DIAGONALS ON THE PERMUTAHEDRA, MULTIPLIHEDRA AND ASSOCIAHEDRA

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Abstract. We construct an explicit diagonal on the permutahedra $\{P_n\}$. Related diagonals on the multiplihedra $\{J_n\}$ and the associahedra $\{K_n\}$ are induced by Tonks' projection $\theta : P_n \to K_{n+1}$ [19] and its factorization through J_n. We use the diagonal on $\{K_n\}$ to define the tensor product of A_∞-(co)algebras. We introduce the notion of a permutahedral set $\mathcal{Z}$, observe that the double cobar construction $\Omega^2 C_*(X)$ is a naturally occurring example and lift the diagonal on $\{P_n\}$ to a diagonal on $\mathcal{Z}$.

1. Introduction

A permutahedral set is a combinatorial object generated by permutahedra $\{P_n\}$ and equipped with appropriate face and degeneracy operators. Permutahedral sets are distinguished from cubical or simplicial set by higher order (non-quadratic) relations among face and degeneracy operators. In this paper we define the notion of a permutahedral set and observe that the double cobar construction $\Omega^2 C_*(X)$ is a naturally occurring example. We construct an explicit diagonal $\Delta_P : C_*(P_n) \to C_*(P_n) \otimes C_*(P_n)$ on the cellular chains of permutahedra and show how to lift Δ_P to a diagonal on any permutahedral set. We factor Tonks' projection $P_n \to K_{n+1}$ through the multiplihedron J_n and obtain diagonals Δ_J on $C_*(J_n)$ and Δ_K on $C_*(K_n)$. We apply Δ_K to define the tensor product of A_∞-(co)algebras in maximal generality; this solves a long-standing open problem in the theory of operads. One setting in which the need to construct the tensor product of two A_∞-algebras arises is the open string field theory of M. Gaberdiel and B. Zwiebach [7].

We mention that Chapoton [4], [5] constructed a diagonal on the direct sum $\oplus_{n\geq 2} C_*(K_n)$ of the form $\Delta : C_*(K_n) \to \oplus_{i+j=n} C_*(K_i) \otimes C_*(K_j)$ that coincides with Loday and Ronco's diagonal on binary trees [13] in dimension zero. While Chapoton's diagonal is primitive on generators, ours is not.

The paper is organized as follows: Section 2 reviews the polytopes we consider and establishes our notation and point-of-view. The diagonal Δ_P is introduced in Section 3; related diagonals Δ_J and Δ_K are obtained in Section 4. Tensor products of A_∞-(co)algebras are defined in Section 5 and permutahedral sets are introduced in Section 6.

Date: November 22, 2002.

1991 *Mathematics Subject Classification.* Primary 55U05, 52B05, 05A18, 05A19; Secondary 55P35.

Key words and phrases. Diagonal, permutahedron, multiplihedron, associahedron.

[1] This research was funded in part by Award No. GM1-2083 of the U.S. Civilian Research and Development Foundation for the Independent States of the Former Soviet Union (CRDF) and by Award No. 99-00817 of INTAS.

[2] This research was funded in part by a Millersville University faculty research grant.

The first author wishes to acknowledge conversations with Jean-Louis Loday from which our representation of the permutahedron as a subdivision of the cube emerged. The second author wishes to thank Millersville University for its generous financial support and the University of North Carolina at Chapel Hill for its kind hospitality during work on parts of this project.

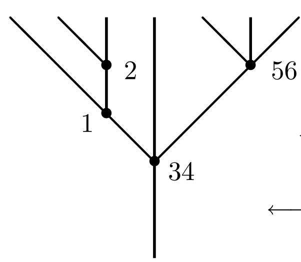
2. Permutahedra, Multiplihedra and Associahedra

We begin with a review of the relevant polytopes and establish our notation. Let S_{n+1} denote the symmetric group on $\underline{n+1} = \{1, 2, \ldots, n+1\}$ and recall that the permutahedron P_{n+1} is the convex hull of $(n+1)!$ vertices $(\sigma(1), \ldots, \sigma(n+1)) \in \mathbb{R}^{n+1}$, $\sigma \in S_{n+1}$ [6], [15]. As a cellular complex, P_{n+1} is an n-dimensional convex polytope whose $(n-k)$-faces are indexed either by (planar rooted) $(k+1)$-leveled trees (PLT's) with $n+2$ leaves or by permutations $M_1|\cdots|M_{k+1}$ of (ordered) partitions of $\underline{n+1}$. Elements of these two indexing sets correspond in the following way: Let T^{k+1}_{n+2} denote a $(k+1)$-leveled tree with root node in level $k+1$ and $n+2$ leaves numbered from left to right. For $1 \leq m \leq n+1$, assign the label m to the node at which branches containing leaves m and $m+1$ meet (a node can have multiple labels) and let $M_j = \{\text{labels assigned to nodes at level } j\}$; we refer to M_j as the *set of j-level meets in* T^{k+1}_{n+2}. Then $M_1|\cdots|M_{k+1}$ is the partition of $\underline{n+1}$ corresponding to T^{k+1}_{n+2} (see Figure 1). In particular, vertices of P_{n+1} are indexed either by binary $(n+1)$-leveled trees or by partitions $M_1|\cdots|M_{n+1}$ of $\underline{n+1}$, i.e., elements of S_{n+1}. The map from S_{n+1} to binary $(n+1)$-leveled trees was constructed by Loday and Ronco [13]; its extension to faces of P_{n+1} was given by Tonks [19].

The associahedra $\{K_{n+2}\}$, which serve as parameter spaces for higher homotopy associativity, are closely related to the permutahedra. In his seminal papers of 1963 [18], J. Stasheff constructed the associahedra in the following way: Let $K_2 = *$; if K_{n+1} has been constructed, define K_{n+2} to be the cone on the set

$$\bigcup_{\substack{r+s=n+3 \\ 1\leq k\leq n-s+3}} (K_r \times K_s)_k \, .$$

Thus, K_{n+2} is an n-dimensional convex polytope.



$\longleftrightarrow$ $256|1|34$

$\longleftrightarrow$ $d_{(0,2)}d_{(0,1)}d_{(1,1)(4,2)}$

$\longleftrightarrow$ $((x_1\,(x_2x_3))\,x_4\,(x_5x_6x_7))$

Figure 1.

Stasheff's motivating example of higher homotopy associativity in [18] is the singular chain complex on the loop space of a connected CW-complex. Here associativity holds up to homotopy, the homotopies between the various associations are homotopic, the homotopies between these homotopies are homotopic, and so on. An abstract A_∞-algebra is a dga in which associativity behaves as in Stasheff's motivating example. If $\varphi^2 : A \otimes A \to A$ is the multiplication on an A_∞-algebra A, the homotopies $\varphi^n : A^{\otimes n} \to A$ are multilinear operations such that φ^3 is a homotopy between the associations $(ab)\,c$ and $a\,(bc)$ thought of as quadratic compositions $\varphi^2\left(\varphi^2 \otimes 1\right)$ and $\varphi^2\left(1 \otimes \varphi^2\right)$ in three variables, φ^4 is a homotopy bounding the cycle of five quadratic compositions in four variables involving φ^2 and φ^3, and so on. Let $C_*(K_r)$ denote the cellular chains on K_r. There is a chain map $C_*(K_r) \to Hom\left(A^{\otimes r}, A\right)$, induced by the natural correspondence between faces of K_r and all possible compositions of the φ^n's in r-variables (modulo an appropriate equivalence), which determines the relations among the compositions of φ^n's. A detailed discussion of A_∞-algebras and their tensor product appears in Section 5.

Now if we disregard levels, a PLT is simply a planar rooted tree (PRT). Quite remarkably, A. Tonks [19] showed that K_{n+2} is the identification space $P_{n+1}/\sim$ obtained from P_{n+1} by identifying all faces indexed by isomorphic PRT's. Since the quotient map $\theta : P_{n+1} \to K_{n+2}$ is cellular, the faces of K_{n+2} are indexed by PRT's with $n+2$ leaves. The correspondence between PRT's with $n+2$ leaves and parenthesizations of $n+2$ indeterminants is simply this: Given a node N, parenthesize all indeterminants corresponding to all leaves on branches that meet at node N.

Example 1. *With one exception, all classes of faces of P_3 consist of a single element. Tonks' projection θ sends elements of the exceptional class*

$$[1|3|2, 13|2, 3|1|2]$$

to the vertex on K_4 represented by the parenthesization $((\bullet\bullet)(\bullet\bullet))$. The permutations $1|3|2 = d_{(1,1)}d_{(0,1)}$ and $3|1|2 = d_{(0,1)}d_{(2,1)}$ insert inner parentheses in the opposite order; the partition $13|2 = d_{(0,1)(2,1)}$ represents a homotopy between them. The classes of faces of P_4 with more than one element and their representations on K_5 are:

$$\begin{aligned}
[12|4|3, 124|3, 4|12|3] &= ((\bullet\bullet\bullet)(\bullet\bullet))\\
[1|3|24, 13|24, 3|1|24] &= ((\bullet\bullet)(\bullet\bullet)\bullet)\\
[1|4|23, 14|23, 4|1|23] &= ((\bullet\bullet)\bullet(\bullet\bullet))\\
[2|4|13, 24|13, 4|2|13] &= (\bullet(\bullet\bullet)(\bullet\bullet))\\
[1|34|2, 134|2, 34|1|2] &= ((\bullet\bullet)(\bullet\bullet\bullet))
\end{aligned}$$

$$\begin{aligned}
&[1|3|2|4, 13|2|4, 3|1|2|4] = (((\bullet\bullet)(\bullet\bullet))\bullet)\\
&[2|4|3|1, 24|3|1, 4|2|3|1] = (\bullet((\bullet\bullet)(\bullet\bullet)))\\
&[1|2|4|3, 1|24|3, 1|4|2|3, 14|2|3, 4|1|2|3] = (((\bullet\bullet)\bullet)(\bullet\bullet))\\
&[1|3|4|2, 13|4|2, 3|1|4|2, 3|14|2, 3|4|1|2] = ((\bullet\bullet)((\bullet\bullet)\bullet))\\
&[1|4|3|2, 14|3|2, 4|1|3|2, 4|13|2, 4|3|1|2] = ((\bullet\bullet)(\bullet(\bullet\bullet)))\\
&[2|1|4|3, 2|14|3, 2|4|1|3, 24|1|3, 4|2|1|3] = ((\bullet(\bullet\bullet))(\bullet\bullet)).
\end{aligned}$$

Elements of the first five classes above correspond to faces and edges that project to an edge; elements of the next six classes correspond to edges and vertices that project to a vertex.

Whereas the correspondence between PRT's and parenthesizations allows us to insert parentheses without regard to order, the analogous correspondence between PLT's and parenthesizations constrains us in the following sense: First, simultaneously insert all level 1 (inner-most) parentheses as indicated by the 1-level meets in M_1; second, simultaneously insert all level 2 parentheses as indicated by the 2-level meets in B, and so on. This procedure suggests a correspondence between partitions $M_1|\cdots|M_{k+1}$ of $\underline{n+1}$ and k-fold compositions of face operators acting on $n+2$ indeterminants, which we now define.

For $s \geq 1$, choose pairs of indices $\{(i_r, \ell_r)\}_{1\leq r\leq s}$ such that $0 \leq i_r < i_{r+1} \leq n$ and $i_r + \ell_r + 1 \leq i_{r+1}$. The face operator

$$d_{(i_1,\ell_1)\cdots(i_s,\ell_s)}$$

acts on $x_1x_2\cdots x_{n+2}$ by inserting s disjoint (non-nested) pairs of parentheses in the following way: The first pair encloses the ℓ_1+1 indeterminants $x_{i_1+1}\cdots x_{i_1+\ell_1+1}$, where $i_1+\ell_1+1 \leq i_2$. The second pair encloses the ℓ_2+1 indeterminants $x_{i_2+1}\cdots x_{i_2+\ell_2+1}$, where $i_2+\ell_2+1 \leq i_3$, and so on. Thus, $d_{(i_1,\ell_1)\cdots(i_s,\ell_s)}$ produces the parenthesization

$$x_1\cdots(x_{i_1+1}\cdots x_{i_1+\ell_1+1})\cdots(x_{i_s+1}\cdots x_{i_s+\ell_s+1})\cdots x_{n+2}.$$

A composition of face operators

$$d_{(0,\ell)}d_{\left(i_1^k,\ell_1^k\right)\cdots\left(i_{s_k}^k,\ell_{s_k}^k\right)}\cdots d_{\left(i_1^1,\ell_1^1\right)\cdots\left(i_{s_1}^1,\ell_{s_1}^1\right)} \tag{2.1}$$

continues this process inductively. If the j^{th} operator has been applied, apply the $(j+1)^{st}$ by treating each pair of parentheses inserted by the j^{th} as a single indeterminant. Since each such composition determines a unique $(k+1)$-leveled tree, the $(n-k)$-faces of P_{n+1} are indexed by compositions of face operators of the form in (2.1) above (see Figure 1). For simplicity we usually suppress the left-most operator $d_{(0,\ell)}$, which inserts a single pair of parentheses enclosing everything.

Conversely, given an operator $d_{\left(i_1^j,\ell_1^j\right)\cdots\left(i_s^j,\ell_s^j\right)}$ with pairs of indices defined as above, let $M_j = \bigcup_{1\leq r\leq s}\left\{i_r^j+1,\ldots,i_r^j+\ell_r^j\right\}$. Then a composition of the form in (2.1) acting on $n+2$ indeterminants corresponds to the partition $M_1|\cdots|M_k$ of $\underline{n+1}$.

Example 2. *Refer to Figure 1 above. The composition* $d_{(0,2)}d_{(0,1)}d_{(1,1)(4,2)}$ *acting on* $x_1\cdots x_7$ *corresponds to the partition* $256|1|34$ *of* $\underline{6}$ *and acts in the following way: First, the operator* $d_{(1,1)(4,2)}$ *inserts two pairs of parentheses*

$$x_1\,(x_2x_3)\,x_4\,(x_5x_6x_7)\,.$$

Note that $\{2,5,6\}$ *decomposes as* $\{2\}\cup\{5,6\}$, *i.e., the union of subsets of consecutive integers with maximal cardinality. Next, the operator* $d_{(0,1)}$ *inserts the single pair*

$$(x_1\,(x_2x_3))\,x_4\,(x_5x_6x_7)\,.$$

Finally, $d_{(0,2)}$ *inserts the single pair*

$$((x_1\,(x_2x_3))\,x_4\,(x_5x_6x_7))\,.$$

We summarize this discussion as a proposition.

Proposition 1. *There exist one-to-one correspondences:*

$$\begin{aligned}\{\textit{Faces of } P_{n+1}\} &\leftrightarrow \{\textit{Leveled trees with } n+2 \text{ leaves}\}\\ &\leftrightarrow \{\textit{Partitions of } \underline{n+1}\}\\ &\leftrightarrow \left\{\begin{array}{c}\textit{Compositions of face operators}\\ \textit{acting on } n+2 \textit{ indeterminants.}\end{array}\right\}\end{aligned}$$

In the discussion that follows, we choose to represent the permutahedra and related polytopes as subdivisions of the n-cube. Although our representations differ somewhat from the classical ones, they are inductively defined and possess the combinatorics we need.

The permutahedron P_{n+1} can be realized as a subdivision of the standard n-cube I^n in the following way: For $\epsilon = 0, 1$ and $1 \leq i \leq n$, let $e_{i,\epsilon}^{n-1}$ denote the $(n-1)$-face $(x_1, \ldots, x_{i-1}, \epsilon, x_{i+1}, \ \ldots, x_n) \subset I^n$. For $0 \leq i \leq j \leq \infty$, let $I_{i,j} = \left[1-2^{-i}, 1-2^{-j}\right] \subset I$, where $2^{-\infty}$ is defined to be 0, and let $\ell_{(k)} = \ell_1 + \cdots + \ell_k$. Set $P_1 = *$ and label this vertex $d_{(0,1)}$. To obtain P_{n+1}, $n \geq 1$, assume that P_n has been constructed. Subdivide and label the $(n-1)$-faces of $P_n \times I$ as indicated below:

Face of P_{n+1}	Label
$e_{n,0}^{n-1}$	$d_{(0,n)}$
$e_{n,1}^{n-1}$	$d_{(n,1)}$
$d_{(i_1,\ell_1)\cdots(i_k,\ell_k)} \times I_{0,n-\ell_{(k)}}$	$d_{(i_1,\ell_1)\cdots(i_k,\ell_k)}$
$d_{(i_1,\ell_1)\cdots(i_k,\ell_k)} \times I_{n-\ell_{(k)},\infty}$	$\begin{cases} d_{(i_1,\ell_1)\cdots(i_k,\ell_k)(n,1)}, & i_k+\ell_k<n \\ d_{(i_1,\ell_1)\cdots(i_k,\ell_k+1)}, & i_k+\ell_k=n \end{cases}$

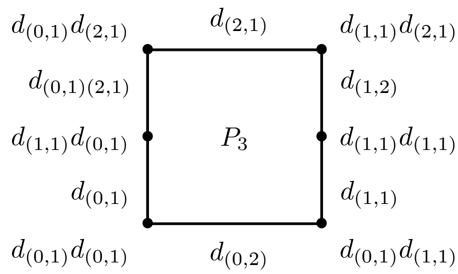


Figure 2: P_3 as a subdivision of $P_2 \times I$.

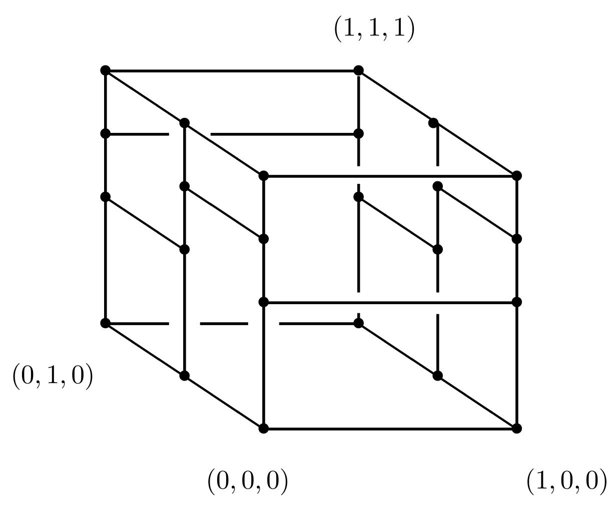


Figure 3a: P_4 as a subdivision of $P_3 \times I$.

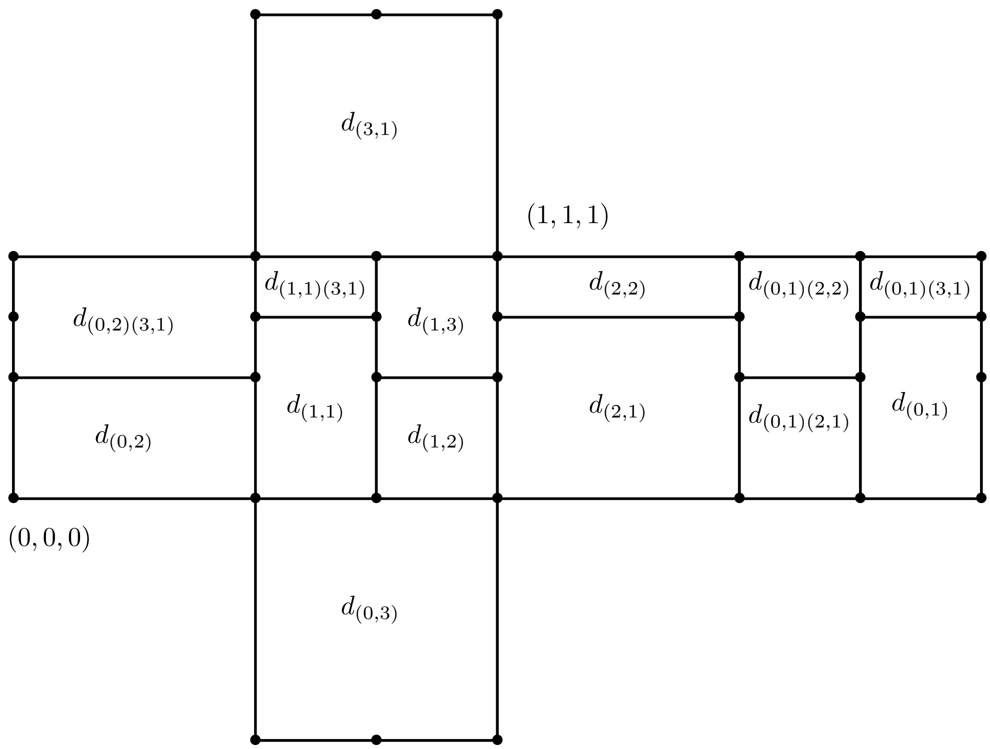


Figure 3b: P_4 as a subdivision of $P_3 \times I$.

The multiplihedra $\{J_{n+1}\}$, which serve as parameter spaces for homotopy multiplicative morphisms of A_∞-algebras, lie between the associahedra and permutahedra (see [18], [8]). If $f^1 : A \to B$ is such a morphism, there is a homotopy f^2 between the quadratic compositions $f^1\varphi_A^2$ and $\varphi_B^2\left(f^1 \otimes f^1\right)$ in two variables, there is a homotopy f^3 bounding the cycle of the six quadratic compositions in three variables involving f^1, f^2, φ_A^2, φ_A^3, φ_B^2 and φ_B^3, and so on. The natural correspondence between faces of J_r and all possible compositions of f^i, φ_A^j and φ_B^k in r-variables (modulo an appropriate equivalence) induces a chain map $C_*(J_r) \to Hom\left(A^{\otimes r}, B\right)$.

The multiplihedron J_{n+1} can also be realized as a subdivision of the cube I^n in the following way: For $n = 0, 1, 2$, set $J_{n+1} = P_{n+1}$. To obtain J_{n+1}, $n \geq 3$, assume that J_n has been constructed. Subdivide and label the $(n-1)$-faces of $J_n \times I$ as indicated below:

Face of J_{n+1}	Label
$e_{n,0}^{n-1}$	$d_{(0,n)}$
$e_{n,1}^{n-1}$	$d_{(n,1)}$
$d_{(0,\ell_1)\cdots(i_k,\ell_k)} \times I_{0,n-k}$	$d_{(0,\ell_1)\cdots(i_k,\ell_k)}$
$d_{(0,\ell_1)\cdots(i_k,\ell_k)} \times I_{n-k,\infty}$	$\begin{cases} d_{(0,\ell_1)\cdots(i_k,\ell_k)(n,1)}, & i_k < n-\ell_k \\ d_{(0,\ell_1)\cdots(i_k,\ell_k+1)}, & i_k = n-\ell_k \end{cases}$
$d_{(i,\ell)} \times I$	$d_{(i,\ell)}, \quad 1 \leq i < n-\ell$
$d_{(i,\ell)} \times I_{0,i}$	$d_{(i,\ell)}, \quad 1 \leq i = n-\ell$
$d_{(i,\ell)} \times I_{i,\infty}$	$d_{(i,\ell+1)}, 1 \leq i = n-\ell$

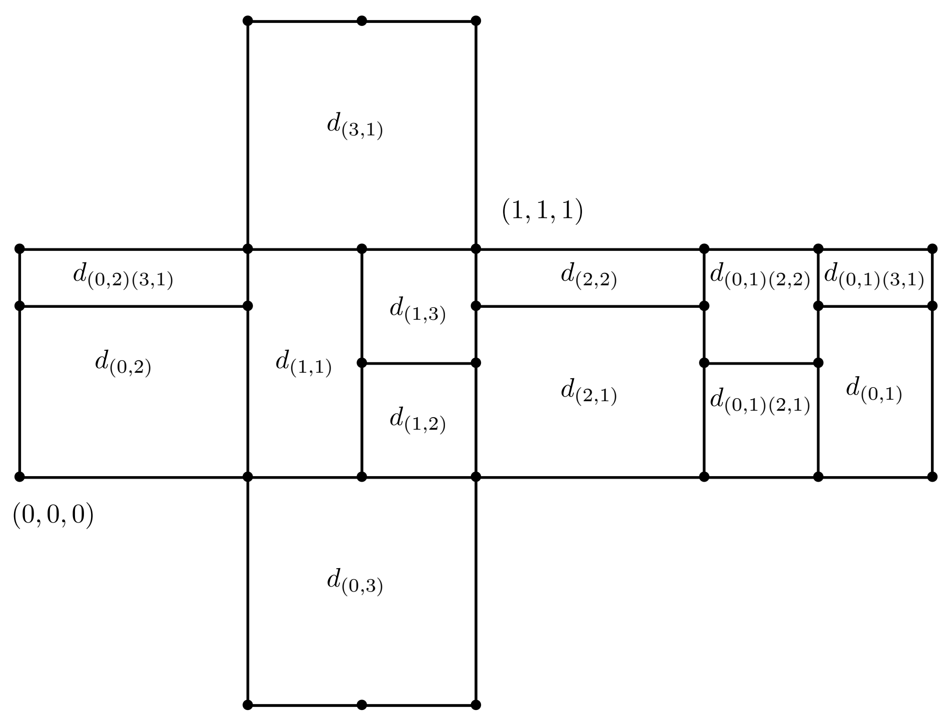


Figure 4: J_4 as a subdivision of $J_3 \times I$.

Thus faces of J_{n+1} are indexed by compositions of face operators of the form

$$d_{(0,\ell)}d_{(i_m,\ell_m)} \cdots d_{(i_{k_1},\ell_{k_1})\cdots(i_{k_s},\ell_{k_s})} \cdots d_{(i_1,\ell_1)}. \tag{2.2}$$

In terms of trees and parenthesizations this says the following: Let T be a $(k+1)$-leveled tree with left-most branch attached at level p. For $1 \leq j < p$, insert level j parentheses one pair at a time without regard to order as in K_{n+2}; next, insert all level p parentheses simultaneously as in P_{n+1}; finally, for $j > p$, insert level j parentheses one pair at a time without regard to order. Thus multiple lower indices in a composition of face operators may only occur when the left-most branch is attached above the root. This suggests the following equivalence relation on the set of $(k+1)$-leveled trees with $n+2$ leaves: Let T and T' be p-leveled trees with $n+2$ nodes whose p-level meets M_p and M'_p contain 1. Then $T \sim T'$ if T and T' are isomorphic as PLT's and $M_p = M'_p$. This equivalence relation induces a cellular projection $\pi : P_{n+1} \to J_{n+1}$ under which J_{n+1} can be realized as an identification space of P_{n+1}. Furthermore, the projection $J_{n+1} \to K_{n+2}$ given by identifying faces of J_{n+1} indexed by isomorphic PLT's gives the following factorization of Tonks' projection:

$$\begin{array}{ccc} P_{n+1} & \xrightarrow{\pi} & J_{n+1} \\ & {\scriptstyle\theta}\searrow & \downarrow \\ & & K_{n+2} \end{array}$$

It is interesting to note the role of the indices ℓ_j in compositions of face operators representing the faces of J_{n+1} as in (2.2). With one exception, each M_j in the corresponding partition $M_1|\cdots|M_{m+1}$ is a set of consecutive integers; this holds without exception for all M_j on K_{n+2}. The exceptional set M_p is a union of s sets of consecutive integers with maximal cardinality, as is typical of sets M_j on P_{n+1}. Thus the combinatorial structure of J_{n+1} exhibits characteristics of both K_{n+2} and P_{n+1}.

In a similar way, the associahedron K_{n+2} can be realized as a subdivision of the cube. For $n = 0, 1$, set $K_{n+2} = P_{n+1}$. To obtain K_{n+2}, $n \geq 2$, assume that K_{n+1} has been constructed. Subdivide and label the $(n-1)$-faces of K_{n+2} as indicated below:

Face of K_{n+2}	**Label**	
$e^{n-1}_{\ell,0}$	$d_{(0,\ell)}$,	$1 \leq \ell \leq n$
$e^{n-1}_{n,1}$	$d_{(n,1)}$,	
$d_{(i,\ell)} \times I$	$d_{(i,\ell)}$,	$1 \leq i < n-\ell$
$d_{(i,\ell)} \times I_{0,i}$	$d_{(i,\ell)}$,	$1 \leq i = n-\ell$
$d_{(i,\ell)} \times I_{i,\infty}$	$d_{(i,\ell+1)}$,	$1 \leq i = n-\ell$

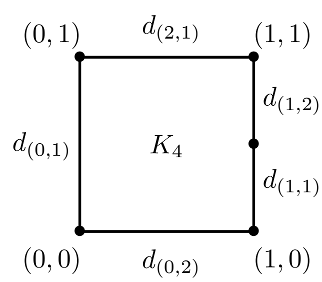


Figure 5: K_4 as a subdivision of $K_3 \times I$.

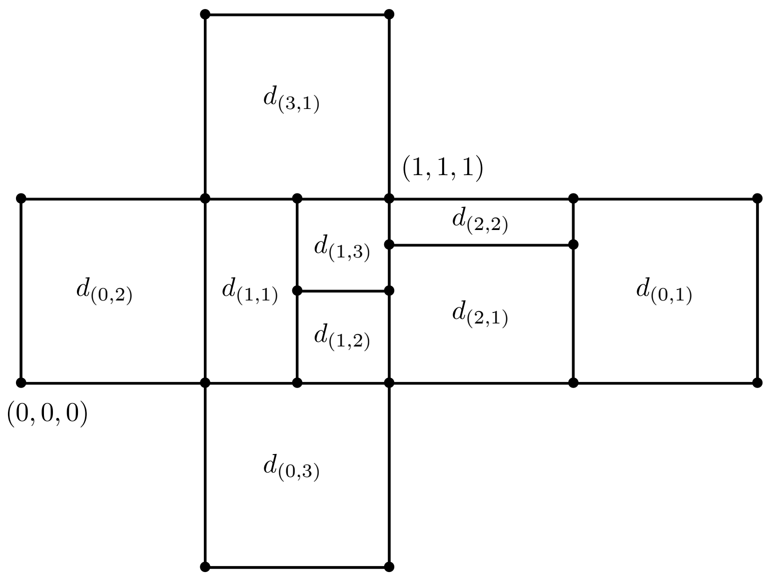


Figure 6: K_5 as a subdivision of $K_4 \times I$.

3. A Diagonal on the Permutahedra

In this section we construct a combinatorial diagonal on the cellular chains of the permutahedron P_{n+1} (up to sign). Issues related to signs will be resolved in Section 6. Given a polytope X, let $(C_*(X), \partial)$ denote the cellular chains on X with boundary ∂.

Definition 1. *A map $\Delta_X : C_*(X) \to C_*(X) \otimes C_*(X)$ is a diagonal on $C_*(X)$ if*

(1) $\Delta_X(C_*(e)) \subseteq C_*(e) \otimes C_*(e)$ *for each cell $e \subseteq X$ and*
(2) $(C_*(X), \Delta_X, \partial)$ *is a dg coalgebra.*

A diagonal Δ_P on $C_*(P_{n+1})$ is unique if the following two additional properties hold:

(1) *The canonical cellular projection $P_{n+1} \to I^n$ induces a dgc map $C_*(P_{n+1}) \to C_*(I^n)$* (see Figures 7 and 8).
(2) *There is a minimal number of components $a \otimes b$ in $\Delta_P(C_k(P_{n+1}))$ for $0 \leq k \leq n$.*

Since the uniqueness of Δ_P is not used in our work, verification of these facts is left to the interested reader.

Certain partitions of $\underline{n+1}$ can be conveniently represented in a tableau, the simplest of which are the "step" tableaux. A step tableau is a matrix whose non-zero entries form a "staircase path" connecting the lower-left and upper-right corners; entries decrease when reading upward along a "riser" and increase when reading from left-to-right along a "tread" (see Example 3 below). More formally,

Definition 2. *A* $q \times p$ *matrix* $E^{q\times p} = (e_{i,j})$ *is a step tableau if and only if*

1. $\{e_{i,j} \in E^{q\times p} \mid e_{i,j} \neq 0\} = \underline{p+q-1}$.
2. $e_{q,1} \neq 0$.
3. *If* $e_{i,j} \neq 0$ *then*
   - (a) $e_{i-1,j-1} = e_{i+1,j+1} = 0$ *and either* $e_{i-1,j} = 0$ *or* $e_{i,j+1} = 0$, *exclusively.*
   - (b) *If* $e_{i,j} = e_{k,\ell}$ *then* $i = k$ *and* $j = \ell$.
   - (c) *If* $e_{i-1,j} = 0$ *then* $e_{i,j} < e_{i,j+1}$.
   - (d) *If* $e_{i,j+1} = 0$ *then* $e_{i-1,j} < e_{i,j}$.

Definition 3. *A partition* $A_1|\cdots|A_p$ *of* $\underline{n}$ *is increasing (resp. decreasing) if* $\min A_j < \max A_{j+1}$ *(resp.* $\max A_{j-1} > \min A_j$*) for all* j.

Given an increasing partition $A_1|\cdots|A_p$ of $\underline{n+1}$, order the elements of each A_j so that $A_j = \left\{a_{1,j} < \cdots < a_{n_j,j}\right\}$; then $\min A_j = a_{1,j}$ and $\max A_j = a_{n_j,j}$. There is a unique $q \times p$ step tableau E associated with $A_1|\cdots|A_p$ determined as follows: Fill in the first column from the bottom row upward, beginning with $a_{n_1,1}$ first, then $a_{n_1-1,1}$, and so on. Inductively, if the $(j-1)^{st}$ column has been filled with $\min A_{j-1}$ in row k, fill in the j^{th} column from row k upward, beginning with $a_{n_j,j}$ first, then $a_{n_j-1,j}$, and so on. This process terminates after p steps with the $q \times p$ step tableau E. For $1 \leq i \leq q$, let $B_i = \{$non-zero entries in row $q-i+1$ of $E\}$; then $B_1|\cdots|B_q$ is a decreasing partition uniquely determined by $A_1|\cdots|A_p$. Similarly, a given decreasing partition $B_1'|\cdots|B_q'$ uniquely determines a $q \times p$ step tableau E' and an increasing partition $A_1'|\cdots|A_p'$ with $A_j' = \{$non-zero entries in column j of $E'\}$.

Definition 4. *Given a step tableau* $E^{q\times p}$, *let* $a = A_1|\cdots|A_p$ *be the increasing partition given by the columns of* E; *let* $b = B_1|\cdots|B_q$ *be the decreasing partition given by the rows of* E. *Then* $a \otimes b$ *is a* (p,q)*-strong complementary pairing (SCP).*

The discussion above establishes the following:

Proposition 2. *There is a one-to-one correspondence*

$$\{\textit{Step tableaux}\} \leftrightarrow \{\textit{Strong complementary pairings}\}.$$

Example 3. *The* $(4,6)$*-SCP* $179|3|48|256 \otimes 9|7|138|46|5|2$ *is associated with the* 6×4 *step tableau*

			2
			5
		4	6
1	3	8	
7			
9			

.

Let $a \otimes b$ be a (p,q)-SCP with $p+q = n+2$. The components of $\Delta_P\left(C_n\left(P_{n+1}\right)\right)$ range over all such SCP's and their "derivatives" produced by right-shift operations on a and left-shift operations on b.

Definition 5. *Let* $a = A_1|\cdots|A_p$ *and* $b = B_1|\cdots|B_q$ *be partitions of* $\underline{n+1}$. *For* $1 \leq i < p$ *and* $1 < j \leq q$, *let* $M \subset A_i$ *and* $N \subset B_j$ *be proper non-empty subsets such that* $\min M > \max A_{i+1}$ *and* $\max B_{j-1} < \min N$. *Define*

$$R_M^i(a) = A_1|\cdots|A_i \setminus M|A_{i+1} \cup M|\cdots|A_p$$

and

$$L_N^j(b) = B_1|\cdots|B_{j-1} \cup N|B_j \setminus N|\cdots|B_q.$$

Also define $R_\varnothing^k = L_\varnothing^k = Id$, *for all* k.

The operators R and L can be thought of as "adjacency" operators on the faces of P_{n+1}. A partition $a = A_1|\cdots|A_p$ of $\underline{n+1}$ corresponds to an $(n-p+1)$-face of P_{n+1}; if M is non-empty, the partition $R_M^i(a)$ corresponds to an $(n-p+1)$-face adjacent to a. Thus, given a partition $a_1 = A_1|\cdots|A_p$ of $\underline{n+1}$, the sequence $\left\{a_1, \ldots, a_{k+1} = R_{M_k}^k(a_k), \ldots, a_p\right\}$ corresponds to a path of adjacent $(n-p+1)$-faces from a_1 to a_p.

Definition 6. *Let $a \otimes b$ be a (p,q)-SCP. A pairing $u \otimes v$ is a (p,q)-complementary pairing (CP) related to $a \otimes b$ if there exist compositions $R_{M_{p-1}}^{p-1}\cdots R_{M_1}^1$ and $L_{N_2}^2\cdots L_{N_q}^q$ such that $u = R_{M_{p-1}}^{p-1}\cdots R_{M_1}^1(a)$ and $v = L_{N_2}^2\cdots L_{N_q}^q(b)$.*

Let $a \otimes b$ be a (p,q)-SCP with associated step tableau $E^{q\times p}$. A related CP $R_{M_{p-1}}^{p-1}\cdots R_{M_1}^1(a) \otimes L_{N_2}^2\cdots L_{N_q}^q(b)$ can be represented by a $q \times p$ tableau derived from $E^{q\times p}$ in the following way: Set $E_1 = E^{q\times p}$. Inductively, for $1 \leq k \leq p-1$ assume that E_k has been obtained. Since $R_{M_k}^k$ is defined, $M_k \subset \{$non-zero entries in column k of $E_k\}$ and either $\min M_k > \max\{$column $k+1$ of $E_k\}$ or $M_k = \varnothing$. Define

$$E_{k+1} = R_{M_k}E_k$$

to be the tableau obtained from E_k by shifting M_k right to column $k+1$ and replacing the cells previously occupied by M_k with zeros. The induction terminates with the tableau

$$E_p = R_{M_{p-1}}\cdots R_{M_1}E.$$

Now continue the induction as follows: For $1 \leq k \leq q-1$, assume that E_{p+k-1} has been obtained. Since $L_{N_k}^k$ is defined, $N_k \subset B_{q-k+1} = \{$non-zero entries in row k of $E_{p+k-1}\}$ and either $\min N_k > \max\{$row $k+1$ of $E_{p+k-1}\}$ or $N_k = \varnothing$. Define

$$E_{p+k} = L_{N_k}E_{p+k-1}$$

to be the tableau obtained from E_{p+k-1} by shifting N_k down to row $k+1$. The induction terminates with the tableau

$$L_{N_{q-1}}\cdots L_{N_1}R_{M_{p-1}}\cdots R_{M_1}E,$$

referred to as a $q \times p$ *derived tableau*, and determines a (p,q)-CP with one of the following forms:

$$\begin{array}{rl} R_{M_{p-1}}\cdots R_{M_1}(a) \otimes L_{N_2}\cdots L_{N_q}(b), & M_i \neq \varnothing \text{ and } N_j \neq \varnothing \text{ for some } i,j \\ R_{M_{p-1}}\cdots R_{M_1}(a) \otimes b, & M_i \neq \varnothing \text{ and } N_j = \varnothing \text{ for some } i, \text{ all } j \\ a \otimes L_{N_2}\cdots L_{N_q}(b), & M_i = \varnothing \text{ and } N_j \neq \varnothing \text{ for all } i, \text{ some } j \\ a \otimes b, & M_i = N_j = \varnothing \text{ for all } i,j. \end{array}$$

For each $n \geq 1$ and $p+q = n+2$, let $\Lambda_{p,q} = \{(p,q)\text{-complementary pairings}\}$. Since a $q \times p$ derived tableau uniquely determines a (p,q)-CP and conversely, we conclude that

Proposition 3. *There is a one-to-one correspondence:*

$$\Lambda_{p,q} \leftrightarrow \{q \times p \text{ derived tableaux}\}.$$

We are ready to define a diagonal on $C_*(P_{n+1})$ up to sign.

Definition 7. *Define $\Delta_P(\underline{1}) = \underline{1} \otimes \underline{1}$. Inductively, assume that $\Delta_P : C_i(P_{i+1}) \to C_*(P_{i+1}) \otimes C_*(P_{i+1})$ has been defined for all $i < n$. For $i = n$, define Δ_P (up to sign) by*

$$\Delta_P(\underline{n+1}) = \sum_{\substack{u \otimes v \in \Lambda_{p,n-p+2} \\ 0 \le p \le n}} u \otimes v. \tag{3.1}$$

Multiplicatively extend Δ_P to all of $C_*(P_{n+1})$ using the fact that each cell of P_{n+1} is a Cartesian product of cells P_{i+1}, $i < n$.

Example 4. *Four tableaux can be derived from the step tableau*

$$E = \begin{array}{|c|c|c|} \hline & 2 & 3 \\ \hline 1 & 5 & \\ \hline 4 & & \\ \hline \end{array} \quad :$$

$$L_\varnothing L_\varnothing R_\varnothing R_\varnothing E = \begin{array}{|c|c|c|} \hline & 2 & 3 \\ \hline 1 & 5 & \\ \hline 4 & & \\ \hline \end{array} \qquad \leftrightarrow \quad 14|25|3 \otimes 4|15|23$$

$$L_\varnothing L_\varnothing R_{\{5\}} R_\varnothing E = \begin{array}{|c|c|c|} \hline & 2 & 3 \\ \hline 1 & & 5 \\ \hline 4 & & \\ \hline \end{array} \qquad \leftrightarrow \quad 14|2|35 \otimes 4|15|23$$

$$L_{\{5\}} L_\varnothing R_\varnothing R_\varnothing E = \begin{array}{|c|c|c|} \hline & 2 & 3 \\ \hline 1 & & \\ \hline 4 & 5 & \\ \hline \end{array} \qquad \leftrightarrow \quad 14|25|3 \otimes 45|1|23$$

$$L_{\{5\}} L_\varnothing R_{\{5\}} R_\varnothing E = \begin{array}{|c|c|c|} \hline & 2 & 3 \\ \hline 1 & & \\ \hline 4 & & 5 \\ \hline \end{array} \qquad \leftrightarrow \quad 14|2|35 \otimes 45|1|23$$

Up to sign, four components of $\Delta_P(\underline{5})$ arise from E, namely,

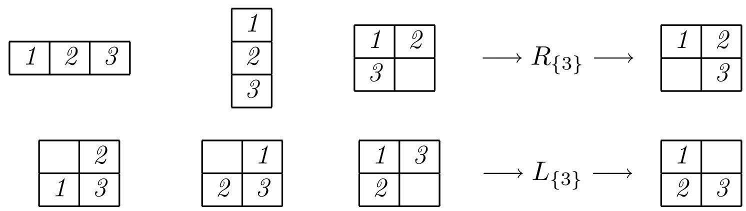
$$(14|2|35 + 14|25|3) \otimes (4|15|23 + 45|1|23).$$

Definition 8. *The transpose of a pairing $A_1|\cdots|A_p \otimes B_1|\cdots|B_q$ is the pairing*

$$(A_1|\cdots|A_p \otimes B_1|\cdots|B_q)^T = B_q|\cdots|B_1 \otimes A_p|\cdots|A_1.$$

Note that the transpose of a step tableau is another step tableau. In fact, $A_1|\cdots|A_p \otimes B_1|\cdots|B_q$ is the (p,q)-SCP associated with the $q \times p$ step tableau E if and only if its transpose $B_q|\cdots|B_1 \otimes A_p|\cdots|A_1$ is the (q,p)-SCP associated with the $p \times q$ step tableau E^T. In general, the transpose of a (p,q)-CP is a (q,p)-CP.

Example 5. *On P_3, all but two of the eight derived tableaux are step tableaux:*



Up to sign,

$$\begin{aligned}\Delta_P(\underline{3}) &= 1|2|3 \otimes 123 + 123 \otimes 3|2|1 \\ &+ 1|23 \otimes 13|2 + 2|13 \otimes 23|1 \\ &+ 13|2 \otimes 3|12 + 12|3 \otimes 2|13 \\ &+ 1|23 \otimes 3|12 + 12|3 \otimes 23|1.\end{aligned}$$

Note that each component in the right-hand column is the transpose of the component to its left.

Example 6. *Up to sign,*

$$\begin{aligned}\Delta_P(\underline{4}) &= 1234 \otimes 4|3|2|1 \\ &+123|4 \otimes (3|2|14 + 3|24|1 + 34|2|1) \\ &+12|34 \otimes (2|14|3 + 24|1|3) \\ &+1|234 \otimes 14|3|2 \\ &+23|14 \otimes (3|24|1 + 34|2|1) \\ &+13|24 \otimes (3|14|2 + 34|1|2) \\ &+(13|24 + 1|234 + 14|23 + 134|2) \otimes 4|3|12 \\ &+(12|34 + 124|3) \otimes (4|2|13 + 4|23|1) \\ &+3|124 \otimes 34|2|1 \\ &+2|134 \otimes (24|3|1 + 4|23|1) \\ &+24|13 \otimes 4|23|1 \\ &+(1|234 + 14|23) \otimes 4|13|2 \\ &+\textit{(all transposes of the above).}\end{aligned}$$

We conclude this section with our main theorem:

Theorem 1. *The cellular boundary map $\partial : C_*(P_{n+1}) \to C_*(P_{n+1})$ is a Δ_P-coderivation for all $n \geq 1$.*

Theorem 1 is an immediate consequence of Lemmas 1 and 2 below. Let $A_1|\cdots|A_p$ be a partition of $\underline{n+1}$. For $1 \leq k \leq p$ and a proper non-empty subset $M \subset A_k$, define a face operator $d_M^k : C_{n-p+1}(P_{n+1}) \to C_{n-p}(P_{n+1})$ by

$$d_M^k(A_1|\cdots|A_p) = A_1|\cdots|A_{k-1}|M|A_k \setminus M|\cdots|A_p;$$

then (up to sign) the cellular boundary $\partial : C_{n-p+1}(P_{n+1}) \to C_{n-p}(P_{n+1})$ decomposes as

$$\partial(A_1|\cdots|A_p) = \sum_{\substack{1\leq k\leq p \\ M\subset A_k}} d_M^k(A_1|\cdots|A_p).$$

In particular, $\partial\left(\underline{n+1}\right) = \sum S_1|S_2$, where $S_2 = \underline{n+1} \setminus S_1$ and ranges over all non-empty proper subsets of $\underline{n+1}$. Thus, $\Delta_P \partial\left(\underline{n+1}\right) = \sum \Delta_P\left(S_1\right)|\Delta_P\left(S_2\right) = \sum\left(u_i \otimes v_j\right)|\left(u^k \otimes v^\ell\right) = \sum u_i|u^k \otimes v_j|v^\ell$, where $u_i \otimes v_j$ and $u^k \otimes v^\ell$ range over all CP's of partitions of S_1 and S_2, respectively. Note that although $u_i|u^k \otimes v_j|v^\ell$ is not a CP, there is the associated block tableau

$$\begin{array}{|c|c|}\hline 0 & E^{\ell\times k} \\ \hline E^{j\times i} & 0 \\ \hline \end{array} \tag{3.2}$$

in which $E^{j\times i}$ and $E^{\ell\times k}$ are the derived tableaux associated with $u_i \otimes v_j$ and $u^k \otimes v^\ell$; in fact, the components of $\Delta_P \partial\left(\underline{n+1}\right)$ lie in one-to-one correspondence with block tableaux of this form. Now consider a particular component $\left(u_i \otimes v_j\right)|\left(u^k \otimes v^\ell\right)$, where $u_i \otimes v_j = U_1|\cdots|U_i \otimes V_1|\cdots|V_j$ is a CP of partitions of S_1 and $u^k \otimes v^\ell = U^1|\cdots|U^k \otimes V^1|\cdots|V^\ell$ is a CP of partitions of S_2. Let $a_i \otimes b_j = A_1|\cdots|A_i \otimes B_1|\cdots|B_j$ and $a^k \otimes b^\ell = A^1|\cdots|A^k \otimes B^1|\cdots|B^\ell$ be the SCP's related to $u_i \otimes v_j$ and $u^k \otimes v^\ell$; then

$$\begin{array}{|c|c|c|}\hline A_1 & \cdots & A_i \\ \hline \end{array} = \begin{array}{|c|}\hline B_j \\ \hline \vdots \\ \hline B_1 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|}\hline A^1 & \cdots & A^k \\ \hline \end{array} = \begin{array}{|c|}\hline B^\ell \\ \hline \vdots \\ \hline B^1 \\ \hline \end{array}$$

are step tableaux involving the elements of S_1 and S_2, respectively, and the block tableau associated with the component $\left(a_i \otimes b_j\right)|\left(a^k \otimes b^\ell\right)$ can be expressed in the following two ways:

$$\begin{array}{|c|c|c|c|c|c|}\hline \multicolumn{3}{|c|}{0} & A^1 & \cdots & A^k \\ \hline A_1 & \cdots & A_i & \multicolumn{3}{|c|}{0} \\ \hline \end{array} = \begin{array}{|c|c|}\hline & B^\ell \\ \cline{2-2} 0 & \vdots \\ \cline{2-2} & B^1 \\ \hline B_j & \\ \cline{1-1} \vdots & 0 \\ \cline{1-1} B_1 & \\ \hline \end{array}.$$

Lemma 1. *Each component of* $\Delta_P \partial\left(\underline{n+1}\right)$ *is also a component of* $\left(1 \otimes \partial + \partial \otimes 1\right) \Delta_P \left(\underline{n+1}\right)$.

Proof. Let $\left(u_i \otimes v_j\right) | \left(u^k \otimes v^\ell\right)$ be a component of $\Delta_P \partial\left(\underline{n+1}\right)$ and consider its related component of SCP's $\left(a_i \otimes b_j\right) | \left(a^k \otimes b^\ell\right)$. There are two cases.

Case 1: *Assume* $\min\left(A_i\right) > \max\left(A^1\right)$.

Then $\min\left(U_i\right) = \min\left(A_i\right) > \max\left(A^1\right) \geq \max\left(U^1\right)$ and

<table>
<tr><td colspan="2">0</td><td>U^1</td><td>$\cdots$</td><td>U^k</td></tr>
<tr><td>U_1</td><td>$\cdots$</td><td>U_i</td><td colspan="2">0</td></tr>
</table>

is the derived tableau associated with the CP $U_1|\cdots|U_i \cup U^1|\cdots|U^k \otimes v_j|v^\ell$ of partitions of $\underline{n+1}$. Hence

$$d^i_{U_i}\left(U_1|\cdots|U_i \cup U^1|\cdots|U^k\right) \otimes v_j|v^\ell = u_i|u^k \otimes v_j|v^\ell$$

is a component of $\left(1 \otimes \partial + \partial \otimes 1\right) \Delta_P \left(\underline{n+1}\right)$.

Case 2: *Assume* $\max\left(B_j\right) < \min\left(B^1\right)$.

Then $\max\left(V_j\right) \leq \max\left(B_j\right) < \min\left(B^1\right) = \min\left(V^1\right)$ and

<table>
<tr><td rowspan="2">0</td><td>V^ℓ</td></tr>
<tr><td>$\vdots$</td></tr>
<tr><td>V_j</td><td>V^1</td></tr>
<tr><td>$\vdots$</td><td rowspan="2">0</td></tr>
<tr><td>V_1</td></tr>
</table>

is the derived tableau associated with the CP $u_i|u^k \otimes V_1|\cdots|V_j \cup V^1|\cdots|V^\ell$ of partitions of $\underline{n+1}$. Again,

$$u_i|u^k \otimes d^j_{V_j}\left(V_1|\cdots|V_j \cup V^1|\cdots|V^\ell\right) = u_i|u^k \otimes v_j|v^\ell$$

is a component of $\left(1 \otimes \partial + \partial \otimes 1\right) \Delta_P \left(\underline{n+1}\right)$ and the conclusion follows. □

Lemma 2. *Each non-vanishing component of* $(1 \otimes \partial + \partial \otimes 1)\,\Delta_P\,(\underline{n+1})$ *is a component of* $\Delta_P \partial\,(\underline{n+1})$*. Specifically, let* $a \otimes b = A_1|\cdots A_p \otimes B_1|\cdots|B_q$ *be a SCP of partitions of* $\underline{n+1}$*, let* $u \otimes v = U_1|\cdots|U_p \otimes V_1|\cdots|V_q$ *be a CP such that* $u = R^{p-1}_{M_{p-1}} \cdots R^1_{M_1}(a)$ *and let* M *be a proper non-empty subset of* U_k*,* $1 \leq k \leq p$*. Then* $d^k_M(u) \otimes v$ *(and dually for* $u \otimes d^k_M(v)$ *with* $M \subset V_q$ *and* $1 \leq k \leq q$*) is a non-vanishing component of* $\Delta_P \partial\,(\underline{n+1})$ *if and only if* $A_k \cap M \neq \varnothing$*,* $\min(U_k \setminus M) < \max A_{k+1}$ *when* $k < p$*,* $\max(U_k \setminus M) < \min M$ *and* $M_k = \varnothing$*.*

Proof. Let $M_0 = M_p = \varnothing$; then clearly, $U_i = (A_i \cup M_{i-1}) \setminus M_i$ for $1 \leq i \leq p$. Under the conditions above, $d^k_M(u) \otimes v$ is a component of

$$\Delta_P(A_1 \cup \cdots \cup A_{k-1} \cup M|(A_k \setminus M) \cup A_{k+1} \cup \cdots \cup A_p).$$

In particular, $a \otimes b = a_{p_1}|a_{p_2} \otimes b_{q_1}|b_{q_2}$ where $a_{p_1} = A_1|\cdots|A_{k-2}|A_{k-1} \cup (A_k \cap M)$ and $a_{p_2} = A_k \setminus M|A_{k+1}|\cdots|A_p$. The dual $u \otimes d^k_M(v)$ follows by "mirror symmetry." Conversely, if the conditions above fail to hold, we exhibit a unique CP $\bar{u} \otimes \bar{v}$ distinct from $u \otimes v$ such that $u \otimes v - \bar{u} \otimes \bar{v} \in \ker(\partial \otimes 1 + 1 \otimes \partial)\,\Delta_P$. For uniqueness, note that if $u \otimes v - \bar{u} \otimes \bar{v} \in \ker(\partial \otimes 1 + 1 \otimes \partial)\,\Delta_P$, then exactly one of the the following relations hold:

(1) $d^k_M(u) = d^\ell_N(\bar{u})$ and $v = \bar{v}$ for some $M \subset U_k$, $N \subset \overline{U}_\ell$, $1 \leq k, \ell \leq p$;

(2) $d^k_M(u) = \bar{u}$ and $v = d^\ell_N(\bar{v})$ for some $M \subset U_k$, $1 \leq k \leq p$, and some $N \subset \overline{V_\ell}$, $1 \leq \ell \leq q-1$;

(3) $u = \bar{u}$ and $d^k_M(v) = d^\ell_N(\bar{v})$ for some $M \subset V_k$, $N \subset \overline{V_\ell}$, $1 \leq k, \ell \leq q$.

For existence, we consider all possible cases.

<u>Case 1</u>: *Assume* $A_k \cap M = \varnothing$.

Then $M \subset M_{k-1} \setminus M_k$ and $M_k \subseteq A_k \cup M_{k-1} \setminus M$. Let

$$\bar{u} = U_1|\cdots|U_{k-1} \cup M|U_k \setminus M|\cdots|U_p;$$

then

$$d^{k-1}_{U_{k-1}}(\bar{u}) = d^k_M(u).$$

Furthermore, since $\min(M_{k-1} \setminus M) \geq \min M_{k-1} > \max A_k$, we may replace $R^{k-1}_{M_{k-1}}$ with $R^{k-1}_{M_{k-1} \setminus M}$ and obtain

$$\bar{u} = R^{p-1}_{M_{p-1}} \cdots R^k_{M_k} R^{k-1}_{M_{k-1} \setminus M} \cdots R^1_{M_1}(a).$$

Therefore $\bar{u} \otimes v$ is a CP related to $a \otimes b$ and $(u - \bar{u}) \otimes v \in \ker(\partial \otimes 1 + 1 \otimes \partial)\,\Delta_P$, i.e., the component $d^k_M(u) \otimes v$ vanishes.

<u>Case 2</u>: *Assume* $A_k \cap M \neq \varnothing$ *and* $\min(U_k \setminus M) > \max A_{k+1}$ *when* $k < p$.

Let

$$\bar{u} = U_1|\cdots|U_{k-1}|M|\,(U_k \setminus M) \cup U_{k+1}|U_{k+2}|\cdots|U_p;$$

then

$$d^{k+1}_{U_k \setminus M}(\bar{u}) = d^k_M(u).$$

Furthermore, $\min[(U_k \setminus M) \cup M_k] > \max A_{k+1}$ by assumption and the fact that $\min M_k > \max A_{k+1}$. Hence we may replace $R^k_{M_k}$ with $R^k_{M_k \cup (U_k \setminus M)}$ and obtain

$$\bar{u} = R^{p-1}_{M_{p-1}} \cdots R^{k+1}_{M_{k+1}} R^k_{M_k \cup (U_k \setminus M)} \cdots R^1_{M_1}(a)$$

so that $\bar{u} \otimes v$ is a CP related to $a \otimes b$. Again, $(u - \bar{u}) \otimes v \in \ker(\partial \otimes 1 + 1 \otimes \partial)\Delta_P$.

<u>Case 3</u>: *Assume* $A_k \cap M \neq \varnothing$, $\min(U_k \setminus M) < \max A_{k+1}$ *when* $k < p$, *and either* $\min M < \max(U_k \setminus M)$ *or* $M_k \neq \varnothing$.

Let

$$\mu = \min\{x \in (A_k \cup M_{k-1}) \setminus M \mid x > \min M\}.$$

<u>Subcase 3A</u>: *Assume* $\mu \notin A_k \setminus M$.

Then either $\mu \in A_i \cap (U_k \setminus M)$ for some $i < k$ or $\mu \in A_i \cap U_j$ for some $i \leq k$ and some $j > k$.

<u>Subcase 3A1</u>: *Assume* $\min A_{i-1} > \max(A_i \setminus \mu)$ *with* $1 < i \leq k$.

First consider $1 < i < k$. Then $\mu = \max A_i > \max A_k$ since $\mu \in U_j$ with $j > k$, and it follows that $A_1|\cdots|A_{i-1} \cup A_i \setminus \mu|A_{i+1}|\cdots|A_k \cup \mu|A_{k+1}|\cdots|A_p$ is increasing. Now $\min(A_k \cap M) < \max[(A_k \setminus M) \cup \mu]$ and furthermore $\max A_k \in M$ (if not, either $\max A_k \in U_k \setminus M$, which is impossible since $\mu \notin A_k$, or $\max A_k \in M_k$, which is also impossible since this would imply $\max A_k \geq \min M_k > \max A_{k+1} > \min(U_k \setminus M) \geq \mu = \max A_i > \max A_k$). Thus $\min A_{k-1} < \max A_k = \max(A_k \cap M)$ and

$$\bar{a} = A_1|\cdots|A_{i-1} \cup A_i \setminus \mu|A_{i+1}|\cdots|A_k \cap M|\,(A_k \setminus M) \cup \mu|A_{k+1}|\cdots|A_p$$

is increasing and uniquely determines a SCP $\bar{a} \otimes \bar{b}$. Let

$$\bar{u} = U_1|\cdots|U_{i-1} \cup U_i|U_{i+1}|\cdots|M|U_k \setminus M|U_{k+1}|\cdots|U_p;$$

note that $\mu \notin M \cup M_j$ and $\mu \in M_s$ for $i \leq s \leq j-1$. Thus

$$\bar{u} = R^{p-1}_{M_{p-1}} \cdots R^k_{M_k} R^{k-1}_{M_{k-1} \setminus \mu \setminus M} R^{k-2}_{M_{k-1} \setminus \mu} \cdots R^{i-1}_{M_i \setminus \mu} R^{i-2}_{M_{i-2}} \cdots R^1_{M_1}(\bar{a})$$

and $\bar{u} \otimes v$ is a CP related to $\bar{a} \otimes \bar{b}$. Finally,

$$d^k_M(u) = d^{i-1}_{U_{i-1}}(\bar{u})$$

so that $(u - \bar{u}) \otimes v \in \ker(\partial \otimes 1 + 1 \otimes \partial)\Delta_P$. On the other hand, if $1 < i = k$, set

$$\bar{a} = A_1|\cdots|A_{k-1} \cup (A_k \cap M)|A_k \setminus M|A_{k+1}|\cdots|A_p$$

and

$$\bar{u}_p = U_1|\cdots|U_{k-1} \cup M|U_k \setminus M|U_{k+1}|\cdots|U_p.$$

<u>Subcase 3A2</u>: *Assume* $\min A_{i-1} < \max(A_i \setminus \mu)$ *with* $1 \leq i \leq k$.

Again, first consider $1 \leq i < k$. Let

$$\bar{a} = A_1|...|A_{i-1}|A_i \setminus \mu|A_{i+1}|...|A_k \cap M|(A_k \setminus M) \cup \mu|A_{k+1}|...|A_p$$

and

$$\bar{u} = d^k_M(u) = U_1|\cdots|M|U_k \setminus M|U_{k+1}|\cdots|U_p.$$

By the argument in Subcase 3A1, $\bar{a}$ is increasing and uniquely determines a SCP $\bar{a}\otimes\bar{b}$. Furthermore, since $\min M_{k-1} = \min M_{k-1}\setminus M = \mu > \max A_k$, both $R^{k-1}_{M_{k-1}\setminus\mu}$ and $R^{k}_{(M_{k-1}\setminus\mu)\setminus M}$ are defined and we have

$$\bar{u} = R^{p-1}_{M_{p-1}}\cdots R^{k+1}_{M_k}R^{k}_{(M_{k-1}\setminus\mu)\setminus M}R^{k-1}_{M_{k-1}\setminus\mu}\cdots R^{i}_{M_i\setminus\mu}R^{i-1}_{M_{i-1}}\cdots R^{1}_{M_1}\left(\bar{a}\right).$$

Choose r such that $\mu\in B_r$ and let

$$\bar{v} = V_1|\cdots|V_{r-1}|V_r\cup V_{r+1}|V_{r+2}|\cdots|V_q.$$

Then $d^{r}_{V_r}(\bar{v}) = v$ and $\bar{u}\otimes\bar{v}$ is a CP related to $\bar{a}\otimes\bar{b}$. Therefore $u\otimes v - \bar{u}\otimes\bar{v}\in\ker\left(\partial\otimes 1 + 1\otimes\partial\right)\Delta_P$. Finally, when $i = k$, define $\bar{v}_q$ as above and set

$$\bar{a} = A_1|\cdots|A_k\cap M|A_k\setminus M|A_{k+1}|\cdots|A_p.$$

Subcase 3B: *Assume* $\mu\in A_k\setminus M$.

Note that $\min M < \mu\leq\max\left(U_k\setminus M\right)$ whether or not $M_k = \varnothing$.

Subcase 3B1: *Assume either* $\mu < \max A_k$ *or* $\min A_{k-1} < \max(A_k\setminus\mu)$.

Note that when $k = 1$, $\mu\in A_1\setminus M$. Let

$$D_k = \{x\in A_k\setminus M \mid x\leq\mu\}.$$

If $\mu < \max A_k$, then $\max\left(A_k\setminus D_k\right) = \max A_k$. On the other hand, if $\mu = \max A_k$, then $\max\left(A_k\setminus\mu\right)\notin D_k$ (otherwise $\max\left(A_k\setminus\mu\right)\in\left(U_k\setminus M\right)\cup\left(A_k\cup U_j\right)$ for some $j > k$, and $\min M < \max\left(A_k\setminus\mu\right) < \mu$ contradicting the choice of μ) and it follows that $\max\left(A_k\setminus D_k\right) = \max\left(A_k\setminus\mu\right)$. In either case, $\min A_{k-1} < \max(A_k\setminus D_k)$ by assumption, and furthermore, $\min(A_k\setminus D_k) = \min\left(A_k\cap M\right) = \min M < \mu = \max D_k$. Finally, either $\min D_k = \min A_k$ (when $\min A_k < \min M$) or $\min D_k = \mu = \min(U_k\setminus M)$ (when $\min A_k = \min M$), and in either case $\min D_k < \max A_{k+1}$. Let

$$\bar{a} = A_1|\cdots|A_{k-1}|A_k\setminus D_k|D_k|A_{k+1}|\cdots|A_p.$$

Then $\bar{a}$ is increasing and uniquely determines a SCP $\bar{a}\otimes\bar{b}$. Since $\min\{[(A_k\setminus D_k)\cup M_{k-1}\setminus M]\} > \mu = \max D_k$, the operator $R^{k}_{((A_k\setminus D_k)\cup M_{k-1})\setminus M}$ is defined and we have

$$R^{p}_{M_{p-1}}\cdots R^{k+1}_{M_k}R^{k}_{[(A_k\setminus D_k)\cup M_{k-1}]\setminus M}R^{k-1}_{M_{k-1}}\cdots R^{1}_{M_1}\left(\bar{a}\right) = d^{k}_{M}(u).$$

Choose r such that $\mu\in B_r$ and let

$$\bar{v} = V_1|\cdots|V_{r-1}|V_r\cup V_{r+1}|V_{r+2}|\cdots|V_q.$$

Then $d^{r}_{V_r}(\bar{v}) = v$ and $\bar{u}\otimes\bar{v} = d^{k}_{M}(u)\otimes\bar{v}$ is a CP related to $\bar{a}\otimes\bar{b}$. Therefore $u\otimes v - \bar{u}\otimes\bar{v}\in\ker\left(\partial\otimes 1 + 1\otimes\partial\right)\Delta_P$.

Subcase 3B2: *Assume* $\mu = \max A_k$ *and* $\min A_{k-1} > \max(A_k\setminus\mu)$, $1 < k\leq p$.

Let

$$\bar{u} = U_1|\cdots|U_{k-1}\cup M|U_k\setminus M|\cdots|U_p;$$

then

$$d^{k-1}_{U_{k-1}}(\bar{u}) = d^{k}_{M}(u).$$

Let

$$\bar{a} = A_1|\cdots|A_{k-1}\cup\left(A_k\cap M\right)|A_k\setminus M|A_{k+1}|\cdots|A_p.$$

We have that $\min(A_{k-1} \cup (A_k \cap M)) = \min A_{k-1} < \max A_k = \max A_k \setminus M$. Also $\min(A_k \setminus M) \leq \min(U_k \setminus M) < \max A_{k+1}$. Then $\bar{a}$ is increasing and uniquely determines a decreasing partition $\bar{b}$ such that $\bar{a} \otimes \bar{b}$ is a SCP. In particular, b can be obtained from $\bar{b}$ by successively shifting μ to the left. Since $\min(M_{k-1} \setminus M) \geq \min M_{k-1} > \max A_k$, the operator $R^{k-1}_{M_{k-1} \setminus M}$ is defined and we obtain

$$\bar{u} = R^{p-1}_{M_{p-1}} \cdots R^{k}_{M_k} R^{k-1}_{M_{k-1} \setminus M} R^{k-2}_{M_{k-2}} \cdots R^{1}_{M_1} (\bar{a}).$$

Then $\bar{u} \otimes v$ is a CP related to $\bar{a} \otimes \bar{b}$. Therefore $(u - \bar{u}) \otimes v \in \ker (\partial \otimes 1 + 1 \otimes \partial) \Delta_P$.

By "mirror symmetry" the case of $u \otimes d^r(v)$ is entirely analogous and the proof is complete. □

4. Diagonals on the Associahedra and Multiplihedra

The diagonal Δ_P on $C_*(P_{n+1})$ descends to diagonals Δ_J on $C_*(J_{n+1})$ and Δ_K on $C_*(K_{n+2})$ under the cellular projections $\pi : P_{n+1} \to J_{n+1}$ and $\theta : P_{n+1} \to K_{n+2}$ discussed in Section 2 above. This fact is an immediate consequence of Proposition 5.

Definition 9. *Let $f : W \to X$ be a cellular map of CW-complexes, let Δ_W be a diagonal on $C_*(W)$ and let $X^{(r)}$ denote the r-skeleton of X. A k-cell $e \subseteq W$ is degenerate under f if $f(e) \subseteq X^{(r)}$ with $r < k$. A component $a \otimes b$ of Δ_W is degenerate under f if either a or b is degenerate under f.*

Let us identify the non-degenerate cells of P_{n+1} under π and θ.

Definition 10. *Let $A_1|\cdots|A_p$ be a partition of $\underline{n+1}$ with $p > 1$ and let $1 \leq k < p$. The subset A_k is exceptional if for $k < j \leq p$, there is an element $a_{i,j} \in A_j$ such that $\min A_k < a_{i,j} < \max A_k$.*

Proposition 4. *Given a face e of P_{n+1}, consider its unique representation as a partition $A_1|\cdots|A_p$ of $\underline{n+1}$ or as a composition of face operators*

$$d_{(0,\ell)} d_{(i_1^{p-1},\ell_1^{p-1})\cdots(i_{s_{p-1}}^{p-1},\ell_{s_{p-1}}^{p-1})} \cdots d_{(i_1^1,\ell_1^1)\cdots(i_{s_1}^1,\ell_{s_1}^1)}(e^n).$$

(1) The following are all equivalent:

(1a) The face e is degenerate under π.

(1b) $\min A_j > \min (A_{j+1} \cup \cdots \cup A_p)$ with A_j exceptional for some $j < p$.

(1c) $i_1^k > 0$ and $s_k > 1$ for some $k < p$.

(2) The following are all equivalent:

(2a) The face e is degenerate under θ.

(2b) A_j is exceptional for some $j < p$.

(2c) $s_k > 1$ for some $k < p$.

Proof. Obvious. □

Example 7. *The subset $A_1 = \{13\}$ in the partition $13|24$ is exceptional; the face e of P_4 corresponding to $13|24$ is degenerate under θ. In terms of compositions of* *face operators, e corresponds to* $d_{(0,2)}d_{(0,1)(2,1)}\,(\underline{4})$ *with* $s_1 = 2$. *Furthermore, since* $i_1^1 = 0$ *(or equivalently* $\min A_1 < \min A_2$*) the face e is non-degenerate under* π.

Next, we apply Tonks' projection and obtain an explicit formula for the diagonal Δ_K on the associahedra.

Proposition 5. *Let* $f : W \to X$ *be a surjective cellular map and let* Δ_W *be a diagonal on* $C_*(W)$. *Then* Δ_W *uniquely determines a diagonal* Δ_X *on* $C_*(X)$ *consisting of the non-degenerate components of* Δ_W *under f. Moreover,* Δ_X *is the unique map that commutes the following diagram:*

$$\begin{array}{ccc} C_*(W) & \overset{\Delta_W}{\longrightarrow} & C_*(W)\otimes C_*(W) \\ f\downarrow & & \downarrow f\otimes f \\ C_*(X) & \underset{\Delta_X}{\longrightarrow} & C_*(X)\otimes C_*(X). \end{array}$$

Proof. Obvious. □

In Section 2 we established correspondences between faces of the associahedron K_{n+2} and PRT's with $n+2$ leaves and between faces of the permutahedron P_{n+1} and PLT's with $n+2$ leaves. Consequently, a face of K_{n+2} can be viewed as a face of P_{n+1} by viewing the corresponding PRT as a PLT.

Definition 11. *For* $n \geq 0$, *let* Δ_P *be the diagonal on* $C_*(P_{n+1})$ *and let* $\theta : P_{n+1} \to K_{n+2}$ *be Tonks' projection. View each face e of the associahedron* K_{n+2} *as a face of* P_{n+1} *and define* $\Delta_K : C_*(K_{n+2}) \to C_*(K_{n+2}) \otimes C_*(K_{n+2})$ *by*

$$\Delta_K(e) = (\theta\otimes\theta)\Delta_P(e).$$

Corollary 1. *The map* Δ_K *given by Definition 11 is the diagonal on* $C_*(K_{n+1})$ *uniquely determined by* Δ_P.

Proof. This is an immediate application of Proposition 5. □

We note that both components of a CP $u \otimes v$ are non-degenerate under θ if and only if $u \otimes v$ is associated with a SCP $a \otimes b$ such that b is non-degenerate and $u = R^{p-1}_{M_{p-1}} \cdots R^1_1(a)$ with each M_j having maximal cardinality.

Choose a system of generators $e^n \in C_n(K_{n+2})$, $n \geq 0$. The signs in formula (4.1) below follow from formula (6.12) in Section 6.

Definition 12. *Define* $\Delta_K\left(e^0\right) = e^0 \otimes e^0$. *Inductively, assume that* $\Delta_K : C_*(K_{i+2}) \to C_*(K_{i+2}) \otimes C_*(K_{i+2})$ *has been defined for all* $i < n$ *and define*

(4.1)
$$\Delta_K(e^n) = \sum_{0\leq p\leq p+q=n+2} (-1)^{\epsilon}\ d_{(i_{p-1},\ell_{p-1})}\cdots d_{(i_1,\ell_1)}\otimes d_{(i'_{q-1},\ell'_{q-1})}\cdots d_{(i'_1,\ell'_1)}\left(e^n\otimes e^n\right),$$

where

$$\epsilon = \sum_{j=1}^{p-1} i_j(\ell_j+1) + \sum_{k=1}^{q-1}(i'_k+k+q)\ell'_k,$$

and lower indices $\left((i_1,\ell_1),\ldots,(i_{p-1},\ell_{p-1});(i'_1,\ell'_1),\ldots,\left(i'_{q-1},\ell'_{q-1}\right)\right)$ *range over all solutions of the following system of inequalities:*

$$(4.2)\qquad \left\{\begin{array}{ll} 1 \leq i'_j < i'_{j-1} \leq n+1 & (1) \\ 1 \leq \ell'_j \leq n+1-i'_j-\ell'_{(j-1)} & (2) \\ 0 \leq i_k \leq \min\limits_{o'(t_k)<r<k} \left\{ i_r,\ i'_{t_k}-\ell_{(o'(t_k))} \right\} & (3) \\ 1 \leq \ell_k = \epsilon_k - i_k - \ell_{(k-1)}, & (4) \end{array}\right\}_{\substack{1\leq k\leq p-1 \\ 1\leq j\leq q-1}}$$

where

$$\left\{\epsilon_1 < \cdots < \epsilon_{q-1}\right\} = \{1,\ldots,n\} \setminus \left\{i'_1,\ldots,i'_{q-1}\right\};$$

$$\epsilon_0 = \ell_0 = \ell'_0 = i_p = i'_q = 0;$$

$$i_0 = i'_0 = \epsilon_q = \ell_{(p)} = \ell'_{(q)} = n+1;$$

$$\ell_{(u)} = \textstyle\sum_{j=0}^{u} \ell_j \text{ for } 0 \leq u \leq p;$$

$$\ell'_{(u)} = \textstyle\sum_{k=0}^{u} \ell'_k \text{ for } 0 \leq u \leq q;$$

$$t_u = \min\left\{ r \mid i'_r + \ell'_{(r)} - \ell'_{(o(u))} > \epsilon_u > i'_r \right\};$$

$$o\left(u\right) = \max\left\{r \mid i'_r \geq \epsilon_u\right\}; \text{ and}$$

$$o'\left(u\right) = \max\left\{r \mid \epsilon_r \leq i'_u\right\}.$$

Multiplicatively extend Δ_K to all of $C_*(K_{n+2})$ using the fact that each cell of K_{n+2} is a product of cells K_{i+2} with $i < n$.

Theorem 2. *The map Δ_K given by Definition 12 is the diagonal induced by θ.*

Proof. If $v = L_\beta(v')$ is non-degenerate in some component $u \otimes v$ of Δ_P, then so is v', and we immediately obtain inequality (1) of (4.2). Next, each non-degenerate decreasing b uniquely determines a SCP $a \otimes b$. Although a may be degenerate, there is a unique non-degenerate $u = R_{M_{p-1}} \cdots R_{M_1}(a)$ obtained by choosing each M_j with maximal cardinality (the case $M_j = \varnothing$ for all j may nevertheless occur); then $u \otimes b$ is a non-degenerate CP associated with $a \otimes b$ in Δ_P. As a composition of face operators, straightforward examination shows that u has form $u = d_{(i_{p-1},\ell_{p-1})} \cdots d_{(i_1,\ell_1)}(e^n)$ and is related to $b = d_{(i'_{q-1},\ell'_{q-1})} \cdots d_{(i'_1,\ell'_1)}\left(e^n\right)$ by

$$i_k = \min_{o'(t_k)<r<k} \left\{ i_r,\ i'_{t_k} - \ell_{(o'(t_k))} \right\},\ 1 \leq k < p;$$

and equality holds in (4) of (4.2). Finally, let $b = L_\beta(\bar{b})$. As we vary $\bar{b}$ in all possible ways, each $\bar{b}$ is non-degenerate and we obtain all possible non-degenerate CP's $\bar{u} \otimes \bar{b}$ associated with $\bar{a} \otimes \bar{b}$ ($\bar{u} = u$ when $\bar{b} = b$ and $\beta = \varnothing$). For each such $\bar{u} = d_{(i_{p-1},\ell_{p-1})} \cdots d_{(i_1,\ell_1)}(e^n)$ we have both inequality (3) and equality in (4) of (4.2). Hence, the theorem is proved. $\square$

Example 8. *On K_4 we obtain:*

$$\Delta_K\left(e^2\right) = \left\{ d_{(0,1)} d_{(0,1)} \otimes 1 + 1 \otimes d_{(1,1)} d_{(2,1)} + d_{(0,2)} \otimes d_{(1,1)} \right.$$
$$\left. + d_{(0,2)} \otimes d_{(1,2)} + d_{(1,1)} \otimes d_{(1,2)} - d_{(0,1)} \otimes d_{(2,1)} \right\} (e^2 \otimes e^2).$$

5. Tensor Products of A_∞-(co)algebras: An Application of Δ_K

In this section, we use Δ_K to define the tensor product of A_∞-(co)algebras in maximal generality. We note that a special case was given by J. Smith [17] for certain objects with a richer structure than we have here. We also mention that Lada and Markl [12] defined a A_∞ tensor product structure on a construct different from the tensor product of graded modules.

We adopt the following notation and conventions: The symbol R denotes a commutative ring with unity; all R-modules are assumed to be $\mathbb{Z}$-graded. The reduced R-module V/V_0 of a connected V is denoted by $\overline{V}$. All tensor products and *Hom*'s are defined over R and all maps are R-module maps unless indicated otherwise. The symbol $1 : V \to V$ denotes the identity map; the suspension and desuspension maps, which shift dimension by $+1$ and -1, are denoted by $\uparrow$ and $\downarrow$, respectively. We let $V^{\otimes n} = V \otimes \cdots \otimes V$ with $n > 0$ factors and define $V^{\otimes 0} = R$; then $TV = \oplus_{n\geq 0} V^{\otimes n}$ and $T^a V$ (respectively, $T^c V$) denotes the free tensor algebra (respectively, cofree tensor coalgebra) of V. Given R-modules $V_1, \ldots, V_n$ and a permutation $\sigma \in S_n$, define the *permutation isomorphism* $\sigma : V_1 \otimes \cdots \otimes V_n \to V_{\sigma^{-1}(1)} \otimes \cdots \otimes V_{\sigma^{-1}(n)}$ by $\sigma(x_1 \cdots x_n) = \pm\, x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(n)}$, where the sign is determined by the Mac Lane commutation rule [14] to which we strictly adhere. In particular, $\sigma_m : (V_1 \otimes V_2)^{\otimes m} \to V_1^{\otimes m} \otimes V_2^{\otimes m}$ is the permutation isomorphism induced by $\sigma_m = (1\ 3\ \cdots\ (2m-1)\ 2\ 4\ \cdots\ 2m)$. If $f : V^{\otimes p} \to V^{\otimes q}$ is a map, we let $f_{i,n-p-i} = 1^{\otimes i} \otimes f \otimes 1^{\otimes n-p-i} : V^{\otimes n} \to V^{\otimes n-p+q}$, where $0 \leq i \leq n-p$. The abbreviations *dgm, dga,*and *dgc* stand for *differential graded R-module, dg R-algebra* and *dg R-coalgebra*, respectively.

We begin with a review of A_∞-(co)algebras paying particular attention to the signs. Let A be a connected R-module equipped with operations $\{\varphi^k \in Hom^{k-2}\left(A^{\otimes k}, A\right)\}_{k\geq 1}$. For each k and $n \geq 1$, linearly extend φ^k to $A^{\otimes n}$ via

$$\sum_{i=0}^{n-k} \varphi^k_{i,n-k-i} : A^{\otimes n} \to A^{\otimes n-k+1},$$

and consider the induced map of degree -1 given by

$$\sum_{i=0}^{n-k} \left(\uparrow \varphi^k \downarrow^{\otimes k}\right)_{i,n-k-i} : \left(\uparrow \overline{A}\right)^{\otimes n} \to \left(\uparrow \overline{A}\right)^{\otimes n-k+1}.$$

Let $\widetilde{B}A = T^c\left(\uparrow \overline{A}\right)$ and define a map $d_{\widetilde{B}A} : \widetilde{B}A \to \widetilde{B}A$ of degree -1 by

$$d_{\widetilde{B}A} = \sum_{\substack{1\leq k\leq n\\ 0\leq i\leq n-k}} \left(\uparrow \varphi^k \downarrow^{\otimes k}\right)_{i,n-k-i}. \tag{5.1}$$

The identities $(-1)^{[n/2]} \uparrow^{\otimes n}\downarrow^{\otimes n} = 1^{\otimes n}$ and $[n/2] + [(n+k)/2] \equiv nk + [k/2] \pmod 2$ imply that

$$d_{\widetilde{B}A} = \sum_{\substack{1\leq k\leq n\\ 0\leq i\leq n-k}} (-1)^{[(n-k)/2]+i(k+1)} \uparrow^{\otimes n-k+1} \varphi^k_{i,n-k-i} \downarrow^{\otimes n}. \tag{5.2}$$

Definition 13. $(A, \varphi^n)_{n\geq 1}$ *is an* A_∞*-algebra if* $d^2_{\widetilde{B}A} = 0$.

Proposition 6. *For each $n \geq 1$, the operations $\{\varphi^n\}$ on an A_∞-algebra satisfy the following quadratic relations:*

$$\sum_{\substack{0\leq\ell\leq n-1\\0\leq i\leq n-\ell-1}} (-1)^{\ell(i+1)}\, \varphi^{n-\ell}\varphi^{\ell+1}_{i,n-\ell-1-i} = 0. \tag{5.3}$$

Proof. For $n \geq 1$,

$$\begin{aligned} 0 &= \sum_{\substack{1\leq k\leq n\\0\leq i\leq n-k}} (-1)^{[(n-k)/2]+i(k+1)} \uparrow \varphi^{n-k+1} \downarrow^{\otimes n-k+1}\uparrow^{\otimes n-k+1} \varphi^k_{i,n-k-i} \downarrow^{\otimes n} \\ &= \sum_{\substack{1\leq k\leq n\\0\leq i\leq n-k}} (-1)^{n-k+i(k+1)}\, \varphi^{n-k-1}\varphi^k_{i,n-k-i} \\ &= -(-1)^n \sum_{\substack{0\leq\ell\leq n-1\\0\leq i\leq n-\ell-1}} (-1)^{\ell(i+1)}\, \varphi^{n-\ell}\varphi^{\ell+1}_{i,n-\ell-1-i}. \end{aligned}$$

□

It is easy to prove that

Proposition 7. *Given an A_∞-algebra $(A,\varphi^n)_{n\geq 1}$, $\left(\widetilde{B}A, d_{\widetilde{B}A}\right)$ is a dgc.*

Definition 14. *Let $(A,\varphi^n)_{n\geq 1}$ be an A_∞-algebra. The tilde bar construction on A is the dgc $\left(\widetilde{B}A, d_{\widetilde{B}A}\right)$.*

Definition 15. *Let A and C be A_∞-algebras. A chain map $f = f^1 : A \to C$ is a map of A_∞-algebras if there exists a sequence of maps $\{f^k \in Hom^{k-1}\left(A^{\otimes k}, C\right)\}_{k\geq 2}$ such that*

$$\widetilde{f} = \textstyle\sum_{n\geq 1}\left(\sum_{k\geq 1} \uparrow f^k \downarrow^{\otimes k}\right)^{\otimes n} : \widetilde{B}A \to \widetilde{B}C$$

is a dgc map.

Dually, consider a sequence of operations $\{\psi^k \in Hom^{k-2}\left(A, A^{\otimes k}\right)\}_{k\geq 1}$. For each k and $n \geq 1$, linearly extend each ψ^k to $A^{\otimes n}$ via

$$\sum_{i=0}^{n-1} \psi^k_{i,n-1-i} : A^{\otimes n} \to A^{\otimes n+k-1},$$

and consider the induced map of degree -1 given by

$$\sum_{i=0}^{n-1} \left(\downarrow^{\otimes k} \psi^k \uparrow\right)_{i,n-1-i} : \left(\downarrow \overline{A}\right)^{\otimes n} \to \left(\downarrow \overline{A}\right)^{\otimes n+k-1}.$$

Let $\widetilde{\Omega}A = T^a\left(\downarrow \overline{A}\right)$ and define a map $d_{\widetilde{\Omega}A} : \widetilde{\Omega}A \to \widetilde{\Omega}A$ of degree -1 by

$$d_{\widetilde{\Omega}A} = \sum_{\substack{n,k\geq 1\\0\leq i\leq n-1}} \left(\downarrow^{\otimes k} \psi^k \uparrow\right)_{i,n-1-i},$$

which can be rewritten as

$$d_{\widetilde{\Omega}A} = \sum_{\substack{n,k\geq 1\\0\leq i\leq n-1}} (-1)^{[n/2]+i(k+1)+k(n+1)} \downarrow^{\otimes n+k-1} \psi^k_{i,n-1-i} \uparrow^{\otimes n}. \tag{5.4}$$

Definition 16. $(A,\psi^n)_{n\geq 1}$ *is an* A_∞*-coalgebra if* $d^2_{\widetilde{\Omega}A}=0$.

Proposition 8. *For each* $n\geq 1$*, the operations* $\{\psi^k\}$ *on an* A_∞*-coalgebra satisfy the following quadratic relations:*

$$\sum_{\substack{0\leq \ell\leq n-1\\ 0\leq i\leq n-\ell-1}} (-1)^{\ell(n+i+1)}\,\psi^{\ell+1}_{i,n-\ell-1-i}\psi^{n-\ell}=0. \tag{5.5}$$

Proof. The proof is similar to the proof of Proposition 6 and is omitted. □

Again, it is easy to prove that

Proposition 9. *Given an* A_∞*-coalgebra* $(A,\psi^n)_{n\geq 1}$*,* $\left(\widetilde{\Omega}A, d_{\widetilde{\Omega}A}\right)$ *is a dga.*

Definition 17. *Let* $(A,\psi^n)_{n\geq 1}$ *be an* A_∞*-coalgebra. The tilde cobar construction on* A *is the dga* $\left(\widetilde{\Omega}A, d_{\widetilde{\Omega}A}\right)$.

Definition 18. *Let* A *and* B *be* A_∞*-coalgebras. A chain map* $g=g^1:A\to B$ *is a map of* A_∞*-coalgebras if there exists a sequence of maps* $\{g^k\in Hom^{k-1}\left(A,B^{\otimes k}\right)\}_{k\geq 2}$ *such that*

$$\widetilde{g}=\textstyle\sum_{n\geq 1}\left(\sum_{k\geq 1}\downarrow^{\otimes k} g^k\uparrow\right)^{\otimes n}:\widetilde{\Omega}A\to\widetilde{\Omega}B, \tag{5.6}$$

is a dga map.

The structure of an A_∞-(co)algebra is encoded by the quadratic relations among its operations (also called "higher homotopies"). Although the "direction," i.e., sign, of these higher homotopies is arbitrary, each choice of directions determines a set of signs in the quadratic relations, the "simplest" of which appears on the algebra side when no changes of direction are made; see (5.1) and (5.3) above. Interestingly, the "simplest" set of signs appear on the coalgebra side when ψ^n is replaced by $(-1)^{[(n-1)/2]}\psi^n$, $n\geq 1$, i.e., the direction of every third and fourth homotopy is reversed. The choices one makes will depend on the application; for us the appropriate choices are as in (5.3) and (5.5).

Let $\mathcal{A}_\infty=\oplus_{n\geq 2}C_*(K_n)$ and let $(A,\varphi^n)_{n\geq 1}$ be an A_∞-algebra with quadratic relations as in (5.3). For each $n\geq 2$, associate $e^{n-2}\in C_{n-2}(K_n)$ with the operation φ^n via

$$e^{n-2}\mapsto(-1)^n\varphi^n \tag{5.7}$$

and each codimension 1 face $d_{(i,\ell)}\left(e^{n-2}\right)\in C_{n-3}(K_n)$ with the quadratic composition

$$d_{(i,\ell)}\left(e^{n-2}\right)\mapsto\varphi^{n-\ell}\varphi^{\ell+1}_{i,n-\ell-1-i}. \tag{5.8}$$

Then (5.7) and (5.8) induce a chain map

$$\zeta_A:\mathcal{A}_\infty\longrightarrow\oplus_{n\geq 2}Hom^*\left(A^{\otimes n},A\right) \tag{5.9}$$

representing the A_∞-algebra structure on A. Dually, if $(A,\psi^n)_{n\geq 1}$ is an A_∞-coalgebra with quadratic relations as in (5.5), the associations

$$e^{n-2}\mapsto\psi^n \text{ and } d_{(i,\ell)}\left(e^{n-2}\right)\mapsto\psi^{\ell+1}_{i,n-\ell-1-i}\psi^{n-\ell}$$

induce a chain map

$$\xi_A:\mathcal{A}_\infty\longrightarrow\oplus_{n\geq 2}Hom^*\left(A,A^{\otimes n}\right) \tag{5.10}$$

representing the A_∞-coalgebra structure on A. The definition of the tensor product is now immediate:

Definition 19. *Let A and B be A_∞-algebras with structural representations ζ_A and ζ_B. The tensor product $A \otimes B$ is the A_∞-algebra whose structural representation $\zeta_{A\otimes B}$ is the sum of the compositions*

$$\begin{array}{ccc} C_*(K_n) & \overset{\zeta_{A\otimes B}}{\longrightarrow} & Hom((A\otimes B)^{\otimes n}, A\otimes B) \\ {}_{\Delta_K}\downarrow & & \uparrow {}_{(\sigma_n)^*} \\ C_*(K_n)\otimes C_*(K_n) & \underset{\zeta_A\otimes\zeta_B}{\longrightarrow} & Hom(A^{\otimes n}, A)\otimes Hom\left(B^{\otimes n}, B\right), \end{array}$$

where $(\sigma_n)^* = Hom(\sigma_n, A\otimes B)$, $n \geq 2$; the operations on $A\otimes B$ are

$$\left\{\zeta_{A\otimes B}\left(e^{n-2}\right) = (\sigma_n)^*\left(\zeta_A\otimes\zeta_B\right)\Delta_K\left(e^{n-2}\right)\right\}_{n\geq 2}.$$

Similarly, let A and B be A_∞-coalgebras with structural representations ξ_A and ξ_B. The tensor product $A\otimes B$ is the A_∞-coalgebra whose structural representation $\xi_{A\otimes B}$ is the sum of the compositions

$$\begin{array}{ccc} C_*(K_n) & \overset{\xi_{A\otimes B}}{\longrightarrow} & Hom(A\otimes B, (A\otimes B)^{\otimes n}) \\ {}_{\Delta_K}\downarrow & & \uparrow {}_{\left(\sigma_n^{-1}\right)_*} \\ C_*(K_n)\otimes C_*(K_n) & \underset{\xi_A\otimes\xi_B}{\longrightarrow} & Hom(A, A^{\otimes n})\otimes Hom\left(B, B^{\otimes n}\right), \end{array}$$

where $\left(\sigma_n^{-1}\right)_* = Hom\left(A\otimes B, \sigma_n^{-1}\right)$, $n\geq 2$; the operations on $A\otimes B$ are

$$\left\{\xi_{A\otimes B}\left(e^{n-2}\right) = \left(\sigma_n^{-1}\right)_*\left(\xi_A\otimes\xi_B\right)\Delta_K\left(e^{n-2}\right)\right\}_{n\geq 2}.$$

Example 9. *Let $(A,\psi^n)_{n\geq 1}$ be an A_∞-coalgebra. The following operations appear on $A\otimes A$:*

$$\begin{aligned} \psi^1_{A\otimes A} = {} & \psi^1\otimes 1 + 1\otimes\psi^1 \\ \psi^2_{A\otimes A} = {} & \sigma_2\left(\psi^2\otimes\psi^2\right), \\ \psi^3_{A\otimes A} = {} & \sigma_3\left(\psi^2_0\psi^2_0\otimes\psi^3 + \psi^3\otimes\psi^2_1\psi^2_0\right), \\ \psi^4_{A\otimes A} = {} & \sigma_4\left(\psi^2_0\psi^2_0\psi^2_0\otimes\psi^4 + \psi^4\otimes\psi^2_2\psi^2_1\psi^2_0 + \psi^3_0\psi^2_0\otimes\psi^2_1\psi^3_0\right. \\ & \left. +\psi^3_0\psi^2_0\otimes\psi^3_1\psi^2_0 + \psi^2_1\psi^3_0\otimes\psi^3_1\psi^2_0 - \psi^2_0\psi^3_0\otimes\psi^2_2\psi^3_0\right) \\ \vdots \qquad & \qquad\qquad \vdots \end{aligned}$$

Whereas Δ_K is homotopy coassociative, the tensor product only iterates up to homotopy. Furthermore, given an arbitrary family of operations $\{\varphi^n \in Hom(A^{\otimes n}, A)\}$, one can form the quadratic compositions in (5.8) and make the same formal identifications with the codimension 1 faces of the associahedra that one would make if they were A_∞-algebra operations. Since Definition 19 makes sense whether or not the quadratic relations in (5.3) hold, we may apply Definition 19 to the φ^n's and obtain iterated operations $\left\{\Phi^n \in Hom\left(\left(A^{\otimes k}\right)^{\otimes n}, A^{\otimes k}\right)\right\}$. The Φ^n's define an

A_∞-algebra structure on $A^{\otimes k}$ whenever $d^2_{\tilde{B}A^{\otimes k}} = 0$. We make extensive use of this fact in the sequel [16].

6. Permutahedral Sets

This section introduces the notion of permutahedral sets, which are combinatorial objects generated by permutahedra and equipped with the appropriate face and degeneracy operators. Naturally occurring examples include the double cobar construction, i.e., the cobar construction on Adams' cobar construction [1] with coassociative coproduct [2], [3], [10] (see Subsection 6.5 below). Permutahedral sets are similar in many ways to simplicial or cubical sets with one crucial difference: Whereas structure relations in simplicial or cubical sets are strictly quadratic, permutahedral sets have higher order structure relations. We note that the exposition on polyhedral sets by D.W. Jones [9] makes no mention of structure relations.

6.1. **Singular Permutahedral Sets.**

To motivate the notion of a permutahedral set, we begin with a construction of our universal example–singular permutahedral sets. For $1 \leq p \leq n$, let $\underline{p} = \{1, \cdots, p\}$ and $\overline{p} = \{n-p+1, \cdots, n\}$, i.e., the first p and last p elements of $\underline{n}$, respectively. Note that $\overline{p} = \{q, \cdots, n\}$ when $p+q = n+1$. Define $\underline{0} = \overline{0} = \varnothing$. If M is a non-empty set, let $\aleph M$ denote its cardinality and define $\aleph\varnothing = 0$. The inductive procedure for labeling the $(n-1)$-faces of P_{n+1} given in Section 2 is conveniently expressed in the following set-theoretic terms:

Face of P_{n+1}	Label
$e^{n-1}_{n,0}$	$\underline{n}\vert n+1$
$e^{n-1}_{n,1}$	$n+1\vert\underline{n}$
$A\vert B \times I_{0,\aleph B}$	$A\vert B \cup \{n+1\}$
$A\vert B \times I_{\aleph B,\infty}$	$A \cup \{n+1\}\vert B.$

For $1 \leq p < n$, let

$$\begin{aligned} \mathcal{Q}_p(n) &= \left\{\text{partitions } A|B \text{ of } \underline{n} \mid \underline{p} \subseteq A \text{ or } \underline{p} \subseteq B\right\} \\ \mathcal{Q}^p(n) &= \left\{\text{partitions } A|B \text{ of } \underline{n} \mid \overline{p} \subseteq A \text{ or } \overline{p} \subseteq B\right\}, \\ \mathcal{Q}^q_p(n) &= \mathcal{Q}_p(n) \cup \mathcal{Q}^q(n), \text{ where } p+q = n+1, \end{aligned}$$

For $1 \leq r \leq n$ and $r+s = n+1$, define canonical projections

$$\Delta_{r,s} : P_n \to P_r \times P_s,$$

mapping each face $A|B \in \mathcal{Q}^s_r(n)$ homeomorphically onto the $(n-2)$-product cell

$$\begin{cases} A \setminus \overline{s-1} \mid B \setminus \overline{s-1} \times \overline{s} & A|B \in \mathcal{Q}^s(n), \\ \underline{r} \times A \setminus \underline{r-1} \mid B \setminus \underline{r-1} & A|B \in \mathcal{Q}_r(n), \end{cases}$$

and each face $A|B \notin \mathcal{Q}^s_r(n)$ onto the $(n-3)$-product cell

$$A \setminus \overline{s-1} \mid B \setminus \overline{s-1} \times A \setminus \underline{r-1} \mid B \setminus \underline{r-1},$$

where $A \setminus \overline{s-1} \mid B \setminus \overline{s-1}$ is a particular partition of $\underline{r}$ and $A \setminus \underline{r-1} \mid B \setminus \underline{r-1}$ is a particular partition of $\overline{s}$ (see Figures 7 and 8).

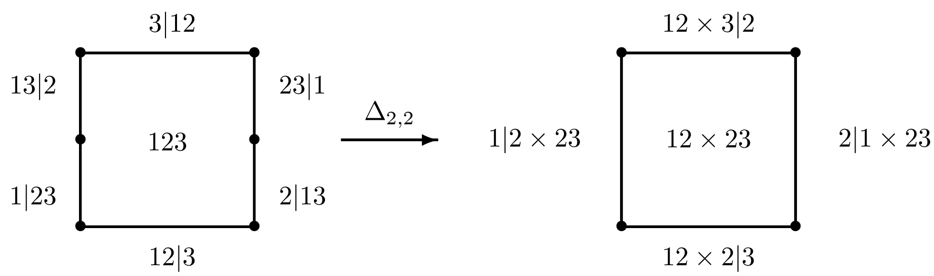


Figure 7: A canonical projection on P_3.

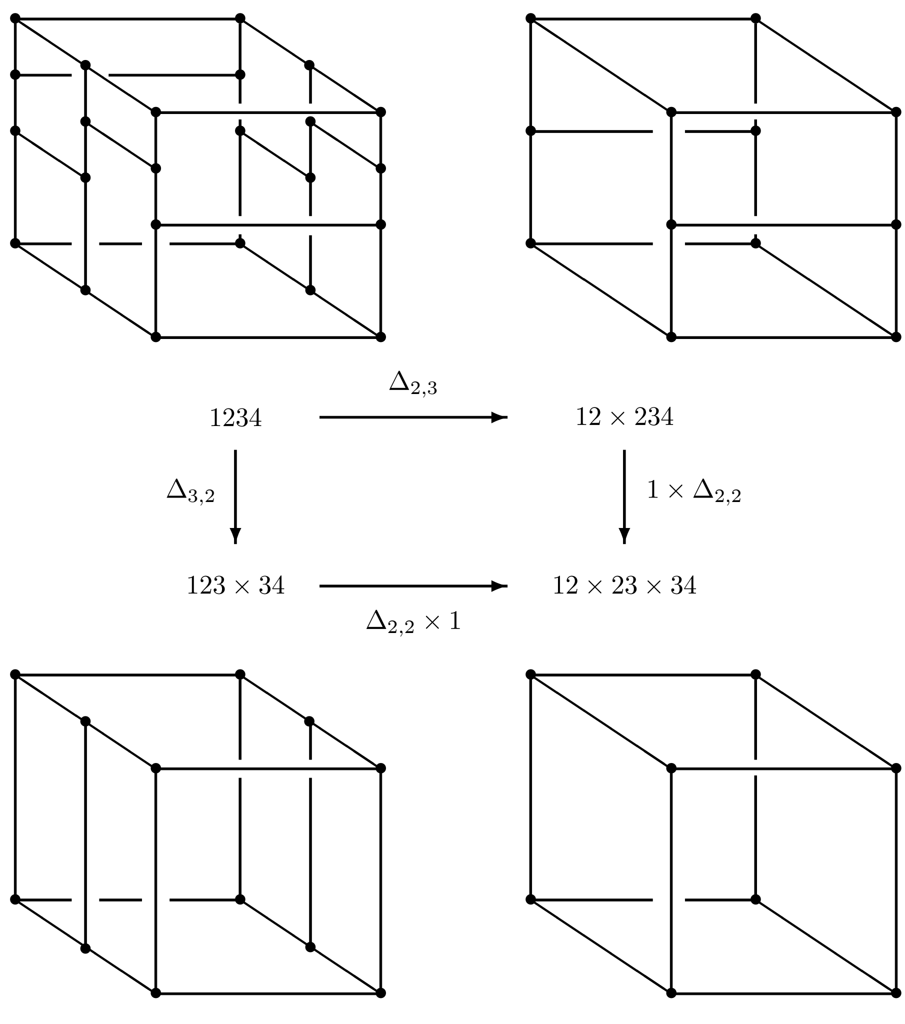


Figure 8: Some canonical projections on P_4.

Given an $(n-1)$-face of $A|B \subset P_{n+1}$, choose a homeomorphism $h_{A|B} : P_r \times P_s \to A|B$. The *singular coface operator associated with* $A|B$

$$\delta_{A|B} : P_n \to P_{n+1}$$

is the composition $h_{A|B} \circ \Delta_{r,s} : P_n \longrightarrow P_r \times P_s \longrightarrow A|B$. Unlike the simplicial or cubical case, $\delta_{A|B}$ is not an inclusion in general. There are two kinds of *singular codegeneracy operators*

$$\alpha_i, \beta_j : P_n \to P_{n-1};$$

α_i is the cellular projection that identifies the faces $\underline{i}|\underline{n}\setminus\underline{i}$ and $\underline{n}\setminus\underline{i}|\underline{i}$, $1 \leq i \leq n-1$; and β_j is the cellular projection that identifies the faces $j|\underline{n}\setminus j$ and $\underline{n}\setminus j|j$, $1 \leq j \leq n$. Note that $\alpha_1 = \beta_1$ and $\alpha_{n-1} = \beta_n$; the projections β_j were first defined by R.J. Milgram in [15] and denoted by D_j.

Definition 20. *Let Y be a topological space. The singular permutahedral set of Y is a tuple $(Sing_*^P Y,\ d_{A|B}, \varrho_i, \varsigma_j)$ in which*

$$Sing_{n+1}^P Y = \{\textit{continuous maps } P_{n+1} \to Y\},\ n \geq 0,$$

singular face operators

$$d_{A|B} : Sing_{n+1}^P Y \to Sing_n^P Y$$

are defined by

$$d_{A|B}(f) = f \circ \delta_{A|B}$$

for each $A|B \subset P_{n+1}$ and singular degeneracy operators

$$\varrho_i, \varsigma_j : Sing_n^P Y \to Sing_{n+1}^P Y$$

are defined by

$$\varrho_i(f) = f \circ \beta_i \textit{ and } \varsigma_j(f) = f \circ \alpha_j$$

for each $1 \leq i \leq n$ and $1 \leq j \leq n-1$.

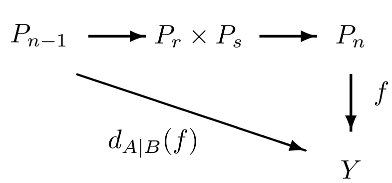


Figure 9: A singular face operator

Although coface operators $\delta_{A|B} : P_n \to P_{n+1}$ need not be inclusions, the top cell of P_n is always non-degenerate under $\delta_{A|B}$ (c.f. Definition 9). However, the top cell may degenerate under compositions of coface operators $\delta_{A|B}\delta_{C|D} : P_{n-1} \to P_{n+1}$. For example, $\delta_{12|34}\delta_{13|2} : P_2 \to P_4$ is a constant map, since $\delta_{12|34} : P_3 \longrightarrow P_2 \times P_2 \longrightarrow P_4$ sends the 1-face $13|2$ to a vertex.

Definition 21. *A quadratic composition of face operators $d_{C|D}d_{A|B}$ acts on P_{n+1} if the top cell of P_{n-1} is non-degenerate under the composition*

$$\delta_{A|B}\delta_{C|D} : P_{n-1} \to P_{n+1}.$$

For comparison, compositions $\partial_i\partial_j$ of face operators in a simplicial set always act on the n-simplex Δ^n and compositions $d_i^\epsilon d_j^\epsilon$ of face operators in a cubical set always act on the n-cube I^n. When $d_{C|D}d_{A|B}$ acts on P_{n+1}, assign the label $d_{C|D}d_{A|B}$ to the $n-2$ face $\delta_{A|B}\delta_{C|D}\,(P_{n-1})$ (see Figure 10).

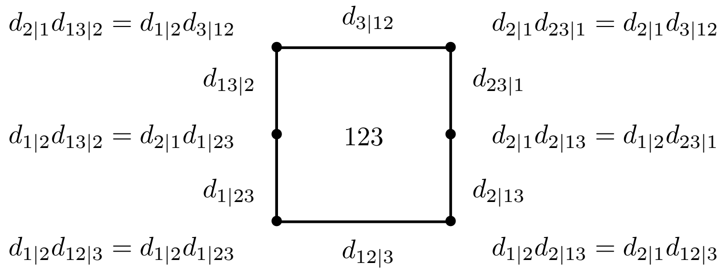


Figure 10: Codimension 2 relations on P_3.

It is interesting to note that singular permutahedral sets have higher order structure relations, an example of which appears in Figure 11 below. This rich structure distinguishes permutahedral sets from simplicial or cubical sets whose structure relations are strictly quadratic.

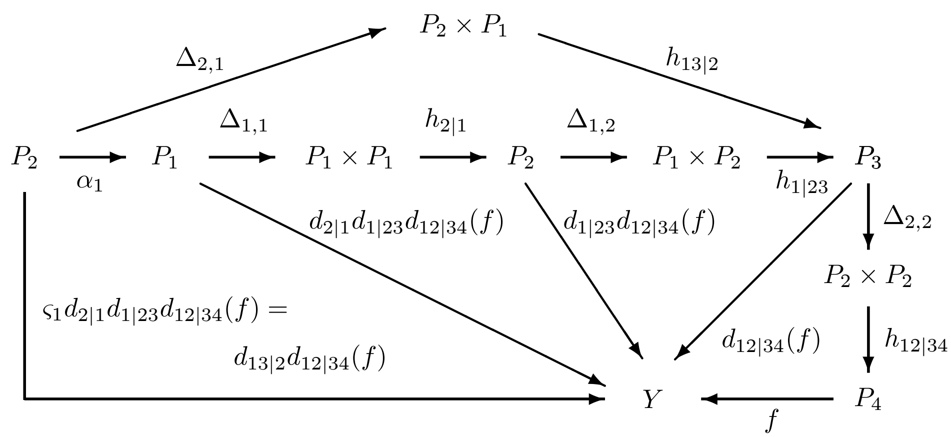


Figure 11: The quartic relation $\varsigma_1 d_{2|1}d_{1|23}d_{12|34} = d_{13|2}d_{12|34}$.

Now $Sing^P Y$ determines the singular (co)homology of Y in the following way: Form the “chain complex” $(C_*(Sing^P Y), d)$ of $Sing^P Y$ with

$$d = \sum_{A|B \in \mathcal{P}_{r,s}(n+1)} (-1)^r sgn(A|B)\ d_{A|B},$$

where $sgn(A|B)$ denotes the sign of the shuffle (A and B are ordered). Note that if $f \in C_*(Sing_4^P Y)$ and $d_{13|2}d_{12|34}\,(f) \neq 0$, the component $d_{13|2}d_{12|34}\,(f)$ of $d^2\,(f) \in C_*(Sing_2^P Y)$ is not cancelled and $d^2 \neq 0$ (see Figure 11). Thus d is not a differential and $(C_*(Sing^P Y), d)$ is not a complex in the classical sense. So form the quotient

$$C_*^{\diamondsuit}(Y) = C_*\left(Sing^P Y\right)/DGN,$$

where DGN is the submodule generated by the degeneracies; then $\left(C_*^{\diamondsuit}(Y), d\right)$ is the complex of singular permutahedral chains on Y. The sequence of canonical cellular projections

$$P_{n+1} \to I^n \to \Delta^n$$

induces a sequence of homomorphisms

$$C_*(SingY) \to C_*(Sing^I Y) \to C_*(Sing^P Y) \to C_*^{\diamondsuit}(Y)$$

whose composition is a chain map that induces a natural isomorphism

$$H_*(Y) \approx H_*^{\diamondsuit}(Y) = H_*(C_*^{\diamondsuit}(Y), d).$$

Although the first two terms in the sequence above are non-normalized chain complexes of singular simplicial and cubical sets, the map between them is not a chain map. In general, a cellular projection between polytopes induces a chain map between corresponding singular complexes if one uses normalized chains in the target. This does not cause a problem here, however, since $df = fd$ and $d^2 = 0$ are independent. Finally, we note that $Sing^P Y$ also determines the singular cohomology ring of Y since the diagonal on the permutahedra and the Alexander-Whitney diagonal on the standard simplex commute with projections.

6.2. **Abstract Permutahedral Sets.**

We begin by identifying each $(n-k)$-face $A_1|\cdots|A_{k+1}$ of P_{n+1} with a composition of face operators $d_{M_{2k-1}|M_{2k}} \cdots d_{M_1|M_2}$ acting on P_{n+1}. First make the identifications $A|B \leftrightarrow d_{A|B}$ (see Figure 10), where $A|B$ ranges over the $(n-1)$-faces of P_{n+1} (see Section 2). Motivated by Definition 21, the conditions under which a quadratic composition $d_{C|D}d_{A|B}$ acts on P_{n+1} can be stated in terms of set operations, which we now define. Given a non-empty ordered set $M = \{m_1 < \cdots < m_k\} \subseteq \mathbb{Z}$, let $I_M : M \to \underline{\aleph M}$ be the index map $m_i \mapsto i$; for $z \in \mathbb{Z}$ let $M + z = \{m_1 + z < \cdots < m_k + z\}$ with the understanding that addition takes preference over set operations. Now think of P_{n+1} as the *ordered* set $\underline{n+1} = \{1 < \cdots < n+1\}$.

Definition 22. *Given non-empty disjoint subsets $A, B \subset \underline{n}$, define the lower and upper disjoint unions*

$$A\underline{\sqcup}B = \begin{cases} I_{\underline{n}\diagdown A}(B) + \aleph A - 1 \cup \underline{\aleph A}, & \text{if } \min B = \min(\underline{n}\diagdown A) \\ I_{\underline{n}\diagdown A}(B) + \aleph A - 1, & \text{if } \min B > \min(\underline{n}\diagdown A); \end{cases}$$

and

$$A\overline{\sqcup}B = \begin{cases} I_{\underline{n}\diagdown B}(A) \cup \overline{\aleph B} - 1, & \text{if } \max A = \max(\underline{n}\diagdown B) \\ I_{\underline{n}\diagdown B}(A), & \text{if } \max A < \max(\underline{n}\diagdown B). \end{cases}$$

If either A or B is empty, define $A\underline{\sqcup}B = A\overline{\sqcup}B = A \cup B$.

In particular, if $A|B$ is a partition of $\underline{n}$, then

$$A\underline{\sqcup}B = A\overline{\sqcup}B = \underline{n-1}.$$

Given a partition $A_1|\cdots|A_{k+1}$ of $\underline{n+1}$, define $A^{(0)} = A^{[k+2]} = \varnothing$; inductively, given $A^{(j)}$, $0 \le j \le k$, let

$$A^{(j+1)} = A^{(j)}\underline{\sqcup}A_{j+1};$$

and given $A^{[j]}$, $2 \leq j \leq k+2$, let

$$A^{[j-1]} = A_{j-1} \sqcap A^{[j]}.$$

And finally, for $1 \leq j \leq k+1$, let

$$A_{(j)} = A_1 \cup \cdots \cup A_j.$$

Now to a given $(n-k)$-face $A_1|\cdots|A_{k+1}$ of P_{n+1}, assign the compositions of face operators

$$\begin{aligned} & d_{A^{(k)}|A^{(k-1)}\sqcup(\underline{n+1}\setminus A_{(k)})} \cdots d_{A^{(1)}|A^{(0)}\sqcup(\underline{n+1}\setminus A_{(1)})} \\ & = d_{A_{(1)}\sqcap A^{[3]}|A^{[2]}} \cdots d_{A_{(k)}\sqcap A^{[k+2]}|A^{[k+1]}} \end{aligned} \tag{6.1}$$

and denote either composition by $d_{A_1|\cdots|A_{k+1}}$.

Note that both sides of relation (6.1) are identical when $k=1$, reflecting the fact that each $(n-1)$-face is a boundary component of exactly one higher dimensional face (the top cell of P_{n+1}). On the other hand, each $(n-2)$-face $A|B|C$ is a boundary component shared by exactly two $(n-1)$-faces. Consequently, $A|B|C$ can be realized as a quadratic composition of face operators in two different ways given by (6.1) with $k=2$:

$$d_{A\sqcup B|A\sqcup C}\ d_{A|B\cup C} = d_{A\sqcap C|B\sqcap C}\ d_{A\cup B|C}. \tag{6.2}$$

Example 10. *In P_8, the 5-face $A|B|C = 12|345|678 = 12|345678 \cap 12345|678$. Since $A\sqcup B = \{1234\}$, $A\sqcup C = \{567\}$, $A\sqcap C = \{12\}$ and $B\sqcap C = \{34567\}$, we obtain the following quadratic relation on $12|345|678$:*

$$d_{1234|567}d_{12|345678} = d_{12|34567}d_{12345|678};$$

similarly, on $345|12|678$ we have

$$d_{1234|567}d_{345|12678} = d_{34567|12}d_{12345|678}.$$

Similar relations on the six vertices of P_3 appear in Figure 10 above.

Given a sequence of (not necessarily distinct) positive integers $\{n_j\}_{1\leq j\leq k}$ such that $n = \sum n_j$, let

$$\mathcal{P}_{n_1,\ldots,n_k}(n) = \{\text{partitions } A_1|\cdots|A_k \text{ of } \underline{n} \mid \aleph A_j = n_j\}.$$

Theorem 3. *Let $A|B \in \mathcal{P}_{p,q}(n+1)$ and $C|D \in \mathcal{P}_{**}(n)$. Then $d_{C|D}d_{A|B}$ denotes an $(n-2)$-face of P_{n+1} if and only if $C|D \in \mathcal{Q}_p^q(n)$.*

Proof. If $d_{C|D}d_{A|B}$ denotes an $(n-2)$-face, say $X|Y|Z$, then according to relation (6.2) we have either

$$A|B = X|\ Y\cup Z \text{ and } C|D = X\sqcup Y|X\sqcup Z$$

or

$$A|B = X\cup Y|Z \text{ and } C|D = X\sqcap Z|\,Y\sqcap Z.$$

Hence there are two cases.

Case 1: $A|B = X|Y\cup Z$. If $\min Y = \min Y\cup Z$, then $\underline{p} \subseteq X\sqcup Y$; otherwise $\min Y\cup Z = \min Z$ and $\underline{p} \subseteq X\sqcup Z$. In either case, $C|D = X\sqcup Y|X\sqcup Z \in \mathcal{Q}_p(n)$.

Case 2: $A|B = X\cup Y|Z$. If $\max X = \max X\cup Y$, then $\overline{q} \subseteq X\sqcap Z$; otherwise $\max(X\cup Y) = \max Y$ and $\overline{q} \subseteq Y\sqcap Z$. In either case, $C|D=$

$X \bar{\sqcup} Z | Y \bar{\sqcup} Z \in \mathcal{Q}^q(n)$.

Conversely, given $A|B \in \mathcal{P}_{p,q}(n+1)$ and $C|D \in \mathcal{Q}_p^q(n)$, let

$$[A|B;\, C|D] = \begin{cases} A|S(C)|S(D), & C|D \in \mathcal{Q}_p(n) \\ \\ T(C)|T(D)|B, & C|D \in \mathcal{Q}^q(n), \end{cases}$$

where

$$S(X) = I_B^{-1}\left(\underline{q} \cap X - p + 1\right) \text{ and } T(X) = I_A^{-1}\left(\underline{p} \cap X\right).$$

A straightforward calculation shows that

$$[X|Y \cup Z;\ X \underline{\sqcup} Y | X \underline{\sqcup} Z] = X|Y|Z = [X \cup Y|Z;\ X \bar{\sqcup} Z | Y \bar{\sqcup} Z].$$

Consequently, if $X|Y|Z = [A|B;\, C|D]$, either

$$A|B = X|\ Y \cup Z \text{ and } C|D = X \underline{\sqcup} Y | X \underline{\sqcup} Z$$

when $C|D \in \mathcal{Q}_p(n)$ or

$$A|B = X \cup Y|Z \text{ and } C|D = X \bar{\sqcup} Z | Y \bar{\sqcup} Z$$

when $C|D \in \mathcal{Q}^q(n)$. □

On the other hand, if $C|D \notin \mathcal{Q}_p^q(n)$, higher order structure relations involving both face and degeneracy operators appear.

We are ready to define the notion of an abstract permutahedral set (c.f. [9]). The relations in an abstract permutahedral set are set-theoretic analogues of those in the singular case. For purposes of applications, only relation (6.3) is essential; the other relations may be assumed modulo degeneracies.

Definition 23. *Let $\mathcal{Z} = \{\mathcal{Z}_{n+1}\}_{n \geq 0}$ be a graded set together with face operators*

$$d_{A|B} : \mathcal{Z}_{n+1} \to \mathcal{Z}_n$$

for each $A|B \in \mathcal{P}_{**}(n+1)$ and degeneracy operators

$$\varrho_i, \varsigma_j : \mathcal{Z}_n \to \mathcal{Z}_{n+1}$$

for each $1 \leq i \leq n+1$, $1 \leq j \leq n$ such that $\varrho_1 = \varsigma_1$ and $\varrho_{n+1} = \varsigma_n$. Then $\left(\mathcal{Z}, d_{A|B}, \varrho_i, \varsigma_j\right)$ is a permutahedral set if the following structure relations hold:

For all $A|B|C \in \mathcal{P}_{***}(n+1)$

$$d_{A \underline{\sqcup} B | A \underline{\sqcup} C}\ d_{A|B \cup C} = d_{A \bar{\sqcup} C | B \bar{\sqcup} C}\ d_{A \cup B|C}. \tag{6.3}$$

For all $A|B \in \mathcal{P}_{r,s}(n+1)$ and $C|D \in \mathcal{P}_{**}(n) \setminus \mathcal{Q}_r^s(n)$

$$d_{C|D} d_{A|B} = \varsigma_j d_{M|N} d_{K|L} d_{A|B} \quad \textit{where} \tag{6.4}$$

$$\textit{either} \quad \begin{cases} K|L = \underline{n} \diagdown (\underline{r} \cap D) | \underline{r} \cap D, \\ M|N = C \bar{\sqcup} (\underline{r} \cap D) | (D \diagdown (\underline{r} \cap D)) \bar{\sqcup} (\underline{r} \cap D), \\ j = \aleph C + \aleph(\underline{r} \cap D) - 1 \ \textit{when } r \in C \\ \qquad \textit{or} \\ K|L = \underline{r} \cap C | \underline{n} \diagdown (\underline{r} \cap C), \\ M|N = (\underline{r} \cap C) \underline{\sqcup} (C \diagdown (\underline{r} \cap C)) | (\underline{r} \cap C) \underline{\sqcup} D, \\ j = \aleph(\underline{r} \cap C) \ \textit{when } r \in D. \end{cases}$$

For all $A|B \in \mathcal{P}_{**}(n+1)$ *and* $1 < j < n$ *(for* $j = 1, n$ *see (6.6) below)*

(6.5)

$$d_{A|B}\varsigma_j = \begin{cases} 1, & \text{if } A = \underline{j} \text{ or } B = \underline{j}, \\ \varsigma_j d_{\underline{j}|\underline{n}\setminus\underline{j}}, & \text{if } A|B \in \mathcal{Q}_j(n+1),\ A \neq \underline{j} \text{ or } B \neq \underline{j}, \\ \varsigma_{j-1} d_{\underline{j-1}|\underline{n}\setminus\underline{j-1}}, & \text{if } A|B \in \mathcal{Q}^{n+1-j}(n+1),\ A \neq \underline{j} \text{ or } B \neq \underline{j}, \\ \varsigma_j\varsigma_j d_{M|N} d_{K|L}, & \text{if } A|B \notin \mathcal{Q}_j^{n+1-j}(n+1) \text{ where} \end{cases}$$

$$\text{either} \begin{cases} K|L = A \boxdot (\underline{j} \cap B) \mid [B \setminus (\underline{j} \cap B)] \boxdot (\underline{j} \cap B), \\ M|N = \overline{\aleph(\underline{j} \cap A)} | \underline{n-1} \setminus \overline{\aleph(\underline{j} \cap A)}, \\ \text{when } j \in A \\ \qquad \text{or} \\ K|L = (\underline{j} \cap A) \sqcup (\underline{j} \cap B) \mid (\underline{j} \cap A) \sqcup B, \\ M|N = \underline{j-1} | \underline{n-1} \setminus \underline{j-1}, \\ \text{when } j \in B. \end{cases}$$

For all $A|B \in \mathcal{P}_{**}(n+1)$ *and* $1 \leq i \leq n+1$

$$d_{A|B}\varrho_i = \begin{cases} 1, & \text{if } A = \{i\} \text{ or } B = \{i\}, \\ \varrho_j d_{C|D}, & \text{where} \end{cases} \tag{6.6}$$

$$\text{either} \begin{cases} C|D = I_{\underline{n+2}\setminus i}(A \setminus i) | I_{\underline{n+2}\setminus i}(B), \\ j = I_A(i) \text{ when } \{i\} \subsetneqq A \\ \qquad \text{or} \\ C|D = I_{\underline{n+2}\setminus i}(A) | I_{\underline{n+2}\setminus i}(B \setminus i), \\ j = I_B(i) + \aleph A \text{ when } \{i\} \subsetneqq B. \end{cases}$$

For all $i \leq j$

$$\begin{aligned} \varrho_i\varrho_j &= \varrho_{j+1}\varrho_i, \\ \varsigma_i\varsigma_j &= \varsigma_{j+1}\varsigma_i, \\ \varsigma_i\varrho_j &= \varrho_{j+1}\varsigma_i, \\ \varrho_i\varsigma_j &= \varsigma_{j+1}\varrho_i. \end{aligned} \tag{6.7}$$

6.3. **The Cartesian product of permutahedral sets.**

Let $\mathcal{Z}' = \{\mathcal{Z}'_r, d'_{A|B}, \varsigma'_i, \varrho'_j\}$ and $\mathcal{Z}'' = \{\mathcal{Z}''_s, d''_{A|B}, \varsigma''_i, \varrho''_j\}$ be permutahedral sets and let

$$\mathcal{Z}' \times \mathcal{Z}'' = \left\{ (\mathcal{Z}' \times \mathcal{Z}'')_n = \bigcup_{r+s=n+1} \mathcal{Z}'_r \times \mathcal{Z}''_s \right\}_{n \geq 1} \Big/ \sim,$$

where $(a, b) \sim (c, d)$ if and only if $a = \varsigma'_r(c)$ and $d = \varsigma''_1(b)$, i.e.,

$$(\varsigma'_r(c), b) = (c, \varsigma''_1(b)) \text{ for all } (c, b) \in \mathcal{Z}'_r \times \mathcal{Z}''_s.$$

Definition 24. *The product of* $\mathcal{Z}'$ *and* $\mathcal{Z}''$*, denoted by* $\mathcal{Z}' \times \mathcal{Z}''$*, is the permutahedral set*

$$\left\{ \mathcal{Z}' \times \mathcal{Z}'', d_{A|B}, \varsigma_i, \varrho_j \right\}$$

with face and degeneracy operators defined by

$$d_{A|B}(a, b) = \begin{cases} \left(d'_{\underline{r} \cap A | \underline{r} \cap B}(a), b\right), & \text{if } A|B \in \mathcal{Q}^s(n), \\ \left(a, d''_{\underline{s} \cap (A-n+s) | \underline{s} \cap (B-n+s)}(b)\right), & \text{if } A|B \in \mathcal{Q}_r(n), \\ \varsigma_i d_{M|N} d_{K|L}(a, b), & \text{otherwise, where} \end{cases} \tag{6.8}$$

$$\textit{either}\left\{\begin{array}{l} K|L = \underline{r}\cap A|\,(\underline{r}\cap B)\cup \underline{s-1}+r \\ M|N = (\underline{r}\cap A)\sqcup(B\diagdown(\underline{r}\cap B))\,|\,(\underline{r}\cap A)\sqcup B \\ i = \aleph(\underline{r}\cap A) \;\; \textit{when}\ r\in B, \\ \qquad\quad \textit{or} \\ K|L = A\cup(B\diagdown(\underline{r}\cap B))\,|\underline{r}\cap B \\ M|N = A\bar{\sqcup}(\underline{r}\cap B)\,|\,(B\diagdown(\underline{r}\cap B))\,\bar{\sqcup}\,(\underline{r}\cap B) \\ i = \aleph A+\aleph(\underline{r}\cap B)-1 \;\; \textit{when}\ r\in A; \end{array}\right.$$

$$\varsigma_i(a,b) = \begin{cases} (\varsigma_i'(a),b), & 1\leq i<r, \\ \left(a,\varsigma_{i-r+1}''(b)\right), & r\leq i\leq n; \end{cases} \tag{6.9}$$

$$\varrho_j(a,b) = \begin{cases} \left(\varrho_j'(a),b\right), & 1\leq j\leq r, \\ \left(a,\varrho_{j-r+1}''(b)\right), & r<j\leq n+1. \end{cases} \tag{6.10}$$

Remark 1. *Note that the right-hand side of the third equality in (6.8) reduces to the first two; indeed, if $r\in B$, then $K|L\in\mathcal{Q}^s(n)$ and $M|N\in\mathcal{Q}_r(n)$; if $r\in A$, $K|L\in\mathcal{Q}^s(n)$ and $M|N\in\mathcal{Q}_r(n)$ if $\underline{m_2}+r-1\subset A\setminus(\underline{r-1}\cap A)$, $m_2=\aleph(\underline{r}\cap B)$, while for $\underline{m_2}+r-1\not\subset A\setminus(\underline{r-1}\cap A)$ one has $K|L\in\mathcal{Q}^s(n)$, $M|N\notin\mathcal{Q}_r(n)$ and $r-1\in L$.*

Example 11. *The canonical map $\iota: Sing^PX\times Sing^PY\to Sing^P(X\times Y)$ defined for $(f,g)\in Sing_r^PX\times Sing_s^PY$ by*

$$\iota(f,g)=(f\times g)\circ\Delta_{r,s}$$

is a map of permutahedral sets. Consequently, if X is a topological monoid, the singular permutahedral complex $Sing^PX$ inherits a canonical monoidal structure.

6.4. **The diagonal on a permutahedral set.**

Let $\mathcal{Z}=(\mathcal{Z}_{n+1},d_{A|B},\varsigma_i,\varrho_i)$ be a permutahedral set. The explicit diagonal

$$\Delta: C_*(\mathcal{Z})\to C_*(\mathcal{Z})\otimes C_*(\mathcal{Z})$$

defined for $a\subset\mathcal{Z}_{n+1}$ is given by the formula

$$\Delta(a)=\sum_{\substack{u\otimes v\in\Lambda_{p,q}\\ p+q=n+2}}\epsilon\cdot d_u(a)\otimes d_v(a), \tag{6.11}$$

where the sign ϵ is defined as follows: For an $(n-m)$-face $u=A_1|\cdots|A_{m+1}\subset P_{n+1}$, define

$$sgn\,u=\prod_{1\leq i\leq m}(-1)^{\aleph A^{(i)}}sgn(A^{(i)}|A^{(i-1)}\sqcup(\underline{n+1}\setminus A_i)).$$

If $u\otimes v$ is a complementary pair, consider the related strong complementary pair $u_x\otimes v_x$, where x is the vertex of P_{n+1} defined by $\max u_x=x=\min v_x$. The cellular projection $\phi: P_{n+1}\to K_{n+2}\to I^n$, i.e., Tonks' projection followed by the "healing map," preserves diagonals in the following way: For a vertex $x\in P_{n+1}$, define a function $g:\{\text{vertices of } P_{n+1}\}\to\{\text{vertices of } P_{n+1} \text{ that coincide with those of } I^n\}$ by

$$g(x)=\min I^{n(x)},$$

where $I^{n(x)} \subset I^n$ denotes the minimal $n(x)$-dimensional face of I^n containing x, i.e., if $x = a_1|\cdots|a_{n+1}$ then $g(x)$ can be obtained from x as the composition $g_n \cdots g_2(x)$ where

$$g_j(b_1|\cdots|b_{n+1}) = \begin{cases} b_1|\cdots|\hat{b}_{n+2-j}|\cdots|b_{i+1}|b_{n+2-j}|b_i|\cdots|b_{n+1}, & \\ \qquad \text{if } b_{n+1-j}, b_{n+3-j}, \ldots, b_{i+1} < b_{n+2-j} < b_i \\ 1, \qquad \text{otherwise.} \end{cases}$$

Assign ϵ_0 to the vertex y of P_{n+1} that coincides with the vertex of I^n determined by

$$\epsilon_0(y) = sgn(u_y^0) \cdot sgn\big((\phi(u_y^0), \phi(v_y)\big) \cdot sgn(v_y),$$

where $u_y^0 \otimes v_y$ is a CP related to the SCP $u_y \otimes v_y$ at vertex y, i.e., $\max u_y = y = \min v_y$, $u_y^0 = R_{M_{p-1}} \cdots R_{M_1}(u_y)$ where each M_j has maximal possible cardinality, and $sgn(\phi(u_y^0), \phi(v_y))$ is the sign of the shuffle, i.e., the sign of the orthogonal pair $\phi(u_y^0) \otimes \phi(v_y)$ in the Serre diagonal on the cube. Define

$$\epsilon = sgn\ u \cdot \epsilon_0(g(x)) \cdot sgn\ v. \tag{6.12}$$

Remark 2. *The components of* $(\phi(u_y^0), \phi(v_y))$ *are non-degenerate under Tonks' projection* $\theta : P_{n+1} \to K_{n+2}$. *Furthermore, the sign* ϵ *defined above is compatible with the Serre diagonal on* I^n *since the components of* $(u, v) = (u_y^0, v_y)$ *are non-degenerate under* ϕ; *thus* $\epsilon = sgn(u_y^0) \cdot sgn(u_y^0) \cdot sgn\big((\phi(u_y^0), \phi(v_y)\big) \cdot sgn(v_y) \cdot sgn(v_y) = sgn\big((\phi(u_y^0), \phi(v_y)\big)$. *Finally, while the sign* ϵ *for* Δ_P *is geometrically motivated, we leave the proof of this fact to the interested reader.*

Remark 3. *The composition of cellular maps*

$$(1 \times \Delta_{2,2}) \cdots (1 \times \Delta_{2,n-1})\Delta_{2,n} : P_{n+1} \to I^n$$

is compatible with diagonals but does not factor through K_{n+2}; *hence this composition differs from* ϕ.

6.5. **The double cobar-construction $\Omega^2 C_*(X)$.**

For a simplicial, cubical or a permutahedral set W, let $C_*(W)$ denote the quotient $C_*(W)/C_{>0}(*)$, where $*$ is a base point of W, and recall that a simplicial or cubical set W is said to be *k-reduced* if W has exactly one element in each dimension $\leq k$. Let ΩC denote the cobar construction on a 1-reduced dg coalgebra C. In [10] and [11], Kadeishvili and Saneblidze construct a functor from the category of 1-reduced simplicial sets to the category of cubical sets and a functor from the category of 1-reduced cubical sets to the category of permutahedral sets for which the following statements hold (c.f. [3], [2]):

Theorem 4. [10] *Given a 1-reduced simplicial set* X, *there is a canonical identification isomorphism*

$$\Omega C_*(X) \approx C_*^{\square}(\mathbf{\Omega} X).$$

Theorem 5. [11] *Given a 1-reduced cubical set* Q, *there is a canonical identification isomorphism*

$$\Omega C_*^{\square}(Q) \approx C_*^{\diamondsuit}(\mathbf{\Omega} Q).$$

For completeness, definitions of these two functors appear in the appendix.

Since the chain complex of any cubical set Q is a dg coalgebra with strict coassociative coproduct, setting $Q = \mathbf{\Omega} X$ in the second theorem immediately gives:

Theorem 6. *For a 2-reduced simplicial set X there is a canonical identification isomorphism*

$$\Omega^2 C_*(X) \approx C_*^{\diamondsuit}(\mathbf{\Omega^2} X).$$

If $X = Sing^1 Y$, then $\Omega C_*(X)$ is Adams' cobar construction for the space Y [1]. Consequently, there is a canonical (geometric) coproduct on $\Omega^2 C_*(Sing^1 Y)$. In the sequel [16] we show that this coproduct extends to an A_∞-Hopf algebra structure.

7. Appendix

For completeness, we review the definition of the functor from the category of 1-reduced simplicial sets to the category of cubical sets and the definition of the functor from the category of 1-reduced cubical sets to the category of permutahedral sets given by Kadeishvili and Saneblidze in [10], [11].

7.1. **The cubical set functor $\mathbf{\Omega} X$.** Given a 1-reduced simplicial set $X = \{X_n, \partial_i, s_i\}_{n\geq 0}$, define the graded set $\mathbf{\Omega} X$ as follows: Let X^c be the graded set of formal expressions

$$X^c_{n+k} = \{\eta_{i_k} \cdots \eta_{i_1}\eta_{i_0}(x) \mid x \in X_n\}_{n\geq 0; k\geq 0},$$

where

$$\eta_{i_0} = 1, i_1 \leq \cdots \leq i_k,\ 1 \leq i_j \leq n+j-1,\ 1 \leq j \leq k,$$

and let $\bar{X}^c = s^{-1}(X^c_{>0})$ be the desuspension of X^c. Define $\Omega' X$ to be the free graded monoid generated by $\bar{X}^c$ and denote elements of $\Omega' X$ by $\bar{x}_1 \cdots \bar{x}_k$, where $x_j \in X_j,\ 1 \leq j \leq k$. Define the (strictly associative) product of two elements $\bar{x}_1 \cdots \bar{x}_k$ and $\bar{y}_1 \cdots \bar{y}_\ell$ by concatenation $\bar{x}_1 \cdots \bar{x}_k \bar{y}_1 \cdots \bar{y}_\ell$ (there are no other relations whatsoever between these expressions). The total degree of an element $\bar{x}_1 \cdots \bar{x}_k$ is the sum $m_{(k)} = m_1 + \cdots + m_k$, where $m_j = |\bar{x}_j|$, and we write $\bar{x}_1 \cdots \bar{x}_k \in (\Omega' X)_{m_{(k)}}$. Let $\mathbf{\Omega} X$ be the monoid obtained from $\Omega' X$ via

$$\mathbf{\Omega} X = \Omega' X / \sim\ ,$$

where $\overline{\eta_{p+1}(x)} \cdot \bar{y} \sim \bar{x} \cdot \overline{\eta_1(y)}$ for $x, y \in X^c$, $|x| = p+1$, and $\eta_n(\bar{x}) \sim \overline{s_n(x)}$ for $x \in X_{>0}$. Clearly, there is an inclusion of graded modules $MX \subset \Omega' X$, where MX denotes the free monoid generated by $\bar{X} = s^{-1}(X_{>0})$.

Apparently $\Omega' X$ canonically admits the structure of a cubical set. Denote the components of Alexander-Whitney diagonal by

$$\nu_i : X_n \to X_i \times X_{n-i},$$

where $\nu_i(x) = \partial_{i+1} \cdots \partial_n(x) \times \partial_0 \cdots \partial_{i-1}(x),\ 0 \leq i \leq n$, and let x^n denote an n-simplex simplex, i.e. $x^n \in X_n$. Then

$$\nu_i(x^n) = ((x')^i,\ (x'')^{n-i}) \in X_i \times X_{n-i}$$

for all $n > 0$. Define face operators $d_i^0, d_i^1 : (\mathbf{\Omega} X)_{n-1} \to (\mathbf{\Omega} X)_{n-2}$ on a (monoidal) generator $\overline{x^n} \in (\bar{X})_{n-1} \subset (\overline{X^c})_{n-1}$ by

$$d_i^0\,(\overline{x^n}) = \overline{(x')^i} \cdot \overline{(x'')^{n-i}}, \quad 1 \leq i \leq n-1$$

$$d_i^1\,(\overline{x^n}) = \overline{\partial_i\,(x^n)}, \qquad 1 \leq i \leq n-1;$$

and extend to elements $\bar{x}_1 \cdots \bar{x}_k \in MX$ by

$$d_i^0\left(\overline{x}_1 \cdots \overline{x}_k\right) = \overline{x}_1 \cdots \overline{(x_q')^{j_q}} \cdot \overline{(x_q'')^{m_q - j_q + 1}} \cdots \overline{x}_k,$$

$$d_i^1\left(\overline{x}_1 \cdots \overline{x}_k\right) = \overline{x}_1 \cdots \overline{\partial_{j_q}\left(x_q\right)} \cdots \overline{x}_k,$$

where $m_{(q-1)} < i \leq m_{(q)},\ j_q = i - m_{(q-1)},\ 1 \leq q \leq k,\ 1 \leq i \leq n-1$. Then the defining identities for a cubical set that involve d_i^0 and d_i^1 can easily be checked on MX. In particular, the simplicial relations between the ∂_i's imply the cubical relations between d_i^1's; the associativity relations between ν_i's imply the cubical relations between d_i^0's, and the commutativity relations between ∂_i's and ν_j's imply the cubical relations between d_i^1's and d_j^0's. Next, define a degeneracy operator $\eta_i : (\mathbf{\Omega} X)_{n-1} \to (\mathbf{\Omega} X)_n$ on a (monoidal) generator $\overline{x} \in (\overline{X^c})_{n-1}$ by

$$\eta_i(\overline{x}) = \overline{\eta_i(x)};$$

and extend to elements $\bar{x}_1 \cdots \bar{x}_k \in \mathbf{\Omega} X$ by

$$\eta_i\left(\overline{x}_1 \cdots \overline{x}_k\right) = \overline{x}_1 \cdots \eta_{j_q}\left(\overline{x_q}\right) \cdots \overline{x}_k,$$

$$\eta_n\left(\overline{x}_1 \cdots \overline{x}_k\right) = \overline{x}_1 \cdots \overline{x}_{m_{k-1}} \cdot \eta_{m_k+1}\left(\overline{x}_k\right),$$

where $m_{(q-1)} < i \leq m_{(q)},\ j_q = i - m_{(q-1)},\ 1 \leq q \leq k,\ 1 \leq i \leq n-1$. Finally, inductively extend the face operators on these degenerate elements so that the defining identities of a cubical set are satisfied. Then in particular, we have the following identities for all $x^n \in X_n$:

$$d_1^0\left(\overline{x^n}\right) = \overline{(x')^1} \cdot \overline{(x'')^{n-1}} = e \cdot \overline{(x'')^{n-1}} = \overline{(x'')^{n-1}} = \overline{\partial_0\left(x^n\right)},$$

$$d_{n-1}^0\left(\overline{x^n}\right) = \overline{(x')^{n-1}} \cdot \overline{(x'')^1} = \overline{(x'')^{n-1}} \cdot e = \overline{(x')^{n-1}} = \overline{\partial_n\left(x^n\right)}.$$

It is easy to see that the cubical set $\{\mathbf{\Omega} X, d_i^0, d_i^1, \eta_i\}$ depends functorially on X.

7.2. **The permutahedral set functor $\mathbf{\Omega} Q$.**

Let $Q = (Q_n, d_i^0, d_i^1, \eta_i)_{n \geq 0}$ be a 1-reduced cubical set. Recall that the diagonal

$$\Delta : C_*(Q) \to C_*(Q) \otimes C_*(Q)$$

on $C_*(Q)$ is defined on $a \in Q_n$ by

$$\Delta(a) = \Sigma\ (-1)^{\epsilon} d_B^0(a) \otimes d_A^1(a),$$

where $d_B^0 = d_{j_1}^0 \cdots d_{j_q}^0$, $d_A^1 = d_{i_1}^1 \cdots d_{i_p}^1$, summation is over all shuffles $\{A, B\} = \{i_1 < \cdots < i_q, j_1 < \cdots < j_p\}$ of $\underline{n}$ and ϵ is the sign of the shuffle. The primitive components of the diagonal are given by the extreme cases $A = \varnothing$ and $B = \varnothing$.

Define the graded set $\mathbf{\Omega} Q$ as follows: Let Q_*^c be the graded set of formal expressions

$$Q_{n+k}^c = \{\varsigma_{i_k} \cdots \varsigma_{i_1} \varsigma_{i_0}(a) |\, a \in Q_n\}_{n \geq 0; k \geq 0},$$

where

$$\varsigma_{i_0}=1,\, i_1\leq\cdots\leq i_k,\, 1\leq i_j\leq n+j-1,\, 1\leq j\leq k,$$

and let $\bar{Q}^c = s^{-1}(Q^c_{>0})$ denote the desuspension of Q^c. Define $\mathbf{\Omega}'Q$ to be the free graded monoid generated by $\bar{Q}^c$. Let $\mathbf{\Omega}Q$ be the monoid obtained from $\mathbf{\Omega}'Q$ via

$$\mathbf{\Omega}Q = \mathbf{\Omega}'Q/\sim\ ,$$

where $\overline{\varsigma_{p+1}(a)}\cdot\bar{b}\sim\bar{a}\cdot\overline{\varsigma_1(b)}$ for $a,b\in Q^c$, $|a|=p+1$. Clearly, there is an inclusion of graded monoids $MQ\subset\mathbf{\Omega}Q$, where MQ denotes the free monoid generated by $\bar{Q}=s^{-1}(Q_{>0})$.

Then $\mathbf{\Omega}Q$ is canonically a permutahedral set in the following way: First, for a monoidal generator $\bar{a}\in\bar{Q}$, define the degeneracy operator ς_i by $\varsigma_i(\bar{a})=\overline{\varsigma_i(a)}$; next, for $\bar{a}\in\bar{Q}\subset\bar{Q}^c$ define $\varrho_j(\bar{a})=\overline{\eta_j(a)}$; and finally, if $\bar{a}$ is any other element of $\bar{Q}^c$ define its degeneracy accordingly to (6.7). Use formulas (6.9) and (6.10) to extend both degeneracy operators on decomposables. Now for $\bar{a}\in\bar{Q}_{n+1}\subset\bar{Q}^c{}_{n+1}$, define the face operator $d_{A|B}$ by

$$d_{A|B}(\bar{a})=\overline{d^0_B(a)}\cdot\overline{d^1_A(a)},\quad A|B\in\mathcal{P}_{*,*}(n+1),$$

while for other elements of $\bar{Q}^c$ and for decomposables in $\mathbf{\Omega}'Q$ use formulas (6.4)-(6.6) and (6.8) to define $d_{A|B}$ by induction on grading. In particular, we have the following identities:

$$d_{i|\underline{n+1}\backslash i}\,(\overline{a})=\overline{d^1_i(a)},\quad 1\leq i\leq n,$$

$$d_{\underline{n+1}\backslash i|i}\,(\overline{a})=\overline{d^0_i(a)},\quad 1\leq i\leq n.$$

It is easy to see that $(\mathbf{\Omega}Q,\, d_{A|B},\, \varsigma_i,\, \varrho_j)$ a permutahedral set, which depends functorially on Q.

Remark 4. *Note that the definition of $\mathbf{\Omega}Q$ uses all cubical degeneracies. This is justified geometrically by the fact that a degenerate singular n-cube in the base of a path space fibration lifts to a singular $(n-1)$-permutahedron in the fibre, which is degenerate with respect to Milgram's projections. We must formally adjoin the other degeneracies to achieve relations (6.4) (c.f., the definition of the cubical set $\mathbf{\Omega}X$ on a simplicial set X).*

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