ON ANALYTICAL APPLICATIONS OF STABLE HOMOTOPY (THE ARNOLD CONJECTURE, CRITICAL POINTS)

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ABSTRACT. We prove the Arnold conjecture for closed symplectic manifolds with $\pi_2(M) = 0$ and $\operatorname{cat} M = \dim M$. Furthermore, we prove an analog of the Lusternik–Schnirelmann theorem for functions with “generalized hyperbolicity” property.

Introduction

Here we show that the technique developed in [R98] can be applied to the Arnold conjecture and to estimation of the number of critical points. For convenience of the reader, this paper is written independently of [R98].

Given a smooth ($= C^\infty$) manifold M and a smooth function $f : M \to \mathbb{R}$, we denote by $\operatorname{crit} f$ the number of critical points of f and set $\operatorname{Crit} M = \min\{\operatorname{crit} f\}$ where f runs over all smooth functions $M \to \mathbb{R}$.

The Arnold conjecture [Ar89, Appendix 9] is a well-known problem in Hamiltonian dynamics. We recall the formulation. Let (M, ω) be a closed symplectic manifold, and let $\phi : M \to M$ be a Hamiltonian symplectomorphism (see [HZ94], [MS95] for the definition). Furthermore, let $\operatorname{Fix} \phi$ denote the number of fixed points of ϕ. Finally, let

$$\operatorname{Arn}(M, \omega) := \min_{\phi} \operatorname{Fix} \phi$$

where ϕ runs over all Hamiltonian symplectomorphisms $M \to M$. The Arnold conjecture claims that $\operatorname{Arn}(M, \omega) \geq \operatorname{Crit}(M)$.

It is well known and easy to see that $\operatorname{Arn}(M, \omega) \leq \operatorname{Crit} M$. Thus, in fact, the Arnold conjecture claims the equality $\operatorname{Arn}(M, \omega) = \operatorname{Crit} M$.

Let $\operatorname{cat} X$ denote the Lusternik–Schnirelmann category of a topological space X (normalized, i.e., $\operatorname{cat} X = 0$ for X contractible).

Given a symplectic manifold (M^{2n}, ω), we define the homomorphisms

$$I_\omega : \pi_2(M) \to \mathbb{Q}, \quad I_\omega(x) = \langle \omega, h(x) \rangle$$
$$I_c : \pi_2(M) \to \mathbb{Z}, \quad I_c(x) = \langle c, h(x) \rangle$$

where $h : \pi_2(M) \to H_2(M)$ is the Hurewicz homomorphism, $c = c_1(\tau M)$ is the first Chern class of M and $\langle -, - \rangle$ is the Kronecker pairing.

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Theorem A (see 3.6). *Let $(M^n, \omega), n = \dim M$ be a closed connected symplectic manifold such that $I_\omega = 0 = I_c$ (e.g., $\pi_2(M) = 0$) and $\operatorname{cat} M = n$. Then $\operatorname{Arn}(M, \omega) \geq \operatorname{Crit} M$, i.e., the Arnold conjecture holds for M.*

It is well known that $\operatorname{Crit} M \geq 1 + \operatorname{cl} M$ for every closed manifold M, where cl denotes the cup-length, i.e., the length of the longest non-trivial cup-product in $\widetilde{H}^*(M)$. So, one has the following weaker version of the Arnold conjecture:

$$\operatorname{Arn}(M, \omega) \geq 1 + \operatorname{cl}(M),$$

and most known results deal with this weak conjecture, see [CZ83], [S85], [H88], [F89-1], [F89-2], [LO96]. (Certainly, there are lucky cases when $\operatorname{Crit} M = 1 + \operatorname{cl} M$, e.g. $M = T^{2n}$, cf. [CZ83].) For example, Floer [F89-1], [F89-2] proved that $\operatorname{Arn}(M) \geq 1 + \operatorname{cl} M$ provided $I_\omega = 0 = I_c$, cf. also Hofer [H88]. So, my contribution is the elimination of the clearance between $\operatorname{Crit} M$ and $1 + \operatorname{cl} M$. (It is easy to see that there are manifolds M as in Theorem A with $\operatorname{Crit} M > 1 + \operatorname{cl} M$, see 3.7 below.)

Let $p : PX \to X$ be the path fibration over a path connected compact metric space X, and let $p_k : P_k(X) \to X$ be the k-fold join over X of p. It is well-known that p_k has a section iff $\operatorname{cat} X < k$. Consider the Puppe sequence

$$P_k(X) \xrightarrow{p_k} X \xrightarrow{j_k} C_k(X) := C(p_k)$$

and set

$$r(X) := \sup\{m | j_m \text{ is stably essential}\}.$$

It is easy to see that $r(X) \leq \operatorname{cat} X$ and, moreover, $r(X) = \operatorname{cat} X$ iff X possesses a detecting element (as defined in [R98]). In particular, $r(M) = \operatorname{cat} M$ for every closed orientable manifold M with $\operatorname{cat} M = \dim M$, Theorem 2.4.

Actually, in Theorem A we prove that $\operatorname{Arn}(M, \omega) \geq 1 + r(M)$. Then we use a result of Takens [T68] which implies that $\operatorname{Crit} M = 1 + \operatorname{cat} M$ provided $\operatorname{cat} M = \dim M$.

After submission of the paper the author and John Oprea proved that $\operatorname{cat} M = \dim M$ for every closed symplectic manifold (M, ω) with $I_\omega = 0$, see [RO97]. So, the condition $\operatorname{cat} M = n$ in Theorem A can be omitted.

Passing to critical points, we prove the following theorem.

Theorem B (see 4.5). *Let M be a closed orientable manifold, and let $g : M \times \mathbb{R}^{p+q} \to \mathbb{R}$ be a C^2-function with the following properties:*

1. *There exist disks $D_+ \subset \mathbb{R}^p$ and $D_- \subset \mathbb{R}^q$ centered in origin such that $\operatorname{int}(M \times D_+ \times D_-)$ contains all critical points of g;*
2. *$\nabla g(x)$ points inward on $M \times \partial D_+ \times \operatorname{int} D_-$ and outward on $M \times \operatorname{int} D_+ \times \partial D_-$.*

Then $\operatorname{crit} g \geq 1 + r(M)$.

In particular, if M is aspherical then $\operatorname{crit} g \geq 1 + \operatorname{cat} M$.[†]

Notice that functions g as in Theorem B are related to the Conley index theory, see [C76]. I remark that Cornea [Co98] have also estimated the number of cririical points of functions as in Theorem B.

[†]Recall that a connected topological space is called aspherical if $\pi_i(X) = 0$ for every $i > 1$.

We reserve the term "map" for continuous functions of topological spaces, and we call a map *inessential* if it is homotopic to a constant map. The disjoint union of spaces X and Y is denoted by $X \sqcup Y$. Furthermore, X^+ denotes the disjoint union of X and a point, and X^+ is usually considered as a pointed space where the base point is the added point.

We follow Switzer [Sw75] in the definition of CW-complexes. A *CW-space* is defined to be a space which is homeomorphic to a CW-complex.

Given a pointed CW-complex X, we denote by $\Sigma^\infty X$ the spectrum $E = \{E_n\}$ where $E_n = S^n X$ for every $n \geq 0$ and $E_n = \text{pt}$ for $n < 0$; here $S^n X$ is the n-fold reduced suspension over X. Clearly, Σ^∞ is a functor from pointed CW-complexes to spectra.

Given any (bad) space X, the cohomology group $H^n(X;\pi)$ is always defined to be the group $[X, K(\pi,n)]$ where $[-,-]$ denotes the set of homotopy classes of maps and the Eilenberg–Mac Lane space $K(\pi,n)$ is assumed to be a CW-space.

"Smooth" always means "C^∞".

"Fibration" always means a Hurewicz fibration.

"Connected" always means path connected.

The sign " $\simeq$" denotes homotopy of maps (morphisms) or homotopy equivalence of spaces (spectra).

§1. Preliminaries on the Lusternik–Schnirelmann category

1.1. Definition. (a) Given a subspace A of a topological space X, we define $\operatorname{cat}_X A$ to be the minimal number k such that $A \subset U_1 \cup \cdots \cup U_{k+1}$ where each U_i is open and contractible in X. We also define $\operatorname{cat}_X A = -1$ if $A = \emptyset$.

(b) Given a map $f : X \to Y$, we define $\operatorname{cat} f$ to be the minimal number k such that $X = U_1 \cup \cdots \cup U_{k+1}$ where each U_i is open in X and $f|U_i$ is inessential for every i.

(c) We define the *Lusternik–Schnirelmann category* $\operatorname{cat} X$ of a space X by setting $\operatorname{cat} X := \operatorname{cat}_X X = \operatorname{cat} 1_X$.

Clearly, $\operatorname{cat} f \leq \min\{\operatorname{cat} X, \operatorname{cat} Y\}$.

The basic information concerning the Lusternik–Schnirelmann category can be found in [Fox41], [J78], [Sv66].

Let X be a connected space. Take a point $x_0 \in X$, set

$$PX = P(X, x_0) = \{\omega \in X^I \bigm| \omega(0) = x_0\}$$

and consider the fibration $p : PX \to X$, $p(\omega) = \omega(1)$ with the fiber ΩX.

Given a natural number k, we use the short notation

$$p_k : P_k(X) \to X. \tag{1.2}$$

for the map

$$\underbrace{p_X *_X \cdots *_X p_X}_{k \text{ times}} : \underbrace{PX *_X \cdots *_X PX}_{k \text{ times}} \longrightarrow X$$

where $*_X$ denotes the fiberwise join over X, see e.g. [J78]. In particular, $P_1(X) = PX$.

1.3. Proposition. *For every connected compact metric space X and every natural number k the following hold:*

(i) $p_k : P_k(X) \to X$ *is a fibration;*

(ii) $\operatorname{cat} P_k(X) < k$*;*

(iii) *The homotopy fiber of the fibration $p_k : P_k(X) \to X$ is the k-fold join $(\Omega X)^{*k}$;*

(iv) *If $\operatorname{cat} X = k$ and X is $(q-1)$-connected then $p_k : P_k(X) \to X$ is a $(kq-2)$-equivalence;*

(v) *If X has the homotopy type of a CW-space then $P_k(X)$ does.*

Proof. (i) This holds since a fiberwise join of fibrations is a fibration, see [CP86].

(ii) It is easy to see that $\operatorname{cat}(E_1 *_X E_2) \leq \operatorname{cat} E_1 + \operatorname{cat} E_2 + 1$ for every two maps $f_1 : E_2 \to X$ and $f_2 : E_2 \to X$. Now the result follows since $\operatorname{cat} P_1(X) = 0$.

(iii) This holds since the homotopy fiber of p_1 is ΩX.

(iv) Recall that $A * B$ is $(a+b+2)$-connected if A is a-connected and B is b-connected. Now, ΩX is $(q-2)$-connected, and so the fiber $(\Omega X)^{*k}$ of p_k is $(kq-2)$-connected.

(v) It is a well-known result of Milnor [M59] that ΩX has the homotopy type of a CW-space. Hence, the space $(\Omega X)^{*k}$ has it. Finally, the total space of any fibration has the homotopy type of a CW-space provided both the base and the fiber do, see e.g. [FP90, 5.4.2]. □

1.4. Theorem ([Sv66, Theorems 3 and 19′])**.** *Let $f : X \to Y$ be a map of connected compact metric spaces. Then $\operatorname{cat} f < k$ iff there is a map $g : X \to P_k(Y)$ such that $p_k g = f$.* □

§2. An invariant $r(X)$

Consider the Puppe sequence

$$P_m(X) \xrightarrow{p_m} X \xrightarrow{j_m} C_m(X) := C(p_m)$$

where $p_m : P_m(X) \to X$ is the fibration (1.2) and $C(p_m)$ is the cone of p_m.

2.1. Definition. Given a connected space X, we set

$$r(X) := \sup\{m | j_m \text{ is stably essential}\}.$$

(Recall that a map $A \to B$ is called stably essential if it is not stably homotopic to a constant map.)

2.2. Proposition. (i) *$r(X) \leq \operatorname{cat} X$ for every connected compact metric space X.*

(ii) *Let X be a connected CW-space, let E be a ring spectrum, and let $u_i \in \widetilde{E}^*(X)), i = 1, \dots, n$ be elements such that $u_1 \cdots u_n \neq 0$. Then $r(X) \geq n$. In other words, $r(X) \geq \operatorname{cl}_E(X)$ for every ring spectrum E.*

It makes sense to remark that $r(X) = \operatorname{cat} X$ iff X possesses a detecting element, as defined in [R96].

Proof. (i) This follows from 1.4.

(ii) Because of 1.3(v), without loss of generality we can and shall assume that $C_n(X)$ is a CW-space. We set $u = u_1 \cdots u_n \in \widetilde{E}^d(X)$. Because of the cup-length estimation of the Lusternik–Schnirelmann category, and by 1.3(ii), we have $p_n^*(u) = 0$. Hence, there is a homotopy commutative diagram

$$\begin{array}{ccc} \Sigma^\infty X & \xrightarrow{\Sigma^\infty j_n} & \Sigma^\infty C_n(X) \\ {\scriptstyle u}\downarrow & & \downarrow \\ \Sigma^d E & = & \Sigma^d E. \end{array}$$

Now, if $r(X) < n$ then $\Sigma^\infty j_n$ is inessential, and so $u = 0$. This is a contradiction. $\square$

2.3. Lemma. *Let $f : X \to Y$ be a map of compact metric spaces with Y connected, and let $j_m : Y \to C_m(Y)$ be as in* 2.1. *If the map $j_m f$ is essential then* $\operatorname{cat} f \geq m$. *In particular, if $r(Y) = r$ and the map*

$$f^\# : [Y, C_r(Y)] \to [X, C_r(Y)]$$

is injective then $\operatorname{cat} f \geq r$.

Proof. Consider the diagram

$$\begin{array}{ccccc} & & X & & \\ & & {\scriptstyle f}\downarrow & & \\ P_m(Y) & \xrightarrow{p_m} & Y & \xrightarrow{j_m} & C_m(Y). \end{array}$$

If $\operatorname{cat} f < m$ then, by 1.4, there is a map $g : X \to P_m(Y)$ with $f = p_m g$, and hence $j_m f$ is inessential. This is a contradiction. $\square$

2.4. Theorem. *Let M^n be a closed oriented connected n-dimensional PL manifold such that* $\operatorname{cat} M = \dim M \geq 4$. *Then* $r(M) = \operatorname{cat} M$.

Proof. Let $MSPL_*(-)$ denote the oriented PL bordism theory. By the definition of $r(X)$, it suffices to prove that $(j_n)_* : MSPL_*(M) \to MSPL_*(C_n(M))$ is a nonzero homomorphism. Hence, it suffices to prove that $(p_n)_* : MSPL_n(P_n(M)) \to MSPL_n(M)$ is not an epimorphism. Clearly, this will be proved if we prove that $[1_M] \in MSPL_n(M)$ does not belong to $\operatorname{Im}(p_n)_*$.

Suppose the contrary. Then there is a map $F : W \to M$ with the following properties:

1. (1) W is a compact $(n+1)$-dimensional oriented PL manifold with $\partial W = M \sqcup V$;
2. (2) $F|M = 1_M$, $F|V : V \to M$ lifts to $P_n(M)$ with respect to the map $p_n : P_n(M) \to M$.

Without loss of generality we can assume that W is connected.

Suppose for a moment that $\pi_1(W, M)$ is a one-point set (i.e., the pair (W, M) is simply connected). Then (W, M) has the handle presentation without handles of indices ≤ 1, see [St68, 8.3.3, Theorem A]. By duality, the pair (W, V) has the handle presentation without handles of indices $\geq n$. In other words, $W \simeq V \sqcup e \sqcup \cdots \sqcup e$

where $e_1, \dots, e_s$ are cells attached step by step and such that $\dim e_i \le n-1$ for every $i = 1, \dots, s$. However, the fibration $p_n : P_n(M) \to M$ is $n-2$ connected. Thus, $F : W \to M$ can be lifted to $P_n(M)$. In particular, p_n has a section. But this contradicts 1.4.

So, it remains to prove that, for every membrane (W, F), we can always find a membrane (U, G) with $\pi_1(U, G) = *$ and $G|\partial U = F|\partial W$. Here $\partial U = \partial W = M \sqcup V$ and $G : U \to M$. We start with an arbitrary connected membrane (W, F). Consider a PL embedding $i : S^1 \to \operatorname{int} W$. Then the normal bundle ν of this embedding is trivial. Indeed, $w_1(\nu) = 0$ because W is orientable.

Since M is a retract of W, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(M) & \longrightarrow & \pi_1(W) & \longrightarrow & \pi_1(W, M) & \longrightarrow & 0 \\ & & \| & & \big\downarrow{\scriptstyle F_*} & & & & \\ & & \pi_1(M) & = & \pi_1(M) & & & & \end{array}$$

where the top line is the homotopy exact sequence of the pair (W, M). Clearly, if F_* is monic then $\pi_1(W, M) = *$.

Let $\pi_1(W)$ be generated by elements $a_1, \dots, a_k$. We set $g_i := F_*(a_i)a_i^{-1} \in \pi_1(W)$ where we regard $\pi_1(M)$ as the subgroup of $\pi_1(W)$. Then $\operatorname{Ker} F_*$ is the smallest normal subgroup of $\pi_1(W)$ contained $g_1, \dots, g_k$. Now we realize $g_1, \dots, g_k$ by PL embeddings $S^1 \to \operatorname{int} W$ and perform the surgeries of (W, F) with respect to these embeddings, see [W70]. The result of the surgery establishes us a desired membrane. $\square$

2.5. Corollary. *Let M be as in 2.4, let X be a compact metric space, and let $f : X \to M$ be a map such that $f^* : H^n(M; \pi_n(C_n(M))) \to H^n(X; \pi_n(C_n(M)))$ is a monomorphism. Then* $\operatorname{cat} f \ge \operatorname{cat} M$.

Notice that, in fact, $\operatorname{cat} f = \operatorname{cat} M$ since $\operatorname{cat} f \le \operatorname{cat} M$ for general reasons.

Proof. We set $\pi = \pi_n(C_n(M))$. It is easy to see that $C_n(M)$ is simply connected. Hence, by 1.3(iv) and the Hurewicz theorem, $C_n(M)$ is $(n-1)$-connected. Thus, $[M, C_n(M)] = H^n(M; \pi)$. Let $\iota : C_n(M) \to K(\pi, n)$ denote the fundamental class. Then f^* can be decomposed as

$$f^* : H^n(M; \pi) = [M, C_n(M)] \xrightarrow{f^\#} [X, C_n(M)] \xrightarrow{\iota_*} [X, K(\pi, n)] = H^n(X; \pi).$$

Since f^* is a monomorphism, we conclude that $f^\#$ is. Thus, by 2.3 and 2.4,

$$\operatorname{cat} f \ge r(M) = \operatorname{cat} M. \quad \square$$

§3. The invariant $r(M)$ and the Arnold conjecture

Recall (see the introduction) that the Arnold conjecture claims that $\operatorname{Arn}(M, \omega) \ge \operatorname{Crit} M$ for every closed symplectic manifold (M, ω).

3.1. Recollection. A *flow* on a topological space X is a family $\Phi = \{\varphi_t\}, t \in \mathbb{R}$ where each $\varphi_t : X \to X$ is a self-homeomorphism and $\varphi_s \varphi_t = \varphi_{s+t}$ for every $s, t \in \mathbb{R}$ (notice that this implies $\varphi_0 = 1_X$).

A flow is called *continuous* if the function $X \times \mathbb{R} \to X, (x,t) \mapsto \varphi_t(x)$ is continuous.

A point $x \in X$ is called a *rest point* of Φ if $\varphi_t(x) = x$ for every $t \in \mathbb{R}$. We denote by $\operatorname{Rest}\Phi$ the number of rest points of Φ.

A continuous flow $\Phi = \{\varphi_t\}$ is called *gradient-like* if there exists a continuous (Lyapunov) function $F : X \to \mathbb{R}$ with the following property: for every $x \in X$ we have $F(\varphi_t(x)) < F(\varphi_s(x))$ whenever $t > s$ and x is not a rest point of Φ.

3.2. Definition (cf. [H88], [MS95])**.** Let X be a topological space. We define an *index function* on X to be any function $\nu : 2^X \to \mathbb{N} \cup \{0\}$ with the following properties:

(1) (monotonicity) If $A \subset B \subset X$ then $\nu(A) \le \nu(B)$;
(2) (continuity) For every $A \subset X$ there exists an open neighbourhood U of A such that $\nu(A) = \nu(U)$;
(3) (subadditivity) $\nu(A \cup B) \le \nu(A) + \nu(B)$;
(4) (invariance) If $\{\varphi_t\}, t \in \mathbb{R}$ is a continuous flow on X then $\nu(\varphi_t(A)) = \nu(A)$ for every $A \subset X$ and $t \in \mathbb{R}$;
(5) (normalization) $\nu(\emptyset) = 0$. Furthermore, if $A \ne \emptyset$ is a finite set which is contained in a connected component of X then $\nu(A) = 1$.

3.3. Theorem. *Let Φ be a gradient-like flow on a compact metric space X. Then*

$$\operatorname{Rest}\Phi \ge \nu(X)$$

for every index function ν on X.

Proof. The proof follows the ideas of Lusternik–Schnirelmann. For X connected see [H88], [MS95, p.346 ff]. Furthermore, if $X = \sqcup X_i$ with X_i connected then

$$\operatorname{Rest}\Phi = \sum \operatorname{Rest}(\Phi|X_i) \ge \sum \nu(X_i) \ge \nu(X). \quad \square$$

3.4. Corollary. *Let Φ be a gradient-like flow on a compact metric space X, let Y be a Hausdorff space which admits a covering $\{U_\alpha\}$ such that each U_α is open and contractible in Y, and let $f : X \to Y$ be an arbitrary map. Then*

$$\operatorname{Rest}\Phi \ge 1 + \operatorname{cat} f.$$

Proof. Given a subspace A of X, we define $\nu(A)$ to be the minimal number m such that $A \subset U_1 \cup \dots \cup U_m$ where each U_i is open in X and $f|U_i$ is inessential. It is easy to see that ν is an index function on X (normalization follows from the properties of Y). But $\nu(X) = 1 + \operatorname{cat} f$, and so, by 3.3, we conclude that $\operatorname{Rest}\Phi \ge 1 + \operatorname{cat} f$. $\square$

3.5. Theorem. *Let (M, ω) be a closed connected symplectic manifold with $I_\omega = 0 = I_c$, and let $\phi : M \to M$ be a Hamiltonian symplectomorphism. Then there exists a map $f : X \to M$ with the following properties:*

(i) *X is a compact metric space;*
(ii) *X possesses a gradient-like flow Φ such that* $\operatorname{Rest}\Phi \le \operatorname{Fix}\phi$*;*
(iii) *The homomorphism $f^* : H^n(M; G) \to H^n(X; G)$ is a monomorphism for every coefficient group G.*

Proof. This can be proved following [F89-2, Theorem 7]. (Note that the formulation

$z^*[P] \neq 0$. Furthermore, the reference [CE] in the proof must be replaced by [F7].) In fact, Floer denoted by $z : \mathscr{S} \to P$ what we denote by $f : X \to M$, and he showed that the homomorphism $z^* : \mathbb{Z} = H^n(P) \to H^n(\mathscr{S})$ is monic. He did it for $\mathbb{Z}$-coefficients, but the proof for arbitrary G is similar.

Also, cf. [H88] and [HZ94, Ch. 6].

In fact, Floer considered Alexander–Spanier cohomology, but for compact metric spaces it coincides with $H^*(-)$. In greater detail, you can find in [Sp66] an isomorphism between Alexander–Spanier and Čech cohomology and in [Hu61] an isomorphism between Čech cohomology and $H^*(-)$. $\square$

Recall that every smooth manifold turns out to be a PL manifold in a canonical way, see e.g. [Mu66].

3.6. Theorem. *Let (M,ω) be a closed connected symplectic manifold with $I_\omega = 0 = I_c$ and such that* $\operatorname{cat} M = \dim M$. *Then* $\operatorname{Arn}(M,\omega) \geq \operatorname{Crit} M$.

Proof. The case $\dim M = 2$ is well known, see [F89-1], [H88], so we assume that $\dim M \geq 4$. Consider any Hamiltonian symplectomorphism $\phi : M \to M$ and the corresponding data Φ and $f : X \to M$ as in 3.5. Then, by 3.5, $\operatorname{Fix} \phi \geq \operatorname{Rest} \Phi$, and hence, by 3.4 and 2.5, $\operatorname{Fix} \phi \geq 1 + \operatorname{cat} M$, and thus $\operatorname{Arn}(M,\omega) \geq 1 + \operatorname{cat} M$. Furthermore, by a theorem of Takens [T68], $\operatorname{Crit} M \leq 1 + \dim M$. Now,

$$1 + \operatorname{cat} M \leq \operatorname{Crit} M \leq 1 + \dim M = 1 + \operatorname{cat} M,$$

and thus $\operatorname{Arn}(M,\omega) \geq \operatorname{Crit} M$. $\square$

3.7. Example ($\operatorname{cat} M > \operatorname{cl} M$)**.** Let M be a four-dimensional aspherical symplectic manifold described in [MS95, Example 3.8]. It is easy to see that $H^1(M) = \mathbb{Z}^3$. Furthermore, $H^*(M)$ is torsion free, and so $a^2 = 0$ for every $a \in H^1(X)$. Hence, $\operatorname{cl} M = 3$. However, $\operatorname{cat} M = 4$ because $\operatorname{cat} V = \dim V$ for every closed aspherical manifold V, see [EG57]. Moreover, for every closed symplectic manifold N we have $\operatorname{cl}(M \times N) < \operatorname{cat}(M \times N)$ because $\operatorname{cat}(M \times N) = \dim N + 4$ accoring to [RO97].

§4. The invariant $r(M)$ and critical points

Let X be a CW-space and let A, B be two CW-subspaces of X. Then for every spectrum E we have the cap-product

$$\cap : E_i(X, A \cup B) \otimes \Pi^j(X, A) \to E_{i-j}(X, B),$$

see [Ad74], [Sw75]. Here $\Pi^*(-)$ denotes stable cohomotopy, i.e., $\Pi^*(-)$ is the cohomology theory represented by the sphere spectrum S.

In particular, if $D = D^k$ is the k-dimensional disk then for every CW-pair (X, A) we have the cup-product

$$\cap : E_i(X \times D, X \times \partial D \cup A \times D) \otimes \Pi^k(X \times D, X \times \partial D) \to E_{i-k}(X \times D, A \times D).$$

Let $a \in \Pi^k(D, \partial D) = \mathbb{Z}$ be a generator. We set $t = p^* a \in \Pi^k(X \times D, X \times \partial D)$ where $p : (X \times D, X \times \partial D) \to (D, \partial D)$ is the projection.

4.1. Lemma. *For every CW-pair (X, A) the homomorphism*

$$\cap t : E_i(X \times D, X \times \partial D \cup A \times D) \to E_{i-k}(X \times D, A \times D)$$

is an isomorphism.

In fact, it is a relative Thom–Dold isomorphism.

Proof. If $A = \emptyset$ then $\cap t$ is the standard Thom–Dold isomorphism for the trivial D^k-bundle (or the suspension isomorphism, if you want), see e.g. [Sw75]. In other words, for $A = \emptyset$ the homomorphism in question has the form $\cap t : E_i(T\alpha) \to E_{i-k}(X \times D)$ where $T\alpha$ is the Thom space of the trivial D^k-bundle α. Furthermore, the homomorphism in question has the form

$$E_i(T\alpha, T(\alpha|A)) \to E_{i-k}(X \times D, A \times D).$$

Considering the commutative diagram

$$\begin{array}{ccccccc}
\cdots \to E_i(T(\alpha|A)) & \longrightarrow & E_i(T\alpha) & \longrightarrow & E_i(T\alpha, T(\alpha|A)) & \to \cdots \\
\downarrow{\scriptstyle \cap(t|A)} & & \downarrow{\scriptstyle \cap t} & & \downarrow{\scriptstyle \cap t} & \\
\cdots \to E_{i-k}(A \times D) & \longrightarrow & E_{i-k}(X \times D) & \longrightarrow & E_{i-k}(X \times D, A \times D) & \to \cdots
\end{array}$$

with the exact rows, and using the Five Lemma, we conclude that the homomorphism in question is an isomorphism. □

4.2. Definition ([CZ83], [MO93])**.** Given a connected closed smooth manifold M, we define $\mathscr{G}H_{p,q}(M)$ to be the set of all C^2-functions $g : M \times \mathbb{R}^{p+q} \to \mathbb{R}$ with the following properties:

(1) There exist disks $D_+ \subset \mathbb{R}^p$ and $D_- \subset \mathbb{R}^q$ centered in origin such that $\operatorname{int}(M \times D_+ \times D_-)$ contains all critical points of g;
(2) $\nabla g(x)$ points inward on $M \times \partial D_+ \times \operatorname{int} D_-$ and outward on $M \times \operatorname{int} D_+ \times \partial D_-$.

4.3. Definition ([CZ83], [MO93])**.** Given $g \in \mathscr{G}H_{p,q}(M)$, consider the gradient flow $\dot{x} = \nabla g(x)$. Let $x\bullet\mathbb{R}$ denote the solution of the flow through x. We choose D_+ and D_- as in 4.2, set $B := M \times D_+ \times D_-$ and define $S_g = S_{g,B} := \{x \in B | x\bullet\mathbb{R} \subset B\}$.

4.4. Theorem (cf [MO93, 4.1])**.** *For every function $g \in \mathscr{G}H_{p,q}(M)$, there is a subpolyhedron K of* $\operatorname{int} B$ *such that $S_g \subset K$ and* $\operatorname{crit} g \geq 1 + \operatorname{cat}_B K$.

Proof. We set $S = S_g$. Because of of 4.2, S is a compact subset of $\operatorname{int} B$. Furthermore, S is an invariant set of the gradient flow $\dot{x} = \nabla g(x)$, and S contains all critical points of g. Given $A \subset S$, we define $\nu(A) = 1 + \operatorname{cat}_B A$. Clearly, ν is an index function on S. Thus, by 3.3, $\nu(S) \leq \operatorname{crit} g$. Now, let $V_1, \ldots V_{\nu(S)}$ be a covering of S such that every V_i is open and contractible in B. Choose any simplicial triangulation of B. Then, by the Lebesgue Lemma, there exists a simplicial subdivision of B with the following property: every simplex e with $e \cap S \neq \emptyset$ is contained in some V_i. Now, we set K to be the union of all simplices e with $e \cap S \neq \emptyset$. Clearly, $1 + \operatorname{cat}_B K \leq \nu(S)$, and thus $\operatorname{crit} g \geq 1 + \operatorname{cat}_B K$. Finally, we can find $K \subset \operatorname{int} B$ because of the collar theorem. □

4.5. Theorem. *For every function $g \in \mathscr{G}H_{p,q}(M)$, the number of critical points of g is at least $1 + r(M)$. In particular,* $\operatorname{crit} g \geq 1 + \operatorname{cat} M$ *if M is aspherical.*

Proof. Here we follow McCord–Oprea [MO93]. However, unlike them, here we use certain extraordinary (co)homology instead of classical (co)homology.

Let $r := r(M)$. We choose K as in 4.4 and prove that $\operatorname{cat}_B K \geq r$. Consider the Puppe sequence

$$P_r(M) \xrightarrow{p_r} M \xrightarrow{j_r} C_r(M).$$

Let $e : M^+ \to C_r(M)$ be a map such that $e|M = j_r$ and e maps the added point to the base point of $C_r(M)$. Let $h : C_r(M) \to C$ be a pointed homotopy equivalence such that C is a CW-complex. We set $E = \Sigma^\infty C$ and let $u_r \in E^0(M)$ be the stable homotopy class of the map $he : M^+ \to C$. Then $u_r \neq 0$ since j_r is stably essential.

We define

$$f : K \subset B = M \times \mathbb{R}^{p+q} \xrightarrow{\text{projection}} M.$$

4.6. Lemma. *If $f^* u_r \neq 0$ then* $\operatorname{cat}_B K \geq r$.

Proof. Since (p_r, j_r) is a Puppe sequence, $p_r^* u_r = 0$. Hence, the map f can't be lifted to $P_r(M)$, and therefore the inclusion $K \subset B$ can't be lifted to $P_r(B)$. So, $\operatorname{cat}_B K \geq r$. The lemma is proved.

We continue the proof of the theorem. Let $j : K \subset B$ be the inclusion. By 4.6, it suffices to prove that $j^* : E^*(B) \to E^*(K)$ is a monomorphism. Notice that if Y is a CW-subspace of $\mathbb{R}^N$ then there is a duality isomorphism

$$E^0(Y) \cong E_{-N}(\mathbb{R}^N, \mathbb{R}^N \setminus Y) := E_{-N}(\mathbb{R}^N \cup C(\mathbb{R}^N \setminus Y))$$

see e.g. [DP84]. So, it suffices to prove that the dual homomorphism

$$D(j^*) : E_*(\mathbb{R}^N, \mathbb{R}^N \setminus B) \to E_*(\mathbb{R}^N, \mathbb{R}^N \setminus K)$$

is monic for a certain (good) embedding $B \to \mathbb{R}^N$.

We have the following commutative diagram:

$$\begin{array}{ccc}
E_*(\mathbb{R}^N, \mathbb{R}^N \setminus B) & = & E_*(\mathbb{R}^N, \mathbb{R}^N \setminus B) \\
{\scriptstyle h}\downarrow \cong & & {\scriptstyle D(j^*)}\downarrow \\
E_*(\mathbb{R}^N, \mathbb{R}^N \setminus \operatorname{int} B) & \longrightarrow & E_*(\mathbb{R}^N, \mathbb{R}^N \setminus K) \\
{\scriptstyle e}\uparrow \cong & & {\scriptstyle e'}\uparrow \cong \\
E_*(B, \partial B) & \xrightarrow{a_*} & E_*(B, B \setminus K)
\end{array}$$

where all the homomorphisms except $D(j^*)$ are induced by the inclusions. Here h is an isomorphism since the inclusion $\operatorname{int} B \to B$ is a homotopy equivalence (the space $B \setminus \{\text{collar}\}$ is a deformation retract of $\operatorname{int} B$). Furthermore, e is an isomorphism since $(B, \partial B)$ and $(\mathbb{R}^N, \mathbb{R}^N \setminus \operatorname{int} B)$ are cofibered pairs, while e' is an isomorphism by Lemma 3.4 from [DP84]. So, $D(j^*)$ is monic if a_* is. Since $B \setminus K \subset B \setminus S$, it suffices to prove that $E_*(B, \partial B) \to E_*(B, B \setminus S)$ is a monomorphism.

Let $B_+ = M \times \partial D_+ \times D_-$, and let $B_- = M \times D_+ \times \partial D_-$. Furthermore, let $A_+ := \{x \in B \mid x\bullet\mathbb{R}_- \in B\}$ and let $A_- := \{x \in B \mid x\bullet\mathbb{R}_+ \in B\}$. Then $B_+ \cap A_- = \emptyset = B_- \cap A_+$, and so there are the inclusions $i_+ : (B, B_+) \to (B, B \setminus A_-)$ and $i_- : (B, B_-) \to (B, B \setminus A_+)$. It turns out to be that both i_+ and i_- are homotopy equivalences, [CZ83, Lemma 3].

Let $t \in \Pi^m(B, B_-)$ be the class as in 4.1, and let $t' := ((i_-)^*)^{-1}(t)$. Since $S = A_+ \cap A_-$, we have the commutative diagram

$$\begin{array}{ccc} E_i(B, \partial B) & \longrightarrow & E_i(B, B \setminus S) \\ \cong \big\downarrow \cap t & & \big\downarrow \cap t' \\ E_{i-q}(B, B_+) & \xrightarrow{\cong} & E_{i-q}(B, B \setminus A_-) \end{array}$$

where the left map is an isomorphism by 4.1 and the bottom map is the isomorphism $(i_+)_*$. (Generally, $(B, B \setminus S)$ is not a CW-pair, but nevertheless in our case the map $\cap t'$ is defined, see [DP84, 3.5].) Thus, the top homomorphism is injective.

Finally, if M is aspherical then $\operatorname{cat} M = \dim M$, [EG57], and so $r(M) = \operatorname{cat} M$ by 2.4. □

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