Commutative Morava homology Hopf algebras

Hal Sadofsky and W. Stephen Wilson

Abstract. We give the Dieudonné module theory for $\mathbf{Z}/2(p^n-1)$-graded bicommutative Hopf algebras over $\mathbf{F}_p$. These objects arise as the Morava K-theory of homotopy associative, homotopy commutative H-spaces.

1. Introduction

In [**DG70**, V], Demazure and Gabriel classify commutative unipotent algebraic groups over a perfect field of characteristic p in terms of Dieudonné modules. Making appropriate translations of terminology, this classifies a category of Hopf algebras in terms of Dieudonné-modules.

Bousfield, in an appendix to [**Bou96b**], proves many of the results of interest in an accessible way, adapting and clarifying [**DG70**, V] for his specific purposes. In particular, he gives much of the structure for $\mathbf{Z}/(2)$-graded bicommutative Hopf algebras over $\mathbf{F}_p$. He needs these results for his work with mod p K-theory. We are interested in similar results for Morava K-theory. The Morava K-theory of homotopy commutative H-spaces gives rise to bicommutative $\mathbf{Z}/2(p^n-1)$-graded Hopf algebras satisfying certain conditions. Having a Dieudonné module theory for these Hopf algebras gives us a great deal of control over their structure. When the grading is forgotten then Bousfield's results can be applied, so our contribution is really to unravel the gradings in our case and make appropriate definitions so that Bousfield's proofs go through with little or no modification.

Specifically, we want to study the category $\mathcal{C}(n)$ of *commutative Morava Hopf algebras*. These are bicommutative, biassociative, Hopf algebras over $\mathbf{F}_p$ which are graded over $\mathbf{Z}/2(p^n-1)$ $(n>0)$ and have an exhaustive primitive filtration. This last condition, *an exhaustive primitive filtration*, means that each element maps trivially under some iterate of the reduced coproduct (into the tensor product of the cokernel of coaugmentation ideal). (Bousfield calls this property "irreducible," a term we wish to reserve for a Hopf algebra with no proper non-trivial sub-Hopf algebras). Bousfield's work analyzes these Hopf algebras quite thoroughly by reducing the $\mathbf{Z}/(2)$-graded case to the ungraded case by splitting off the sub-algebra generated by odd-dimensional primitives. The study of these Hopf algebras in [**HRW98**] concentrates on important consequences of the grading. Our goal here is to put the grading into Bousfield's results.

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[**HRW98**] shows $\mathcal{C}(n)$ splits (for odd primes) as the product of two categories; $\mathcal{OC}(n)$, whose objects are Hopf algebras generated by primitives in odd degrees, and hence primitively generated exterior algebras, and $\mathcal{EC}(n)$, whose objects are all concentrated in even degrees. (Note that [**Bou96b**] proves the same result for the $\mathbf{Z}/(2)$-graded case, and his proof also holds without alteration for $\mathcal{C}(n)$.) So, our real category of study will be $\mathcal{EC}(n)$.

In [**Bou96a**], Bousfield shows the $\mathbf{Z}/(2)$-graded case is an abelian category, but again, his proof works just as well for $\mathcal{C}(n)$. Our contribution is to define a functor to Dieudonné modules, to which we again appeal to [**Bou96b**] with small alterations.

We define a *Morava-Dieudonné module* to be a $\mathbf{Z}/(p^n-1)$-graded abelian group M_* with endomorphisms $F : M_t \to M_{pt}$, and $V : M_t \to M_{t/p}$ (note that in $\mathbf{Z}/(p^n-1)$ every t/p exists) such that:

(i) $FV = p = VF$;
(ii) for each $x \in M$, there exists a $q \geq 1$ with $V^q(x) = 0$.

We denote the category of Morava-Dieudonné modules by $\mathcal{MD}(n)_*$. Note the definition implies the modules are all p-torsion, i.e. $p^q(x) = (FV)^q(x) = F^qV^q(x) = 0$ where we have used (i) and (ii).

Our main theorem is:

THEOREM 1.1. *There is a functor $m_* : \mathcal{EC}(n) \to \mathcal{MD}(n)_*$ which is an equivalence of abelian categories.*

Let $\mathbf{Z}/(n)$ act on $\mathbf{Z}/(p^n-1)$ by $k * j = jp^k$. The orbits are given by $\gamma = \{j, jp, jp^2, jp^3, \ldots, jp^n = j\}$. We can see with no work that $\mathcal{MD}(n)_*$ splits as a product of categories $\mathcal{MD}_\gamma(n)_*$ where all elements are in the degrees contained in γ. This translates into a similar theorem, a major result of [**HRW98**], about $\mathcal{EC}(n)$.

COROLLARY 1.2 ([**HRW98**], Theorem 1.13). *As categories:*

$$\mathcal{EC}(n) \simeq \prod_\gamma \mathcal{EC}(n)_\gamma$$

where $\mathcal{EC}(n)_\gamma$ consists of Hopf algebras in $\mathcal{EC}(n)$ with all primitives in degrees 2γ.

This result was proven in [**HRW98**] so it could be combined with results of [**RW80**] to show that there is only the trivial map from $K(n)_*(K(\mathbf{Z}/(p^i),s))$ to $K(n)_*(K(\mathbf{Z}/(p^j),t))$ when $s \neq t$.

The motivation for us is Morava K-theory, $K(n)_*(-)$, which is a generalized homology theory with a Künneth isomorphism. The coefficient ring is $K(n)_* \simeq \mathbf{F}_p[v_n, v_n^{-1}]$ where the degree of v_n is $2(p^n-1)$. By setting $v_n = 1$ we get a $\mathbf{Z}/2(p^n-1)$-graded theory, $\overline{K(n)}_*(-)$. If a space X is connected and the loop space of an H-space then $\overline{K(n)}_*(X)$ is in $\mathcal{C}(n)$ for odd primes (there are some complications with commutativity for $p=2$).

In [**RW80**] the Morava K-theory of all (abelian) Eilenberg-Mac Lane spaces was computed. They all, with the exception of $K(\mathbf{Z},1)$, the circle, lie in $\mathcal{EC}(n)$. Let $I = (i_1, \ldots, i_q)$, $0 \leq i_1 < i_2 < \cdots < i_q < n$. When $i_1 > 0$ define $s(I) = (i_1-1, i_2-1, \ldots, i_q-1)$ and when $i_q < n-1$ define $s^{-1}(I) = (i_1+1, i_2+1, \ldots, i_q+1)$. We can just read off the Morava-Dieudonné module stucture from [**RW80**].

THEOREM 1.3. *The Morava-Dieudonné module,* $m_*(\overline{K(n)}_*(K(\mathbf{Z}/(p^j),q)))$, *is a free* $\mathbf{Z}/(p^j)$ *module on generators* $a_I = a_{(i_1)}a_{(i_2)}\dots a_{(i_q)}$, $I = (i_1,\dots,i_q)$, $0 \leq i_1 < i_2 < \cdots < i_q < n$ *in degree* $p^{i_1}+p^{i_2}+\cdots+p^{i_q}$ *with*

$$V(a_I) = \begin{cases} a_{s(I)} & \text{if } i_1 > 0 \\ (-1)^{q-1} p a_{(i_2-1)} a_{(i_3-1)} \dots a_{(n-1)} & \text{if } i_1 = 0 \end{cases}$$

and

$$F(a_I) = \begin{cases} p a_{s^{-1}(I)} & \text{if } i_q < n-1 \\ (-1)^{q-1} a_{(0)} a_{(i_1+1)} \dots a_{(i_{q-1}+1)} & \text{if } i_q = n-1. \end{cases}$$

We have, from [**RW80**, Corollary 13.1], a short exact sequence of Hopf algebras:

$$\mathbf{F}_p \to \overline{K(n)}_*(K(\mathbf{Z}/(p^i),q)) \to \overline{K(n)}_*(K(\mathbf{Z}/(p^{i+j}),q)) \to \overline{K(n)}_*(K(\mathbf{Z}/(p^j),q)) \to \mathbf{F}_p.$$

This translates into the most obvious exact sequence in the Morava-Dieudonné modules. For the surjection, [**RW80**, Proposition 11.4] is enough and for the injection we need that and [**RW80**, Theorem 11.1(c)].

Since

$$\lim_{\to} \overline{K(n)}_*(K(\mathbf{Z}/(p^i),q)) \simeq \overline{K(n)}_*(K(\mathbf{Q}/\mathbf{Z}_{(p)},q)) \simeq \overline{K(n)}_*(K(\mathbf{Z},q+1))$$

the above map shows the Morava-Dieudonné module for this is a bunch of copies of $\mathbf{Q}/\mathbf{Z}_{(p)}$ and gives both F and V.

The Morava-Dieudonné module description of the Morava K-theory of Eilenberg-Mac Lane spaces is much simpler than the Hopf algebra description of [**RW80**]. This is because it is no longer necessary to know the order of truncation of the algebra generators to give a precise description (as it is for Hopf algebras). The order of F on generators can be calculated from the description of the Morava-Dieudonné module structure, but it is not necessary to know it to give the description. This is a definite improvement because the orders are probably wrong in [**RW80**].

Note that because Morava K-theory has a Künneth isomorphism the Morava-Dieudonné module of the Morava K-theory of a finite product of Eilenberg-Mac Lane spaces is just the direct product of the respective Morava-Dieudonné modules.

We define an *irreducible* Morava Hopf algebra as one with no proper non-trivial sub-Hopf algebras. Since we are working with an abelian category, it is clear that any finite Morava Hopf algebra can be filtered so that the quotients are irreducible. The remaining interesting problem is to identify these finite irreducibles. This is easiest to do in the category of Morava-Dieudonné modules. We need only work in the category $\mathcal{EC}(n)_\gamma$ because of Corollary 1.2. Let $|\gamma|$ be the order of the orbit γ.

THEOREM 1.4. *Every finite irreducible Morava-Dieudonné module can be filtered so the quotients are irreducible Morava-Dieudonné modules. The finite irreducible Morava-Dieudonné modules in* $\mathcal{EC}_\gamma(n)$ *are:*

(i) *An* $\mathbf{F}_p$ *(in any degree in* γ*) with* F *and* V *trivial.*
(ii) *A* q *dimensional vector space over* $\mathbf{F}_p$ *in each degree in* γ *with* V *and* $f(F^{|\gamma|})$ *acting trivially (*F *is an isomorphism) where* f *is a primitive polynomial over* $\mathbf{F}_p$ *(of degree* q*).*

These irreducibles correspond to the following Morava Hopf algebras. The first is the truncated polynomial algebra $P(x)/(x^p)$, where x is primitive. The second is the more complicated $P(x)/(f(F^{|\gamma|})(x))$ where again x is primitive. This last relation just tells us how to write $x^{p^{q|\gamma|}}$ in terms of $x^{p^{i|\gamma|}}$ for $i < q$. The primitive nature of the polynomial prevents us from finding sub-Hopf algebras of this.

This theory can be generalized to a work over a perfect field of characteristic p, see [**Rav75**]. If we do this over the agebraic closure of $\mathbf{F}_p$, then the irreducibles are much simplified. In fact, none of the more exotic irreducible Hopf algebras have been spotted in nature, raising the question of whether they can arise as part of the Morava K-theory of a space or not.

There is an additional theorem worth pointing out. Under very mild conditions, see [**Bou96b**, Appendix B, Theorem B.4], Sweedler proves what we would call a dual Borel theorem (about the coalgebra structure) in [**Swe67a**] and [**Swe67b**].

We remark that our classification of Hopf algebras constructs Dieudonné modules by, as usual, considering maps of some variant of the Witt ring (see section 2 for the variant suited to the problem in this paper) into the Hopf algebra under consideration. There is another context familiar to algebraic topologists where objects called Dieudonné modules arise. This is in the classification of formal groups over a field k of characteristic p. We would like to point out the similarities of these situations as the authors have not seen this made explicit elsewhere.

Recall that bicommutative Hopf k-algebras are abelian group objects in the category of commutative k-algebras. The Witt algebra (or as in this paper, some variant of it) is a particular bicommutative Hopf k-algebra that serves as a generator for this category. Our classification comes about by taking maps of Hopf algebras from the Witt algebra to some other Hopf algebra and considering this group as a "Diedonné module" - a module over the ring of self maps of the Witt Hopf algebra.

Commutative formal groups over k are, on the other hand, abelian group objects in the category of formal schemes over k. There is a special commutative formal group (infinite dimensional) called the Witt vectors. (The Witt algebra is essentially the representing ring of the Witt vectors in the category of k-algebras.) To classify formal groups [**Laz75**], we use the "curves" functor which is maps of formal groups from the Witt vectors (as a formal group) to the formal group under consideration. This gives us a "Diedonné module" which is a module over the self maps of the Witt vectors. Because the Witt vectors are represented by the Witt Hopf algebra, the Diedonné module is also a module over the self maps of the Witt Hopf algebra.

Background. A Dieudonné module theory has long been developed for graded Hopf algebras, [**Sch70**]. The authors had been talking about this project for some time when the second author was decimated by the chairmanship of his department. By the time he had recovered, the two papers, [**Bou96b**] and [**Bou96a**], were available and the project was easy to complete by co-opting the proofs from Bousfield, [**Bou96b**]. Pete Bousfield politely but firmly declined the authors' invitation to be a coauthor. The authors regret his decision and make no claim to originality. We feel the theorems should be made available however. Following a suggestion of Igor Kriz's we were developing the general Dieudonné module theory for Hopf rings only to find that Paul Goerss had already done it. We thank the referee for his helpful suggestions.

2. Differences from Bousfield

At the suggestion of the referee we assume the basic facts about the elementary Hopf algebras which we use without a detailed review. Some basic and specific references are [**Wit36**], [**Ser67**], [**Kra72**], [**HRW98**], and [**Bou96b**].

There is a bicommutative Hopf algebra $W = \mathbf{Z}[x_0, x_1, \dots]$ with $x_0^{p^n} + px_1^{p^{n-1}} + \cdots + p^{n-1}x_{n-1}^p + p^n x_n$ primitive. We can grade W by setting $|x_0| = 2t$ and $|x_i| = p^i 2t$. The primitives are homogeneous, so we get a graded Hopf algebra which we call W_t. From it we get Hopf algebras graded over $\mathbf{Z}/2(p^n - 1)$ by defining

$$W(s,t) \equiv \mathbf{F}_p[x_0, x_1, \dots, x_s] \subset W_t \otimes \mathbf{F}_p.$$

Bicommutative Hopf algebras over perfect fields of characteristic p come with two self maps, the *Frobenius* F and the *Verschiebung*, V. The Frobenius is the p^{th} power map, i.e. $F(x) = x^p$, which is a map of Hopf algebras since we are in characteristic p, but not of graded Hopf algebras. The Verschiebung can be defined as the dual of the Frobenius map on the dual Hopf algebra, as in [**HRW98**], or in terms of the coproduct, as in [**Bou96b**]. It gives a map of Hopf algebras, but not a map of graded Hopf algebras since $|V(x)| = |x|/p$. We have $V(x_i) = x_{i-1}$ in $W(s,t)$. When x is primitive, $V(x) = 0$, so V is 0 on primitively generated bicommutative Hopf algebras. Clearly $FV = VF$ since V is a map of algebras. $\operatorname{Hom}_{\mathcal{EC}(n)}(A, B)$ is an abelian group with addition of f and g defined using the coproduct on A and the product on B:

$$A \xrightarrow{\psi} A \otimes A \xrightarrow{f \otimes g} B \otimes B \longrightarrow B.$$

It is standard that $FV = p$ in the group $\operatorname{Hom}_{\mathcal{EC}(n)}(A, B)$.

The proof of Theorem 1.1 is the same as Bousfield's, [**Bou96b**], and at the suggestion of the referee has not been included here. We must include the definition of the functor though because here we do differ from Bousfield.

There is a surjective map $W(s, t/p) \to W(s-1, t)$ which takes the primitive generator in degree $2t/p$ to zero.

We use the sequence:

$$W(0,t) \leftarrow W(1, t/p) \leftarrow \cdots \leftarrow W(s, t/p^s) \leftarrow W(s+1, t/p^{s+1}) \quad \cdots$$

to define

$$m_t : \mathcal{EC}(n) \longrightarrow \mathcal{MD}(n)_t$$

by

$$m_t(A) = \varinjlim \operatorname{Hom}(W(s, t/p^s), A).$$

Remark 2.1. To get the equivalence of $\mathcal{EC}(n)_\gamma$ and $\mathcal{MD}(n)_\gamma$ we need only restrict our attention to the t in γ. In particular, when $\gamma = \{0\}$ we are precisely in the ungraded case of [**Bou96b**] except with a slightly modified definition of the functor between them.

To complete our definition of $m_* : \mathcal{EC}(n) \to \mathcal{MD}(n)_*$ we need to define the action of F and V on $m_*(A)$. We require a minor departure from the standard definition in order to preserve grading. In our case we use the fact that the image of F on $W(s,t)$ is canonically isomorphic to $W(s, pt)$, i.e. the composite

$$W(s,t) \xrightarrow{\simeq} W(s, pt) \hookrightarrow W(s,t)$$

that sends $x_i \mapsto x_i \mapsto x_i^p$ is F. The second map here preserves degree, so is in our category.

We get a map of sequences

$$\begin{array}{ccccccccc}
W(0,t) & \leftarrow & W(1,t/p) & \leftarrow & \cdots & \leftarrow & W(s,t/p^s) & \leftarrow & W(s+1,t/p^{s+1}) & \cdots \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow & \\
W(0,pt) & \leftarrow & W(1,t) & \leftarrow & \cdots & \leftarrow & W(s,t/p^{s-1}) & \leftarrow & W(s+1,t/p^s) & \cdots
\end{array}$$

which induces the map F

$$m_t(A) = \lim_{\rightarrow} \mathrm{Hom}(W(s,t/p^s),A) \longrightarrow m_{pt}(A) = \lim_{\rightarrow} \mathrm{Hom}(W(s,pt/p^s),A).$$

We now need the map V. The image of V acting on $W(s,t)$ is the sub-Hopf algebra $W(s-1,t)$. There is a canonical map of $W(s,t/p)$ which surjects to this image. In particular, this gives us maps

$$W(s,t/p) \to W(s-1,t) \subset W(s,t).$$

The corresponding map of sequences

$$\begin{array}{ccccccccc}
W(0,t) & \leftarrow & W(1,t/p) & \leftarrow & \cdots & \leftarrow & W(s,t/p^s) & \leftarrow & W(s+1,t/p^{s+1}) & \cdots \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow & \\
W(0,t/p) & \leftarrow & W(1,t/p^2) & \leftarrow & \cdots & \leftarrow & W(s,t/p^{s+1}) & \leftarrow & W(s+1,t/p^{s+2}) & \cdots
\end{array}$$

induces V

$$m_t(A) = \lim_{\rightarrow} \mathrm{Hom}(W(s,t/p^s),A) \longrightarrow m_{t/p}(A) = \lim_{\rightarrow} \mathrm{Hom}(W(s,(t/p)/p^s),A).$$

3. Irreducibles

PROOF OF THEOREM 1.4. We construct the filtration of an arbitrary finite Morava-Dieudonné module and see what we get. Assuming that M_* is finite, we look at the kernel of V on it. This is always a nontrivial sub-Morava-Dieudonné module because V iterated enough times must be zero. Of course this kernel has trivial V action so we have reduced our problem to filtering a Morava-Dieudonné module with $V = 0$. This is a $\mathbf{F}_p$ vector space because $p = FV = 0$.

If F has a kernel then the kernel is a sub-Morava-Dieudonné module with trivial F and V action. Any element of this generates a sub-module of type (i) of the Theorem. We have now reduced our problem to the case where V acts trivially and F has no kernel, i.e. where F is an isomorphism of finite dimensional vector spaces. Thus the dimensions of M_t for each $t \in \gamma$ must be the same.

Pick any $x \in M_*$. Because M_* is finite dimensional there must be some minimal u such that $F^u(x)$ can be written in terms of $F^i(x)$ for $i < u$. We first claim that u is a multiple of $|\gamma|$. If not, then $F^u(x)$ is not in the same degree as x and the above relation does not include $x = F^0(x)$. Since F is an isomorphism, we can divide the relation by some power of F until x is in the relation. This contradicts the minimality of u.

Our relation can now be written as $f(F^{|\gamma|})(x) = 0$ because elements in dimensions other than that of x cannot be part of the relation. Factor f into primitive polynomials $f_1 f_2 \dots f_k$. The element $y = f_2 \dots f_k(F^{|\gamma|})(x)$ is in $M_{|x|}$. It has the property that $f_1(F^{|\gamma|})(y) = 0$ and the vector space $\{y, F^{|\gamma|}(y), F^{2|\gamma|}(y), F^{3|\gamma|}(y), \dots, F^{(q-1)|\gamma|}(y)\}$ (where q is the degree of f_1) is an irreducible sub-vector space of $M_{|x|}$ under the action of $F^{|\gamma|}$. We now take the sub-Morava-Dieudonné module generated by y. This completes our proof and gives our modules of type (ii).

Note that we have just reproven the fundamental structure theorem for finite modules over the P.I.D. $\mathbf{F}_p[z]$ (where $z = F^{|\gamma|}$) and we could have just referred to it instead. □

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University of Oregon, Eugene, OR, 97403
E-mail address: `sadofsky@math.uoregon.edu`

Johns Hopkins University, Baltimore, Maryland 21218
E-mail address: `wsw@math.jhu.edu`