THE CLASSIFICATION OF p-LOCAL FINITE GROUPS OVER THE EXTRASPECIAL GROUP OF ORDER p^3 AND EXPONENT p

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1. Introduction

The concept of p-local finite group arise in the work of Broto-Levi-Oliver [2] as a generalization of the classical concept of finite group. These purely algebraic objects, defined in Section 2, are the culmination of a research programme, carried by several authors over the last three decades, in two apparently unrelated areas: Group Theory and Homotopy Theory.

Back in the 70's, and influenced by the celebrated work of Alperin, Puig [11] set out a systematic framework for the study of a finite group G in terms of what is known as the p-local structure of G or its p-fusion: its p-subgroups, their normalizers, and the relations given by conjugation in G. This axiomatic development of p-fusion system, reviewed in Section 2, has been proved to be an useful tool for determine many of the properties of G and the p-completion of its classifying space BG (see for example [9]). Unfortunately, p-fusion is not enough to determine all the homotopy theoretic information related to $BG^\wedge_p$ as the results in [1] show. The complete description of $BG^\wedge_p$ requires the use of an additional structure, the centric linking system, and leads to the definition of p-local finite group (Definition 2.7).

The theory of p-local finite groups is a generalization of the classical theory of finite groups in the sense that every finite group leads to a p-local finite group, but there exist exotic p-local finite groups which are not associated to any finite group as it can be read in [2, Sect. 9], [8], or Lemma 4.16. Therefore, the classification of p-local finite groups has interest, not only by itself but, as an opportunity to enlighten one of the highest mathematical achievements in the last decades: The Classification of Finite Simple Groups [6]. This Classification of Finite Simple Groups provides 26 mathematical gems, 26 sporadic finite simple groups that enjoy an intriguing property: if G is an sporadic finite simple group with p-Sylow $S \leq G$, $p > 2$, of order p^3, then S is isomorphic to the extraspecial group of order p^3 and exponent p, denoted by p^{1+2}_+, and $p \leq 13$. This fact, partially explained in Corollary 4.11, is the start point for the classification of p-local finite group over the p-groups of type p^{1+2}_+.

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Theorem 1.1. *Let p be an odd prime. If $(p_+^{1+2}, \mathcal{F}, \mathcal{L})$ is a p-local finite group then it is completely determined by the saturated fusion system $(p_+^{1+2}, \mathcal{F})$ and it corresponds to one in the following tables:*

$\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$	$\sharp\mathcal{F}^{ec}$-rad	$\mathrm{Aut}_{\mathcal{F}}(V)$	Group	Condition
W	0	$\emptyset$	$p_+^{1+2} : W$	$p \nmid \lvert W \rvert$
$(p-1) \times r$	1	$\mathrm{SL}_2(p).r$	$p^2{:}(\mathrm{SL}_2(p).r)$	$r \mid (p-1)$
$(p-1) \times \frac{(p-1)}{3}$	$1+1$	$\mathrm{SL}_2(p) : \frac{(p-1)}{3}$	$L_3(p)$	$3 \mid (p-1)$
$(p-1) \times \frac{(p-1)}{3} : 2$	2	$\mathrm{SL}_2(p) : \frac{(p-1)}{3}$	$L_3(p) : 2$	
$(p-1)^2$	$1+1$	$\mathrm{GL}_2(p)$	$L_3(p) : 3$	
$(p-1)^2 : 2$	2	$\mathrm{GL}_2(p)$	$L_3(p) : S_3$	
$(p-1)^2$	$1+1$	$\mathrm{GL}_2(p)$	$L_3(p)$	$3 \nmid (p-1)$
$(p-1)^2 : 2$	2	$\mathrm{GL}_2(p)$	$L_3(p) : 2$	

TABLE 1. Semidirect products and extensions of linear groups

$\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$	$\sharp\mathcal{F}^{ec}$-rad	$\mathrm{Aut}_{\mathcal{F}}(V)$	Group	Condition
D_8	$2+2$	$\mathrm{GL}_2(p)$	${}^2F_4(2)'$	$p=3$
SD_{16}	4	$\mathrm{GL}_2(p)$	J_4	
$4S_4$	6	$\mathrm{GL}_2(p)$	Th	$p=5$
$S_3 \times 3$	3	$\mathrm{SL}_2(p)$	He	$p=7$
$S_3 \times 6$	3	$\mathrm{SL}_2(p) : 2$	$He : 2$	
$S_3 \times 6$	$3+3$	$\mathrm{SL}_2(p) : 2$	Fi_{24}'	
$6^2 : 2$	6	$\mathrm{SL}_2(p) : 2$	Fi_{24}	
$6^2 : 2$	$6+2$	$\mathrm{SL}_2(p){:}2, \mathrm{GL}_2(p)$		
$D_8 \times 3$	$2+2$	$\mathrm{SL}_2(p) : 2$	$O'N$	
$D_{16} \times 3$	4	$\mathrm{SL}_2(p) : 2$	$O'N : 2$	
$D_{16} \times 3$	4+4	$\mathrm{SL}_2(p) : 2$		
$SD_{32} \times 3$	8	$\mathrm{SL}_2(p) : 2$		
$3 \times 4S_4$	6	$\mathrm{SL}_2(p).4$	M	$p=13$

TABLE 2. Sporadic groups and exotic p-local finite groups

where $\mathcal{F}^{ec}$-rad is the set of elementary abelian $\mathcal{F}$-centric and $\mathcal{F}$-radical p-subgroups, its cardinal is separated by $\mathcal{F}$-conjugation classes, and $\mathrm{Aut}_{\mathcal{F}}(V)$ is the group of $\mathcal{F}$-automorphisms for each representative V of the $\mathcal{F}$-conjugacy classes in $\mathcal{F}^{ec}$-rad.

Proof. See Section 4. □

Remark 1.2. Notice that the classification above provides three new examples of 7-local finite groups. These are the simplest examples of exotic p-local finite groups known so far ([2, Section 9], [8]) and are intensively studied in a sequel paper [12]. The proof of the nonexistence of a finite group whose 7-fusion data is isomorphic to any of those examples (Lemma 4.16) is heavily based in the Classification of Simple Groups [6]. Therefore, within our philosophy, a more conceptual proof would be desirable.

Remark 1.3. Notice also that the groups in both tables are not the only ones with these fusion systems, even if we consider just simple finite groups. The following table completes the list of sporadic simple groups, whose p-Sylow is also of the form p_+^{1+2}, and their corresponding fusion system.

$\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$	$\sharp\mathcal{F}^{ec}$-rad	$\mathrm{Aut}_{\mathcal{F}}(V)$	Group	p
2^2	$1+1$	$\mathrm{GL}_2(3)$	M_{12}	3
8	0	$\emptyset$	J_2	3
D_8	2	$\mathrm{GL}_2(3)$	M_{24}, He	3
SD_{16}	4	$\mathrm{GL}_2(3)$	Ru	3
$4^2:2$	2	$\mathrm{GL}_2(5)$	Ru	5
$24:2$	0	$\emptyset$	Co_3	5
$4S_4$	0	$\emptyset$	Co_2	5
$8:2$	0	$\emptyset$	HS	5
$3:8$	0	$\emptyset$	McL	5
$5\times 2S_4$	0	$\emptyset$	J_4	11

Finally, Theorem 1.1 is all we need to classify all p-local finite groups (p-odd) over p-groups of order p^3:

Corollary 1.4. *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group, $p>2$, such that $|S|=p^3$. Then $(S,\mathcal{F},\mathcal{L})$ is completely determined by the saturated fusion system $(S,\mathcal{F})$ and it corresponds to one of the following:*

- *The fusion system of the group $S:\mathrm{Aut}_{\mathcal{F}}(S)$ if $S\ncong p_+^{1+2}$.*
- *One of the fusion systems in the tables in Theorem 1.1 if $S\cong p_+^{1+2}$.*

Proof. If $|S|=p^3$, then S is either abelian, or generalized extraspecial [13, Definition 3.1]. Therefore, if $S\ncong p_+^{1+2}$, then S is a resistant group [13, Theorem 4.2], that is, $\mathcal{F}=N_{\mathcal{F}}^{\mathrm{Aut}_{\mathcal{F}}(S)}(S)$ [2, Definition A.3] and $\mathcal{F}$ is the saturated fusion system of the group $S:\mathrm{Aut}_{\mathcal{F}}(S)$. Finally, the obstruction classes to the existence and uniqueness of centric linking systems associated to the saturated fusion system of a group of type $S:W$, where $p\nmid |W|$, live in $H^*(Z(S);W)=0$. If $S\cong p_+^{1+2}$ we apply Theorem 1.1. □

Organization of the paper: In Section 2, we briefly review the general theory of p-local finite groups. In Section 3, we describe the basic properties of the group p_+^{1+2} that are going to be used along of the last section, where the proof of Theorem 1.1 is worked out.

Notation: By p we always denote an odd prime. The group theoretical notation used along this paper is that described in the Atlas [4, 5.2]. For a group G, and $g \in G$, we denote by c_g the conjugation morphism $x \mapsto gxg^{-1}$. If $P, Q \leq G$, the set of G-conjugation morphisms from P to Q is denoted by $\mathrm{Hom}_G(P, Q)$, so if $P = Q$ then $\mathrm{Aut}_G(P) = \mathrm{Hom}_G(P)$. Notice that $\mathrm{Aut}_G(G)$ is then the set of inner automorphisms of G.

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2. p-local finite groups

Along this section, we quickly review the basic definitions and results related to the theory of p-local finite groups we shall use. We refer to [2] and [3] for a more complete description and properties of these objects.

Definition 2.1. A fusion system $\mathcal{F}$ over a finite p-group S is a category whose objects are the subgroups of S, and whose morphisms sets $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ satisfy the following two conditions:

(a) $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$ for all P, Q subgroups of S, where Hom_G means the morphisms induced by conjugation in G and Inj the injective morphisms.

(b) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

We say that two subgroups P,$Q \leq S$ are *$\mathcal{F}$-conjugate* if there is an isomorphism between them in $\mathcal{F}$. As all the morphisms are injective, and by condition (b) we denote by $\mathrm{Aut}_{\mathcal{F}}(P, Q)$ the morphisms in $\mathcal{F}$ from P to Q when $|P| = |Q|$ and by $\mathrm{Aut}_{\mathcal{F}}(P)$ the group $\mathrm{Aut}_{\mathcal{F}}(P, P)$. We denote by $\mathrm{Out}_{\mathcal{F}}(P)$ the quotient group $\mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Aut}_P(P)$.

The fusion systems that we will consider satisfy the following conditions:

Definition 2.2. Let $\mathcal{F}$ a fusion system over a p-group S.

- A subgroup $P \leq S$ is *fully centralized in* $\mathcal{F}$ if $|C_S(P)| \geq |C_S(P')|$ for all P' which is $\mathcal{F}$-conjugate to P.
- A subgroup $P \leq S$ is *fully normalized in* $\mathcal{F}$ if $|N_S(P)| \geq |N_S(P')|$ for all P' which is $\mathcal{F}$-conjugate to P.
- $\mathcal{F}$ is a *saturated fusion system* if the following two conditions hold:
  (I) Each fully normalized subgroup $P \leq S$ is fully centralized and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.
  (II) If $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ are such that φP is fully centralized, and if we set
  $$N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi P)\},$$
  then there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\overline{\varphi}|_P = \varphi$.

Remark 2.3. From the definition of fully normalized and the condition (I) of saturated fusion system we get that if $\mathcal{F}$ is a saturated fusion system over a p-group S then p cannot divide the outer automorphism group $\mathrm{Out}_{\mathcal{F}}(S)$.

As expected, every finite group G gives rise to a saturated fusion system [2, Proposition 1.3], which provides valuable information about $BG_p^\wedge$ [9]. Some classical results for finite groups can be generalized to saturated fusion systems, as for example, Alperin's fusion theorem [2, Theorem A.10]:

Definition 2.4. Let $\mathcal{F}$ be any fusion system over a p-group S. A subgroup $P \leq S$ is:

- $\mathcal{F}$*-centric* if P and all its $\mathcal{F}$-conjugates contain their S-centralizers. Let $\mathcal{F}^c$ denote the full subcategory of $\mathcal{F}$ whose objects are the $\mathcal{F}$-centric subgroups of S.
- $\mathcal{F}$*-radical* if $\mathrm{Out}_{\mathcal{F}}(P)$ is p-reduced, that is, if $\mathrm{Out}_{\mathcal{F}}(P)$ has no proper normal p-subgroup.

Theorem 2.5. *Let $\mathcal{F}$ be a saturated fusion system over S. Then for each morphism $\psi \in \mathrm{Aut}_{\mathcal{F}}(P, P')$, there exists a sequence of subgroups of S*

$$P = P_0, P_1, \dots, P_k = P' \quad \text{and} \quad Q_1, Q_2, \dots, Q_k,$$

and morphisms $\psi_i \in \mathrm{Aut}_{\mathcal{F}}(Q_i)$, such that

- *Q_i is fully normalizer in $\mathcal{F}$, $\mathcal{F}$-radical and $\mathcal{F}$-centric for each i;*
- *$P_{i-1}, P_i \leq Q_i$ and $\psi_i(P_{i-1}) = P_i$ for each i; and*
- *$\psi = \psi_k \circ \psi_{k-1} \circ \cdots \circ \psi_1$.*

Nevertheless, the definition of p-local finite group still needs of some extra structure:

Definition 2.6. Let $\mathcal{F}$ be a fusion system over the p-group S. A *centric linking system associated to* $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor

$$\pi \colon \mathcal{L} \longrightarrow \mathcal{F}^c$$

and "distinguished" monomorphisms $P \xrightarrow{\delta_P} \mathrm{Aut}_{\mathcal{L}}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$, which satisfy the following conditions:

(A) π is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $p, Q \in \mathcal{L}$, $Z(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leq \mathrm{Aut}_{\mathcal{L}}(P)0$, and π induces a bijection

$$\mathrm{Mor}_{\mathcal{L}}(P, Q)/Z(P) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{F}}(P, Q)$$

(B) For each $\mathcal{F}$-centric subgroup $P \leq S$ and each $g \in P$, π sends $\delta_P(g) \in \mathrm{Aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.

(C) For each $f \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ and each $g \in P$, the equality $\delta_Q(\pi(f)(g)) \circ f = f \circ \delta_P(g)$ holds in $\mathcal{L}$

Finally, the definition of p-local finite group is:

Definition 2.7. A *p-local finite group* is a triple $(S, \mathcal{F}, \mathcal{L})$, where S is a p-group, $\mathcal{F}$ is a saturated fusion system over S and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The *classifying space* of the p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is the space $|\mathcal{L}|^{\wedge}_p$.

Given a fusion system $\mathcal{F}$ over the p-group S, there exists an obstruction theory for the existence and uniqueness of a centric linking system, i.e. p-local finite groups, associated to $\mathcal{F}$. The question is solved for p-groups of small rank by the following result [2, Theorem E]:

Theorem 2.8. *Let $\mathcal{F}$ be any saturated fusion system over a p-group S. If $\mathrm{rk}(S) < p^3$, then there exists a centric linking system associated to $\mathcal{F}$. And if $\mathrm{rk}_p(S) < p^2$, then there exists a unique centric linking system associated to $\mathcal{F}$.*

All p-local finite groups studied in this work are over the p-group p_+^{1+2}. As the p-rank of p_+^{1+2} is 2, we obtain:

Corollary 2.9. *Let p be an odd prime. Then the set of p-local finite groups over the p-group p_+^{1+2} is in bijective correspondence with the set of saturated fusion systems over p_+^{1+2}.*

3. The group p_+^{1+2}

In this section, we collect the basic properties of p_+^{1+2}, the extraspecial group of order p^3 and exponent p (p odd), that we shall use along the paper.

The group p_+^{1+2} is usually presented as

$$p_+^{1+2} \cong \langle A, B, C \mid A^p = B^p = C^p = [A, C] = [B, C] = [A, B]C^{p-1} = 1\rangle$$

so p_+^{1+2} is generated by the elements A and B, and C appears as a generator of $Z(p_+^{1+2}) = [p_+^{1+2}, p_+^{1+2}]$. Therefore any automorphism of p_+^{1+2} is determined by the images of the elements A and B. We can read in [7], that the automorphisms of p_+^{1+2} are indeed those endomorphisms which map A to $A^{r'}B^{s'}C^{t'}$ and B to $A^rB^sC^t$. The inner automorphisms are then automorphisms of the form $A \mapsto AC^{t'}$ and $B \mapsto BC^t$, so we can describe the outer automorphisms of p_+^{1+2} as:

Lemma 3.1. *The outer automorphism group of p_+^{1+2} is isomorphic to $\mathrm{GL}_2(p)$: the matrix $\left(\begin{smallmatrix} r' & r \\ s' & s \end{smallmatrix}\right)$ with determinant j corresponds to the class of the automorphism which sends A to $A^{r'}B^{s'}$, sends B to A^rB^s and C to C^j.*

The following Lemma gives us the structure of the $\mathcal{F}$-centric subgroups of p_+^{1+2}, which is a fundamental tool in the study of the possible saturated fusion systems.

Lemma 3.2.

(a) *There are exactly $(p+1)$ rank two elementary abelian subgroups in p_+^{1+2} that we label as $V_i = \langle C, AB^i\rangle$ for $i = 0, \dots, (p-1)$ and $V_p = \langle C, B\rangle$.*
(b) *Every inner automorphism of p_+^{1+2} restricts to an automorphism of V_i for all i.*
(c) *Assume $\mathcal{F}$ is a fusion system over p_+^{1+2}. Then the $\mathcal{F}$-centric subgroups are the total and the rank two elementary abelian subgroups.*

Proof. Statements (a) and (b) follow from the presentation of p_+^{1+2} and the description of its automorphisms given above.

In order to obtain (c), recall that P is $\mathcal{F}$-centric if and only if P and all its $\mathcal{F}$-conjugates contain their p_+^{1+2}-centralizers. Let P be $\mathcal{F}$-centric. As $\langle C\rangle = Z(p_+^{1+2})$, then $\langle C\rangle \lneq P$. If we add just one element to $\langle C\rangle$, we will have a rank two elementary abelian self-centralizing p-subgroup. As the $\mathcal{F}$-conjugates of a rank two elementary abelian p-subgroup must be again a rank two elementary abelian p-subgroup, then it will be also self-centralizing and that means that the rank two elementary abelian p-subgroups are the smaller $\mathcal{F}$-centric subgroups.

If we add any element to a rank two elementary abelian p-subgroup, then we will have the total, so the result follows. □

4. Proof of the classification

The aim of this section is to provide a proof of Theorem 1.1. This proof appears at the end of the section and is subdivided in the following series of lemmas. The notation used for the elements and subgroups of p_+^{1+2} is the one described in the previous section, and $\mathcal{F}$ will always denote a saturated fusion system over p_+^{1+2}.

According with *Alperin's theorem for fusion systems*, Theorem 2.5, the fusion system $\mathcal{F}$ is determined by the full subcategory of $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroups.

In Lemma 3.2 we show that p_+^{1+2} and its rank two elementary abelian p-subgroups are all $\mathcal{F}$-centric. As p_+^{1+2} is always $\mathcal{F}$-radical, it remains to see which of these rank two elementary abelian p-subgroups can be $\mathcal{F}$-radical. Let $\mathcal{F}^{ec}$-rad denote the set of elementary abelian $\mathcal{F}$-centric and $\mathcal{F}$-radical p-subgroups, that is, the $\mathcal{F}$-radical rank two elementary abelian p-subgroups.

We begin with a characterization of the elements in $\mathcal{F}^{ec}$-rad.

Lemma 4.1. *Let V be a rank two elementary abelian p-subgroup of p_+^{1+2}. Then $V \in \mathcal{F}^{ec}$-rad if and only if $\mathrm{SL}_2(p) \leq \mathrm{Aut}_{\mathcal{F}}(V)$.*

Proof. If $\mathrm{SL}_2(p) \leq \mathrm{Aut}_{\mathcal{F}}(V) = \mathrm{Out}_{\mathcal{F}}(V) \leq \mathrm{GL}_2(p)$, then $\mathrm{Aut}_{\mathcal{F}}(V)$ is p-reduced and V is $\mathcal{F}$-radical. So $V \in \mathcal{F}^{ec}$-rad

Now, let V be in $\mathcal{F}^{ec}$-rad. As it is a rank two elementary abelian p-subgroup, it must be one of the V_i's, generated by C and $X = A^iB^j$ with $(i,j) \neq (0,0)$. We think of $\{C, X\}$ as a $\mathbb{F}_p$-basis of V. In this fixed basis, the matrix $m = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, which generates a subgroup of order p, is in $\mathrm{Aut}_{\mathcal{F}}(V)$, because is the restriction of an inner automorphism of p_+^{1+2}. As V is abelian, hence $\mathrm{Out}_{\mathcal{F}}(V) = \mathrm{Aut}_{\mathcal{F}}(V)$, and $\mathcal{F}$-radical, then there exist matrices $m' \in \mathrm{Out}_{\mathcal{F}}(V)$ conjugated to m which are not in $\langle m\rangle$.

We have that $\mathrm{Aut}_{\mathcal{F}}(V)$ acts on the projective space L of lines in $\mathbb{F}_p^2$, and that $\langle m\rangle$ divides L in two orbits, one of them with the single element $(1,0)$. Consider an element $m' \in \mathrm{Out}_{\mathcal{F}}(V)$ conjugated to m, which will take the element $(1,0)$ to another, making the action of $\langle m, m'\rangle$ transitive on L. Then there exists $n \in \langle m, m'\rangle$ which maps $(1,0) \mapsto (1,1)$. As $m, m' \in \mathrm{SL}_2(p)$, then $n \in \mathrm{SL}_2(p)$, and n is a matrix of the form:

$$\begin{pmatrix} 1 & x \\ 1 & x+1 \end{pmatrix}.$$

Finally, $nm^{-x} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \langle m, m' \rangle \leq \mathrm{Aut}_{\mathcal{F}}(V)$. But $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ generate $\mathrm{SL}_2(p)$, hence $\mathrm{SL}_2(p) \leq \mathrm{Aut}_{\mathcal{F}}(V)$ □

Fix a a primitive $(p-1)$ root of unity in $\mathbb{F}_p^*$, and denote by $(p-1)_{-2}$ the subgroup generated by the matrix

$$\begin{pmatrix} a & 0 \\ 0 & a^{-2} \end{pmatrix}.$$

The next result shows how the number of subgroups of $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$ which are in the $\mathrm{GL}_2(p)$-conjugacy class of $(p-1)_{-2}$ provides an upper bound for $\sharp\mathcal{F}^{ec}$-rad.

Lemma 4.2. *For each $V \in \mathcal{F}^{ec}$-rad there is a different subgroup $(p-1)_V \leq \mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$ in the $\mathrm{GL}_2(p)$-conjugacy class of $(p-1)_{-2}$.*

Moreover $(p-1)_V$ breaks the set of rank two elementary abelian p-subgroups of p_+^{1+2} in the following orbits: two of one element, corresponding to the eigenvectors of the generator of $(p-1)_V$ and:

(a) *one of $(p-1)$ elements if $3 \nmid (p-1)$,*
(b) *three orbits of $(p-1)/3$ elements otherwise.*

Proof. Let V be in $\mathcal{F}^{ec}$-rad, with $\mathbb{F}_p$ basis $\{C, X\}$ as in the proof of Lemma 4.1. Let a be the primitive $(p-1)$ root of the unity fixed above.

As $\mathrm{SL}_2(p) \leq \mathrm{Aut}_{\mathcal{F}}(V)$, then the map φ corresponding to the matrix $\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right)$ will be in $\mathrm{Aut}_{\mathcal{F}}(V)$, and as all the rank two elementary abelian p-subgroups are fully centralized, it will extend to a map in N_φ (Definition 2.2), which is all p_+^{1+2}. Consider now this extension in $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2}) \leq \mathrm{GL}_2(p)$, so we can think of it as a matrix with X as eigenvector of eigenvalue a and determinant a^{-1}, so it has a^{-2} as the other eigenvalue, different from a. We see that just one matrix with this properties can be in $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2}) \leq \mathrm{GL}_2(p)$: if we had two different matrices such that X is an eigenvector of eigenvalue a and determinant a^{-2}, we can conjugate one of them to be diagonal $M_1 = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-2} \end{smallmatrix}\right)$, and by the properties assumed, the other must be $M_2 = \left(\begin{smallmatrix} a & b \\ 0 & a^{-2} \end{smallmatrix}\right)$, with $b \neq 0$. Now we have that $M_1^{-1}M_2$ has order p, getting that $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$ has an element of order p, what cannot be true by Remark 2.3. With this the result about the existence of as many different subgroups conjugated to $(p-1)_{-2}$ as $\mathcal{F}$-radical rank two elementary abelian p-groups follows.

Finally we take directly the subgroup $(p-1)_{-2}$ to check the action of its generator on the set $\{V_i\}_{i=0..p}$, equivalent to the projective space of lines in $\mathbb{F}_p^2$, and we obtain the invariant spaces of the statement. □

Next lemma shows how $\mathrm{Aut}_{\mathcal{F}}(p_+^{1+2})$ and $\mathcal{F}^{ec}$-rad determine $\mathrm{Aut}_{\mathcal{F}}(V)$ for every rank two elementary abelian subgroup $V \leq p_+^{1+2}$.

Lemma 4.3. *Let V be a rank two elementary abelian subgroup of p_+^{1+2}. Then the group $\mathrm{Aut}_{\mathcal{F}}(V)$ is determined by $\mathrm{Aut}_{\mathcal{F}}(p_+^{1+2})$ and the property of being or not $\mathcal{F}$-radical.*

Proof. If V is not $\mathcal{F}$-radical, by Theorem 2.5, we get that the elements in $\mathrm{Aut}_{\mathcal{F}}(V)$ are restrictions of elements in $\mathrm{Aut}_{\mathcal{F}}(p_+^{1+2})$.

Assume now that V is $\mathcal{F}$-radical. We have that $\mathrm{SL}_2(p) \leq \mathrm{Aut}_{\mathcal{F}}(V) \leq \mathrm{GL}_2(p)$, so there exists r, divisor of $(p-1)$, such that $\mathrm{Aut}_{\mathcal{F}}(V) \cong \mathrm{SL}_2(p).r$. To see that r depends only on $\mathrm{Aut}_{\mathcal{F}}(p_+^{1+2})$ consider the elements in $\mathrm{Aut}_{\mathcal{F}}(p_+^{1+2})$ which restrict to an element in $\mathrm{Aut}_{\mathcal{F}}(V)$. The image by the determinant of this restriction is an ideal I' generated by r' in $\mathbb{Z}/(p-1)$. By definition r is one generator of the ideal I generated by the image of the determinant applied to $\mathrm{Aut}_{\mathcal{F}}(V)$ in $\mathbb{Z}/(p-1)$. We claim that these two ideals are the same: we have $I' \subset I$, so we have just to prove $I \subset I'$. Given $r'' \in I$, and using that $\mathrm{SL}_2(p) \leq \mathrm{Aut}_{\mathcal{F}}(V)$, we have that there are diagonal matrices in $\mathrm{Aut}_{\mathcal{F}}(V)$ with determinant r''. These diagonal matrices extend to an element in $\mathrm{Aut}_{\mathcal{F}}(p_+^{1+2})$ (we are using the same argument as in the proof of Lemma 4.2 about the extensions of maps to N_φ), so $r'' \in I'$, and $I = I'$, that is, r can be deduced from $\mathrm{Aut}_{\mathcal{F}}(p_+^{1+2})$. □

A consequence of the previous lemma is:

Corollary 4.4. *The category $\mathcal{F}$ is characterized by $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$ and the set $\mathcal{F}^{ec}$-rad.*

Proof. First observe that $\mathrm{Aut}_{\mathcal{F}}(p_+^{1+2}) = \mathrm{Aut}_{p_+^{1+2}}(p_+^{1+2}) : \mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$. Thus $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$ determines $\mathrm{Aut}_{\mathcal{F}}(p_+^{1+2})$. By Lemma 4.3, $\mathrm{Aut}_{\mathcal{F}}(p_+^{1+2})$ and $\mathcal{F}^{ec}$-rad determine $\mathrm{Aut}_{\mathcal{F}}(V)$ for all $V \in \mathcal{F}^{ec}$-rad. But, according to *Alperin's theorem for fusion systems*, Theorem 2.5, that is all the information we need to reconstruct the category $\mathcal{F}$. □

Therefore, in accordance with the corollary above, we proceed to classify the possible $\mathcal{F}$ in terms of the possible combinations of $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$ and $\mathcal{F}^{ec}$-rad. The easiest case is:

Lemma 4.5. *If $\mathcal{F}^{ec}$-rad $= \emptyset$, then $\mathcal{F}$ is the fusion system of the group $p_+^{1+2} : W$, where W is $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2}) \leq \mathrm{GL}_2(p)$ of order coprime to p.*

Proof. Using Corollary 4.4, $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$ and the fact that $\mathcal{F}^{ec}$-rad $= \emptyset$ give all the information and determines $\mathcal{F}$.

The fact that p cannot divide the order of W follows from Remark 2.3.

Fix now $W \leq \mathrm{GL}_2(p)$ of order coprime to p and consider the group $G \overset{\mathrm{def}}{=} p_+^{1+2} : W$. Let $\mathcal{F}$ be the saturated fusion system associated to G. We have to check that $\mathcal{F}^{ec}$-rad $= \emptyset$. But, if $V \in \mathcal{F}^{ec}$-rad, then $\mathrm{SL}_2(p) \leq \mathrm{Aut}_{\mathcal{F}}(V)$, by Lemma 4.1, and that is impossible as $Z(p_+^{1+2}) < V$ is invariant by G-conjugation. □

We next study the case of $\sharp\mathcal{F}^{ec}$-rad $= 1$.

Lemma 4.6. *If $\sharp\mathcal{F}^{ec}$-rad $= 1$ then $\mathcal{F}$ is the fusion system of the group $p^2 : (\mathrm{SL}_2(p).r)$ where $\mathrm{SL}_2(p).r \leq \mathrm{GL}_2(p)$ is the normal subgroup of index $(p-1)/r$ which contains $\mathrm{SL}_2(p)$.*

Proof. Assume that $\sharp\mathcal{F}^{ec}$-rad $= 1$. We can assume the only element in $\mathcal{F}^{ec}$-rad is $V_0 = \langle A, C \rangle$ and therefore $(p-1)_{-2} \leq \mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$.

As, V_0 cannot be conjugated to any other rank two p-local finite subgroup, then $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$ must be contained in the upper triangular matrices, but if this subgroup contains any non-diagonal matrix, as it also contains $(p-1)_{-2}$, then it will contain an element or order p. This implies that $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$ must be contained in the subgroup of diagonal matrices which is of type $(p-1)^2$.

Then $(p-1)_{-2} \leq \mathrm{Out}_{\mathcal{F}}(p_+^{1+2}) \leq (p-1)^2$, and it follows that for each r, divisor of $(p-1)$, we have one subgroup isomorphic to $(p-1) \times r$ and that these are all the possibilities for $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$.

Using the Corollary 4.4 we have that the category $\mathcal{F}$ is characterized by $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$ and the fact that V_0 is the only radical.

Consider now the fusion system of the group $G \stackrel{\text{def}}{=} p^2 : \mathrm{SL}_2(p).r$. It's clear that, choosing $V_0 = p^2 : 1$ we obtain that $\mathrm{Aut}_{\mathcal{F}}(V_0) = \mathrm{SL}_2(p).r$, $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2}) = (p-1) \times r$, and that there are no more $\mathcal{F}$-radical rank two elementary abelian p-subgroups. $\square$

The study of the case $\sharp\mathcal{F}^{ec}$-rad $= 2$ requires a more complicated analysis that depends on the existence and uniqueness (up to conjugation) of some subgroups of $\mathrm{GL}_2(p)$.

Lemma 4.7.

(a) *Up to conjugation, there is just one subgroup isomorphic to* $(p-1)^2 : 2$ *in* $\mathrm{GL}_2(p)$.
(b) *If* $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2}) = (p-1)^2 : 2$ *and* $p \neq 7$, *then* $\sharp\mathcal{F}^{ec}$-rad $= 0$ *or* 2.

Proof. To prove (a) we think of $(p-1)^2 < (p-1)^2 : 2$ as a rank two abelian group generated by two matrices x and y. Any element of order $(p-1)$ is conjugated to a diagonal matrix, so we can assume that x is a diagonal matrix. If y is also diagonal, we have finish, using that the diagonal matrices in $\mathrm{GL}_2(p)$, as a group, are isomorphic to $(p-1)^2$. If y is not diagonal, using that x is diagonal and y must commute with x, we get that x is of the form $a\,\mathrm{Id}$, and as x has order $(p-1)$, then a is a primitive $(p-1)$ root of the unity and $\langle x\rangle$ is the center of $\mathrm{GL}_2(p)$, so we can conjugate y to a diagonal matrix without changing x and we get that, again, it is conjugated to the subgroup of diagonal matrices.

Assuming now that $(p-1)^2 \subset (p-1) : 2$ is the subgroup of diagonal matrices, we need a non-diagonal matrix z of order 2 in $N_{\mathrm{GL}_2(p)}((p-1)^2)$, and we get that $z = \pm\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Therefore there is just one possibility, up to conjugation, for a group of the form $(p-1) : 2$.

To prove (b), we think of $\mathrm{Out}(p_+^{1+2}) = (p-1)^2 : 2$ as the group of the diagonal matrices and the twisting $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We can see that this group breaks the rank two elementary abelian p-subgroups of p_+^{1+2} in two $\mathcal{F}$-conjugacy classes, one with 2 elements (V_0 and V_p) and another with $(p-1)$ elements (V_i for $i = 1, \dots, p-1$).

If $\mathcal{F}^{ec}$-rad $= \{V_0, V_p\}$, then $\mathrm{Out}_{\mathcal{F}} V_0 = \mathrm{Out}_{\mathcal{F}} V_p = \mathrm{GL}_2(p)$ by Lemma 4.3, and $\mathcal{F}$ is the fusion system of $L_3(p) : S_3$ if $3|(p-1)$ and $L_3(p) : 2$ otherwise.

Finally, if $\mathcal{F}^{ec}$-rad $= \{V_1, \dots, V_{p-1}\}$, then it would imply that conjugating by the matrix $\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ the group $\mathrm{Out}(p_+^{1+2})$ we should obtain upper triangular matrices conjugated to $(p-1)_{-2}$, corresponding to $(p-1)_{V_{p-1}}$ and we can check that we get just matrices of the form $\lambda\,\mathrm{Id}$ and $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & -\lambda \end{smallmatrix}\right)$, which generates one $(p-1)_{-2}$ only if $\lambda^3 = -1$ and then p must be 7. $\square$

Finally a classification for the case $\sharp\mathcal{F}^{ec}$-rad $= 2$.

Lemma 4.8. *If* $\sharp\mathcal{F}^{ec}$-rad $= 2$ *then* $\mathcal{F}$ *is one of the following saturated fusion systems:*

<table>
<tr><th>$\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$</th><th>$\sharp\mathcal{F}^{ec}$-rad</th><th>$\mathrm{Aut}_{\mathcal{F}}(V)$</th><th>Group</th><th>Condition</th></tr>
<tr><td>$(p-1)\times\frac{(p-1)}{3}$</td><td>$1+1$</td><td>$\mathrm{SL}_2(p):\frac{(p-1)}{3}$</td><td>$L_3(p)$</td><td rowspan="4">$3\vert(p-1)$</td></tr>
<tr><td>$(p-1)\times\frac{(p-1)}{3}:2$</td><td>$2$</td><td>$\mathrm{SL}_2(p):\frac{(p-1)}{3}$</td><td>$L_3(p):2$</td></tr>
<tr><td>$(p-1)^2$</td><td>$1+1$</td><td>$\mathrm{GL}_2(p)$</td><td>$L_3(p):3$</td></tr>
<tr><td>$(p-1)^2:2$</td><td>2</td><td>$\mathrm{GL}_2(p)$</td><td>$L_3(p):S_3$</td></tr>
<tr><td>$(p-1)^2$</td><td>$1+1$</td><td>$\mathrm{GL}_2(p)$</td><td>$L_3(p)$</td><td rowspan="2">$3\nmid(p-1)$</td></tr>
<tr><td>$(p-1)^2:2$</td><td>2</td><td>$\mathrm{GL}_2(p)$</td><td>$L_3(p):2$</td></tr>
</table>

where $\sharp\mathcal{F}^{ec}$-rad *is the cardinal of* $\sharp\mathcal{F}^{ec}$-rad *separated by* $\mathcal{F}$*-conjugacy classes and* $\mathrm{Aut}_{\mathcal{F}}(V)$ *is the group of* $\mathcal{F}$*-automorphisms for each representative of the* $\mathcal{F}$*-conjugacy classes in* $\sharp\mathcal{F}^{ec}$-rad.

Proof. Assume that $\mathcal{F}^{ec}$-rad $=\{V,V'\}$, and let w and w' be matrices in $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})\leq \mathrm{GL}_2(p)$ corresponding to the generators of the groups $(p-1)_V$ and $(p-1)_{V'}$ defined in Lemma 4.2 We see that w and w' have the same the eigenvectors (otherwise, the $\mathcal{F}$-class distribution given by Lemma 4.2 would imply $\sharp\mathcal{F}^{ec}$-rad >2) so in an appropriate basis we get that $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$ is contained in the subgroup generated by diagonal matrices and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ (again, any other non diagonal matrix would imply an $\mathcal{F}$-class distribution such that $\sharp\mathcal{F}^{ec}$-rad >2).

The difference between being $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$ included in the diagonal matrices or not will be that the two $\mathcal{F}$-radical will appear in just one conjugation class (denoted as 2 in the table) or in two (denoted as $1+1$ in the table).

Assume first that $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$ is included in the group of diagonal matrices, as $\sharp\mathcal{F}^{ec}$-rad $=2$ then $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$ must contain the subgroup generated by:

$$\begin{pmatrix} a & 0\\ 0 & a^{-2}\end{pmatrix} \text{ and } \begin{pmatrix} a^{-2} & 0\\ 0 & a\end{pmatrix}.$$

This subgroup generates all the diagonal matrices if $3\nmid(p-1)$ and a subgroup isomorph to $(p-1)\times\frac{(p-1)}{3}$ if $3|(p-1)$. In this second case we can also consider all the group $(p-1)^2$, getting the two different cases of the table.

Corollary 4.4 tells us that the category $\mathcal{F}$ is characterized by this data, and Lemma 4.3 tells us how to compute the column $\mathrm{Aut}_{\mathcal{F}}(V)$ in the table.

Now, if $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\in\mathrm{Aut}_{\mathcal{F}}(p_+^{1+2})$, the same arguments work, getting the information with the 2 in the table.

Finally we have to check that these fusion systems correspond to the groups in the table. Consider S the p-Sylow of $L_3(p)$ as the represented by the classes of the upper triangular matrices with 1 in the diagonal. If we regard the conjugations in S by elements in $L_3(p)$ as a subgroup of $\mathrm{GL}_2(p)$, we obtain that these are the matrices of the form $\left(\begin{smallmatrix}a&0\\0&ab^3\end{smallmatrix}\right)$, where a and b are invertible elements in $\mathbb{F}_p$, so the group $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$ is $(p-1)\times\frac{(p-1)}{3}$ or $(p-1)^2$ depending whether 3 divides $(p-1)$ or not.

To check that $\sharp\mathcal{F}^{ec}$-rad $= 2$ consider elementary abelian rank two p-subgroup as the classes of matrices in the p-Sylow above with a 0 in the position $(2,3)$ and respectively the other p-subgroup with a zero in the position $(2,1)$. Using the symmetry of this two subgroups we have just to check that one of this is $\mathcal{F}$-radical. To see these, check that $\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&1&1\end{smallmatrix}\right)$ (respectively $\left(\begin{smallmatrix}1&0&0\\0&1&1\\0&0&1\end{smallmatrix}\right)$) acts as $\left(\begin{smallmatrix}1&-1\\0&1\end{smallmatrix}\right)$ (respectively $\left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right)$) so, as they generate $\mathrm{SL}_2(p)$, these are $\mathcal{F}$-radical. Moreover, applying the Lemma 4.7, out of $p = 7$ we cannot have more than 2 $\mathcal{F}$-radical, and for $p = 7$ we can check the Atlas [4] information about $L_3(7)$.

To complete the other cases just add that the action of the order two element in $\mathrm{Out}(L_3(p))$ can be regarded as the matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ in $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$ and, if 3 is a divisor of $(p-1)$, we have also a cyclic group of order 3 in $\mathrm{Out}(L_3(p))$, which makes $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2})$ to be $(p-1)^2$. □

Lemma 4.7 suggest that the analysis of the case $p = 7$ cannot fit in a general framework. The next results identify the other anomalous primes that we have to check case by case. We need the following results concerning the subgroups of $\mathrm{PGL}_2(p)$.

Lemma 4.9. *Let p an odd prime. The subgroups of* $\mathrm{PGL}_2(p)$ *of order coprime to p are isomorphic to one of the following groups:*

(a) *A subgroup of the dihedral groups of order $(p^2 \pm 1)$.*
(b) *A_4, S_4.*
(c) *A_5 if and only if $5 \mid p(p^2-1)$.*

Proof. Theorems 3.6.25 and 3.6.26 in [14] describe the possible subgroups of $\mathrm{PSL}_2(q)$ for $q = p^n$. Then the result follows as a consequence of the inclusion of $\mathrm{PGL}_2(p)$ in $\mathrm{PSL}_2(p^2)$ given in [14, 3.6.26(v)], and the fact that A_5 is simple. □

The relation about anomalous primes and the subgroups described in the previous lemma appears clear in the next result.

Lemma 4.10. *Assume $\sharp\mathcal{F}^{ec}$-rad > 0 and denote by $\tilde{W}$ the projection in* $\mathrm{PGL}_2(p)$ *of* $\mathrm{Out}_{\mathcal{F}}(p_+^{1+2}) \leq \mathrm{GL}_2(p)$. *Then:*

(a) *If $p > 7$ then $\tilde{W}$ cannot be in $D_{p^2\pm1}$.*
(b) *If $p > 5$ and $3 \nmid (p-1)$ then $\tilde{W}$ cannot be neither in A_4, nor in S_4, nor in A_5.*
(c) *If $p > 16$ and $3|(p-1)$ then $\tilde{W}$ cannot be neither in A_4, nor in S_4, nor in A_5.*

Proof. Assume that $\{V, V', V'\} \subset \mathcal{F}^{ec}$-rad. Then the projection in $\mathrm{PGL}_2(p)$ of the groups $(p-1)_V$, $(p-1)_{V'}$ and $(p-1)_{V'}$ will generate, at least, two different subgroups of order $\frac{(p-1)}{3}$ or $(p-1)$, depending if $3|(p-1)$ or not. Such situation can happen in $D_{p^2\pm1}$ if and only if either $\frac{(p-1)}{3}$ or $(p-1)$ equals 2, that is, when $p = 3$ or $p = 7$, what proves (a). Notice also that the order of the elements in the groups A_4, S_4 and A_5 is less or equal than 5, hence:

(i) if $3 \nmid (p-1)$, then $p - 1 \leq 5$ what proves (b), and
(ii) if $3|(p-1)$, then $\frac{p-1}{3} \leq 5$ what proves (c).

□

We can now give a partial answer, but independent of [6], to the question of why sporadic finite simple groups with p-Sylow of order p^3, $p > 2$, have p-Sylow of type p_+^{1+2} and occur only for $p \leq 13$.

Corollary 4.11. *Let p be an odd prime and let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group over S, a finite p group of order p^3. If either $S \ncong p_+^{1+2}$, or $p \neq 3, 5, 7, 13$, then the p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is completely determined by the saturated fusion system $(S, \mathcal{F})$, which is also the saturated fusion system associated to a group of type either*

- *$S : W$, where $W \leq \mathrm{Aut}(S)$ and $p \nmid |W|$,*
- *$p^2 : W$, where $\mathrm{SL}_2(p) \leq W \leq \mathrm{GL}_2(p)$, or*
- *$L_3(p) : W$, where $W \leq \mathrm{Out}(L_3(p))$ and $p \nmid |W|$.*

Proof. The argument is the one used to prove Corollary 1.4. If $|S| = p^3$, then S is either abelian, or generalized extraspecial [13, Definition 3.1]. Therefore, if $S \ncong p_+^{1+2}$, then S is a resistant group [13, Theorem 4.2], that is, $\mathcal{F} = N_{\mathcal{F}}^{\mathrm{Aut}_{\mathcal{F}}(S)}(S)$ [2, Definition A.3] and $\mathcal{F}$ is the saturated fusion system of the group $S : \mathrm{Aut}_{\mathcal{F}}(S)$. Finally, the obstruction classes to the existence and uniqueness of centric linking systems associated to the saturated fusion system of a group of type $S : W$, where $p \nmid |W|$, live in $H^*(Z(S); W) = 0$, what proves the corollary under the first assumption.

If $S \cong p_+^{1+2}$, then $p \neq 3, 5, 7, 13$, and the result follows from the series of previous lemmas. □

Therefore it remains to study the cases $p = 3$, $p = 5$, $p = 7$ and $p = 13$, when $\sharp\mathcal{F}^{ec}$-rad > 2.

The case $p = 3$ is worked out in the next lemma.

Lemma 4.12. *Let $\mathcal{F}$ a saturated fusion system over 3_+^{1+2} such that $\sharp\mathcal{F}^{ec}$-rad > 2. Then $\mathcal{F}$ is one in the following list:*

$\mathrm{Out}_{\mathcal{F}}(3_+^{1+2})$	$\sharp\mathcal{F}^{ec}$-rad	$\mathrm{Aut}_{\mathcal{F}}(V)$	*Group*
$2^2 : 2$	$2+2$	$\mathrm{GL}_2(3), \mathrm{GL}_2(3)$	${}^2F_4(2)'$
SD_{16}	4	$\mathrm{GL}_2(3)$	J_4

where $\sharp\mathcal{F}^{ec}$-rad is the cardinal of $\mathcal{F}^{ec}$-rad separated by $\mathcal{F}$-conjugacy classes and $\mathrm{Aut}_{\mathcal{F}}(V)$ is the group of $\mathcal{F}$-automorphisms for each representative of the $\mathcal{F}$-conjugacy classes in $\sharp\mathcal{F}^{ec}$-rad.

Proof. We have to identify the possible $\mathrm{Out}_{\mathcal{F}}(3_+^{1+2}) \leq \mathrm{GL}_2(3)$ of order coprime with 3 (Remark 2.3) and containing more than 3 different elements of determinant -1 and trace 0 (Lemma 4.2). There are two subgroups (up to conjugation) verifying those properties: the group generated by diagonal matrices and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, isomorphic to $2^2 : 2 \cong D_8$, and a 2-Sylow of $\mathrm{GL}_2(3)$, isomorphic to SD_{16}.

In the first case, $\mathrm{Out}_{\mathcal{F}}(3_+^{1+2}) \cong 2^2 : 2$ divides the 4 rank two elementary abelian 3-subgroups in 2 $\mathcal{F}$-conjugacy classes, both of them with 2 elements. We have that all of the elements A^iB^j are eigenvectors of eigenvalue -1 in a matrix conjugated to $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so all the rank two elementary abelian p-subgroups can be $\mathcal{F}$-radical. If we look at the restriction of $\mathrm{Aut}_{\mathcal{F}}(3^{1+2}_+)$ to any rank two elementary abelian p-subgroup, there are matrices of determinant -1, so $\mathrm{Aut}_{\mathcal{F}}(V) \cong \mathrm{GL}_2(3)$. Looking to the information of the Atlas [4] of the Tits group at $p = 3$ we deduce that there are two conjugation classes of rank two elementary abelian 3-subgroups and both of them are $\mathcal{F}$-radical.

In the case $\mathrm{Out}_{\mathcal{F}}(3^{1+2}_+) \cong SD_{16}$, the action on the rank two elementary abelian 3-subgroups is transitive, so there is just one $\mathcal{F}$-conjugacy class. As there is a matrix conjugated to $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then we can make all of them to be radical in our saturated fusion system. Finally, the same argument given in the previous case applies to obtain that $\mathrm{Aut}_{\mathcal{F}}(V) \cong \mathrm{GL}_2(3)$. The information in [15, Table (4.1)] tell us that $\mathcal{F}$ is the fusion system of the Janko's biggest group. □

The study of the case $p = 5$ requires the analysis of one more subgroup of $\mathrm{GL}_2(5)$. The analysis is done for any p

Lemma 4.13.

(a) *Up to conjugation, there exists just one $H \leq \mathrm{GL}_2(p)$ with the following presentation:*

$$\langle x, y \mid x^2 = y^{p^2-1} = 1, xyx = y^p \rangle .$$

(b) *If $p \neq 3$ and $p \neq 7$ and $\mathrm{Out}_{\mathcal{F}}(p^{1+2}_+) \cong H$, then $\sharp\mathcal{F}^{ec}$-rad $= 0$*

Proof. First we prove (a): The group H has one element $y \in \mathrm{GL}_2(p)$ of order p^2-1 so y will have two different conjugated eigenvalues in $\mathbb{F}_{p^2}$, which will be (p^2-1) primitive roots of the unity. All the cyclic subgroups of $\mathrm{GL}_2(p)$ of order p^2-1 are conjugated, so we fix one of them. Notice that $y^{p+1} = b\,\mathrm{Id}$, where b is a primitive $(p-1)$ root of the unity in $\mathbb{F}_p$.

We have to prove that fixed y there exists just one possible extension to H. Therefore we need a matrix x such that $x^2 = 1$ and $yxy = bx$, an easy calculation tell us that x has determinant -1 and is unique, modulus $-\mathrm{Id}$, but as $y^{\frac{p-1}{2}} = -\mathrm{Id}$, the extension H is the same.

In order to prove (b), observe first that we can conjugate H such that it contains the matrix $x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (x is an involution with determinant -1, and all of them are conjugate). Consider y the generator of the cyclic group (p^2-1) in H, so the eigenvalues of y are two conjugated primitive roots of unity in $\mathbb{F}_{p^2}$.

Now, if $\sharp\mathcal{F}^{ec}$-rad > 0 then a subgroup of type $(p-1)_{-2}$ must be inside this H. But the matrices in $\langle y \rangle$ whose eigenvalues are in $\mathbb{F}_p$ are of the form $\lambda\,\mathrm{Id}$, so they don't generate any $(p-1)_{-2}$. So we have to look to the matrices of the form $y'_j = y^j \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. One can check that $(y'_j)^2 = b^j Id$, where b is the primitive $(p-1)$ root of the unity in $\mathbb{F}_p$ that appears in the proof of (a). So if y'_j is conjugate to a matrix of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-2} \end{smallmatrix}\right)$ for a a primitive $(p-1)$ root of unity in $\mathbb{F}_p$, computing the square we have $a^6 = 1$ and therefore $p = 3$ or $p = 7$. □

Now we can study the case $p = 5$.

Lemma 4.14. *Let $\mathcal{F}$ be a saturated fusion system over 5_+^{1+2} such that $\sharp\mathcal{F}^{ec}$-rad > 2. Then $\mathcal{F}$ is one in the following list:*

$\mathrm{Out}_{\mathcal{F}}(5_+^{1+2})$	$\sharp\mathcal{F}^{ec}$-rad	$\mathrm{Aut}_{\mathcal{F}}(V)$	*Group*
$4S_4$	6	$\mathrm{GL}_2(5)$	Th

where $\sharp\mathcal{F}^{ec}$-rad *is the cardinal of* $\sharp\mathcal{F}^{ec}$-rad *separated by $\mathcal{F}$-conjugacy classes and* $\mathrm{Aut}_{\mathcal{F}}(V)$ *is the group of $\mathcal{F}$-automorphisms for each representative of the $\mathcal{F}$-conjugacy classes in* $\sharp\mathcal{F}^{ec}$-rad.

Proof. According Lemmas 4.7, 4.13 and 4.10, we have to consider subgroups of $\mathrm{GL}_2(5)$ of order prime to 5 such that they project onto A_4, S_4 or A_5 in $\mathrm{PGL}_2(5)$. We get that there is just one of those subgroups, isomorphic to $4S_4$, which projects onto S_4.

This group has matrices conjugated to $\left(\begin{smallmatrix} 2 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, so we can consider a saturated fusion system with radicals, and as all of them are in the same $\mathcal{F}$-conjugation class, all 6 will be $\mathcal{F}$-radical.

If we compute the determinant of the restriction of the matrices in $\mathrm{Out}_{\mathcal{F}}(5_+^{1+2})$ to any rank two elementary abelian p-subgroup, we get that is surjective in $\mathbb{F}_5^*$, so $\mathrm{Aut}_{\mathcal{F}}(V) = \mathrm{GL}_2(5)$.

Finally, according to [15, Table (4.1)], $\mathcal{F}$ is the fusion system of the Thompson group. □

The following case, $p = 7$, must be separated in two different results. The first one, giving a similar classification as in the previous case, but leaving 3 blank spaces in the *Group*'s column. This is because these three saturated fusion systems are exotic, that is, there is no finite group with any of these three fusion systems, as it is proved in the second result, Lemma 4.16.

Lemma 4.15. *Let $\mathcal{F}$ be a saturated fusion system over 7_+^{1+2} such that $\sharp\mathcal{F}^{ec}$-rad > 2. Then $\mathcal{F}$ is one in the following list:*

$\mathrm{Out}_{\mathcal{F}}(7_+^{1+2})$	$\sharp\mathcal{F}^{ec}$-rad	$\mathrm{Aut}_{\mathcal{F}}(V)$	*Group*
$S_3 \times 3$	3	$\mathrm{SL}_2(7)$	He
$D_8 \times 3$	$2+2$	$\mathrm{SL}_2(7):2, \mathrm{SL}_2(7):2$	$O'N$
$S_3 \times 6$	3	$\mathrm{SL}_2(7):2$	$He:2$
$S_3 \times 6$	$3+3$	$\mathrm{SL}_2(7):2, \mathrm{SL}_2(7):2$	Fi_{24}'
$6^2:2$	6	$\mathrm{SL}_2(7):2$	Fi_{24}
$6^2:2$	$6+2$	$\mathrm{SL}_2(7):2, \mathrm{GL}_2(7)$	
$D_{16} \times 3$	4	$\mathrm{SL}_2(7):2$	$O'N:2$
$D_{16} \times 3$	$4+4$	$\mathrm{SL}_2(7):2, \mathrm{SL}_2(7):2$	
$SD_{32} \times 3$	8	$\mathrm{SL}_2(7):2$	

where $\sharp\mathcal{F}^{ec}$-rad *is the cardinal of* $\mathcal{F}^{ec}$-rad *separated by* $\mathcal{F}$*-conjugacy classes and* $\mathrm{Aut}_{\mathcal{F}}(V)$ *is the group of* $\mathcal{F}$*-automorphisms for each representative of the* $\mathcal{F}$*-conjugacy classes in* $\sharp\mathcal{F}^{ec}$-rad.

Proof. For shortness we do not give details about the computation of the column $\mathrm{Aut}_{\mathcal{F}}(V)$ in each case, but recall that the results are inferred from the method described in Lemma 4.3.

As $\mathrm{Out}_{\mathcal{F}}(7_+^{1+2}) \leq \mathrm{GL}_2(7)$ is of order coprime $p = 7$, we first check the maximal subgroups with this property. From the study of the maximal solvable subgroups of $\mathrm{PGL}_2(p)$ in [10, Proposition 3.3], and Lemma 4.9 we obtain that the maximal subgroups of $\mathrm{GL}_2(7)$ of order coprime with 7 are the preimages of S_4 and $D_{2(p\pm1)}$ by the projection $\mathrm{GL}_2(7) \to \mathrm{PGL}_2(7)$, so obtaining the subgroups $6S_4$, $6^2 : 2$ and $48 : 2$, which are unique up to conjugation.

Now, we proceed to work with explicit subgroups of $\mathrm{GL}_2(7)$, checking in each case the matrices which can generate subgroups of the form 6_V, for $V \in \mathcal{F}^{ec}$-rad, as described in Lemma 4.2.

The subgroup $6S_4$ is then generated by the matrices

$$\left\{ \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 6 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 3 & 6 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 6 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \right\}.$$

Observe that the determinant of any of those matrices is a square in $\mathbb{F}_7$, so any saturated fusion system $\mathcal{F}$ over 7_+^{1+2} such that $\mathrm{Out}_{\mathcal{F}}(7_+^{1+2}) \leq S_4$ verifies $\sharp\mathcal{F}^{ec}$-rad $= 0$.

Now, think of $6^2 : 2$ as the subgroup generated by the diagonal matrices and the twisting $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The subgroups in $6^2 : 2$ of the form 6_V are given in the following table:

V	V_0	V_1	V_2	V_3	V_4	V_5	V_6	V_7
6_V	$\langle\left(\begin{smallmatrix} 3 & 0 \\ 0 & 4 \end{smallmatrix}\right)\rangle$	$\langle\left(\begin{smallmatrix} 0 & 3 \\ 3 & 0 \end{smallmatrix}\right)\rangle$	$\langle\left(\begin{smallmatrix} 0 & 5 \\ 6 & 0 \end{smallmatrix}\right)\rangle$	$\langle\left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right)\rangle$	$\langle\left(\begin{smallmatrix} 0 & 6 \\ 5 & 0 \end{smallmatrix}\right)\rangle$	$\langle\left(\begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix}\right)\rangle$	$\langle\left(\begin{smallmatrix} 0 & 4 \\ 4 & 0 \end{smallmatrix}\right)\rangle$	$\langle\left(\begin{smallmatrix} 4 & 0 \\ 0 & 3 \end{smallmatrix}\right)\rangle$

Then the subgroups of $6^2 : 2$ generated by, at least, three elements in the table above are of the form $S_3 \times 3$, $D_8 \times 3$, $S_3 \times 6$ and $6^2 : 2$, and again are unique up to conjugation. Therefore, we fix generators for each case and study the specific cases.

The group $\mathrm{Out}_{\mathcal{F}}(7_+^{1+2}) = S_3 \times 3$ is generated by 6_{V_1}, 6_{V_2} and 6_{V_4}, and then $\mathcal{F}^{ec}$-rad $= \{V_1, V_2, V_3\}$, all of them in the same $\mathcal{F}$-conjugacy class. The information in [15, Table (4.1)] tell us that $\mathcal{F}$ is the fusion system of the Held group.

The group $\mathrm{Out}_{\mathcal{F}}(7_+^{1+2}) = D_8 \times 3$ is generated by 6_{V_0}, 6_{V_7}, 6_{V_1} and 6_{V_6}, and these V's are the only elements in the set $\mathcal{F}^{ec}$-rad, which is divided into two $\mathcal{F}$-conjugacy classes of two elements each one. Again the information in [15, Table (4.1)] tell us the group with this fusion system: the O'Nan's group.

The group $\mathrm{Out}_{\mathcal{F}}(7_+^{1+2}) = S_3 \times 6$ is generated by 6_{V_1}, 6_{V_2}, 6_{V_3}, 6_{V_4}, 6_{V_5} and 6_{V_6}. This is also the list of possible elements in $\mathcal{F}^{ec}$-rad, and is divided into two $\mathcal{F}$-conjugacy classes of three elements each, so we must consider the possibility of just one of these conjugacy classes to be $\mathcal{F}$-radical, or both two. In the first case we get the fusion system of the automorphism group of the Held group, because $\mathrm{Out}(He)$ does not enlarge the cardinal of $\mathcal{F}$-rad, but adds an involution to $\mathrm{Aut}_{He}(V)$. Assume now that the we have six elements in $\mathcal{F}^{ec}$-rad, then by the information in the Atlas [4], $\mathcal{F}$ is the fusion system of the derived Fisher subgroup.

The group $\mathrm{Out}_{\mathcal{F}}(7_+^{1+2}) = 6^2 : 2$ breaks the rank two elementary abelian 7-subgroups into two conjugation classes, one with 6 elements, and another with 2. Again the information of the Atlas [4] tell that this corresponds to the fusion system of the Fisher group Fi_{24}.

Now we begin the study of the subgroups of $48 : 2 \cong SD_{32} \times 3$. Observe that we are in the case of Lemma 4.13, so we consider y an element in $\mathrm{GL}_2(7)$ of order 48, and x an involution. We see that the determinant of y must be a generator of $\mathbb{F}_7^*$, and we can suppose y has determinant 3 (if not, the determinant must be 3^{-1}, and we can take y^{-1} as generator). Moreover, we can assume $x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Now we must look to the matrices in H with eigenvalues 3 and -3: the matrices in $\langle y \rangle$ with eigenvalues in $\mathbb{F}_7$ are the ones of the form $y^{8i} = 3^i \,\mathrm{Id}$, so if there are matrices in $48 : 2$ with these properties are of the form $y^j x$. Now using that the determinant must be -2, we get that $j \in J \stackrel{\text{def}}{=} \{2, 8, 14, 20, 26, 32, 38, 44\}$. A direct computation using that $xyx = y^7$, we get $xy = 3y^{-1}x$, and so $y = \left(\begin{smallmatrix} r & s \\ -s & t \end{smallmatrix}\right)$. Then we see that the trace of $y^2 x$, which is equal to the trace of yxy must be 0. Similar arguments show that the trace of $y^j x$ is zero for all $j \in J$, so all that matrices have the desired eigenvalues to generate a subgroup of the form 6_V. The fact that 7 does not divide the order of $48 : 2$, implies that all that matrices have different eigenvector of eigenvalue 3 (same argument as the proof of Lemma 4.2), so each one generates a different V_i, for all $0 \leq i \leq 7$.

Now we have to assign to each element $y^j x$ a V_{i_j}. This will depend on the choice of y, and we can fix one, for example $y = \left(\begin{smallmatrix} 3 & 2 \\ 5 & 2 \end{smallmatrix}\right)$. With this we have the following table:

V	V_0	V_1	V_2	V_3	V_4	V_5	V_6	V_7
6_V	$\langle \left(\begin{smallmatrix} 3 & 5 \\ 0 & 4 \end{smallmatrix}\right) \rangle$	$\langle \left(\begin{smallmatrix} 0 & 3 \\ 3 & 0 \end{smallmatrix}\right) \rangle$	$\langle \left(\begin{smallmatrix} 2 & 4 \\ 3 & 5 \end{smallmatrix}\right) \rangle$	$\langle \left(\begin{smallmatrix} 4 & 2 \\ 0 & 3 \end{smallmatrix}\right) \rangle$	$\langle \left(\begin{smallmatrix} 5 & 3 \\ 4 & 2 \end{smallmatrix}\right) \rangle$	$\langle \left(\begin{smallmatrix} 3 & 0 \\ 2 & 4 \end{smallmatrix}\right) \rangle$	$\langle \left(\begin{smallmatrix} 0 & 4 \\ 4 & 0 \end{smallmatrix}\right) \rangle$	$\langle \left(\begin{smallmatrix} 4 & 0 \\ 5 & 3 \end{smallmatrix}\right) \rangle$

Now we proceed as in the previous case, checking all the subgroups of $48 : 2$ generated by at least three matrices in the list. If we check all the possibilities we get $D_8 \times 3$ which has been studied previously, $D_{16} \times 3$ and $SD_{32} \times 3$. These last two corresponds to the normal subgroup of $48 : 2$ generated by $\langle x, y^2 \rangle$ and the total group. In the first case, checking the action of the outer automorphism group of the O'Nan's group, we get that the two conjugacy classes in $\mathcal{F}^{ec}$-rad fuse in just one, with 4 elements. Finally $SD_{32} \times 3$ acts transitively on the rank two elementary abelian 7-groups, getting the result in the list. $\square$

Lemma 4.16. *There is no finite group with fusion system over 7_+^{1+2} isomorphic to any of the following cases:*

$\mathrm{Out}_{\mathcal{F}}(7_+^{1+2})$	$\sharp\mathcal{F}^{ec}$-rad	$\mathrm{Aut}_{\mathcal{F}}(V)$
$D_{16} \times 3$	$4 + 4$	$\mathrm{SL}_2(7) : 2, \mathrm{SL}_2(7) : 2$
$6^2 : 2$	$6 + 2$	$\mathrm{SL}_2(7) : 2, \mathrm{GL}_2(7)$
$SD_{32} \times 3$	8	$\mathrm{SL}_2(7) : 2$

Proof. According to [2, Lemma 9.2], we have to check finite simple groups and their possible extensions. Checking the Classification of Finite Simple Groups from [6], we have to look to the fusion systems of the following cases:

(i) Simple groups of Lie type: if the characteristic of this group is 7, then, looking to the orders, there must be $L_3(7)$ or $U_3(7)$. The linear one is considered in the list and the unitary one doesn't have any radical elementary rank two abelian 7-subgroup.

If the characteristic of the field of the simple group of Lie type is different from 7, then we can use the result of [5, 10-2], which says that the 7-Sylow in that situation has a unique elementary abelian subgroup of maximal rank, and that is not true for 7_+^{1+2}.

(ii) Sporadic groups: one can check the list one by one, just in the cases that the 7-Sylow is 7_+^{1+2} and all of them appear in the list of Lemma 4.15.

Finally one has to check the possible extensions, and all the possibilities which give new information are also in the list. □

Lemma 4.17. *If* $p = 13$ *there is just one saturated fusion system* $\mathcal{F}$ *over* p_+^{1+2} *such that* $\sharp\mathcal{F}^{ec}\text{-rad} \geq 2$*, corresponding to the Fischer-Griess monster:*

$\mathrm{Out}_{\mathcal{F}}(13_+^{1+2})$	$\sharp\mathcal{F}^{ec}$-rad	$\mathrm{Aut}_{\mathcal{F}}(V)$	*Group*
$3 \times 4S_4$	6	$\mathrm{SL}_2(13.4)$	M

where $\sharp\mathcal{F}^{ec}$-rad *is the number of rank two* $\mathcal{F}$*-radical elementary abelian* 13*-subgroups in just one conjugation class and* $\mathrm{Aut}_{\mathcal{F}}(V)$ *the automorphism group of those.*

Proof. To check all the possible saturated fusion systems with more than two $\mathcal{F}$-radical, using the Lemma 4.10, we get that we need that the projection to $\mathrm{PGL}_2(13)$ must be S_4 (it cannot be $(D_{13\pm1})$ and we need an element of order 4). The inverse image of this group, which is unique up to conjugation [10, Proposition 3.3], is a subgroup in $\mathrm{GL}_2(p)$ isomorphic to $3 \times 4S_4$ and also unique up to conjugation. The action of this group over the rank two elementary abelian 13-groups divide them in two conjugacy classes, with 6 and 8 elements and the elements in the first class can be $\mathcal{F}$-radical. The image of the determinant for the matrices which induce an automorphism in the $\mathcal{F}$-radical rank two elementary abelian 13-group is the ideal generated by 3 in $\mathbb{Z}/12$, so the automorphisms group for those is $\mathrm{SL}_2(13).4$.

To see that it corresponds to the fusion system of M, the Fischer-Griess Monster, we use the information in the Atlas [4]. There is a subgroup $13A$ in M of order 13 such that $N(13A) \cong 13_+^{1+2} : (3 \times 4S_4)$, so we get the desired $\mathrm{Out}_{\mathcal{F}}(13_+^{1+2})$. Once we know that, as there is just one conjugation class of such a group in $\mathrm{GL}_2(13)$, then it must break the rank two elementary abelian p-subgroups in 2 orbits, with 6 and 8 elements. So, we have just to prove that there is one $\mathcal{F}$-radical rank two elementary abelian p-subgroup, and it follows from that there is $13B^2$, a rank two elementary p-subgroup in M, such that $N(13B^2) \cong 13^2 : 4L_2(13).2$. The order of $4L_2(13).2$ is the same as $\mathrm{SL}_2(13).4$ and there is just one subgroup with this order in $\mathrm{GL}_2(13)$, so it contains $SL_2(13)$. That means that there is at least one $\mathcal{F}$-radical rank two elementary abelian p-subgroup and the result follows. □

Notice that the lemma above corrects a minor error in the description of the 13-fusion of M given in [15, Table (4.1)].

Finally a summary of the arguments followed to prove Theorem 1.1:

Proof of Theorem 1.1. According to Corollary 2.9, it is enough to classify saturated fusion systems over p_+^{1+2}.

For getting the result for the Table 1, apply Lemmas 4.5, 4.6 and 4.8.

To obtain the data in Table 2, apply Lemmas 4.12, 4.14, 4.15 and 4.17. Lemma 4.16 tell us that the 3 blank spaces left in the "Group" column cannot be filled.

Finally, to see that there is a complete classification, use that the primes which doesn't apply the Lemma 4.10 are the ones in the Table 2 and apply again Lemmas 4.12, 4.14, 4.15 and 4.17. □

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