ON STRICT CATEGORY WEIGHT AND THE ARNOLD CONJECTURE

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Abstract. In [R2] and [RO] the Arnold conjecture for symplectic manifolds (M, ω) with $\pi_2(M) = 0$ was proved. This proof used surgery and cobordism theory. Here we give a purely cohomological proof of this result.

Introduction

Given a smooth (=C^∞) manifold M, we set $\operatorname{Crit} M := \min\{\operatorname{crit} f\}$ where f runs over all smooth functions $M \to \mathbb{R}$.

Let (M, ω) be a symplectic manifold. Given a smooth function $f : M \to \mathbb{R}$, let $\operatorname{sgrad} f$ denote the symplectic gradient of f, i.e., the vector field defined as follows:

$$\omega(\operatorname{sgrad} f, \xi) = -df(\xi)$$

for every vector field ξ.

In [A] Arnold proposed the following remarkable conjecture. Let (M, ω) be a closed symplectic manifold, and let $H : M \times \mathbb{R} \to \mathbb{R}$ be a smooth function such that $H(m, t) = H(m, t + 1)$ for every $m \in M, t \in \mathbb{R}$. We define $H_t : M \to \mathbb{R}$ by setting $H_t(m) := H(m, t)$. Consider the time-dependent differential equation

$$\dot{x} = \operatorname{sgrad} H_t(x(t)). \tag{*}$$

The Arnold conjecture claims that the number of 1-periodic solutions of (*) is at least $\operatorname{Crit} M$.

This conjecture admits another interpretation. The equation (*) yields a family $\varphi_t : M \to M, t \in \mathbb{R}$, where, for every $x \in M$, $\varphi_t(x)$ is the integral curve of (*). A *Hamiltonian symplectomorphism* is a diffeomorphism $\phi : M \to M$ which has the form $\phi = \varphi_1$ for some function $H : M \times \mathbb{R} \to \mathbb{R}$ as above. So, the Arnold conjecture can be refolmulated as follows:

$$\operatorname{Fix} \phi \geq \operatorname{Crit} M$$

for every Hamiltonian symplectomorphism ϕ, where $\operatorname{Fix} \phi$ denotes the number of fixed points of ϕ.

This conjecture was proved for many special cases, see [MS] and [HZ] for a survey. Here we notice the following result of Floer [Fl] and Hofer [H]: the number $\operatorname{Fix} \phi$ can be estimated from below by the cup-lenght of M. So, here we have a weak form of the Arnold conjecture.

In [R2] and [RO] the Arnold conjecture was proved for every closed connected symplectic manifold (M,ω) with $\omega|_{\pi_2(M)} = 0 = c_1|_{\pi_2(M)}$. In greater detail, in [R2] the conjecture was proved under the additional condition $\operatorname{cat} M = \dim M$, and it was proved in [RO] that the condition $\omega|_{\pi_2(M)} = 0$ implies the condition $\operatorname{cat} M = \dim M$. Because of the last result, it turns out to be that $\operatorname{Crit} M = 2n+1$ provided $\omega|_{\pi_2(M)} = 0$, and actually we have the inequality $\operatorname{Fix}\phi \geq 2n+1$.

The proof of the Arnold conjecture in [R] uses surgery and cobordism theory. Here we give another proof of the Arnold conjecture (under the same restriction $\omega|_{\pi_2(M)} = 0 = c_1|_{\pi_2(M)}$). This proof uses the ordinary cohomology H^* only; probably, it is more convenient for people which work in the area of dynamical systems and are not very familiar with extraordinary cohomology. The main line of the proof follows Rudyak–Oprea [RO], but here we use the strict category weight instead the category weight.

Remarks. 1. Hofer and Zehnder [HZ, p.250] mentioned that the Theorem 2.2 below is true without the restriction $c_1|_{\pi_2(M)} = 0$. Because of this, the Arnold conjecture turns out to be valid for all closed connected symplectic manifolds with $\omega|_{\pi_2(M)} = 0$.

2. Actually, in Theorem 2.2 below the number $\operatorname{Fix}\phi$ is the number of 1-periodic solutions of the equation $(*)$, while $\operatorname{Rest}\Phi$ is the number of contractble 1-periodic solutions of this equation. So, here (as well as in [R2] and [RO]) it is proved that the number of contractible 1-periodic solutions of $(*)$ is at least $2n+1$.

The paper is organized as follows. In §1 we discuss strict category weight, in §2 we use Floer's results in order to reduce the Arnold conjecture to a certain topological problem, in §3 we prove main results, in Appendix we discuss an analog of the Arnold conjecture for locally Hamiltonian symplectomorphisms.

The cohomology group $H^n(X;G)$ is always defined to be the Alexander–Spanier cohomology group with coefficient group G, see [M] or [S] for the details.

We reserve the term "map" for continuous functions.

"Connected" always means path connected.

§1. Strict category weight

1.1. Definition ([LS], [Fox], [F], [BG])**.** Given a map $\varphi : A \to X$, we define the *Lusternik–Schnirelmann category* $\operatorname{cat}\varphi$ of φ to be the minimal number k with the following property: A can be covered by open sets $A_1, \dots, A_{k+1}$ such that $\varphi|A_i$ is null-homotopic for every i. Furthrmore, we define the *Lusternik–Schnirelmann category* $\operatorname{cat} X$ of a space X by setting $\operatorname{cat} X := \operatorname{cat} 1_X$.

1.2. Proposition ([BG])**.** (i) *For every diagram* $A \xrightarrow{\varphi} Y \xrightarrow{f} X$ *we have* $\operatorname{cat} f\varphi \leq \min\{\operatorname{cat}\varphi, \operatorname{cat} f\}$. *In particular,* $\operatorname{cat} f \leq \min\{\operatorname{cat} X, \operatorname{cat} Y\}$.

(ii) *If* $\varphi \simeq \psi : A \to X$ *then* $\operatorname{cat}\varphi = \operatorname{cat}\psi$.

(iii) *If* $h : Y \to X$ *is a homotopy equivalence then* $\operatorname{cat}\varphi = \operatorname{cat} h\varphi$ *for every* $\varphi : A \to X$. □

Given a connected pointed space X, let $\varepsilon : S\Omega X \to X$ be the map adjoint to $1_{\Omega X}$, here ΩX is the loop space of X and S denotes the suspension, see e.g. [Sw].

1.3. Theorem ([Sv, Theorems 3, 19′ and 21]). *Let $\varphi : A \to X$ be a map of connected Hausdorff paracompact spaces. Then* $\operatorname{cat} \varphi < 2$ *iff there is a map* $\psi : A \to S\Omega X$ *such that* $\varepsilon\psi = \varphi$. $\square$

1.4. Definition ([R1]). Let X be a Hausdorff paracompact space, and let $u \in H^q(X; G)$ be an arbitrary element. We define the *strict category weight* of u (denoted by $\operatorname{swgt} u$) by setting

$$\operatorname{swgt} u = \sup\{k \mid \varphi^* u = 0 \text{ for every map } \varphi : A \to X \text{ with } \operatorname{cat} \varphi < k\}$$

where A runs over all Hausdorff paracompact spaces.

We use the term "strict category weight", since the term "category weight" is already used (introduced) by Fadell–Hussein [FH]. Concerning the relation between category weight and strict category weight, see [R1].

We remark that $\operatorname{swgt} u = \infty$ if $u = 0$.

1.5. Theorem. *Let X and Y be two Hausdorff paracompact spaces. Then for every $u \in H^*(X)$ the following hold:*

(i) *for every map $f : Y \to X$ we have* $\operatorname{cat} f \geq \operatorname{swgt} u$ *provided $f^* u \neq 0$. Furthermore, if X is connected then* $\operatorname{swgt} u \geq 1$ *whenever $u \in \widetilde{H}^*(X)$;*

(ii) *for every map $f : Y \to X$ we have* $\operatorname{swgt} f^* u \geq \operatorname{swgt} u$*;*

(iii) *for every $u, v \in H^*(X)$ we have* $\operatorname{swgt}(uv) \geq \operatorname{swgt} u + \operatorname{swgt} v$*.*

Proof. (i) This follows from the definition of swgt.

(ii) This follows from 1.2(i).

(iii) Let $\operatorname{swgt} u = k$, $\operatorname{swgt} v = l$ with $k, l < \infty$. Given $f : A \to X$ with $\operatorname{cat} f < k + l$, we prove that $f^*(uv) = 0$. Indeed, $\operatorname{cat} f < k + l$, and so $A = A_1 \cup \cdots \cup A_{k+l}$ where each A_i is open in A and $f|A_i$ is null-homotopic. Set $B = A_1 \cup \cdots \cup A_k$ and $C = A_{k+1} \cup \cdots \cup A_{k+l}$. Then $\operatorname{cat} f|B < k$ and $\operatorname{cat} f|C < l$. Hence $f^* u|B = 0 = f^* v|C$, and so $f^*(uv)|(B \cup C) = 0$. i.e., $f^*(uv) = 0$.

The case of infinite category weight is leaved to the reader. $\square$

§2. Floer's reduction and related things.

2.1. Recollection. A *flow* on a topological space X is a family $\Phi = \{\varphi_t\}, t \in \mathbb{R}$ where each $\varphi_t : X \to X$ is a self-homeomorphism and $\varphi_s \varphi_t = \varphi_{s+t}$ for every $s, t \in \mathbb{R}$ (notice that this implies $\varphi_0 = 1_X$).

A flow is called *continuous* if the function $X \times \mathbb{R} \to X, (x, t) \mapsto \varphi_t(x)$ is continuous.

A point $x \in X$ is called a *rest point* of Φ if $\varphi_t(x) = x$ for every $t \in \mathbb{R}$. We denote by $\operatorname{Rest} \Phi$ the number of rest points of Φ.

A continuous flow $\Phi = \{\varphi_t\}$ is called *gradient-like* if there exists a continuous (Lyapunov) function $F : X \to \mathbb{R}$ with the following property: for every $x \in X$ we have $F(\varphi_t(x)) < F(\varphi_s(x))$ whenever $t > s$ and x is not a rest point of Φ.

The following Theorem can be found in [Fl, Th. 7], cf. also [HZ].

2.2. Theorem. *Let (M,ω) be a closed connected symplectic manifold such that $\omega|_{\pi_2(M)} = 0 = c_1|_{\pi_2(M)}$, and let $\phi : M \to M$ be a Hamiltonian symplectomorphism. Then there exists a map $f : X \to M$ with the following properties:*

(i) *X is a compact metric space;*

(ii) *X possesses a continuous gradient-like flow Φ such that* $\operatorname{Rest}\Phi \leq \operatorname{Fix}\phi$*;*

(iii) *The homomorphism $f^* : H^n(M;G) \to H^n(X;G)$ is a monomorphism for every coefficient group G.* □

The following theorem (of Lusternik–Schnirelmann type) is proved in [R2].

2.3. Theorem. *Let Φ be a continuous gradient-like flow on a compact metric space X, let Y be a Hausdorff space which can be covered by open and contractible in Y subspaces, and let $f : X \to Y$ be a map. Then*

$$\operatorname{Rest}\Phi \geq 1 + \operatorname{cat} f. \quad \square$$

We need also the following well-known fact which follows from [LS] and [T].

2.4. Theorem. *For every closed smooth manifold M we have*

$$1 + \operatorname{cat} M \leq \operatorname{Crit} M \leq 1 + \dim M. \quad \square$$

§3. Proof of the Arnold conjecture

3.1. Theorem (cf. [FH], [RO], [St]). *Let π be a discrete group. Then for every $u \in H^k(K(\pi,1);G)$ with $k > 1$ we have* $\operatorname{swgt} u \geq 2$.

(Actually, Strom [St] proved that $\operatorname{swgt} u \geq k$. Moreover, it is easy to see that $\operatorname{swgt} u \leq k$ provided $u \neq 0$, and so $\operatorname{swgt} u = k$ if $u \neq 0$.)

Proof. Because of 1.3, it suffices to prove that $\varepsilon^* u = 0$ where ε is a map from 1.3 and $\varepsilon^*; H^*(K(\pi,1);G) \to H^*(S\Omega K(\pi,1);G)$ is the induced homomorphism. But $\Omega K(\pi,1)$ is homotopy equivalent to the discrete space π, and so $S\Omega K(\pi,1)$ is homotopy equivalent to a wedge of circles. Hence, $H^i(K(\pi,1(;G) = 0$ for $k > 1$, and thus $\varepsilon^* u = 0$. □

3.2. Theorem (cf. [RO]). *Let Y be a connected finite CW-space, and let $y \in H^2(Y;G)$ be such that $y|\pi_2(Y) = 0$. Then* $\operatorname{swgt} y \geq 2$.

Proof. Let $\pi = \pi_1(Y)$, and let $g : Y \to K(\pi,1)$ be a map which induces an isomorphism of fundamental groups. First, we prove that

$$y \in \operatorname{Im}\{g^* : H^2(K(\pi,1);R) \to H^2(Y;R)\}.$$

Indeed, since Y is a finite CW-space, its singular cohomology coincides with H^*, and so we have the universal coefficient sequence

$$0 \to \operatorname{Ext}(H_1(Y),R) \to H^2(Y;R) \xrightarrow{l} \operatorname{Hom}(H_2(Y),R) \to 0.$$

On the other hand, there is a Hopf exact sequence

$$\pi_2(Y) \to H_2(Y) \to H_2(K(\pi,1)) \to 0$$

and so we have the following commutative diagram with exact rows and colomn:

$$\begin{array}{ccccccc}
\mathrm{Ext}(H_1(K),R) & \longrightarrow & H^2(K);R) & \longrightarrow & \mathrm{Hom}(H_2(K),R)) & \longrightarrow & 0 \\
{\scriptstyle g'}\big\downarrow{\scriptstyle \cong} & & \big\downarrow{\scriptstyle g^*} & & \big\downarrow{\scriptstyle g''} & & \\
\mathrm{Ext}(H_1(Y),R) & \longrightarrow & H^2(Y;R) & \xrightarrow{\ l\ } & \mathrm{Hom}(H_2(Y),R) & \longrightarrow & 0 \\
 & & & & \big\downarrow & & \\
 & & & & \mathrm{Hom}(\pi_2(Y),R) & &
\end{array}$$

where K denotes $K(\pi,1)$. Now, since $y|\pi_2(Y)=0$, we conclude that $l(y)\in \operatorname{Im} g''$. Since g' is an isomorphism, an easy diagram hunting shows that $y\in \operatorname{Im} g^*$.

Thus, by 3.1 and 1.5(ii), $\operatorname{swgt} y \geq 2$. □

3.3. Corollary. *Let Y be a connected finite CW-space, let R be a commutative ring, let $y\in H^2(Y;R)$ be such that $y|\pi_2(Y)=0$, and let X be a Hausdorff paracompact space. If $f: X\to Y$ is a map with $f^*(y^n)\neq 0$, then* $\operatorname{cat} f \geq 2n$.

Proof. By 1.5,

$$\operatorname{cat} f \geq \operatorname{swgt} y^n \geq n \operatorname{swgt} y \geq 2n. \quad \square$$

3.4. Corollary ([RO]). *Let (M^{2n},ω) be a closed connected symplectic manifold with $\omega|_{\pi_2(M)}=0$. Then* $\operatorname{cat} M = 2n$ *and* $\operatorname{Crit} M = 2n+1$.

Proof. Since $\omega^n\neq 0$, we conclude that, by 3.3, $\operatorname{cat} M = \operatorname{cat} 1_M \geq 2n$. So, $\operatorname{cat} M = 2n$ since $\operatorname{cat} M \leq \dim M$. The second equality follows from 2.4. □

3.5. Corollary. *Let (M^{2n},ω) be a closed symplectic manifold with $\omega|_{\pi_2(M)} = 0 = c_1|_{\pi_2(M)}$, and let $\phi: M\to M$ be a Hamiltonian symplectomorphism. Then* $\operatorname{Fix}\phi \geq 2n+1$. *In particular, the Arnold conjecture holds for (M,ω).*

Proof. Let $f: X\to M$ and Φ be a map as in 2.2. Since every closed connected smooth manifold is a finite polyhedron, and since ω^n yields a non-trivial cohomology class in $H^*(X;\mathbb{R})$, we conclude that, by 3.3, $\operatorname{cat} f\geq 2n$. Now, by 2.2 and 2.3,

$$\operatorname{Fix}\phi \geq \operatorname{Rest}\Phi \geq 1+\operatorname{cat} f \geq 2n+1.$$

Thus, because of 3.4, the Arnold conjecture holds for (M,ω). □

References

[**A**] V. I. Arnold:*Mathematical Methods in Classical Mechanics.* Springer, Berlin Heidelberg New York 1989

[**BG**] I. Berstein, T. Ganea: *The category of a map and of a cohomology class.* Fund. Math. **50** (1961/2), 265–279

[**FH**] E. Fadell, S. Husseini: Category weight and Steenrod operations, Boletin de la Sociedad Matemática Mexicana, **37** (1992), 151–161

[**F**] A. I. Fet: *A connection between the topological properties and the number of extremals on a manifold* (Russian). Doklady AN SSSR, **88** (1953), 415–417

[**Fl**] A. Floer: *Symplectic fixed points and holomorphic spheres,* Commun. Math. Phys. **120** (1989), 575–611

[**Fox**] R. Fox: *On the Lusternik–Schnirelmann category*, Ann. of Math. **42** (1941), 333-370

[**H**] H. Hofer: *Lusternik–Schnirelmann theory for Lagrangian intersections.* Annales de l'inst. Henri Poincaré – analyse nonlineare, **5** (1988), 465–499

[**HZ**] H. Hofer, E. Zehnder: *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser, Basel, 1994

[**LS**] L. A. Lusternik, L. G. Schnirelmann, *Methodes topologiques dans le problèmes variationels.* Hermann, Paris 1934

[**M**] W. Massey: *Homology and cohomology theory*, Marcel Dekker, INC, New York and Basel, 1978

[**MS**] D. McDuff, D. Salamon: *Introduction to Symplectic Topology,* Clarendon Press, Oxford 1995

[**R1**] Yu. B. Rudyak: *On category weight and its applications,* to appear in Topology, (1998)

[**R2**] Yu. B. Rudyak: *On analytical applications of stable homotopy* (*the Arnold conjecture, critical points*) to appear in Math. Z., available as dg-ga/9708008

[**RO**] Yu. B. Rudyak, J. Oprea: *On the Lusternik–Schnirelmann Category of Symplectic Manifolds and the Arnold Conjecture,* to appear in Math. Z., available as dg-ga/9708007

[**S**] E. H. Spanier: *Algebraic Topology.* McGraw-Hill, New York 1966.

[**St**] J. Strom: *Essential category weight and classifying spaces*, Preprint Univ. of Madison-Wisconsin, 1997.

[**Sv**] A. Svarc: *The genus of a fiber space.* Amer. Math. Soc. Translations **55** (1966), 49–140

[**Sw**] R. Switzer: *Algebraic Topology – Homotopy and Homology.* Springer, Berlin Heidelberg New York (1975)

[**T**] F. Takens: *The minimal number of crirical points of a function on a compact manifold and the Lusternik–Schnirelmann category,* Invent. Math. **6** (1968), 197–244

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