Homology equivalences inducing an epimorphism on the fundamental group

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Abstract

Quillen's plus construction is a topological construction that kills the maximal perfect subgroup of the fundamental group of a space without changing the integral homology of the space. In this paper we show that there is a topological construction that, while leaving the integral homology of a space unaltered, kills even the intersection of the transfinite lower central series of its fundamental group. Moreover, we show that this is the maximal subgroup that can be factored out of the fundamental group without changing the integral homology of a space.

0 Introduction

As explained in [8], [9], Bousfield's $H\mathbb{Z}$-localization $X_{H\mathbb{Z}}$ of a space X ([2]) is homotopy equivalent to its localization with respect to a map $Bf\colon B\mathcal{F}_1 \to B\mathcal{F}_2$ induced by a certain homomorphism $f\colon \mathcal{F}_1 \to \mathcal{F}_2$ between free groups. This means that a space X is $H\mathbb{Z}$-local if and only if the induced map $Bf^*\colon \mathrm{map}(B\mathcal{F}_2, X) \to \mathrm{map}(B\mathcal{F}_1, X)$ is a weak homotopy equivalence. Moreover, the effect of Bf-localization on the fundamental group produces precisely the group-theoretical $H\mathbb{Z}$-localization (i.e. f-localization) of the fundamental group, i.e. $\pi_1 L_{Bf} X \cong L_f(\pi_1 X) \cong (\pi_1 X)_{H\mathbb{Z}}$ for all spaces X.

A universal acyclic space for $H\mathbb{Z}$-localization (i.e. Bf-localization), in the sense of Bousfield ([4]), was studied by Berrick and Casacuberta in [1]. They show that nullification with respect to such a universal acyclic space coincides with Quillen's plus construction X^+ for a space X. Moreover, this universal acyclic space can be taken to be a two-dimensional Eilenberg–Mac Lane space $K(A(f), 1)$, where $A(f)$ is a locally free, universal f-acyclic group, in the sense of [12]. The effect of $K(A(f), 1)$-nullification on the fundamental group is precisely given by its $A(f)$-nullification. This $A(f)$-nullification factors out the perfect radical of a group, i.e. the maximal perfect subgroup, which can be obtained as the intersection of the (transfinite) derived series of the group ([7]).

The intersection ΓG of the transfinite lower central series of a group G is also a radical, and, as observed by Bousfield ([3]), it is in fact the maximal G-perfect normal subgroup of G (where a normal subgroup H of G is called G-perfect if $H \cong [H, G]$). As explained in [7], there is an epireflection that corresponds to this radical ΓG. By adapting the methods used by Berrick and Casacuberta in [1], we describe in Theorem 2.6 an epimorphism g such that $L_g G \cong G/\Gamma G$ for all groups G.

We further construct a localization functor of topological spaces that is intermediate between Quillen's plus construction and Bousfield's homological localization, and induces localization with respect to g on the fundamental group. More precisely, we describe in Theorem 2.10 a map φ such that φ-localization of spaces factors out the intersection of the transfinite lower central series of the fundamental group, while preserving the integral homology of the space.

We furthermore show that this is in some sense the best possible result. Indeed, we show in Proposition 3.2 that the maximal subgroup that can be factored out of the fundamental group without altering the integral homology of a space, is precisely the intersection of the transfinite lower central series of the fundamental group.

Finally, we turn our attention to the question whether there exists a localization functor of spaces which induces localization with respect to a universal epimorphic $H\mathbb{Z}$-equivalence of groups (in the sense of [12]) on the fundamental group, without changing the integral homology of a space. In fact, we show in Proposition 3.3 that the answer to this question is affirmative if and only if, for all groups G, the kernel of the $H\mathbb{Z}$-localization homomorphism $G \to G_{H\mathbb{Z}}$ coincides with ΓG.

1 Preliminaries

For the convenience of the reader, we recall some terminology and bacis facts about localization with respect to a given continuous map (see e.g. [4], [8]). Given a map $f\colon A \to B$, a space X is called *f-local* if the induced map

$$f^*\colon \mathrm{map}(B, X) \to \mathrm{map}(A, X)$$

is a weak homotopy equivalence. For every space X there is a map $l_X \colon X \to L_f X$, which is initial among all maps from X into f-local spaces. L_f is called the localization functor with respect to f. A map ϕ is called an *f-equivalence* if $L_f\phi$ is a homotopy equivalence. Further a space X is called *f-acyclic* if $L_f X$ is contractible. In the special case where f is of the form $f\colon A \to *$, the f-localization of a space X is also denoted by $P_A X$ and it is called the *A-nullification* of X. The *localization class* of f, denoted by $\langle f \rangle$, is defined as the collection of all maps g such that L_g is naturally equivalent to L_f. When f is of the form $f\colon A \to *$, the class of f is simply denoted by $\langle A \rangle$ and is called the *nullification class* of A. One says that $\langle f \rangle \leq \langle g \rangle$ if and only if there is a natural transformation of localization functors $L_f \to L_g$. Recall that this is equivalent to every g-local space being f-local, to every f-equivalence being a g-equivalence, or to f being a g-equivalence. As shown by Bousfield in [4], the collection of localization functors with respect to maps is a small-complete lattice for this partial order relation. Note that a space X is f-acyclic if and only if $\langle X \rangle \leq \langle f \rangle$. Furthermore, in [4] Bousfield proved that every localization class has a best possible approximation by a nullification class. More precisely, for any map f, there is a maximal nullification class $\langle A(f) \rangle$ which is smaller than $\langle f \rangle$ in the lattice of localization classes. The space $A(f)$ is called a *universal f-acyclic space*, since a space X is f-acyclic if and only if X is $A(f)$-acyclic.

An important algebraic tool in studying localization with respect to a given map is given by its discrete analogue in the category of groups (see e.g. [4], [5]). For a given group homomorphism $f\colon A \to B$, a group G is called *f-local* if the induced map of sets

$$f^*\colon \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G)$$

is a bijection. For every group G there is a homomorphism $l_G\colon G \to L_f G$, which is initial among all homomorphisms from G into f-local groups. This allows to introduce the localization functor L_f with respect to f. In an obvious way, one speaks of f-equivalences and f-acyclic groups, and, if f is of the form $f : A \to 1$, of A-nullification. It makes also sense to speak of the localization class of a homomorphism f, denoted by $\langle f \rangle$, which in the special case when f is of the form $f : A \to 1$, is simply denoted by $\langle A \rangle$ and is called the nullification class of A. Furthermore, the collection of all localization classes of homomorphisms is again a small-complete lattice for an obvious partial order relation. In [12] it was proved that every localization class of a homomorphism has a best possible approximation by a nullification class and by the localization class of an epimorphism.

More precisely, for any homomorphism f, there is a maximal nullification class $\langle A(f)\rangle$ and a maximal class $\langle \mathcal{E}(f)\rangle$ where $\mathcal{E}(f)$ is an epimorphism such that

$$\langle A(f)\rangle \leq \langle \mathcal{E}(f)\rangle \leq \langle f\rangle.$$

The group $A(f)$ is called a *universal f-acyclic group*, since a group G is f-acyclic if and only if G is $A(f)$-acyclic. Accordingly, $\mathcal{E}(f)$ is called a *universal epimorphic f-equivalence*, since an epimorphism g is an f-equivalence if and only if g is an $\mathcal{E}(f)$-equivalence.

It was shown in [12] (see also [7]) that to any localization class $\langle f\rangle$ of an epimorphism f, there is associated a radical R_f on the category of groups such that $L_fG \cong G/R_fG$ for all groups G. (Recall that a radical R is a functor assignating to every group G a normal subgroup RG in such a way that every homomorphism $G \to K$ restricts to $RG \to RK$ and such that $R(G/RG) = 1$.) In fact, there is a bijective correspondence between epireflections (i.e. idempotent functors L on the category of groups for which $G \to LG$ is an epimorphism for any group G) and radicals. Furthermore, if the class $\langle f\rangle$ is actually a nullification class, then the associated radical R_f is idempotent, meaning that $R_fR_fG = R_fG$ for all groups G.

2 A universal epimorphic $H\mathbb{Z}$-map

It is well known that Bousfield's $H\mathbb{Z}$-localization $X_{H\mathbb{Z}}$ of a space X ([2]) is homotopy equivalent to its localization with respect to a map $Bf\colon B\mathcal{F}_1 \to B\mathcal{F}_2$ induced by a certain homomorphism $f\colon \mathcal{F}_1 \to \mathcal{F}_2$ between free groups ([8], [9]). In fact, the homomorphism f can be taken to be the free product of a set of representatives of isomorphism classes of homomorphisms between countable, free groups inducing an isomorphism on the first integral homology group. Furthermore, the effect of Bf-localization on the fundamental group is to produce its f-localization, which coincides with the group-theoretical $H\mathbb{Z}$-localization of the fundamental group. In [1, Proposition 4.2] the authors show that a universal f-acyclic group $A(f)$ (in the sense of [12]) can be taken to be the free product of a set of representatives of all isomorphism classes of countable, locally free, perfect groups. The key lemma here is a result due to Heller ([10]), which states that for every element x in any perfect group P, there exists a countable, locally free, perfect group D and a homomorphism $D \to P$ containing x in its image. We first prove a "relative" version of [10, Lemma 5.7]. Recall that a normal subgroup H of a group G is called *G-perfect* if

$H \cong [H, G]$.

Lemma 2.1 *Let A be any group with an A-perfect normal subgroup K. Let F be a countable free group and let F' be a subgroup. Let $\psi : F \to A$ be a homomorphism such that the restriction $\psi|F'$ is a homomorphism into K. Then there exists a countable, locally free group D with a D-perfect normal subgroup D' such that ψ can be factorized as $F \to D \to A$ and moreover this factorization restricts to a factorization of $\psi|F'$ as $F' \to D' \to K$.*

Proof. Choose free generators x_i, y_j for F such that the generators x_i freely generate F'. For each generator x_i of F', we can find finitely many elements $a_{i,\ell}$ of A and $k_{i,\ell}$ of K, such that $\psi(x_i) = \prod_\ell [a_{i,\ell}, k_{i,\ell}]$. Let F_1 be the free group generated by the set $\{x(i,\ell,1), x(i,\ell,2), y(j)\}$, and let F_1' be the free subgroup generated by the subset $\{x(i,\ell,1), x(i,\ell,2)\}$. Define a homomorphism $\psi_1 \colon F_1 \to A$, by setting $\psi_1(x(i,\ell,1)) = a_{i,\ell}$, $\psi_1(x(i,\ell,2)) = k_{i,\ell}$ and $\psi_1(y(j)) = \psi(y_j)$. Define a homomorphism $\varphi : F \to F_1$ by $\varphi(x_i) = \prod_\ell [x(i,\ell,1), x(i,\ell,2)]$ and $\varphi(y_j) = y(j)$. Then $\psi_1 \circ \varphi = \psi$ and the image of ψ_1 restricted to F_1' is contained in K. Iterating this construction by similarly factorizing ψ_1, we arrive at a sequence $F \to F_1 \to F_2 \to ...$ and a sequence of subgroups $F' \to F_1' \to F_2' \to ...$, whose colimits are the required D, resp. D'. $\square$

This immediately implies the following result.

Lemma 2.2 *Let A be any group with an A-perfect normal subgroup K. Let x be any element in K. Then there exist a countable, locally free group D with a D-perfect normal subgroup D' and a homomorphism $\psi \colon D \to A$ such that x belongs to the image of the restriction $\psi|D'$.* $\square$

Recall ([3]) that any group G has a maximal G-perfect subgroup, which we denote by ΓG, and that can be obtained as the intersection of the transfinite lower central series of G. Lemma 2.2 enables us to prove the following result.

Proposition 2.3 *Let G be any group. Then the following assertions are equivalent:*

(i) *For every group A and every A-perfect normal subgroup K of A, the restriction of any homomorphism $A \to G$ to K is trivial;*

(ii) *For every countable, locally free group A and every A-perfect normal subgroup K of A, the restriction of any homomorphism $A \to G$ to K is trivial;*

(iii) *For every countable, locally free group A, the restriction of any homomorphism $A \to G$ to ΓA is trivial.* $\square$

The subgroup ΓG actually defines a radical on the category of groups, and, hence, by [7], the assignation $G \to LG = G/\Gamma G$ is an epireflection. Our aim is to show that this epireflection is singly generated. More precisely, we want to exhibit a homomorphism g such that $L_g G \cong LG = G/\Gamma G$ for any group G. Moreover, by [12] we know that it is possible to choose g to be an epimorphism.

Let Λ be a set of representatives of isomorphism classes of countable, locally free groups. If $\mathcal{D}'$ denotes the normal closure of $\bigcup_\Lambda \Gamma D$ in the free product $\mathcal{D}$ of all groups D in Λ, which we denote by $\mathrm{Fr}_\Lambda D$, then we have a short exact sequence (cf. [11, Exercise 6.2.5])

$$\mathcal{D}' \rightarrowtail \mathcal{D} = \mathrm{Fr}_\Lambda D \twoheadrightarrow \mathcal{D}/\mathcal{D}' \cong \mathrm{Fr}_\Lambda(D/\Gamma D), \tag{2.1}$$

where $\mathrm{Fr}_\Lambda(D/\Gamma D)$ denotes the free product of all groups $D/\Gamma D$ for which (the isomorphism class of) D belongs to Λ.

Proposition 2.4 *Localization with respect to the epimorphism*

$$h\colon \mathcal{D} \twoheadrightarrow \mathcal{D}/\mathcal{D}'$$

given in (2.1) satisfies $L_h G \cong G/\Gamma G$ for all groups G.

Proof. Observe that a group G is f-local for any given epimorphism $f\colon A \twoheadrightarrow B$ if and only if the restriction of any homomorphism $A \to G$ to the kernel of f is trivial. The proof is now completed by using Proposition 2.3. $\square$

Note that we can partition Λ into Λ_1, containing all the representatives of the isomorphism classes of countable, locally free, perfect groups, and its complement Λ_1^c. We then can write

$$\begin{aligned}
\langle h \rangle &= \langle \mathrm{Fr}_{\Lambda_1}(D \to D/\Gamma D)\rangle * \langle \mathrm{Fr}_{\Lambda_1^c}(D \to D/\Gamma D)\rangle \\
&= \langle \mathrm{Fr}_{\Lambda_1}(D \to 1)\rangle * \langle \mathrm{Fr}_{\Lambda_1^c}(D \to D/\Gamma D)\rangle \\
&= \langle \mathcal{F} \rangle * \langle \mathrm{Fr}_{\Lambda_1^c}(D \to D/\Gamma D)\rangle,
\end{aligned}$$

where $\mathcal{F}$ is the universal $H\mathbb{Z}$-acyclic group defined in [1]. (Here we denote by $\langle f_1 \rangle * \langle f_2 \rangle$ the least upper bound of the classes $\langle f_1 \rangle$ and $\langle f_2 \rangle$ in the lattice of localization classes, and we have used the fact that the free product $f_1 * f_2$ is a representative of this least upper bound.)

However, it is possible to give another description of $\langle h \rangle$, which will be more useful later on. Indeed, note that, if $f_1 : A \twoheadrightarrow B$ is an epimorphism, and $f_2 : B \to C$ is any homomorphism, then $\langle f_2 \circ f_1 \rangle = \langle f_1 * f_2 \rangle = \langle f_1 \rangle * \langle f_2 \rangle$. This enables us to prove the following preliminary result.

Lemma 2.5 *Let $f\colon A \twoheadrightarrow B$ be any epimorphism. Then $\langle f \rangle = \langle l_A \rangle$, where $l_A\colon A \to L_f A$ denotes the localization homomorphism.*

PROOF. Since f is an epimorphism, we infer from [12, Theorem 2.1] that l_A is an epimorphism. Hence, $\langle f \rangle = \langle f \rangle * \langle l_B \rangle = \langle l_B \circ f \rangle = \langle L_f f \circ l_A \rangle = \langle L_f f \rangle * \langle l_A \rangle = \langle l_A \rangle$. □

The above lemma now proves the following alternative description of $\langle h \rangle$.

Theorem 2.6 *Consider the natural homomorphism $g\colon \mathcal{D} \to \mathcal{D}/\Gamma\mathcal{D}$, where $\mathcal{D}$ is as defined in (2.1). Then $\langle g \rangle = \langle h \rangle$, where h is defined in (2.1). In other words, $L_g G \cong G/\Gamma G$ for all groups G.* □

In fact, the homomorphism g is a "universal epimorphic $H\mathbb{Z}$-map", as we next show. Recall from [3] that a group homomorphism g is called an *$H\mathbb{Z}$-map* if $H_1 g$ (i.e. the homomorphism induced by g on the first integral homology group) is an isomorphism and $H_2 g$ is an epimorphism. We first need a characterization of the epimorphisms that are $H\mathbb{Z}$-maps (cf. [3]).

Lemma 2.7 *Let h be an epimorphism $h\colon A \twoheadrightarrow B$ with kernel K. Then h is an $H\mathbb{Z}$-map if and only if K is A-perfect.*

PROOF. This is an obvious consequence of the 5-term exact sequence

$$H_2(A) \to H_2(B) \to K/[K, A] \to H_1(A) \to H_1(B) \to 0. \quad \square$$

Proposition 2.8 *Let G be any group and let $g : \mathcal{D} \to \mathcal{D}/\Gamma\mathcal{D}$, where $\mathcal{D}$ is as defined in (2.1). The homomorphism $G \to L_g G \cong G/\Gamma G$ is terminal among all epimorphic $H\mathbb{Z}$-maps going out of G.* □

Observe that the epimorphism g that we have constructed is not a universal epimorphic $H\mathbb{Z}$-equivalence (in the sense of [12]). Indeed, there are "more" epimorphic $H\mathbb{Z}$-equivalences than epimorphic $H\mathbb{Z}$-maps. However, if we denote by $A(f)$, resp. $\mathcal{E}(f)$ a universal $H\mathbb{Z}$-acyclic group, resp. a universal epimorphic $H\mathbb{Z}$-equivalence, then there are natural homomorphisms

$$G \to P_{A(f)}G \to L_g G \cong G/\Gamma G \to L_{\mathcal{E}(f)}G \to G_{H\mathbb{Z}},$$

for any group G, where $G \to L_{\mathcal{E}(f)}G \to G_{H\mathbb{Z}}$ is an epi-mono factorization. Moreover, for many groups G (e.g. finite groups, nilpotent groups, or more generally, groups for which the lower central series stabilizes), we have isomorphisms $L_g G \cong G/\Gamma G \cong L_{\mathcal{E}(f)}G \cong G_{H\mathbb{Z}}$ (cf. [3]).

We now want to realize localization with respect to $g\colon \mathcal{D} \twoheadrightarrow \mathcal{D}/\Gamma\mathcal{D}$ topologically, by exhibiting a localization functor of spaces that induces g-localization on the fundamental group and which is intermediate between Quillen's plus construction and Bousfield's $H\mathbb{Z}$-localization (and, hence, does not change the integral homology of a space).

We will need the following proposition, which is similar to results obtained in [5] and [6].

Proposition 2.9 *Let $\psi\colon A \to B$ be any map which induces an epimorphism $\psi_* = \pi_1(\psi)\colon \pi_1(A) \twoheadrightarrow \pi_1(B)$ and suppose that A is a CW-complex of dimension at most two. Then ψ-localization of spaces is π_1-compatible, i.e.*

$$\pi_1(L_\psi X) \cong L_{\psi_*}(\pi_1 X),$$

for all spaces X.

Proof. For any space X, the map $X \to L_\psi X$ is a ψ-equivalence. Hence, by [5, Proposition 3.3], the induced homomorphism $\pi_1(X) \to \pi_1(L_\psi X)$ is a ψ_*-equivalence. Moreover, we claim that $\pi_1(L_\psi X)$ is ψ_*-local. To see this, it suffices to prove that the restriction of any homomorphism $\ell\colon \pi_1 A \to \pi_1(L_\psi X)$ to $\ker \psi_*$ is trivial. However, since the dimension of A is at most two, there exists a map $\xi\colon A \to L_\psi X$ inducing ℓ on the fundamental group. Since $L_\psi X$ is ψ-local, we infer that there exists a map $\chi\colon B \to L_\psi X$ such that $\chi \circ \psi \simeq \xi$, which implies that the restriction $\ell | \ker \psi_* = (\pi_1(\chi) \circ \psi_*) | \ker \psi_*$ is trivial. □

Now choose a two-dimensional CW-complex $M\mathcal{D}$ such that $\pi_1 M\mathcal{D} = \mathcal{D}$. Attach 2-cells to $M\mathcal{D}$, thereby obtaining a map $i\colon M\mathcal{D} \hookrightarrow C$ which induces $g\colon \mathcal{D} \twoheadrightarrow \mathcal{D}/\Gamma\mathcal{D}$ on the fundamental group. We then obtain a diagram

$$\begin{array}{ccccccccc}
 & & \pi_2 M\mathcal{D} & \to & \pi_2 C & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & H_2 M\mathcal{D} & \to & H_2 C & \to & H_2(C, M\mathcal{D}) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & H_2(\pi_1 M\mathcal{D}) & \to & H_2(\pi_1 C) & \to & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Moreover, since $H_2(\pi_1 C) = H_2(\mathcal{D}/\Gamma\mathcal{D}) = 0$, we infer that $H_2 C$ is an epimorphic image of $\pi_2 C$. This means that we can kill $H_2 C$ by attaching 3-cells to C, through a map $j\colon C \to C'$. It is now easily verified that the composition φ of

$$M\mathcal{D} \xrightarrow{i} C \xrightarrow{j} C' \tag{2.2}$$

is an integral homology equivalence and that φ induces the homomorphism $g\colon \mathcal{D} \twoheadrightarrow \mathcal{D}/\Gamma\mathcal{D}$ on the fundamental group (cf. [2, Lemma 6.1]).

Theorem 2.10 *Let $\varphi\colon M\mathcal{D} \to C'$ be the composition given in (2.2). Then φ-localization of spaces induces g-localization on the fundamental group. Moreover, for any space X, there are natural maps*

$$X \to X^+ \to L_\varphi X \to X_{H\mathbb{Z}}.$$

Proof. Since $M\mathcal{D}$ is a two-dimensional CW-complex and since φ induces an epimorphism on the fundamental group, we know by Proposition 2.9 that φ-localization is π_1-compatible, so that it induces g-localization on the fundamental group. The second claim is obvious, since φ is clearly an integral homology equivalence, and the perfect radical of $\pi_1(L_\varphi X)$ being trivial (cf. [1]). □

To see that the natural maps given in Theorem 2.10 are not equivalences in general, and thus that we have constructed a functor which is really different from Quillen's plus construction and from $H\mathbb{Z}$-localization, observe the following. On one hand, $L_\varphi(S^1 \vee S^1) \simeq S^1 \vee S^1 \simeq (S^1 \vee S^1)^+$. Indeed, the fact that $\mathbb{Z} * \mathbb{Z}$ is g-local implies that $S^1 \vee S^1$ is φ-local. On the other hand, $\pi_1(L_\varphi B\Sigma_3) \cong L_g(\pi_1 B\Sigma_3) \cong \mathbb{Z}/2$, while $\pi_1 B\Sigma_3^+ \cong P_{A(f)}(\pi_1 B\Sigma_3) \cong \Sigma_3$.

3 Homology equivalences inducing an epimorphism on the fundamental group

In this section we want to explore some immediate consequences of our results. In particular, we want to show that there are some restrictions on (integral) homology equivalences that induce an epimorphism on the fundamental group.

Proposition 3.1 *Let $\psi : X \to Y$ be an integral homology equivalence of spaces such that the induced homomorphism $f = \pi_1\psi$ is an epimorphism. Then $R_f G \subset \Gamma G$ for all groups G, where R_f denotes the radical associated to the epireflection class $\langle f \rangle$.*

PROOF. By hypothesis we know that f is an epimorphic $H\mathbb{Z}$-map. Hence, $\langle f \rangle \leq \langle g \rangle$, where g denotes the universal epimorphic $H\mathbb{Z}$-map of Theorem 2.6. Hence, there are natural homomorphisms $G \twoheadrightarrow L_f G \cong G/R_f G \twoheadrightarrow L_g G \cong G/\Gamma G$, for any group G. □

In particular, we have the following result.

Proposition 3.2 *Let $\psi\colon X \to Y$ be an integral homology equivalence of spaces such that the induced homomorphism $f = \pi_1\psi$ is an epimorphism. Then $\ker f \subset \Gamma\pi_1 X$.*

PROOF. Since f is an epimorphic $H\mathbb{Z}$-map, we know that $\ker f$ is $\pi_1 X$-perfect. □

In other words, for any space X, the maximal subgroup that can be factored out of $\pi_1 X$ without altering the integral homology of X, is precisely $\Gamma\pi_1 X$. In particular, this implies that there is a restriction on the possibility of realizing topologically a universal epimorphic $H\mathbb{Z}$-equivalence of groups (i.e. of finding an integral homology equivalence of spaces which induces localization with respect to a universal epimorphic $H\mathbb{Z}$-equivalence of groups on the fundamental group).

Proposition 3.3 *Let $\mathcal{E}(f)$ denote a universal epimorphic $H\mathbb{Z}$-equivalence of groups. Then the following assertions are equivalent:*

(i) *There exists an integral homology equivalence $\psi\colon X \to Y$ of spaces such that $\pi_1\psi = \mathcal{E}(f)$ and L_ψ is π_1-compatible;*

(ii) $\ker(l_{H\mathbb{Z}}\colon G \to G_{H\mathbb{Z}}) = \Gamma G$ *for all groups G.*

PROOF. To see that (i) implies (ii), it suffices to show that $\ker l_{H\mathbb{Z}} \subset \Gamma G$. However, this is an immediate consequence of Proposition 3.2, since $L_{\pi_1\psi} G \cong L_{\mathcal{E}(f)} G \cong G/\ker l_{H\mathbb{Z}}$ for all groups G. Finally (ii) implies (i), as is shown by our construction of φ in (2.2). □

References

[1] A. J. Berrick and C. Casacuberta, *A universal space for plus-constructions*, Topology (to appear).

[2] A. K. Bousfield, *The localization of spaces with respect to homology*, Topology **14** (1977), 133–150.

[3] A. K. Bousfield, *Homological localization towers for groups and* Π*-modules*, Mem. Amer. Math. Soc. **186** (1977).

[4] A.K. Bousfield, *Homotopical localizations of spaces*, Amer. J. Math. **119** (1997), 1321–1354.

[5] C. Casacuberta, *Anderson localization from a modern point of view*, in: The Čech Centennial, Contemporary Math. **181**, Amer. Math. Soc., Providence (1995), 35–44.

[6] C. Casacuberta, J. L. Rodríguez, and Jin-Yen Tai, *Localizations of abelian Eilenberg–Mac Lane spaces of finite type*, preprint (1998).

[7] C. Casacuberta, J. L. Rodríguez, and D. Scevenels, *Singly generated radicals associated with varieties*, in: Proceedings of Groups St Andrews 1997 in Bath, London Math. Soc. Lecture Note Series, Cambridge University Press (to appear).

[8] E. Dror Farjoun, *Cellular Spaces, Null Spaces and Homotopy Localization*, Lecture Notes in Math. vol. 1622, Springer-Verlag, Berlin Heidelberg New York (1996).

[9] E. Dror Farjoun, K. Orr and S. Shelah, *Bousfield localization as an algebraic closure of groups*, Israel J. Math. **66** (1989), 143–153.

[10] A. Heller, *On the homotopy theory of topogenic groups and groupoids*, Illinois J. Math. **24** (1980), 576–605.

[11] D. J. S. Robinson, *A course in the theory of groups*, 2nd ed., Graduate Texts in Mathematics vol. 80, Springer-Verlag, Berlin Heidelberg New York (1996).

[12] J. L. Rodríguez and D. Scevenels, *Universal epimorphic equivalences for group localizations*, preprint (1998).

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