (\hat{p,a})$. Now $[-\hat{1},a]$ and $(-\hat{1},a)$ are the formulas in $W^{\otimes 2}$ for the symmetric and admissible Adem relations. So specializing Equation (18) to $p = -1$ shows by induction on a that

$$[-\hat{1},a] + (-\hat{1},a) = X_{-1,a} \in R. \quad \square$$

The Tangora recursion relations (15) are a theoretical improvement over the usual recursion formula (which arose in the proof of Lemma 7.3)

$$C_{a+1,k} = C_{a,k} + C_{a,k-1} + C_{a-2,k-2}, \tag{19}$$

because it was clear that we had a sum of admissibles, and it was easy to see that $C_{a,k} = \binom{a-k-1}{k}$.

In doing hand calculations, the Tangora recursion relations also give a big improvement over the usual recursion scheme, because Tangora's involves 2 terms instead of 3, and the calculation stays on the same "page." For instance, we quickly and independently obtain on the λ_0 and λ_1 pages:

$$\begin{aligned}
\lambda_0\lambda_1 &= 0 & \lambda_1\lambda_3 &= 0\\
\lambda_0\lambda_2 &= \lambda_1\lambda_1 & \lambda_1\lambda_4 &= \lambda_2\lambda_3\\
\lambda_0\lambda_3 &= \lambda_2\lambda_1 & \lambda_1\lambda_5 &= \lambda_3\lambda_3\\
\lambda_0\lambda_4 &= \lambda_3\lambda_1 + \lambda_2\lambda_2 & \lambda_1\lambda_6 &= \lambda_4\lambda_3 + \lambda_3\lambda_4\\
\lambda_0\lambda_5 &= \lambda_4\lambda_1 & \lambda_1\lambda_7 &= \lambda_5\lambda_3\\
\lambda_0\lambda_6 &= \lambda_5\lambda_1 + \lambda_4\lambda_2 + \lambda_3\lambda_3 & \lambda_1\lambda_8 &= \lambda_6\lambda_3 + \lambda_5\lambda_4 + \lambda_4\lambda_5\\
\lambda_0\lambda_7 &= \lambda_6\lambda_1 + \lambda_4\lambda_3 & \lambda_1\lambda_9 &= \lambda_7\lambda_3 + \lambda_5\lambda_5\\
\lambda_0\lambda_8 &= \lambda_7\lambda_1 + \lambda_6\lambda_2 + \lambda_4\lambda_4 & \lambda_1\lambda_{10} &= \lambda_8\lambda_3 + \lambda_7\lambda_4 + \lambda_5\lambda_6\\
\lambda_0\lambda_9 &= \lambda_8\lambda_1 & \lambda_1\lambda_{11} &= \lambda_9\lambda_3
\end{aligned}$$

In usual recursive scheme, based on (19), one applies D to each equation to get the next one. In the Steenrod algebra [M-T], this works OK. To compute $Sq^3Sq^4 = Sq^7$ on the Sq^4 page, we need the Sq^i page for $i = 1, 2, 3$, and this presents no hardship. But in Λ, the λ_0 page requires part of the λ_1 page, which requires part of the λ_2 page, etc. For instance, to compute $\lambda_0\lambda_9$, we apply D to the equations for $\lambda_0\lambda_8$, $\lambda_1\lambda_7$ and $\lambda_2\lambda_6$ to obtain

$$\lambda_0\lambda_9 = \big(\lambda_8\lambda_1 + \lambda_6\lambda_3 + \lambda_5\lambda_4 + \lambda_4\lambda_5\big) + \big(\lambda_6\lambda_3 + \lambda_5\lambda_4\big) + \lambda_4\lambda_5 = \lambda_8\lambda_1.$$

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