STABLY DUALIZABLE GROUPS

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Abstract. We extend the duality theory for topological groups from the classical theory for compact Lie groups, via the topological study by (Dwyer and) J.R. Klein [Kl01] and the p-complete study for p-compact groups by T. Bauer [Ba04], to a general duality theory for stably dualizable groups in the E-local stable homotopy category, for any spectrum E. The principal new examples occur in the $K(n)$-local category, where the Eilenberg–Mac Lane spaces $G = K(\mathbb{Z}/p, q)$ are stably dualizable for all $0 \le q \le n$. We show how to associate to each E-locally stably dualizable group G a stably defined representation sphere S^{adG}, called the dualizing spectrum, which is dualizable and invertible in the E-local category. Each stably dualizable group is Atiyah–Poincaré self-dual in the E-local category, up to a shift by S^{adG}. There are dimension-shifting norm- and transfer maps for spectra with G-action, again with a shift given by S^{adG}. The stably dualizable group G also admits a kind of framed bordism class $[G] \in \pi_*(L_E S)$, in degree $\dim_E(G) = [S^{adG}]$ of the Pic_E-graded homotopy groups of the E-localized sphere spectrum.

Contents

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1. Introduction

1.1. The symmetry groups of stable homotopy theory.

Compact Lie groups occur naturally as the symmetry groups of geometric objects, e.g. as the isometry groups of Riemannian manifolds [MS39]. Such geometric objects can usefully be viewed as equivariant objects, i.e., as a spaces with an action by a Lie group. The homotopy theory of such equivariant spaces is quite well approximated by the corresponding stable equivariant homotopy theory, which in its strong "genuine" form relies, already in its construction, on the good representation theory for actions by Lie groups on finite-dimensional vector spaces.

As a first example of a useful stable result, consider the Adams equivalence $Y/G \simeq (\Sigma^{-adG} Y)^G$ of [LMS86, II.7]. Here Y is any free G-spectrum, adG denotes the adjoint representation of G on its Lie algebra and $\Sigma^{-adG} Y$ is the stably defined desuspension of Y with respect to this G-representation.

As a second example, Atiyah duality [At61] asserts that if M is a smooth closed manifold with stable normal bundle ν, the functional (Spanier–Whitehead) dual $DM_+ = F(M_+, S)$ of M_+ is equivalent to the Thom spectrum $Th(\nu \downarrow M)$. When $M = G$ is a compact Lie group, and thus parallelizable, we can write this as a stable Poincaré duality equivalence $DG_+ \simeq Th(\epsilon^{-n} \downarrow G) = \Sigma^{-n}\Sigma^\infty(G_+)$. But G acts on itself both from the left and the right, and the bi-equivariant form of this equivalence takes the more precise form

$$DG_+ \wedge S^{adG} \simeq \Sigma^\infty(G_+)$$

where G acts by conjugation from the left on the one-point compactification S^{adG} of the adjoint representation and trivially from the right. See Theorem 3.1.4 below.

As a third example, the left-invariant framing of an n-dimensional compact Lie group G gives it an associated stably framed cobordism class $[G]$ in $\Omega_n^{fr} \cong \pi_n(S)$, the n-th stable stem. For example $[S^1] = \eta \in \pi_1(S)$ realizes the stable class of the Hopf fibration $\eta \colon S^3 \to S^2$. It is of interest to see which stable homotopy classes actually occur in this way [Os82].

The formulation of these three results may appear to require that G admits a geometric representation theory, with tangent spaces, adjoint representations, etc., but in fact much less is required, and that is the main thrust of the present article.

1.2. Algebraic localizations.

Homotopy-theoretically, the main properties of compact Lie groups are that they are compact manifolds, hence admit the structure of a finite CW complex, and that they are topological groups, hence are (homotopy equivalent to) loop spaces. Browder [Br61, 7.9] showed that all finite H-spaces are Poincaré complexes, and recently Bauer, Kitchloo, Notbohm and Pedersen [BKNP04] showed that all finite loop spaces are indeed manifolds (but not generally Lie groups [ABGP04]). A standard method in homotopy theory, and a key ingredient in [BKNP04], is the possibility to study homotopy types locally, say with respect to a Serre class, an algebraic localization in the sense of Sullivan and Bousfield–Kan, or a Bousfield localization with respect to a homology theory [Bo75], [Bo79].

In the p-complete category, where a map (of spaces or spectra) is considered to be an equivalence if it induces an isomorphism on ordinary homology with $\mathbb{F}_p$-coefficients, the local incarnations of finite loop spaces are the p-compact groups of Dwyer and Wilkerson [DW94]. These are topological groups $G \simeq \Omega BG$ with (totally) finite mod p homology $H_*(G; \mathbb{F}_p)$, such that the classifying space BG is p-adically complete. We consider a compact Lie group G as a geometric, integrally defined object, which can be analyzed one rational prime p at a time by way of its homotopy-theoretic, locally defined p-compact pieces, namely the p-compact groups $\Omega(BG)^\wedge_p$ obtained by p-completing the classifying space BG at p and looping. There are also other more exotic examples of p-compact groups, which only exist locally at one or more primes p, without the global, geometric origin of a compact Lie group [DW93].

In his Ph.D.-thesis, T. Bauer [Ba04] showed that for each p-compact group G one can produce a p-complete stable replacement for the adjoint representation sphere S^{adG}, for the purposes of p-complete stable homotopy theory. It suffices to work G-equivariantly in the "naive" sense, where the objects are spectra equipped with a G-action, and the (weak) equivalences are G-equivariant maps that are stable equivalences in the underlying non-equivariant category. Bauer showed that for a p-compact group G, analogous results to the Adams equivalence and the Atiyah–Poincaré duality equivalences above hold, with S^{adG} reinterpreted as the dualizing spectrum $(\Sigma^\infty G_+)^{hG} = F(EG_+, \Sigma^\infty G_+)^G$ of W. Dwyer (unpublished) and J.R. Klein [Kl01], but formed in the p-complete category. Bauer also showed that a p-compact group G has the analogue of a framed bordism class $[G]$ in $\pi_*(S^\wedge_p)$. For example, the Sullivan spheres (see Example 2.3.5) are examples of p-compact groups, and represent the generators $\alpha_1 \in \pi_{2p-3}(S^\wedge_p)$.

1.3. Chromatic localizations.

In stable homotopy theory it is well-known (following [Ra84]) that it is possible to localize much further than to the (algebraic) p-local or p-complete situations, by way of the chromatic Bousfield localizations with respect to the Morava- and Johnson–Wilson spectra $K(n)$ and $E(n)$, for $n \geq 0$. See e.g. [HS99]. We can therefore analyze compact Lie groups and p-compact groups in even finer detail, focusing only on the p-primary v_n-periodic parts of their homotopy theory, by working in the p-primary $K(n)$-local category. The topological groups G that have the finiteness property that $K(n)_*(G)$ is finite in each degree will be called *$K(n)$-locally stably dualizable* groups, and among these we can single out the *$K(n)$-compact groups* as those whose classifying space BG is a $K(n)$-local space. See Section 2.3 below. Again, there are now new, exotic, examples of $K(n)$-locally stably dualizable groups that only exist $K(n)$-locally for some (p, n), without even the intermediary origin of a p-compact group. The simplest, abelian, examples are provided by the Eilenberg–Mac Lane spaces $G = K(\pi, q)$, e.g. for $\pi = \mathbb{Z}/p$, $0 \le q \le n$ [RW80], which are not p-compact for $q \neq 0$, and these are $K(n)$-compact for $q \neq n$.

In this paper we show that also for a $K(n)$-locally stably dualizable group G, the *dualizing spectrum* $S^{adG} = L_{K(n)}\Sigma^\infty(G_+)^{hG}$ formed in the $K(n)$-local stable category has the properties that make it a stable substitute for the adjoint representation sphere of a compact Lie group. The dualizing spectrum S^{adG} is a *dualizable* and *invertible* spectrum in the $K(n)$-local category, cf. Theorem 3.3.4, which means that it has an equivalence class $[S^{adG}] \in \mathrm{Pic}_{K(n)}$ in the $K(n)$-local Picard group [HMS94]. In particular, suspending (smashing) by S^{adG} is an invertible self-equivalence of the $K(n)$-local category.

We show that there is a natural *norm map*

$$N\colon (X \wedge S^{adG})_{hG} \to X^{hG}$$

for any spectrum X with G-action, which is a $K(n)$-local equivalence under slightly different conditions on X than those of the Adams equivalence. See Theorem 5.2.4.

We also show that there is an (implicitly $K(n)$-local) natural *Atiyah–Poincaré duality equivalence*

$$DG_+ \wedge S^{adG} \simeq \Sigma^\infty G_+ ,$$

which is G-equivariant from both the left and the right. See Theorem 3.1.4.

Finally, we combine the norm map $N\colon BG^{adG} = (S^{adG})_{hG} \to S^{hG} = D(BG_+)$ for $X = S$ with a bottom cell inclusion $i\colon S^{adG} \to BG^{adG}$ and the projection $p\colon S^{hG} \to S$ to obtain a natural map

$$pNi\colon S^{adG} \to S ,$$

representing a homotopy class $[G] \in \pi_*(L_{K(n)}S)$ in the $\mathrm{Pic}_{K(n)}$-*graded homotopy groups* of the $K(n)$-local sphere spectrum. See Definition 5.4.1. We informally think of this as the $K(n)$-locally framed bordism class of G.

The results discussed up to now hold in a uniform manner in the E-local stable category, for each fixed spectrum E and suitably defined E-locally stably dualizable groups. This is how the main body of the paper is written.

In Chapter 4 we develop calculational tools to study E-locally stably dualizable groups, mostly particular to the case $E = K(n)$. The group structure on G makes $H = K(n)_*(G)$ a graded Frobenius algebra over $R = K(n)_*$ (Proposition 4.2.4), for the R-dual $H^* = K(n)^*(G)$ is a free graded H-module of rank 1. There is a strongly convergent homological spectral sequence of Eilenberg–Moore type

$$E^2_{s,t} = \mathrm{Tor}^H_{s,t}(R, H^*) \Longrightarrow K(n)^{-(s+t)}(S^{adG})$$

(Proposition 4.1.1). It collapses at the E^2-term to the line $s = 0$, and its dual identifies $K(n)_*(S^{adG})$ with the H^*-comodule primitives $P_{H^*}(H) \cong \mathrm{Hom}_H(H^*, R)$ in $\mathrm{Hom}_R(H^*, R) \cong H = K(n)_*(G)$ (Theorem 4.2.6). For example, when $G = K(\mathbb{Z}/p, n)$ is viewed as a $K(n)$-locally stably dualizable group, it follows that $[G]\colon S^{adG} \to S$ is an equivalence in the $K(n)$-local category (Example 5.4.6), so the Atiyah–Poincaré duality equivalence takes the untwisted form

$$F(K(\mathbb{Z}/p, n)_+, L_{K(n)}S) \simeq L_{K(n)}\Sigma^\infty K(\mathbb{Z}/p, n)_+ .$$

1.4. Applications.

It is conceivable that more invertible spectra in the $K(n)$-local category can be constructed in the form S^{adG} for $K(n)$-locally stably dualizable groups G, than just the localized integer sphere spectra $L_{K(n)}\Sigma^d S$ for $d \in \mathbb{Z}$. There are no such examples in the p-complete setting, but the $K(n)$-local Picard group is more subtle. Likewise, it is conceivable that the associated homotopy classes $[G] \in \pi_*(L_{K(n)}S)$ can realize more homotopy classes than those that appear from Lie groups and p-compact groups. However, so far we have mostly studied the abelian examples of $K(n)$-locally stably dualizable groups given by Eilenberg–Mac Lane spaces, where this added potential is not realized. We think of these abelian groups as playing the analogous role of tori in the theory of compact Lie groups, and expect to develop a richer supply of non-abelian examples in joint work with T. Bauer, cf. Remark 2.3.7.

This work was partially motivated by the author's formulation [Ro:g] of Galois theory of E-local commutative S-algebras. If $A \to B$ is an E-local G-Galois extension there is a useful norm equivalence $N\colon (B \wedge S^{adG})_{hG} \to B^{hG}$, with $A \simeq B^{hG}$. For finite groups G this follows as in [Kl01], but the natural generality for the theory appears to be to allow topological Galois groups G that are E-locally stably dualizable, as considered here. The constructions in Chapters 3 and 5 of the present paper will find applications in the cited Galois theory.

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2. The dualizing spectrum

2.1. The E-local stable category.

As our basic model for spectra we shall take the bicomplete, bitensored closed symmetric monoidal category $\mathcal{M}_S$ of *S-modules* from [EKMM97]. The symmetric monoidal pairing is the smash product $X \wedge Y$, the unit object is the sphere spectrum S, and the internal function object is the mapping spectrum $F(X, Y)$. We write $DX = F(X, S)$ for the functional dual. For a based topological space T we write $X \wedge T = X \wedge \Sigma^\infty T$ and $F(T, X) = F(\Sigma^\infty T, X)$ for the resulting bitensors.

Let E be any S-module. It induces the (generalized) homology theory E_* that takes an S-module X to the graded abelian group $E_*(X) = \pi_*(E \wedge X)$. A map $f\colon X \to Y$ of S-modules is said to be an *E-equivalence* if the induced homomorphism $f_*\colon E_*(X) \to E_*(Y)$ is an isomorphism, and an S-module Z is *E-local* if for each E-equivalence $f\colon X \to Y$ the induced homomorphism $f^\#\colon [Y, Z]_* \to [X, Z]_*$ is an isomorphism.

Let $\mathcal{M}_{S,E}$ be the full subcategory of $\mathcal{M}_S$ of *E-local S-modules*. There is a Bousfield localization functor $L_E\colon \mathcal{M}_S \to \mathcal{M}_{S,E}$ [Bo79], [EKMM97, Ch. VIII] that comes equipped with a natural E-equivalence $X \to L_E X$ for each S-module X (with $L_E X$ E-local). Let $\mathcal{D}_S = \bar{h}\mathcal{M}_S$ be the homotopy category of $\mathcal{M}_S$, i.e., the *stable category*, and let $\mathcal{D}_{S,E} = \bar{h}\mathcal{M}_{S,E}$ be the homotopy category of $\mathcal{M}_{S,E}$, i.e., the *E-local stable category*. It is a stable homotopy category in the sense of [HPS97, 1.2.2].

The induced E-localization functor $L_E\colon \mathcal{D}_S \to \mathcal{D}_{S,E}$ is left adjoint to the forgetful functor $\mathcal{D}_{S,E} \to \mathcal{D}_S$.

The E-local category $\mathcal{M}_{S,E}$ inherits the structure of a bicomplete, bitensored closed symmetric monoidal category from $\mathcal{M}_S$ by applying L_E to each construction formed in $\mathcal{M}_S$. The symmetric monoidal pairing takes X and Y to $L_E(X \wedge Y)$, and the unit object is the E-local sphere spectrum $L_E S$. The internal function object $F(X,Y)$ is already E-local when Y is E-local, hence does not change when E-localized. In a similar fashion the (limits and) colimits in $\mathcal{M}_{S,E}$ are obtained from those formed in $\mathcal{M}_S$ by applying the E-localization functor, and likewise for tensors (and cotensors).

Example 2.1.1. We may take $E = S$, in which case every spectrum is S-local, $\mathcal{M}_{S,S} = \mathcal{M}_S$ and the S-local stable category is the whole stable category.

Example 2.1.2. For a fixed rational prime p and number $0 \le n < \infty$ we may take $E = E(n)$, the n-th p-primary Johnson–Wilson spectrum, with

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n, v_n^{-1}]\,.$$

When $n = 0$, $E(0) = H\mathbb{Q}$ is the rational Eilenberg–Mac Lane spectrum and E-equivalence means rational equivalence. In each case $L_n = L_{E(n)}$ is a smashing localization, $L_n S$ is a commutative S-algebra and the $E(n)$-local category $\mathcal{L}_n = \mathcal{M}_{S,E(n)}$, as studied in [HS99], is equivalent to the category $\mathcal{M}_{L_n S}$ of $L_n S$-modules. In this case the forgetful functor $\mathcal{M}_{S,E(n)} \to \mathcal{M}_S$ preserves the symmetric monoidal pairing, but not the unit object.

Example 2.1.3. For each prime p and number $0 \le n \le \infty$ we may alternatively take $E = K(n)$, the n-th p-primary Morava K-theory spectrum. When $n = 0$, $K(0) = E(0) = H\mathbb{Q}$, as discussed above. When $0 < n < \infty$,

$$K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$$

is a graded field, and $\mathcal{K}_n = \mathcal{D}_{S,K(n)}$ is the $K(n)$-local stable category, again studied in [HS99]. When $n = \infty$, $K(\infty) = H\mathbb{F}_p$ and E-equivalence means p-adic equivalence, so $\mathcal{M}_{S,H\mathbb{F}_p}$ is the category of p-complete S-modules. For $0 < n \le \infty$ the forgetful functor to $\mathcal{M}_S$ neither preserves the symmetric monoidal pairing nor the unit object.

Convention 2.1.4. Hereafter we shall work entirely within the E-local category $\mathcal{M}_{S,E}$. We refer to the objects of $\mathcal{M}_{S,E}$ as *(E-local) S-modules*, or simply as *spectra*. For brevity we shall write $X \wedge Y$ for the smash product, S for the sphere spectrum and $F(X,Y)$ for the function spectrum within this category. The same applies to the functional dual DX, limits, colimits, tensors and cotensors, all of which then take values in $\mathcal{M}_{S,E}$.

2.2. Dualizable spectra.

Following Dold–Puppe [DP80], Lewis–May–Steinberger [LMS86, III.1] observe that in any closed symmetric monoidal category there are natural canonical maps $\rho\colon X \to DDX$, $\nu\colon F(X,Y) \wedge Z \to F(X, Y \wedge Z)$ and $\wedge\colon F(X,Y) \wedge F(Z,W) \to$

$F(X \wedge Z, Y \wedge W)$. We follow Hovey–Strickland [HS99, 1.5] and say that a spectrum X is *(E-locally) dualizable* if the canonical map

$$\nu \colon DX \wedge X \to F(X, X)$$

(in the special case $X = Z$, $Y = S$) is an equivalence in $\mathcal{M}_{S,E}$. Lewis *et al* then show [LMS86, III.1.2, 1.3]:

Lemma 2.2.1.

(1) *The canonical map $\rho \colon X \to DDX$ is an equivalence if X is dualizable.*
(2) *The canonical map $\nu \colon F(X,Y) \wedge Z \to F(X, Y \wedge Z)$ is an equivalence if X or Z is dualizable.*
(3) *The smash product map $\wedge \colon F(X,Y) \wedge F(Z,W) \to F(X \wedge Z, Y \wedge W)$ is an equivalence if X and Z are dualizable, or if X is dualizable and $Y = S$.*

It follows that the function spectrum $F(X, Y)$ and smash product $X \wedge Y$ are dualizable when X and Y are dualizable. In particular, DX is dualizable when X is dualizable.

Example 2.2.2. For $E = S$, a spectrum X is dualizable if and only if it is stably equivalent to a finite CW spectrum [M96, XVI.7.4], i.e., if and only if $X \simeq \Sigma^\infty \Sigma^d K$ for some finite CW complex K and integer $d \in \mathbb{Z}$.

Example 2.2.3. For $E = K(n)$ with $0 \le n \le \infty$, Hovey–Strickland [HS99, 8.6] show that a $K(n)$-local S-module X is dualizable if and only if $K(n)_*(X)$ is a finitely generated $K(n)_*$-module. Note that this includes the cases $n = 0$ with $K(0) = H\mathbb{Q}$ and $n = \infty$ with $K(\infty) = H\mathbb{F}_p$. In each case $K(n)_*$ is a graded field, so $K(n)_*(X)$ will automatically be free.

Lemma 2.2.4. *If a spectrum X is $H\mathbb{F}_p$-locally dualizable then $L_{K(n)}X$ is $K(n)$-locally dualizable for each $0 < n < \infty$.*

Proof. The Atiyah–Hirzebruch spectral sequence

$$E^2_{s,t} = H_s(X; \pi_t K(n)) \Longrightarrow K(n)_{s+t}(X)$$

shows that if $H_*(X; \mathbb{F}_p)$ is a (totally) finite $\mathbb{F}_p$-module, then $K(n)_*(X)$ is a finitely generated $K(n)_*$-module for each $0 < n < \infty$. □

2.3. Stably dualizable groups.

Let G be a topological group. We write

$$S[G] = S \wedge G_+ = L_E \Sigma^\infty G_+$$

for the E-localization of the unreduced suspension spectrum on G, and $DG_+ = F(S[G], S) = F(G_+, L_E \Sigma^\infty S^0)$ for its functional dual. We may always suppose that G is cofibrantly based and of the homotopy type of a based CW-complex.

Definition 2.3.1. A topological group G is *(E-locally) stably dualizable* if $S[G] = L_E \Sigma^\infty G_+$ is dualizable in $\mathcal{M}_{S,E}$.

Lemma 2.3.2. *The product $G = G_1 \times G_2$ of two E-locally stably dualizable groups is again E-locally stably dualizable.*

Proof. If $S[G_1]$ and $S[G_2]$ are dualizable, then so is $S[G] \cong S[G_1] \wedge S[G_2]$. $\square$

For the following definition we shall also need to refer to Bousfield's homological localization for spaces [Bo75]. A map of spaces $f\colon X \to Y$ is an *E-equivalence* if the induced homomorphism $f_*\colon E_*(X) \to E_*(Y)$ is an isomorphism, and a space Z is *E-local* if for each E-equivalence $f\colon X \to Y$ the induced map of mapping spaces $f^\#\colon \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ is a weak homotopy equivalence.

Definition 2.3.3. An *E-compact group* is an E-locally stably dualizable group G whose classifying space BG is an E-local space.

Example 2.3.4. If $E = S$, then G is a stably dualizable group if and only if G_+ is stably equivalent to a finite CW complex, up to an integer suspension, cf. Example 2.2.2. So each compact Lie group G is stably dualizable, since G itself then is a finite CW complex. If BG is nilpotent as a space then it is S-local, so in this case G is also an S-compact group.

Example 2.3.5. For $E = H\mathbb{F}_p$, a topological group G is stably dualizable if and only if $H_*(G; \mathbb{F}_p)$ is a (totally) finite $\mathbb{F}_p$-module. The group G is $H\mathbb{F}_p$-compact if and only if $G \simeq \Omega BG$ is a *p-compact group* in the sense of Dwyer–Wilkerson [DW94].

The loop space of the p-completed classifying space of a compact Lie group provides a standard example of a p-compact group, but there are also exotic examples, such as the p-compact Sullivan sphere $(S^{2p-3})^\wedge_p = \Omega(B((\mathbb{Z}/p)^\times \ltimes B\mathbb{Z}_p)^\wedge_p)$ for p odd, and the 2-compact Dwyer–Wilkerson group $DI(4)$ [DW93]. These only exist locally, in the sense that they do not extend to integrally defined stably dualizable groups.

Example 2.3.6. For $E = K(n)$, a topological group G is stably dualizable if and only if $K(n)_*(G)$ is a finitely generated $K(n)_*$-module.

By the calculations of Ravenel–Wilson [RW80, 11.1] for p odd, and [JW85, Appendix] for $p = 2$, each Eilenberg–Mac Lane space $G = K(\pi, q) = B^q\pi$ for π a finite abelian group is a stably dualizable group. The classifying space $BG = K(\pi, q+1)$ is $K(n)$-local if and only if π is a (finite abelian) p-group and $0 \le q < n$, hence in all these cases G is $K(n)$-compact.

More generally, by [HRW98, 1.1] any topological group G with only finitely many nonzero homotopy groups, each of which is a finite abelian p-group, has finite $K(n)$-homology, hence is stably dualizable.

Once again, compact Lie groups or p-compact groups provide examples of $K(n)$-compact groups through $K(n)$-localization, but the Eilenberg–Mac Lane space examples above do not arise in this fashion. They are only defined in the chromatically most local context, i.e., in the $K(n)$-local category, and do not extend to stably dualizable groups in the p-complete or integral category.

Remark 2.3.7. These examples are all abelian topological groups, and can be expected to play a similar role to that of tori in the theory of compact Lie groups.

For non-abelian examples it is natural to look to finite Postnikov systems, as in [HRW98], or to looped localized Borel constructions of the form

$$G = \Omega L_{K(n)}(EW \times_W BA)$$

where A is an abelian topological group, such as $A = K(\pi, q)$, the Weyl group W is a finite group acting on A, $EW \times_W BA = B(W \ltimes A)$ is the classifying space of the semi-direct product $W \ltimes A$ and $L_{K(n)}$ denotes the Bousfield $K(n)$-localization of spaces. To analyze the $K(n)$-homology of G it is necessary to study the convergence properties of the $K(n)$-based Eilenberg–Moore spectral sequence in the path–loop fibration of $L_{K(n)}B(W \ltimes A)$. This is joint work in progress with T. Bauer.

2.4. The dualizing and inverse dualizing spectra.

Let $EG = B(*, G, G)$ be the usual free, contractible right G-space. Let X be a spectrum with right G-action, and let Y be a spectrum with left G-action. We define the *G-homotopy fixed points* of X to be

$$X^{hG} = F(EG_+, X)^G$$

and the *G-homotopy orbits* of Y to be

$$Y_{hG} = EG_+ \wedge_G Y\,.$$

In all cases G acts on EG from the right. These constructions only involve naive G-equivariant spectra, or spectra with G-action, in the sense that no deloopings with respect to non-trivial G-representations are involved. Each G-equivariant map $X_1 \to X_2$ that is an equivalence induces an equivalence $X_1^{hG} \to X_2^{hG}$ of homotopy fixed points, and similarly for homotopy orbits.

Definition 2.4.1. Let G be an E-locally stably dualizable group. The group multiplication provides the suspension spectrum $S[G] = L_E\Sigma^\infty G_+$ with mutually commuting *standard* left and right G-actions. We define the *dualizing spectrum* S^{adG} of G to be the G-homotopy fixed point spectrum

$$S^{adG} = S[G]^{hG} = F(EG_+, S[G])^G$$

of $S[G]$, formed with respect to the standard right G-action. The standard left action on $S[G]$ induces a left G-action on S^{adG}.

Remark 2.4.2. A discrete group G of type FP (e.g. the classifying space BG is finitely dominated) is called a *duality group* if $H^*(G; \mathbb{Z}[G])$ is concentrated in a single degree n and torsion free. The G-module $D = H^n(G; \mathbb{Z}[G])$ is then called the *dualizing module* of G, cf. [Br82, VIII.10]. The spectrum level construction above is clearly analogous to this algebraic notion, and was previously considered for topological groups by Dwyer and by J.R. Klein [Kl01, §1], and for p-compact groups by T. Bauer [Ba04, 4.1]. In the latter case the finite domination hypothesis on BG is usually unreasonable. Klein writes D_G and Bauer writes S_G for the dualizing spectrum of G. We use D for the functional dual and S for the sphere spectrum, so we prefer to write S^{adG} instead, in view of the compact Lie group example recalled immediately below. Our construction differs a little from that of Dwyer and Klein, due to our implicit E-localization.

Examples 2.4.3. (a) When G is a finite group, there is a canonical equivalence $S[G] = S \wedge G_+ \simeq F(G_+, S)$, so $S[G]^{hG} \simeq F(G_+, S)^{hG} \cong F(EG_+, S) \simeq S$ is naturally equivalent to the sphere spectrum.

(b) More generally, when G is a compact Lie group Klein [Kl01, 10.1] shows that the dualizing spectrum S^{adG} is equivalent as a spectrum with left G-action to the suspension spectrum of the representation sphere associated to the adjoint representation adG of G, i.e., the left conjugation action of G on its tangent space T_eG at the identity.

(c) In the case of a p-compact group G, Bauer [Ba04] shows that $S^{adG} \simeq (S^d)^\wedge_p$ for some integer $d = \dim_p G$ called the *p-dimension* of G, and that S^{adG} takes over the role of the representation sphere in the duality theory in that context. The present paper extends some of Bauer's work to the E-local stable category.

Lemma 2.4.4. *When G is abelian, the left G-action on S^{adG} is homotopically trivial, in the sense that it extends over the inclusion $G \subset EG$ to an action by the contractible topological group EG.*

Proof. When G is abelian, the left and right G-actions on $S[G]$ agree. In $S^{adG} = F(EG_+, S[G])^G$ the right action on $S[G]$ is equal to the right action on EG_+, which in the commutative case factors as

$$EG_+ \wedge G_+ \subset EG_+ \wedge EG_+ \to EG_+ \,. \quad \square$$

Remark 2.4.5. It can be inconvenient to study the E-homology of S^{adG} directly from its definition as a homotopy fixed point spectrum. We shall soon see that this dualizing spectrum is the functional dual of another spectrum S^{-adG}, which we call the inverse dualizing spectrum, and which admits a computationally more convenient construction as a homotopy orbit spectrum. Once we know that these two spectra are indeed dualizable, and mutually dual, this provides a convenient route to E-homological calculations.

Definition 2.4.6. Let G be a stably dualizable group. The left and right G-actions on $S[G]$ induce *standard* right and left G-actions on its functional dual $DG_+ = F(S[G], S)$, respectively, by acting in the source of the mapping spectrum. We define the *inverse dualizing spectrum* S^{-adG} of G to be the G-homotopy orbit spectrum

$$S^{-adG} = (DG_+)_{hG} = EG_+ \wedge_G DG_+$$

of DG_+, formed with respect to the standard left G-action. These left and right actions commute, so the standard right action on DG_+ induces a right G-action on S^{-adG}.

Proposition 2.4.7. *There is a natural equivalence*

$$S^{adG} \simeq DS^{-adG}$$

between the dualizing spectrum and the functional dual of the inverse dualizing spectrum, as spectra with left G-action.

Proof. The canonical equivalence $\rho\colon S[G] \to DDG_+ = F(DG_+, S)$ induces an equivalence ρ^{hG} of G-homotopy fixed points, from S^{adG} to

$$F(DG_+, S)^{hG} = F(EG_+, F(DG_+, S))^G \cong F(EG_+ \wedge_G DG_+, S) = DS^{-adG} \,. \quad \square$$

3. Duality theory

3.1. Poincaré duality.

Let G be a stably dualizable group. The topological group structure on G makes $S[G]$ a cocommutative Hopf S–algebra, with product $\phi\colon S[G] \wedge S[G] \to S[G]$, unit $\eta\colon S \to S[G]$, coproduct $\psi\colon S[G] \to S[G] \wedge S[G]$, counit $\epsilon\colon S[G] \to S$ and conjugation (antipode) $\chi\colon S[G] \to S[G]$, induced by the group multiplication $m\colon G \times G \to G$, unit inclusion $\{e\} \to G$, diagonal map $\Delta\colon G \to G \times G$, collapse map $G \to \{e\}$ and group inverse $i\colon G \to G$, respectively.

The product ϕ and unit η makes $S[G]$ an E-local S-algebra in $\mathcal{M}_{S,E}$, while the coproduct, counit and conjugation need only be defined in the E-local stable category $\mathcal{D}_{S,E}$.

The standard right G-action on DG_+ makes DG_+ a right $S[G]$-module. The module action is given by the map

$$\alpha\colon DG_+ \wedge S[G] \to DG_+$$

that in symbols takes $\xi \wedge x$ to $\xi * x\colon y \mapsto \xi(xy)$. Inspired by [Ba04, §4.3], we consider the following shearing equivalence. Its definition is simpler than that considered by Bauer, but the key idea is the same.

Definition 3.1.1. Let the *shear map* $sh\colon DG_+ \wedge S[G] \to DG_+ \wedge S[G]$ be the composite map

$$sh\colon DG_+ \wedge S[G] \xrightarrow{1\wedge\psi} DG_+ \wedge S[G] \wedge S[G] \xrightarrow{\alpha\wedge 1} DG_+ \wedge S[G]\,.$$

Algebraically, $sh\colon \xi \wedge x \mapsto \sum(\xi * x') \wedge x''$ where $\psi(x) = \sum x' \wedge x''$.

The standard left and right G-actions on $S[G]$ (and DG_+) can be converted into right and left G-actions on $S[G]$ (and DG_+), respectively, by way of the group inverse $i\colon G \to G$. We refer to these non-standard actions as *actions through inverses*. For example, the left G-action through inverses on DG_+ is given by the composite map

$$S[G] \wedge DG_+ \underset{\cong}{\xrightarrow{\gamma}} DG_+ \wedge S[G] \xrightarrow{1\wedge\chi} DG_+ \wedge S[G] \xrightarrow{\alpha} DG_+\,,$$

where $\gamma\colon X \wedge Y \to Y \wedge X$ denotes the canonical twist map. Algebraically, this action takes (x, ξ) to $\xi * \chi(x)\colon y \mapsto \xi(\chi(x)y)$.

Lemma 3.1.2. *The shear map sh is equivariant with respect to each of the following three mutually commuting G-actions:*

1. *The first, left G-action given by the action through inverses on DG_+ and the standard action on $S[G]$ in the source, and the standard action on $S[G]$ in the target;*
2. *The second, right G-action given by the action through inverses on DG_+ in the source, and the action through inverses on DG_+ in the target;*
3. *The third, right G-action given by the standard action on $S[G]$ in the source and by the standard actions on DG_+ and $S[G]$ in the target.*

Each action is trivial on the remaining smash factors.

Proof. In each case this is clear by inspection. □

Lemma 3.1.3. *The shear map sh is an equivalence, with homotopy inverse given by the composite map*

$$DG_+ \wedge S[G] \xrightarrow{1\wedge\psi} DG_+ \wedge S[G] \wedge S[G] \xrightarrow{1\wedge\chi\wedge 1} DG_+ \wedge S[G] \wedge S[G] \xrightarrow{\alpha\wedge 1} DG_+ \wedge S[G]\,.$$

Proof. This is an easy diagram chase, using coassociativity of ψ, the fact that α is a right $S[G]$-module action with respect to the product ϕ on $S[G]$, the Hopf conjugation identities $\phi(\chi \wedge 1)\psi \simeq \eta\epsilon \simeq \phi(1 \wedge \chi)\psi$, counitality for ψ and unitality for α. □

Theorem 3.1.4. *Let G be a stably dualizable group. There is a natural equivalence*

$$DG_+ \wedge S^{adG} \xrightarrow{\simeq} S[G]\,.$$

It is equivariant with respect to the first, left G-action through inverses on DG_+, the standard left action on S^{adG} and the standard left action on $S[G]$. It is also equivariant with respect to the second, right G-action through inverses on DG_+, the trivial action on S^{adG} and the standard right action on $S[G]$.

Proof. The shear equivalence $sh\colon DG_+ \wedge S[G] \to DG_+ \wedge S[G]$ induces a natural equivalence

$$(sh)^{hG}\colon (DG_+ \wedge S[G])^{hG} \xrightarrow{\simeq} (DG_+ \wedge S[G])^{hG}$$

of G-homotopy fixed points with respect to the third, right G-action. Note that this action is different in the source and in the target of sh.

There is a natural equivalence to the source of $(sh)^{hG}$:

$$DG_+ \wedge S^{adG} = DG_+ \wedge S[G]^{hG} \xrightarrow{\simeq} (DG_+ \wedge S[G])^{hG}\,.$$

To see that this map is an equivalence, consider the commutative square

$$\begin{array}{ccc} DG_+ \wedge S[G]^{hG} & \longrightarrow & (DG_+ \wedge S[G])^{hG} \\ \downarrow \simeq & & \downarrow \simeq \\ F(G_+, S[G]^{hG}) & \xrightarrow{\cong} & F(G_+, S[G])^{hG}\,. \end{array}$$

The vertical maps are equivalences, because $S[G]$ is dualizable and passage to homotopy fixed points respects equivalences. Hence the upper horizontal map is also an equivalence.

There is also a (composite) natural equivalence from the target of $(sh)^{hG}$:

$$(DG_+ \wedge S[G])^{hG} \xrightarrow{\simeq} F(G_+, S[G])^{hG} \xrightarrow{\simeq} S[G]\,.$$

The left hand map is an equivalence because $S[G]$ is dualizable, by the same argument as above. The right hand map is the composite equivalence

$$F(G_+, S[G])^{hG} \cong F(EG_+ \wedge G_+, S[G])^G \cong F(EG_+, S[G]) \xrightarrow{\simeq} S[G]\,.$$

Here the middle isomorphism uses that G acts freely on G_+ in the source.

The composite of these three natural equivalences is the desired natural equivalence $DG_+ \wedge S^{adG} \to S[G]$. The equivariance statements follow by inspection. □

Remark 3.1.5. We call $DG_+ \wedge S^{adG} \simeq S[G]$ the *Poincaré duality equivalence*. It shows how $S[G]$ is functionally self-dual, up to a shift by the dualizing spectrum. See also Remark 3.3.5. The equivariance statements in the theorem express the standard left and trivial right G-actions on S^{adG} in terms of the more familiar G-actions on DG_+ and $S[G]$.

Lemma 3.1.6. *Let G_1 and G_2 be stably dualizable groups. There is a natural equivalence*

$$S^{adG_1} \wedge S^{adG_2} \simeq S^{ad(G_1 \times G_2)}$$

of spectra with standard left and trivial right $(G_1 \times G_2)$-actions.

Proof. The Poincaré duality equivalences for G_1, G_2 and $(G_1 \times G_2)$ compose to an equivalence

$$\begin{aligned} DG_{1+} \wedge S^{adG_1} \wedge DG_{2+} \wedge S^{adG_2} &\simeq S[G_1] \wedge S[G_2] \\ &\simeq S[G_1 \times G_2] \simeq D(G_1 \times G_2)_+ \wedge S^{ad(G_1 \times G_2)} . \end{aligned}$$

It is equivariant with respect to the first, left $(G_1 \times G_2)$-action that involves the standard left action on S^{adG_1}, S^{adG_2} and $S^{ad(G_1 \times G_2)}$, as well as with respect to the second, right $(G_1 \times G_2)$-action through inverses on $DG_{1+} \wedge DG_{2+}$ and $D(G_1 \times G_2)_+$. Taking homotopy fixed points with respect to the second, right action we obtain the desired equivalence, which is equivariant with respect to the first, left action. Any equivalence is equivariant with respect to the trivial right action. □

Remark 3.1.7. A similar relation $S^{adG} \simeq S^{adH} \wedge S^{adQ}$ is likely to hold for an extension $1 \to H \to G \to Q \to 1$ of stably dualizable groups, cf. [Kl01, Thm. C], but for simplicity we omit the then necessary discussion of how to promote S^{adH} to a spectrum with G-action, etc.

3.2. Inverse Poincaré duality.

The aim of this section is to establish an inverse Poincaré equivalence

$$S[G] \wedge S^{-adG} \simeq DG_+ .$$

The initial idea is to functionally dualize the construction of the shear map in Section 3.1, and to apply homotopy orbits in place of homotopy fixed points. Following Milnor–Moore [MM65, §3], we identify the functional dual of a smash product $X \wedge Y$ of dualizable spectra with the smash product $DX \wedge DY$, in that order, via the canonical equivalence

$$DX \wedge DY = F(X, S) \wedge F(Y, S) \xrightarrow[\simeq]{\wedge} F(X \wedge Y, S \wedge S) = D(X \wedge Y) .$$

However, to form homotopy orbits we need genuine G-equivariant maps, and it is generally not the case that a G-equivariant inverse can be found for the (weak) equivalence displayed above. Thus some care will be in order.

Working for a moment in the E-local stable category $\mathcal{D}_{S,E} = \bar{h}\mathcal{M}_{S,E}$, let

$$\beta\colon S[G] \to S[G] \wedge DG_+$$

be dual to the module action map $\alpha\colon DG_+ \wedge S[G] \to DG_+$. It makes $S[G]$ a right DG_+-comodule spectrum, up to homotopy, where DG_+ has the weakly defined coproduct $\psi'\colon DG_+ \to DG_+ \wedge DG_+$ that is dual to ϕ. Furthermore, let

$$\phi'\colon DG_+ \wedge DG_+ \to DG_+$$

be the (strictly defined) product on DG_+ that is dual to ψ. The functional dual $sh^{\#}$ of the shear map is then the composite

$$sh^{\#}\colon S[G] \wedge DG_+ \xrightarrow{\beta\wedge 1} S[G] \wedge DG_+ \wedge DG_+ \xrightarrow{1\wedge\phi'} S[G] \wedge DG_+\,,$$

which is an equivalence by Lemma 3.1.3 and duality.

Returning to the category $\mathcal{M}_{S,E}$, we shall now obtain G-equivariant representatives for these maps.

Definition 3.2.1. Let $\tilde{\phi}\colon S[G] \to F(S[G], S[G])$ be right adjoint to the opposite product map $\phi\gamma\colon S[G] \wedge S[G] \to S[G]$. Algebraically, $\tilde{\phi}\colon x \mapsto (y \mapsto yx)$. Let $\psi^{\#}\colon F(S[G] \wedge S[G], S[G] \wedge S) \to F(S[G], S[G])$ be given by precomposition by $\psi\colon S[G] \to S[G] \wedge S[G]$ and postcomposition with $S[G] \wedge S \cong S[G]$.

The *dual shear map* $sh'\colon S[G] \wedge DG_+ \to F(S[G], S[G])$ is defined to be the composite map:

$$\begin{aligned} sh'\colon S[G] \wedge DG_+ &\xrightarrow{\tilde{\phi}\wedge 1} F(S[G], S[G]) \wedge DG_+ \\ &\xrightarrow[\simeq]{\wedge} F(S[G] \wedge S[G], S[G] \wedge S) \xrightarrow{\psi^{\#}} F(S[G], S[G])\,. \end{aligned}$$

It is equivariant with respect to the left G-action given by the standard left actions on $S[G]$ and DG_+ on the left hand side, and the left action through the standard right action on the $S[G]$ in the source of the mapping spectrum.

Theorem 3.2.2. *The dual shear map sh' is homotopic to the composite map*

$$S[G] \wedge DG_+ \xrightarrow[\simeq]{sh^{\#}} S[G] \wedge DG_+ \xrightarrow[\simeq]{\nu\gamma} F(S[G], S[G])\,.$$

In particular, sh' is an equivalence. On G-homotopy orbit spectra it induces an equivalence

$$DG_+ \simeq S[G] \wedge S^{-adG}\,.$$

Proof. The right action map α factors up to homotopy as the composite

$$\begin{aligned} DG_+ \wedge S[G] &\xrightarrow{\psi'\wedge 1} DG_+ \wedge DG_+ \wedge S[G] \\ &\xrightarrow{1\wedge\gamma} DG_+ \wedge S[G] \wedge DG_+ \xrightarrow{\epsilon\wedge 1} S \wedge DG_+ = DG_+\,. \end{aligned}$$

Here $\epsilon\colon DG_+ \wedge S[G] \to S$ is the pairing that evaluates a function on an element in its source. Let $\eta\colon S[G] \wedge DG_+ \to S$ be its functional dual, in the homotopy category. Then the dual map β factors up to homotopy as

$$S[G] \cong S \wedge S[G] \xrightarrow{\eta\wedge 1} S[G] \wedge DG_+ \wedge S[G] \xrightarrow{1\wedge\gamma} S[G] \wedge S[G] \wedge DG_+ \xrightarrow{\phi\wedge 1} S[G] \wedge DG_+ \,.$$

A diagram chase then verifies that $\tilde{\phi}$ is homotopic to the composite

$$S[G] \xrightarrow{\beta} S[G] \wedge DG_+ \underset{\cong}{\xrightarrow{\gamma}} DG_+ \wedge S[G] \underset{\simeq}{\xrightarrow{\nu}} F(S[G], S[G])\,.$$

A similar chase shows that the diagram

$$\begin{array}{ccccc} S[G] \wedge DG_+ \wedge DG_+ & & \xrightarrow{1\wedge\phi'} & & S[G] \wedge DG_+ \\ \simeq \big\downarrow \nu\gamma\wedge 1 & & & & \simeq \big\downarrow \nu\gamma \\ F(S[G], S[G]) \wedge DG_+ & \underset{\simeq}{\xrightarrow{\wedge}} & F(S[G] \wedge S[G], S[G] \wedge S) & \xrightarrow{\psi^{\#}} & F(S[G], S[G]) \end{array}$$

homotopy commutes.

Taken together, these diagrams show that $\nu\gamma \circ sh^{\#} \simeq sh'$. Applying G-homotopy orbits to the chain of equivalences

$$S[G] \wedge DG_+ \underset{\simeq}{\xrightarrow{sh'}} F(S[G], S[G]) \underset{\simeq}{\xleftarrow{\nu\gamma}} S[G] \wedge DG_+$$

we obtain the desired chain of equivalences

$$DG_+ \simeq (S[G] \wedge DG_+)_{hG} \underset{\simeq}{\xrightarrow{(sh')_{hG}}} F(S[G], S[G])_{hG} \underset{\simeq}{\xleftarrow{(\nu\gamma)_{hG}}} (S[G] \wedge DG_+)_{hG} \simeq S[G] \wedge S^{-adG}\,. \quad \square$$

Proposition 3.2.3. *Let G be a stably dualizable group. The dualizing spectrum S^{adG} and the inverse dualizing spectrum S^{-adG} are both dualizable spectra. Hence*

$$S^{-adG} \simeq DS^{adG}$$

as spectra with right G-action. The inverse Poincaré equivalence

$$S[G] \wedge S^{-adG} \simeq DG_+$$

is equivariant with respect to the dual G-actions to those of Theorem 3.1.4: The first of these is the right G-action through inverses on $S[G]$, the standard right action on S^{-adG} and the standard right action on DG_+. The second is the left G-action *through inverses on $S[G]$, the trivial action on S^{-adG} and the standard left action on DG_+.*

Proof. It suffices to prove that S^{-adG} is dualizable, in view of Proposition 2.4.7 and Theorem 3.1.4. We must show that the canonical map

$$\nu\colon DS^{-adG} \wedge S^{-adG} \to F(S^{-adG}, S^{-adG})$$

is an equivalence. We first check that ν smashed with the identity map of $S[G]$ is an equivalence. This map factors as the composite

$$\begin{aligned} DS^{-adG} \wedge S^{-adG} \wedge S[G] \simeq DS^{-adG} \wedge DG_+ \xrightarrow[\simeq]{\nu} F(S^{-adG}, DG_+) \\ \simeq F(S^{-adG}, S^{-adG} \wedge S[G]) \xleftarrow[\simeq]{\nu} F(S^{-adG}, S^{-adG}) \wedge S[G]\,. \end{aligned}$$

Here the first and third equivalences follow from the inverse Poincaré equivalence, while the second and fourth equivalences are consequences of the dualizability of DG_+ and $S[G]$, respectively. Thus $\nu \wedge 1_{S[G]}$ is an equivalence. Since S is a retract of $S[G]$, it follows that also ν itself is an equivalence. Hence S^{-adG} is dualizable. □

3.3. The Picard group.

The Picard group of the category of E-local S-modules was introduced by M. Hopkins; see [HMS94].

Definition 3.3.1. An E-local S-module X is *invertible* if there exists a spectrum Y with $X \wedge Y \simeq S$ in $\mathcal{M}_{S,E}$. Then Y is also invertible. The smash product $X \wedge X'$ of two invertible spectra X and X' is again invertible.

The *E-local Picard group* $\mathrm{Pic}_E = \mathrm{Pic}(\mathcal{M}_{S,E})$ is the set of equivalence classes of invertible E-local S-modules. We write $[X] \in \mathrm{Pic}_E$ for the equivalence class of X. The abelian group structure on Pic_E is defined by $[X] + [X'] = [X \wedge X']$ and $-[X] = [Y]$, with X, Y and X' as above.

Example 3.3.2. The only invertible spectra in $\mathcal{M}_S$ are the sphere spectra $S^d = \Sigma^d S$ for integers $d \in \mathbb{Z}$, so $\mathrm{Pic}_S \cong \mathbb{Z}$. Similarly, in the p-complete category $\mathcal{M}_{S,H\mathbb{F}_p}$ the invertible spectra are precisely the p-completed sphere spectra $(S^d)^\wedge_p$ for $d \in \mathbb{Z}$, so $\mathrm{Pic}_{H\mathbb{F}_p} \cong \mathbb{Z}$ too.

Example 3.3.3. By Hopkins, Mahowald and Sadofsky [HMS94, 1.3], a $K(n)$-local spectrum X is invertible if and only if $K(n)_*(X)$ is free of rank one over $K(n)_*$. These authors show [HMS94, 2.1, 2.7, 3.3] that for $n = 1$ and $p \neq 2$ there is a non-split extension

$$0 \to \mathbb{Z}_p^\times \to \mathrm{Pic}_{K(1)} \to \mathbb{Z}/2 \to 0$$

while for $n = 1$ and $p = 2$ there is a non-split extension

$$0 \to \mathbb{Z}_2^\times \to \mathrm{Pic}_{K(1)} \to \mathbb{Z}/8 \to 0\,.$$

Furthermore, they show [HMS94, 7.5] that when $n^2 \leq 2p - 2$ and $p > 2$ there is an injection $\alpha\colon \mathrm{Pic}_{K(n)} \to H^1(S_n; \pi_0(E_n)^\times)$, where E_n is the Hopkins–Miller commutative S-algebra and S_n is (one of the variants of) the n-th Morava stabilizer group. This permits an algebraic identification of $\mathrm{Pic}_{K(2)}$ for p odd. The homomorphism α seems to have a non-trivial kernel for $n = 2$ and $p = 2$, cf. [HMS94, §6].

Theorem 3.3.4. *Let G be a stably dualizable group. Then*

$$S^{adG} \wedge S^{-adG} \simeq S$$

so S^{adG} and S^{-adG} are mutually inverse invertible spectra in the E-local stable category. Hence the equivalence classes $[S^{adG}]$ and $[S^{-adG}]$ represent inverse elements in the E-local Picard group Pic_E.

Proof. The Poincaré duality equivalence and the inverse Poincaré equivalence provide a chain of equivalences

$$S[G] \wedge S^{-adG} \wedge S^{adG} \simeq DG_+ \wedge S^{-adG} \simeq S[G]\,,$$

which is equivariant with respect to the standard left action on both copies of $S[G]$, the trivial action on S^{-adG} and the standard left action on S^{adG}. Taking G-homotopy orbits of both sides yields the required equivalence

$$S^{-adG} \wedge S^{adG} \simeq S[G]_{hG} \simeq S\,. \quad \square$$

Remark 3.3.5. These results show that the shift given by smashing with S^{adG}, as in the Poincaré duality equivalence, is really an invertible self-equivalence of the stable homotopy category of spectra with G-action, in that it can be undone by smashing with $S^{-adG} \simeq DS^{adG}$.

Definition 3.3.6. Let the *E-dimension* of G be the equivalence class $\dim_E(G) = [S^{adG}] \in \mathrm{Pic}_E$ of the dualizing spectrum S^{adG} in the E-local Picard group.

Example 3.3.7. For $E = S$ the S-dimension of a compact Lie group G equals its manifold dimension in $\mathrm{Pic}_S \cong \mathbb{Z}$. Similarly, for $E = H\mathbb{F}_p$ the $H\mathbb{F}_p$-dimension of a p-compact group G is the same as its p-dimension.

4. Computations

4.1. A spectral sequence for E-homology.

Suppose that the S-module E is an S-algebra. The standard left G-action α' on DG_+ makes $E_*(DG_+) = E^{-*}(G)$ a left $E_*(G)$-module via the composite action map

$$E_*(G) \otimes E_*(DG_+) \to E_*(S[G] \wedge DG_+) \xrightarrow{\alpha'_*} E_*(DG_+)\,.$$

Proposition 4.1.1. *Let E be an S-algebra and let G be a stably dualizable group. There is a spectral sequence*

$$E^2_{s,t} = \mathrm{Tor}^{E_*(G)}_{s,t}(E_*, E^{-*}(G)) \Longrightarrow E_{s+t}(S^{-adG})$$

converging strongly to $E_*(S^{-adG}) \cong E^{-*}(S^{adG})$.

Proof. This is the E-homology homotopy orbit spectral sequence, which is a special case of the Eilenberg–Moore type spectral sequence [EKMM97, IV.6.4] for the E-homology of

$$S^{-adG} = EG_+ \wedge_G DG_+ \cong S[EG] \wedge_{S[G]} DG_+\,.$$

Here $E_*(S[EG]) \cong E_*$, $E_*(S[G]) \cong E_*(G)$ and $E_*(DG_+) \cong E^{-*}(G)$. The duality $S^{-adG} \simeq DS^{adG}$ from Proposition 3.2.3 relates the abutment to the E-cohomology of S^{adG}. $\square$

4.2. Morava K-theories.

In this and the following section (4.3) we specialize to the case when $E = K(n)$, for some fixed prime p and number $0 \le n \le \infty$. Hence stably dualizable means $K(n)$-locally stably dualizable, etc.

Lemma 4.2.1. *Let G be a stably dualizable group, so that $H = K(n)_*(G)$ is a finitely generated (free) module over $R = K(n)_*$. Then H is a graded cocommutative Hopf algebra over R, and its R-dual $H^* = K(n)^*(G) = \mathrm{Hom}_R(H, R)$ is a graded commutative Hopf algebra over R.*

Proof. By [HS99, 8.6], a topological group G is stably dualizable if and only if $H = K(n)_*(G)$ is finitely generated over $R = K(n)_*$. The group multiplication and diagonal map on G induce the Hopf algebra structure on H, in view of the Künneth isomorphism

$$K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y) \xrightarrow{\cong} K(n)_*(X \wedge Y)$$

in the case $X = Y = S[G]$. The identity $K(n)^*(G) \cong \mathrm{Hom}_R(H, R)$ is a case of the universal coefficient theorem

$$K(n)^*(X) \xrightarrow{\cong} \mathrm{Hom}_{K(n)_*}(K(n)_*(X), K(n)_*)\,.$$

This also leads to the Hopf algebra structure on H^*. □

Proposition 4.2.2. *Let G be a stably dualizable group. Then $K(n)_*(S^{adG}) \cong \Sigma^d R$ for some integer d, and $K(n)_*(S^{-adG}) \cong \Sigma^{-d} R$.*

Proof. By Theorem 3.3.4, S^{adG} is an invertible $K(n)$-local spectrum with inverse S^{-adG}, so by the Künneth theorem

$$K(n)_*(S^{adG}) \otimes_R K(n)_*(S^{-adG}) \cong K(n)_*(S) = R\,.$$

This implies that $K(n)_*(S^{adG})$ and $K(n)_*(S^{-adG})$ both have rank one over R. (Alternatively, use Theorem 3.1.4 and the Künneth theorem to obtain the isomorphism

$$H^* \otimes_R K(n)_*(S^{adG}) \cong H\,.$$

The total ranks of H^* and H as R-modules are equal, and finite, so $K(n)_*(S^{adG})$ must have rank one. In view of [HMS94, 1.3] or [HS99, 14.2], this also provides an alternative proof that S^{adG} is invertible in the $K(n)$-local category.) □

Definition 4.2.3. Let the integer $d = \deg_{K(n)}(G)$ such that $K(n)_*(S^{adG}) \cong \Sigma^d R$ be the $K(n)$-*degree* of G. When $0 < n < \infty$ this number is only well-defined modulo $|v_n| = 2(p^n - 1)$.

Remark 4.2.4. The evident homomorphism $\deg\colon \mathrm{Pic}_{K(n)} \to \mathbb{Z}/|v_n|$ takes the $K(n)$-dimension of G to its $K(n)$-degree. By [HMS94, 1.3] or [HS99, 14.2] we also have $\widehat{E}(n)^*(S^{adG}) \cong \Sigma^d \widehat{E}(n)^*$, where $\widehat{E}(n) = L_{K(n)}E(n)$. Similarly $E_n^*(S^{adG}) \cong \Sigma^d E_n^*$, where E_n is the Hopkins–Miller commutative S-algebra. Taking into account the action of the n-th Morava stabilizer group on $E_n^*(S^{adG})$ it is in principle possible to recover much more information about the $K(n)$-dimension of G than just the $K(n)$-degree.

For any graded commutative ring R and R-algebra H, we may consider both H and its R-dual $H^* = \mathrm{Hom}_R(H, R)$ as left H-modules in the standard way. Recall from e.g. [Pa71, §4] that H is called a *(graded) Frobenius algebra* over R if

(1) H is finitely generated and projective as an R-module, and
(2) H and some suspension $\Sigma^d H^*$ are isomorphic as left H-modules.

It follows that H is also isomorphic to $\Sigma^d H^*$ as right H-modules, and conversely. A (left or right) module M over a Frobenius algebra H is projective if and only if it is injective.

Proposition 4.2.4. *Let G be a stably dualizable group. Then $H = K(n)_*(G)$ is a Frobenius algebra over $R = K(n)_*$. In particular, $H^* = K(n)^*(G)$ is an injective and projective (left) H-module. In fact, it is free of rank one.*

Proof. Applying $K(n)$-homology to the equivalence of Theorem 3.1.4 gives an isomorphism

$$H^* \otimes_R \Sigma^d R = K(n)_*(DG_+) \otimes_{K(n)_*} K(n)_*(S^{adG}) \cong K(n)_*(S[G]) = H\,.$$

Here H acts from the left via the inverse of the second G-action, i.e., by the standard left action on H^*, the trivial action on $K(n)_*(S^{adG}) = \Sigma^d R$, and the left action through inverses on H. We continue with the isomorphism

$$\chi_* \colon H = K(n)_*(G) \xrightarrow{\cong} K(n)_*(G) = H$$

induced by the conjugation χ on $S[G]$, which takes the left action through inverses to the standard left action. Then the composite of these two isomorphisms exhibits H as a Frobenius algebra over R.

It is a formality that H^* is injective as a left H-module, so the general theory implies that it is also projective. But we can also see this directly in our case, since $H^* \cong \Sigma^{-d} H$ is an isomorphism of left H-modules, and the right hand side is free of rank one and thus obviously projective. □

Theorem 4.2.5. *Let G be a $K(n)$-locally stably dualizable group. The spectral sequence*

$$E^2_{s,t} = \mathrm{Tor}^H_{s,t}(R, H^*) \Longrightarrow K(n)_{s+t}(S^{-adG})$$

collapses to the line $s = 0$ at the E^2-term. The natural map $i \colon DG_+ \to S^{-adG}$ identifies

$$\Sigma^{-d} R = K(n)_*(S^{-adG}) \cong R \otimes_H H^*$$

with the left $H = K(n)_*(G)$-module indecomposables of $H^* = K(n)^{-*}(G)$. Dually, the natural map $p \colon S^{adG} \to S[G]$ identifies

$$\Sigma^d R = K(n)_*(S^{adG}) \cong \mathrm{Hom}_H(H^*, R)$$

with the left H^*-comodule primitives in H.

Proof. The spectral sequence is that of Proposition 4.1.1 in the special case $E = K(n)$. By Proposition 4.2.4, H^* is a free left H-module of rank one, hence flat. Thus $\mathrm{Tor}^H_{s,t}(R, H^*) = 0$ for $s > 0$, while for $s = 0$, $\mathrm{Tor}^H_{0,*}(R, H^*) = R \otimes_H H^*$. Hence the spectral sequence collapses to the line $s = 0$, and the edge homomorphism corresponding to the inclusion $i\colon DG_+ \to EG_+ \wedge_G DG_+ = S^{-adG}$ is the surjection $H^* = K(n)_*(DG_+) \to K(n)_*(S^{-adG}) = R \otimes_H H^*$. Thinking of H^* as a left H-module, these are the H-module coinvariants, or indecomposables, of H^*.

Passing to duals, the projection $p\colon S^{adG} = F(EG_+, S[G]) \to S[G]$ is functionally dual to the inclusion above, hence induces the R-dual injection $\mathrm{Hom}_R(R \otimes_H H^*, R) \to \mathrm{Hom}_R(H^*, R)$ in $K(n)$-homology. Thus $K(n)_*(S^{adG})$ is identified with $\mathrm{Hom}_R(R \otimes_H H^*, R) \cong \mathrm{Hom}_H(H^*, R)$, sitting inside $\mathrm{Hom}_R(H^*, R) \cong H$. The left H-module structure on H^* dualizes to a left H^*-comodule structure on H. The inclusion $\mathrm{Hom}_H(H^*, R) \to \mathrm{Hom}_R(H^*, R) \cong H$ then identifies $\mathrm{Hom}_H(H^*, R)$ with the H^*-comodule primitives in H. □

Remark 4.2.6. We sometimes write $Q_H(H^*) = R \otimes_H H^*$ for the left H-module indecomposables of H^*, and dually $P_{H^*}(H) = \mathrm{Hom}_H(H^*, R)$ for the left H^*-comodule primitives in H. Then $K(n)_*(S^{-adG}) \cong Q_H(H^*)$ and $K(n)_*(S^{adG}) \cong P_{H^*}(H)$.

To be explicit, an element $x \in H \cong \mathrm{Hom}_R(H^*, R)$ lies in $\mathrm{Hom}_H(H^*, R)$ if and only if $(y * \xi)(x) = \xi(xy)$ equals $\epsilon(y)\xi(x) = \xi(x\epsilon(y))$ for each $y \in H$ and $\xi \in H^*$. Here $\epsilon\colon H \to R$ is the augmentation. This condition is equivalent to asking that $xy = 0$ for each $y \in \ker(\epsilon)$, i.e., $x \in H$ multiplies to zero with each element in the augmentation ideal of H. So $P_{H^*}(H)$ is the left annihilator ideal of the augmentation ideal of H.

4.3. Eilenberg–Mac Lane spaces.

We can make the identifications in Theorem 4.2.5 explicit in the cases when $G = K(\mathbb{Z}/p, q)$ is an Eilenberg–Mac Lane space. For p an odd prime the $K(n)$-homology $H = K(n)_*K(\mathbb{Z}/p, q)$ was computed by Ravenel–Wilson in [RW80, 9.2], as we now recall:

Writing $K(n)_*K(\mathbb{Z}, 2) \cong K(n)_*\{\beta_m \mid m \geq 0\}$ with $|\beta_m| = 2m$ there are classes $a_m \in K(n)_*K(\mathbb{Z}/p, 1)$ in degree $|a_m| = 2m$ for $0 \leq m < p^n$ such that the Bockstein map $K(\mathbb{Z}/p, 1) \to K(\mathbb{Z}, 2)$ takes each a_m to β_m. Let $a_{(i)} = a_{p^i}$ in degree $|a_{(i)}| = 2p^i$ for $0 \leq i < n$. The q-fold cup product $K(\mathbb{Z}/p, 1) \wedge \cdots \wedge K(\mathbb{Z}/p, 1) \to K(\mathbb{Z}/p, q)$ takes $a_{(i_1)} \otimes \cdots \otimes a_{(i_q)}$ to a class $a_I \in K(n)_*K(\mathbb{Z}/p, q)$, where $I = (i_1, \dots, i_q)$ and $|a_I| = 2(p^{i_1} + \cdots + p^{i_q})$.

For $q = 0$, $G = K(\mathbb{Z}/p, 0) = \mathbb{Z}/p$ is a finite group and not very special to the $K(n)$-local situation. For each $q > n$, $K(\mathbb{Z}/p, q)$ has the $K(n)$-homology of a point. The intermediate cases $0 < q \leq n$ are more interesting.

For $0 < q < n$ there is an algebra isomorphism

$$K(n)_*K(\mathbb{Z}/p, q) \cong \bigotimes_I K(n)_*[a_I]/(a_I^{p^{\rho(I)}}),$$

where $I = (i_1, \dots, i_q)$ ranges over all integer sequences with $0 < i_1 < \cdots < i_q < n$, and $\rho(I) = s + 1$ where $s \in \{0, 1, \dots, q\}$ is maximal such that the final s-term subsequence has the form

$$(i_{q-s+1}, \dots, i_q) = (n - s, \dots, n - 1).$$

Equivalently, s is minimal such that $i_{q-s} < n-s-1$.

For $q = n$ there is an algebra isomorphism

$$K(n)_*K(\mathbb{Z}/p,n) \cong K(n)_*[a_I]/(a_I^p + (-1)^n v_n a_I)\,,$$

where $I = (0,1,\dots,n-1)$. Here $|a_I| = 2(1+p+\dots+p^{n-1}) = 2(p^n-1)/(p-1)$.

Proposition 4.3.1. *For $G = K(\mathbb{Z}/p,q)$ with $0<q<n$, $K(n)_*(S^{adG})$ is generated over $K(n)_*$ by the product $\pi = \prod_I a_I^{p^{\rho(I)}-1}$. Its $K(n)$-degree is 0 modulo $2(p^n-1)$.*

Proof. By Theorem 4.2.5 we identify $K(n)_*(S^{adG})$ with the left H^*-comodule primitives in H, which consists of the elements of H that multiply to zero with every element in the augmentation ideal of H. These are generated by the product π above. Its degree $\deg_{K(n)}(G) \equiv |\pi|$ can be computed by grouping together the integer sequences with the same value of $\rho(I) = s+1$:

$$\begin{aligned}
|\pi| &= \sum_I 2(p^{i_1} + \dots p^{i_q})(p^{\rho(I)}-1) \\
&= \sum_{\substack{0\le s\le q \\ 1\le i_1<\dots<i_{q-s}\le n-s-2}} 2(p^{i_1}+\dots+p^{i_{q-s}}+p^{n-s}+\dots+p^{n-1})(p^{s+1}-1) \\
&\equiv \sum_{\substack{0\le s\le q \\ s+2\le j_{s+1}<\dots<j_q\le n-1}} 2(p^{j_{s+1}}+\dots+p^{j_q}+p^1+\dots+p^s) \\
&\quad - \sum_{\substack{0\le s\le q \\ 1\le i_1<\dots<i_{q-s}\le n-s-2}} 2(p^{i_1}+\dots+p^{i_{q-s}}+p^{n-s}+\dots+p^{n-1}) \\
&= \sum_{1\le j_1<\dots<j_q\le n-1} 2(p^{j_1}+\dots+p^{j_q}) - \sum_{1\le i_1<\dots<i_q\le n-1} 2(p^{i_1}+\dots+p^{i_q}) = 0
\end{aligned}$$

modulo $2(p^n-1)$. □

Proposition 4.3.2. *For $G = K(\mathbb{Z}/p,n)$, $K(n)_*(S^{adG})$ is generated over $K(n)_*$ by $\pi = a_I^{p-1} + (-1)^n v_n$. Its $K(n)$-degree is 0 modulo $2(p^n-1)$.*

Proof. In this case the primitives in $K(n)_*(S^{adG})$ are generated by

$$\pi = a_I^{p-1} + (-1)^n v_n$$

in degree $|v_n| = 2(p^n-1)$. So also in this case $\deg_{K(n)}(G) \equiv 0$. □

Remark 4.3.3. It would be interesting to produce exotic elements in the $K(n)$-local Picard group $\mathrm{Pic}_{K(n)}$ as the class $[S^{adG}]$ of the dualizing spectrum of a $K(n)$-locally stably dualizable group G. The Eilenberg–Mac Lane examples above, together with Lemmas 2.3.2 and 3.1.6, seem to indicate that the stably dualizable group would have to be non-abelian, and not even homotopy-commutative by [HRW98]. This adds interest to the construction suggested in Remark 2.3.7.

5. Norm and transfer maps

5.1. Thom spectra.

The Thom space of a G-representation V is the reduced Borel construction, or homotopy orbit space, $BG^V = EG_+ \wedge_G S^V$, where S^V is the representation sphere of V. Generalizing the compact Lie group case, when S^{adG} is the (suspension spectrum of) the representation sphere of the adjoint representation adG, we make the following definition:

Definition 5.1.1. Let G be a stably dualizable group. The *Thom spectrum* BG^{adG} of its dualizing spectrum is the homotopy orbit spectrum

$$BG^{adG} = (S^{adG})_{hG} = EG_+ \wedge_G S^{adG} .$$

The inclusion $G \subset EG$ induces the *bottom cell inclusion* $i \colon S^{adG} \to BG^{adG}$.

Note that for G abelian, S^{adG} is a spectrum with EG-action by Lemma 2.4.4, so in these cases

$$BG^{adG} \simeq BG_+ \wedge S^{adG} .$$

As in Proposition 4.1.1, when E is an S-algebra there is a strongly convergent spectral sequence

$$\begin{aligned} E^2_{s,t} &= \mathrm{Tor}^{E_*(G)}_{s,t}(E_*, E_*(S^{adG})) \\ &\Longrightarrow E_{s+t}(BG^{adG}) . \end{aligned} \tag{5.1.2}$$

When $E = K(n)$ we have $K(n)_*(S^{adG}) \cong \Sigma^d K(n)_*$ by Proposition 4.2.2, with $d = \deg_{K(n)}(G)$. Thus the spectral sequence takes the form

$$\begin{aligned} E^2_{s,t} &= \mathrm{Tor}^{K(n)_*(G)}_{s,t}(K(n)_*, \Sigma^d K(n)_*) \\ &\Longrightarrow K(n)_{s+t}(BG^{adG}) . \end{aligned} \tag{5.1.3}$$

When S^{adG} is $K(n)$-orientable, so that the bottom cell inclusion $i \colon S^{adG} \to BG^{adG}$ induces a nonzero homomorphism $i_* \colon \Sigma^d K(n) \cong K(n)_*(S^{adG}) \to K(n)_*(BG^{adG})$, then this spectral sequence 5.1.3 is a free comodule over the corresponding bar spectral sequence for $K(n)_*(BG)$, on a single generator in degree d.

5.2. The norm map and Tate cohomology.

Let X be a spectrum with left G-action. We give it the trivial right G-action. The smash product $X \wedge S[G]$ then has the diagonal left G-action, as well as the right G-action that only affects $S[G]$. Consider forming homotopy orbits with respect to the left action, and forming homotopy fixed points with respect to the right action. As usual, there is a canonical colimit/limit exchange map

$$\kappa \colon ((X \wedge S[G])^{hG})_{hG} \to ((X \wedge S[G])_{hG})^{hG}$$

induced by the familiar map

$$EG_+ \wedge F(EG_+, Y) \to F(EG_+, EG_+ \wedge Y) ,$$

in the case $Y = X \wedge S[G]$.

On the left hand side, there is also a natural map $\nu\colon X \wedge S^{adG} = X \wedge S[G]^{hG} \to (X \wedge S[G])^{hG}$, since G acts trivially on X from the right. It induces a map

$$\nu_{hG}\colon (X \wedge S^{adG})_{hG} \to ((X \wedge S[G])^{hG})_{hG}$$

on homotopy orbits (with respect to the left actions). In the special case $X = S$, the maps ν and ν_{hG} are isomorphisms

On the right hand side, there is an untwisting equivalence $\zeta\colon (X \wedge S[G])_{hG} \to X \wedge S[G]_{hG} \simeq X \wedge S \cong X$, cf. [LMS86, p. 76], that takes the remaining right action on $(X \wedge S[G])_{hG}$ to the right action on X through the inverse of the left action. Hence there is an equivalence

$$\zeta^{hG}\colon ((X \wedge S[G])_{hG})^{hG} \xrightarrow{\simeq} X^{hG}$$

of homotopy fixed points, formed with respect to these right actions.

Definition 5.2.1. Let X be a spectrum with left G-action. The *(homotopy) norm map*

$$N\colon (X \wedge S^{adG})_{hG} \to X^{hG}$$

is the composite of the natural maps:

$$(X \wedge S^{adG})_{hG} \xrightarrow{\nu_{hG}} ((X \wedge S[G])^{hG})_{hG} \xrightarrow{\kappa} ((X \wedge S[G])_{hG})^{hG} \xrightarrow{\zeta^{hG}} X^{hG} .$$

The *G-Tate cohomology spectrum* X^{tG} of X is the cofiber of the norm map:

$$(X \wedge S^{adG})_{hG} \xrightarrow{N} X^{hG} \to X^{tG} .$$

Remark 5.2.2. In view of [GM95, 3.5], it is reasonable to expect that if X is an S-algebra with G-action, then X^{tG} is an S-algebra and $X^{hG} \to X^{tG}$ is a map of S-algebras. We do not know how to give a direct model for X^{tG}, say as the "G-fixed points" of the spectrum $\widetilde{E}G \wedge F(EG_+, X)$ with G-action, so it is not so easy to verify our expectation.

In the special case when $X = S$ with trivial G-action, the norm map simplifies to the canonical colim/lim exchange map

$$BG^{adG} = (S^{adG})_{hG} = (S[G]^{hG})_{hG} \xrightarrow{\kappa} (S[G]_{hG})^{hG} \simeq S^{hG} \cong D(BG_+) .$$

Here we use that $S^{hG} = F(EG_+, S)^G \cong F(BG_+, S) = D(BG_+)$ is the functional dual of BG_+, since S has trivial G-action. Hence there is a cofiber sequence

$$BG^{adG} \xrightarrow{N} D(BG_+) \to S^{tG} .$$

In the case of a compact Lie group G, the G-Tate cohomology X^{tG} is the same as that denoted $t_G(X)^G$ by Greenlees–May [GM95] and $\hat{\mathbb{H}}(G, X)$ by Bökstedt–Madsen [BM94].

Definition 5.2.3. A spectrum with G-action X is in the *thick subcategory* generated by spectra of the form $G_+ \wedge W$, if X can be built from $*$ in finitely many steps by (1) attaching cones on induced G-spectra of the form $G_+ \wedge W$, with W any spectrum, (2) passage to (weakly) equivalent spectra with G-action and (3) passage to retracts. For instance, any finite G-cell spectrum has this form.

Theorem 5.2.4. *Let G be a stably dualizable group. If a spectrum with G-action X is in the thick subcategory generated by spectra of the form $G_+ \wedge W$, then:*

1. *The norm map $N\colon (X \wedge S^{adG})_{hG} \to X^{hG}$ for X is an equivalence.*
2. *The G-Tate cohomology $X^{tG} \simeq *$ is contractible.*

Proof. If $X = G_+ \wedge W$ is induced up from a spectrum W with trivial G-action, the source of the norm map can be identified with

$$(G_+ \wedge W \wedge S^{adG})_{hG} \simeq W \wedge S^{adG}$$

while the target of the norm map can be identified with

$$(G_+ \wedge W)^{hG} \simeq (DG_+ \wedge S^{adG} \wedge W)^{hG} \simeq F(G_+, W \wedge S^{adG})^{hG} \simeq W \wedge S^{adG} .$$

These identifications are compatible, as can be checked by starting with the case $W = S$, hence in this case the norm map is itself an equivalence. The general case follows by induction on the number of attachments made. □

Remark 5.2.5. This result generalizes the third case of [Kl01, Thm. D], from compact Lie groups to stably dualizable groups. For compact Lie groups G this norm equivalence can be compared with the genuinely G-equivariant Adams equivalence $Y/G \simeq (Y \wedge S^{-adG})^G$ for Y a free G-spectrum [LMS86, II.7]. Any such Y is a filtered colimit of finite, free G-spectra, which are in the thick subcategory generated by $S[G] \cong G_+ \wedge S$. But, while genuine G-fixed points $(Y \mapsto Y^G)$ commute with filtered colimits, this is not generally the case for G-homotopy fixed points $(Y \mapsto Y^{hG})$. Therefore we cannot extend Theorem 5.2.4 to all spectra X with free G-action.

Dually, suppose that X is a spectrum with right G-action, and give it the trivial left G-action. The smash product $DG_+ \wedge X$ has the left G-action that only affects DG_+, and the diagonal right G-action. There is a canonical colim/lim exchange map

$$\kappa\colon ((DG_+ \wedge X)^{hG})_{hG} \to ((DG_+ \wedge X)_{hG})^{hG} .$$

The source of κ receives an equivalence from X_{hG} obtained by applying homotopy orbits to the equivalence $X \to F(G_+, X)^{hG}$. The target of κ admits a map to $(S^{-adG} \wedge X)^{hG}$ obtained by taking homotopy fixed points of the map $(DG_+ \wedge X)_{hG} \to (DG_+)_{hG} \wedge X = S^{-adG} \wedge X$, which exists because G acts trivially on X from the left.

Definition 5.2.6. Taken together, these maps yield the alternate norm map N', defined as the composite:

$$N'\colon X_{hG} \xrightarrow{\simeq} ((DG_+ \wedge X)^{hG})_{hG} \xrightarrow{\kappa} ((DG_+ \wedge X)_{hG})^{hG} \to (S^{-adG} \wedge X)^{hG} .$$

Its homotopy fiber X_{tG} is the *G-Tate homology spectrum*, and sits in a cofiber sequence

$$X_{tG} \to X_{hG} \xrightarrow{N'} (S^{-adG} \wedge X)^{hG} .$$

If X is a co-S-algebra with G-action, it again appears likely that X_{tG} is such a coalgebra and that $X_{tG} \to X_{hG}$ is a map of co-S-algebras.

5.3. The G-transfer map.

Definition 5.3.1. Let X be a spectrum with left G-action. The *G-transfer map*

$$\mathrm{trf}_G \colon (X \wedge S^{adG})_{hG} \to X$$

is the composite of the norm map $N \colon (X \wedge S^{adG})_{hG} \to X^{hG}$ and the forgetful map $p \colon X^{hG} \to X$.

When $X = S[Y]$ is the unreduced suspension spectrum of a G-space Y, this is the *dimension-shifting G-transfer map* associated to the principal G-bundle $Y \simeq EG \times Y \to EG \times_G Y$.

5.4. E-local homotopy classes.

Let G be a stably dualizable group, with dualizing spectrum S^{adG}. The composite of the bottom cell inclusion $i \colon S^{adG} \to BG^{adG}$ and the dimension-shifting G-transfer $\mathrm{trf}_G \colon BG^{adG} \to S$ is the composite map

$$S^{adG} \xrightarrow{i} (S^{adG})_{hG} = (S[G]^{hG})_{hG} \xrightarrow{\kappa} (S[G]_{hG})^{hG} \simeq S^{hG} \xrightarrow{p} S .$$

Noting that the projection p amounts to forgetting G-homotopy invariance, this map can also be expressed as the composite

$$S^{adG} = S[G]^{hG} \xrightarrow{p} S[G] \xrightarrow{i} S[G]_{hG} \simeq S .$$

Definition 5.4.1. The composite map $p\kappa i \colon S^{adG} \to S$ represents a class denoted $[G] \in \pi_*(L_E S)$ in the Pic_E-graded homotopy groups of the E-local sphere spectrum, in grading $* = \dim_E(G) = [S^{adG}] \in \mathrm{Pic}_E$. We might call $[G]$ the (E-local) *stably framed bordism class* of G.

Example 5.4.2. For the circle group $G = S^1$ and $E = S$ we have $[G] = \eta \in \pi_1(S)$. For the p-complete Sullivan sphere $G = (S^{2p-3})^\wedge_p$ and $E = H\mathbb{F}_p$ we have $[G] = \alpha_1 \in \pi_{2p-3}(S^\wedge_p)$, when p is odd. These examples are also detected $K(1)$-locally, i.e., for G considered as a $K(1)$-locally stably dualizable group.

Lemma 5.4.3. *In the case $E = K(n)$, the induced homomorphism*

$$[G]_* \colon \Sigma^d K(n)_* \cong K(n)_*(S^{adG}) \to K(n)_*(S) = K(n)_*$$

takes a generator of the $K(n)^*(G)$-comodule primitives in $K(n)_*(G)$ to its image under the augmentation $\epsilon \colon K(n)_*(G) \to K(n)_*$.

Proof. Recall from Theorem 4.2.5 that $K(n)_*(S^{adG})$ is identified with the $H^* = K(n)^*(G)$-comodule primitives $\mathrm{Hom}_H(H^*, R)$, and the projection $S^{adG} \to S[G]$ induces the forgetful inclusion into $\mathrm{Hom}_R(H^*, R) \cong H = K(n)_*(G)$. The inclusion $S[G] \to S[G]_{hG} \simeq S$ induces the augmentation ϵ, which establishes the claim □

Example 5.4.4. When $E = K(n)$ and G is a finite discrete group we get $H = R[G]$ and $P_{H^*}(H) \cong R\{N\}$, where $N = \sum_{g \in G} g$ is the norm element in H. Then $\epsilon(N) = |G|$ equals the order of G, so $[G]_*$ multiplies by $|G|$ in $R = K(n)_*$.

Example 5.4.5. When $E = K(n)$ and $G = K(\mathbb{Z}/p, q)$ for $0 < q < n$, the H^*-comodule primitives were found in Proposition 4.3.1 to be generated by an element π in the augmentation ideal $\ker(\epsilon)$, so the induced homomorphism $[G]_*$ is zero and $[G]\colon S^{adG} \to S$ has positive $K(n)$-based Adams filtration.

Example 5.4.6. When $E = K(n)$ and $G = K(\mathbb{Z}/p, n)$, with $q = n$, Proposition 4.3.2 exhibited a generating element $a_I^{p-1} + (-1)^n v_n$ for $P_{H^*}(H)$, which augments to the unit $(-1)^n v_n \in K(n)_*$. Hence in this case $[G]\colon S^{adG} \to S$ induces an isomorphism on $K(n)$-homology, and so $S^{adG} \simeq S$ in the $K(n)$-local category. By Lemma 2.4.4, the G-action on S^{adG} is homotopy trivial in this case. Hence the Poincaré duality equivalence 3.1.4 amounts to a $K(n)$-local self-duality equivalence

$$F(G_+, L_{K(n)}S) = DG_+ \simeq S[G] = L_{K(n)}\Sigma^\infty G_+$$

which is left and right G-equivariant up to homotopy, and which may be compared with [HS99, 8.7].

References

[ABGP04] K. Andersen, T. Bauer, J. Grodal, E. Pedersen, *A finite loop space not rationally equivalent to a compact Lie group*, Invent. Math. **157** (2004), 1–10.

[At61] M.F. Atiyah, *Thom complexes*, Proc. Lond. Math. Soc., III. Ser. **11** (1961), 291–310.

[Ba04] T. Bauer, *p-compact groups as framed manifolds*, Topology **43** (2004), 569–597.

[BKNP04] T. Bauer, N. Kitchloo, D. Notbohm and E.K. Pedersen, *Finite loop spaces are manifolds*, Acta Math. **192** (2004), 5–31.

[Bo75] A.K. Bousfield, *The localization of spaces with respect to homology*, Topology **14** (1975), 133–150.

[Bo79] A.K. Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), 257–281.

[BM94] M. Bökstedt and I. Madsen, *Topological cyclic homology of the integers*, Astérisque **226** (1994), 57–143.

[Br61] W. Browder, *Torsion in H-spaces*, Ann. Math. (2) **74** (1961), 24–51.

[Br82] K.S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer, 1982.

[DP80] A. Dold and D. Puppe, *Duality, trace, and transfer*, Proceedings of the International Conference on Geometric Topology (Warsaw, 1978), PWN, Warsaw, 1980, pp. 81–102.

[DW93] W.G. Dwyer and C.W. Wilkerson, *A new finite loop space at the prime two*, J. Am. Math. Soc. **6** (1993), 37–64.

[DW94] W.G. Dwyer and C.W. Wilkerson, *Homotopy fixed-point methods for Lie groups and finite loop spaces*, Ann. Math. (2) **139** (1994), 395–442.

[EKMM97] A.D. Elmendorf, I. Kriz, M.A. Mandell and J.P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, 1997.

[GM95] J.P.C. Greenlees and J.P. May, *Generalized Tate cohomology*, Mem. Am. Math. Soc., vol. 543, 1995.

[HMS94] M.J. Hopkins, M. Mahowald and H. Sadofsky, *Constructions of elements in Picard groups*, Topology and representation theory, Contemp. Math., vol. 158, 1994, pp. 89–126.

[HRW98] M.J. Hopkins, D.C. Ravenel and W.S. Wilson, *Morava Hopf algebras and spaces K(n) equivalent to finite Postnikov systems*, Stable and unstable homotopy, Fields Institute communications, vol. 19, 1998.

[HPS97] M. Hovey, J.H. Palmieri and N.P. Strickland, *Axiomatic stable homotopy theory*, Mem. Am. Math. Soc., vol. 610, 1997.

[HS99] M. Hovey and N.P. Strickland, *Morava K-theories and localisation*, Mem. Am. Math. Soc., vol. 666, 1999.

[JW85] D.C. Johnson and W.S. Wilson, *The Brown–Peterson homology of elementary p-groups*, Am. J. Math. **107** (1985), 427–453.

[Kl01] J.R. Klein, *The dualizing spectrum of a topological group*, Math. Ann. **319** (2001), 421–456.

[LMS86] L.G. Lewis, jun., J.P. May and M. Steinberger, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics, vol. 1213, 1986.

[M96] J.P. May, *Equivariant homotopy and cohomology theory*, CBMS Regional Conference Series in Mathematics, vol. 91, 1996.

[MM65] J.W. Milnor and J.C. Moore, *On the structure of Hopf algebras*, Ann. Math. (2) **81** (1965), 211–264.

[MS39] S.B. Myers and N.E. Steenrod, *The group of isometries of a Riemannian manifold*, Ann. of Math. (2) **40** (1939), 400–416.

[Os82] E. Ossa, *Lie groups as framed manifolds*, Topology **21** (1982), 315–323.

[Pa71] B. Pareigis, *When Hopf algebras are Frobenius algebras*, J. Algebra **18** (1971), 588–596.

[Ra84] D.C. Ravenel, *Localization with respect to certain periodic homology theories*, Am. J. Math. **106** (1984), 351–414.

[RW80] D.C. Ravenel and W.S. Wilson, *The Morava K-theories of Eilenberg–Mac Lane spaces and the Conner-Floyd conjecture*, Am. J. Math. **102** (1980), 691–748.

[Ro:g] J. Rognes, *Galois extensions of structured ring spectra*, preprint (2005).

[St00] N.P. Strickland, *Gross–Hopkins duality*, Topology **39** (2000), 1021–1033.

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